Mindset Makers

UPSC CSE MAIN 2024

Mathematics Optional

PAPER-1: DETAILED ANSWERS

- Well Structured
- Addressing Demand of IAS/IFoS
- Systematically Designed
- Video Discussion Upendra Sir

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SECTION-A

1. (a) Let H be a subspace of R^4 spanned by the vectors $v_1 = (1,-2,5,-3)$, $v_2 = (2,3,1,-4), v_3 = (3,8,-3,-5)$. Then find a basis and dimension of H, and extend the basis of H to a basis of R^4.

(b) Let $T:\mathrm{R}^3 \to \mathrm{R}^3$ be a linear operator and $B = \{v_1, v_2, v_3\}$ be a basis of R^3 over R. Suppose that

$T\mathrm{v}_1 = (1,1,0), T\mathrm{v}_2 = (1,0,-1), T\mathrm{v}_3 = [2,1,-1)$. Find a basis for the range space and null space of T.

(c) Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{1}{1-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for all values of x.

(d) Expand $ln(x)$ in powers of $(x-1)$ by Taylor's theorem and hence find the value of $ln(1.1)$ correct up to four decimal places.

(e) Find the equation of the right circular cylinder which passes through the circle $x^2 + y^2 + z^2 = 9, x - y + z = 3$.

2. (a) Consider a linear operator T on R^3 over R defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Is T invertible? If yes. justify your answer and find T^{-1}.

(b) If $u = (x + y)/(1 - xy)$ and $v = \tan^{-1} x + \tan^{-1} y$, then find $\partial(u, v)/\partial(x, y)$. Are u and v functionally related? If yes, find the relationship.

(c) Find the image of the line $x = 3 - 6t, y = 2t, z = 3 + 2t$ in the plane $3x + 4y - 5z + 26 = 0$.

3. (a) Let $V = M_{2\times2}(R)$ denote a vector space over the field of real numbers. Find the matrix of the linear mapping $\phi: V \to V$ given by $\phi(v) = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} v$ with respect to standard basis of $M_{2\times2}(R)$, and hence find the rank of ϕ. Is ϕ invertible? Justify your answer.

(b) Find the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α.

(c) Find the vertex of the cone $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$.

4. (a) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ be a 3×3 matrix. Find the eigenvalues and the corresponding eigenvectors of A. Hence find the eigenvalues and the corresponding eigenvectors of A^{-15}, where $A^{-15} = \left(A^{-1}\right)^{15}$.

(b) Using double integration, find the area lying inside the cardioid $r = a(1 + cos\theta)$ and outside the circle $r = a$.

(c) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally, the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

SECTION-B

5. (a) Find the orthogonal trajectories of the family of curves $r = c(sec\theta + tan\theta)$ where c is a parameter.

(b) Solve the integral equation $y(t) = cost + \int_0^t y(x) cos(t - x) dx$ using Laplace transform.

(c) At any time t (in seconds), the coterminous edges of a variable parallelepiped are represented by the vectors

$\vec{a} = t\hat{i} + (t+1)\hat{j} + (2t+1)\hat{k}$

$\vec{\beta} = 2t\hat{i} + (3t-1)\hat{j} + t\hat{k}$

$\vec{\gamma} = \hat{i} + 3t\hat{j} + \hat{k}$

What is the rate of change of the vectorial area of the parallelogram, whose coterminous edges are $\vec{\alpha}$ and $\vec{\gamma}$? Also find the rate of change of the volume of the parallelepiped at $t = 1$ second.

(d) A solid hemisphere rests in equilibrium on a solid sphere of equal radius. Determine the stability of the equilibrium in the two situations-(i) when the curved surface and (ii) when the flat surface of the hemisphere rests on the sphere.

(e) (i) Let C be a plane curve $\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}$, where f and g have second-order derivatives. Show that the curvature at a point is given by

$$K=\frac{\left|f'(t)g''(t)-g'(t)f''(t)\right|}{\left(\left[f'(t)\right]^2+\left[g'(t)\right]^2\right)^{3/2}}$$

What is the value of torsion τ at any point of this curve?

(ii) Show that the principal normals at two consecutive points of a curve do not intersect unless torsion τ is zero.

6. (a) A regular tetrahedron, formed of six light rods, each of length l, rests on a smooth horizontal plane. A ring of weight W and radius r is supported by the - slant sides. Using the principle of virtual work, find the stress in any of the horizontal sides.

(b) A particle executes simple harmonic motion such that in two of its positions, velocities are u and v, and the two corresponding accelerations are f_1 and f_2. For what value(s) of *k*, the distance between the two positions is $k\left(v^2-u^2\right)$? Show also that the amplitude of the motion is

$$\frac{1}{f_2^2-f_1^2}\left[\left(u^2-v^2\right)\left(u^2 f_2^2-v^2 f_1^2\right)\right]^{1/2}$$

(c) (i) Find the second solution of the differential equation $xy''+(x-1)y'-y=0$ using $u(x)=-e^{-x}$ as one of the solutions.

(ii) Find the general solution of the differential equation $x^2y''-2xy'+2y=x^3 sinx$ by the method of variation of parameters.

7. (a) State uniqueness theorem for the existence of unique solution of the initial value problem $\frac{dy}{dx}=f(x,y), y(x_0)=y_0$ in the rectangular region $R:|x-x_0|\le a$, $|y-y_0|\le b$. Test the existence and uniqueness of the solution of the initial value problem $\frac{dy}{dx}=2\sqrt{y}, y(1)=0$, in a suitable rectangle R. If more than one solution exist, then find all the solutions.

(b) A heavy particle hanging vertically from a fixed point by a light inextensible string of length / starts to move with initial velocity u in a circle so as to make a complete revolution in a vertical plane. Show that the sum of tensions at the ends of any diameter is constant.

(c) State Stokes' theorem and verify it for the vector field $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$ over the surface S, which is the upwardly oriented part of the cylinder $z = 1 - x^2$, for $0 \le x \le 1, -2 \le y \le 2$.

8. (a) Using Laplace transform, solve the initial value problem

$$y'' + 2y' + 5y = \delta(t-2),\ y(0) = 0,\ y'(0) = 0$$

where $\delta(t-2)$ denotes the Dirac delta function.

(b) Using Gauss divergence theorem, evaluate the integral

$$\iint_S \left(y^2 i + xz^3 \hat{j} + (z-1)^2 \hat{k}\right).\hat{n}dS$$

over the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 1$ and $z = 5$.

(c) A particle moves with a central acceleration $\mu\left(\frac{3}{r^3} + \frac{d^2}{r^5}\right)$ being projected from a distance d at an angle 45° with a velocity equal to that in a circle at the same distance. Prove that the time it takes to reach the centre of force is $\frac{d^2}{\sqrt{2\mu}}\left(2 - \frac{\pi}{2}\right)$

SECTION-A

1. (a) Given, H is a subspace of $\mathbb{R}^4$ & spanned by

$S=\{v_1,v_2,v_3\}$; where $v_1=(1,-2,5,-3), v_2=(2,3,1,-4), v_3=(3,8,-3,-5)$

i.e. $H=L(S)\subseteq \mathbb{R}^4$ 5

Now, $A=\begin{bmatrix}1 & -2 & 5 & -3\\ 2 & 3 & 1 & -4\\ 3 & 8 & -3 & -5\end{bmatrix}$

$R_2\to -2R_1+R_2,\ R_3\to -3R_1+R_3$

$$A\sim\begin{bmatrix}1 & -2 & 5 & -3\\ 0 & 7 & -9 & 2\\ 0 & 14 & -18 & 4\end{bmatrix} R_3\to -2R_2+R_3; \sim\begin{bmatrix}1 & -2 & 5 & -3\\ 0 & 7 & -9 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

As, A is in echelon form & have two non-zero rows $\therefore$ dim.(H) = 2

$\therefore\ S_1=\{(1,-2,5,-3),(0,7,-9,2)\}$ is a basis of H.

We know the standard basis of $\mathbb{R}^4$ is $S_2=\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

So, clearly,

$S_3=\{(1,-2,5,-3),(0,7,-9,2),(0,0,1,0),(0,0,0,1)\}$;is a basis of $\mathbb{R}^4$ which is extended form of S_1

1. (b) Given $T=\mathbb{R}^3\to\mathbb{R}^3$ & $B=\{v_1,v_2,v_3\}$ be a basis of $\mathbb{R}^3$.

$T(v_1)=(1,1,0), T(v_2)=(1,0,-1), T(v_3)=(2,1,-1)$

So, if $\alpha\in\mathbb{R}^3$ then $\alpha=av_1+bv_2+cv_3$

$\Rightarrow T(\alpha)=T\{av_1+bv_2+cv_3\}$

$\Rightarrow T(\alpha)=aT(v_1)+bT(v_2)+cT(v_3)$

$\Rightarrow T(\alpha)=a(1,1,0)+b(1,0,-1)+c(2,1,-1)$

$\Rightarrow T(\alpha)=(a+b+2c,a+c,-b-c)$

We know, if $B=\{v_1,v_2,v_3\}$ spans $\mathbb{R}^3$ then $\{T(v_1),T(v_2),T(v_3)\}$ spans Range set of T.

$\therefore\ A=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & -1\\ 2 & 1 & -1\end{bmatrix}$

$R_2\to -R_1+R_2,\ R_3\to -2R_1+R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}; R_3 \to -R_2 + R_3; A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ dim (Range set of T) = 2

A basis of Range set of T = {(1, 1, 0), (0, –1, –1)}

Now, Null sp. $T = \left\{\alpha \in \mathbb{R}^3 : \quad T(\alpha) = (0,0,0)\right\}$

So, $T(\alpha) = (0,0,0)$

$\Rightarrow (a+b+2c, ab+c, -b-c) = (0,0,0)$

$\Rightarrow \quad a+b+2c = 0 \quad$ ....(1)

$a+bc = 0 \quad$ ....(2)

$-b-c = 0 \quad$ ....(3)

From (2) & (3); $a = b$ *and* From (1); $2c = -2b$ i.e. $c = -b$

$\therefore a = -c$

$\therefore$ Number of free variable = 1 (say c)

$\therefore$ dim (Null sp T) = 1.

$\therefore$ A basis of Null sp T = {(–1, –1, 1)}

1. (c) $f(x) = \begin{cases} \dfrac{1}{1-e^{-\frac{1}{x}}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Let check the continuity of $f(x)$ at $x = 0$.

LHL = $\lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0} \dfrac{1}{1-e^{1/h}} = \lim\limits_{h\to 0} \dfrac{e^{-1/h}}{e^{1/h}-1} = 0$

RHL = $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} \dfrac{1}{1-e^{-1/h}} = \dfrac{1}{1-0} = 1$

$\therefore$ The function $f(x)$ is not continuous at $x = 0$ and as the function is a combination of exponential function for all non zero x, so it will be continuous at all other points except 0.

1. (d) $f(x) = \log x$

By Tylor's theorem we have

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots$$

Takin a = 1,

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \qquad \text{...(1)}$$

$\because \; f(x) = \log x \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$\therefore$ From (1),

$$f(x) = 0 + (x-1) + \frac{(x-1)^2}{2!} \times (-1) \frac{(x-1)^3}{3!} \times 2 + --$$

$$\log x = (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + ---$$

Now, $\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{2(1.1-1)^3}{6} = 0.1 - 0.005 + \frac{1}{3} \times 0.001$

log(1.1) = 0.1 – 0.005 + 0.0003=**0.0953**

1. (e) The given equation of circle & plane is

$x^2 + y^2 + z^2 = 9$, $x - y + z = 3$

As, the axis of the cylinder is perpendicular to the plane

$\therefore$ Direction ratios of axis are 1, –1, 1

Let (α, β, γ) be a point on the cylinder then the equation of the generator through this point is

$\frac{x-\alpha}{1} = \frac{y-\beta}{-1} = \frac{y-\gamma}{1} = r$ (say)

$\therefore$ $(\alpha + r, \beta - r, \gamma + r)$ is a point on this generator

Now, this point will also lie on the given curve i.e.,

$$(\alpha + r)^2 + (\beta - r)^2 + (\gamma + r)^2 = 9$$

$$\Rightarrow 3r^2 + 2r(\alpha - \beta + \gamma) + \alpha^2 + \beta^2 + \gamma^2 - 9 = 0 \qquad \text{.....(1)}$$

On satisfying in given plane by this point, we get $r=\dfrac{3-\alpha+\beta-\gamma}{3}$ .....(2)

Using (2) in (1).

$$\alpha^2+\beta^2+\gamma^2+\frac{2}{3}(3-\alpha+\beta-\gamma)(\alpha-\beta+\gamma)+\frac{1}{3}(3-\alpha+\beta-\gamma)^2=9$$

$$\Rightarrow 3\left(\alpha^2+\beta^2+\gamma^2\right)+2(\alpha+\beta+\gamma)(3-\alpha+\beta-\gamma)+(3-\alpha+\beta-\gamma)^2=27$$

$$\Rightarrow 3\left(\alpha^2+\beta^2+\gamma^2\right)+(3-\alpha+\beta-\gamma)(2\alpha-2\beta+2\gamma+3-\alpha+\beta-\gamma)=27$$

$$\Rightarrow 3\left(\alpha^2+\beta^2+\gamma^2\right)+(3-\alpha+\beta-\gamma)(3\alpha+\beta-\gamma)=27$$

$$\Rightarrow 3\left(\alpha^2+\beta^2+\gamma^2\right)+9-(\alpha-\beta+\gamma)^2=27$$

$$\Rightarrow 2\alpha^2+2\beta^2+2\gamma^2+2\alpha\beta+2\beta\gamma-2\alpha\gamma=18$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma-\alpha\gamma-9=0$$

So, the locus of (α,β,γ) is

$x^2+y^2+z^2+xy+yz-xz-9=0$; Which is equation of cylinder.

2. **(a)** Given $T:\mathbb{R}^3\to\mathbb{R}^3$; $T(x,y,z)=(2x,4x-y,2x+3y-z)$

Let $(x,y,z)\in\ker(T)$

$$\Rightarrow T(x,y,z)=(0,0,0)$$

$$\Rightarrow (2x,4x-y,2x+3y-z)=(0,0,0)$$

$$\Rightarrow 2x=0,\quad 4x-y=0\quad 2x+3y-z=0$$

$\Rightarrow x=0, y=0, z=0 \quad \therefore Ker\ (T)=\{(0,\ 0,\ 0)\}$. So, T is invertible.

Now as T is invertible so T must be onto

$\therefore$ Let $X=(a,b,c)\in\mathbb{R}^3$, then there exists some $(x,y,z)\in\mathbb{R}^3$ s.t

$$T(x,y,z)=(a,b,c)$$

$$\Rightarrow (2x,4x-y,2x+3y-z)=(a,b,c)$$

$$\Rightarrow 2x=a,\ 4x-y=b,\quad 2x+3y-z=c$$

$$\Rightarrow x=\frac{a}{2},\quad y=-b+4\times\frac{a}{2}\quad \&\quad z=2\times\frac{9}{2}+3\times(2a-b)$$

$$y = 2a - b \qquad\qquad z = a + 6a - 3b - c$$

$$z = 7a - 3b - c$$

As; $T(x, y, z) = (a, b, c)$

$\Rightarrow T^{-1}(a, b, c) = (x, y, z)$

$\Rightarrow T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a - b, 7a - 3b - c\right); \forall (a, b, c) \in \mathbb{R}^3.$

2. (b) $u = \frac{x + y}{(1 - xy)}, \quad v = \tan^{-1} x + \tan^{-1} y$

We have,

$$\frac{\partial u}{\partial x} = \frac{(1 - xy) - (x + y)(-y)}{(1 - xy)^2} = \frac{1 + y^2}{(1 - xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1 - xy) - (x + y)(-x)}{(1 - xy)^2} = \frac{1 + x^2}{(1 - xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1 + x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1 + y^2}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1 + y^2}{(1 - xy)^2} & \frac{1 + x^2}{(1 - xy)^2} \\ \frac{1}{1 + x^2} & \frac{1}{1 + y^2} \end{vmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{(1 - xy)^2} - \frac{1}{(1 - xy)^2} = 0$$

$\therefore$ *u* & *v* are related functionally.

As, $u = \frac{x + y}{1 - xy}$

$\therefore \tan^{-1} u = \tan^{-1}\left\{\frac{x + y}{1 - xy}\right\}$

$\Rightarrow \tan^{-1} u = \tan^{-1}(x) + \tan^{-1} y$

$\Rightarrow \tan^{-1} u = v \Rightarrow u = \tan v$; Which is the required relationship between $u \,\&\, v$.

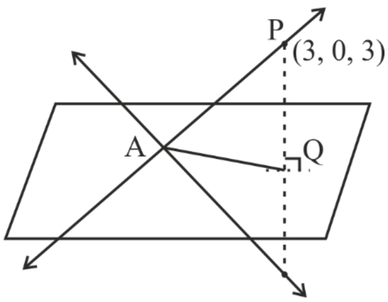
2. **(c)** The given line is,

$$x = 3 - 6t, y = 2t, z = 3 + 2t \Rightarrow \frac{x-3}{-6} = \frac{y-0}{2} = \frac{z-3}{2} = t$$

$\therefore$ The line is passing through (3, 0, 3).

The given equation of the plane is $3x + 4y - 5z + 26 = 0$. So, direction ratios are: 3,4,-5

Now, we have the equation of *PQ*.



$$\frac{x-3}{3} = \frac{y-0}{4} = \frac{z-3}{-5} = r(\text{say})$$

$\Rightarrow x = 3r + 3, y = 4r, z = -5r + 3$

As, the coordinate of Q i.e., (*x*, *y*, *z*) must satisfy the equation of plane.

$\therefore\ 3(3r+3) + 4 \times 4r - 5(-5r+3) + 26 = 0$

$9r + 9 + 16r + 25r - 15 + 26 = 0$

$50r + 20 = 0$

$$r = \frac{-2}{5}$$

$\therefore$ Coordinate of $Q = \left(\frac{-6}{5} + 3, \frac{-8}{5}, 5\right) = \left(\frac{9}{5}, \frac{-8}{5}, 5\right)$

Now the any point on the line is,

$x = -6t + 3, \quad y = 2t, \ z = 2t + 3$

$\therefore$ The intersecting point of the line & the plane is

$3(3-6t) + 4 \times 2t - 5(2t+3) + 26 = 0$

$9 - 18t + 8t - 10t - 15 + 26 = 0$

$-20t = -20; t = 1$

$\therefore$ The interesting point A is (-3, 2, 5)

$\therefore$ Direction ration of line $AQ = -3 - \frac{9}{5}, 2 + \frac{8}{5}, 5 - 5 = \frac{-24}{5}, \frac{18}{5}, 0$

$\therefore$ The equation of line which is the image on the plane is

$$\frac{x+3}{-\frac{24}{5}}=\frac{y-2}{\frac{18}{5}}=\frac{z-5}{0}\Rightarrow\frac{x+3}{-24}=\frac{y-2}{18}=\frac{z-5}{0}.$$

3. (a) Let $S=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ be a basis of $\mathrm{M}_{2\times 2}(\mathbb{R})$

Now $\phi(v_1)=\begin{bmatrix}1&2\\3&-1\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\3&0\end{bmatrix}=1.w_1+0.w_2+3.w_3+0.w_4$

$\phi(v_2)=\begin{bmatrix}1&2\\3&-1\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\0&3\end{bmatrix}=0.w_1+1.w_2+0.w_3+3.w_4$

$\phi(v_3)=\begin{bmatrix}1&2\\3&-1\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}2&0\\-1&0\end{bmatrix}=2.w_1+0.w_2-1.w_3+0.w_4$

$\phi(v_4)=\begin{bmatrix}1&2\\3&-1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&2\\0&-1\end{bmatrix}=0.w_1+2.w_2-0.w_3+1.w_4$

$\therefore$ Matrix representation of ϕ is given by.

$$A=\begin{bmatrix}1&0&3&0\\0&1&0&3\\2&0&-1&0\\0&2&0&-1\end{bmatrix}$$

$$\Rightarrow A\sim\begin{bmatrix}1&0&3&0\\0&1&0&3\\0&0&-7&0\\0&2&0&-1\end{bmatrix}\Rightarrow A\sim\begin{bmatrix}1&0&3&0\\0&1&0&3\\0&0&1&0\\0&0&0&-7\end{bmatrix}\Rightarrow A\sim\begin{bmatrix}1&0&3&0\\0&1&0&3\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$\therefore$ Rank of A = 4.

We know,

dim $\mathrm{M}_{2+2}(\mathbb{R})=2\times 2=4$

$\therefore$ By Rank Nullity Theorem,

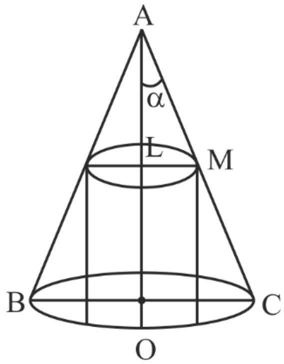
Rank ϕ + Nullity ϕ = $\dim \mathrm{M}_{2\times 2}(\mathbb{R})$

4 + Nullity ϕ = 4

Nullity ϕ = 0 $\quad\therefore$ ϕ is invertible.

3. **(b)** Given, OA = h & $\angle LAM = \alpha$

Let the radius of cylinder be *x* & height of cylinder be H.



From ΔALM,

$$\tan\alpha = \frac{x}{h-H}$$

$$x = (h-H)\tan\alpha \qquad .....(1)$$

$\therefore$ Volume of cylinder $(v) = \pi x^2 H$

$$= \pi(h-H)^2 H\tan^2\alpha \qquad \text{\{Using (1)\}}$$

Now,

$$\frac{dv}{dH} = \pi\tan^2\alpha\left[(h-H)^2 \times 1 + H\times 2(h-H)\times(-1)\right]$$

$$= \pi\tan^2\alpha\left[(h-H)^2 - 2H(h-H)\right]$$

$$\frac{dv}{dH} = \pi\tan^2\alpha\left[(h-H)(h-3H)\right]$$

Now, For maximum volume, $\frac{dv}{dH} = 0$

$\Rightarrow (h-H)(h-3H) = 0$

$\Rightarrow$ *h* = H(not possible) or *h* = 3H

$$\therefore\ H = \frac{h}{3} \qquad .....(2)$$

$\therefore$ Maximum volume (v) = $\pi\left(h-\frac{h}{3}\right)^2 \frac{h}{3}\tan^2\alpha$ {Using(1) & (2)}

Max. Volume $= \frac{4}{27}\pi h^3 \tan^2\alpha$

3. **(c)** Given $\phi(x,y,z)=4x^2-y^2+2z^2+2xy-3xyz+12x-11y+6z+4=0$

Let's make the given equation as homogenous equation introducing a new variable. '*t*'.

$$\therefore\ \phi(x,y,z,t)=4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2=0$$

Now, $\dfrac{\partial\phi}{\partial x}=8x+2y+12t=0$

$$\frac{\partial\phi}{\partial y}=-2y+2x-11t=0$$

$$\frac{\partial\phi}{\partial z}=4z-3y+6t=0$$

$$\frac{\partial\phi}{\partial t}=12x-11y+6z+8t=0$$

For *t* = 1, we get

$8x+2y=-12$ ....(1)

$2x-2y=11$ .....(2)

$4z-3y=-6$ .....(3)

$12x-11y+6z=-8$ .....(4)

From (1) & (2), $x=\dfrac{-1}{10}$

And $2y=2x-11$; $2y=\dfrac{-1}{5}-11$ i.e. $y=\dfrac{-28}{5}$

From (3)

$$4z=3y-6$$

$$4z=-\frac{84}{5}-6;\quad z=\frac{-57}{10}$$

Therefore, the required vertex of cone is $\left(\dfrac{-1}{10},\dfrac{-28}{5},\dfrac{-57}{10}\right)$

4. **(a)** $A=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}$

Let λ be the eigenvalues of A, then by necessary condition.

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\{-\lambda(3-\lambda)-4\}-2\{2(3-\lambda)-8\}+4\{4+4\lambda\}=0$$

$$(3-\lambda)\{-3\lambda+\lambda^2-4\}-2\{-2\lambda-2\}+16+16\lambda=0$$

$$-\lambda^3+3\lambda^2-9\lambda+3\lambda^2-12+4\lambda+4\lambda+4+16+16\lambda=0$$

$$\Rightarrow -\lambda^3+6\lambda^2+15\lambda+8=0$$

$$\Rightarrow \lambda^3-6\lambda^2-15\lambda-8=0$$

$$\Rightarrow \lambda^2(\lambda+1)-7\lambda(\lambda+1)-8(\lambda+1)=0$$

$$\Rightarrow (\lambda+1)(\lambda^2-7\lambda-8)=0$$

$$\Rightarrow (\lambda+1)(\lambda+1)(\lambda-8)=0$$

$$\therefore \lambda=-1,-1,8$$

Let $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to $\lambda = -1$

$$\therefore (A+1.I)v=0$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1+2x_2+2x_3=0$$

$$\therefore v_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \ \& \ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to $\lambda = 8$

$$\therefore (A - 8I)v = 0$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 1 \\ -5 & 2 & 4 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 1 \\ 0 & -18 & 9 \\ 0 & 18 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 4x_2 + x_3 = 0$$

$$-2x_2 + x_3 = 0$$

$$\therefore \; v_1 = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

Now, we know, if λ is an eigenvalue of A then the eigenvalue of A^{-1} is $\frac{1}{\lambda}$.

$\therefore$ Also, if λ be an eigenvalue of A then λ^n be the eigenvalue of A^n.

$\therefore$ Eigenvalue of $\left(A^{-1}\right)^{15} = \left(\frac{1}{8^{15}}, (-1)^{15}, (-1)^{15}\right) = \left(\frac{1}{8^{15}}, -1, -1\right)$

And eigenvectors of A^{-15} corresponding to eigenvalue -1; same as $\lambda = -1$

$\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & for $\lambda = 8$ will be $\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

4. (b) The given cardioid is $r = a(1+\cos\theta)$ & circle is $r = a$

$\therefore\ a = a(1+\cos\theta) \Rightarrow \cos\theta = 0\ \therefore\ \theta = \pm\frac{\pi}{2}$

$\therefore$ The required area = $\int_{r=a}^{a(1+\cos\theta)} \int_{-\frac{\pi}{2}}^{\pi/2} r\,d\theta\,dr = \int_{-\frac{\pi}{2}}^{\pi/2} \left[\frac{r^2}{2}\right]_{r=a}^{a(1+\cos\theta)} d\theta$

$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\pi/2} \left((1+\cos\theta)^2 - 1\right)d\theta = \frac{a^2}{2}\times 2\int_0^{\pi/2}\left(\cos^2\theta + 2\cos\theta\right)d\theta$

$= a^2\left[\frac{\sqrt{3/2}\sqrt{1/2}}{2\sqrt{2}} + 2[\sin\theta]_0^{\pi/2}\right] = a^2\left[\frac{1}{4}\pi + 2\right]$

The required area = $\frac{a^2}{4}(\pi + 8)$

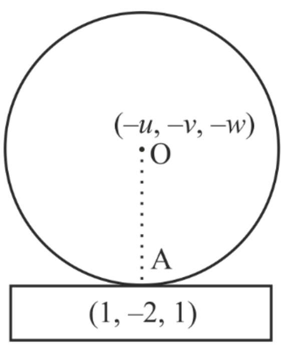
4. (c) The given equation of plane is $3x + 2y - z + 2 = 0$ .....(A)

Let the equation of sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .....(1)

As, the sphere (1) & $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ cuts orthogonally, so by condition of orthogonality ; $2(u_1u_2 + v_1v_2 + w_1w_2) = d_1 + d_2$, we have

$2(u\times(-2) + v(3) + w\times 0) = d + 4$

$2(-2u + 3v) = d + 4$ .....(2)



Also, the point (1, –2, 1) satisfies the equation (1).

1 + 4 + 1 + 2*u* + 4*v* + 2*w* + *d* = 0

2*u* – 4*v* + 2*w* + 6 + *d* = 0 .....(3)

Now, the equation of the line OA,

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-1}{-1}=r \text{ (say)}$$

$\therefore\ x=3r+1, \qquad y=2r-2, \qquad z=-r+1$

$\therefore$ Let $-u=3r+1, \qquad -v=2r-2, \qquad -w=-r+1$

$\Rightarrow u=-3r-1, \qquad v=2-2r, \qquad w=r-1$

Using the values of u, v & w in (3), we have,

$2\{-2(-3r-1)+3(-2r+2)\}=d+4$

$2\{6r+2-6r+6\}=d+4$

d = 12

Now, from (3),

$2\{-3r-1\}+4\{2r-2\}+2(r-1)+6+12=0$

$-6r-2+8r-8+2r-2+18=0$

$$4r+6=0$$

$$r=\frac{-3}{2}$$

$\therefore\ u=\frac{9}{2}-1=\frac{7}{2}, \qquad v=3+2=5, \qquad w=\frac{-3}{2}-1=\frac{-5}{2}$

$\therefore$ The required equation of sphere is,

$x^2+y^2+z^2+7x+10y-5z+12=0$

SECTION-B

5. (a) Given $r=c(\sec\theta+\tan\theta)$ .....(1)

Differentiating (1) w.r.t θ,

$$\frac{dr}{d\theta}=c\left(\sec\theta\tan\theta+\sec^2\theta\right)$$

$$\frac{dr}{d\theta}=c\sec\theta(\tan\theta+\sec\theta)$$

$$\frac{dr}{d\theta}=r\sec\theta \qquad \{\text{Using (1)}\}$$

$$\Rightarrow r\frac{d\theta}{dr}=\cos\theta$$

Now, to get orthogonal trajectory replacing $r\frac{d\theta}{dr}$ by $-\frac{1}{r\frac{d\theta}{dr}}$

$$\therefore \frac{-1}{r\frac{d\theta}{dr}} = \cos\theta$$

$\Rightarrow -\cos\theta d\theta = \frac{dr}{r}$;On integrating, we get

$$\Rightarrow -\int \cos\theta d\theta = \int \frac{dr}{r}$$

$$\Rightarrow -\sin\theta = \log r + \log c_1 \Rightarrow -\sin\theta = \log(rc_1)$$

$\Rightarrow rc_1 = e^{-\sin\theta}$; is required orthogonal trajectories.

5. (b) Given $y(t) = \cos t + \int_0^t y(x)\cos(t-x)dx$ .....(1)

As we know that the convolution is defined as $(f*g)(t) = \int_0^t f(x)g(t-x)dx$.

So, $\int_0^t y(x)\cos(t-x)dx = (y*\cos)(t)$ and using it, for Laplace by convolution theorem

$$L[(y*\cos)(t)] = L[y(t)]\times L[\cos t]$$

Now, Taking hap lace on both sides of (1), we get

$$L\{y(t)\} = L\{\cos t\} + L\{y(t)\times\cos(t)\}$$

$$L\{y(t)\} = \frac{p}{p^2+1} + L\{y(t)\}.\frac{p}{p^2+1}$$

$$L\{y(t)\}\left[1-\frac{p}{p^2+1}\right] = \frac{p}{p^2+1}$$

$$L\{y(t)\}\left(\frac{p^2+1-p}{p^2+1}\right) = \frac{p}{(p^2+1)}$$

$$L\{y(t)\} = \frac{p}{p^2-p+1}$$

$$y(t) = L^{-1}\left\{\frac{P}{P^2-P+1}\right\}$$

$$y(t)=L^{-1}\left\{\frac{p}{\left(p-\frac{1}{2}\right)+\frac{3}{4}}\right\}=e^{\frac{t}{2}}L^{-1}\left\{\frac{p}{p^2+\frac{3}{4}}\right\}$$

$y(t)=e^{t/2}\cos\left(\frac{\sqrt{3}}{2}\right)t$ is required solution of given integral equation.

5. (c) Given $\bar{\alpha}=t\hat{i}+(t+1)\hat{j}+(2t+1)\hat{k}$

$\bar{\beta}=2t\hat{i}+(3t-1)\hat{j}+t\hat{k}$

$\bar{\gamma}=\hat{i}+3t\hat{j}+\hat{k}$

Area of the parallel gem = $|\bar{\alpha}\times\bar{\gamma}|$ .....(1)

Now, $\bar{\alpha}\times\bar{\gamma}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t & t+1 & 2t+1\\ 1 & 3t & 1\end{vmatrix}$

$=\hat{i}\left(t+1-6t^2-3t\right)-\hat{j}\left(t-2t-1\right)+\hat{k}\left(3t^2-t-1\right)$

$(\bar{\alpha}\times\bar{\gamma})=\left(-6t^2-2t+1\right)\hat{i}+(t+1)\hat{j}+\left(3t^2-t-1\right)\hat{k}$

Now, Volume of the parallelopiped = $\bar{\alpha}.(\bar{\beta}\times\bar{\gamma})$

$=\begin{vmatrix}t & t+1 & 2t+1\\ 2t & 3t-1 & t\\ 1 & 3t & 1\end{vmatrix}$

= $t\left\{3t-1-3t^2\right\}-(t+1)\{t\}+(2t+1)\left\{6t^2-3t+1\right\}$

= $-3t^3+3t^2-t-t^2-t+12t^3-6t^2+6t^2+2t-3t+1$

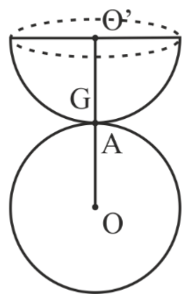
Volume = $9t^3+2t^2-3t+1$

Rate of change of volume = $\frac{dV}{dt}=27t^2+4t-3$

Now, At t = 1 second.

Rate of charge of volume = $\left(\frac{dV}{dt}\right)_{t=1}=27+4-3=28$

5. (d) (1) Consider a solid hemisphere rests on a solid sphere of equal radius a in equilibrium



Let G is the centre of gravity & A is the point of contact

Also, here the concave part of the hemisphere rests on the sphere.

$r = a$, R = a

$$h = AG = O'A - O'G = a - \frac{3a}{8} = \frac{5a}{8}$$

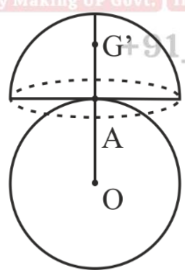
We have, $\frac{1}{R} + \frac{1}{r} = \frac{1}{a} + \frac{1}{a} = \frac{2}{a}$

And, $\frac{1}{h} = \frac{8}{5a}$

Clearly $\frac{1}{R} + \frac{1}{r} > \frac{1}{h}$ $\left\{ \because \frac{2}{a} > \frac{8}{5a} \right\}$

$\therefore$ Equilibrium is unstable

Similarly, when the flat part of the hemisphere rests on the sphere.



$r = \infty$, R = a

$$h = AG' = \frac{3a}{8}$$

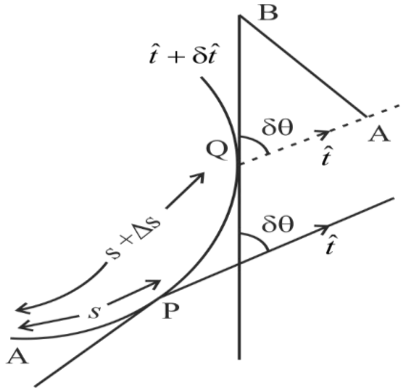
$$\therefore \frac{1}{r} + \frac{1}{R} = \frac{2}{a}$$

$$\frac{1}{h} = \frac{8}{3a}$$

Clearly, $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$ $\left\{ \because \frac{8}{3a} > \frac{2}{a} \right\}$

$\therefore$ Equilibrium is stable.

5. (e)



If $\vec{r}$ is position vector of any point P one the curve, then.

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}.\frac{ds}{dt} = \vec{r}\,'\frac{ds}{dt} = \vec{r}\,'\dot{s} = \hat{t}\,\dot{s} \text{ ....(1)} \quad \therefore \ |\dot{\vec{r}}| = |\hat{t}\,\dot{s}| = \dot{s} \text{ .....(2)}$$

Diff. w.r.t *t*, $\ddot{\vec{r}} = \hat{t}\,\ddot{s} + \hat{t}\,'(\dot{s})^2 \Rightarrow \ddot{\vec{r}} = \hat{t}\,\ddot{s} + (\kappa\hat{n})(\dot{s})^2$ .....(3)

Taking cross product of (1) & (3), we have,

$$\dot{\vec{r}}\times\ddot{\vec{r}} = \hat{t}\dot{s}\times\left\{\kappa\hat{n}(\dot{s})^2 + \hat{t}\ddot{s}\right\} \Rightarrow \dot{\vec{r}}\times\ddot{\vec{r}} = \kappa(\dot{s})^3(\hat{t}\times\hat{n}) \quad \left\{\because\ \hat{t}\times\hat{t} = \vec{0}\right\}$$

$$\dot{\vec{r}}\times\ddot{\vec{r}} = \kappa(\dot{s})^3\hat{b} \quad \text{.....(4)} \Rightarrow |\dot{\vec{r}}\times\ddot{\vec{r}}| = \left|\kappa(\dot{s})^3\right|.1 \quad \left\{\because\ |\hat{b}| = 1\right\}$$

$$\boxed{\kappa = \frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}} \text{ ....(3)} \qquad \left\{\dot{\vec{r}} = \dot{s}\ (\text{from (2)})\right\}$$

Given, $\vec{r} = f(t)\hat{i} + g(t)\hat{j}$

So, $\dot{\vec{r}} = f\,'(t)\hat{i} + g\,'(t)\hat{j}$, $\ddot{\vec{r}} = f\,''(t)\hat{i} + g\,''(t)\hat{j}$, $|\dot{\vec{r}}| = \sqrt{(f\,'(t))^2 + (g\,'(t))^2}$,

$\dot{\vec{r}}\times\ddot{\vec{r}} = f\,'(t)g\,''(t) - g\,'(t)f\,''(t)$

Now, on using in (3), we get $K = \dfrac{|f'(t)g''(t) - g'(t)f''(t)|}{\left[f'(t)\right]^2 + \left[g'(t)\right]^2)^{3/2}}$

(ii) The necessary & sufficient condition for a given curve to be a plane curve is that $\tau = 0$ at all points of the curve.

Sol. Let if curve is place then we have to prove $\tau = 0$

$\therefore$ By a plane curve means the tangents & normals at all points of the curve is in the plane of curve.

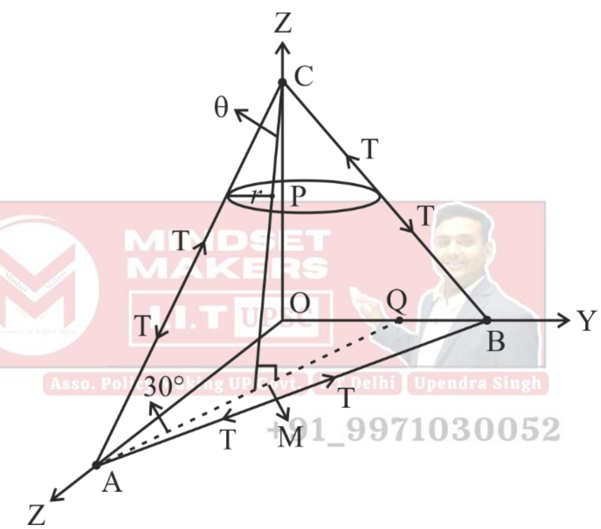
So, we can conclude that osculating plane at all points of the curve, is the plane of the curve

$\therefore$ $\hat{b}$; the unit vector along binormal is constant.

$\hat{b}$ = constant $\Rightarrow \frac{d\hat{b}}{ds} = 0 \quad \therefore \quad \tau = 0$

So, if we take two consecutive points and then find change in unit binormal w.r.t. s i.e. $\frac{d\hat{b}}{ds}$. We found $\frac{d\hat{b}}{ds}$ =0, i.e. $\tau = 0$

6. (a)



Let's consider this regular tetrahedron consists of light six rods (Equal and opposite stress/tension in each rod). Length of each rod is *l*.

Let θ be the angle which slant (rod) side makes with OC. Ring of radius *r* and weight W.

$\because$ By principle of virtual work,

$$-W.\delta(\text{PM}) - 3\text{T}.\delta(\text{AB}) = 0 \qquad \text{.....(1)}$$

Now by geometry, we wish to find PM.

$\because$ PM = CM – CP, But $\left.\begin{array}{l}\text{CP} = r\cot\theta \\ \text{CM} = l\cos\theta\end{array}\right]$ .....(2)

$$AM = \frac{2}{3}AQ = l\sin\theta \Rightarrow AQ = \frac{3}{2}l\sin\theta$$

$$OA = OB = AB = AB/\cos 30° = \frac{2}{\sqrt{3}} \times \frac{3}{2}l\sin\theta \quad \Rightarrow AB = \sqrt{3}l\sin\theta \text{ .....(3)}$$

Now, on using (2) and (3) in (1), we get

$$-W\delta(l\cos\theta - r\cot\theta) - 3T\delta\left(l\sqrt{3}\sin\theta\right) = 0$$

$$\therefore\ W\left(-l\sin\theta + r\,\text{cosec}^2\theta\right) - 3\sqrt{3}l\,T\cos\theta = 0$$

$$\therefore\ W = \frac{T \times 3\sqrt{3}l\cos\theta}{\left(r\,\text{cosec}^2\theta - l\sin\theta\right)} \quad \text{.....(4)}$$

Now, using the condition for equilibrium; $\sin\theta = \frac{1}{\sqrt{3}}$ .....(5)

$\therefore$ Using (5) in (4), we get

$$T = \frac{W\left(r \times 3 - l \times \frac{1}{\sqrt{3}}\right)}{3\sqrt{3}l \times \frac{\sqrt{2}}{\sqrt{3}}}$$

$$T = \frac{W\left(3\sqrt{3}r - l\right)}{3\sqrt{3}.\sqrt{2}.l}$$

6. (b) As, the particle is executing SHM

So, the diff. equation of SHM is

$$\frac{d^2x}{dt^2} = -\mu x \quad \text{.....(1)}$$

Let the amplitude of the SHM he a then,

$$\left(\frac{dx}{dt}\right)^2 = \mu\left(a^2 - x^2\right)$$

Let us consider two positions at distances x_1 & x_2

Where velocities are u & v & acceleration are f_1 & f_2

$$\therefore\ f_1 = \mu x_1 \quad \text{.....(1)}$$

$$f_2 = \mu x_2 \qquad .....(2)$$

Also, $u^2 = \mu\left(a^2 - x_1^2\right) \qquad .....(3)$

$$v^2 = \mu\left(a^2 - x_2^2\right) \qquad .....(4)$$

Now, $v^2 - u^2 = \mu\left(x_1^2 - x_2^2\right)$

$$v^2 - u^2 = \mu\left(x_1 + x_2\right)\left(x_1 - x_2\right)$$

$$x_1 - x_2 = \frac{v^2 - u^2}{\left(f_1 + f_2\right)} \qquad \text{\{Using (1) \& (2)\}}$$

$$\therefore\ k = \frac{1}{f_1 + f_2} \qquad \text{\{On comparing with, } \left(v^2 - u^2\right)k\text{\}}$$

Using (1) in (3),

$$u^2 = \mu\left(a^2 - \frac{f_1^2}{\mu^2}\right)$$

$$\Rightarrow a^2\mu^2 - f_1^2 = u^2\mu$$

$$\Rightarrow a^2\mu^2 - u^2\mu - f_1^2 = 0 \qquad .....(5)$$

Similarly, from (2) & (4)

$$a^2\mu^2 - v^2\mu - f_2^2 = 0 \qquad .....(6)$$

Solving (5) & (6);

$$\frac{\mu^2}{u^2 f_2^2 - v^2 f_1^2} = \frac{\mu}{-a^2 f_1^2 + a^2 f_2^2} = \frac{1}{a^2u^2 - a^2v^2}$$

Equating the values of μ^2 from above equation.

$$\frac{u^2 f_2^2 - v^2 f_1^2}{a^2\left(v^2 - u^2\right)} = \left[\frac{a^2\left(f_2^2 - f_1^2\right)}{a^2\left(v^2 - u^2\right)}\right]^2$$

$$\frac{f_1^2v^2 - f_2^2u^2}{a^2\left(v^2 - u^2\right)} = \frac{\left(f_1^2 - f_2^2\right)^2}{\left(v^2 - u^2\right)^2}$$

$$a^2 = \frac{\left(f_1^2v^2 - f_2^2u^2\right)\left(v^2 - u^2\right)}{\left(f_1^2 - f_2^2\right)^2}$$

$$a = \frac{\left\{\left(f_1^2 v^2 - f_2^2 u^2\right)\left(v^2 - u^2\right)\right\}^{1/2}}{\left(f_1^2 - f_2^2\right)}$$

6. (c) (i) The given differential equation is $xy'' + (x-1)y' - y = 0$ .....(1)

Given $f(x) = -e^{-x}$ is one of the solution

Comparing the given equation with $p(x)y'' + q(x)y' + r(x)y = 0$,

we have $p(x) = x$ and $q(x) = (x-1)$.

Also here $f(x) = -e^{-x}$.

Hence the second solution $= ve^{x}$, where v is found by using formula

$$v = \int \frac{\exp\left[-\int \{q(x)/p(x)\}\,dx\right]}{\{f(x)\}^2}dx, \qquad \text{....(i)}$$

Now, $\int \frac{q(x)}{p(x)}dx = \int \frac{x-1}{x}dx = \int (1 - 1/x)\,dx = (x - \log x)$

$\therefore \exp\left[-\int \{q(x)/p(x)\}\,dx\right] = e^{-(x-\log x)} = e^{-x+\log x} = e^{-x}e^{\log x} = xe^{-x}$

$\therefore$ from(i), $v = \int \frac{xe^{-x}}{\left(-e^{-x}\right)^2}dx = \int xe^{x}dx = xe^{x} + e^{x}$

$\therefore$ Required second solution = $v(-e^{-x}) = (x+1)e^{x}(-e^{-x}) = -(x+1)$

6. (c) (ii) Given differential equation is.

$x^2y'' - 2xy' + 2y = x^3 \sin x$ .....(1)

This is cauchy euler differential equation

Let $x = e^{z} \Rightarrow z = \log x$

$\therefore$ equation (1) is reduced to,

$$\left(D(D-1) - 2D + 2\right)y = e^{3z}\sin\left(e^{z}\right); \quad where\ D = \frac{d}{dz}$$

$$\left(D^2 - 3D + 2\right)y = e^{3z}\sin\left(e^{z}\right) \qquad \text{.....(2)}$$

Equation (2) is in the form of differential equation with constant coefficients.

For C.F, the homogenous differential equation for (2) is,

$$\left(D^2 - 3D + 2\right)y = 0 \quad ...(3)$$

Auxiliary equation is,

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1, 2$$

$\therefore$ C.F = $c_1e^z + c_2e^{2z}$

So, $u = e^z$ & $v = e^{2z}$ are two LI solutions of (3).

Wronskian of above two solutions: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^z & e^{2z} \\ e^z & 2e^{2z} \end{vmatrix} = 2e^{3z} - e^{3z} = e^{3z}$

$$\therefore \ A = -\int \frac{vR}{w} = -\int \frac{e^{2z}e^{3z}\sin\left(e^z\right)}{e^{3z}}dz = -\int e^{2z}\sin\left(e^z\right)dz$$

$$= -\int x^2 \sin x \times \frac{1}{x}dx$$

$$= -(-x\cos x + \sin x)$$

$$A = x\cos x - \sin x$$

$$B = \int \frac{uR}{w} = \int \frac{e^z e^{3z}\sin\left(e^z\right)dz}{e^{3z}}$$

$$= \int x\sin x + \frac{1}{x}dx$$

$$B = \ -cosx$$

$\therefore$ Required P.I. = $\mathrm{Au} + \mathrm{Bv}$

$$= (x\cos x - \sin x)x + (-\cos x) \times x^2$$

$\therefore$ Required general solution is,

$$y = c_1e^z + c_2e^{2z} + (x\cos x - \sin x)x - x^2\cos x$$

$$y = c_1x + c_2x^2 + (x\cos x - \sin x)x - x^2\cos x$$

$$y = c_1x + c_2x^2 - x\sin x$$

7. (a) Existence and Uniqueness Theorem

Let's consider an IVP $\frac{dy}{dx} = f(x,y);\ y(x_0) = y_0$ if $f(x,y)$ is continuous in a closed domain $R:\{(x,y) \| |x| \le h,\ |y| \le k\}$ i.e. it is bounded in R.

- If $|f(x,y)| \le M$, then $\exists$ **at least one solution** of $y' = f(x,y);\ y(x_0) = y_0$ defined for $|x - x_0| \le \alpha$, where $\alpha = \min\left\{h, \frac{k}{M}\right\}$....(1)

- If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in the closed domain R i.e., $f(x,y)$ and $\frac{\partial f}{\partial y}$ are bounded in R.

Then $\exists$ **at most one solution** of $y' = f(x,y);\ y(x_0) = y_0$ defined for $|x - x_0| \le \alpha$ where $\alpha = \min\left\{h, \frac{k}{M}\right\}$....(2)

*On combining above two statements ; If $f(x,y)$, $\frac{\partial f}{\partial y}$ are bounded in R

Then $y' = f(x,y);\ y(x_0) = y_0$ has unique solution defined for $|x - x_0| \le \alpha$; where $\alpha = \min\left\{h, \frac{k}{M}\right\}$

Note: (x_0, y_0) is some initial point i.e. $x_0, y_0 \in R$

i.e. R is of the form $R \cdot \{(x,y) : |x - x_0| \le h,\ |y - y_0| \le k\}$

Let's consider an IVP, $y' = y^\alpha$; $y(0) = 0$ *where* $0 < \alpha < 1$ *This IVP has infinite no. of solution.*

E.g. $y' = 2\sqrt{y}$; $y(0) = 0$

Let $y(x) = \begin{cases} 0\ ;\ x \le r \\ (x-r)^2\ ;\ x > r \end{cases}$

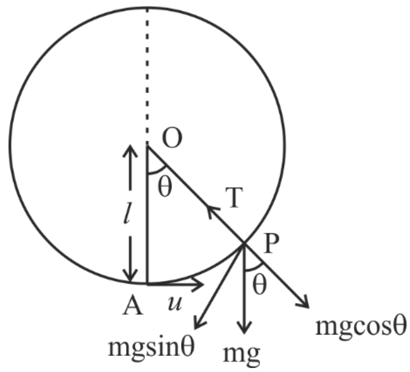
where $r \in \mathbf{R}^+$ i.e. r is any positive integer

$\therefore y'(x) = \begin{cases} 0\ ;\ x \le r \\ 2(x-r)\ ;\ x > r \end{cases}$

$\because$ $y(x)$ satisfies the given IVP also r is arbitrary

i.e., the given IVP has infinitely many solutions. We can different such solutions $y(x)$ by taking different positive real numbers in place of r in above expression of $y(x)$.

7. **(b)** When the particle is moving in circular motion with initial velocity u from A then at any point P the tension in the string is given by, $T = \frac{m}{l}\left(u^2 - 2lg + 3lg\cos\theta\right)$...(1)



If we consider any diameter with center O, at any angle α w.r.t OA, the tension is,

$$T_1 = \frac{m}{l}\left(u^2 - 2lg + 3lg\cos\alpha\right)$$

And at the other end of the diameter the tension is given by

$$T_2 = \frac{m}{l}\left(u^2 - 2lg + 3l\cos(\pi + \alpha)\right)$$

$$T_2 = \frac{m}{l}\left(u^2 - 2lg - 3l\cos\alpha\right)$$

$\therefore\ T_1 + T_2 = \frac{2m}{l}\left(u^2 - 2lg\right)$, which is constant.

$\therefore$ Sum of the tension at the ends of any diameter of the center is constant.

Note: for the formula; equation of motion is

- $m\frac{d^2s}{dt^2}$ = –mg sinθ.....(1)
  - $m\frac{v^2}{l}$ = T – mg cos θ....(2)

□direction of mg sinθ is in direction of s decreasing. So, negative sign is attached

□$s = l\theta \Rightarrow \dfrac{ds}{dt} = l\dfrac{d\theta}{dt}$ and $\dfrac{d^2s}{dt^2} = l\dfrac{d^2\theta}{dt^2}$

By using this, Now, to solve differential equation (1) and (2);

$\therefore\ m\left(l\dfrac{d^2\theta}{dt^2}\right) = -mg\sin\theta$

$l\dfrac{d^2\theta}{dt^2} = -g\sin\theta.$

Multiplying both side by $2l\dfrac{d\theta}{dt}$ and the integrating

$\left(l\dfrac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$; where A is constant of integration

i.e., $v^2 = 2lg\cos\theta + A \quad \left[\because v = \dfrac{ds}{dt} = l\dfrac{d\theta}{dt}\right]$

Initially at point A;

$\theta = 0,\ v = u.$

$\therefore$ we get, $u^2 = 2lg + A \Rightarrow A = u^2 - 2lg$

So, we have $v^2 = u^2 - 2lg + 2lg\cos\theta$ ....(3)

Now, we discuss about tension in string; T

$\therefore$ from (2), we have, $\dfrac{mv^2}{l} = T - mg\cos\theta$

$\dfrac{m}{l}\left\{2lg\cos\theta + u^2 - 2lg\right\} = T - mg\cos\theta$

$$\boxed{T = \frac{m}{l}\left\{u^2 - 2lg + 3lg\cos\theta\right\}} \quad ...(4)$$

7. (c) • **Statement:** If S is a surface with closed boundary C, then $\int_C \vec{F}.\vec{dr} = \int_S \text{Curl}\,\vec{F}.\hat{n}\,dS$.

Now, decoding boundary C of the given surface S.

Parameterize: **Note:** Here $z = 1 - x^2$ is the surface with four boundary segments;

Curve 1; $< 1-x, 2, 1-(1-x)^2 >$ where $0 \le x \le 1$

Curve 2; $< x, -2, 1-x^2 >$ where $0 \le x \le 1$

Curve 3; $< 0, -y, 1 >$ where $-2 \le y \le 2$

Curve 4; $<1, y, 0>$ where $-2 \le y \le 2$

So, the boundary is consisting of four curves 1, 2, 3, 4,

Finding line integral around C;

- Around curve (1): $\int_{C_1} \vec{F}.\overrightarrow{dr} = \int_{x=0}^{1} \vec{F}.\overrightarrow{dr}$

$\because \ \vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$

$\therefore \ \vec{F}.\overrightarrow{dr} = xydx + yzdy + zxdz$

$\therefore \ \vec{F}.\overrightarrow{dr} = -(1-x)2dx + 2\left(1-(1-x)^2\right)\times 0 + \left(1-(1-x)^2\right)x.2(1-x)dx$

$\therefore \ \int_{x=0}^{1} \vec{F}.\overrightarrow{dr} = \int_{x=0}^{1} -2(1-x)dx + 0 + 2\left(1-(1-x)^2\right)2x(1-x)dx$

$= \int_0^1 -2(1-x) + 2(1-x)^2\left(2x.x^2\right)dx = -\dfrac{11}{15}$

- Around curve 2 (C_2)

$\vec{F}.\overrightarrow{dr} = x(-2)dx + (-2)\left(1-x^2\right)\times 0 + \left(1-x^2\right)x(-2x2dx)$

$\therefore \ \int_{C_2} \vec{F}.\overrightarrow{dr} = \int_{x=0}^{1} -2x - 2x^2\left(1-x^2\right)dx = \dfrac{-19}{15}$

- Around Curve 3 (C_3)

$\vec{F}.\overrightarrow{dr} = 1\times y\times 0 + 0 + 0$

$\therefore \ \int_{C_3} \vec{F}.\overrightarrow{dr} = \int_{y=-2}^{2} ydy = 0$

- Around Curve 4 (C_4)

$\vec{F}.\overrightarrow{dr} = 1\times y\times 0 + 0 + 0$

$\therefore \ \int_{C_4} \vec{F}.\overrightarrow{dr} = 0$

So, $\int_{C} \vec{F}.\overrightarrow{dr} = \int_{C_1} \vec{F}.\overrightarrow{dr} + \int_{C_2} \vec{F}.\overrightarrow{dr} + \int_{C_3} \vec{F}.\overrightarrow{dr} + \int_{C_4} \vec{F}.\overrightarrow{dr} = -\dfrac{11}{15} - \dfrac{19}{15} = -2$ .....(1)

Finding surface integral $\int_S \text{Curl } \vec{F}.\hat{n} dS$

$$\therefore \text{ Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\hat{i} - z\hat{j} - x\hat{k}$$

$$\hat{n} = \frac{2x\hat{i} + 0\,\hat{j} + 2.\hat{k}}{\sqrt{1+4x^2}} \quad \because \; z = 1 - x^2 \Rightarrow z + x^2 + 1 = 0 = \phi \,; \quad \hat{n} = \frac{\text{grad}\,\phi}{|\text{grad }\phi|}$$

$\therefore$ Curl $\vec{F}.\hat{n} = \dfrac{-2xy - x}{\sqrt{1+4x^2}}$

$\therefore$ Curl $\vec{F}.\hat{n}dS = \dfrac{-2xy - x}{\sqrt{1+4x^2}} \times \sqrt{1+4x^2}\, dx$

Curl $\vec{F}.\hat{n}dS = \dfrac{-2xy - x}{\sqrt{1+4x^2}} \times \sqrt{1+4x^2}\, dx = -2xy - x$

$\because \; dS = \dfrac{dxdy}{|\hat{n}.\hat{k}|} \quad \because \; \hat{n}.\hat{k} = \dfrac{1}{\sqrt{1+4x^2}}$

$\therefore \int_S \text{Curl}\,\vec{F}.\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{2} (-2xy - x)bndy = \int_0^1 -4rd\,x = -2$ .....(2)

Therefore, from (1) & (2), we have,

$\int_C \vec{F}.\vec{dr} = \int_S \text{Curl}\,\vec{F}.\,\hat{n}\,dS = -2$ Hence Stoke's theorem is verified.

8. **(a)** Given $y'' + 2y' + 5y = \delta(t-2), \;\; y(0) = 0, \,\&\, y'(0) = 0$

Taking Laplace on both sides,

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{\delta(t-2)\}$$

$$p^2L\{y\} - py(0) - y'(0) + 2\{pL\{y\} - y(0)\} + 5\{y\} = e^{-2p} \qquad \{\because L(\delta(t-2) = e^{-2p}\}$$

$$p^2L\{y\} - 2pL\{y\} + 5L\{y\} = e^{-2y}$$

$$\left(p^2 + 2p + 5\right)L\{y\} = e^{-2p}$$

$$L\{y\}=\frac{e^{-2p}}{\left(p^2+2p+5\right)}$$

$$L\{y\}=\frac{e^{-2p}}{(p+1)^2+4}$$

$$y=L^{-1}\left\{\frac{e^{-2p}}{(p+1)^2+2^2}\right\}$$

Let $f(p)=\frac{1}{(P+1)^2+2^2}$

$$L^{-1}\{f(p)\}=e^{-t}L^{-1}\left\{\frac{1}{P^2+2^2}\right\}$$

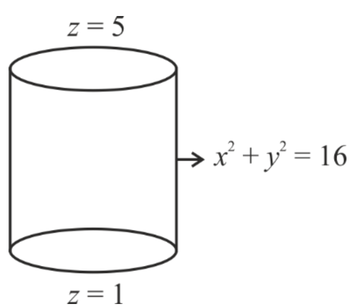
$$L^{-1}\{f(p)\}=e^{-t}\times\frac{\sin 2t}{2}=F(t) \qquad \text{(say)}$$

$$\therefore\ y=L^{-1}\left\{\frac{e^{-2p}}{(p+1)^2+2^2}\right\}=\begin{cases}F(t-2) & ,t>2\\ 0 & ,t<2\end{cases}$$

$$y=\begin{cases}\frac{e^{-(t-2)}\sin 2(t-2)}{2} & ,t>2\\ 0 & ,t<2\end{cases}$$

8. **(b)** Given $\vec{F}=y^2\hat{i}+xz^3\hat{j}+(z-1)^2\hat{k}$

As, the given surface S is bounded & closed enclosing a volume V.



So, Here Gauss divergence theorem is applicable.

By Gauss divergence Theorem,

$$\iint_S \vec{F}.\hat{n}ds=\iint_V \vec{\nabla}.\vec{F}\,dV \qquad .....(1)$$

$$\vec{\nabla}.\vec{F}=0+0+2(z-1)=2z-2$$

Let's take the cylinder in polar coordinate.

$x = r\cos\theta, y = r\sin\theta, \quad z = z$

Limits of r is 0 to 4

Limits $0 < \theta < 2\pi$

$$\therefore \iint_v \vec{\nabla}.\vec{F}dv = \int_{r=0}^{4}\int_{\theta=0}^{2\pi}\int_{z=1}^{5}(\nabla.F)\,rd\theta drdz$$

$$= \int_{\theta=0}^{2\pi}\int_{z=1}^{5} 2(z-1)\left[\frac{r^2}{2}\right]_0^4 d\theta dz$$

$$= 8\times 2\times 2\pi\left[\frac{z^2}{2}-z\right]_1^5$$

$$= 32\pi\times\left[\frac{25}{2}-5-\frac{1}{2}+1\right]$$

$$= 32\pi\times\frac{16}{2}$$

$$\iiint_v \vec{\nabla}.\vec{F}dv = 256\pi$$

$\therefore$ From (1),

$$\iint_s \vec{F}.\hat{n}ds = \iint_s\left(y^2\hat{i}+xz^3\hat{j}+(z-1)^2\hat{k}\right).\hat{n}ds = 256\pi$$

8. (c) Given, $P = \mu\left(\frac{3}{r^3}+\frac{d^2}{r^5}\right)$

$P = \mu\left(3u^3 + d^2u^5\right) \qquad \left\{\because u = \frac{1}{r}\right\}$

$\therefore$ The differential equation of the central orbit is.

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(3u^3+d^2u^5\right)$$

$$h^3\left[u+\frac{d^2u}{d\theta^2}\right] = \mu\left(3u+d^2u^3\right)$$

Multiplying both sides by $\frac{2du}{d\theta}$ & integrating

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2 + \frac{d^2}{2}u^4\right) + A \qquad .....(1)$$

Now, If v is the velocity in a circular path at a distance d, then

$$\frac{V^2}{d} = [\mathrm{P}]_{r=d}$$

$$\frac{V^2}{d} = \mu\left(\frac{3}{d^3} + \frac{1}{d^3}\right)$$

$$\frac{V^2}{d} = \frac{4\mu}{d^3}$$

$$V = \frac{2\sqrt{\mu}}{d}$$

From (1), Initially when $r = d$, $u = \frac{1}{d}$, $v = \frac{2\sqrt{\mu}}{d}$, $\phi = 45°$

$\therefore \; p = r\sin\phi$

$$p = d\sin\frac{\pi}{4}$$

$$p = \frac{d}{\sqrt{2}}$$

And, $\frac{1}{P^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2}{d^2}$

$\therefore$ From (1)

$$\frac{4\mu}{d^2} = h^2\frac{2}{d^2} = \mu\left(\frac{3}{d^2} + \frac{d^2}{2d^4}\right) + A$$

$$\Rightarrow h^2 = 2\mu \;\; \& \;\; A = \frac{\mu}{d^2}\left(4 - 3 - \frac{1}{2}\right)$$

$$A = \frac{\mu}{2d^2}$$

Using h^2 & A in (1),

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2 + \frac{d^2}{2}u^4\right) + \frac{\mu}{2d^2}$$

$$2\left(\frac{du}{d\theta}\right)^2 = u^2 + \frac{d^2}{2}u^4 + \frac{1}{2d^2}$$

Putting $u = \frac{1}{r}$ & $\frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}$

$$\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{d^2}{2r^4} + \frac{1}{2d^2}$$

$$\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2r^2d^2 + d^{64} + r^4}{2r^4d^2}$$

$$4d^2\left(\frac{dr}{d\theta}\right)^2 = \left(r^2 + d^2\right)^2$$

$$\frac{dr}{d\theta} = -\frac{\left(r^2 + d^2\right)}{2d}$$ {-ve sign taken as *r* decreases when θ increases}

We know,

$$h = \frac{r^2 d\theta}{dt}$$

$$h = r^2\frac{d\theta}{dr}.\frac{dr}{dt}$$

$$\Rightarrow \sqrt{2\mu} = -r^2\frac{2d}{\left(r^2 + d^2\right)}\frac{dr}{dt}$$

$$\Rightarrow dt = \frac{-2d}{\sqrt{2\mu}}.\frac{r^2}{r^2 + d^2}dr$$

Time taken to reach the centre of force be t_1

$$\therefore\ t_1 = -\frac{2d}{\sqrt{2\mu}}\int_{r=d}^{0}\frac{r^2}{r^{2+d^2}}dr$$

$$t_1 = -\frac{2d}{\sqrt{2\mu}}\int_{r=d}^{0}\left(1 - \frac{d^2}{r^2 + d^2}\right)$$

$$t_1 = -\frac{2d}{\sqrt{2\mu}}\left[r - d\tan^{-1}\left(\frac{r}{d}\right)\right]_{r=d}^{0}$$

$$t_1 = \frac{-2d}{\sqrt{2\mu}}\left[0 - \left\{d - d\tan^{-1}(1)\right\}\right]$$

$$t_1 = \frac{-2d^2}{\sqrt{2\mu}}\left\{1 - \frac{\pi}{4}\right\}$$

$$t_1 = \frac{d^2}{\sqrt{2\mu}}\left[2 - \frac{\pi}{2}\right]$$

Although the reference could be given with each answer from the books written by Upendra Sir, but the screenshots below are more than enough to give a glimpse abot the systematically designed books. This topic Existence and uniqueness, not asked before. Still it was in books with appropriate remarks from Upendra Sir. Rest students themselves can guess the usefulness of these books. Download books from website mindsetmakers.in

Existence and Uniqueness of solution of 1[st] order ODE's(**It's just to have some idea and not diretly asked in UPS optional. But still we need to go through it at least one)**

Let's consider a definition of equation (IVP) $y' = f(x,y)\ ;\ y(x_0) = y_0$

Now we are interested to know

(1) The given IVP has some solution or not?i.e. existence of solution of IVP

(2) if solution exists then ; is it unique?

e.g. (i) $y' = 2x\ ,\ y(0) = 1$ (ii) $xy' = y - 1\ ;\ y(0) = 1$ (iii) $y'^2 + 1 + y^2 = 0\ ;\ y(0) = 1$

(i) $\frac{dy}{dx} = 2x \Rightarrow dy = 2xdx \Rightarrow y = x^2 + c$ at $x = 0, y = 1 \Rightarrow 1 = 0 + c \Rightarrow = 1\ \therefore\ \boxed{y = x^2 + 1}$

i.e. the given IVP has unique solution.

(ii) $\frac{dy}{dx} - \frac{y}{x} = \frac{-1}{x}$ I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x} \Rightarrow \frac{y}{x} = \int \frac{-1}{x^2}dx + c = \frac{1}{x} + c \Rightarrow \boxed{y = cx + 1} \rightarrow$ infinitely

many solutions.

Note: Let's consider an IVP $y' = y^{\alpha}\ ;\ 0 < \alpha < 1$, with $y(0) = 0$. This IVP has infinite no. of solution.

e.g. $y' = 2\sqrt{y}\ ;\ y(0) = 0$

Let $y(x) = \begin{cases} 0\ ;\ x \le r \\ (x-r)^2\ ;\ x > r \end{cases}$

$r \in \mathbf{R}^+$

$y'(x) = \begin{cases} 0\ ;\ x \le r \\ 2(x-r)\ ;\ x > r \end{cases}$

$\because\ y(x)$ satisfies the given IVP also r is arbitrary

i.e., the given IVP has infinitely many solutions.

Q. The IVP $y' = \sqrt{y},\ y(0) = \alpha,\ \alpha \ge 0$ has

(a) At least two solution if $\alpha = 0$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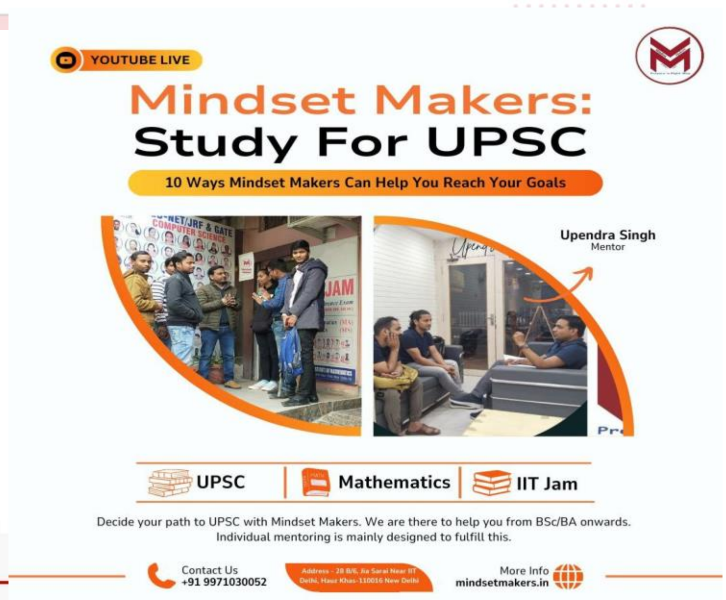
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