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PAPER - 01

# FULL ANALYSIS

# MATHEMATICS OPTIONAL

## UPSC CSE 2026

By Upendra Singh Sir

See teaching methodology https://www.youtube.com/@MindsetMakersPrepareInRightWay

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# UPSC CSE 2026: Mathematics Optional PAPER-01

***Observations of 2026 mathematics Optional papers UPSC CSE***

**#hard** attempts by UPSC to change the trend of maths optional of years. **Not tough questions*: but the way.*

**#Earlier** hardly any one or two maximum questions used to be of nature like open ended. But this year it happened with almost 70% questions and in both papers.

As a person indulged into higher mathematics since a long, I'm sure that paper setting done by professors who are having clarity of concepts and depth of every topic.

**#Resulting** as: No question was tough but needed a conceptual understanding. If you don't have that conceptual clarity: you'll get panicked and starts moving another to another question in a hope that you'll get question in your zone and as I said it was done with every question in one or two sub parts; so students found it very difficult. Many lost their temperament.

As I've been consistently trying to read patterns by UPSC; yes they do it randomly with some optional. The year 2025 papers were easiest and it was well settled news in air that maths optional is easier to do now compared to other Optionals. UPSC body observes everything. So they played this year. It's not only with maths but it happens with every optional in some years.

**#above** point is very crucial: Because it might give you a switching to other optional and what's the guarantee that the next year's new optional will not be on same cycle of gambling optionals. Yes it happens.

**#Point:** we all know UPSC luck factors: let's say you started preparing in 2024 and you got very good in maths in 2025. A person who started preparing in 2019 in increasing difficulty level of maths trend till 2023, so he/she might be trapped but he/also done very good in 2024/25 because of not changing the optional but being in cycle and finally got.
So changing optional just because of trend is not a good idea.

**#Analysis of questions:**
I am sharing my side here -
No need to change the resources but change the mindset. In place of focusing just on a large of questions solving from n number of different books;

you need to be conceptually clear and then attempting problems of different categories from one systematically designed resource/book/teacher. You may get mindset makers books at your address from here https://mindsetmakers.in/upsc-study-material/

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**Track record;** when first time in year 2024, upsc asked about Existence and Uniqueness of IVPs of first order, it was already given in mindset makers books and this year as well the kind of conceptual clarity is focused in teaching methodology, given the confidence.

I'm sure that if you follow my words, you'll get what you are required. The way I try to decode the concept and if students follow properly their home work part, I assure, even in this year's paper pattern, you'll be in situation to try problems of such nature and yes it may not be the case that you get same problem but the attitude towards any open problem after conceptual clarity with categorical problems will give you a comfortable situation and you won't be panicked.

Also; it's an alarming situation for me as a teacher that yes coaching/batch/organisational SoPs have their timelines to complete the syllabus but I'll be consistently forcing students to complete their tasks which I give after the class. Earlier I used to be like okay it's students choice if they want to go for books problems or not but not I'll be forcefully implementing it with serious students. Because I'm getting confidence about my teaching methodology year by year.

Suggestions are welcome.
*Let's learn together and perform better*

keep learning keep smiling
Upendra Singh

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# Initiative: For those who've done the syllabus/written multiple main/s CSE/IFoS

*Problem Solving Techniques*
UPSC CSE IFoS 2027/28
Session-1
Mathematics Optional
YouTube live Saturday 7-11 pm at See teaching methodology
https://www.youtube.com/@MindsetMakersPrepareInRightWay

Let's develop mathematical skills to hit new problems based on conceptual clarity.

I've been working over this plan since a long and wanted to implement only when it's in concrete shape. So now on every Saturday; we can plan such sessions.

**UPSC CSE MAIN EXAMINATION 2026**

**MATHEMATICS OPTIONAL PAPER-01**

**Section A**

Q1. (a) Find all the values of a for which the system of homogeneous equations

$$ax + y + z = 0$$
$$x + y - z = 0$$
$$x + y - az = 0$$

has non-trivial solution, and hence determine all the solutions for each value of $a$

(b) If A and B are similar matrices, then show that A and B have the same rank, trace, characteristic polynomial and eigenvalues.

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(c) if $u(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$ , then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

(d) Using Maclaurin's theorem, obtain the first four terms of the expansion of $e^{x\cos x}$ in ascending powers of $x$. Hence or otherwise find the limit of $\frac{e^x - e^{x\cos x}}{x - \sin x}$ as $x$ tends to zero.

(e) Obtain the equation of the sphere having its centre on the line $5y + 2z = 0 = 2x - 3y$ and passing through the points $(0, -2, -4)$ and $(2, -1, -1)$.

Q2. (a) Let $T : P_2[x] \to P_2[x]$ b e a linear transformation defined by

$T\left(a_0 + a_1 x + a_2 x^2\right) = -\left(a_0 + 2a_1 + a_2\right) + \left(2a_0 + 3a_1\right)x - 2\left(a_0 + a_1\right)x^2$ where $P_2[x]$ denotes the set of all polynomials in $x$ of degree $\le 2$ and $a_0, a_1, a_2 \in \mathbb{R}$. Find the eigenvalues and corresponding eigenvectors of T using the matrix representation of T with respect to the standard basis of $P_2[x]$.

(b) If the sum of the lengths of the hypotenuse the another side of a right-angled triangle be given, then find the angle between these sides so that the area of the triangle is maximum.

(c) Find the points on the lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ which are nearest to each other. Hence find the shortest distance between the lines and its equation.

Q3. (a) Given $P_3[x]=\{a_0+a_1x+a_2x^2+a_3x^3 \mid a_0,a_1,a_2,a_3\in\mathbb{R}\}$ as a vector space over the field $\mathbb{R}$.

(i) Let $H=\left\{p(x)\in P_3[x] \mid \int_{-1}^{1} p(x)\,dx=0\right\}$. Prove that H is a subspace of $P_3[x]$ and find a basis of H.

(ii) Find the subspace K of $P_3[x]$ such that $P_3[x]=H\oplus K$.

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(b) (i) Evaluate the double integral $\int_0^2\int_{x^2+1}^{2x+1} x^2 y\,dy\,dx$ by reversing the order of integration.

(ii) Find the volume of the solid, which is formed from the intersection of the coordinate planes $x=0, y=0, z=0$ and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(c) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, then find the equations of the other two.

Q4. (a) Let F be a subfield of complex numbers and let T be the linear transformation from $F^3$

into $F^3$, defined by $T(x,y,z)=(x-y+2z,2x+y,-x-2y+2z)$.

If $(a,b,c)$ is a vector in $F^3$, then answer the following:

(i) Find the condition on $a,b,c$ such that the vector $(a,b,c)$ be in the null space of T. What us the nullity of T?

(ii) Find the condition on $a, b, c$ such that vector $(a, b, c)$ be in the range of T. What is the rank of T?

(b) Trace the curve $y^2(a+x) = x^2(3a-x)$.

(c) (i) Find the equations of the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which

pass through the points $(2,3,-4)$ and $\left(2,-1,\frac{4}{3}\right)$.

(ii) Find the equation of the right circular cylinder whose axis is $x = 2y = -z$ and radius is 4. Prove that the area of the section of this cylinder by the plane $z = 0$ is $24\pi$.

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**Section B**

Q5. (a) If $y_1$ and $y_2$ are two solutions of $\frac{dy}{dx}+P(x)y=Q(x)$, and $y_2=y_1u$, then show then

$u=1+ae^{-\int\left(\frac{Q}{y_1}\right)dx}$, where $a$ is a non-zero constant.

(b) Apply Laplace transform to solve $\left(D^2+D\right)y(t)=2, D\equiv\frac{d}{dt}$, subject to the conditions $y(0)=3, y'(0)=1$.

(c) A smooth paraboloid of revolution is fixed with its axis vertical and vertex upwards; on it is placed a heavy elastic string of unstretched length $2\pi b$. When the string is in equilibrium, apply the principle of virtual work to show that it rests in the form of a circle of radius $\frac{4\pi ab\lambda}{4\pi ab-Wb}$, where $W$ is the weight of the string, $\lambda$ is the modulus of elasticity and $4a$ is the latus rectum of the generating parabola.

(d) A particle falls from rest at the vertex of an inverted catenary. Prove that the particle will leave the curve when the path described is $\sqrt{3}$ times the vertical distance through which the particle has fallen.

(e) Given $\vec{F}=(2x+3y)\hat{i}-4z\hat{j}-5x\hat{k}$ and S is the surface $2x+4y+4z=5$, bounded by $x=0$, $x=2$, $y=0$ and $y=3$. Evaluate $\iint_S\left(\nabla\times\vec{F}\right).\hat{n}\,dS$.

Q6. (a) Solve $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\sin(\log x)+1}{x}\log x$

(b) (i) A battleship is steaming ahead with velocity $u$. A gun is mounted on the ship so as to point straight backwards and is set at an angle of elevation $\alpha$. If $v$ be the velocity of projection relative to the gun, show that the range is $\frac{2v}{g}\sin\alpha\left(v\cos\alpha-u\right)$ and the angle for maximum range is $\cos^{-1}\left(\frac{u+\sqrt{u^2+8v^2}}{4v}\right)$ where $g$ stands for acceleration due to gravity.

(ii) A uniform chain of length $l$ has one end fixed at a height $h$ above a rough table and rests in a vertical plane so that a portion of it lies in a straight line on the table. Prove that if the chain is on the point of slipping, the length on the table is $(l+\mu h)-\sqrt{(\mu^2+1)h^2+2\mu lh}$, where $\mu$ is the coefficient of friction.

(c) Determine $\hat{T},\hat{N},\hat{B},\kappa$ and $\tau$ for the parabola $y^2=4ax$, where $a$ is a constant. Here $\hat{T},\hat{N},\hat{B},\kappa$ and $\tau$ denote unit tangent vector, unit principal normal vector, unit binormal vector, curvature and torsion respectively.

Q7. (a) A light elastic string of natural length $l$ has one extremity fixed at a point A and the other attached to a stone (mass $m$) the weight of which in equilibrium would extend the string to a length $b$. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{b^2-l^2}$ below the equilibrium position and this depth is attained in time $\sqrt{\frac{2l}{g}}+\sqrt{\frac{b-l}{g}}\left\{\pi-\cos^{-1}\sqrt{\frac{b-l}{b+l}}\right\}$. Prove also that if the greatest depth below A be $l\cot^2\frac{\theta}{2}$, then the modulus of elasticity is $\frac{1}{2}mg\tan^2\theta$.

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(b) If $\vec{F}=4xz\hat{i}-y^2\hat{j}+yz\hat{k}$, then evaluate $\iint_S \vec{F}.\hat{n}\,dS$, where S is the surface of a unit cube with two opposite corners at $(0,0,0)$ and $(1,1,1)$ respectively. Hence verify the divergence theorem.

(c) (i) Show that $(x+y+1)^{-4}$ is an integrating factor of

$$(2x-y-1)y\,dx+(2y-x-1)x\,dy=0$$

Hence solve it.

(iii) If $L\{f(t)\}=\overline{f}(s)$, then show that $L\left\{\frac{f(t)}{t}\right\}=\int_S^\infty \overline{f}(u)\,du$. Hence evaluate $L\left\{\frac{\cos at-\cos bt}{t}\right\}$.

Q8. (a) Solve $\frac{d^2y}{dx^2} + a^2 y = \sec ax$ by the method of variation of parameters.

(b) AB is a uniform rod of length $l$ and weight W, which can turn freely about a fixed point in its length distant $\frac{l}{3}$ from A. AC and BC are light strings each of length $\frac{5}{6}l$ m attached to a particle C of weight $w$. Prove that if W is less than $2w$, there will be stable equilibrium with AB inclined to the horizontal at an angle $\tan^{-1}\left(\frac{W+w}{4w}\right)$.

(c) (i) For a scalar point function $\phi$ and a vector point function $\vec{F}$, prove that $\nabla \times \left(\phi \vec{F}\right) = (\nabla \phi) \times \vec{F} + \phi\left(\nabla \times \vec{F}\right)$.

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \left|\vec{r}\right|$, then verify the above identity for $\phi = \frac{1}{r}$ and $\vec{F} = \vec{r}$.

(ii) If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$, then show that $\left|\vec{r}\,' \times \vec{r}\,''\right|^2 = a^2\left(a^2 + b^2\right)$ and $\left|\vec{r}\,', \vec{r}\,'', \vec{r}\,'''\right|^2 = a^2 b$

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**Section A**

**Sol. 1 (a)** $ax+y+z=0$

$x+y-z=0$

$x+y-az=0$

Comparing with $AX=0$;

$$A=\begin{bmatrix} a & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -a \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let's reduce matrix A into echelon form by applying elementary row operations.

$$A=\begin{bmatrix} a & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -a \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 1 & -1 \\ a & 1 & 1 \\ 1 & 1 & -a \end{bmatrix} \begin{matrix} R_2 \to -aR_1+R_2 \\ R_3 \to -R_1+R_3 \end{matrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & (1-a) & (a+1) \\ 0 & 0 & (1-a) \end{bmatrix} \quad (1)$$

$\because$ For $AX=0$; to have non-trivial (non-zero) Solution: $|A|=0$ $(\rho(A)<3)$

For this, we must have $(1-a)^2=0 \Rightarrow a=1$

On putting $a=1$; in echelon form (1) of A, we get $B=\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $BX=0$ gives $x+y-z=0$

$2z=0 \quad \therefore z=0$

$\therefore$ We have $x+y=0$

All solutions of given system of equations are of the form $X=\begin{bmatrix} \alpha \\ -\alpha \\ 0 \end{bmatrix}; \alpha \in \mathbb{R}$.

**Sol. 1 (b)** By definition, we know that A and B are similar matrices, if there exists an invertible matrix P s.t. $B=P^{-1}AP$ (1)

For rank: If we multiply some matrix with an invertible matrix then rank of that matrix remains same.

$\therefore$ Rank (B) = rank ( $P^{-1}AP$ ) = rank (AP) = rank (A)

For char. Poly:

$\because$ $|B-\lambda I|=|P^{-1}AP-\lambda I|=|P^{-1}AP-P^{-1}\lambda I\, P|$

$=|P^{-1}(A-\lambda I)P|$



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$$= \left|P^{-1}\right| \left|A-\lambda I\right| \left|P\right|$$

$$= \frac{1}{|P|} \left|A-\lambda I\right| \left|P\right|$$

$\Rightarrow$ $\quad |B-\lambda I| = |A-\lambda I|$

$\Rightarrow$ $\quad$ Char. Polynomial. B = char. Poly. A

For eigenvalues: For $\lambda$ to be an eigenvalues of matrix A the necessary condition is $|A-\lambda I| = 0$

$\Rightarrow$ $\quad |B-\lambda I| = 0$ from above step if B is similar to A.

$\Rightarrow$ $\quad \lambda$ is also an eigenvalue of B. (2)

<u>For trace:</u> $\because$ Trace of A = sum of eigenvalues of A

= sum of eigenvalues of B

= trace B.

**Sol. 1 (c)** Given $u = \left(x^2+y^2+z^2\right)^{-1/2} = \left(r^2\right)^{-1/2} = \frac{1}{r}$ (1)

$\because \quad r^2 = x^2+y^2+z^2$

$\therefore \quad 2r = \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$ (2)

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$\because \quad u_x = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial r} = \frac{\partial x}{\partial r} = \left(-\frac{1}{r^2}\right).\frac{x}{r} = \frac{-x}{r^3}$

$\therefore \quad u_{xx} = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = \frac{-\partial}{\partial x}\left(x.r^{-3}\right)$

$$= -1r^{-3} + (-x).\left(-3r^{-4}.\frac{\partial r}{\partial x}\right)$$

$$= -\frac{1}{r^3} + \frac{3x^2}{r^5} \text{ ; using } \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly (symmetric):

$$u_{yy} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$$

$$u_{zz} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$$

$$\therefore \quad u_{xx}+u_{yy}+u_{zz}=\frac{-3}{r^3}+\frac{3\left(x^2+y^2+z^2\right)}{r^5}$$

$$=\frac{-3}{r^3}+3.\frac{r^2}{r^5}=0$$

**Sol. 1 (d)** In powers of $x$; $\therefore$ Putting $a=0$ in Taylor's expansion about point a.

$$\because \quad y(x)=e^{x\cos x} \quad \therefore y(0)=e^{0\cos 0}=1$$

$$y'(x)=e^{\cos x}.\frac{d}{dx}(x\cos x)=e^{\cos x}(\cos x-x\sin x)$$

$$\therefore \quad y'(0)=1.\ (\cos 0-0)=1$$

$$\because \quad y'=y(\cos x-x\sin x)$$

$$\therefore \quad y''=y'(\cos x-x\sin x)+y(-\sin x-\sin x-x\cos x)$$

$$\Rightarrow \quad y''(x)=y'(\cos x-x\sin x)-y(x\cos x+2\sin x)$$

$$\therefore \quad y''(0)=y'(0)(\cos 0-0)-y(0)(0\cos 0+2\sin 0)=1$$

$$\therefore \quad y'''(x)=y''(\cos x-x\sin x)+y'(-\sin x-\sin x-x\cos x)$$

$$-y'(2\sin x+x\cos x)-y(2\cos 2x+\cos x-x\sin x)$$

$$\therefore \quad y'''(0)=1.1-2.1.0-1.3=-2$$

By Maclaurins; we have $f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$

$$\therefore \quad y(x)=e^{x\cos x}=1+x.1+\frac{x^2}{2}.1+\frac{x^3}{6}.(-2)+...$$

$$\Rightarrow \quad e^{x\cos x}=1+x+\frac{x^2}{2!}-2\frac{x^3}{3!}+... \qquad (1)$$

$$\because \quad e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+....... \qquad (2)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-........ \qquad (3)$$

$$\therefore \quad x-\sin x=\frac{x^3}{3!}-\frac{x^5}{5!}+....... \qquad (4)$$

$$e^x-e^{x\cos x}=3.\frac{x^3}{3!}+0\left(x^4\right) \qquad (5)$$

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On dividing (5) and (4), we get $\dfrac{e^x - e^{x\cos x}}{x - \sin x} = \dfrac{3\dfrac{x^3}{3!} + o(x^4)}{\dfrac{x^3}{3!} + o(x^5)} = \dfrac{3 + o(x)}{1 + o(x^2)}$

$\therefore \quad \lim\limits_{x \to 0} \dfrac{e^x - e^{x\cos x}}{x - \sin x} = \dfrac{3+0}{1+0} = 3$

**Sol. 1 (e)** $\because$ Given line in symmetric form, can be written as $x = \dfrac{3}{2}y,\ z = -\dfrac{5}{2}y$

$\dfrac{x}{3} = \dfrac{y}{2} = \dfrac{z}{-5}$ (1)

$\therefore$ In parameter $r$, we have $x = 3r,\ z = -5r,\ y = 2r$

$\therefore$ Centre of asked sphere is taken as: $(3r, -5r, 2r)$ (2)

$\because$ Sphere Passes: through points $(0,-2,-4)$ and $(2,-1,-1)$

$\therefore$ The distances of centre of sphere from both points must be same.

$\therefore$ We have

$\sqrt{(0-3r)^2 + (-2-5r)^2 + (-4-2r)^2} = \sqrt{(2-3r)^2 + (-1-5r)^2 + (-1-2r)^2}$

$\Rightarrow \quad 38r^2 - 32r = 38r^2 - 18r + 6 - 20$

$\Rightarrow \quad 14r = 14 \Rightarrow r = 1$

$\therefore$ Centre of sphere is $(3,-5,2)$ (3)

Radius of sphere = distance of either point from centre

$= \sqrt{38 \times 1^2 - 32 \times 1 + 20} = \sqrt{26}$

$\therefore$ Required equation of sphere is: $(x-3)^2 + (y-2)^2 + (z+5)^2 = \left(\sqrt{26}\right)^2$

$\Rightarrow \quad x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$

**Sol. 2 (a)** $T : P_2[x] \to P_2[x]$

$T\left(a_0 + a_1x + a_2x^2\right) = -\left(a_0 + 2a_1 + a_2\right) + \left(2a_0 + 3a_1\right)x - 2\left(a_0 + a_1\right)x^2$

Standard basis of $P_2[x]$

$B = \{1, x, x^2\}$

$T(1) = T\left(1.1 + 0.x + 0.x^2\right)$

$= -(1 + 2\times 0 + 0) + (2\times 1 - 3\times 0)x - 2(1+0)x^2$

$\Rightarrow \quad T(1) = -1.1 + 2x - 2x^2 \quad \therefore [T(1)]_B = [-1 \quad 2 \quad -2]^T$

$T(x) = T\left(0.1 + 1.x + 0.x^2\right)$

$$=-(0+2\times1+0)+(2\times0+3\times1)x-2(0+1)x^2$$

$$=-2.1+3.x-2.x^2 \quad \therefore [T(x)]_B=[-2 \quad 3 \quad -2]^T$$

$$T(x^2)=T(0.1+0.x+1.x^2)$$

$$=-(0+2\times0+1)+2(2\times0+3\times0)x-2(0+0)x^2$$

$$=-1.1+0.x+0.x^2 \quad \therefore [T(x^2)]_B=[-1 \quad 0 \quad 0]^T$$

$\therefore$ Matrix of w.r.t B is $A=\begin{bmatrix}-1 & -2 & -1\\ 2 & 3 & 0\\ -2 & -2 & 0\end{bmatrix}$

Finding eigenvalues of A:

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix}1-\lambda & -2 & -1\\ 2 & 3-\lambda & 0\\ -2 & -2 & -\lambda\end{vmatrix}=0$$

$$\Rightarrow -1[2(-2)-(-2)(3-\lambda)]-\lambda[(1-\lambda)(3-\lambda)-2(-2)]=0$$

$$\Rightarrow -\lambda^3+2\lambda^2+\lambda-2=0$$

$$\Rightarrow (\lambda+1)(\lambda+1)(\lambda-2)=0$$

$$\Rightarrow \lambda_1=-1, \lambda_2=1, \lambda_3=2$$

Eigenvectors corresponding to $\lambda_1=-1$

Solving $[A+1.I]X=0; X_F\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}0 & -2 & -1\\ 2 & 4 & 0\\ -2 & -2 & 1\end{bmatrix}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow$ $x=-2y$ and $z=-2y$

$\therefore$ Choosing $y=1 \quad \therefore x=-2; -z=-2$

$\therefore$ $X_1=[-2 \quad 1 \quad 2]^T$

Eigenvector corresponding to $\lambda_2=1$

Solving $[A-1.I]X_2=0$

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$$\Rightarrow \quad \begin{bmatrix} -2 & -2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad y = -x,\ z = 0$

$\therefore \quad X_2 = [1 \quad -1 \quad 0]^T$

Eigenvector corresponding to $\lambda_3 = 2$

Solving $[A - 2I]X_3 = 0$

$$\Rightarrow \quad \begin{bmatrix} -3 & -2 & -1 \\ 2 & 1 & 0 \\ -2 & -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad y = -2x,\ z = x$

$\therefore \quad X_3 = [1 - 2\ 1]^T$

$\therefore$ Required eigenvectors are:

$V_1 = 2 - x + 2x^2; X_1$

$V_2 = 1 - x\ ;\ X_2$

$V_3 = 1 - 2x + x^2;\ X_3$

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**Sol. 2 (b)** Let's try by Lagrange's method of undermined multiplier method.

Let $\theta$ be the angle between hypotenuse $(h)$ and given side $(x)$

$\therefore \quad \cos\theta = \dfrac{x}{h}$

$\therefore$ Third side $= \sqrt{h^2 - x^2}$

Let sum $x + h = k$

Area $= \dfrac{1}{2}x\sqrt{h^2 - x^2} = A$

To get maxima; Let's go by $f = 4A^2$ (just to keep simple calculation and at last well manage).

Let $f(h,x) = x^2(h^2 - x^2) = x^2h^2 - x^2$ (1)

Given condition $\phi : h + x - k = 0$ (2)

$\therefore$ Considering $F = f + \lambda\phi$ (3)

$F = (x^2h^2 - x^4) - \lambda(h + x - k)$

For points of maxima/minima:

$$\frac{\partial F}{\partial h}=0 \Rightarrow 2x^2h-\lambda=0 \Rightarrow \lambda=2x^2h \qquad (4)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2xh^2-4x^3-\lambda=0 \Rightarrow \lambda=2xh^2-4x^3 \qquad (5)$$

On solving (4) and (5) we get (to eliminate $\lambda$ )

$2x^2h=2xh^2-4x^3$

$\Rightarrow \quad xh=h^2-2x^2 \Rightarrow h^2-xh-2x^2=0$

$\Rightarrow (h-2x)(h+x)=0$

For $x \,\&\, h$ positive, $h-2x=0$

$$h=2x \qquad (6)$$

On using (6) in $\cos\theta=\frac{x}{2h}=\frac{x}{2x}=\frac{1}{2}$

$\therefore$ Required angle is $\theta=\frac{\pi}{3}$.

**Sol. 2 (c).** $\because$ Point on line (1) is $P(r+3;-2r+5,r+7)$

Point on line (2) is $Q(7s-1,-6s-1,s-1)$

$\because$ $dr's$ of line PQ are: $7s-r-4,-6s+2r-6,s-r-8$

$\because$ PQ is perpendicular to line (1) with d.c's $1,-2,1$

$\therefore$ $1.(78-r-4)-2.(-6s+2r-6)+1.(s-r-8)=0$

$$\Rightarrow \quad 20s-6r=0 \Rightarrow 10s-3r=0 \qquad (3)$$

`Similarly with line (2), we get $7.(7s-r-4)-6(-6s+2r-6)+1.(s-r-8)=0$

$$\Rightarrow \quad 86s-20r=0 \Rightarrow 43s-10r=0 \qquad (4)$$

$\therefore$ On solving (3) and (4), we get $r=0, s=0$

$\therefore$ P is $(3,5,7)$ and Q is $(-1,-1,-1)$

$\therefore$ Shortest distance $=\sqrt{(1-3)^2+(-1-5)^2+(-1-7)^2}=\sqrt{116}=2\sqrt{29}$.

$\because$ d.r's of line PQ are $P:-1-3,\ -1-5,-1-7$ i.e. $-4,-6,-8$. i.e. $2,3,4$

$\because$ Required equation of line is $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$

**Sol. 3 (a)** $\because\ H=\left\{p(x)\in\mathbb{P}_3[x]:\int_{-1}^{1}p(x)\,dx=0\right\}$

$$=\left\{a_0+a_1x+a_2x^2+a_3x^3:\int_{-1}^{1}\left(a_0+a_1x+a_2x^2+a_3x^3\right)dx=0\right\}$$

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$$= \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 : \int_{-1}^{1} \left( a_0 + a_2 x^2 \right) x dx = 0 \right\}$$

$$= \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 : 2\left( a_0 + \frac{a_2}{3} \right) = 0 \right\}$$

$$= \left\{ a_0 + a_1 x - 3a_0 x^2 + a_3 x^3 : a_0, a, a_3 \in \mathbb{R} \right\} \quad (1)$$

$\because$ $\int_{-1}^{1} 0(x)\, dx = \int_{-1}^{1} 0\, dx = 0 \quad \therefore 0(x) \in H$

$\therefore$ H is non-empty subset of $P_3[x]$

Also for $\alpha, \beta \in \mathbb{R}$, $u(x), v(x) \in H$

We have $\int_{-1}^{1} \left( \alpha u(x) + \beta v(x) \right) dx = \alpha \int_{-1}^{1} u(x)\, dx + \int_{-1}^{1} v(x)\, dx$

$= \alpha .0 + \beta .0 = 0$

$\Rightarrow$ $\alpha u(x) + \beta v(x) \in H$

$\therefore$ H is a subspace of $P_3[x]$ over $\mathbb{R}$.

From (1), A basis of H is $B = \left\{ 1 - 3x^2, x, x^3 \right\}$ : using (1)

(ii) Finding subspace K:

$\because$ $P_3 = H \oplus K$ i.e. $H \cap K = \{0\}$ and $\because \dim P_3 = 3 + 1 = 4$

$\dim H = 3$

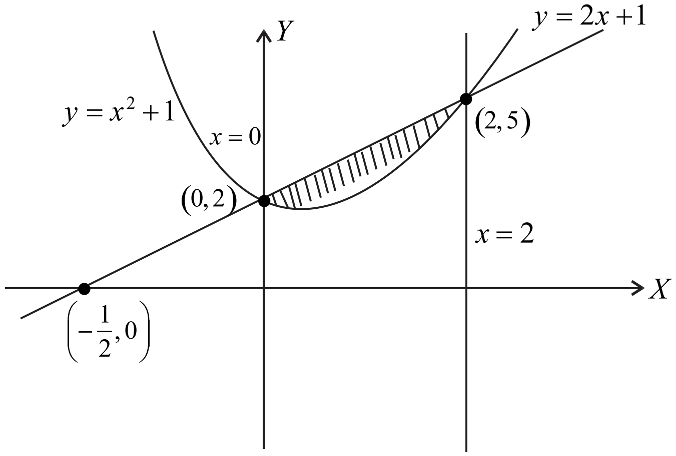
$\therefore$ $\dim K = 4 - 3 = 1$

$\therefore$ In polynomial space : a subspace with dimension 1 is 0+1 i.e. K = spanned by 1.

Or any non-zero real number only.

$\therefore$ $K = \{ \alpha : \alpha \in \mathbb{R} \}$; non-zero constant polynomials.

**Sol. 3 (b) (i)** $I = \int_{x=0}^{2} \int_{y=x^2+1}^{2x+1} x^2 y\, dx\, dy$




Solving $y = x^2+1$ and $y = 2x+1$, we get $x^2+1 = 2x+1$

$\Rightarrow \quad x^2 - 2x = 0$

$\Rightarrow \quad x = 0, 2$

$\therefore \quad y = 1, 5$

$\therefore$ Points are $(0,1)$ and $(2,5)$

$\therefore$ we have limit of integration as when $y=1$ to 5 then $x = \dfrac{y-1}{2}$ to $\sqrt{(y-1)}$

$$\therefore \quad I = \int_{y=1}^{5}\int_{x=\frac{y-1}{2}}^{\sqrt{y-1}} x^2\, y\, dy\, dx$$

$$= \int_{y=1}^{5} y\left[\frac{x^3}{3}\right]_{\frac{y-1}{2}}^{\sqrt{y-1}} = \int_{y=1}^{5}\frac{y}{3}\left[(y-1)^{3/2} - \frac{(y-1)^3}{8}\right]dy$$

Put $y-1 = u \Rightarrow dy = du$

$$\therefore \quad I = \frac{1}{3}\int_0^4 (u+1)\left(u^{1/3} - \frac{u^3}{8}\right)du$$

$$= \frac{1}{3}\int_0^4\left(u^{5/2} + u^{3/2} - \frac{u^4}{8} - \frac{u^3}{8}\right)du$$

$$= \frac{1}{3}\left(\frac{2}{7}u^{7/2} + \frac{2}{5}u^{5/2} - \frac{u^5}{40} - \frac{u^4}{32}\right)_0^4$$

$$= \frac{1}{3}\left(\frac{256}{7} + \frac{64}{3} - \frac{1024}{40} - \frac{256}{32}\right)$$

$$= \frac{1}{3}\left(\frac{256}{7} + \frac{64}{5} - \frac{128}{5} - 8\right) = \frac{1}{3}\times\frac{552}{35} = \frac{184}{35}$$

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**Sol. 3 (b) (ii)** Required volume is given by $V=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dx\,dy\,dz$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c.\left(1-\frac{x}{a}-\frac{y}{b}\right)dx\,dy$$

$$=c\int_0^a\left[\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{b}\left(b^2\left(1-\frac{x}{a}\right)^2\right)\right]dx$$

$$=c\left[\frac{b}{a}.a^2\frac{1}{a^2}.\frac{a^3}{3}-b\left(a+\frac{1}{a}\frac{a^2}{2}-\frac{2}{a}a^2\right)\right]$$

$$=\frac{1}{6}abc$$

**Sol. 3 (c)** $\because$ Given cone has vertex $(0,0,0)$.

(Homogenous equation of second degree represent a cone with vertex at origin)

One given generator is $\frac{x}{1}-\frac{y}{1}=\frac{z}{-1}$ (1)

Now Let's find equation of plane passing through $(0,0,0)$ and perpendicular to line (1), we have $1(x-0)+2(y-0)+3(z-0)=0$

$\Rightarrow$ $x+2y+3z=0$ (2)

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ given required generators

(lie on plane (2) as well)

$\therefore$ We have $(lr,mr,nr)$ lies on (2) and given equation of cone.

$\therefore$ we have $(l+2m+3n)r=0$

$(5mn-8nl-3mn)r^2=0$

Let's solve $l+2m+3n=0$ (3)

$5mn-8nl-3mn=0$ (4)

From (3), $l=-2m-3n$; using in (4), we get $6m^2+30mn+24h^2=0$

$\Rightarrow$ $(m+n)(m+4n)=0$

For $m+n=0\Rightarrow m=-n\Rightarrow\frac{m}{1}=\frac{n}{-1}=\frac{2m-3n}{-2.1+3}=\frac{l}{1}$

$\therefore$ $d\,r's$ are: $(1,1,-1)$

For $m+4n=0\ \Rightarrow\frac{m}{4}=\frac{n}{-1}=\frac{-2m-3n}{-2\times4+3}=\frac{l}{-5}$

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$\therefore$ $dr's$ are $(-5,4,-1)$ or $(5,-4,1)$

$\therefore$ Required other two generators are:

$\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-0}{-1}$ i.e. $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ and $\frac{x-0}{5}=\frac{y-0}{-4}=\frac{z}{1}$ i.e. $\frac{x}{5}=\frac{y}{4}=\frac{z}{1}$

**Sol. 4 (a)** $T(x,y,z)=(x-y+2z,2x+y,-x-2y+2z)$

Let $(a,b,c)\in NullSPT$

$\Rightarrow$ $T(a,b,c)=(0,0,0)$

$\Rightarrow$ $(a-b+2c,2a+b,-a-2b+2c)=(0,0,0)$

$\Rightarrow$ $a-b+2c=0$

$2a+b=0$

$-a-2+2c=0$

$$A=\begin{bmatrix}1&-1&2\\2&1&0\\-1&-2&2\end{bmatrix}\sim\begin{bmatrix}1&-1&2\\0&3&-4\\0&-3&4\end{bmatrix}\sim\begin{bmatrix}1&-1&2\\0&3&-4\\0&0&0\end{bmatrix}$$

$\therefore$ $3b-4c=0\Rightarrow b=\frac{4}{3}c$

$a-b+2c=0$

$\therefore$ $a=-\frac{2}{3}c$

So; above are required conditions on $a,b,c$

Nullity $=1$

For range space: $(a,b,c)\in$ Range SPT

If $\exists$ some $(x,y,z)\in F^3$ such that $(a,b,c)=T(x,y,z)$

$\Rightarrow$ $(a,b,c)=(c-y+2z,2x+y,-x-2y+2z)$

$\Rightarrow$ $x-y+2z=a$

$2x+y=b$ (4)

$-x-2y+2z=c$

$$[A:B]=\left[\begin{array}{ccc|c}1&-1&2&a\\2&1&0&b\\-1&-2&2&c\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-1&2&a\\0&3&-4&b-2a\\0&-3&4&c+a\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-1&2&a\\0&3&-4&b-2a\\0&0&0&a-b-c\end{array}\right]$$

For above system (4) to be consistent.

$\therefore$ We must have $a-b-c=0$ is the required condition.

**Sol. 4 (b)** $y^2(a+x)=x^2(3a-x)$

We have $y^2(a+x)-x^2(3a-x)=0$ (1)

- Curve is symmetrical about x-axis.
- Putting lowest degree terms to zero to get tangent at origin; ($\because$ (1) passes through (0,0))

$y^2-3ax^2=0 \Rightarrow y-3x^2=0 \Rightarrow y=\pm\sqrt{3}x$

Two real and distinct tangents at $(0,0)$ $\therefore (0,0)$ is node.

- Parallel asymptotes:

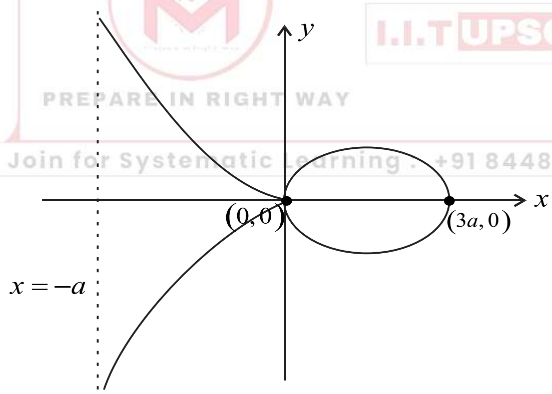
Putting coefficient of highest power of $y$ as zero; $a+x=0 \Rightarrow x=-a$; asymptote parallel to y-axis.

$\because$ $1\neq 0$ $\therefore$ No asymptote parallel to x-axis.

$\because$ $y^2=\dfrac{x^2(3a-x)}{a+x} \Rightarrow y=\pm x\sqrt{\dfrac{3a-x}{x+a}}$ (2)

Observe from (2); for $\left(\dfrac{3a-x}{x+a}\right)<0$; $y$ will be imaginary i.e. $x<3a$ for real $y$ and $x>-a$ $\therefore -a<x\leq 3a$

Intersection on y-axis at $(0,0)$ only x-axis at $(0,0)$ and $(3a,0)$.




**Sol. 4 (c) (i)** Let $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$; required (1)

Generating lies on given hyperboloid.

$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1 \quad (2)$$

$\therefore$ Arbitrary point line on (1) is $(\alpha+lr,\beta+mr,\gamma+nr)$

$\because$ Line (1): is generator, so lies completely on (2).

$\therefore$ We have $\dfrac{(\alpha+lr)^2}{4}+\dfrac{(\beta+mr)^2}{9}-\dfrac{(\gamma-nr)^2}{16}=1$ (3)

$\because$ Quadratic in $r$.

$\because$ Above holds for arbitrary values of $r$

$\therefore$ On putting: coefficient of $r^2=0$

Coefficient of $r=0$; we get ($\because$ (3) must be on identity in $r$)

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad (4)$$

$$\& \ \frac{l.\alpha}{a^2}+\frac{m.\beta}{b^2}-\frac{n.\gamma}{c^2}=0 \quad (5)$$

$\therefore$ On using $(\alpha,\beta,\gamma)$ as $(2,3,-4)$ in (4), (5) we get $\dfrac{2l}{4}+\dfrac{3m}{9}-\dfrac{4n}{16}=0$.

Let if $l=0$ $\therefore$ $m=\dfrac{-3n}{4}$

$\therefore$ $d.r's$: $0,3,4$

Lies is $\dfrac{x-2}{0}=\dfrac{y-3}{3}=\dfrac{z+4}{-4}$ (6)

On using $(\alpha,\beta,\gamma)$ as $\left(2,-1,\dfrac{4}{3}\right)$ in (4) & (5)

$$\frac{l}{2}-\frac{m}{a}-\frac{n}{12}=0.$$

Let if $l=0$, $m=\dfrac{-3n}{4}$

$\therefore$ $d.r's$: $0,3,-4$; same

$\therefore$ For $l\neq 0$; we use $l=\dfrac{2m}{9}+\dfrac{n}{6}$ in other equation.

We have $\dfrac{1}{4}.\left(\dfrac{2m}{9}+\dfrac{n}{6}\right)^2+\dfrac{m^2}{9}-\dfrac{n^2}{16}=0$

$\Rightarrow$ $m=\dfrac{3n}{5}$ and $l=\dfrac{3n}{10}$

$\therefore$ $d.r's$: $3,6,10$

$\therefore$ Line $\dfrac{x-2}{3}=\dfrac{y+1}{6}=\dfrac{z-4/3}{10}$ (7)

(6) & (7) are required generating lines.

**Sol. 4 (c) (ii)** $\because$ Axis is $\dfrac{x}{2}=\dfrac{y}{1}=\dfrac{z}{-2}$ (1)



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Clearly (1) passes through (0,0,0)

$d.e's$ of line (1): $\frac{2}{3},\frac{1}{3},\frac{-2}{3}$

Let $P(x,y,z)$ be an arbitrary point on cylinder

$\therefore$ $OP^2 = x^2 + y^2 + z^2$

Length of projection of OP on axis (1) is (L)

$lx + my + nz = \frac{2x + y - 2z}{3}$

$\therefore$ $PL^2 = OP^2 - OL^2$

$\Rightarrow$ $\left(x^2 + y^2 + z^2\right) - \left(\frac{2x + y - 2z}{3}\right)^2 = 4^z$

$\Rightarrow$ $5x^2 + 8y^2 + 5z^2 - 4xy + 4yz + 8zx - 144 = 0$

Section of (2) by $z = 0$; is given by $5x^2 - 4xy + 8y^2 - 144 = 0$; ellipse

Area of this ellipse $= \frac{2\pi D}{\sqrt{4AC - B^2}}$ for $Ax^2 + Bxy + cy^2 = D$

$\therefore$ Required area $\frac{2\pi(144)}{\sqrt{4\times 5\times 8 - (-4)^2}} = \frac{2\pi \times 144}{\sqrt{144}} = 24\pi$.

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**Section B**

**Sol. 5 (a)** $\because$ $y_1$ and $y_2$ are solutions of given ODE

$\therefore$ On satisfying, we have $y_1' + P(x)y_1 = Q(x)$ (1)

$y_2' + P(x)y_2 = Q(x)$ (2)

$\because$ Given $y_2 = y_1 u$ ; on using in (2), we get $(y_1u)' + P(x)(y_1u) = Q(x)$

$\Rightarrow$ $(y_1'u + y_1u') + P(x)(y_1u) = Q(x)$ (3)

Using (1); in (3)

$uQ(x) + y_1u' = Q(x)$

$$\Rightarrow \quad y_1\frac{du}{dx} = Q(x)(1-u)$$

$\Rightarrow$ $\dfrac{du}{1-u} = \dfrac{Q(x)}{y_1}dx$ on integrating $\displaystyle\int\frac{du}{1-u} = \int\frac{Q(x)}{y_1}dx$

$$\Rightarrow \quad -\log|(1-u)| = -\int\frac{Q(x)}{y_1}dx + \log c$$

$\therefore$ $u = 1 + a\,e^{-\int\frac{Q(x)}{y_1}dx}$ a for arbitrary constant.

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**Sol. 5 (b)** $y''(t) + y'(t) = 2$ (1)

Taking Laplace on both sides of (1) $L\{y''(t)\} + L\{y'(t)\} = L\{2\}$

$$s^2L\{y\} - s\,y(0) - y'(0) + s\,L\{y\} - y(0) = \frac{2}{s}$$

$$s^2L\{y\} - s\times 3 - 1 + s\,L\{y\} - 3 = \frac{2}{s}$$

$$\Rightarrow \quad (s^2+s)L\{y\} - 3s + 4 + \frac{2}{s}$$

$$\Rightarrow \quad L\{y\} = \frac{3s^2+4s+2}{s(s^2+s)} = \frac{3s^2+4s+2}{s^2(s+1)} \qquad (2)$$

Let $\dfrac{3s^2+4s+2}{s^2(s+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+1}$

$\Rightarrow$ $3s^2 + 4s + 2 = As(s+1) + B(s+1) + Cs^2$

$\Rightarrow$ $A + C = 3, A + B = 4, B = 3$

$\therefore$ $A = 2,\ C = 1$

$\therefore$ Using A,B,C in (2), we get $L\{y\} = \frac{2}{s} + \frac{2}{s^2} + \frac{1}{s+1}$

$\Rightarrow$ $y(t) = L^{-1}\left\{\frac{2}{s}\right\} + L^{-1}\left\{\frac{2}{s^2}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\}$

$y(t) = 2 + 2t + e^{-t}$

**Sol. 5 (c)** $z = \frac{r^2}{4a}$; equation of paraboloid with vertex origin, axis (1) vertical (pointing downwards) here $4a$: latus rectum of parabola $r$: radius of circular string at depth $z$.

$\therefore$ Weight (W) acts vertically downwards

Tension (T): acting along the circumference of the string.

$\therefore$ By Hook's law $T = \lambda . \frac{2\pi r - 2\pi b}{2\pi b} = \pi \frac{(r-b)}{b}$ (2)

Let small displacement is $r$ to $r + sr$

Due to gravity $= W.\delta z = W\left(\frac{r}{2a}\right)\delta r$

Due to internal tension: work $= -T.\,\delta l = -T(2\pi\delta r)$

$\therefore$ By principle of virtual work:

$\frac{W}{2a} r - 2\pi s\left(\frac{r-b}{b}\right) = 0$

$\Rightarrow$ $W\,rb - 4\pi a\,\lambda r + 4\pi\lambda b = 0$

$\Rightarrow$ $r = \frac{4\pi a\,b\,\lambda}{4\pi a\,\lambda - Wb}$

**Sol. 5 (d)** Constrained motion;

$\because$ For catenary; $s = c\tan\psi$ (1)

The vertical distance $y$ dropped from the vertex is $y = c(\sec\psi - 1)$ (2)

Equations of motion: $\frac{mv^2}{\rho} = mg\cos\psi - R$

$\because$ $\rho = \frac{ds}{d\psi} = c\sec^2\psi$

For separation: The particle will leave the curve when no reaction R i.e. $R = 0$

Also by conservation of energy : $\frac{1}{2}mv^2 = mg\,y$

$\Rightarrow \quad v^2 = 2gy$

$\therefore$ We have $\dfrac{m(2gy)}{\rho} = mg\cos\psi$

$\Rightarrow \quad \dfrac{2y}{\rho} = \cos\psi$

$\Rightarrow \quad \dfrac{2c(\sec\psi - 1)}{c\sec^2\psi} = \cos\psi$

$\Rightarrow \quad 2\sec\psi - 2 = \sec\psi$

$\Rightarrow \quad \sec\psi = 2$

$\tan\psi = \sqrt{4-1} = \sqrt{3}$

$\therefore$ We have $y = e(2-1) = c,\ s\,c\tan\psi = \sqrt{3}\,c$

$\therefore \quad s = \sqrt{3}\,y$.

**Sol. 5 (e)** $Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+3y & -4z & -5x \end{vmatrix}$

$(4)\hat{i} + 5\,\hat{j} - 3\,\hat{k}$ (1)

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$\because \quad z = -\dfrac{1}{2}x - y$; To take projection on XY-plane

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$\hat{n} = \left(-\dfrac{\partial z}{\partial x}\hat{i} - \dfrac{\partial z}{\partial y}\hat{j} + \hat{k}\right)$

$= \dfrac{1}{2}\hat{i} + \hat{j} + \hat{k}$

$\therefore \quad \hat{n}\,dS = \left(\dfrac{1}{2}\hat{i} + \hat{j} + \hat{k}\right)dx\,dy$

$\therefore \quad \left(Curl\,\vec{F}\right).\hat{n}\,dS = \left(4\hat{i} + 5\hat{j} - 3\hat{k}\right).\left(\dfrac{1}{2}\hat{i} + \hat{j} + \hat{k}\right)dx\,dy$

$= 4\,dx\,dy$

$\therefore \quad \iint_S Curl\,\vec{F}.\hat{n}\,dS = \int_{x=0}^{2}\int_{y=0}^{3} 4\,dx\,dy$

$\int_0^2 12\,dx = 24$.

**Sol. 6 (a)** Putting $x=e^z$ in given ODE, we get $\left(D(D-1)y-3Dy+y\right)=\frac{\sin z+1}{e^z}z$

$\Rightarrow \quad \left(D^2-4D+1\right)y=e^{-z}\left(z\sin z+z\right)$

Finding C.F.; $m^2-4m+1=0$

$m=2\pm\sqrt{3}$

$\therefore$ C.F. $=C_1x^{2+\sqrt{3}}+C_2x^{2-\sqrt{3}}$

Finding PI. $=\frac{1}{\left(D^2-4D+1\right)}\left(e^{-z}\left(2\sin z+z\right)\right)$

$$=e^{-z}.\frac{1}{(D-1)^2-4(D-1)+1}(z\sin z+z)$$

$$=e^{-z}\frac{1}{D^2-6D+6}(z\sin z+z)$$

$$=e^{-z}\left\{z.\frac{1}{\left(D^2-6D+6\right)}\sin z-\frac{2D-6}{\left(D^2-6D+6\right)}\sin z\right\}+\frac{1}{D^2-6D+6}z$$

$$=e^{-z}\left\{(5\sin z+6\cos z)+\frac{1}{3721}(54\sin 2+382\cos 2)\right\}+\frac{1}{6}\left(1-\frac{6D-D^2}{6}\right)^{-1}z$$

$$=e^{-z}\left[\frac{z+1}{6}+\frac{z}{61}(5\sin z+6\cos z)+\frac{1}{3+21}(54\sin z+382\cos z)\right]$$

Putting $t=\log x$ in above.

C.F. + P.I.

$$y=C_1x^{2+\sqrt{3}}+C_2x^{2-\sqrt{3}}+\frac{1}{x}\left[\frac{\log x+1}{6}+\frac{\log x}{61}\left(5\sin(\log x)+6\cos(\log x)+\frac{1}{3721}(54\sin(\log x)+382\cos)\log x\right)\right]$$

**Sol. 6 (b) (i)** Ship moves forward: with constant horizontal velocity $u$.

Gun is pointed straight backwards; tilted at angle $\alpha$.

Velocity of projection relative to gun is $v$.

$v_x=v\cos\alpha-u$ (1) $\because$ Projectile is shot backwards

$v_y=v\sin\alpha$ (2)

Total time for which projectile is in air is purely due to vertical component of velocity.

$$y=v_yT-\frac{1}{2}gT^2$$

$\because$ at ground $y=0$

$\therefore$ We have $T\left(v\sin\alpha-\frac{1}{2}gT\right)=0$

$\therefore$ $T=\frac{2v\sin\alpha}{g}$ (3)

Horizontal range $R=v_x.T$

$$R=(v\cos\alpha-u)\left(\frac{2v\sin\alpha}{g}\right)$$

$$R=\frac{2v}{g}\sin\alpha(v\cos\alpha-u)$$

**Sol. 6 (b) (ii)** Hanging portion; treating as a common catenary

Let $x$ : length of chain resting in a straight line on table (span)

$l$ : length of chain, $s$ : are length (hanging part)

$\mu$ : Coefficient of friction

$w$: weight per unit length

Hanging part is $s=l-x$

$\because$ At lowest point $T_0=w.c$ (1)

For a segment of length $x$ on rough table normal $=w.x$

Maximum limiting force: $F=\mu.wx$ (2)

$\therefore$ In position of slipping $T_0=\mu wx\Rightarrow w.c=\mu wx\Rightarrow c=\mu x$ (3)

$\because$ $y^2=c^2+s^2\Rightarrow(c+h)^2=c^2+s^2$

$\Rightarrow$ $2ch+h^2=s^2$

Using $c=\mu x$ from (3)

$$2\mu x(h)+h^2=(l-x)^2$$

$\Rightarrow$ $x^2-2lx+2\mu hx+l^2-h^2=0$

$$x=(l+\mu h)\pm\sqrt{(l+\mu h)^2-(l^2-h^2)}$$

$\because$ Length $x$ constant exceed $l$.

$\therefore$ $x=l+\mu h-\sqrt{(\mu^2+1)h^2-2\mu lh}$

**Sol. 6 (c)** $\vec{r}(t)=(at^2,2at,0)=at^2\hat{i}+2at\hat{j}+0\hat{k}$

$$\hat{t}=\left(\frac{d\vec{r}}{dt}\right)/\left|\frac{d\vec{r}}{dt}\right|$$

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$$= 2a\frac{1}{\sqrt{t^2+1}}\left(2at\,\hat{i}+2a\,\hat{j}\right)$$

$$= \frac{1}{\sqrt{t^2+1}}\left(t\,\hat{i}+\hat{j}\right)$$

$$= \left(0\,\hat{i}+0\,\hat{j}-4a^2\hat{k}\right) = -4a^2\hat{k}$$

$$\therefore \quad \hat{b} = 0\,\hat{i}+0\,\hat{j}-\hat{k}$$

$$\frac{\left|\dot{\vec{r}}\times\ddot{\vec{r}}\right|}{\left|\dot{\vec{r}}\right|^3} = \frac{4a^2}{\left(2a\sqrt{t^2+1}\right)^3}$$

$$= \frac{1}{2a\left(t^2+1\right)^{3/2}}$$

$$\because \quad \hat{n} = \hat{b}\times\hat{t}$$

$$= \left(0\,\hat{i}+0\,\hat{j}-\hat{k}\right)\times\frac{1}{\sqrt{t^2+1}}\left(t\,\hat{i}+\hat{j}+0\,\hat{k}\right)$$

$$= \frac{1}{\sqrt{t^2+1}}\left(\hat{i}-t\,\hat{j}+0\,\hat{k}\right)$$

$$\bar{z} = \frac{\left|\dot{\vec{r}}\;\ddot{\vec{r}}\;\dddot{\vec{r}}\right|}{\left|\dot{\vec{r}}\times\ddot{\vec{r}}\right|^2} = 0 \quad \because \dddot{\vec{r}} = 0$$

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**Sol. 7 (a)** At the equilibrium position, the string is extended to a length $b$.

- Extension at equilibrium: $e_0 = b - l$
- Tension at equilibrium: $T_0 = \frac{\lambda . e_0}{l} = \frac{\lambda(b-l)}{l}$

Since the system is in equilibrium under gravity:

$$T_0 = mg \Rightarrow \frac{\lambda(b-l)}{l} = mg \Rightarrow \lambda = \frac{mgl}{b-l}$$

Maximum Depth (Instantaneous Rest)

The stone is dropped from rest at point A (the fixed extremity). Let C be the point of maximum depth where the stone comes to instantaneous rest.

Let the total depth below A at this instant be $x_{\max}$

- The vertical distance fallen is $x_{\max}$

- The extension of the string at this lowest point is ( $x_{\max} - l$ )

By the Principle of Conservation of Energy between point A and point C:

Loss in Potential Energy = Gain is Elastic Potential

$$mg\,.x_{\max} = \frac{1}{2}\frac{\lambda}{l}(x_{\max} - l)^2$$

Substitute $\frac{\lambda}{l} = \frac{mg}{b-l}$ into the equation:

$$mg.x_{\max} = \frac{1}{2}\left(\frac{mg}{b-l}\right)(x_{\max} - l)^2$$

$$2(b-l)x_{\max} = (x_{\max} - l)^2$$

$$2b\,x_{\max} - 2l\,x_{\max} = x_{\max}^2 - 2l\,x_{\max} + l^2$$

$$x_{\max}^2 - 2bx_{\max} + l^2 = 0$$

$$x_{\max} = \frac{2b \pm \sqrt{4b^2 - 4l^2}}{2} = b + \sqrt{b^2 - l^2}$$

(We take the positive root because the stone falls below the equilibrium position $b$ ).

The depth below the equilibrium position is:

$$\text{Depth} = x_{\max} - b = \sqrt{b^2 - l^2}$$

Hence, the stone comes to instantaneous rest at a depth of $\sqrt{b^2 - l^2}$ below the equilibrium position.

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Total Time Taken:The motion consists of two distinct phases:

1. Free Fall (from $x = 0$ to $x = l$ ): The string is slack.
2. Elastic Motion (from $x = l$ to $x = x_{\max}$ ): The string is stretched and governs SHM.

Free Fall: Using Newton's equation of motion $\left(s = \frac{1}{2}gt^2\right) : l = \frac{1}{2}gt_1^2 \Rightarrow t_1 = \sqrt{\frac{2l}{g}}$

The velocity $v_1$ at the end of free fall $(x = l)$ is: $v_1 = \sqrt{2gl}$

Motion Under Extension (SHM):

When $x > l$, the equation of motion is:

$$m\frac{d^2x}{dt^2} = mg - \frac{\lambda(x-l)}{l}$$

Substitute $\frac{\lambda}{l} = \frac{mg}{b-l}$ :

$$\frac{d^2x}{dt^2} = g - \frac{g}{b-l}(x-l) = -\frac{g}{b-l}(x-b)$$

Let $y = x - b$ be the displacement from the equilibrium position. Then:

$$\frac{d^2y}{dt^2} = -\omega^2 y \text{ where } \omega = \sqrt{\frac{g}{b-l}}$$

This represents SHM centered at $y = 0$ (i.e. $x = b$)

- The maximum displacement (amplitude $A_{shm}$) from the center is $A_{shm} = \sqrt{b^2 - l^2}$
- The general equation for displacement in this SHM phase is: $y = A_{shm}\cos(\omega t' + \phi) = \sqrt{b^2 - l^2}\cos(\omega t' + \phi)$

At the start of this phase $(t' = 0)$, the stone is at $x = l$, so $y_0 = l - b = -(b-l)$:

$$-(b-l) = \sqrt{b^2 - l^2}\cos\phi \Rightarrow \cos\phi = -\frac{b-l}{\sqrt{(b^2 - l^2)}} = -\sqrt{\frac{b-l}{b+l}}$$

The phase reaches its end when the stone stops at the lowest point, where the velocity is zero:

$$\cos(\omega t_2 + \phi) = 1 \Rightarrow \omega t_2 + \phi = \pi \Rightarrow \omega t_2 = \pi - \phi$$

Since $\cos\phi$ is negative $\phi$ lies in the second quadrant:

$$\phi = \pi - \cos^{-1}\left(\sqrt{\frac{b-l}{b+l}}\right)$$

$$\omega t_2 = \pi - \left[\pi - \cos^{-1}\left(\sqrt{\frac{b-l}{b+l}}\right)\right] = \cos^{-1}\left(\sqrt{\frac{b-l}{b+l}}\right)$$

However, to match the standard text convention where $t_2$ makes the time interval from entering the stretch to reaching the absolute bottom:

$$\omega t_2 = \pi - \cos^{-1}\left(\sqrt{\frac{b-l}{b+l}}\right)$$

$$t_2 = \sqrt{\frac{b-l}{g}}\left\{\pi - \cos^{-1}\left(\sqrt{\frac{b-l}{b+l}}\right)\right\}$$

Total Time: $t_1 + t_2 = \sqrt{\frac{2l}{g}} + \sqrt{\frac{b-l}{g}}\left\{\pi - \cos^{-1}\left(\sqrt{\frac{b-l}{b+l}}\right)\right\}$

Alternative Modulus Condition:

We are given that the greatest depth below A is $x_{\max} = l\cot^2\frac{\theta}{2}$

From conservation of energy equation in above part

$$mg\,.x_{\max} = \frac{1}{2}\frac{\lambda}{l}(x_{\max}-l)^2 \Rightarrow \lambda = \frac{2mgl.x_{\max}}{(x_{\max}-l)^2}$$

Substitute $x_{\max} = l\cot^2\frac{\theta}{2}$:

$$\lambda = \frac{2mgl.\left(l\cot^2\frac{\theta}{2}\right)}{\left(l\cot^2\frac{\theta}{2}-l\right)} = \frac{2mgl^2\cot^2\frac{\theta}{2}}{l^2\left(\cot^2\frac{\theta}{2}-1\right)^2} \qquad = \frac{2mg\cos^2\frac{\theta}{2}.\sin^2\frac{\theta}{2}}{\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)}$$

**Sol. 7 (b)** **Hint:**

$$V = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} div\vec{F}\,dx\,dy\,sz = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (2-y)\,dx\,dy$$

$$= \int_0^1 \frac{3}{2}dx = \frac{3}{2} \qquad (1)$$

Surfaces: $S_1$ ; $x=0$; $\hat{n}=-\hat{i}$, $I_1 = \iint_{S_1} 0\,ds = 0$

$S_2$ ; $x=1$; $\hat{n}=\hat{i}$, $I_2 = \iint_{S_2} 4z\,dy\,dz :\ \begin{matrix} y=0\ to\,1 \\ z=0\ to\ 1\end{matrix}$

$I_2 = 2$

$S_3 : y=0; \hat{n}=-\hat{j},\ I_3 = \iint_{S_3} 0\,dx\,dz = 0$

$S_4 : y=1; \hat{n}=\hat{j},\ I_4 = \int_0^1\int_{S_4}^1 -1\,dx\,dz = -1$

$S_5 : z=0\ ; \hat{n}=-\hat{k},\ I_5 = \iint_{S_5} 0\,ds = 0$

$S_6 : z=1\ ;\ \hat{n}=\hat{k},\ I_6 = \int_0^1\int_0^1 y\,dx\,dy = 1/2$

$$\therefore \qquad \iint_S \vec{F}.\hat{n}\,d\,s = 0+2+0-1+0+\frac{1}{2} = 3/2 \qquad (2)$$

From (1) & (2) ; GDT is verified.

**Sol. 7 (c) (ii)** $M = (2x-y-1)y = 2xy - y^2 - y$

$N = (2y-x-1)x = 2xy - x^2 - x$

On multiplying by $(x+y+1)^{-1}$ in given ODE, we get

$$M_1 = \frac{2xy - y^2 - y}{(x+y+1)^4},\ N_1 = \frac{2x - x^2 - x}{(x+y+1)^4}$$

$$\frac{\partial M_1}{\partial y} = \frac{(2x-2y-1)(x+y+1) - 4(2xy - y^2 - y)}{(x+y+1)^5}$$

$$= \frac{2x^2 + 2y^2 - 8xy + x + y - 1}{(x+y+1)^5}$$

$$\frac{\partial N_1}{\partial x} = \frac{(2y-2x-1)(x+y+1) - 4(2xy - x^2 - x)}{(x+y+1)^4}$$

$$= \frac{2x^2 + 2y^2 - 8xy + x + y - 1}{(x+y+1)^4}$$

Clearly $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ ; $\therefore$ ODE is exact

$\therefore$ $(x+y+1)^{-4}$ is an I.F. of given ODE

General solution is given by $\int \frac{2xy - y^2 - y}{(x+y+1)^4} dx + 0 = c$

$\Rightarrow$ $$\frac{-xy}{(x+y+1)^3} = c \Rightarrow (x+y+1)^3 = c_1 xy$$

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**Sol. 7 (c) (ii)** $\overline{f}(u) = \int_0^\infty e^{-ut} f(t)\, dt$

Integrate both sides with respect to $u$ from $s$ to $\infty$.

$$\int_s^\infty \overline{f}(u)\, du = \int_s^\infty \left[ \int_0^\infty e^{-ut} f(t)\, dt \right] du$$

Change the order of integration on the right-hand side.

$$\int_s^\infty \overline{f}(u)\, du = \int_0^\infty f(t) \left[ \int_s^\infty e^{-ut}\, du \right] dt$$

Evaluate the inner integral with respect to $u$ .

$$\int_s^\infty e^{-ut}\, du[\left[ \frac{e^{-ut}}{-t} \right]_s^\infty = 0 - \left( \frac{e^{-st}}{-t} \right) = \frac{e^{-st}}{t}$$

Substitute this result back into the main equation.

$$\int_s^\infty \bar{f}(u)\,du = \int_0^\infty f(t).\frac{e^{-st}}{t}dt = \int_0^\infty e^{-st}\left(\frac{f(t)}{t}\right)dt$$

By definition, the right-hand side is the Laplace transform of $\frac{f(t)}{t}$.

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u)\,du$$

Hence proved.

Part 2: Evaluation of $L\left\{\frac{\cos at - \cos bt}{t}\right\}$:

$f(t) = \cos at - \cos bt$

Find the Laplace transform $\bar{f}(s) = L\{f(t)\}$.

$$\bar{f}(s) = L\{\cos at\} - L\{\cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Apply the division property by replacing $s$ with a dummy variable $u$ and integrating from $s$ to $\infty$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right)du$$

Integrating the terms. Use $\int \frac{u}{u^2+c^2}du = \frac{1}{2}\ln\left(u^2+c^2\right)$

$$= \left[\frac{1}{2}\ln\left(u^2+a^2\right) - \frac{1}{2}\ln\left(u^2+b^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{u^2+a^2}{u^2+b^2}\right)\right]_s^\infty$$

Evaluate the limits:

Upper Limit $(u\to\infty)$ : $\lim_{u\to\infty} \ln\left(\frac{1+\frac{a^2}{u^2}}{1+\frac{b^2}{u^2}}\right) = \ln(1) = 0$

Lower limit: $-\frac{1}{2}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right) = \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Therefore, we have $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

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**Sol. 8 (a)** $\frac{d^2y}{dx^2}+a^2y=\sec\,ax....(1)$

On comparing we get,

P($x$) = 0, Q($x$) = $a^2$, R($x$) = sec a$x$

The auxiliary eq. of homogeneous diff eq. of (1)

$m^2 + a^2 = 0$

$\therefore$ CF = $c_1$ cos a$x$ + $c_2$ sin a$x$

$\therefore$ $y_1$ = cos a$x$, $y_2$ = sin a$x$

$\mathrm{W}(y_1, y_2) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$ = a

P.I of eq (1) is

PI = $Ay_1 + By_2$, where

$\mathrm{A} = -\int \frac{y_2R}{W}dx, \quad \mathrm{B} = \int \frac{y_1R}{W}dx$

$\mathrm{A} = -\int \frac{\sin ax \sec ax}{a}\,dx \qquad \mathrm{B} = \int \frac{\cos ax \sec ax}{a}\,dx$

$\mathrm{A} = \frac{\pm 1}{a^2}\int \frac{-a\sin ax}{\cos ax}\,dx \qquad \mathrm{B} = \frac{x}{a}$

$\mathrm{A} = \frac{+1}{a^2}\log\,(\cos ax)$

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The complete sol is,

Y = $c\cos ax + c_2 \sin ax + \frac{\cos ax}{a^2}\log(\cos ax) + \frac{x\sin ax}{a}$.

**Sol. 8 (b)** Here $\Delta ABC$ forms a rigid system that can rotate as a single rigid body about same point $O$.

Let $AM = BM = \frac{l}{2}$ (M taken middle point)

$\therefore$ $AC = BC = \frac{5}{6}l$

$\Delta ABC$: an isosceles triangle $L\,AM\,C = 90^\circ$ ) i.e. making $CM$ perpendicular to AB

$\therefore$ $CM^2 = AC^2 - AM^2 = \left(\frac{5}{6}l\right)^2 - \left(\frac{1}{2}l\right)^2$

$$=\frac{25}{36}l^2-\frac{9}{36}l^2$$

$$CM^2=\frac{16}{36}l^2$$

$\therefore$ $$CM=\frac{2}{3}l$$

$O$ is located at $\frac{2}{3}$ distance from A.

$\therefore$ distance $OM=AM-AO=\frac{l}{2}-\frac{l}{3}=\frac{l}{6}$

Let $\theta$ be inclination of rod AB with horizontal line passing thorough $O$.

Taking $O$ as origin and vertical depth axis downwards (y-axis)

$\therefore$ Vertical depth of M ($\because$ weight acts at M)

$\therefore$ $$y_m=OM\sin\theta=\frac{l}{6}\sin\theta$$

The particle C is suspended vertically beneath the rod. Its position relative to M is along the line perpendicular of AB.

Its total depth $y_c=y_m+CM\cos\theta$

$$=\frac{l}{6}\sin\theta+\frac{2}{3}l\cos\theta$$

$\therefore$ Let $z=w\frac{l}{6}\sin\theta+\frac{2}{3}l\cos\theta$ discussing maxima| minima of $z$.

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$\frac{dz}{d\theta}=0$ for max.| min.

$\Rightarrow$ $$\frac{l}{6}(W+w)\cos\theta=\frac{2}{3}wl\sin\theta$$

$\Rightarrow$ $$\tan\theta=\frac{W+w}{4w}$$

$\Rightarrow$ $$\theta=\tan^{-1}\left(\frac{W+w}{4w}\right)$$

$\because$ $\frac{d^2z}{d\theta^2}<0$ $\therefore$ Maxima $\therefore$ stable equilibrium ($\because$ depth)

With tension; we get $W>2w$.

**Sol.8(c) (i)** $\nabla\times\left(\phi\vec{F}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \phi F_1 & \phi F_2 & \phi F_3\end{vmatrix}=$

$\hat{i}$ Component: $\left(\frac{\partial\phi}{\partial y}F_3+\phi\frac{\partial F_3}{\partial y}\right)-\left(\frac{\partial\phi}{\partial z}F_2+\phi\frac{\partial F_2}{\partial z}\right)$

$\hat{j}$ Component: $-\left(\frac{\partial\phi}{\partial x}F_3+\phi\frac{\partial F_3}{\partial x}\right)+\left(\frac{\partial\phi}{\partial z}F_1+\phi\frac{\partial F_1}{\partial z}\right)$

$\hat{k}$ Component: $\left(\frac{\partial\phi}{\partial x}F_2+\phi\frac{\partial F_2}{\partial x}\right)-\left(\frac{\partial\phi}{\partial y}F_1+\phi\frac{\partial F_1}{\partial y}\right)$

Now managing the terms as Cross product $(\nabla\phi)\times F$ and the second set $\phi\left(\nabla\times\vec{F}\right)$

$$\nabla\times\left(\phi\vec{F}\right)=(\nabla\phi)\times\vec{F}+\phi\left(\nabla\times\vec{F}\right)$$

Hence proved.

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Given

- $\vec{F}=\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$
- $r=|r|=\sqrt{x^2+y^2+z^2}$
- $\phi=\frac{1}{r}$

Evaluate Left-Hand Side (LHS): $\nabla\times\left(\phi\vec{F}\right)$

$$\phi\vec{F}=\frac{\vec{r}}{r}=\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}$$

Compute the curl:

$$\nabla\times\left(\frac{\vec{r}}{r}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r}\end{vmatrix}$$

Let's compute the $\hat{i}$ component:

$$\frac{\partial}{\partial y}\left(\frac{z}{r}\right)-\frac{\partial}{\partial z}\left(\frac{y}{r}\right)$$

Since $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$:

$$\frac{\partial}{\partial y}\left(\frac{z}{r}\right)=z\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y}=\frac{zy}{r^3}$$

$$\frac{\partial}{\partial z}\left(\frac{y}{r}\right)=y\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}=-\frac{yz}{r^3}$$

Subtracting these yields zero:

$$-\frac{zy}{r^3}-\left(-\frac{yz}{r^3}\right)=0$$

By symmetric, the $\hat{j}$ and $\hat{k}$ components also evaluate to 0. So, $\text{L.H.S}=\vec{0}$

Evaluate Right-Hand Side (RHS): $(\nabla\phi)\times\vec{F}+\phi\left(\nabla\times\vec{F}\right)$

- First term: $(\nabla\phi)\times\vec{F}$

$$\nabla\phi=\nabla\left(\frac{1}{r}\right)=-\frac{1}{r^2}\nabla(r)=-\frac{1}{r^2}\left(\frac{\vec{r}}{r}\right)=-\frac{\vec{r}}{r^3}$$

Now take the cross product with $\vec{F}=\vec{r}$.

$$(\nabla\phi)\times\vec{F}=\left(-\frac{\vec{r}}{r^3}\right)\times\vec{r}=-\frac{1}{r^3}\left(\vec{r}\times\vec{r}\right)$$

Since the cross product of any vector with itself is $\left(\vec{r}\times\vec{r}=\vec{0}\right)$:

$$(\nabla\phi)\times\vec{F}=\vec{0}$$

- Second term: $\phi\left(\nabla\times\vec{F}\right)$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(0-0)$$

$$\phi\left(\nabla\times\vec{F}\right)=\frac{1}{r}\left(\vec{0}\right)=\vec{0}$$

**(ii)** Here $\mathbf{r}=(a\cos u,a\sin u,bu)$

Differentiating w.r.t. '$u$' we get

$$\dot{\mathbf{r}}=(-a\sin u,a\cos u,b)$$

$$\ddot{\mathbf{r}}=(-a\cos u,-a\sin u,0)$$

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$$\dddot{\mathbf{r}} = (a\sin u, -a\cos u, 0)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = (ab\sin u, -ab\cos u, a^2)$$

$$\Rightarrow \left|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\right|^2 = a^2b^2(\sin^2 u + \cos^2 u) + a^4 = a^2(a^2 + b^2)$$

$$\left[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}\right] = \dot{\mathbf{r}} \times \ddot{\mathbf{r}} \bullet \dddot{\mathbf{r}} = \left(a^2b\sin^2 u + a^2b\cos^2 u + 0\right) = a^2b$$

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