Mindset Makers

PAPER-2: Paper Analysis

UPSC CSE MAIN 2024

Mathematics Optional

- Well Structured
- Addressing Demand of IAS/IFoS
- Systematically Designed
- Video Discussion by Upendra Sir

UPSC CSM 2024 MATHEMATICS OPTIONAL PAPER-2 ANALYSIS

1. (a) Let G be a finite group of order *mn*, where m and n are prime numbers with $m > n$. Show that G has at most one subgroup of order *m*.

(b) If $w = f(z)$ is an analytic function of *z*, then show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$$

(c) Test the convergence of $\int_0^2 \frac{\log x}{\sqrt{(2-x)}}dx$

(d) If ▯ and ψ are function of *x* and *y* satisfying Laplace equation, then show that $f(z) = p + iq, i = \sqrt{-1}$ is an analytic function, where $p = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}$ and $q = \frac{\partial\phi}{\partial x} -+ \frac{\partial\psi}{\partial y}$

(e) Use two phase method to solve the following linear programming problem:

Maximize $z = x_1 + 2x_2$

Subject to $x_1 - x_2 \geq 3$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0$

2. (a) Using Cauchy's general principle of convergence, examine the convergence of the sequence $< f_n >$, where $f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$.

(b) Show that every homomorphic image of an abelian group is abelian, but the converse in not necessarily true.

(c) Find the function which is analytic inside and on the circle $C : z = e^{i\theta}, 0 \leq \theta \leq 2\pi$ and has the value $\frac{(a^2-1)\cos\theta + i(a^2+1)\sin\theta}{a^4 - 2a^2\cos 2\theta + 1}$ on the circumference of C, where $a^2 > 1$.

3. (a) Locate the poles and their order for the function $f(z) = \frac{1}{z(\sin \pi z)\left(z+\frac{1}{2}\right)}$ Also, find the residue of f(z) at these poles.

(b) Consider the series $\sum_{n=1}^{\infty} U_n(x), 0 \le x \le 1$, the sum of whose first n terms is given by $S_n(x)$

$S_n(x) = \frac{1}{2n^2}\log\left(1+n^4x^2\right), x \in [0, 1]$. Show that the given series can be differentiated term-by-term, though $\sum_{n=1}^{\infty} U_n'(x)$, does not converge uniformly on [0, 1].

(c) Using duality principle, solve the following linear programming problem:

Minimize $z = 4x_1 + 3x_2 + x_3$

Subject to $x_1 + 2x_2 + 4x_3 \ge 12$

$3x_1 + 2x_2 + x_3 \ge 8$

$x_1, x_2, x_3 \ge 0$

4. (a) Consider the polynomial ring Z[x] over the ring Z of integers. Let S be an ideal of Z[x] generated by x. Show that S is prime but not a maximal ideal of Z[x].

(b) Find the upper and lower Riemann integrals for the function *f* defined on [0, 1] as follows:

$$f(x) = \begin{cases} \left(1-x^2\right)^{1/2}, & \text{if } x \text{ is rational} \\ (1-x), & \text{if } x \text{ is rational} \end{cases}$$

Hence, show that f is not Riemann integrable on [0, 1].

(c) The personnel manager of a company wants to assign officers A, B and C to the regional offices at Delhi, Mumbai, Kolkata and Chennai. The cost of relocation (in thousand Rupees) of the three officers at the four regional offices are given below:

Officer	Delhi	Mumbai	Kolkata	Chennai
A	16	22	24	20
B	10	32	26	16
C	10	20	46	30

Find the assignment which minimizes the total cost of relocation and also determine the minimum cost.

SECTION B

5. (a) Show that if f and g are arbitrary function of their respective arguments, then $u = f(x-kt+iay) + g(x-kt-iay)$, is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}$, where $\alpha^2 = 1 - \frac{k^2}{C^2}$

.

(b) Solve the following system of linear equation by Gauss-Jordan method:

$$2x+3y-z=5$$

$$4x+4y-3z=3$$

$$2x-3y+2z=2$$

(c) (i) Determine the decimal equivalent in sign magnitude form of $(8D)_{16}$ and $(FF)_{16}$.

(ii) Determine the decimal equivalent of $(9B2.1A)_{16}$.

(d) A rough uniform board of mass m and length 2a rests on a smooth horizontal plane and a man of mass M walks on it form on end to the other. Find the distance covered by the board during this time.

(e) The velocity potential ▯ of a flow is given by

$$\phi=\frac{1}{2}\left(x^2+y^2-2z^2\right)$$

Determine the streamlines.

6. (a) Show that the solution of the two-dimensional Laplace's equation

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial y^2}=0, \qquad x\in(-\infty,\infty), y\geq 0$$

Subject to the boundary condition

$$\phi(x,0)=f(x), x\in(-\infty,\infty)$$

Along with $\phi(x,y)\to 0$ for $|x|\to\infty$ and $y|\to\infty$ can be written in the form

$$\phi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{y^2+(x-\xi)^2}.$$

(b) Draw the logical circuit for the Boolean expression

$Y=AB\bar{C}+B\bar{C}+\bar{A}B$. Also obtain the output Y (truth table) for the three input bit sequences:

A = 10001111, B = 00111100, C = 11000100

(c) Find the moment of inertia of a quadrant of a elliptic disk $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of mass M about the line passing through its centre and perpendicular to its plane. Given that the density at any point is proportional to xy.

7. (a) Find the integral surface of the following quasi-linear equation

$$(y-\phi)\frac{\partial\phi}{\partial x}+(\phi-x)\frac{\partial\phi}{\partial y}=x-y.$$

Which passe through the curve $\phi = 0$, $xy = 1$ and through the circle $x + y + \phi = 0, x^2 + y^2 + \phi^2 = a^2$

(b) Integrate $f(x) = 5x^3 - 3x^2 + 2x + 1$ from $x = -2$ to $x = 4$ using

(i) Simpson's $\frac{3}{8}$ rule with width h = 1, and

(ii) Trapezoidal rule with width h = 1.

(c) Let the velocity field

$$u(x,y) = \frac{B\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}, \quad v(x,y) = \frac{2Bxy}{\left(x^2 + y^2\right)^2}, \quad w(x,y) = 0$$

Satisfy the equation of motion for inviscid incompressible flow, where B is a constant. Determine the pressure associated with this velocity field.

8. (a) Solve the partial differential equation

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \phi\right) + 2x^2 y\left(\frac{\partial \phi}{\partial x} + \phi\right) = 0$$

By transforming it to the canonical form.

(b) Using Newton's forward difference formula for interpolation, estimate the value of f(2.5) from the following data:

x:	1	2	3	4	5	6
f(*x*):	0	1	8	27	64	123

(c) Suppose an infinite liquid contain two parallel, equal and opposite rectilinear vortices at a distance 2a. Show that the streamlines relative to the vortex and given by the equation

$$\log \frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} + \frac{y}{a} = C$$

Where C is a constant, the origin is the middle point of the join, and the line joining the vortices is the axis of *y*.

ANALYSIS & ANSWERS

1. (a)

We know that; For two finite subgroups H and K of a group,

define the set $HK = \{hk \mid h \in H,\ k \in K\}$. Then,

$|HK| = |H||K| / |H \cap K|$. i.e. $\boxed{o(HK) = \dfrac{o(H).o(K)}{o(H \cap K)}}$ ...(1)

Now let if G has two distinct subgroups H and K of order m; then,

By (1); $o(HK) = \dfrac{o(H).o(K)}{o(H \cap K)} = \dfrac{m.m}{1} = m^2$;....(2)

But $o(G) = m.n$, so order of subgroup of G must be less than equal to order of G.

So, $o(HK) \le m.n$ ..(3)

i.e. (2) is not possible because $m > n \Rightarrow m^2 > m.n$

1. (b) Let if *w* = *f*(*z*) = *u* + *iv* & $z = x + iy,\ \overline{z} = x - iy$

$\therefore x = \dfrac{z + \overline{z}}{2},\ y = \dfrac{z - \overline{z}}{2i}$

Now using: If : ϕ is function of x and y and x,y are functions of $z \,\&\, \overline{z}$; then

$$\partial\phi = \frac{\partial\phi}{\partial x}.\partial x + \frac{\partial\phi}{\partial y}.\partial y$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial x}.\frac{\partial x}{\partial z} + \frac{\partial\phi}{\partial y}.\frac{\partial y}{\partial z} \Rightarrow \frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial x}.\frac{1}{2} + \frac{\partial\phi}{\partial y}.\frac{1}{2i} \Rightarrow \frac{\partial}{\partial z} = \frac{\partial}{\partial x}.\frac{1}{2} + \frac{\partial}{\partial y}.\frac{1}{2i} \text{ ....(1)}$$

And, $$\frac{\partial\phi}{\partial \overline{z}} = \frac{\partial\phi}{\partial x}.\frac{\partial x}{\partial \overline{z}} + \frac{\partial\phi}{\partial y}.\frac{\partial y}{\partial \overline{z}} \Rightarrow \frac{\partial\phi}{\partial \overline{z}} = \frac{\partial\phi}{\partial x}.\frac{1}{2} - \frac{\partial\phi}{\partial y}.\frac{1}{2i} \Rightarrow \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x}.\frac{1}{2} - \frac{\partial}{\partial y}.\frac{1}{2i} \text{ ......(2)}$$

Using (1) and (2), we get

$$\frac{\partial^2}{\partial z \partial \overline{z}} = \left(\frac{\partial}{\partial x}.\frac{1}{2} + \frac{\partial}{\partial y}.\frac{1}{2i}\right)\left(\frac{\partial}{\partial x}.\frac{1}{2} - \frac{\partial}{\partial y}.\frac{1}{2i}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)$$

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\partial \bar{z}}$$

$$\therefore\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\left|f'(z)\right|=4\frac{\partial^2}{\partial z\partial \bar{z}}\times\frac{1}{2}\log\left|f'(z)\right|^2$$

$$=2\frac{\partial^2}{\partial z\partial \bar{z}}\log\left\{f'(z)\cdot f'(\bar{z})\right\}$$ logic always works for $\because\left|\phi(z)\right|^2=\phi(z)\overline{\phi(z)}\,\phi(z)\phi(\bar{z})$

$$=2\frac{\partial^2}{\partial z\partial \bar{z}}\left[\log f'(z)+\log f'(\bar{z})\right]$$

$$=2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right]=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\left|f'(z)\right|=0$$

1. (c) given, $I=\int_0^2\frac{\log x}{\sqrt{2-x}}dx$

As, the given improper integral is of 2nd kind & gives infinity at x = 0 & at x = 2.

$$\therefore I=\int_0^1\frac{\log x}{\sqrt{2-x}}dx+\int_1^2\frac{\log x}{\sqrt{2-x}}dx\quad ...(1)$$

- checking the convergence of $\int_0^1\frac{\log x}{\sqrt{2-x}}dx$

$$\lim_{x\to0}(x-0)^{\mu}\times\frac{\log x}{\sqrt{2-x}}=\lim_{x\to0}\frac{x^{\mu}\log x}{\sqrt{2-x}}=\frac{1}{\sqrt{2}}\lim_{x\to0}x^{\mu}\log x=0\ \text{ for }\ 0<\mu<1$$

As, μ < 1, so, $\int_0^1\frac{\log x}{\sqrt{2-x}}dx$ is convergent ...(1)

- checking the convergence of $\int_1^2\frac{\log x}{\sqrt{2-x}}dx$

Take $g(x) = \dfrac{1}{\sqrt{2-x}}$

$$\therefore \lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{\frac{\log x}{\sqrt{2-x}}}{\frac{1}{\sqrt{2-x}}} = \lim_{x\to 2} \log x = \log 2$$, which is finite & non-zero

$\therefore$ By comparison test both $\int_1^2 \frac{\log x}{\sqrt{2-x}}$ & $\int_1^2 \frac{1}{\sqrt{2-x}}dx$ converges or diverges together.

As, $\int_1^2 \frac{1}{\sqrt{2-x}}dx$ is convergent $\left[\because \int_a^b \frac{dx}{(x-a)^n} \text{ is convergent for } n<1\right]$

$\therefore \int_1^2 \frac{\log x}{\sqrt{2-x}}dx$ is also convergent....(2)

$\therefore$ From (1) & (2), $\int_0^2 \frac{\log x}{\sqrt{2-x}}dx$ is convergent.

1. **(d)** given, φ(*x*, *y*) & ψ(*x*, *y*) satisfies Laplace equation.

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad .....(1)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad .....(2)$$

Also, given,

$$p = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \quad \& \qquad .....(3)$$

$$q = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y} \qquad .....(4)$$

Now,

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}\right) = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\frac{\partial q}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial q}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2}$$

Now,

$$\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} = 0 \{\text{ using } (2)\}$$

$$\boxed{\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}}$$

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \{\text{using } (1)\}$$

$$\therefore \boxed{\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}}$$

Also, $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial q}{\partial x} \& \frac{\partial q}{\partial y}$ are continuous

So, $f(z) = p + iq$ is an analytic function.

1. **(e)** Max $z = x_1 + 2x_2$

sub to, $x_1 - x_2 \geq 3$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0$

Standard form:

Max $z = x_1 + 2x_2$

sub to, $x_1 - x_2 - s_1 + a_1 = 3$

$2x_1 + x_2 + s_2 = 10$

s.t $x_1, x_2, s_1, s_2, \geq 0$

Here, a_1 is artificial variable.

Phase I

max $z^* = 0x_1 + 0x_2 + 0 \cdot s_1 + 0s_2 - 1 \cdot a_1$

sub to, $x_1 - x_2 - s_1 + a_1 = 3$

$2x_1 + x_2 + s_2 = 10$

s.t $x_1, x_2, s_1, s_2, a_1 \geq 0$

	$C_j \rightarrow$	0	0	0	0	−1	
B, C_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Min $\frac{X_B}{Y_1}$
a_1, −1	3	[1]	−1	−1	0	1	3 →
S_2, 0	10	2	1	0	1	0	5
	$\Delta j \rightarrow$	1	−1	−1	0	0	
		↑					

	$C_j \rightarrow$	0	0	0	0	−1	
B, C_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Min X_B
x_1, 0	3	1	−1	−1	0	1	
S_2, 0	4	0	3	2	1	−2	
	$\Delta j \rightarrow$	0	0	0	0	−1	

∵ all Δj's ≤ 0 & No artificial variable, is present in phase I.

So, it works as initial bfs for phase II.

Phase II

$C_j \rightarrow$	5	8	0	0

B, C_B	X_B	Y_1	Y_2	Y_3	Y_4	Min
x_1, 5	3	1	-1	-1	0	-ve
S_2, 0	4	0	[3]	2	1	$\frac{4}{3}$ →
	Δj →	0	13	5	0	

	C_j →	5	8	0	0	
B, C_B	X_B	Y_1	Y_2	Y_3	Y_4	Min
x_1, 5	$\frac{13}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	
x_2, 8	$\frac{4}{3}$	0	1	$\frac{2}{3}$	$\frac{1}{3}$	
	$\Delta_j \rightarrow$	0	0	-ve	-ve	

$\because$ all Δ_j's ≤ 0

$\therefore$ Current bfs is optimal

$\therefore$ Required optimal sol. is $x_1 = \frac{13}{3}$ & $x_2 = \frac{4}{3}$

Optimal value = $\frac{13}{3} + 2 \times \frac{4}{3} = \frac{21}{3} = 7$

2. (a) Given, $f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots \frac{1}{n!}$

• For $n \geq m$,

$$| f_n - f_m | = \left| 1 + \frac{1}{1!} + \ldots\ldots + \frac{1}{m!} + \frac{1}{(m+1)!} \ldots\ldots + \frac{1}{n!} - \left(1 + \frac{1}{1!} + \ldots\ldots + \frac{1}{m!}\right) \right|$$

$$= \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \ldots. + \frac{1}{n!}$$

$$\leq \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$\leq \frac{\frac{1}{2^m}\left(1 - \left(\frac{1}{2}\right)^{n-m}\right)}{1 - \frac{1}{2}}$$

$\leq \frac{2}{2^m}$

$|f_n - f_m| \leq \frac{1}{2^{m-1}}$

• Now for $\in > 0, |f_n - f_m| < \in$ if $\frac{1}{2^{m-1}} < \in$

$\Rightarrow 2^{m-1} < \frac{1}{\in}$

$\Rightarrow (m-1)\log 2 > \log\left(\frac{1}{\in}\right)$

$\Rightarrow m-1 > \log\left(\frac{1}{\in}\right).(\log 2)^{-1}$

$\Rightarrow m > 1 + \log\left(\frac{1}{\in}\right)(\log 2)^{-1}$ ...(1)

• Therefore, For each $\in > 0$, there exists $m \in N$ s.t

$|f_n - f_m| < \in \forall n \geq m$

So, $< f_n >$ is a convergent sequence by Cauchy's general principle of convergence.

2. **(b)** Homomorphic image of abelian group is abelian i.e.

$f : G \to G'$ is an onto homomorphism, if G is abelian then G' is abelian.

Let $f : G \to G'$ is onto homomorphism and G is abelian then $xy = yx, \forall x, y \in G$

Let $f(x) \in G', f(y) \in G'$

$f(x) \cdot f(y) = f(xy)$ [$\because f$ is homomorphism]

$= f(yx)$ $[xy = yx, \because \text{G is abelian}]$

$= f(y) \cdot f(x)$

$\Rightarrow f(x) \cdot f(y) = f(y) \cdot f(x), \forall f(x), f(y) \in G'$ then G' is abelian. [Proved]

<u>Converse of the above theorem need not be true</u>

$f : S_3 \to Z_2$ with $\ker f = A_3$ then

$\frac{S_3}{A_3} \approx Z_2$

i.e. $f(S_3) \approx Z_2$ with $\ker f = A_3$

Z_2 is abelian but S_3 is non-abelian.

2. **(c)** $\because$ Given that *f*(*z*) at |*z*| = 1 is $\frac{(a^2-1)\cos\theta + 1(a^2+1)\sin\theta}{a^4 - 2a^2\cos\theta + 1}$

$$\therefore \; f = \frac{a^2.e^{i\theta} - e^{-i\theta}}{(a^2 - e^{i2\theta})(a^2 - e^{-i2\theta})}$$

$\because$ *f*(2) is analytic for $|z| \le 1$.

$\therefore$ By Taylor's theorem, we have

$f(z) = \sum_{n=0}^{\infty} a_n (z-0)^n$; where $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-0)^{n+1}} dz$

Now, let's find a_n ;

$$a_n = \frac{1}{2\pi i}\int_c \frac{a^2.e^{i\theta} - e^{-i\theta}}{(e^{i\theta})^{n+1}} . \frac{ie^{i\theta} d\theta}{(a^2 - e^{i2\theta})(a^2 - e^{-i2\theta})}$$

$$= \frac{1}{2\pi i}\int_c \frac{e^{-i(n-1)\theta}}{a^2}\left(1 - \frac{e^{i2\theta}}{a^2}\right)^{-1} d\theta$$

$$= \frac{1}{2\pi a^2}\int_{\theta=0}^{2\pi} \frac{e^{-i(n-1)\theta}}{a^2}\left(1 - \frac{e^{i2\theta}}{a^2}\right)^{-1} d\theta \qquad .....(1)$$

$\because \int_0^{2\pi} e^{i\alpha\theta} d\theta = 0$ for $\alpha \in$ □ , ...(2)

$\therefore \; a_n = \frac{1}{2\pi a^2}\int_0^{2\pi} e^{-i2m\theta} . \frac{e^{i2m\theta}}{a^{2m}} d\theta$; using (2) in (1), putting for *n* odd, *n*-1= 2m

$$= \frac{1}{2\pi a^2} \cdot \frac{2\pi}{a^2 m} = \frac{1}{a^2 a^{n-1}}$$

$\therefore \quad a_n = \dfrac{1}{a^{n+1}}$ where n is dd.

$$\therefore \; f(z) = \sum_{n=0}^{\infty} a_n z^n . = \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} z^n = \frac{1}{a} \sum_{n=1,3,5,7,...} \left(\frac{z}{a}\right)^n$$

$$\Rightarrow f(z) = \frac{1}{a} \frac{z}{a\left(1 - \left(\frac{z}{a}\right)^2\right)}$$

$$\Rightarrow f(z) = \frac{z}{a^2 - z^2}$$

3. (a) $f(z) = \dfrac{1}{z(\sin \pi z)\left(z + \frac{1}{2}\right)}$

- For poles of $f(z)$, lets put denominator as zero.

$$z \sin \pi z \left(z + \tfrac{1}{2}\right) = 0$$

$z \sin \pi z = 0 \Rightarrow \; z = 0, \pm 1 \pm 2, ...$ It must be noticed here that as the denominator is consisting of $z.\sin \pi z$, so $z = 0$ will become a pole of order 2. $\lim_{z \to 0} z^2 f(z) \neq 0$.

$z = \pm 1 \pm 2, ...$ are simple poles

or $z + \dfrac{1}{2} = 0 \Rightarrow \; z = \dfrac{-1}{2}$ is a pole of order 1.

- Now, Residue of $f(z)$ at $z = 0$ is:

$$\lim_{z \to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z \to 0} \frac{d}{dz} \frac{z^2}{z(\sin \pi z)\left(z + \frac{1}{2}\right)} = \lim_{z \to 0} \frac{d}{dz} \frac{z}{(\sin \pi z)\left(z + \frac{1}{2}\right)}$$

$$\lim_{z \to 0} \frac{d}{dz} \frac{z}{(\sin \pi z)\left(z + \frac{1}{2}\right)} = \lim_{z \to 0} \left\{ \frac{(\sin \pi z)\left(z + \frac{1}{2}\right).1 - z\left(\sin \pi z.1 + \pi \cos \pi z\left(z + \frac{1}{2}\right)\right)}{\left(\sin^2 \pi z\right)\left(z + \frac{1}{2}\right)^2} \right\} =$$

$$\lim_{z\to 0}\left\{\frac{(\sin \pi z)\left(z+\frac{1}{2}\right)}{(\sin^2 \pi z)\left(z+\frac{1}{2}\right)^2}-\frac{z\sin \pi z}{(\sin^2 \pi z)\left(z+\frac{1}{2}\right)^2}-\frac{\pi z\cos \pi z\left(z+\frac{1}{2}\right)}{(\sin^2 \pi z)\left(z+\frac{1}{2}\right)^2}\right\}$$

$$\lim_{z\to 0}\left\{\frac{1}{(\sin \pi z)\left(z+\frac{1}{2}\right)}-\frac{z}{(\sin \pi z)\left(z+\frac{1}{2}\right)^2}-\frac{\pi z\cos \pi z}{(\sin^2 \pi z)\left(z+\frac{1}{2}\right)}\right\}=\lim_{z\to 0}\left\{\frac{1}{2(\sin \pi z)\left(z+\frac{1}{2}\right)^2}-\frac{\pi z\cos \pi z}{(\sin^2 \pi z)\left(z+\frac{1}{2}\right)}\right\}$$

$$=\lim_{z\to 0}\left\{\frac{(\sin \pi z)\left(z+\frac{1}{2}\right)-\pi z\cos \pi z}{2(\sin^2 \pi z)\left(z+\frac{1}{2}\right)^2}\right\}$$

$$\lim_{z\to 0}\left\{\frac{\left\{\pi z-\frac{\pi^3 z^3}{3!}+\frac{\pi^5 z^5}{5!}-\ldots\right\}-\pi z\left\{1-\frac{\pi^2 z^2}{2!}+\frac{\pi^4 z^4}{4!}\right\}}{2\left\{\pi z-\frac{\pi^3 z^3}{3!}+\frac{\pi^5 z^5}{5!}-\ldots\right\}^2\left(z+\frac{1}{2}\right)^2}\right\}=\frac{-4}{\pi}$$

- For residue at $z=-1/2$; $\lim_{z\to\frac{-1}{2}}\frac{1}{z\sin \pi z}=-\frac{1}{1/2(-1)}=2$

- For residue at $z=n\pi$; $\lim_{z\to n\pi}\frac{z-n\pi}{z\sin \pi z(z+1/2)}=(-1)^n\frac{1}{n\pi(n+\frac{1}{2})}$

3. (b) Consider the series $\sum_{n=1}^{\infty}U_n(x), 0\le x\le 1,$ $S_n(x)=\frac{1}{2n^2}\log\left(1+n^4x^2\right), x\in[0,1].$

- For uniform convergence:

$$S'_n(x)=\frac{1}{2n^2}\cdot\frac{1}{(1+n^4x^2)}.2xn^4=\frac{n^2x}{(1+n^4x^2)}$$

$$Max.S'_n(x);\quad at\ x=\frac{1}{n^2}.\quad Also,\ \lim_{n\to\infty}S'_n(x)=0$$

$$M_n=Sup\left|S'_n(x)-0\right|=\frac{n^2.\frac{1}{n^2}}{1+\frac{n^4}{n^4}}=\frac{1}{2}.\qquad \lim_{n\to\infty}M_n=1/2\neq 0.$$

$By\ M_n - Test;\quad \sum_{n=1}^{\infty} U'_n(x)$ does not converge uniformly in [0,1].

• To show that the given series can be differentiated term-by-term, though $\sum_{n=1}^{\infty} U'_n(x)$, does not converge uniformly on [0, 1]:

$\because S'_n(x) = \frac{1}{2n^2}.\frac{1}{(1+n^4x^2)}.2xn^4 = \frac{n^2x}{(1+n^4x^2)}$ and $Sum\ function\ S'(x) = 0$

$\because \frac{d}{dx}\left\{\lim_{n\to\infty} S'_n(x)\right\} = \frac{d}{dx}\left\{\lim_{n\to\infty} S(x)\right\} = 0$ shows that yes term by term differentiation is possible.

3. (c) Dual is.

max $Z_D = 12w_1 + 8w_2$

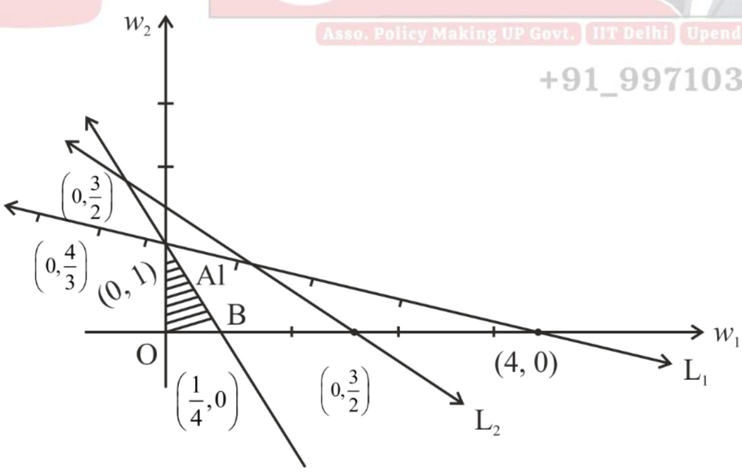
Sub to,

$w_1 + 3w_2 \le 4$ ....(1)

$2w_1 + 2w_2 \le 3$ ....(2)

$4w_1 + w_2 \le 1$ ....(3)

$w_1, w_2 \ge 0$



So, here, the optimal region is OAB,

$\therefore\ (Z_D)_O = 0,\quad (Z_D)_A = 8\quad (Z_D)_B = 12\times\frac{1}{4} = 3$

$\therefore$ max $Z_D = 8$ at A(0, 1)

$\therefore$ min z = 8

4. (a)

Let Z [x] be the ring of polynomials over the ring of integers Z.

Let S be the principal ideal of Z [x] generated by x i.e., let S = (x).

- We shall show that (x) is prime but not maximal.

We have S = (x) = {x f(x): f(x) $\in$ Z [x]}.

First we shall prove that S is prime.

Let a(x), b(x) $\in Z$ [x] be such that a(x) b(x) $\in$ S. Then there exists a polynomial c (x) $\in$ Z [x] such that

xc (x) = a (x) b (x) ...(1)

Let a (x) $= a_0 + a_1 x + a_1 x^2 + ..., b(x) = b_0 + b_1 x + b_1 x^2 + ...$

$c(x) = (c_0 + c_1 x + ...) = (a_0 + a_1 x + ...)(b_0 + b_1 x + ...)$ Then (1) becomes

$x(c_0 + c_1 x + ...) = (a_0 + a_1 x + ...)(b_0 + b_1 x + ...)$

Equating the constant term on both sides, we get

$$a_0 b_0 = 0$$

$\Rightarrow a_0 - 0 \text{ or } b_0 = 0$ [$\because$ Z is without zero divisors]

Now $a_0 = 0 \Rightarrow a(x) = a_1 x + a_2 x^2 + ...$

$\Rightarrow a(x) = (x)(a_1 + a_2 x + ...) \Rightarrow a(x) \in (x)$

Similarly $b_0 = 0 \Rightarrow b(x) = b_1 x + b_2 x^2 + ...$

$\Rightarrow b(x) = (x)(b_1 + b_2 x + ...) \Rightarrow b(x) \in (x)$

Thus a (x) b (x) $\in$(x) $\Rightarrow$either $a(x) \in (x)$ or $b(x) \in (x)$

Hence (x) is a prime ideal.

Now we shall show that (x) is not a maximal ideal of Z [x].

For this we must show an ideal N of Z [x] such that (x) is properly contained in N, while N itself is properly contained in Z [x].

The ideal N = (x, 2) serves this purpose.

Obviously (x)$\subseteq$(x, 2).

In order to show that (x) is properly contained in (x, 2) we must show an element of (x, 2) which is not in (x).

Clearly 2 $\in$ (x, 2). We shall show that 2 $\notin$ (x).

Let $2 \in (x)$. Then we can write,

$2 = xf(x)$ for some $f(x)$ in Z [x].

Let $f(x) = a_0 + a_1 x + ...$

Then $2 = xf(x) \Rightarrow 2 = x(a_0 + a_1 x + ...)$

$$\Rightarrow 2 = a_0 x + a_1 x^2 + ...$$

$$\Rightarrow 2 = 0 + a_0 x + a_1 x^2 + ...$$

$\Rightarrow 2 = 0$ [by equality of two polynomials]

But 2 ≠ 0 in the ring of integers. Hence 2 $\notin$ (x). Thus (x) is properly contained in (x,2).

Now obviously (x,2) $\subseteq Z$ [x].

In order to show that (x,2) is properly contained in Z [x], we must show

an element of Z [x], which is not in (x, 2).

Clearly 1 $\in Z$ [x].

We shall show that $1 \notin (x, 2)$. Let $1 \in I[x]$ Then we have a relation of the form $1 = xf(x) + 2g(x)$ where $f(x), g(x) \in Z$ [x].

Let $f(x) = a_0 + a_1 x + ..., g(x) = b_0 + B_1 x +$

Then $1 = xf(a_0 + a_1 x + ...) + 2(b_0 + B_1 x + ...)$

$\Rightarrow 1 = 2b_0$ [Equating constant term on both sides]

But there is no integer b_0 such that 1 = $2b_0$

Hence $1 \notin (x, 2)$ Thus $(x, 2)$ is properly contained in Z [x].

Therefore (x) is not a maximal ideal of Z [x].

4. (c)

Upper sum *(U (P,f)) is defined as*

$$f(x_1)\Delta x_1 + f(x_1)\Delta x_2 + ... + f(x_n)\Delta x_n$$

; Where $f(x_i)$ is the maximum value of function in particular i-th subinterval

Lower sum *(L (P,f)) is defined as*

$$f(x_1)\Delta x_1 + f(x_1)\Delta x_2 + \ldots\ldots + f(x_n)\Delta x_n$$

;Where $f(x_i)$ is the minimum value of function in particular i-th subinterval

Let's take a partition of [0,1] by dividing into n sub intervals

i.e., $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$$x_0 = 0,\ x_1 = 0 + \frac{1}{n},\ x_2 = 0 + \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$$

Partition P is :

$$\begin{array}{cccc} [x_0, x_1]\cup & [x_1, x_2]\cup & [x_2, x_3]\cup\ldots\ldots\cup & [x_{n-1}, x_n] \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \text{length of subinterval} & \Delta x_2 = \frac{1}{n} & \Delta x_3 = \frac{1}{n} & \Delta x_n = \frac{1}{n} \end{array}$$

- $\Delta x_i = \frac{1}{n}$
- $f(x) = \begin{cases} \sqrt{1-x^2}\,; x \in \mathbf{Q} \\ 1-x \quad ; x \in \mathbf{Q}^c \end{cases}$

For max value of function in i-th interval, $M_i = 1 - x = 1 - \frac{i}{n}$

$$U(p.f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (1-\frac{i}{n}).\frac{1}{n} = \frac{1}{n}\left(1-\left(\frac{1}{n}+\frac{2}{n}+\ldots.+\frac{n-1}{n}+\frac{n}{n}\right)\right) = \frac{1}{n}.\frac{1}{n}(1+2+3+\ldots\ldots+(n-1)) = \frac{1}{2}\left(1-\frac{1}{n}\right)$$

So, $U(P,f) = \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n}\right) = \frac{1}{2}.$

For min value of function in i-th interval, $m_i = \sqrt{1-x^2} = \sqrt{1-\left(\frac{i}{n}\right)^2}$

$$L(p.f) == \sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}.\frac{1}{n} = \frac{1}{n}\left(\sqrt{1--\left(\frac{1}{n^2}+\frac{2^2}{n^2}+\ldots.+\frac{(n-1)^2}{n}+\frac{n^2}{n^2}\right)}\right);$$ So, $L(P,f) = \pi/4$ for $n \to \infty$.

$\because U(P,f) \neq L(P,f)$; So, given function is not Riemann Integrable.

4. (c) As. The given assignment produce is not balanced so, adding a dummy row, we get the problem as.

	Delhi	Mumbai	Kolkata	Chanai
A	16	22	24	20
B	10	32	26	16
C	10	20	46	30
D	0	0	0	0

A	[0]	6	8	4	✓ (4)
B	⊗	22	16	6	✓ (1)
C	⊗	10	36	20	✓ (2)
D	⊗	[0]	⊗	⊗	
	✓ (3)				

A	⊗	2	4	[0]	
B	[0]	18	12	2	✓ (3)
C	⊗	6	32	16	✓ (1)
D	4	[0]	⊗	⊗	
	✓ (2)				

A	2	2	4	[0]	✓ (4)
B	[0]	16	10	⊗	✓ (3)
C	⊗	4	30	14	✓ (1)
D	6	[0]	⊗	⊗	
	✓ (2)			✓ (5)	

	Delhi	Mumbai	Kolkata	Chanai
A	2	[0]	2	⊗
B	⊗	14	8	[0]
C	[0]	2	28	14

D	8	$\otimes$	$\boxed{0}$	2

So, the optimal assignment is

A $\rightarrow$ Mumbai

B $\rightarrow$ Chaenai

C $\rightarrow$ Delhi

D $\rightarrow$ Kolkata

The minimum cost = 22 + 16 + 10 + 0 = 48

5. (a) Given, $u = f(x-kt+i\alpha y)+g(x-kt-i\alpha y)$

$$\therefore \frac{\partial u}{\partial x} = f'(x-kt+i\alpha y)+g'(x-kt-i\alpha y)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-kt+i\alpha y)+g'(x-kt-i\alpha y)$$

And $\dfrac{\partial u}{\partial y} = f'(x-kt+i\alpha y)\times i\alpha - g'(x-kt-i\alpha y)\times i\alpha$

$$\frac{\partial^2 u}{\partial y^2} = f''(x-kt+i\alpha)\times i^2\alpha^2 + g''(x-kt-i\alpha y)\times i^2\alpha^2$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2\left[f''(x-kt+i\alpha)+g''(x-kt-i\alpha y)\right]$$

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \{f''(x-kt+i\alpha)+g''(x-kt-i\alpha)\}(1-\alpha^2) \qquad .....(1)$$

And, $\dfrac{\partial u}{\partial t} = -k\left[f'(x-kt+i\alpha y)+g'(x-kt-i\alpha y)\right]$

$$\frac{\partial^2 u}{\partial t^2} = k^2\left[f''(x-kt+i\alpha y)+g''(x-kt-i\alpha y)\right] \qquad ....(2)$$

$\therefore$ Using (2) in (1)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = (1-\alpha^2)\times\frac{1}{k^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2}\left\{1-\left(1-\frac{k^2}{C^2}\right)\right\}\frac{\partial^2 u}{\partial t^2} \qquad \left[\because \text{ Given } \alpha^2 = 1-\frac{k^2}{C^2}\right]$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{C^2}\frac{\partial^2 v}{\partial t^2}$$

5. **(b)** The given system of linear equation in motion notation form is written as AX = B, where

$$A=\begin{bmatrix}2 & 3 & -1\\ 4 & 4 & -3\\ 2 & -3 & 2\end{bmatrix},\qquad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\qquad B=\begin{bmatrix}5\\ 3\\ 2\end{bmatrix}$$

$$[\mathrm{A:B}]=\left[\begin{array}{ccc|c}2 & 3 & -1 & 5\\ 4 & 4 & -3 & 3\\ 2 & -3 & 2 & 2\end{array}\right]$$

$$\mathrm{R_1 \leftrightarrow R_2}$$

$$[\mathrm{A:B}]=\left[\begin{array}{ccc|c}4 & 4 & -3 & 3\\ 2 & 3 & -1 & 5\\ 2 & -3 & 2 & 2\end{array}\right]$$

$$\mathrm{R_1 \to \frac{1}{4}R_1}$$

$$[\mathrm{A:B}]\sim\left[\begin{array}{ccc|c}1 & 1 & \frac{-3}{4} & \frac{3}{4}\\ 2 & 3 & -1 & 5\\ 2 & -3 & 2 & 2\end{array}\right]$$

$$\mathrm{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 2R_1}$$

$$[\mathrm{A:B}]\sim\left[\begin{array}{ccc|c}1 & 1 & \frac{-3}{4} & \frac{3}{4}\\ 0 & 1 & \frac{1}{2} & \frac{7}{2}\\ 0 & -5 & \frac{7}{2} & \frac{1}{2}\end{array}\right]$$

$$\mathrm{R_3 \to R_3 + 5R_2,\ R_1 \to R_1 - R_2}$$

$$[\mathrm{A:B}]\sim\left[\begin{array}{ccc|c}1 & 0 & \frac{-5}{4} & \frac{-11}{4}\\ 0 & 1 & \frac{1}{2} & \frac{7}{2}\\ 0 & 0 & 6 & 8\end{array}\right]$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & \frac{-5}{4} & \frac{-11}{4} \\ 0 & 1 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_2 \to R_2 - \frac{1}{2}R_3, \qquad R_1 \to R_1 + \frac{5}{4}R_3$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

∴ From AX = B, we get

$x = 1$

$y = 2$

$z = 3$

5. (c)

(i) As we know that in sign magnitude form, the first bit of binary representation represents the sign. If it is 1 then negative sign and 0 then positive sign. Rest of the bits represent the magnitude of that number. So,

$(8D)_{16} = (1000\,1101)_2 = (-13)_{10}$ [∵ D = 13 in decimal]

$$\underset{-ve}{1}\ \underbrace{000\,1101}_{magnitude} = (-13)_{10}$$

$$(FF)_{16} = (11111111)_2 = \underset{-ve}{1}\ \underbrace{1111111}_{magnitude} = (-127)_{10}$$

(ii) $(9B2.\ 1A)_{16}$ = $9\times16^2 + B\times16 + 2\times16^\circ + 1\times\frac{1}{16} + A\times\frac{1}{16^2}$

$= 9\times256 + 11\times16 + 2 + 0.0625 + 10\times0.00390625$

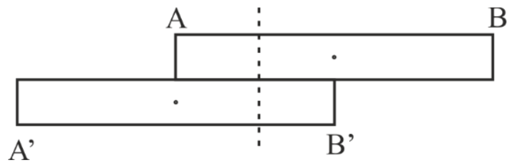
= 2304 + 176 + 2 + 0.1015625

= $(2482.1015625)_{10}$

5. (d) Here, there is us internal forces in horizontal direction only the weight of board & man are acting downward & the reaction of the horizontal plane acting upward.

So, By D'Alembert's Principle the C.G. of the system does not move, as the man moves from A to B.

In the initial position,



Distance of CG from A = $\dfrac{M\times 0+m\times AC}{M+m}=\dfrac{ma}{M+m}$

In final position, when the man reaches the point B,

Distance of CC from $A=\dfrac{M\times AB'+mAC'}{M+m}$

= $\dfrac{M(2a-x)+m(a-x)}{M+m}$

Now, as the position of C of the system remains unchanged

$\therefore\ \dfrac{ma}{M+m}=\dfrac{M(2a-x)+m(a-x)}{M+m}$

$\Rightarrow ma = 2Ma - Mx + ma - ma$

$\Rightarrow x(M+m)=2Ma$

$\Rightarrow x=\dfrac{2Ma}{M+m}$

5. **(e)** Given $\phi=\dfrac{1}{2}\left(x^2+y^2-2z^2\right)$

We know the velocity 'q' can be written as,

$$q=-\nabla\phi$$

$$\Rightarrow u\hat{i}+v\hat{j}+w\hat{k}=-\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left\{\frac{1}{2}\left(x^2+y^2-2z^2\right)\right\}$$

$$\Rightarrow u\hat{i}+v\hat{j}+w\hat{k}=-\left[x\hat{i}+y\hat{j}-2z\hat{k}\right]$$

On comparing we get,

$u=-x$, $v=-y$, $w=2z$

$\therefore$ The striations are given by,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

$$\frac{dx}{-x}=\frac{dy}{-y}=\frac{dz}{2z} \qquad .....(1)$$

Taking 1st two fraction, of (1)

$$\frac{dx}{x} = \frac{dy}{y}$$

On integrating,

$\log x = \log y + \log c,$ where C_1 is arbitrary constant

$x = c_1 y$ .....(A)

Taking last two fraction, of (1)

$$\frac{dy}{-y} = \frac{dz}{2z}$$

$$\frac{dy}{y} + \frac{dz}{2z} = 0$$

On integrating

$\log y + \frac{1}{2}\log z = \log c_2,$ where c_2 is arbitrary constant

$y\sqrt{z} = c_2$ .....(B)

Equations (A) & (B) gives the equation of streamlines or streamlines are given by

$$\phi\left(\frac{x}{y}, y\sqrt{z}\right) = 0$$

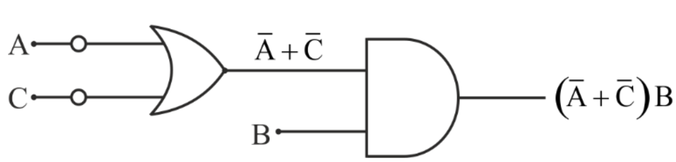
6. (b) $y = AB\bar{C} + B\bar{C} + \bar{A}B$

Step (I): Simplifying the Boolean expression

$$y = AB\bar{C} + B\bar{C} + \bar{A}B$$

$$= (A + 1)B\bar{C} + \bar{A}B$$

Step (II): $y = B(\bar{C} + \bar{A})$ .....(1)



Part (II): $\because$ Given

A = 1001111, B = 00111100, C = 11000100

$$\therefore \quad y = B(\bar{C} + \bar{A}) = (00111100)(01110000 + 00111011)$$

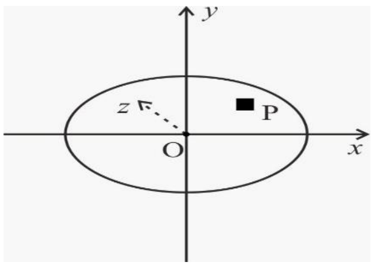
= (00111100) (1001001010000011)

= 0100100000011100011101

Truth Table for the given input is.

A	B	C	$AB\bar{C}$	$B\bar{C}$	$\bar{A}B$	$y = AB\bar{C} + B\bar{C} + \bar{A}B$
1	0	1	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	1	1	1
0	1	0	0	1	1	1
1	1	0	1	1	0	1
1	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0

6. (c)



Let P(*x*, *y*) be the small element with mass dm.

O; the centre of the elliptic disk

Let the *z*-axis be the line passing through the centre of elliptic disk and perpendicular to plane of it

∴ Moment of inertia about OZ, of small element dm at P.

$$dI = \left(x^2 + y^2\right) dm$$

$\therefore$ Total moment of inertia of disk about $OZ = \int dI = I$

$$\therefore \; I = \iint dm\left(x^2 + y^2\right)$$

$$= \iint \rho dxdy\left(x^2 + y^2\right) \qquad .....(1)$$

$\because$ For rectangular element dm = $\rho dxdy$

For limits of integration for *x* & *y* in (1)

We have $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$

$$I = \iint \alpha xy\left(x^2 + y^2\right)$$

α is proportional constant as ρ is given proportional to *xy*.

$$I = \alpha \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} r^2 ab \sin\theta\cos\theta.\left(a^2\cos^2\theta + b^2\sin^2\theta\right) r^2 abrd\theta dr$$

$$I = \alpha a^2 b^2 \int_{r=0}^{1} r^5 dr\left(\frac{a^2}{4} + \frac{b^2}{4}\right) = \frac{a}{6} a^2 b^2 \frac{\left(a^2 + b^2\right)}{4}$$

$$I = \frac{\alpha}{24} a^2 b^2 \left(a^2 + b^2\right) \qquad .....(2)$$

$\because$ Total mass $M = \iint \rho dxdy = \iint \rho dxdy$

$$\therefore \; M = a\int_{r=0}^{1}\int_{\theta=0}^{\pi/2} a^2 b^2 r^2 \sin\theta\cos\theta rd\theta dr = \frac{\alpha}{8} a^2 b^2$$

$$\Rightarrow \; \alpha = \frac{8M}{a^2 b^2} \qquad ......(3)$$

So, on using (3) in (2), we get required moment of inertia = $\frac{8M}{a^2b^2} \cdot \frac{a^2b^2}{24}\left(a^2 + b^2\right) = \frac{1}{3} M\left(a^2 + b^2\right)$

7. (a) On comparing given differential equation with P*p* + Q.*q* = R

We get $P = y - \phi$, $Q = \phi - x$, $R = x - y$

$\therefore$ Lagrange's auxiliary equations are

$$\frac{dx}{y-\phi} = \frac{dy}{\phi - x} = \frac{d\phi}{x-y} \quad .....(1)$$

By taking first two fractions of (1), adding then taking with the third fraction, we get,

$$\frac{dx+dy}{y-x} = \frac{d\phi}{x-y} \Rightarrow x + y + \phi = C_1$$

$$\therefore\ u(x, y, \phi) = x + y + \phi \quad .....(2)$$

Choosing x, y, ϕ multipliers and then taking the third fraction with it in (1), we get

$$\frac{xdx + ydy + \phi d\phi}{\phi(x-y)} = \frac{d\phi}{x-y} \Rightarrow x^2 + y^2 + \phi^2 = C_2$$

$$\therefore\ v(x, y, \phi) = x^2 + y^2 + \phi \quad .....(3)$$

Now using the given conditions in (2) and (3), we get

$$c_1 = 0, c_2 = a^2$$

Parameterizing $\phi = 0,\ xy = 1$ we get

$x = t,\ y = \frac{1}{t}, \phi = 0$ and using in given condition, and then eliminating t, we get $c_1^2 - 2 = c_2$ .....(4)

$\therefore$ Required integral surface is $(x + y + \phi)^2 - 2 = x^2 + y^2 + \phi^2$

7. (b) Given $f(x) = 5x^3 - 3x^2 + 2x + 1$

Given *a* = –2 & *b* = 4. & *h* = 1

x	$f(x) = y$
$x_0 = -2$	$y_0 = -55$

$$x_1 = a + h = -1 \qquad y_1 = -9$$

$$x_2 = 0 \qquad y_2 = 1$$

$$x_3 = 1 \qquad y_3 = 5$$

$$x_4 = 2 \qquad y_4 = 33$$

$$x_5 = 3 \qquad y_5 = 115$$

$$x_6 = 4 \qquad y_6 = 281$$

Now, By sump son's $\frac{3^{\text{th}}}{8}$ rule,

$$I = \int_{-2}^{4} f(x) = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2 \times y_3\right]$$

$$= \frac{3}{8}\left[(-55 + 281) + 3(-9 + 1 + 33 + 115) + 2 \times 5\right]$$

$$= \frac{3}{8} \times [226 + 420 + 10]$$

I = 246

(ii) For trapezoidal rule,

Given h = 1

So, the values of $x_0, x_1, \ldots\ldots\ldots, x_6$ & $y_0, y_1, \ldots\ldots\ldots, y_6$ will be same.

The formula for trapezoidal rule is given by,

$$\text{I} = \int_{-2}^{4} f(x) = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(-55 + 281) + 2(-9 + 1 + 5 + 33 + 115)\right]$$

$$= \frac{1}{2}[226 + 290]$$

= 258

7. **(c)** Given the velocity field

$$u(x, y) = \frac{B\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}, v(x, y) = \frac{2Bxy}{\left(x^2 + y^2\right)^2}, w(x, y) = 0$$

Satisfies the equation of motion for inside incompressible flow

The equation of motion are,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial x} \qquad .....(1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial x} \qquad .....(2)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial z} \qquad .....(3)$$

Now,

$$\frac{\partial u}{\partial x}=\frac{\left(x^2+y^2\right)^2\times 2Bx-B\left(x^2-y^2\right)2\left(x^2+y^2\right)\times 2x}{\left(x^2+y^2\right)^4}=\frac{2Bx\left(3y^2-x^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial u}{\partial y}=\frac{\left(x^2+y^2\right)\times(-2By)-B\left(x^2-y^2\right)\times 2\left(x^2+y^2\right)\times 2y}{\left(x^2+y^2\right)^4}=\frac{-2By\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial u}{\partial z}=0$$

$$\frac{\partial v}{\partial x}=\frac{\left(x^2+y^2\right)^2\times 2By-2Bxy\times 2\left(x^2+y^2\right)\times 2x}{\left(x^2+y^2\right)^4}=\frac{2By\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial v}{\partial y}=\frac{\left(x^2+y^2\right)\times 2Bx-2Bxy\times 2\left(x^2+y^2\right)\times 2y}{\left(x^2+y^2\right)^4}=\frac{2Bx\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial v}{\partial z}=0$$

$$\frac{\partial w}{\partial x}=0,\ \frac{\partial w}{\partial y}=0\ \&\ \frac{\partial w}{\partial z}=0$$

$\therefore$ From (1),

$$\frac{B\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\times\frac{2Bx\left(3y^2-3x^2\right)}{\left(x^2+y^2\right)^3}-\frac{2Bxy}{\left(x^2+y^2\right)^2}\times\frac{2By\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3}=\frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{2B^2x}{\left(x^2+y^2\right)^5}\left[\left(x^2-y^2\right)\left(3y^2\right)-2y^2\left(3x^2-y^2\right)\right]=\frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{2B^2x}{\left(x^2+y^2\right)^5}\left(-x^4-2x^2y^2-y^4\right)=\frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{-2B^2x}{\left(x^2+y^2\right)^5}\times\left(x^2+y^2\right)=\frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial x}=\frac{2B^2x\rho}{\left(x^2+y^2\right)^3}$$

From (2),

$$\frac{B\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\cdot\frac{2By\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}+\frac{2Bxy}{\left(x^2+y^2\right)^2}\frac{2Bx\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}=\frac{-1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{2B^2y}{\left(x^2+y^2\right)^5}\left[\left(x^2-y^2\right)\left(y^2-3x^2\right)+2x^2\left(x^2-3y^2\right)\right]=\frac{-1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{2B^2y}{\left(x^2+y^2\right)^5}\left(-x^4-2x^2y^2-y^4\right)=\frac{-1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial y}=\frac{2B^2y\rho}{\left(x^2+y^2\right)^3}$$

From (3),

$$0=\frac{\partial p}{\partial z}$$

So, the pressure is only dependent on *x* & *y*.

$$\therefore\ dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy$$

$$dp=\frac{2B^2x\rho}{\left(x^2+y^2\right)^3}dx+\frac{2B^2y\rho}{\left(x^2+y^2\right)^3}dy$$

$$dp=\frac{B^2\rho}{\left(x^2+y^2\right)^3}\left(2dx+2dy\right)$$

$$dp=\frac{B^2\rho}{\left(x^2+y^2\right)^3}d\left(x^2+y^2\right)$$

On integrating we get

$$P = \frac{-B^2\rho}{2\left(x^2+y^2\right)^2} + c$$, when c is integrate constant,

8 (a)

Given PDE is

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}+\phi\right)+2x^2y\left(\frac{\partial\phi}{\partial x}+\phi\right)=0$$

$$\Rightarrow \frac{\partial 2\phi}{\partial y\partial x}+\frac{\partial\phi}{\partial y}+2x^2y\frac{\partial\phi}{\partial x}+2x^2y\phi=0 \qquad .....(1)$$

▯ - quadratic equation of (1) is $R\lambda^2+S\lambda+T=0$,

Where R = 0, S = 1, T = 0

$\therefore$ We have $o.\lambda^2+1.\lambda+0=0\Rightarrow\lambda=0$

$\therefore$ Characteristic curve: $\frac{dy}{dx}+0=0\Rightarrow y=C_1$

$$\therefore\ u(x,y)=y \qquad .....(2)$$

Choosing $v(x,y)=x$

Verifying Jacobians of u & v.

$$J(u,v)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}\neq 0$$

$\therefore$ u & v are independent.

Step (II): Now finding p, q, r, s, t by using (2) and (3).

$$p=\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial u}{\partial x}=\frac{\partial\phi}{\partial v}\Rightarrow\frac{\partial}{\partial x}=\frac{\partial}{\partial v}$$

$$p=\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial u}{\partial y}=\frac{\partial\phi}{\partial u}.1+\frac{\partial\phi}{\partial v}.0=\frac{\partial\phi}{\partial u}\Rightarrow\frac{\partial}{\partial}=\frac{\partial}{\partial u}$$

$$s=\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\phi}{\partial u\partial v}$$

Now using p,q,s in (1), we get required canonical form of (1) as

$$\frac{\partial^2\phi}{\partial u\partial v}+\frac{\partial\phi}{\partial u}+2v^2u\frac{\partial\phi}{\partial v}+2v^2u=0 \qquad .....(4)$$

Step (iii)

To solve (4):

$$\Rightarrow \frac{\partial^2\phi}{\partial u\partial v}+\frac{\partial\phi}{\partial v}+2v^2u\left(\frac{\partial\phi}{\partial v}+\phi\right)=0 \qquad .....(4)$$

Now, Let's take $\frac{\partial^2\phi}{\partial v}+\phi=z$

$$\Rightarrow \frac{\partial^2\phi}{\partial u\partial v}+\frac{\partial\phi}{\partial u}=\frac{\partial z}{\partial u} \qquad .....(5)$$

On using (5) in (4), we get

$$\frac{\partial z}{\partial u}+2v^2uz=0$$

$$\Rightarrow \frac{1}{z}\frac{\partial z}{\partial u}=-2v^2u$$

On integrating w.r.t *u*, we get

$\log z+v^2.u^2=\alpha(v);\ \alpha(v)$ is integration constant

$$\Rightarrow z=e^{\alpha(v)-v^2u^2}$$

$$\Rightarrow \frac{\partial\phi}{\partial v}+\phi=e^{\alpha(v)-v^2u^2}$$

$$\Rightarrow \frac{\partial}{\partial v}\left(\phi e^v\right)=e^{\alpha(v)-v^2u^2}$$

on integrating w.r.t v

$\phi e^{v} = \int e^{\alpha(v)-v^2u^2} dv + \beta(u);\ \beta(u)$ is integrating

8. (b) Constructing the different table for the given problem.

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1	0	1			
2	1	7	6	6	0
3	8	19	12	6	0
4	27	37	18	6	0
5	64	61	24		
6	125				

Let $x = x_0 + uh$

$$\Rightarrow u = \frac{x - x_0}{h}$$

$$\Rightarrow u = \frac{x-1}{1} = (x-1)$$

By, Newtons formed interpolation formula, we have

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)$$

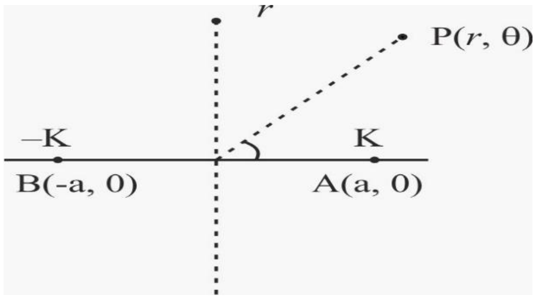
$$f(x) = 0 + (x-1)1 + \frac{(x-1)(x-2)}{2}\times 6 + \frac{(x-1)(x-2)(x-3)}{3!}\times 6$$

$$f(2.5) = 1.5 + \frac{1.5\times(0.5)}{2}\times 6 + \frac{1.5\times 0.5(-0.5)}{6}\times 6$$

$$f(2.5) = 1.5 + 2.25 - 0.375$$

$$f(2.5) = 3.375$$

8. (c)



Due to vortices at A and B; the complex potential at P(r, θ) is

$$W = \frac{ikz}{4\pi a} + \frac{ikz}{2\pi}\log(z-a) - \frac{ik}{2\pi}\log(z+a) \quad .....(1)$$

$$\phi + i\psi = \frac{ikz}{4\pi a} + \frac{ik}{2\pi}\log(z-a) - \frac{ik}{2\pi}\log(z+a)$$

$\therefore$ Streamlines are given by ψ = constant.

$$\Rightarrow \frac{ky}{4\pi a} + \frac{k}{2\pi}\left[\log|z-a| - \log|z+a|\right] = \text{constant}$$

$$\Rightarrow \frac{y}{a} + \left[\log|x+iy-a| - \log|x+iy+a|\right] = \text{constant}$$

$$\Rightarrow \log\frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a} = \text{constant}$$

As it's a long answer, so we must justify the formula:

- Let $P(r, \theta)$ be an arbitrary point.

$$\rightarrow \frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial r}$$

Also, $\because$ Outside the vortex $\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

$\because \nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$

Note:- There is symmetry about origin, ψ must be independent of θ

We have

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + 0 = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 0$$

On integrating ; $r\dfrac{\partial\psi}{\partial r}$ = c; c is integration constant ...(6)

ψ = c log r ...(7)

$$\frac{\partial\psi}{\partial r}=\frac{c}{r}\Rightarrow\frac{-1}{r}\frac{\partial\phi}{\partial\theta}=\frac{c}{r}$$

On integrating; $\boxed{\phi=-c\theta}$ ...(8)

Now, summarizing above discussion; we have

The complex potential $w = \varphi + i\,\psi = -c\theta + ic\log r$ ...(9)

Let K be the 'circulation' in the circuit embracing the vortex (strength of vortex)

Remember

$$\boxed{K=\int_{\theta=0}^{2\pi}\left(\frac{-1}{r}\frac{\partial\phi}{\partial\theta}\right)rd\theta} = c\int_{\theta=0}^{2\pi}d\theta = 2\pi c \Rightarrow c = \frac{K}{2\pi} \quad ...(10)$$

Using (1) in (9), we have

$\therefore\ \varphi = -\dfrac{K}{2\pi}\theta,\ \psi = \dfrac{K}{2\pi}\log r$

$\boxed{W=i\dfrac{K}{2\pi}\log z}$ As; $z = re^{i\theta}$; $\log z = \log(re^{i\theta})$ $=\log r + i\,\theta \therefore i\log z = i(\log r + i\theta)$

Note:-

i. If vortex is not at origin but at some point $z = z_0$; then $\boxed{w=\dfrac{iK}{2\pi}\log(z-z_0)}$

ii. If there are several rectilinear vortices, then

$$\boxed{w=\frac{iK_1}{2\pi}\log(z-z_1)+\frac{iK_2}{2\pi}\log(z-z_2)+\ldots+\frac{iK_n}{2\pi}\log(z-z_n)}$$

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