PAPER - 02

FULL ANALYSIS

MATHEMATICS OPTIONAL

UPSC CSE 2026

Mindset Makers
U.P.S.C. I.I.T.
Prepare in Right Way

By Upendra Singh Sir

UPSC CSE 2026: Mathematics Optional PAPER-02

Observations of 2026 mathematics Optional papers UPSC CSE

#hard attempts by UPSC to change the trend of maths optional of years. **Not tough questions*: but the way.*

#Earlier hardly any one or two maximum questions used to be of nature like open ended. But this year it happened with almost 70% questions and in both papers.

As a person indulged into higher mathematics since a long, I'm sure that paper setting done by professors who are having clarity of concepts and depth of every topic.

#Resulting as: No question was tough but needed a conceptual understanding. If you don't have that conceptual clarity: you'll get panicked and starts moving another to another question in a hope that you'll get question in your zone and as I said it was done with every question in one or two sub parts; so students found it very difficult. Many lost their temperament.

As I've been consistently trying to read patterns by UPSC; yes they do it randomly with some optional. The year 2025 papers were easiest and it was well settled news in air that maths optional is easier to do now compared to other Optionals. UPSC body observes everything. So they played this year. It's not only with maths but it happens with every optional in some years.

#above point is very crucial: Because it might give you a switching to other optional and what's the guarantee that the next year's new optional will not be on same cycle of gambling optionals. Yes it happens.

#Point: we all know UPSC luck factors: let's say you started preparing in 2024 and you got very good in maths in 2025. A person who started preparing in 2019 in increasing difficulty level of maths trend till 2023, so he/she might be trapped but he/also done very good in 2024/25 because of not changing the optional but being in cycle and finally got.
So changing optional just because of trend is not a good idea.

#Analysis of questions:
I am sharing my side here -
No need to change the resources but change the mindset. In place of focusing just on a large of questions solving from n number of different books;

you need to be conceptually clear and then attempting problems of different categories from one systematically designed resource/book/teacher. You may get mindset makers books at your address from here https://mindsetmakers.in/upsc-study-material/;

order and get at your home SYSTEMATICALLY DESIGNED BOOKS theory+examples+PYQs(with answers chapter wise)+short revision document https://mindsetmakers.in/upsc-study-material/

Track record; when first time in year 2024, upsc asked about Existence and Uniqueness of IVPs of first order, it was already given in mindset makers books and this year as well the kind of conceptual clarity is focused in teaching methodology, given the confidence.

I'm sure that if you follow my words, you'll get what you are required. The way I try to decode the concept and if students follow properly their home work part, I assure, even in this year's paper pattern, you'll be in situation to try problems of such nature and yes it may not be the case that you get same problem but the attitude towards any open problem after conceptual clarity with categorical problems will give you a comfortable situation and you won't be panicked.

Also; it's an alarming situation for me as a teacher that yes coaching/batch/organisational SoPs have their timelines to complete the syllabus but I'll be consistently forcing students to complete their tasks which I give after the class. Earlier I used to be like okay it's students choice if they want to go for books problems or not but not I'll be forcefully implementing it with serious students. Because I'm getting confidence about my teaching methodology year by year.

Suggestions are welcome.
Let's learn together and perform better

keep learning keep smiling
Upendra Singh

Initiative: For those who've done the syllabus/written multiple main/s CSE/IFoS

Problem Solving Techniques
UPSC CSE IFoS 2027/28
Session-1
Mathematics Optional
YouTube live Saturday 7-11 pm at See teaching methodology https://www.youtube.com/@MindsetMakersPrepareInRightWay

Let's develop mathematical skills to hit new problems based on conceptual clarity.

I've been working over this plan since a long and wanted to implement only when it's in concrete shape. So now on every Saturday; we can plan such sessions.

UPSC CSE MAIN EXAMINATION 2026

MATHEMATICS OPTIONAL PAPER-02

Section A

Q1. (a) Let G be a group of order pq, where p and q are primes. Show that either G is an abelian group or no non-identity elements commutes with every element of G .

(b) Show that every non-zero prime ideal in a Euclidean Domain is maximal.

(c) Find the points of discontinuity of the function $f:(0,1)\to\mathbb{R}$ defined by

$$f(x)=\lim_{n\to\infty}\frac{\left(1+\sin\frac{\pi}{x}\right)^n-1}{\left(1+\sin\frac{\pi}{x}\right)^n+1},x\in(0,1)$$

(d) Show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin, although Cauchy-Riemann equation are satisfied at the point.

(e) Solve the dual problem of the following linear programming problem by the graphical method:

Maximize $Y:10y_1+6y_2+2y_3+y_4$

Subject to the constraints:

$$\begin{aligned} y_1+y_2+y_3+y_4 &\ge 2\\ 2y_1+y_2-y_3-2y_4 &\ge 1\\ y_1,y_2,y_3,y_4 &\ge 0 \end{aligned}$$

Q2. (a) Let G be a finite group generated by two elements u and v of order 2 each. Show that G is isomorphic to the dihedral group of order $2n$, where $O(uv)=n$.

(b) Prove that the sequence (u_n) defined by the recursion formula $u_{n+1}=\sqrt{7+u_n},u_1=\sqrt{7}$, convergence to the positive root of $x^2-x-7=0$.

(c) Use the method of contour integration to prove:

$$\int_0^{2\pi} \frac{\sin^2\theta}{(a+b\cos\theta)} d\theta = \frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}; \quad a>b>0$$

Q3. (a) Prove that $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$, where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)\, d\theta \text{, for } n = 0,1,2,\ldots$$

(b) A function f is defined on $[0,1]$ by $f(x) = \begin{cases} x^2, & \textit{when } x \textit{ is rational} \\ x^3, & \textit{when } x \textit{ is irrational} \end{cases}$

Evaluate lower integral $\int_0^1 f$ and upper integer $\int_0^1 f$. Does the integral $\int_0^1 f$ exist?

(c) The following table represents the estimated costs of assigning jobs to machines. Solve this assignment problem to minimize the total cost.

Machine

Job	M_1	M_2	M_3	M_4
J_1	8	24	28	32
J_2	8	13	17	19
J_3	8	15	19	22

Is the assignment unique for the total minimum cost? Justify your answer and obtain an alternate assignment, if it exists.

Q4(a) Let n be a positive integer. Show that the following statements are equivalent:

(i) $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.

(ii) $\mathbb{Z}/n\mathbb{Z}$ is a field.

(iii) n is prime.

(b) Prove that the improper integral $\int_0^{\infty} \frac{1}{1+x^2\sin^2 x}dx$ is divergent.

(c) Solve the following linear programming problem by the Simplex method:

Maximize $Z = 4x_1 + 10x_2$

Subject to the constraints

$$2x_1 + x_2 \leq 10$$
$$2x_1 + 5x_2 \leq 20$$
$$x_1, x_2 \geq 0$$

Obtain an alternate optimal solution, if the exists, with explanation.

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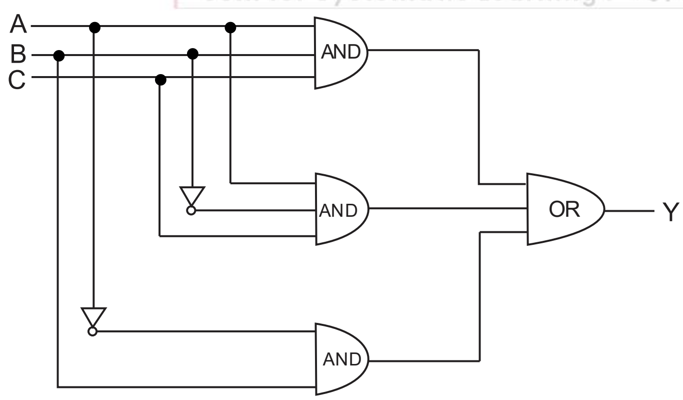
SECTION-B

Q5. (a) Find the integral surface of the equation $\dfrac{dx}{xz-y} = \dfrac{dy}{yz-x} = \dfrac{dz}{1-z^2}$.

(b) By using the Newtow-Raphson iterative formula, establish the formula $x_{i+1} = \dfrac{1}{3}\left(2x_1 + \dfrac{N}{x_i^2}\right)$ to find the cube root of N.

Use this formula, if established, to find the cube root of 63 correct to four decimal places with the initial approximation 3.9.

(c) Consider the logic circuit L:



(i) Express Y as a Boolean expression

(ii) Write 8-bit special sequence for A,B and C.

(iii) Find the truth table L using 8 bits special sequences.

(d) A simple source of strength m is fixed at the origin O in a uniform stream of incompressible fluid moving with velocity $U\hat{i}$. Find out the velocity potential ϕ at any point P of the stream where $OP = r$ and θ is the angle $\overrightarrow{OP}$ makes with the direction $\hat{i}$. Find the differential equation of the stream lines and show that they lie on the surface $U r^2 \sin^2\theta - 2m\cos\theta =$ constant.

(e) A uniform heavy solid hemisphere of radius 'a' is held at rest with is base vertical and its curved surface in contact with a horizontal plane. If the hemisphere is released when plan is rough enough to prevent slipping, show that the angle θ that the base makes with the horizontal at time t, is such that $\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{a(28-15\cos\theta)}$.

Q6.

(a) Reduce the partial differential equation $y\frac{\partial^2 z}{\partial x^2} + (x+y)\frac{\partial^2 z}{\partial x\,\partial y} + x\frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

(b) (i) Find the decimal equivalent of the following floating point 10-bit number with 5 bits as fractional prat:
1010111011 and 0100101010

(ii) Evaluate $(11001.1011)_{10} + (33.24)_8 + (101101)_2$ and convert the final result in hexadecimal system.

(c) Suppose that there is an infinitely long cylinder of radius 'a' placed in a uniform stream having velocity $-V\hat{i}$. Then using Milen-Thomson's circle theorem, what will be its complex velocity potential ? If in addition, a circulation round the cylinder of k is produced, find out the stagnation points. Do they produce a lifting tendency in the vertical direction?

Explain how.

7(a) Solve the partial differential equation

$$\left(2D^2 - 5DD' + 2D'^2\right)z = 24(y-x) + \sin(y-x) \text{ where } D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}, z = z(x,y).$$

(b) Evaluate the integral

$\int_{0}^{0.8} f(x)dx$ where $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

by using single application of Simpson's $\frac{3}{8}$ rule.

(c) A smooth uniform rod, say OA, of length 2a and mass m is pivoted at one end to a fixed point O. The rod is inclined at an angle θ with the downward vertical line OZ and the plane OAZ makes an angle ϕ with a fixed vertical plane. A bead of mass λm slide smoothly on the rod and is connected to O by a light elastic string of modulus nmg and natural length a.

Show that the kinetic energy T of system is given by

$$2T = \frac{4}{3}ma^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin\theta\right) + \lambda m\left(\dot{x}^2 + x^2\,\dot{\theta}^2 + x^2\,\dot{\theta}^2 \sin^2\theta\right),$$

where x is the stretched length of the string, and derive the equations of motion.

8(a) Two-dimensional harmonic equation in plane polar coordinates (r,θ) takes the form:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

Deduce that it has solutions of the form $\left(Ar^n + Br^{-n}\right)e^{\pm in\theta}$, where A,B, n are constants. Determine V if it satisfies the two-dimensional harmonic equation in the region $0 \le r \le a,\ 0 \le \theta \le 2\pi$ and satisfies the conditions:

(i) V remains finite as $r \to 0$

(ii) $V = \sum_{n} c_n \cos(n\theta)$ on $r = a$.

(b) Consider the initial value problem:

$$\frac{dy}{dx} = 1 + \frac{y}{x}, y(1) = 1,\ h = 1 \text{ (step-size)}$$

(i) Apply the classical fourth order Runge-Kutta method to approximate the solution $y(2)$.

(ii) Use Euler's method to find $y(2)$ and $y(3)$.

(c) A Viscous incompressible fluid is filled between two concentric cylinders of radius $a, b(b > a)$. The flow is steady and no body forces are taken into consideration. Discuss and formulate the velocity of the fluid if:

(i) The inner cylinder is given angular velocity Ω while the outer one is held at rest:

(ii) The outer cylinder is rotated with angular velocity Ω while the inner one is held at rest.

Section A

Sol. 1 (a) Let $Z(G)$ be the centre of G

$\therefore$ By Lagrange's theorem $(\because Z(G) \le G)$

$o(Z(G)) = 1/p/q/pq$

Case (i): Let if $o(Z(G)) = p$

$\therefore$ By $o(G/Z(G)) = \frac{pq}{p} = q$

$\Rightarrow$ $G/Z(G)$; group of prime order

$\therefore$ $G/Z(G)$ is cyclic group

$\therefore$ G is abelian

Case (ii): Let if $o(Z(G)) = q$

Similarly; G is abelian

Case (iii): Let $o(Z(G)) = pq = O(G)$

$\therefore$ G is abelian

Case (iv): Let if $o(Z(G)) = 1$

Therefore; if $o(Z(G)) \neq 1$; then G is abelian.

Proved.

Sol. 1 (b) Let R be an ED $\therefore$ R is a PID

$\therefore$ Let $P = \langle p \rangle$ for some elements $p \in R$

(1)

Let M be some order ideal such that $P \subset M \subset R$ (2)

$\because$ R is a PID $\therefore M = \langle q \rangle$ for some element $q \in R$.

$\therefore$ We have $\langle p \rangle \subset \langle q \rangle \subset R$ (3)

$\because$ $p \in \langle p \rangle$ $\therefore$ by divisibility $p = qx$ for some $x \in R$

But $p = \langle p \rangle$ is a prime ideal

$\therefore$ $q\,x \in P \Rightarrow$ either $q \in P$ or $x \in P$

- If $q \in P$, then $\langle q \rangle \subseteq \langle p \rangle$ (4)

$\therefore$ From (3) & (4) $\langle p \rangle = \langle q \rangle$

- If $x \in P$ then $x = p\,y$ for some $y \in R$

$\therefore \quad p = q(py) = p(qy)$

$\Rightarrow \quad p(1-qy) = 0 \Rightarrow qy = 1$

$\Rightarrow \quad q$ is unit in R

$\Rightarrow \quad \langle q \rangle = R \qquad \therefore M = R$ (5)

Therefore; if P is a prime ideal then it is maximal ideal as well here.

Sol. 1 (c) Let $1+\sin\frac{\pi}{x} = u$

$\therefore \quad f(x) = \lim_{n\to\infty} \frac{u^n - 1}{u^n + 1}$

$\because \quad \sin\frac{\pi}{x} \in [-1,1] \quad \therefore u \in [0,2]$

- For $0 \le u < 1$, $\lim_{n\to\infty} u^n = 0$

$\therefore \quad f(x) = \frac{0-1}{0+1} = -1$

- For $u = 1$, $u^n = 1$

$\therefore \quad f(x) = \frac{1-1}{1+1} = 0$

- For $1 < u \le 2$, $\lim_{n\to\infty} u^n = \infty$

$$f(x) = \lim_{n\to\infty} \frac{1-\frac{1}{u^n}}{1+\frac{1}{u^n}} = 1$$

Therefore, we have

$$f(x) = \begin{cases} -1 \ ; \ 0 \le u < 1 \ \Rightarrow \sin\frac{\pi}{x} < 0 \\ 0 \ ; \quad u = 1 \ \Rightarrow \sin\frac{\pi}{x} = 0 \\ 1 \ ; \quad u > 1 \ \Rightarrow \sin\frac{\pi}{x} > 0 \end{cases}$$

$\Rightarrow \quad f(x) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{x}\right)\right)$

For points of discontinuity of $f(x)$.

Note that when $\sin\frac{\pi}{x} = 0$; then left and Right $f(x)$ goes for $-1+1$

$\therefore$ Discontinuous at; $\sin\frac{\pi}{x}=0 \Rightarrow \frac{\pi}{x}=n\pi \Rightarrow x=\frac{1}{n}$

But $\because x\in(0,1)$ $\therefore n>1$.

Therefore, required points of discontinuity of $f(x)$ are; $\frac{1}{n}$; $n>1,\ n\in\mathbb{N}$

Sol. 1 (d) Let *f (z) = u (x, y) + iv (x, y).*

Then *u (x, y) = √|xy|, v (x, y) = 0.*

At the origin, we have

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}\lim_{y\to 0}\frac{0-0}{y}=0.$$

$$Similarly\frac{\partial v}{\partial x}=0,\frac{\partial v}{\partial y}=0$$

Hence Cauchy-Riemann equations are satisfied at origin.

Now $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}\lim_{z\to 0}\frac{\sqrt{|xy|}-0}{x+iy}$.

Suppose z→ 0 along y = mx, then we have

$$f'(0)=\lim_{z\to 0}\frac{\sqrt{|mx^2|}}{x+imx}=\lim_{z\to 0}\frac{\sqrt{|m|}}{(1+im)},$$

which is not unique, since it depends on m.

∴ *f'(0)* does not exist.

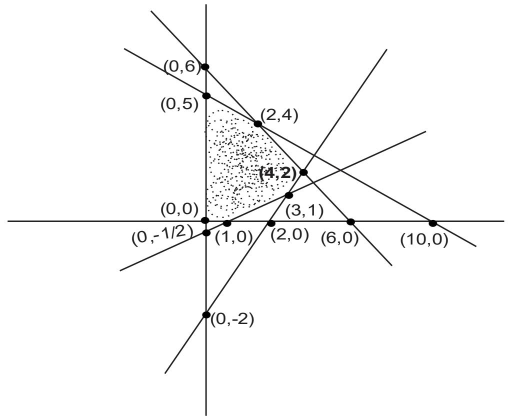
Sol. 1 (e) Dual LPP is:

Maximize $Z=2w_1+2w_2$

Subject to

$$w_1+2w_2\le 10$$
$$w_1+w_2\le 6$$
$$w_1-w_2\le 2$$
$$w_1-2w_2\le 1$$

s.t. $w_1,w_2\ge 0$



Clearly; we have optimal solution:

$w_1 = 4,\ w_2 = 2$

Optimal value $z = 10$

Sol. 2 (a) To show G is isomorphic to D_n.

Here D_n represents dihedral group of order $2n$.

Note that; to show above; we have to Show; that properties shown by G (i.e. its elements) are same as of D_n.

For this: Definition of D_n;

$D_n = \left\{x^i y^j : x^2 = e, y^n = e, xy = y^{-1}x : i = 0,1; j = 0,1,\ldots,(n-1)\right\}$

e: identity element of D_n. Reflections and rotations are elements of D_n (group of symmetries of a regular n-gon)

$\because$ $G = \langle u, v\rangle,\ o(u) = 2 \Rightarrow u^2 = 1, v^2 = 1$ here 1; identity elements of G.

Also $u = u^{-1},\ v - v^{-1}$.

Let $\alpha = u\,v$ and $o(\alpha) = n \Rightarrow (u\,v)^n = 1$

$\alpha^{-1} = (u\,v)^{-1} = v^{-1}\,u^{-1} = v\,u$

$u\,\alpha\,u^{-1} = u(u\,v)u = (u\,u)(\,v u) = 1.v u = v u = \alpha^{-1}$

$u\,\alpha\,u^{-1} = \alpha^{-1}$ (1)

$\because$ $(u\,v)^k = \alpha^k$ and $(u\,v)^k\,u = \alpha^k\,u$

$\therefore$ $G = \left\{\alpha^k : 0 \le k < n\right\} \cup \left\{\alpha^k u : 0 \le k < n\right\}$

$\Rightarrow$ $|G| = 2n$ (2)

Let's define $\phi : D_n \to G$ such that $\phi(x) = \alpha,\ \phi(y) = u$

$\because$ $\alpha^n = 1,\ u^2 = 1$ and $u\,\alpha\,u = \alpha^{-1}$

$\therefore$ ϕ is operation preserving & bijective.

$\therefore$ $D_n \cong G$

Note: here $o(D_n) = 2n$.

Sol. 2 (b) Given $u_{n+1} = \sqrt{7+u_n}$ (1)

$u_1 = \sqrt{7}$ (2)

$\therefore$ $u_2 = \sqrt{7+u_1} > u_1$

Let $u_{k+1} > u_k \Rightarrow 7+u_{k+1} > 7+u_k$

$\Rightarrow$ $\sqrt{7+u_{k+1}} > \sqrt{7+u_k}$

$\Rightarrow$ $u_{k+2} > u_{k+1}$

i.e. we have if $u_{k+1} > u_k \Rightarrow u_{k+2} > u_{k+1}$

$\therefore$ By mathematical induction method, $u_{n+1} > u_n$; holds for all but infinitely many $n \in \mathbb{N}$.

$\therefore$ $\{u_n\}$ is a monotonically increasing sequence (3)

Also $\because u_1 = \sqrt{7} < \alpha$

$\therefore$ $u_1 + 7 = \sqrt{7} + 7$

$\Rightarrow$ $\sqrt{u_1+7} = \sqrt{7+\sqrt{7}} \Rightarrow u_2 = \sqrt{7+\sqrt{7}} < \alpha$

Let if $u_k < \alpha$

$\Rightarrow$ $7 + u_k < \alpha + 7$

$\Rightarrow$ $\sqrt{7+u_k} < \sqrt{7+\alpha}$

$\Rightarrow$ $u_{k+1} < \alpha$

$\therefore$ $\{u_n\}$ is bounded above by α.

$\therefore$ By monotone convergence theorem, $\{u_n\}$ is convergent.

Let $\{u_n\}$ converges to l.

$\therefore$ $\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \sqrt{7+u_n}$

$\Rightarrow$ $l = \sqrt{l+7}$

$\Rightarrow$ $l^2 - l - 7 = 0$

$\Rightarrow$ l is a root of $x^2 - x - 7 = 0$.

Sol. 2 (c) Let $C : |z| = 1 \quad \therefore z = e^{i\theta}$ for $0 \le \theta \le 2\pi$

$$\therefore \qquad d\theta = \frac{1}{iz}dz$$

$$\therefore \qquad I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}d\theta = \int_C \frac{-\frac{1}{4}\left(\frac{z^2-1}{z}\right)^2}{a + \frac{b}{2}\left(\frac{z^2+1}{2}\right)} \frac{dz}{iz}$$

$$= \int_C \frac{-\left(z^2-1\right)^2}{4z^2} \cdot \frac{2z}{\left(bz^2+2az+b\right)} \frac{dz}{iz}$$

$$= \frac{1}{2}\int_C \frac{\left(z^2-1\right)^2}{z^2\left(bz^2+2az+b\right)}dz$$

$$I = \frac{i}{2b}\int_C \frac{\left(z^2-1\right)^2}{\left(z^2+\frac{2a}{b}z+1\right)}dz \qquad (1)$$

Let $f(x) = \dfrac{\left(z^2-1\right)^2}{z^2\left(z^2+\frac{2a}{b}+1\right)}$

Poles of $f(z)$ are given by:

$z^2 = 0 \Rightarrow z = 0$ is a pole of order 2.

$z^2 + \dfrac{2a}{b}z + 1 = 0$: simple poles.

$$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}, \; z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$$

So, $z = 0,\ z_1$; Poles inside c.

Residue at $z = 0$;

$$\mathrm{Res}\left(f_0\right) = \lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right)$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{\left(z^2-1\right)^2}{\left(z^2+\frac{2a}{b}z+1\right)}\right]$$

$$= \lim_{z\to 0} \frac{4z\left(z^2-1\right)\left(z^2+\frac{2a}{b}z+1\right)-\left(z^2-1\right)^2\left(2z+\frac{2a}{b}\right)}{\left(z^2+\frac{2a}{b}z+1\right)^2}$$

$$= \frac{0\times 1 - 1\times \frac{2a}{b}}{(1)^2} = \frac{-2a}{b}$$

Res. At z_1

$$\operatorname{Res}(f, z_1) = \lim_{(z\to z_1)} (z-z_1) f(z) = \frac{\left(z_1\left(z_1-z_2\right)\right)^2}{z_1^2\left(z_1-z_2\right)}$$

$$= z_1 - z_2 = \frac{2\sqrt{a^2-b^2}}{b}$$

On applying Cauchy Residue Theorem

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{2a}{b}+\frac{2\sqrt{a^2-b}}{b}\right) \qquad (2)$$

On using (2) in (1), we get $I = \frac{i}{2b}.\frac{4\pi i}{b}\left(\sqrt{a^2-b^2}-a\right)$

$$= \frac{-2\pi}{b^2}\left(\sqrt{a^2-b^2}-a\right)$$

$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

Sol. 3 (a) The function $\cosh\left(z+\frac{1}{z}\right)$ is analytic in every finite part of the z-plane except at $z = 0$.

Thus the given function is analytic in the annulus $r \le |z| \le R$ where r is small and R is large so that we can expand $f(z)$ in a Laurent's series in the annulus $r < |z| < R$.

$$\therefore \quad \cosh\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where $a_n = \frac{1}{2\pi i}\int_C \cosh\left(z+\frac{1}{z}\right)\frac{dz}{z^{n+1}}$

and $b_n = \frac{1}{2\pi i}\int_C \cosh\left(z+\frac{1}{z}\right) z^{n-1}dz$

where C is a circle with centre at origin.

Let C be the unit circle defined by $|z| = 1$. Then $z = e^{i\theta}$, $dz = ie^{i\theta}\, d\theta$.

$$\therefore \quad a_n = \frac{1}{2\pi i}\int_0^{2\pi}\cosh\left(e^{i\theta}+e^{-i\theta}\right)\frac{ie^{i\theta}}{e^{i(n+1)\theta}}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cosh\left(2\cos\theta\right)e^{-in\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cosh\left(2\cos\theta\right)\left(\cos n\theta - i\sin n\theta\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cosh\left(2\cos\theta\right)\cos n\theta\, d\theta$$

{since the other integral becomes zero by the property $\int_0^{2a} f(\theta)d\theta = 0$ if $f(2a-\theta) = -f(\theta)$}

Now the function $\cosh\left(z+\frac{1}{z}\right)$ remains unchanged by replacing z by $1/z$, therefore we have

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi}\cosh\left(2\cos\theta\right)\cos\left(-n\theta\right)d\theta = a_n$$

Hence $\cosh\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty}a_n z^n + \sum_{n=1}^{\infty}b_n z^{-n}$

$$= \sum_{n=0}^{\infty}a_n z^n + \sum_{n=1}^{\infty}a_n z^{-n} \qquad = a_0 + \sum_{n=1}^{\infty}a_n\left(z^n + z^{-n}\right), \qquad [\because b_n = a_n]$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi}\cosh\left(2\cos\theta\right)\cos n\theta\, d\theta$.

Sol. 3 (b) Hint: Lower integral; on $[0,1]$

$\because$ $x^3 \le x^2$ holds for $x \in [0,1]$

$\therefore$ $\int_0^1 f\, dx = \int_0^1 x^3\, dx = \frac{1}{4}$

Upper integral $\int_0^1 f(x)dx = \int_0^1 x^2\, dx = \frac{1}{3}$

$\because$ Lower integral $\neq$ Upper integral

$\therefore$ $f(x)$ is not Riemann integrable.

Sol. 3 (c) Balancing the given problem.

8	29	28	32
8	13	17	19
8	15	19	22
0	0	0	0

Balanced

0	16	20	24
0	5	9	11
0	7	11	14
0	0	0	0

Row & Column reduction

0	16	20	24
0	5	9	11
0	7	11	14
0	0	0	0

All zeros covered by two lines column 1 and row 4

0	9	9	13
2	0	0	2
0	0	0	3
11	4	0	0

On subtracting 5 from all uncovered elements and adding 5 to intersection elements

Optimal assignment

$M_1 \to J_1, M_2 \to J_2, M_3 \to J_3, M_4 \to J_4$ (dummy)

$\therefore$ Total minimum cost $= 8+13+19+0 = 40$

Another optimal assignment, $M_1 \to J_1, M_2 \to J_2, M_3 \to J_3, M_4 \to J_4$

$\therefore$ Total minimum cost $= 8+17+15+0 = 40$

$\therefore$ In any case Job 4 is left unassigned. (as its for dummy)

Sol. 4 (a) To show (i) $\Rightarrow$ (ii)

Let $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.

Let $a \in \mathbb{Z}/n\mathbb{Z}$: a is non-zero, where $1 \le a < n$.

Consider $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ defined by $f(x) = ax$.

$\because$ $\mathbb{Z}/n\mathbb{Z}$ is an integral domain and $a \ne 0$ $\therefore f(x) = f(y) \Rightarrow a(x-y) = 0$

$\Rightarrow x = y$

$\therefore$ f is one-one

$\because$ $\mathbb{Z}/n\mathbb{Z}$ is finite set $\therefore$ f is onto.

$\therefore$ $\exists$some $b \in \mathbb{Z}/n\mathbb{Z}$ such that $f(b) = ab = 1$

$\Rightarrow$ Every non-zero element has its multiplicative inverse in $\mathbb{Z}/n\mathbb{Z}$.

$\therefore$ $\mathbb{Z}/n\mathbb{Z}$ is a field.

(ii) $\Rightarrow$ (iii)

Let $\mathbb{Z}/n\mathbb{Z}$ is field.

$\therefore$ $O(\mathbb{Z}/n\mathbb{Z})>1$ (it must have at least two elements)

Let if n is not a prime say $n=ab$ for some integers a,b with $1<a<n,\ 1<b<n$

$\because$ In $\mathbb{Z}/n\mathbb{Z}$; it means $a\neq 0$, $b\neq 0$ but their product $ab=n=0$; contradiction of the fact that a field has no zero divisor.

$\therefore$ n must be prime.

(iii) $\Rightarrow$(i)

Let n is prime.

$\therefore$ $\mathbb{Z}/n\mathbb{Z}$ has at least two elements.

Let $a,b\in\mathbb{Z}/n\mathbb{Z}$ such that $ab=0$

$\Rightarrow$ n divides ab

$\therefore$ By Euclid's lemma for prime numbers, if a prime (n) divides ab, then n divides a or n divides b.

$\therefore$ $a=0$ or $b=0$

It shows $\mathbb{Z}/n\mathbb{Z}$ is an integral domain.

Sol. 4 (b) $\displaystyle\int_0^{\infty}\frac{1}{1+x^2\sin^2 x}dx=\sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{1}{\left(1+x^2\sin^2 x\right)}dx$

$\therefore$ For each $x\in\left[n\pi,\ (n+1)\pi\right]$

$\because$ $x\leq(n+1)\pi$ $\quad\therefore$ on replacing x by $(n+1)\pi$, denominator is maximum and so given lower bound of integral

$$\therefore\quad \int_{n\pi}^{(n+1)}\frac{1}{1+x^2\sin^2 x}dx\geq\int_{n\pi}^{(n+1)\pi}\frac{1}{1+(n+1)^2\pi^2\sin^2 x}\frac{dx}{}\geq 1 \qquad (1)$$

Let's evaluate I; substituting $x=n\pi+t$

$\therefore$ $\sin^2(n\pi+t)=\sin^2 t$

$$I=\int_0^{\pi}\frac{1}{1+(n+1)^2\pi^2\sin^2}\frac{dt}{t}$$

$$\because\quad \int_0^{\pi}\frac{1}{1+a^2\sin^2 t}dt=\frac{\pi}{\sqrt{1+a^2}}$$

$\therefore \qquad \int_0^{\pi} \frac{1}{1+(n+1)^2 \pi^2 \sin^2 t} dt = \frac{\pi}{\sqrt{1+(n+1)^2 \pi^2}}$

Using this in (1) and then applying comparison test, we get for large values of n.

$$\frac{\pi}{\sqrt{1+(n+1)^2 \pi^2}} \geq \frac{\pi}{\sqrt{4n^2\pi^2}} = \frac{1}{2n}$$

$\therefore$ On summing these lower bounds from $n=1$ to ∞.

$$\sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)} \frac{1}{1+n^2 \sin^2 x} dx \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n}$; divergent

$\therefore$ By comparison test given integral is divergent.

Sol. 4 (c) $\max . z = 4x_1 + 10x_2 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to

$$\begin{aligned} 2x_1 + x_2 + s_1 \quad &= 10 \\ 2x_1 + 5x_2 \quad + s_2 \quad &= 20 \\ 2x_1 + 3x_2 \quad + s_3 &= 18 \end{aligned}$$

Such that $x_1, x_2, s_1, s_2, s_3 \geq 0$

	$C_j \to$	4	10	0	0	0		
B	C_B	y_1	y_2	y_3	y_4	y_5	X_B	θ
s_1	0	2	1	1	0	0	10	$10/1=10$
s_2	0	2	5	0	1	0	20	$20/5=4$ min.
s_3	0	2	3	0	0	1	18	$18/3=6$
	$\Delta_j \to$	4	10	0	0	0		

s_1	0	1.6	0	1	-0.2	0	6
x_2	10	0.4	1	0	0.2	0	4
s_3	0	0.8	0	0	-0.6	1	6
		0	0	0	-2	0	

$\Delta_j \le 0$ current solution is optimal.

$\therefore$ First optimal solution $x_1 = 0, x_2 = 4, z = 40$

We know that if a non-basic variable has $\Delta = 0$ in final simplex table. It shows that by taking that variable into basis will change the coordinates of optimal solution without changing the optimal value of z. So we have here alternate optimal solutions.

Finding alternate optimal solution; x_1 as enterring variable in final table.

x_1	4	1	0	0.625	-0.125	0	3.75
x_2	10	0	1	-0.25	0.25	0	2.5
s_3	0	0	0	-0.5	-0.5	1	3
		0	0	0	-2	0	$z = 40$

SECTION-B

Sol. 5 (a) Choosing $1,1,0$ as multipliers, we get each fraction is equal to

$$\frac{1.dx + 1.dy + 0.dz}{2(xz - y) + 1(yz - x) + 0(1 - z^2)} = \frac{dx + dy}{(x+y)(z-1)}$$

$\therefore$ Taking $\dfrac{dx+dy}{(x+y)(z-1)}=\dfrac{dz}{(1-z^2)}$

$$\frac{dx+dy}{x+y}=\frac{-dz}{1+z}$$

$\Rightarrow$ $\log(x+y)=-\log(1+z)+\log C_1$

$\Rightarrow$ $(x+y)(z+1)=C_1$

- Choosing $(1,-1,0)$ as multipliers

$$\frac{1.dx-1.dy+0.dz}{1.(xz-y)-1(yz-x)+0(1-z^2)}=\frac{dx-dy}{(x-y)(z+1)}$$

$\therefore$ Taking $\dfrac{dx-dy}{(x-y)(z+1)}=\dfrac{dz}{(1-z^2)}$

$\Rightarrow$ $\dfrac{dx-dy}{x-y}=\dfrac{dz}{1-z}$

$\Rightarrow$ $\log(x-y)=-\log(1-z)+\log C_2$

$\Rightarrow$ $(x-y)(1-z)=C_2$.

$\therefore$ The general solution defining the integral surface is

$\phi\left((x+y)(z+1),(x-y)(1-z)\right)>0$ where ϕ is an arbitrary function.

Sol. 5 (b) Let $x=\sqrt[3]{N}$

$\therefore$ $f(x)=x^3-N$ (1)

$\therefore$ $f'(x)=3x^2$

By Newton-Raphson method,

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

$$=x_i-\frac{x_i^3-N}{3x_i^2}$$

$$=\frac{3x_i^3-(x_i^3-N)}{3x_i^2}$$

$\Rightarrow$ $x_{i+1}=\dfrac{1}{3}\left(2x_i+\dfrac{N}{x_i^2}\right)$; required formula

Finding cube root of 63. By above formula:

Taking $x_0=3.9$

$$x_1 = \frac{1}{3}\left(2\times 3.9 + \frac{63}{(9.3)^2}\right) = \frac{1}{3}(7.8 + 4.1420118)$$

$$\Rightarrow \quad x_1 = 3.9806706$$

$$x_2 = \frac{1}{3}\left(2\times 3.9806706 + \frac{63}{(3.9806706)^2}\right)$$

$$\Rightarrow \quad x_2 = \frac{1}{3}\left(7.9613412 + \frac{63}{15.8457385}\right)$$

$$x_2 = 3.9790578$$

$$x_3 = \frac{1}{3}\left(7.9581156 + \frac{63}{(3.9790578)^2}\right)$$

$$= \frac{1}{3}(7.9581156 + 3.9790559)$$

$$x_3 = 3.9790572$$

Required cube root of 63 is 3.9791.

Sol. 5 (c) (i) $Y = A.C.(B+\overline{B}) + A.B.\overline{C}$

Simplifying $Y = A.C.1 + A.B.\overline{C}$

$= A.(C + B\overline{C})$

Using distributive law $(C+B)(C+\overline{C}) = (C+B).1$

We get $Y = A.(B+C)$

(ii) Writing 8-bit special sequences for A, B, C

$A = 00001111$

$B = 00110011$

$C = 01010101$

(iii) Truth Table

A	0	0	0	0	1	1	1	1
B	0	0	1	1	0	0	1	1
C	0	1	0	1	0	1	0	1
$\overline{B}$	1	1	0	0	1	1	0	0
$\overline{C}$	1	1	1	0	1	0	1	0
$A.B.C$	0	0	0	0	0	0	0	1
$A.\overline{B}.C$	0	0	0	0	0	1	0	0
$A.B.\overline{C}$	0	0	0	0	0	0	1	0
Y	0	0	0	0	0	1	1	1

Sol. 5 (d) Flow is axisymmetric; direction of uniform stream $(\hat{i})$.

Let's work with spherical coordinates (r,θ,ϕ).

$\because$ $\vec{q} = \nabla\phi$

$\because$ Uniform stream moving with velocity $u\hat{i}$, $\therefore$ it has a velocity potential

$\phi_1 = u_x = u\,r\cos\theta$

A point source of strength m at origin: velocity potential $\phi_2 = \frac{-m}{r}$

$\therefore$ By principle of super position, total velocity potential

$$\phi = \phi_1 + \phi_2 = u\,r\cos\theta - \frac{m}{r} \quad (1)$$

Differential equation of the stream lines, $q_r = \frac{\partial\phi}{\partial r} = u\cos\theta + \frac{m}{r^2}$ (2)

$$q_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -u\sin\theta \quad (3)$$

For axisymmetric flow, differential equation governing stream lines

$$\frac{dr}{q_r} = \frac{rd\theta}{q_\theta}$$

$\Rightarrow$ $\dfrac{dr}{u\cos\theta + \frac{m}{r^2}} = \dfrac{r\,d\theta}{-u\sin\theta}$ $\therefore$ Using (2) & (3)

$\Rightarrow$ $-u\,r\sin\theta\,dr = u\,r^2\cos\theta\,d\theta + m\,d\theta$

$\Rightarrow$ $u\,r^2\cos\theta\,d\theta + u\,r\sin\theta\,dr + m\,d\theta = 0$

On multiplying by $2\sin\theta$ in above equation

$$2ur^2\sin\theta\cos\theta\,d\theta + 2u\,r\sin^2\theta\,dr + 2m\sin\theta\,d\theta = 0$$

$\Rightarrow$ $u\left(d\left(r^2\sin^2\theta\right)+2m\sin\theta\, d\theta=0\right)$

On integrating, we get $u\,r^2\sin^2\theta-2m\cos\theta=\text{constant}$.

Sol. 5 (e) $\because$ Plane is rough enough to prevent slipping.

So the hemisphere rolls without sliding, i.e. point P of contact (say P) acts the instantaneous centre of rotation.

- Hemisphere: mass M, radius a ; G:centre to mass.

Let O; centre of the flat base

$\because$ $OG=\dfrac{3a}{8}$

$\because$ The centre O remains at a constant height a above the ground, the height of G at any instant is $y_a=a-\dfrac{3a}{8}\cos\theta$

θ: angle bases makes with horizontal OG (axis of symmetry) makes θ with vertical.

$\because$ Hemisphere released from rest when its base is vertical i.e. $\theta=\dfrac{\pi}{2}$

$\therefore$ Initial height of G: $y_1=a-\dfrac{3a}{8}\cos\left(\dfrac{\pi}{2}\right)=a$ at angle θ; $y_G=a-\dfrac{3a}{8}\cos\theta$

$\therefore$ Loss in potential energy $=M\left(y_i-y_G\right)$

$AV=\dfrac{3}{8}aM\,g\cos\theta$

Moment of inertial about instantaneous axis (using parallel axis theorem about point P)

$\because$ $I_{base}=\dfrac{2}{5}Ma^2$

$\therefore$ $I_G=I_{base}-M\left(OG\right)^2=\dfrac{2}{5}Ma^2-M\left(\dfrac{3}{8}a\right)^2=\dfrac{83}{320}Ma^2$

$\because$ $GP^2=\left(\dfrac{3a}{8}\sin\theta\right)^2+\left(a-\dfrac{3a}{8}\right)^2=a^2\left(\dfrac{73}{64}-\dfrac{3}{4}\cos\theta\right)$

About P; $I_P=I_G+M\left(GP\right)^2=\dfrac{83}{320}Ma^2+Ma^2\left(\dfrac{73}{64}-\dfrac{3}{4}\cos\theta\right)$

$\Rightarrow$ $I_p=\dfrac{Ma^2}{20}\left(28-15\cos\theta\right)$

$\because$ The total kinetic energy (T) of the rolling hemisphere is purely rotational about P.

$$T = \frac{1}{2} I_P \left(\frac{d\theta}{dt}\right)^2 = \frac{Ma^2}{40}(28 - 15\cos\theta)\left(\frac{d\theta}{dt}\right)^2$$

On equating $T = \Delta V$, we get

$$\frac{Ma^2}{40}(28 - 15\cos\theta)\left(\frac{d\theta}{dt}\right)^2 = \frac{3}{8} Mga\cos\theta$$

$$\Rightarrow \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{a(28 - 15\cos\theta)}.$$

Sol. 6 (a)

Given $yr + (x+y)s + xt = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y, S = (x+y)$ and $T = x$ so that $S^2 - 4RT = (x+y)^2 - 4xy = (x-y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $y\lambda^2 + (x+y)\lambda + x = 0$ or $(y\lambda + x)(\lambda + 1) = 0$ so that $\lambda = -1, -x/y$.

Then the corresponding characteristic equations are given by

$(dy/dx) - 1 = 0$ and $(dy/dx) - (x/y) = 0$

Integrating these, $y - x = c_1$ and $y^2/2 - x^2/2 = c_2$

In order to reduce (1) to its canonical form, we choose

$u = y - x$ and $v = y^2/2 - x^2/2$ ....(2)

Final required form is: $u^2 \dfrac{\partial^2 z}{\partial u \partial v} + u\dfrac{\partial z}{\partial v} = 0$, by (2) or $u\dfrac{\partial^2 z}{\partial v \partial v} + \dfrac{\partial z}{\partial v} = 0$, as $u \neq 0$ ....(3)

Solution of (3). Multiplying both sides of (3) by v, we get

$uv(\partial^2 z/\partial u \partial v) + v(\partial z/\partial v) = 0$ or $(uv\,DD' + vD')z = 0$ ....(4)

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. To reduce (4) into linear equation with constant coefficients, we take new variables X and Y as follows.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$ ....(5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$ .Then (4) reduces to

$\left(D_1D_1' + D_1'\right)z = 0$ or $D_1'\left(D_1 + 1\right)z = 0$

Its general solution is $z = e^{-X}\phi_1(Y) + \phi_2(X) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$

or $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1\left(y^2 - x^2\right) + \psi_2(y-x)$,

where ψ_1 and ψ_2 are arbitrary functions.

Sol. 6 (b) Let's write each point in base 10

$\because$ $(33.24)_8 = 3\times 8^1 + 3\times 8^0 + 2\times 8^{-1} + 4\times 8^{-2}$

$= 24 + 3 + \dfrac{2}{8} + \dfrac{4}{64}$

$= 27 + 0.25 + 0.625 = 0.3125$

$\because$ $(101101)_2 = 1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0 = 32 + 0 + 8 + 4 + 0 + 1 = 45$

So required sum $= 11001.1011 + 27.3125 + 45$

$= 11073.4136 = N$

Converting (N) to hexadecimal from decimal.

16	11073	1
16	692	4
16	43	$11 = B$
	2	2

$0.4136\times 16 = 6.6176$

$0.6176\times 16 = 9.8816$

$0.8816\times 16 = 14.1056; 14 = E$

$0.1056\times 16 = 1.6896$

$0.6896\times 16 = 11.0336\ ;\ 11 = B$

$\therefore$ $(11073.4136)_{10} = (2B41.69E1B)_{16}$

Sol. 6 (c) • Complex potential without circulation $W_0(z) = -Vz$

Applying Milne-Thomson circle Theorem

$W(z) = W_0(z) + \left(\overline{W_0\left(\dfrac{a^2}{\bar{z}}\right)}\right)$; for a cylinder of radius a

$\Rightarrow$ $W(z) = -V\, z + \left(\overline{-V\left(\frac{a^2}{\overline{z}}\right)}\right)$

$W(z) = -V\left(z + \frac{a^2}{z}\right)$

- Complex potential with circulation $W(z) = -V\left(z + \frac{a^2}{z}\right) + \frac{i\,k}{2\pi}\log z$

For stagnation points: Putting $\frac{dW}{dz} = 0$

$\Rightarrow$ $-V\left(1 - \frac{a^2}{z^2}\right) + \frac{i\,k}{2\pi z} = 0$

$\Rightarrow$ $V\,z^2 - \frac{i\,k}{2\pi} z - Va^2 = 0$ (1)

$\because$ Stagnation points lie on the boundary of cylinder $z = a\,e^{i\theta} - \frac{i\,k}{2\pi} a\,e^{i\theta} - V\,a^2 = 0$

$\Rightarrow$ $e^{i\theta} - e^{-i\theta} = \frac{i\,k}{2\pi\,Va} \Rightarrow \sin\theta = \frac{k}{4\,\pi\,Va}$ (2)

Case (i) If $|k| < 4\pi\,Va$; then two distinct stagnation point exist on the cylinder at angles $\theta = \sin^{-1}\left(\frac{k}{4\,\pi V\,a}\right)$

Case (ii) if $|k| = 4\,\pi V\,a$; single stagnation point at top $\theta = \frac{\pi}{2}$ or bottom $\theta = \frac{-\pi}{2}$ of cylinder.

Case (iii) If $|k| > 4\,\pi V\,a$; The stagnation points leave the cylinder surface: one moves into the fluid flow, and one goes inside the cylinder.

Lifting tendency (Magnus Effect)

Blacius theorem

Force $X - i\,Y = \frac{i\,\rho}{2}\oint_C \left(\frac{dW}{dz}\right)^2 dz$

Evaluating residue at $z = 0$ for $\left(\frac{dW}{dz}\right)^2$

$X - i\,Y = i\,\rho V\,k \Rightarrow X = 0,\ Y = -\rho V\,k$

To get an upward lifting tendency, the circulation k must be clockwise $(k < 0)$, making the top a low-pressure zone.

Sol. 7 (a) For C.F. ; $2m^2 - 5m + 2 = 0$

$(2m-1)(m-2) = 0$

$$m_1 = \frac{1}{2}, m_2 = 2$$

$\because$ C.F. $= \phi_1\left(y + \frac{x}{2}\right) + \phi_2(y - 2x)$

$= \phi_1(2y + x) + \phi_2(y + 2x)$ where ϕ_1 and ϕ_2 are arbitrary functions:

For P.I.

- $I_1 = \frac{1}{9}\iint 24u\,du\,du$; $F(-1,1) = 9$

$= \frac{1}{9}\int 24u\,du = \frac{1}{9}12u^2 = \frac{4}{9}u^3 = \frac{4}{9}(y-x)^3$

- $I_2 = \frac{1}{9}\iint \sin u\,du\,du = \frac{1}{9} - \int \cos u\,du = -\frac{1}{9}\sin u = \frac{-1}{9}\sin(y-x)$

Therefore, required general solutions.

$$z = \phi_1(2y + x) + \phi_2(y + 2x) + \frac{4}{9}(y-x)^3 - \frac{1}{9}\sin(y-x)$$

Sol. 7 (b) $h - \frac{b-a}{3} = \frac{0.8-0}{3} = 0.26667$

Taking $n = 3$ subintervals

x	$f(x)$
$x_0 = 0$	0.2
$x_1 = 0.2667$	1.43272
$x_2 = 0.53333$	3.48718
$x_3 = 0.8$	0.232

$\therefore$ $I = \frac{3 \times 0.2667}{8}\left[0.2 + 3(1.43272) + 3(3.48918) + 0.232\right]$

$= 0.1 \times 15.1917 = 1.51917$

Sol. 7 (c) Kinetic Energy of the Uniform Rod $(\mathbf{T}_{\mathbf{rod}})$

The rod OA has length $2a$, mass m, and is pivoted at one rod O.

- The moment of inertia of a uniform rod of mass m and length $2a$ about an axis passing through its end O is:

$$I = \frac{1}{3}m(2a)^2 = \frac{4}{3}ma^2$$

The orientation of the rod is described using spherical polar coordinates (θ, ϕ) relative to the downward vertical line OZ:

- θ is the angle with the vertical.

- ϕ is the azimuthal angle representing the rotation of the vertical plane OAZ.

  The angular velocity vector components in spherical coordinates yield a total squared angular velocity of:

$$\omega^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin\theta$$

Therefore, the rotational kinetic energy of the rod pivoting about fixed point O is:

$$T_{rod} = \frac{1}{2} I\,\omega^2 = \frac{1}{2}\left(\frac{4}{3}ma^2\right)\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

$$2T_{rod} = \frac{4}{3}ma^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

Kinetic Energy of the Bead $\left(\mathbf{T}_{\mathbf{bead}}\right)$

The bead has mass λm. Its position on the rod is at a distance x from the pivot O (the stretched length of the string).

In spherical polar coordinates (r, θ, ϕ), then coordinates of the bead are:

$$r = x,\ \theta = \theta,\ \phi = \phi$$

The square of the velocity vector in spherical coordinates is given by:

$$v^2 = \dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2$$

Substituting $r = x$ and $\dot{r} = \dot{x}$:

$$v^2 = \dot{x}^2 + x^2\,\dot{\theta}^2 + x^2\,\dot{\phi}^2 \sin^2\theta$$

Thus, the kinetic energy of the bead is:

$$T_{bead} = \frac{1}{2}(\lambda m)v^2 = \frac{1}{2}\lambda m\left(\dot{x}^2 + x^2\,\dot{\theta}^2 + x^2\,\dot{\phi}^2 \sin^2\theta\right)$$

$$2T_{bead} = \lambda\, m\left(\dot{x}^2 + x^2\,\dot{\theta}^2 + x^2\,\dot{\phi}^2 \sin^2\theta\right)$$

Total Kinetic Energy of the System (T)

Combining both components $\left(T = T_{rod} + T_{bead}\right)$

$$2T = \frac{4}{3}ma^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \lambda m\left(\dot{x}^2 + x^2\,\dot{\theta}^2 + x^2\,\dot{\phi}^2 \sin^2\theta\right)$$

Derivation of the Equations of Motion:

To find the equations of motion, we use Lagrange's equations. First, we define the Potential Energy (V) of the system.

Potential Energy (V)

The potential energy consists of gravitational potential energy (taking the pivot O as the zero-energy reference level down-positive) and the elastic potential energy of the string:

Gravitational PE of Rod (center of mass is at distinct a):

$V_{rod} = -mga\cos\theta$

Gravitational PE of Bead: $V_{bead} = -\lambda mg\, x\cos\theta$

Elastic PE of String (modulus nmg natural length a, stretched length x):

$$V_{elastic} = \frac{Modulus \times (extension)^2}{2 \times natural\, length} = \frac{nmg(x-a)^2}{2a}$$

Total Potential Energy V: $V = -mga\cos\theta - \lambda mgx\cos\theta + \dfrac{nmg(x-a)^2}{2a}$

Lagrangian (L = T – V)

$$L = \frac{2}{3}ma^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}\lambda m\left(\dot{x}^2 + x^2\dot{\theta}^2 + x^2\dot{\phi}^2\sin^2\theta\right)$$

We solve Lagrange's equation $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} = 0$ for each generalized coordinate (x,θ,ϕ):

Equation for ϕ:

Since ϕ does not explicitly appear in L, it is a cyclic coordinate. The conjugate momentum is conserved:

$$\frac{\partial L}{\partial \dot{\phi}} = \left(\frac{4}{3}ma^2\sin^2\theta + \lambda mx^2\sin^2\theta\right)\dot{\phi} = \text{Constant.}$$

$$\frac{d}{dt}\left[m\sin^2\theta\left(\frac{4}{3}a^2 + \lambda x^2\right)\dot{\phi}\right] = 0$$

Equation for x:

$$\frac{\partial L}{\partial \dot{x}} = \lambda m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \lambda m\ddot{x}$$

$$mx\dot{\theta}^2 + \lambda mx\dot{\phi}^2\sin^2\theta + \lambda mg\cos\theta - \frac{nmg(x-a)}{a} = 0$$

$$\lambda m\ddot{x} - \lambda m\dot{\theta}^2 - \lambda mx\dot{\phi}^2\sin^2\theta - \lambda mg\cos\theta + \frac{nmg(x-a)}{a} = 0$$

Equation for θ:

$$\frac{\partial L}{\partial \dot{\theta}} = \left(\frac{4}{3}ma^2 + \lambda m x^2\right)\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\frac{4}{3}ma^2 + \lambda m x^2\right)\ddot{\theta} + 2\lambda m x \dot{x}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \frac{4}{3}ma^2\dot{\phi}^2 \sin\theta\cos\theta + \lambda m x^2 \dot{\phi}^2 \sin\theta\cos\theta - mga\sin\theta - \lambda m g x\sin\theta$$

$$\left(\frac{4}{3}ma^2 + \lambda m x^2\right)\ddot{\theta} + 2\lambda m x \dot{x}\dot{\theta} - \left(\frac{4}{3}ma^2 + \lambda m x^2\right)\dot{\phi}^2 \sin\theta\cos\theta + mg\left(a + \lambda x\right)\sin\theta = 0$$

Sol. 8 (a) $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

Assume a separable solution of the form: $V(r,\theta) = R(r)\Theta(\theta)$

Substitute this into the partial differential equation:

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

Divide the entire equation by $\frac{R(r)\Theta(\theta)}{r^2}$:

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

Separate the variable by setting each part equal to a constant λ:

$$\frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

Solving the Angular Equation:

For a physically meaningful single-valued solution in the full region $0 \le \theta \le 2\pi$, $V(r,\theta)$ must be periodic such that $\Theta(\theta) = \Theta(\theta + 2\pi)$. This requires λ to be positive. Let $\lambda = n^2$, where n is a constant (typically an integer):

$$\Theta''(\theta) + n^2\Theta(\theta) = 0$$

The general solution for the angular part is:

$\Theta(\theta) = e^{\pm in\theta}$

Solving the Radial Equation:

With $\lambda = n^2$, the radial equation becomes:

$$r^2 R''(r) + r R'(r) - n^2 R(r) = 0$$

This is a Cauchy-Euler equation. Assume a solution of the form $R(r) = r^m$:

$$m(m-1)r^m + mr^m - n^2 r^m = 0 \Rightarrow m^2 - n^2 = 0$$

Thus, the general solution for the radial part (for $n \neq 0$) is

$$R(r) = Ar^{n} + Br^{-n}$$

Combining $R(r)$ and $\Theta(\theta)$, we successfully deduce the form of the solutions:

$$V(r,\theta) = \left(Ar^{n} + Br^{-n}\right)e^{\pm in\theta}$$

The total solution is a superposition of all possible states:

$$V(r,\theta) = \left(A_0 + B_0 \ln r\right) + \sum_{n=1}^{\infty}\left(A_n r^{n} + B_n r^{-n}\right)e^{\pm in\theta}$$

Condition (i): V remains finite as $r \to 0$

- The terms $\ln r$ and r^{-n} approach infinity set $r \to 0$
- To keep V finite in the region containing the origin $(0 \leq r \leq a)$, their coefficients must be zero:

  $$B_0 = 0, \quad B_n = 0, \quad D_n = 0$$

  This simplifies our potential function to:

  $$V(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{n} \cos(n\theta) + \sum_{n=1}^{\infty} C_n r^{n} \sin(n\theta)$$

Condition (ii): $V = \sum_n c_n \cos(n\theta)$ on $r = a$

Substitute $r = a$ into our simplified expression for V:

$$V(a,\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^{n} \cos(n\theta) + \sum_{n=1}^{\infty} C_n a^{n} \sin(n\theta)$$

By comparing coefficient on both sides:

$$\sum_{n=1}^{\infty} A_n a^{n} \cos(n\theta) + \sum_{n=1}^{\infty} C_n a^{n} \sin(n\theta) = \sum_{n} c_n \cos(n\theta)$$

By comparing coefficient on both sides:

- There are no sine terms on the right-hand side, so:

  $C_n = 0$ for all n

- For the constant term $(n = 0)$:

  $$A_0 = c_0$$

- For the cosine terms $(n \geq 1)$:

  $$A_n a^{n} = c_n \Rightarrow A_n = c_n a^{-n}$$

Substituting the determined constants back into the equation for $V(r,\theta)$ yields the specific solution:

$$V(r,\theta) = c_0 + \sum_{n=1}^{\infty} c_n \left(\frac{r}{a}\right)^n \cos(n\theta)$$

Sol. 8 (b) $f(x,y) = 1 + \frac{y}{x}$; $x_0 = 1,\ y_0 = 1,\ h = 1$

$$k_1 = h\,f(x_0, y_0) = 1\left(1 + \frac{1}{1}\right) = 2.0$$

$$k_2 - h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 1.f\left(1 + 0.5, 1 + \frac{2}{2}\right) = f(1.5, 2) = 1 + \frac{2}{1.5} = 2.3333$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k^2}{2}\right) = 1.f\left(1.5, 1 + \frac{2.3333}{2}\right) = 1 + \frac{2.1667}{1.5} = 1 + \frac{13}{9} = 2.4444$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 1\ f\left(2, 1 + \frac{22}{9}\right) = f\left(2, \frac{31}{9}\right) = 1 + \frac{31}{18} = 2.7222$$

$\therefore$ $$y(2) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 3k_3 + k_4)$$

$$= 1 + \frac{1}{6}\left(2 + 2\times\frac{7}{3} + 2\times\frac{22}{9} + \frac{44}{18}\right)$$

$$= 1 + \frac{257}{108} = 3.3796.$$

(ii) By Euler's method

$\because$ $$y_{n+1} = y_n + h\,f(x_n, y_n)$$

$\therefore$ $$y(2) = y_1 = y_0 + h\,f(x_0, y_0) = 1 + 1.\left(1 + \frac{1}{1}\right) = 1 + 2 = 3$$

$$y(3) = y_2 = y_1 + h.f(x_1, y_1) = 3 + 1.\left(1 + \frac{3}{2}\right) = 5.5$$

Sol. 8 (c) $\because$ The velocity field for a steady, incompressible fluid between two concentric circles is $v_\theta = r\,\Omega(r)$, driven by rotational shear.

Let's work with cylindrical coordinates; for navier stokes equations.

$\because$ Steady $\therefore \frac{\partial}{\partial t} = 0$, axisymmetric $\therefore \frac{\partial}{\partial \theta} = 0$ and no axial flow $\therefore v_z = 0$

The redial momentum equation (balancing between centrifugal force and pressure)

$$\frac{\partial p}{\partial r} = \frac{\rho v_\theta^2}{r}$$

$\therefore$ Equation is: $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\,v_\theta)\right] = 0$

On integrating, we get $v_\theta(r) = c_1 r + \frac{c_2}{r}$; c_1 and c_2 are arbitrary constants. (1)

Case (i) Boundary conditions:

At $r = a$ (inner circle); $v_\theta(a) = a\Omega$

At $r = b$ (outer circle) ; $v_\theta(b) = 0$ (2)

On using (2) in (1)

$$c_1 a + \frac{c_2}{a} = a\Omega,\ c_1 b + \frac{c_2}{b} = 0$$

$$\therefore \quad c_2 = -c_1 b^2 \quad \therefore c_1 a - c_1 \frac{b^2}{2} = a\Omega$$

$$\therefore \quad c_1 = \frac{a^2\Omega}{a^2 - b^2}$$

$\therefore$ Velocity distribution is $v_\theta(r) = \frac{a^2\Omega}{a^2 - b^2}\left(r - \frac{b^2}{r}\right)$

Case (ii) Boundary Conditions:

At $r = a;\ v_\theta(a) = 0$

At $r = b$; $v_\theta(b) = b\Omega$ (3)

On using (3) in (1)

$$c_1 a + \frac{c_2}{a} = 0 \Rightarrow c_2 = -c_1 a^2$$

$$c_1 b + \frac{c_2}{b} = b\Omega \therefore c_1 = \frac{b^2\Omega}{b^2 - a^2}$$

$$\therefore \quad v_\theta(r) = \frac{b^2\Omega}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)$$