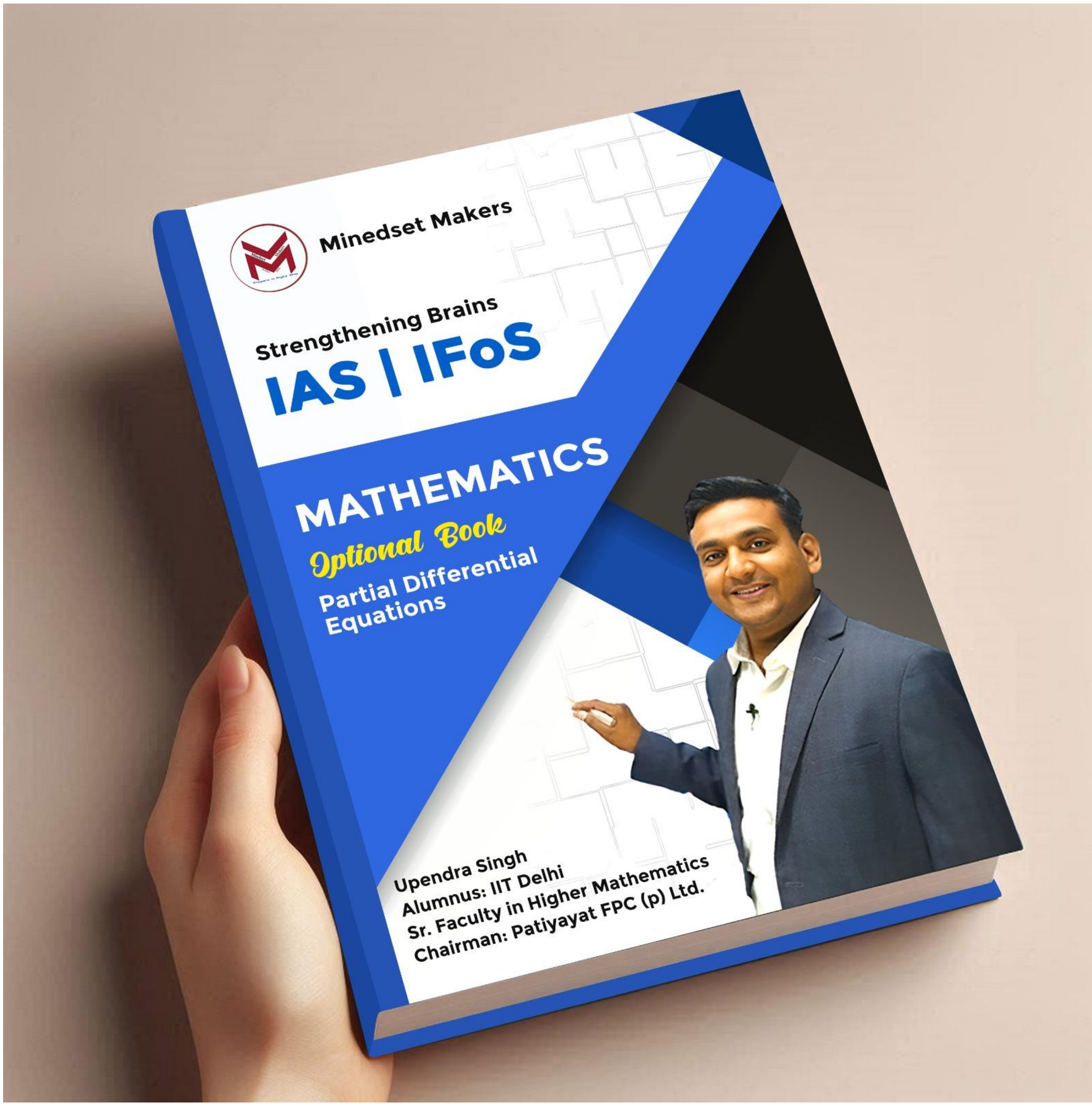
Minedset Makers
Strengthening Brains
IAS | IFoS
MATHEMATICS
Optional Book
Partial Differential Equations
Upendra Singh
Alumnus: IIT Delhi
Sr. Faculty in Higher Mathematics
Chairman: Patiyayat FPC (p) Ltd.

WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFOS :

01 Conceptual Development

02 Problem Solving Techniques

03 Assignments

04 Chapter wise PYQs Analysis

05 Test

MINDSET MAKERS

Upendra Singh

WhatsApp contact

This book conations following chapters:

1. **Introduction to Partial Differential Equations: Formation of PDEs by eliminating-Arbitrary constants, Arbitrary functions.**

2. **Solving Linear, Semi-linear, Quasi-linear PDEs of order 1 by Lagrange's method, Finding Integral surfaces passing through given curves represented by solution of a given PDE.**

3. **Finding Complete Integral, Singular Integral, general integral of Non-Linear PDEs of order 1 by Charpit's method, Finding Integral surfaces passing through given curves represented by solution of a given PDE: Cauchy-Characteristic Method.**

4. **Solving Lineah Homogeneous and Non-Homogeneous Linear PDEs of Higher order with constant coefficients: finding C.F. & P.I., Cauchy-Euler PDEs.**

5. **Classification of PDEs of second order, reduction into Canonical form and solving those Hyperbolic, Parabolic and Elliptic PDEs.**

6. **Solving Initial Boundary Value Problems(IBVPs): Method of separation of variables, Eigenvalues and eigen functions of a Boundary value Problem, Using initial ccondiion and by Fourier series getting required solution. Heat Equation, Laplace Equation and Wave equation**

PARTIAL DIFFERENTIAL EQUATIONS

- **Chapter 1: Formation of PDEs**
  By removing arbitrary constants
  By removing arbitrary functions
- **Chapter 2: PDE of order 1**

  Linear: Lagrange's Auxiliary equations method.

  Non-linear: Charpit's method, Characteristic method.

- **Chapter 3**: **ODEs of higher order** (i.e. order greater than 1) Linear homogeneous and non-homogeneous PDEs
  Linear ODEs with constant coefficients
  General solution & Particular solution; Complete sol $\rightarrow$ C.F + P.I
  Finding CF. & P.I. by reducing into PDE with constant coefficients.

Chapter 4:- Classification of PDEs & their application

- Reducing into canonical form
- Solving PDEs of different types

$\Downarrow$

Heat Eq. Wave Eq.
Solution. By variable separable method

$$R(x,y)\frac{\partial^2 z}{\partial x^2}+S(x,y)\frac{\partial^2 z}{\partial x\partial y}+T(x,y)\frac{\partial^2 z}{\partial y^2}=0 \quad \text{Or} \quad Rr+Ss+Tt=0 \qquad \text{..............} (1)$$

$\downarrow$

S^2-4RT

- Here (1) is parabolic PDE if $S^2-4RT=0$
- Then (1) is a hyperbolic PDE if $S^2-4RT>0$
- Then (1) is a elliptic PDE if $S^2-4RT<0$

BASIC DEFINITIONS

Differential equation: An equation which involves derivatives, dependent variable/s, independent variable/s.

e.g. (1) $\frac{dy}{dx}+2x^2y=x^3e^x,\ x\in(-1,1)$. Here x is independent variable and y is depending on x so dependent variable.

(2) $\frac{d^3y}{dx^3}+2\frac{dy}{dx}-e^x\frac{d^2y}{dx^2}=\cos x,\ x\in(a,b)$

(3) $\frac{dz}{dt}+e^t=y,\ \frac{dy}{dt}-2z=\cos t;t\in(a,b)$ is a system of differential equations.

(4) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}+x^2y=x^3\ ;\ x\in(a,b),y\in(c,d)$

here x and y are independent variables and z is dependent.

Note: Variables for which domain is defined are called independent variables.

Classification of differential equations

ODE: If number of independent variable is one.

PDE: If number of independent variables are more than one.

e.g. (1), (2), (3) are ODEs, (4) is a PDE.

From calculus:

If *y* is *function* of *x* alone; then we talk ; like total derivative or simply derivative of y w.r.t *x*.
If *y* is *function* of *u*, *v* then we talk; Partial derivative of y w.r.t u or v.

- **Order of differential equation**: The order of the highest order derivative present in that differential equation is called order of that differential equation.

e.g. Differential equation (1) is of order 1.

Differential equation (2) is of order 3.

- **Degree of a Differential equation**: The power of the highest order derivative, present in that differential equation is called degree of that differential equation provided that derivatives are radical free.

e.g. (1) $\left(\frac{dy}{dx}\right)^{3/4} = 1 - y$, order = 1

For degree, we try to make derivatives radical free.

$\because$ given differential equation can be written as

$\frac{dy}{dx} = (1-y)^{4/3}$ $\therefore$ degree = power of highest order derivative = 1

e.g. (2) $e^{\frac{d^2y}{dx^2}} + \cos x \frac{d^3y}{dx^3} + 2y = \left(\frac{d^4y}{dx^4}\right)^2$

here order of differential equation = 4

degree of differential equation is **NOT DEFINED** (since derivatives are functions of exponential trigonometric, so not radical free)

e.g. (3) $\cos\left(\frac{d^5y}{dx^5}\right)^2 + x^2y = \cos y$

$\because$ given differential equation can be managed as $\left(\frac{d^5y}{dx^5}\right)^2 = \cos^{-1}\left(-x^2y + \cos y\right)$

now derivatives are radical free. So degree of differential equation is 2.

• **Linear differential equation**: A differential equation is said to be linear 'in the variable *z* if it satisfies following three conditions:

(i) power of *z* does not exceed by 1

(ii) power of partial derivatives of *z* does not exceed by 1

(iii) No term present as product of partial derivatives or derivatives and *z*

Note: here powers are either 0 or 1.

e.g. (1) $\frac{dy}{dx} + p(x)y = Q(x)$; is a liner differential equation of order one in the variable y.

(2) $\frac{d^4y}{dx^4} - \frac{dy}{dx} + 2y = e^{x^2}$; is a linear differential equation of order 4 in the variable y.

(3) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial y}{\partial x} = \frac{\partial^4 z}{\partial x^4 \partial y}$: is a linear PDE in z. z dependent variable, *x* & *y* independent

(4) $\left(\frac{\partial z}{\partial x}\right)^2 + z = x^2 y$ Not linear in z.

Note: Linearity has nothing to do with order of derivatives. It only cares about degree of the variable and its derivatives for which we are checking the linearity.

Notations used in PDEs:

$$z_x = \frac{\partial z}{\partial x} = p,\ z_y = \frac{\partial z}{\partial y} = q$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = r, \quad z_{yy} = \frac{\partial^2 z}{\partial y^2} = t, \quad z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = s$$

Classification of first order PDEs:

Linear: A differential equation $f(x, y, z, p, q) = 0$ is said to be linear if it is linear in p, q, z. i.e. given equation is of the form $P(x, y).p + Q(x, y).q = R(x, y)$.

Semi-linear: A differential equation $f(x, y, z, p, q) = 0$ is said to be semi-linear if it is linear in p, q. and coefficients of $p \,\&\, q$ are functions of $x \,\&\, y$ only. i.e. given equation is of the form $P(x, y).p + Q(x, y).q = R(x, y, z)$.

Quasi-linear: A differential equation $f(x, y, z, p, q) = 0$ is said to be quasi-linear if it is linear in p, q. i.e. given equation is of the form $P(x, y, z).p + Q(x, y, z).q = R(x, y, z)$.

Non-linear: A differential equation $f(x, y, z, p, q) = 0$ is said to be non-linear.

• **<u>Homogeneous and non-homogeneous linear PDEs Higher order</u>** :

A partial differential equation is said to be linear homogeneous when all the deriavtives in that equation are of same order. Otherwise it's linear non homogeneous.

General from of higher order linear homogeneous PDE

$$a_0 \frac{\partial^n z}{\partial x^n} + +\alpha_1 \frac{\partial^n z}{\partial y \partial x^{n-1}} + \alpha_2 \frac{\partial^n z}{\partial y^2 \partial y^{n-2}} + \dots\dots\dots\dots + a_n \frac{\partial^n z}{\partial y^n} = f(x, y)$$

General from of higher order linear non-homogeneous PDE

$$a_0 \frac{\partial^n z}{\partial x^n} + \beta_0 \frac{\partial^n z}{\partial y^n} + \alpha_1 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \beta_1 \frac{\partial^{n-1} z}{\partial y^{n-1}} + \dots\dots\dots\dots = f(x, y)$$

<u>Partial Differential equation represented by a given family of surfaces</u>

E.g. Find differential equation represented by z = *ax + y*; where a is arbitrary constant z is depending on *x* and *y.*

Ans. z = *ax + y.....(1)*

Differentiating (1) we partially w.r.t *x*, we get

$\frac{\partial z}{\partial x} = a \Rightarrow a = \frac{\partial z}{\partial x}$

Now, using (2) in (1) we get

z = $\left(\frac{\partial z}{\partial x}\right)$*x+ y ;* No arbitrary constant present ....(3)

So, (3) is differential equation of (1)

Or different (1) partially w.r.t y, we get. $\frac{\partial z}{\partial y}$ = 0 + 1

$\frac{\partial z}{\partial y}$ = 1 .... (4) ; is also a differential equation of (1).

Detailed about formation of PDE:

Let's consider a given relation as $F(x_1, x_2......x_n, z, a_1, a_2.....a_n) = 0$ where $a_1, a_2.....a_n$ are arbitrary constants. z is dependent variable on independent variables $x_1, x_2......x_n$.

Case-1: If number of independent variables is same as number of arbitrary constants in given relation then we get unique PDE of order one on eliminating arbitrary constants.

Case-2: If number of independent variables is less than the number of arbitrary constants in given relation then we usually get a PDE of order greater than one on eliminating arbitrary constants.

Case-3: If number of independent variables is greater than the number of arbitrary constants in given relation then we get more than one PDEs of order one on eliminating arbitrary constants.

Initial Value Problem (IVP): A differential equation with given conditions at initial point of domain of independent variable.

e.g. ODE (1) $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$ where $x_0 \in (a,b)$ with $x \in [a,b]$ is an IVP e.g. PDE $\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2+z^2=0; \, with\ z(x_0,y_0)=z_0$ with $(x_0,y_0)\in X\times Y=[a,b;c,d]$

Boundary Value Problems: A differential equation with conditions at boundary points.

e.g. $\frac{d^3y}{dx^3}+2\frac{dy}{dx}+y=e^x$; $x\in[-2,2]$ with $y(-2)=0, y'(2)=0$

• **Solution of a differential equation**: A function or curve which satisfies/represents the given differential equation, is known as solution of that differential equation. Satisfies means if on differentiating the given function and using derivatives and function in given differential equation, we get both sides LHS and RHS as equal.

e.g. (1) $\because\ \frac{x^2+y^2}{2}=1$ satisfies the differential equation $xdx+ydy=0$. So it is a solution of it.

(2) $y(x)=c_1\cos x+c_2\sin x$ with c_1 and c_2 as arbitrary constants, satisfies $\frac{d^2y}{dx^2}+y=0$. So $y(x)$ is the solution of this differential equation.

Let us now discuss **different categories of solutions of PDEs of order one:**

Complete Solution/Integral:

$F(x,y,z,a,b)=0$ is known as complete solution of $f(x,y,z,p,q)=0;$ where $z: dependent\ on\ x\ \&\ y$.

A Solution which has that many arbitrary constants as the number of independent variables.

Particular Solution/Integral: by giving some particular values to arbitrary constants in complete solution, we get a particular solution of that PDE.

Singular Solution/Integral: The envelope of surfaces given by complete integral, which also satisfy given PDE; is known as singular solution of that PDE.

Found by eliminating arbitrary constants from $F(x,y,z,a,b)=0,\quad \frac{\partial F}{\partial a}=0,\quad \frac{\partial F}{\partial b}=0$

General Solution/Integral: Assume $b=\phi(a)$ in $F(x,y,z,a,b)=0$, i.e. we have $F(x,y,z,a,\phi(a))=0$,

One parameter family of surfaces. So it's envelope gives the General solution.

Found by eliminating a from $F(x,y,z,a,\phi(a))=0,\quad \frac{\partial F}{\partial a}=0$

Solution of an IVP: A solution / curve which satisfies the differential equation as well as given initial conditions is known as solution of that IVP.

Formation of PDE

Category I: By removing arbitrary constants from given family of curves.

Ex-1 Find PDE represented by $z = ax + y + 2;$ a is arbitrary constant.

Ans. $\because\ z = ax + y + 2\ \therefore\ \frac{\partial z}{\partial x} = a + 0 + 0 \therefore$ we have, $z = \frac{\partial z}{\partial x} x + y + z$ is req. PDE

Or, $z = ax + y + 2 \qquad \therefore\ \frac{\partial z}{\partial y} = a.o + 1 + 0 \Rightarrow \frac{\partial z}{\partial y} = 1$ is also a PDE rep. by given family.

Observe: No. of arbitrary constant = 1 < no. of independent variables 2. (x & y). So we get more than one PDEs of order 1.

Ex-2. Find PDE of family of spheres having radius λ & centre in *xy* plane.

Ans. $\because$ Centre of sphere: in *xy* plane i.e. (h, k, o); where *h* & *k* are arbitrary constants; representing sphere for given family.

i.e., We have, given family as, $(x-h)^2 + (y-k)^2 + z^2 = \lambda^2$ ......(1)

Observe:- No. of arb. Constants = No. of ind. variables = 2

Partially diff. (1) w.r.t. x; $2(x-h) + 0 + 2z\frac{\partial z}{\partial x} = 0$ ....(2)

Partially diff. (1) w.r.t y, $0 + 2(y-k) + 2z\frac{\partial z}{\partial y} = 0$ ......(3)

Now, using (2) & (3) in (1), we get req. PDE

$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 + z^2 = \lambda^2$$

Ex.3 Find PDE rep. by $z = ax + by + c,$ a, b, c are arbitrary constants.

Ans. $\because\ z = ax + by + c\,;\quad \frac{\partial z}{\partial x} = a + 0 + 0 \Rightarrow a = \frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y} = 0 + b + 0 \Rightarrow b = \frac{\partial z}{\partial y}$

$\therefore\ z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + c\,;$ *c* is not removed yet

Now, diff. again partially w.r.t x, $\frac{\partial z}{\partial x} = x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.1 + \left(y\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}.0\right) + 0$

$\because$ *x* & *y* are independent var. $\partial y / \partial x = 0$

$$\Rightarrow \frac{\partial z}{\partial x} = x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y}$$ is req. PDE.

Observe: No. of arb. const = 3 > No. of indep. variable = 2 (*x* & *y*).

Category II:- Formation of PDE by removing arbitrary function from the eq. $\phi(u,v)=0$; where *u, v* are functions of *x, y, z*.

Note: we treat z as dependent variables on independent variables *x* & *y*.

Derivation : $\because\ \phi(u,v)=0$ .... (1); Where ϕ is arbitrary constant.

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}du + \frac{\partial \phi}{\partial v}dv = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right) = 0 \text{ ....(2)}$$

{ $\because$ *u* itself is a function of x, y, z $\therefore\ du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}du + \frac{\partial u}{\partial z}dz$ }

- Diff. (2) partially w.r.t x; we get,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}.1 + \frac{\partial u}{\partial y}.0 + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}.1 + \frac{\partial v}{\partial y}.0 + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right) = 0 \text{ ....(3)}$$

- Diff. (2) partially w.r.t. y; we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}.0 + \frac{\partial u}{\partial y}.1 + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}.0 + \frac{\partial v}{\partial y}.1 + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial y}\right) = 0 \text{ ....(4)}$$

{ $\because$ *x* & *y* are ind. Variables $\therefore \frac{dy}{dx} = 0$ and $\because$ *z* is function of *x* & *y*, we'll have $\frac{\partial z}{\partial x}$ in 3}

$\because$ From (3) we have $$-\frac{\partial \phi / \partial u}{\partial \phi / \partial v} = \frac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}}{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}} = \frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} \text{ ....(5)}$$

From (4) we have $$-\frac{\partial \phi / \partial u}{\partial \phi / \partial v} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}} \text{ ......(6)}$$

Now, eliminating ϕ from (5) & (6) we have,

$$\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}=\frac{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}$$ $\therefore$ is req. PDE.

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \boxed{P.p+Q.q=R}:PDE \quad \text{i.e. } \frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

Here $\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

i.e., we have $\boxed{\frac{\partial(u,v)}{\partial(y,z)}.p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}}$ .... (7)

Exam point (1)

To get PDE by eliminating ϕ from $\phi(u,v)=0;$

Way (1): Following the procedure:

- Writing $\phi(u,v)=0;$
- Differentiating partially w.r.t x, then y & then eliminating ϕ.

Way 2

Find:- $\frac{\partial(u,v)}{\partial(x,y)},\frac{\partial(u,v)}{\partial(y,z)},\frac{\partial(u,v)}{\partial(z,x)}$ with the given *u* & *v* from $\phi(u,v)=0$

Using these in above final form (7)

Req. PDE $\rightarrow$ $P(x,y,z)P+Q(x,y,z).q=R(x,y,z)$

Ex-4 Find PDE by removing arbitrary function from $\phi(x+y+z,x^2+y^2-z^2)=0$

Ans. **Way 1**

$\because$ $\phi(x+y+z,x^2+y^2-z^2)=0$ ... (1)

For if $\phi(u,v)=0$ $\therefore$ $u=x+y+z,$ $v=x^2+y^2-z^2$

$\Rightarrow \quad d\phi = -0$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right) = 0 \qquad \text{... (2)}$$

Diff. (2) partially w.r.t x,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}.1 + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}(1 + 1.p) + \frac{\partial \phi}{\partial v}(2x + (-2z).p) = 0 \qquad \text{......(3)}$$

Diff. (2) partially w.r.t y,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial \phi}{\partial u}(1 + 1.q) + \frac{\partial \phi}{\partial u}(2y + (-2z)q) = 0 \qquad \text{......(4)}$$

Now, eliminating ϕ from (3) & (4)

$$\frac{2(x - zp)}{1 + p} = \frac{2(4 - zq)}{1 + q}$$

$$(x - zp)(1 + q) = (y - zq)(1 + p)$$

$$x + qx - zp - zpq = y + py - zq - pqz$$

$(y + z)p - (x + z)q = x - y$ is the req. PDE.

Ans. Way 2:-

$\because$ We have been given

$$\phi(x + y + z, x^2 + y^2 - z^2) = 0$$

$\therefore \quad u = x + y + z, \; v = x^2 + y^2 - z^2 \qquad \text{..... (1)}$

Now, we know that, $\phi(u, v) = 0$ is representing the PDE;

$$\frac{\partial(u,v)}{\partial(u,z)}p + \frac{\partial(u,v)}{\partial(z,x)}q = \frac{\partial(u,v)}{\partial(x,y)} \qquad \text{......(2)}$$

$\therefore$ From (1);

$$\frac{\partial(u,v)}{\partial(y,z)}=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2y & -2z\end{vmatrix}=-2(y+z)$$

$$\frac{\partial(u,v)}{\partial(z,x)}=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial u}{\partial x}\end{vmatrix}=\begin{vmatrix}1 & 1\\ -2z & 2x\end{vmatrix}=2(x+z)$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2x & 2y\end{vmatrix}=2(y-x)$$

Using the above in equation (2), we get the equation PDE as,

$-2(y + z)\, p + 2\,(z + x)\, q = 2(y - x) \Rightarrow (y + z)\, p - (z + x)\, q = x - y$.

Miscelleneous: category I & category II Through process

Ex. 1. Find the PDE represented by $z = f(x^2 - y) + g\,(x^2 + y)$; by removing f & g.

Ans: □$z = f(x^2 - y) + g(x^2 + y)$ .....(1)

Now, we try to eliminate f & g; by thought process of category (I).

Different (1) partially w.r.t. x;

$$\frac{\partial z}{\partial x}=f'\left(x^2-y\right)\,.\,2x+g'\left(x^2+y\right)\times 2x \Rightarrow \frac{1}{2x}\frac{\partial z}{\partial x}=f'\left(x^2-y\right)+g'(x^2+y) \qquad ...(2)$$

Diff. (1) partially w.r.t y;

$$\frac{\partial z}{\partial y}=f'(x^2-y)\times(-1)+g'(x^2+y)\times 1 \Rightarrow \frac{\partial z}{\partial y}=-f'(x^2-y)+g'(x^2+y) \;.....(3)$$

Diff. (2) partially w.r.t x,

$$\frac{1}{2x}\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}\left(\frac{-1}{2x^2}\right)=f''\left(x^2-y\right)\times 2x+g''\left(x^2+y\right)\times 2x$$

$$\frac{1}{4x^2}\left\{\frac{\partial^2 z}{\partial x^2}-\frac{1}{x}\frac{\partial z}{\partial x}\right\}=f''\left(x^2-y\right)+g''\left(x^2+y\right) \qquad ...(4)$$

Different (3) partially w.r.t y; $\quad \frac{\partial^2 z}{\partial y^2}=f''(x^2-y)+g''(x^2+y)$ ....(5)

Now, subtracting (5) from (4), we get

$\frac{1}{4x^2}\left\{\frac{\partial^2 z}{\partial x^2}-\frac{1}{x}\frac{\partial z}{\partial x}\right\}-\frac{\partial^2 z}{\partial y^2}=0$ is required PDE.

Ex. 2. Find the differential equation of surfaces of revolution whose axis of rotation is z-axis.

Ans: Surface of revolution whose axis of rotation is z–axis, given by $z^2 = x^2 + y^2$. Recall from 3D; cone $z = \phi(\sqrt{x^2+y^2})$ ...(1), clearly, we don't have $\phi(u, v) = 0$ type, so, we follow:

Diff. (1) partially w.r.t. x, $\frac{\partial z}{\partial x} = \phi'\left(\sqrt{x^2+y^2}\right) \times \frac{2x}{2\sqrt{x^2+y^2}}$ ....(2)

Diff. (i) partially w.r.t. y, $\frac{\partial z}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} \phi'(\sqrt{x^2+y^2})$ ....(3)

Now, from (2)/(3) $\frac{p}{q} = \frac{x}{y} \Rightarrow$ $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$ is required PDE.

Ex 3. Find the PDE by removing arbitrary functions f & g from $z = y f(x) + xg(y)$.

Ans: We have $z = yf(x) + xg(y)$ .....(1)

Partially diff. (1) w.r.t x, $\frac{\partial z}{\partial x} = yf'(x) + g(y) \times 1 \Rightarrow \frac{\partial z}{\partial x} = yf'(x) + g(y)$ ...(2)

Partially different (1) w.r.t. y, $\frac{\partial z}{\partial y} = f(x) \times 1 + x.g'(y) \Rightarrow \frac{\partial z}{\partial y} = f(x) + xg'(y)$ ...(3)

Partially different (2) w.r.t. y, $\frac{\partial^2 z}{\partial y \partial x} = f'(x) + g'(y)$ ....(4)

From (2) × x + (3) × y; $xyf'(x) + xg(y) + yf(x) + xyg'(y) = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

$x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = \{xg(y) + yf(x)\} + xy\{f'(x) + g'(y)\} \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 z}{\partial x \partial y}$; required PDE

Examples to Practice

Type-1 Problems

Example.1. Eliminate a and b from $az + b = a^2x + y$ ....(1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$a(\partial z/\partial x) = a^2$ ....(2), $a(\partial z/\partial y) = 1$....(3)

Eliminating a from (2) and (3), we have $(\partial z/\partial x)(\partial z/\partial y) = 1$,

which is the unique partial differential equation of order one.

Example.2. Eliminate a, b and c from $z = ax + by + cxy$ ....(1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$\partial z/\partial x = a + cy$ ....(2), $\partial z/\partial y = b + cx$ ....(3)

From (2) and (3), $\partial^2 z/\partial x^2 = 0$, $\partial^2 z/\partial y^2 = 0$ ....(4) and $\partial^2 z/\partial x \partial y = c$ ....(5)

Now, (2) and (3) $\Rightarrow x(\partial z/\partial x) = ax + cxy$ and $y(\partial z/\partial y) = by + cxy$

$\therefore\ x(\partial z/\partial x) + y(\partial z/\partial y) = ax + by + cxy + cxy$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = z + xy(\partial^2 z/\partial x \partial y)$, using (1) and (5) ....(6)

Thus, we get three partial differential equations given by (4), (5) and (6), which are all of order two.

Type-2 Problems

Ex. 3. Find a partial differential equation by eliminating a and b from $z = ax + by + a^2 + b^2$.

Solution. Given $z = ax + by + a^2 + b^2$ ....(1)

Differentiating (1) partially with respect to x and y, we get; $\partial z/\partial x = a$ and $\partial z/\partial y = b$.

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain, $z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2$; which is the required partial differential equation.

Ex. 4. Eliminate arbitrary constants a and b from $z = (x-a)^2 + (y-b)^2$ to form the partial differential equation.

Solution. Given $z = (x-a)^2 + (y-b)^2$. ....(1)

Differentiating (1) partially with respect to a and b, we get

$\partial z/\partial x = 2(x-a)$ and $\partial z/\partial y = 2(y-b)$.

Squaring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x-a)^2 + 4(y-b)^2 = 4\left[(x-a)^2 + (y-b)^2\right]$$

$(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z$, using (1).

Ex. 5. Eliminate a and b from $z = axe^y + (1/2) \times a^2 e^{2y} + b$.

Solution. Given $z = axe^y + (1/2) \times a^2 e^y + b$ .....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = ae^y$ ....(2) and $\partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2$ .....(3)

Substituting the value of ae^y from (2) in (3), we get $\partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2$.

Ex. 6. Form the differential equation by eliminating a and b from $z = (x^2 + a)(y^2 + b)$. **[I.A.S. 1997]** Solution. Given $z = (x^2 + a)(y^2 + b)$ .....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = 2x(y^2 + b)$ or $(y^2 + b) = (1/2x) \times (\partial z/\partial x)$ ....(2)

and $\partial z/\partial y = 2y(x^2 + a)$ or $(x^2 + a) = (1/2y) \times (\partial z/\partial y)$. ....(3)

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1) gives

$z = (1/2y) \times (\partial z/\partial y) \times (1/2x) \times (\partial z/\partial x)$ or $4xyz = (\partial z/\partial x)(\partial z/\partial y)$,

which the required partial differential equation.

Ex. 7. Form differential equation by eliminating constants A and p from $z = Ae^{pt} \sin px$.

Solution. Given $z = Ae^{pt} \sin px$ . ....(1)

Differentiating (1) partially with respect to x and t, we get

$\partial z/\partial x = Ap\,e^{pt} \cos px$ ....(2), $\partial z/\partial t = Ap\,e^{pt} \sin px$ ....(3)

Differentiating (2) and (3) partially with respect to x and t respectively gives

$\partial^2 z/\partial x^2 = -Ap^2\,e^{pt} \sin px$ ....(4) and $\partial^2 z/\partial t^2 = Ap^2\,e^{pt} \sin px$ ....(5)

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$,which is the required partial differential equation.

Ex. 8. Find the differential equation of the set of all right circular cones whose axes coincide with z -axis.

Solution. The general equation of the set of all right circular cones whose axes coincide with z -axis, having semi-vertical angle α and vertex at $(0,0,c)$ is given by

$x^2 + y^2 = (z - c)^2 \tan^2 \alpha$ ....(1); in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t x and y, we get

$2x = 2(z - c)(\partial z/\partial x)\tan^2 \alpha$ and $2y = 2(z - c)(\partial z/\partial y)\tan^2 \alpha$

$\Rightarrow y(z-c)(\partial z/\partial x)\tan^2\alpha = xy$ and $x(z-c)(\partial z/\partial y)\tan^2\alpha = xy$

$\Rightarrow y(z-c)(\partial z/\partial x)\tan^2\alpha = x(z-c)(\partial z/\partial y)\tan^2\alpha$

Thus, $y(\partial z/\partial x) = x(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that $yz + zx + xy = 0$ is a surface satisfying the above equation. **[I.A.S. 1979, 2009]**

Solution. The equation of any cone with vertex at origin is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$,....(1) where a, b, c, f, g, h are parameters.
Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have

$2ax + 2czp + 2fyp + 2g(px + z) + 2hy = 0$ or $ax + gz + hy + p(cz + gx + fy) = 0$ ....(2)

and $2by + 2czq + 2f(yq + z) + 2gxq + 2hx = 0$ or $by + fz + hx + q(cz + fy + gx) = 0$ ....(3)

Multiplying (2) by x and (3) by y and adding, we have

$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)(px + qy) = 0$

$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0$ using (1)

or $(cz + fy + gx)(px + qy - z) = 0$ or $px + qy - z = 0$,....(4) required partial differential equation.

Second Part : Given surface is $yz + zx + xy = 0$ ....(5)

Differentiating (5) partially w.r.t 'x' and 'y' by turn, we get

$yp + px + z + y = 0$ and $z + qy + xq + x = 0$ ....(6)

Solving (6) for p and q, $p = -(z+y)/(x+y)$ and $q = -(z+x)/(x+y)$.

$\therefore \quad px + qy - z = -\dfrac{x(z+y)}{x+y} - \dfrac{y(z+x)}{x+y} - z = -\dfrac{2(xy+yz+zx)}{x+y} = 0$, using (5)

Hence (5) is a surface satisfying (4).

Type-3 Problems

Ex. 11. Form a partial differential equation by eliminating the arbitrary function f from the equation $x + y + z = f(x^2 + y^2 + z^2)$.

Solution. Given $x+y+z=f\left(x^2+y^2+z^2\right)$.....(1)

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$1+p=f'\left(x^2+y^2+z^2\right)\cdot(2x+2zp)$. ....(2)

and $1+q=f'\left(x^2+y^2+z^2\right)\cdot(2y+2zq)$ ....(3)

Eliminating $f'\left(x^2+y^2+z^2\right)$ from (2) and (3), we obtain

$(1+p)/(2x+2zp)=(1+q)/(2y+2zq)$ or $(1+p)(y+zq)=(1+q)(x+zp)$

or $(y-z)p+(z-x)q=x-y$, which is the required partial differential equations.

Ex. 12. Eliminate the arbitrary functions *f* and F from $y=f(x-at)+F(x+at)$.

Solution. Given $y=f(x-at)+F(x+at)$.....(1)

From (1), $\partial y/\partial x=f'(x-at)+F'(x+at)$ and hence $\partial^2 y/\partial x^2=f''(x-at)+F''(x+at)$.....(2)

Also, $\partial y/\partial t=f'(x-at)\cdot(-a)+F'(x+at)\cdot(a)$

and hence $\partial^2 y/\partial t^2=f''(x-at)\cdot(-a)^2+F''(x+at)\cdot(a)^2$

or $\partial^2 y/\partial t^2=a^2\left[f''(x-at)+F''(x+at)\right]$.....(3)

Then, (2) and (3) $\Rightarrow \partial^2 y/\partial t^2=a^2\left(\partial^2 y/\partial x^2\right)$.

Ex. 13. Eliminate arbitrary function *f* from (i) $z=f\left(x^2-y^2\right)$. (ii) $z=f\left(x^2+y^2\right)$.

Solution. (i) Given $z=f\left(x^2-y^2\right)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x=f'\left(x^2-y^2\right)\times 2x$ so that $f'\left(x^2-y^2\right)=(1/2x)\times(\partial z/\partial x)$ ....(2)

and $\partial z/\partial y=f'\left(x^2-y^2\right)\times(-2y)$ so that $f'\left(x^2-y^2\right)=-(1/2y)\times(\partial z/\partial y)$. ....(3)

Eliminating $f'\left(x^2-y^2\right)$ between (2) and (3), we have $\dfrac{1}{2x}\dfrac{\partial z}{\partial x}=-\dfrac{1}{2y}\dfrac{\partial z}{\partial y}$ or $y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=0$.

(ii) **Ans.** $y(\partial z/\partial x)-x(\partial z/\partial y)=0$.

Ex. 14. Form a partial differential equation by eliminating the function *f* from

(i) $z = f(y/x)$. (ii) $z = x^n f(y/x)$.

Solution. Given $z = f(y/x)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(y/x)\times(-y/x^2)$ or $f'(y/x) = -(x^2/y)\times(\partial z/\partial x)$ ....(2)

and $\partial z/\partial y = f'(y/x)\times(1/x)$ or $f'(y/x) = x(\partial z/\partial y)$. ....(3)

Eliminating $f'(y/x)$ between (2) and (3), we have

$-\dfrac{x^2}{y}\dfrac{\partial z}{\partial x} = x\dfrac{\partial z}{\partial y}$ or $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$,which is the required partial differential equation.

(ii) Given $z = x^n f(y/x)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = nx^{n-1} f(y/x) + x^n f'(y/x)\times(-y/x^2)$ ....(2)

and $\partial z/\partial y = x^n f'(y/x)\times(1/x)$. ....(3)

Multiplying both sides of (2) by x, we have $x(\partial z/\partial x) = nx^n f(y/x) - yx^{n-1} f'(y/x)$. ....(4)

Multiplying both sides of (3) by y, we have $y(\partial z/\partial y) = yx^{n-1} f'(y/x)$. ....(5)

Adding (4) and (5), $x(\partial z/\partial x) + y(\partial z/\partial y) = nx^n f(y/x)$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = nz$, by (1)

Ex. 15. Form a partial differential equation by eliminating the function ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$.

Solution. Given $lx + my + nz = \phi(x^2 + y^2 + z^2)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$l + n(\partial z/\partial x) = \phi'(x^2 + y^2 + z^2)\times\{2x + 2z(\partial z/\partial x)\}$ ....(2)

and $m + n(\partial z/\partial y) = \phi'(x^2 + y^2 + z^2)\times\{2y + 2z(\partial z/\partial y)\}$ ....(3)

Dividing (2) by (3), we get $\dfrac{l+n(\partial z/\partial x)}{m+n(\partial z/\partial y)}=\dfrac{2\{x+z(\partial z/\partial x)\}}{2\{y+z(\partial z/\partial y)\}}$

or $(ny-mz)(\partial z/\partial x)+(lz-nx)(\partial z/\partial y)=mx-ly$, which is the required partial differential equation.

Ex. 16. Form partial differential eqn. by eliminating the function *f* from $z=e^{ax+by}f(ax-by)$.

Solution. Given $z=e^{ax+by}f(ax-by)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x=e^{ax+by}af'(ax-by)+ae^{ax+by}f(ax-by)$ ....(2)

and $\partial z/\partial y=e^{ax+by}\{-bf'(ax-by)\}+b\,e^{ax+by}f(ax-by)$. ....(3)

Multiplying (2) by b and (3) by a and adding, we get

$b(\partial z/\partial x)+a(\partial z/\partial y)=2ab\,e^{ax+by}f(ax-by)$ or $b(\partial z/\partial x)+a(\partial z/\partial y)=2abz$, by (1)

Ex. 17. Form a partial differential equation by eliminating the arbitrary functions *f* and F from $z=f(x+iy)+F(x-iy)$, where $i^2=-1$.

Solution. Given $z=f(x+iy)+F(x-iy)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x=f'(x+iy)+F'(x-iy)$....(2) and $\partial z/\partial y=if'(x+iy)-iF'(x-iy)$.....(3)

Differentiating (2) and (3) partial w.r.t. x and y respectively, we get

$\partial^2 z/\partial x^2=f''(x+iy)+F''(x-iy)$ ....(4)

and $\partial^2 z/\partial y^2=i^2f''(x+iy)+i^2F''(x-iy)=-\{f''(x+iy)+F''(x+iy)\}$. ....(5)

Adding (4) and (5), $\partial^2 z/\partial x^2+\partial^2 z/\partial y^2=0$, which is the required equation.

Ex. 18. Form partial differential equation by eliminating arbitrary functions f and g from $z=f(x^2-y)+g(x^2+y)$. **[I.A.S. 1996]**

Solution. Given $z=f(x^2-y)+g(x^2+y)$. ....(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}$. ....(2)

and $\partial z/\partial y = -f'(x^2 - y) + g'(x^2 + y)$. ....(3)

Differentiating (2) and (3) w.r.t. x and y respectively, we get

$\partial^2 z/\partial x^2 = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\}$ ....(4)

and $\partial^2 z/\partial y^2 = f''(x^2 - y) + g''(x^2 + y)$. ....(5)

Again (2) $\Rightarrow f'(x^2 - y) + g'(x^2 + y) = (1/2x) \times (\partial z/\partial x)$. ....(6)

Substituting the values of $f''(x^2 - y) + g''(x^2 + y)$ and $f'(x^2 - y) + g'(x^2 + y)$ from (5) and (6) in (4), we have $\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2 \frac{\partial^2 z}{\partial y^2}$ or $x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3 \frac{\partial^2 z}{\partial y^2}$,required pde.

Type-4 Problems

Extra Exam Point

Ex.1 State the properties of $\Phi(x, y)$ if there exists a surface $z = \Phi(x, y)$ which passes through the curve C with parametric equations $x = x_0(\mu), y = y_0(\mu), z = z_0(\mu)$ and at every point of which the direction $(p, q, -1)$ of the normal is such that $f(x, y, z, p, z) = 0$.

Approach:

*Let $z = \phi(x, y)$ ....(1) be the equation of the given surface

Let $F(x, y, z) = \phi(x, y) - z$. ....(2)

From (1) and (2), $\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial z}{\partial x} = p$, $\frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial z}{\partial y} = q$, $\frac{\partial F}{\partial z} = -1$

Since ∇F is normal to the surface $F(x, y, z) = 0$, $\partial F/\partial x, \partial F/\partial y, \partial F/\partial z$ i.e. $p, q - 1$ are direction ratios of the normal to $F(x, y, z) = 0$ or $\phi(x, y)$.

For more: See the explanation example below

Ex. 19. Solve the Cauchy's problem for $zp + q = 1$, when the initial data curve is $x_0 = \mu, y_0 = \mu, z_0 = \mu/2, 0 \le \mu \le 1$.

Solution. Given $f(x,y,z,p,q) = zp + q - 1 = 0$ ....(1)

Given initial data curve $x_0 = \mu,\ y_0 = \mu,\ z_0 = \mu/2,\ 0 \le \mu \le 1$ ....(2)

From (1), $\partial f/\partial p = z,\ \ \partial f/\partial q = 1$, and $\dfrac{\partial f}{\partial q}\dfrac{dx_0}{d\mu} - \dfrac{\partial f}{\partial p}\dfrac{dy_0}{d\mu} = 1\times 1 - z\times 1 = 1 - \dfrac{1}{2}\ \ \mu \ne 0$, for $0 \le \mu \le 1$.

Now, we have the following ordinary differential equations:

$\dfrac{dx}{dt} = \dfrac{\partial f}{\partial p}, \dfrac{dy}{dt} = \dfrac{\partial f}{\partial q}$ and $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$ or $dx/dt = z,\ dy/dt = 1$ ....(3)

and $dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = pz + q = 1$, by (1) ....(4)

Integrating (3) and (4), $y = t + C_1$ and $z = t + C_2$ ....(5)

From (2), at $t = 0$, $x(\mu,0) = \mu,\ y(\mu,0) = \mu$ and $z(\mu,0) = \mu/2$ ....(6)

Using (6), (5) reduces to $y = t + \mu$ and $z = t + \mu/2$ ....(7)

Then, from (3) and (7), $dx/dt = t + \mu/2$ so that $x = (1/2)\times t^2 + (1/2)\times \mu t + C_3$ ....(8)

Using (6), (8) reduces to $x = (1/2)\times t^2 + (1/2)\times \mu t + \mu$ ....(9)

Solving $y = t + \mu$ with (9) for μ and t in terms of x and y, we get

$$t = \frac{y - x}{1 - (y/2)} \quad \text{and} \quad \mu = \frac{x - (y^2/2)}{1 - (y/2)}$$

Putting these values in $z = t + \mu/2$, the required solution passing through the initial data curve is

$$z = \{2(y - x) + x - y^2/2\}/(2 - y).$$

Assignment: Questions

Q.1. Eliminate a, b and c from $z = a(x+y) + b(x-y) + abt + c$. **[I.A.S. 1998]**

Q.2. Form the partial differential equation by eliminating the arbitrary constants a and b from $\log(az - 1) = x + ay + b$. **[I.A.S. 2002]**

Q.3. Find a partial differential equation by eliminating a, b, c from $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Q.4. Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.

Q.5. Show that the partial differential equation obtained by eliminating the arbitrary constants a and c from $z = ax + g(a)y + c$, where $g(a)$ is an arbitrary function of a, is free of the variables x, y, z.

Q.6. Show that the partial differential equation obtained by eliminating the arbitrary constants a and b from $z = ax + by + f(a,b)$ is given by $z = px + qy + f(p,q)$.

Q.7. Form a partial differential equation by eliminating a, b and c from the relation $ax^2 + by^2 + cz^2 = 1$.

Q.8. Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$.

Q.9. Eliminate the arbitrary function f and obtain the partial differential equation from $z = e^y f(x+y)$.

Q.10. Equation of any cone with vertex at $P(a,b,c)$ is of the form $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$. Find the differential equation of the cone.

Solution.1 Given $z = a(x+y) + b(x-y) + abt + c$ ....(1)

Differentiating (1) partially w.r.t. $'x', 'y'$ and $'t'$, we get

$\partial z/\partial x = a + b$ ....(2), $\partial z/\partial y = a - b$ ....(3) $\partial z/\partial t = ab$ ....(4)

Using $(a+b)^2 - (a-b)^2 = 4ab \;\therefore\; (\partial z/\partial x)^2 - (\partial z/\partial y)^2 = 4(\partial z/\partial t)$, using (2), (3) and (4).

Solution.2 (a) Given $\log(az-1) = x + ay + b$ ....(1)

Diffe (1) partially w.r.t. $'x'$ and 'y', we get $\frac{a}{az-1}\frac{\partial z}{\partial x} = 1$ ....(2), $\frac{a}{az-1}\frac{\partial z}{\partial y} = a$ ....(3)

From (3), $az - 1 = \frac{\partial z}{\partial y}$ so that $a = \frac{1 + (\partial z/\partial y)}{z}$ ....(4)

Putting the above values of $az-1$ and a in (2), we have $\frac{1+(\partial z/\partial y)}{z(\partial z/\partial y)}\frac{\partial z}{\partial x} = 1$ or $\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}$

Solution.3 Given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$....(1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{dz}{dx}=0 \text{ or } c^2x+a^2z\frac{dz}{dx}=0 \qquad ....(2)$$

and $\frac{2y}{b^2}+\frac{2x}{c^2}\frac{\partial z}{\partial y}=0$ or $c^2y+b^2z\frac{\partial z}{\partial y}=0$. ....(3)

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \qquad ....(4)$$

$$c^2+b^2\left(\frac{\partial z}{\partial x}\right)^2+b^2z\frac{\partial^2 z}{\partial y^2}=0 \qquad ....(5)$$

From (2), $c^2=-\left(a^2z/x\right)\times\left(\partial z/\partial x\right)$ ....(6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0 \text{ or } zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0. \qquad ....(7)$$

Similarly, from (3) and (5), $zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0$. ....(8)

Differentiating (2) partially w.r.t. y, $0+a^2\left\{\left(\partial z/\partial y\right)\left(\partial z/\partial x\right)+z\left(\partial^2 z/\partial x\partial y\right)\right\}=0$

or $\left(\partial z/\partial x\right)\left(\partial z/\partial y\right)+z\left(\partial^2 z/\partial x\partial y\right)=0$ ....(9)

(7), (8) and (9) are three possible forms of the required partial differential equations.

Solution.4

Let $lx+my+nz=a$ ....(1)

be the equation of the given plane where l,m,n are direction cosines of the normal to the plane so that $l^2+m^2+n^2=1,\ l,m,n$ being parameters ....(2)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$l+np=0$ ....(3), $\qquad m+nq=0$,....(4)

where $p=\partial z/\partial x$ and $q=\partial z/\partial y$. From (3) and (4), $l=-np$ and $m=-nq$. Substituting these values in (2), we have $n^2\left(p^2+q^2+1\right)=1$ so that $n=\left(p^2+q^2+1\right)^{-1/2}$ ....(5)

$\therefore \;\; l = -np = -p\left(p^2+q^2+1\right)^{-1/2}$ and $m = -nq = -q\left(p^2+q^2+1\right)^{-1/2}$ ....(6)

Substituting the values of l, m, n given by (5) and (6) in (1), we get

$$-px\left(p^2+q^2+1\right)^{-1/2} - qy\left(p^2+q^2+1\right)^{-1/2} + z\left(p^2+q^2+1\right)^{-1/2} = a$$

or $z = px + qy + a\left(p^2+q^2+1\right)^{1/2}$, which is the required partial differential equation.

Solution.5 Differentiating $z = ax + g(a)y + c$ partially w.r.t $'x'$ and $'y'$ yields

$p = a$ and $q = g(a)$. Eliminating a between them leads to $q = g(p)$ or $f(p,q) = 0$, where f is an arbitrary function of p and q. Clearly, the resulting partial differential equation contains p and q but none of the variables x, y, z.

Solution.6 Differentiating $z = ax + by + f(a,b)$ ....(1)

partially with respect to $'x'$ and $'y'$, we get $p = a$ and $q = b$ ....(2)

Eliminating a and b from (1) and (2) yields $z = px + qy + f(p,q)$.

Solution.7 Given $ax^2 + by^2 + cz^2 = 1$. ....(1)

Differentiating (1) partially w.r.t. $'x'$ and $'y'$, we have

$2ax + 2cz(\partial z/\partial x) = 0$ ....(2), $\quad 2by + 2cz(\partial z/\partial y) = 0$ ....(3)

Differentiating (2) partially w.r.t. $'y'$, we get

$0 + 2c\left\{(\partial z/\partial y)(\partial z/\partial x) + z\left(\partial^2 z/\partial y\partial x\right)\right\} = 0$ or $(\partial z/\partial x)(\partial z/\partial y) + z\left(\partial^2 z/\partial x\partial y\right) = 0$, ...(4)

since c is an arbitrary constant. (4) is the desired partial differential equation.

Again, differentiating partially (2) w.r.t. x and (3) w.r.t. y, we get

$2a + 2c\left\{(\partial z/\partial x)^2 + z\left(\partial^2 z/\partial x^2\right)\right\} = 0$ ....(5), $\quad 2b + 2c\left\{(\partial z/\partial y)^2 + z\left(\partial^2 z/\partial y^2\right)\right\} = 0$ ....(6)

From (2), $a = -(cz/x)\times(\partial z/\partial x)$. Putting this in (5), we get

$-(cz/x)\times(\partial z/\partial x) + c\left\{(\partial z/\partial x)^2 + z\left(\partial^2 z/\partial x^2\right)\right\} = 0$ or $zx\left(\partial^2 z/\partial x^2\right) + x(\partial z/\partial x)^2 - z(\partial z/\partial x) = 0$ ....(7)

Similarly, from (3) and (6), we get $zy\left(\partial^2 z/\partial y^2\right) + y(\partial z/\partial y)^2 - z(\partial z/\partial y) = 0$ ....(8)

(4), (7) and (8) are three possible forms of the required partial differential equations.

Solution.8 Given $\phi\left(x^2+y^2+z^2, z^2-2xy\right)=0$. ....(1)

Let $u=x^2+y^2+z^2$ and $v=z^2-2xy$. ....(2)

Then, (1) becomes $\phi(u,v)=0$. ....(3)

Differentiating (3) partially w.r.t. $'x'$, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0, \quad ....(4)$$

where $p=\partial z/\partial x$ and $q=\partial z/\partial y$. Now, from (2), we have

$$\partial u/\partial x=2x,\ \partial u/\partial y=2y,\ \partial u/\partial z=2z,\ \partial v/\partial x=-2y,\ \partial v/\partial y=-2x,\ \partial v/\partial z=2z \quad ....(5)$$

Using (5), (4) reduces to $(\partial\phi/\partial u)(2x+2pz)+(\partial\phi/\partial v)(-2y+2pz)=0$

or $(x+pz)(\partial\phi/\partial u)=(y-pz)(\partial\phi/\partial v)$. ....(6)

Again, differentiating (3) partially w.r.t. $'y'$, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial y}\right)=0$$

or $(\partial\phi/\partial u)(2y+2qz)+(\partial\phi/\partial v)(-2x+2qz)=0$, by (5)

or $(y+qz)(\partial\phi/\partial u)=(x-qz)(\partial\phi/\partial v)$. ....(7)

Dividing (6) by (7), $(x+pz)/(y+qz)=(y-pz)/(x-qz)$

or $pz(y+x)-qz(y+x)=y^2-x^2$ or $(p-q)z=y-x$.

Solution.9 Given $z=e^y f(x+y)$ ....(1)

Differentiating (1) partially w.r.t. x and y, we get

$\partial z/\partial x=e^y f'(x+y)$ and $\partial z/\partial y=e^y f(x+y)+e^y f'(x+y)$ ....(2)

From (1) and (2), we have $\partial z/\partial y=z+\partial z/\partial x$

Solution.10 $(x-a)/(z-c)=u$ and $(y-b)/(z-c)=v$ ....(1)

Then, the equation of the given cone becomes $f(u,v)=0$ ....(2)

Differentiating (2) partially with respect to $'x'$, we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=0 \text{ or } \frac{\partial f}{\partial u}\left(\frac{1-0}{z-c}-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right)=0, \text{ using (1)}$$

$$\text{or } \frac{\partial f}{\partial u}\left(\frac{1}{z-c}-p\frac{x-a}{(z-c)^2}\right)-\frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right)=0, \text{ where } p=\frac{\partial z}{\partial x} \qquad ....(3)$$

Differentiating (2) partially with respect to $'y'$, we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0 \text{ or } \frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{1-0}{z-c}-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right)=0, \text{ using (1)}$$

$$\text{or } -\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z-c}-q\frac{y-b}{(z-c)^2}\right)=0, \text{ where } q=\frac{\partial z}{\partial y} \qquad ....(4)$$

Eliminating $\partial f/\partial u$ and $\partial f/\partial v$ from (3) and (4), we have

$$\begin{vmatrix} \frac{1}{z-c}-p\frac{x-a}{(z-c)^2} & -p\frac{y-b}{(z-c)^2} \\ -q\frac{x-a}{(z-c)^2} & \frac{1}{z-c}-q\frac{y-b}{(z-c)^2} \end{vmatrix}=0 \Rightarrow \begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix}=0$$

$$\Rightarrow \{z-c-p(x-a)\}\{z-c-q(y-b)\}-pq(x-a)(y-b)=0$$

$$\Rightarrow (z-c)^2-p(x-a)(z-c)-q(y-b)(z-c)=0 \text{ or } (x-a)p+(y-b)q=z-c.$$

which in the required partial differential equation of the given cone.

PREVIOUS YEARS QUESTIONS ANALYSIS

Q1. Show that if f and g are arbitrary function of their respective arguments, then $u=f(x-kt+iay)+g(x-kt-iay)$, is a solution of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2}$, where $\alpha^2=1-\frac{k^2}{C^2}$.

[5a UPSC CSE 2021] **Eliminate f and g and get the PDE. Take help from Ex 12**

Q 1.1. By eliminating the arbitrary functions f and g from $z=f(x^2-y)+g(x^2+y)$, form partial differential equation. **Refer example 18 page no. 20** **[5a UPSC CSE 2023]**

Q 2. It is given that the equation of any cone with vertex at (a,b,c) is $f\left(\frac{x-a}{z-a},\frac{y-b}{z-c}\right)=0$. Find the differential equation of the cone. **[5a UPSC CSE 2022, IFoS 2022]**

Refer Solution 10 page no. 20 assignment

Q 3. Obtain the partial differential equation by eliminating arbitrary function f from the equation $f\left(x+y+z,x^2+y^2+z^2\right)=0$. **Refer example 4 page no. 10** **[5a UPSC CSE 2021]**

Q 4. Find the partial differential equation of the family of all tangent planes to the ellipsoid: $x^2+4y^2+4z^2=4$, which are not perpendicular to the *xy*-plane. **[5a UPSC CSE 2018]**

Hint: family of all tangent planes **to** $ax^2+by^2+cz^2=1$ at the point (α,β,γ) is given by

$a\alpha x+b\beta y+c\gamma z=1$. So for given ellipsoid: $a=1/4, b=4/4, c=4/4$; family of all tangent planes; Passing through (α,β,γ) is given by

$\frac{\alpha x}{4}+\frac{\beta y}{1}+\frac{\gamma z}{1}=1 \Rightarrow \alpha x+4\beta y+4\gamma z=4$ ....(1) Here α,β,γ are arbitrary constants.

Also it is given that these are not perpendicular to xy plane i.e. plane $z=0 \Rightarrow 0.x+0.y+1.z=0$.

So the product $\alpha.0+4\beta.0+4\gamma.1 \neq 0 \Rightarrow \gamma \neq 0$...(2)

So, we have to find PDE for $z=\frac{4-(\alpha x+4\beta y)}{4\gamma};\gamma \neq 0$. Here α,β,γ are arbitrary constants.

Q5. Find the partial differential equation of all planes which are at a constant distance *a* from the origin. **[(5d) 2018 IFoS]**

Hint: Equation representing such planes : $lx+my+nz=a$; where l,m,n are arbitrary constants.

Refer Assignment Solution 4 on page no. 24

Q6. Obtain the partial differential equation by eliminating arbitrary function *f* from the equation $f\left(x+y+z,x^2+y^2+z^2\right)=0$. **[5a UPSC CSE 2021]** **Refer example 4 page no. 10**

Q7. Form a partial differential equation by eliminating the arbitrary functions $f(x)$ and $g(y)$ from $z=yf(x)+xg(y)$ and specify its nature (elliptic, hyperbolic or parabolic in the region $x>0, y>0$. **[5a UPSC CSE 2020]** **Refer example 3 page no. 13**

Q8. Construct a partial differential equitation of all surfaces of revolution having the *z*-axis as the axis of rotation. **[(5a) 2020 IFoS]** **Refer example 2 page no. 13**

Q9. Form a partial differential equation of the family of surfaces given by the following expression: $\psi\left(x^2+y^2+2z^2, y^2-2zx\right)=0$. **[1a UPSC CSE 2019] Refer example 4 page no. 10**

Q10. Form the partial differential equitation by eliminating arbitrary functions φ and ψ from the relation $z=\varphi\left(x^2-y\right)+\psi\left(x^2+y\right)$. **[(5a) 2017 IFoS] Refer example 18 page no. 20**

Q11. Obtain the partial differential equation governing the equations

$\phi(u,v)=0,\ u=xyz,\ v=x+y+z$. **[(5a) 2016 IFoS] Refer example 4 page no. 10**

Q12. Form a partial differential equation by eliminating the arbitrary functions *f* and *g* from $z=yf(x)+xg(y)$. **[5a UPSC CSE 2013] Refer example 3 page no. 13**

Q13. Eliminate the arbitrary function *f* from the given equation $f\left(x^2+y^2+z^2, x+y+z\right)=0$.
[(5b) 2013 IFoS] Refer example 4 page no. 10

FIRST ORDER LINEAR PDEs

Lagrange's method

An equation of the form $\boxed{P(x,y,z)p+Q(x,y,z)q=R(x,y,z)}$ is known as Lagrange's equation. Also it's written as $\boxed{Pp+Qq=R}$ ...(1)

Base behind Lagrange's method

If $u(x, y, z) = c_1$, $v(x, y, z) = c_2$ are two independent solutions of systems of differential equations $\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$: where c_1 & c_2 are arbitrary constant.

Then $\phi(u, v) = 0$ is a solution of $Pp+Qq=R$

Where ϕ is an arbitrary function(Proof can be easily done as category II of prev. Chapter Not required; eliminating arbitrary function ϕ.

Lagrange's method: To solve a PDE of the form $Pp+Qq=R$

Step (1): Write system of differential equations as

$\frac{dx}{p}=\frac{dy}{Q}=\frac{dz}{R}$; by using *P, Q, R* form given PDE...(2)

Step (2): Find two linearly independent solutions by solving differential equations out of above system (in step (i)).

E.g. Let's say by solving $\frac{dx}{P}=\frac{dx}{Q}$, we $u(x,y,z)=c_1$

By solving $\frac{dx}{P}=\frac{dz}{R}$, we get $v(x,y,z)=c_2$

$\therefore$ Required solution of given PDE is $\phi(u, v) = 0$; where *u* & *v* are from step (ii)

or $v = \phi(u)$ or $u = \phi(v)$ where $\phi$is arbitrary *f*unction.

Type I Problem

Where by taking any two fractions out of system of differential equations, solving we get,

$u(x,y,z)=c_1$ & $v(x,y,z)=c_2$

Type II Problem Let's say if we $u(x,y,z)=c_1$ by taking any two fractions out of system of differential equations, But we're not getting directly $v(x,y,z)=c_2$ by other combination of fractions.

In this case ; we use $u(x,y,z)=c_1$ to find $v(x,y,z)=c_2$

Type III Problem: Choosing Multipliers

By algebra ; each fraction of (2) can be equal to $\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$

Where P_1, Q_1, R_1 are called multipliers i.e. $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$

Note: For multipliers selection

We choose P_1, Q_1, R_1 in such a why the $P_1P+Q_1Q+R_1R=0$

Because in this case : We have ; $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{P_1dx+Q_1dy+R_1dz}{0}$

Now let's say if we take first & last fraction

We have , $\frac{dx}{P}=\frac{P_1dx+Q_1dy+R_1dz}{0} \Rightarrow P_1dx+Q_1dy+R_1dz=0$ ....(3)

Now suppose if P_1 is a function of *x* alone Q_1 is function of *y*- alone , R_1 is function of z- alone; means we integrate (3) and easily get the solution.

Ex 1. Solve $\left(\frac{y^2z}{x}\right)p+xzq=y^2$.

Solution. On comparing given PDE with $Pp+Qq=R$ we get $P=\frac{y^2z}{x}$, $Q=zx, R=y^2$

Lagrange's system of differential equations is , $\frac{dx}{\left(\frac{y^2z}{x}\right)}=\frac{dy}{xz}=\frac{dz}{y^2}$ .... (1)

By taking first two fractions of (1); $\frac{dx}{\left(\frac{y^2z}{x}\right)}=\frac{dy}{xz} \Rightarrow \frac{xdx}{y^2}=dy \Rightarrow x^2dx-y^2dy=0$

On integrating ; $\frac{x^3}{3}-\frac{y^3}{3}=c_1$; c_1 is an integration constant.

We get u (*x,y,z*) = c_1 where $u(x,y,z) = \frac{x^3 - y^3}{3}$

By taking 1st & 3rd fraction of (1); $\frac{xdx}{z}$ = *dz*

xdx = *zdz*; On integrating $\frac{x^2}{2} - \frac{z^2}{2} = c_2 \Rightarrow \frac{x^2 - z^2}{2} = c_2$. So, $v(x,y,z) = \frac{x^2 - z^2}{2}$

Required solution of given PDE is $\phi\left(\frac{x^3 - y^3}{3}, \frac{x^2 - z^2}{2}\right) = 0$

The linear partial differential equation with n independent variables and its

solution. Let $x_1, x_2 \ldots, x_n$ be the *n* independent variables and let $p_1 = \partial z/\partial x_1$, $p_2 = \partial z/\partial x_2, \ldots, p_n = \partial z/\partial x_n$, where *z* is the dependent variable. Consider the general linear partial differential equation with *n* independent variables

$$P_1 p_1 + P_2 p_2 + \ldots + P_n p_n = R \qquad \ldots(1)$$

where $P_1, P_2, .., P_n$ are functions of $x_1, x_2 \ldots, xn$.

Let $u_1 = c_1, u_2 = c_2, \ldots, un = cn$ be any *n* independent integrals of the auxiliary equations

$$(dx_1)/P_1 = (dx_2)/P_2 = \ldots = (dx_n)/Pn$$

Then the general solution of (1) is given by $\varphi(u_1, u_2, \ldots un) = 0$.

Note that the above procedure is generalization of Lagrange's method.

Integral surfaces passing through a given curve. Till now, we discussed about general integral of *Pp* + *Qq* = *R*. We'll now learn two Categories of problems of using such a general solution for getting the integral surface which passes through a given curve.

Category-I. Let $Pp + Qq = R \qquad \ldots(1)$

be the given equation. Let its auxiliary equations give the following two independent solutions

$$u(x,y,z) = c_1 \;\; and \;\; v(x,y,z) = c_2. \qquad \ldots(2)$$

Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by $x = x(t),\ y = \ (t), z = z(t),$ ...(3) where *t* is a parameter.

Then (2) may be expressed as

$$u[x(t), y(t), z(t)] = c_1 \quad and \quad v[x(t), y(t), z(t)] = c_2. \quad \ldots(4)$$

We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

Category II. Let $Pp + Qq = R$ ...(1)

be the given equation. Let is Lagrange's auxiliary equations give the following two independent integrals

$$u(x, y, z) = c_1 \text{ and } v(x, y, z) = c_2. \quad ...(2)$$

Suppose we wish to obtain the integral surface passing though the curve which is determined by the following two equations

$$\phi(x, y, z) = 0 \text{ and } \psi(x, y, z) = 0. \quad ...(3)$$

We eliminate x, y, z from four equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

Let $f(x, y, z) = C$ ...(1)

represents a system of surfaces where C is parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

[from vector calculus gradient]

Let the surface, $z = \varphi(x, y)$ ...(2)

cuts each surface of (1) at right angles.

Then the normal at (x, y, z) to (2) has direction ratios

$\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1.

Since normals at (x, y, z) to (1) and (2) are at right angles, we have

$$p(\partial f / \partial x) + q(\partial f / \partial y) - (\partial f / \partial z) = 0 \Rightarrow p(\partial f / \partial x) + q(\partial f / \partial y) = \partial f / \partial z \quad ...(3)$$

which is of the form $Pp + Qq = R$.

Conversely, we easily verify that any solution of (3) is orthogonal to every surface of (1).

Geometrical description of the solutions of $Pp + Qq = R$ **and of the system of equations** $dx/P = dy/Q = dz/R$ **and to establish relationship between the two.**

Proof. Consider $Pp + Qq = R$ ...(1)

and $(dx)/P = (dy)/Q = (dz)/R$ ...(2)

where P, Q and R are functions of x.

Let $z = \varphi(x,y)$ ...(3)

represent the solution of (1). Then (3) represents a surface whose normal at any point (x, y, z) has direction ratios $\partial z/\partial x, \partial z/\partial y, -1$ i.e., $p, q, -1$. Also we know that the simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we have

$$Pp + Qq + R(-1) = 0 \qquad ...(4)$$

showing that the normal to surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence the member must touch the surface at that point. Since this holds for each point on (3), we conclude that the curves (2) lie completely on the surface (3) whose differential equation is (1).

Another geometrical interpretation of Lagrange's equation $Pp+Qq=R$.

To show that the surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad ...(1)$$

are orthogonal to the system of the surfaces whose equation satisfies

$$Pdx + Qdy + Rdz = 0 \qquad ...(2)$$

Again from previous discussion, the curves of (1) lie completely on the surface represented by

$$Pp + Qq = R \qquad ...(3)$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

Type-1 Problems

Ex. 1. Solve $p\tan x + q\tan y = \tan z$.

Solution. Given $(\tan x)p + (\tan y)q = \tan z$. ....(1)

The Lagrange's auxiliary equations for (1) are $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$. ....(2)

Taking the first two fractions of (2), $\cot x\,dx - \cot y\,dy = 0$.

Integrating, $\log\sin x - \log\sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1$ ....(3)

Taking the last two fractions of (2), $\cot y\,dy - \cot z\,dz = 0$.

Integrating, $\log\sin y - \log\sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2$. ....(4)

From (3) and (4), the required general solution is

$\sin x/\sin y = \phi(\sin y/\sin z)$, ϕ being an arbitrary function.

Ex. 2. Solve $y^2 p - xyq = x(z-2y)$.

Solution. Here Lagrange's auxiliary equations are $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$. ....(1)

Taking the first two fractions of (1) and re–writing, we get

$2xdx + 2ydy = 0$ so that $x^2 + y^2 = c_1$. ....(2)

Now, taking the last two fractions of (1) and re–writing, we get

$\frac{dz}{dy} = -\frac{z-2y}{y}$ or $\frac{dz}{dy} + \frac{1}{y}z = 2$ ....(3)

which is linear in z and y. Its I.F. = $e^{\int (1/y)dy} = e^{\log y} = y$. Hence solution of (3) is

$z \cdot y = \int 2y\,dy + c_2$ or $zy - y^2 = c_2$. ....(4)

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired solution, where ϕ is an arbitrary function.

Ex.3. Solve $p + 3q = 5z + \tan(y - 3x)$.

Solution. Given $p + 3q = 5z + \tan(y - 3x)$. ....(1)

The Lagrange's subsidiary equations for (1) are $\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}$. ....(2)

Taking the first two fractions, $dy-3dx=0$. ....(3)

Integrating (3), $y-3x=c_1, c_1$ being an arbitrary constant. ....(4)

Using (4), from (2) we get $\frac{dx}{1}=\frac{dz}{5z+\tan c_1}$. ....(5)

Integrating (5), $x-(1/5)\times\log(5z+\tan c_1)=(1/5)\times c_2, c_2$ being an arbitrary constant.

or $5x-\log[5z+\tan(y-3x)]=c_2$, using (4) ....(6)

From (4) and (6), the required general integral is

$5x-\log[5z+\tan(y-3x)]=\phi(y-3x)$, where ϕ is an arbitrary function.

Ex. 4. Solve $z(z^2+xy)(px-qy)=x^4$.

Solution. Given $xz(z^2+xy)p-yz(z^2+xy)q=x^4$. ....(1)

The Lagrange's subsidiary equations for (1) are $\frac{dx}{xz(z^2+xy)}=\frac{dy}{-yz(z^2+xy)}=\frac{dz}{x^4}$ .....(2)

Cancelling $z(z^2+xy)$, the first two fractions give

$(1/x)dx=-(1/y)dy$ or $(1/x)dx+(1/y)dy=0$. ....(3)

Integrating (3), $\log x+\log y=\log c_1$ or $xy=c_1$. ....(4)

Using (4), from (2) we get $\frac{dx}{xz(z^2+c_1)}=\frac{dz}{x^4}$

or $x^3dx=z(z^2+c_1)dz$ or $x^3dz-(z^3+c_1z)dz=0$. ....(5)

Integrating (5), $x^4/4-z^4/4-(c_1z^2)/2=c_2/4$ or $x^4-z^4-2c_1z^2=c_2$

or $x^4-z^4-2xy\,z^2=c_2$, using (4) ....(6)

From (4) and (6), the required general integral is

$\phi(xy, x^4-z^4-2xy\,z^2)=0$, ϕ being an arbitrary function.

Ex. 5. Solve $xyp + y^2 q = zxy - 2x^2$.

Solution. Given $xyp + y^2 q = zxy - 2x^2$. ....(1)

The Lagrange's subsidiary equations for (1) are $\frac{dx}{dy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$. ....(2)

Taking the first two fractions of (2), we have

$(dx)/xy = (dy)/y^2$ or $(1/x)dx - (1/y)dy = 0$ ....(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ....(4)

From (4), $x = c_1 y$. Hence from second and third fractions of (2), we get

$\frac{dy}{y^2} = \frac{dz}{c_1 zy^2 - 2c_1^2 y^2}$ or $c_1 dy - \frac{dz}{z - 2c_1^2} = 0$. ....(5)

Integrating (5), $c_1 y - \log\left(z - 2c_1^2\right) = c_2$ or $x - \log\left[z - 2\left(x^2/y^2\right)\right] = c_2$, using (4) ....(6)

From (4) and (6), the required general solution is

$x - \log\left[z - 2\left(x^2/y^2\right)\right] = \phi(x/y)$, ϕ being an arbitrary function.

Type-2 Problems

Ex.1. Solve $\{(b-c)/a\} yzp + \{(c-a)/b\} zxq = \{(a-b)/c\} xy$.

Solution. Given $\{(b-c)/a\} yzp + \{(c-a)/b\} zxq = \{(a-b)/c\} xy$. ....(1)

The Lagrange's subsidiary equations of (1) are $\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$ .....(2)

• Choosing x, y, z as multipliers, each fraction for (2)

$$= \frac{a\,xdx + by\,dy + cz\,dz}{xyz\left[(b-c)+(c-a)+(a-b)\right]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

$\therefore ax\,dx + by\,dy + cz\,dz = 0$ or $2axdx + 2bydy + 2czdz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1, c_1$ being an arbitrary constant. ....(3)

• Again, choosing ax, by, cz as multipliers, each fraction of (2)

$$= \frac{a^2xdx + b^2ydy + c^2zdz}{xyz\left[a(b-c) + b(c-a) + c(a-b)\right]} = \frac{a^2xdx + b^2ydy + c^2zdz}{0}$$

$\therefore\ a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2, c_2$ being an arbitrary constant. ....(4)

From (3) and (4), the required general solution is given by

$\phi\left(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2\right) = 0$, where ϕ is an arbitrary function.

Ex. 2. Solve $z(x+y)p + z(x-y)q = x^2 + y^2$.

Solution. Given $z(x+y)p + z(x-y)q = x^2 + y^2$. ....(1)

The Langrange's subsidiary equations for (1) are $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. ....(2)

• Choosing $x, -y, -z$, as multipliers, each fraction

$$= \frac{xdx - ydy - zdz}{xz(x+y) - yz(x-y) - z(x^2 - y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$\therefore\ x\,dx - y\,dy - z\,dz$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1, c_1$ being an arbitrary constant. ....(3)

• Again, choosing $y, x, -z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2 + y^2)} = \frac{y\,dx - x\,dy - z\,dz}{0}$$

$\therefore\ y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2zdz = 0$.

Integrating, $2xy - z^2 = c_2, c_2$ being an arbitrary constant. ....(4)

From (3) and (4), the required general solution is given by

$\phi\left(x^2 - y^2 - z^2, 2xy - z^2\right) = 0$, ϕ being an arbitrary function.

Ex. 3. Solve $(mz - ny)p + (nx - lz)q = ly - mx$. **[I.A.S. 1977]**

Solution. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}. \qquad ....(1)$$

• Choosing x, y, z as multipliers, each fraction of (1)

$$=\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{xdx+ydy+zdz}{0}$$

$\therefore\ xdx+ydy+zdz=0$ or $2xdx+2ydy+2zdz=0$

Integrating, $x^2+y^2+z^2=c_1, c_1$ being an arbitrary constant. ....(2)

• Again, choosing l, m, n as multipliers, each fraction of (1)

$$=\frac{ldx+mdy+ndz}{l(mx-ny)+m(nx-lz)+n(ly-mx)}=\frac{ldx+mdy+ndz}{0}.$$

$\therefore\ ldx+mdy+ndz=0$ so that $lx+my+nz=c_2$. ....(3)

From (2) and (3), the required general solution is given by

$\phi\left(x^2+y^2+z^2, lx+my+nz\right)=0,\ \phi$ being an arbitrary function.

Ex. 4. Solve $x\left(y^2-z^2\right)p-y\left(z^2+x^2\right)q=z\left(x^2+y^2\right)$.

Solution. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{-y\left(z^2+x^2\right)}=\frac{dz}{z\left(x^2+y^2\right)}. \qquad ....(1)$$

• Choosing x, y, z, as multipliers, each fraction of (1)

$$=\frac{xdx+ydy+zdz}{x^2\left(y^2-z^2\right)-y^2\left(z^2+x^2\right)+z^2\left(x^2+y^2\right)}=\frac{xdx+ydy+zdz}{0}$$

$\Rightarrow xdx+ydy+zdz=0$ so that $x^2+y^2+z^2=c_1$. ....(2)

• Choosing $1/x, -1/y, -1/z$ as multipliers, each fraction of (1)

$$=\frac{(1/x)dx-(1/y)dy-(1/z)dz}{y^2-z^2+z^2+x^2-\left(x^2+y^2\right)}=\frac{(1/x)dx-(1/y)dy-(1/z)dz}{0}$$

$\Rightarrow (1/x)dx-(1/y)dy-(1/z)dz=0$ so that $\log x-\log y-\log z=\log c_2$

$\Rightarrow \log\{x/(yz)\}=\log c_2 \Rightarrow x/yz=c_2$. ....(3)

$\therefore$ The required solution is $\phi\left(x^2+y^2+z^2, x/yz\right)=0$, ϕ being an arbitrary function.

Ex. 5. Solve $(y-zx)p+(x+yz)q=x^2+y^2$.

Solution. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y-zx}=\frac{dy}{x+yz}=\frac{dz}{x^2+y^2}. \qquad ....(1)$$

• Choosing $x, -y, z$ as multipliers, each fraction of (1)

$$=\frac{xdx-ydy+zdz}{x(y-zx)-y(x+yz)+z\left(x^2+y^2\right)}=\frac{xdx-ydy+zdz}{0}$$

$\Rightarrow 2xdx-2ydy+2zdz=0$ so that $x^2-y^2+z^2=c_1$. ....(2)

• Choosing $y, x, -1$ as multipliers, each fraction of (1)

$$=\frac{ydx+xdy-dz}{y(y-zx)+x(x+yz)-\left(x^2+y^2\right)}=\frac{d(xy)-dz}{0}\Rightarrow d(xy)-dz=0 \text{ so that } xy-z=c_2. \qquad ....(3)$$

$\therefore$ From (2) and (3) solution is $\phi\left(x^2-y^2+z^2, xy-z\right)=0$, ϕ being an arbitrary function.

Ex. 6. Solve $x\left(y^2+z\right)p-y\left(x^2+z\right)q=z\left(x^2-y^2\right)$. **[I.A.S. 2004]**

Solution. Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{z\left(x^2-y^2\right)}. \qquad ....(1)$$

• Choosing $1/x, 1/y, 1/z$ as multipliers, each fraction of (1)

$$=\frac{(1/x)dx+(1/y)dy+(1/z)dz}{y^2+z-\left(x^2+z\right)+x^2-y^2}=\frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow (1/x)dx+(1/y)dy+(1/z)dz=0$ so that $\log x+\log y+\log z=\log c_1$

$\Rightarrow \log(xyz)=\log c_1 \quad \Rightarrow xyz=c_1$. ....(2)

• Choosing $x, y, -1$ as multipliers, each fraction of (1)

$$=\frac{xdx+ydy-dz}{x^2\left(y^2+z\right)-y^2\left(x^2+z\right)-z\left(x^2-y^2\right)}=\frac{xdx+ydy-dz}{0}$$

$\Rightarrow x\,dx + y\,dy - z\,dz = 0$ so that $x^2 + y^2 - 2z = c_2$. ....(3)

$\therefore$ From (2) and (3), solution is $\phi\left(x^2 + y^2 - 2z, xyz\right) = 0$, ϕ is being an arbitrary function.

Ex. 7. Solve $(x+2z)q + (4zx - y)q = 2x^2 + y$.

Solution. Here Lagrange's auxiliary equations are $\dfrac{dx}{x+2z} = \dfrac{dy}{4zx-y} = \dfrac{dz}{2x^2+y}$. ....(1)

- Choosing $y, x, -2z$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - 2zdz}{y(x+2z) + x(4zx-y) - 2z(2x^2+y)} = \frac{d(xy) - 2zdz}{0}$$

$\Rightarrow d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ....(2)

- Choosing $2x, -1, -1$ as multipliers, each fraction of (1)

$$= \frac{2xdx - dy - dz}{2x(x+2z) - (4zx-y) - (2x^2+y)} = \frac{2xdx - dy - dz}{0}$$

$\Rightarrow 2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ....(3)

$\therefore$ From (2) and (3), solution is $\phi\left(xy - z^2, x^2 - y - z\right) = 0$, ϕ being an arbitrary function.

Ex. 8. Solve $\left(z^2 - 2yz - y^2\right)p + (xy + zx)q = xy - zx$

Solution. Here Lagrange's auxiliary equations for given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}. \quad ....(1)$$

- Taking the last two fractions of (1), we have

$(y-z)dy = (y+z)dz$ or $2ydy - 2zdz - 2(zdy + ydz) = 0$.

Integrating, $y^2 - z^2 - 2yz = c_1, c_1$ being an arbitrary constant. ....(2)

- Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x\left(z^2 - 2yz - y^2\right) + xy(y+z) + xz(y-z)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow 2xdx + 2ydy + 2zdz = 0$ so that $x^2 + y^2 + z^2 = c_2$. ....(3)

From (2) and (3), solution is $\phi\left(y^2 - z^2 - 2yz, x^2 + y^2 + z^2\right) = 0$, ϕ being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at $(0,0,0)$, i.e., origin.

Ex. 9. Solve $\left(y^3x - 2x^4\right)p + \left(2y^4 - x^3y\right)q = 9z\left(x^3 - y^3\right)$.

Solution. Here Lagrange's auxiliary equations for the given equation are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z\left(x^3 - y^3\right)}. \qquad ....(1)$$

• Taking first two fractions of (1), we have $\left(2y^4 - x^3y\right)dx = \left(y^3x - 2x^4\right)dy$

Dividing both sides by x^3y^3 gives $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$

or $\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$ or $d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$.

Integrating, $\left(y/x^2\right) + \left(x/y^2\right) = c_1, c_1$ being an arbitrary constant. ....(2)

• Choosing $1/x, 1/y, 1/3z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{\left(y^3 - 2x^3\right) + \left(2y^3 - x^3\right) + 3\left(x^3 - y^3\right)} = \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/3)dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_2$

$\Rightarrow \log\left(xy\,z^{1/3}\right) = \log c_2 \Rightarrow xyz^{1/3} = c_2$. ....(3)

From (2) and (3) solution is $\phi\left(xyz^{1/3}, y/x^2 + x/y^2\right) = 0$, ϕ being an arbitrary function.

Ex. 10. Solve $x^2p + y^2q = nxy$.

Solution. Here Lagrange's auxiliary equations are $(dx)/x^2 = (dy)/y^2 = (dz)/nxy$ ....(1)

• Taking the first two fractions of (1), we get $x^{-2}dx - y^{-2}dy = 0$.

Integrating, $-1/x + 1/y = -c_1$ so that $(y - x)/xy = c_1$. ....(2)

• Choosing $1/x, -1/y, c_1/n$ as multipliers, each fraction of (2)

$$= \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + c_1 xy} = \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + y - x}, \text{ by (2)}$$

$$= \frac{(1/x)dx + (1/y)dy + (c_1/n)dz}{0} \text{ so that } \frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0.$$

Integrating, $\log x - \log y + (c_1/n)z = (c_1/n)c_2, c_2$ being an arbitrary constant.

or $z - (n/c_1)(\log y - \log x) = c_2$ or $z - (n/c_1)\log(y/x) = c_2$

or $z - \frac{nxy}{y-x}\log\frac{y}{x} = c_2$, using (2). ....(3)

From (2) and (3), the required general solution is

$\phi\left(\frac{y-x}{xy}, z - \frac{nxy}{y-x}\log\frac{y}{x}\right) = 0$, ϕ being an arbitrary function.

Type-3 Problems

Ex. 1. Solve $(y+z)p + (z+x)q = x+y$.

Solution. Here the Lagrange's auxiliary equations are $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$. ....(1)

• Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \frac{dx - dy}{(y+z)-(z+x)} = \frac{d(x-y)}{-(x-y)}$. ....(2)

• Again, choosing 0, 1, –1 as multipliers, each fraction of (1) $= \frac{dy - dz}{(z+x)-(x+y)} = \frac{d(y-z)}{-(y-z)}$ ....(3)

• Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)}. \quad ....(4)$$

(2), (3) and (4) $\Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}$. ....(5)

Taking the first two fractions of (5), $\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$.

Integrating, $\log(x-y) = \log(y-z) + \log c_1, c_1$ being an arbitrary constant.

or $\log\{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$. ....(6)

Taking the first and the third fractions of (5), $2\dfrac{d(x-y)}{(x-y)} + \dfrac{d(x+y+z)}{x+y+z} = 0$

Integrating, $2\log(x-y) + \log(x+y+z) = \log c_2$ or $(x-y)^2 + (x+y+z) = c_2$. ....(7)

From (6) and (7), the required general solution is

$\phi\left[(x-y)^2(x+y+z), (x-y)/(y-z)\right] = 0$, ϕ being an arbitrary function.

Ex. 2. Solve $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$.

Solution. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}. \qquad ....(1)$$

• Taking the first two fractions of (1), $x^2 dx = -y^2 dy$ or $3x^2 dx + 3y^2 dy = 0$.

Integrating, $x^3 + y^3 = c_1, c_1$ being an arbitrary as constant. ....(2)

• Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \qquad ....(3)$$

• Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \text{ or } \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x-y) - \log z = \log c_2$ or $(x-y)/z = c_2$. ....(4)

From (3) and (4), solution is $\phi(x^3+y^3, (x-y)/z) = 0$, ϕ being an arbitrary function.

Ex. 3. Solve $(x^2-y^2-z^2)p + 2xyq = 2xz$ or $(y^2+z^2-x^2)p - 2xyq = -2xz$.

[I.A.S. 1973; P.C.S. (U.P.) 1991]

Solution. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \qquad ....(1)$$

• Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1$. ....(2)

• Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x\left(x^2 + y^2 + z^2\right)} \quad ....(3)$$

• Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx + y\,dy + z\,dz}{-x\left(x^2 + y^2 + z^2\right)} = \frac{dz}{-2xz} \text{ or } \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log\left(x^2 + y^2 + z^2\right) - \log z = \log c_2$ or $\left(x^2 + y^2 + z^2\right)/z = c_2$. ....(4)

From (2) and (4) solution is $\phi\left(y/z, \left(x^2 + y^2 + z^2\right)/z\right) = 0$, ϕ being an arbitrary function.

Ex. 4. Solve $(1+y)p + (1+x)q = z$.

Solution. Here the Lagrange's auxiliary equations are $\dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$. ....(1)

• Taking the first two fractions of (1), we have

$(1+x)dx = (1+y)dy$ or $2(1+x)dx - 2(1+y)dy = 0$

Integrating, $(1+x)^2 - (1+y)^2 = c_1, c_1$ being an arbitrary constant. ....(2)

• Taking 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx + dy}{1+y+1+x} = \dfrac{d(2+x+y)}{2+x+y}$. ....(3)

Combining the last fraction of (1) with fraction (3), we get

$$\frac{d(2+x+y)}{2+x+y} = \frac{dz}{z} \text{ or } \frac{d(2+x+y)}{2+x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log(2+x+y) - \log z = \log c_2$ or $(2+x+y)/z = c_2$ ....(4)

From (2) and (4), the required general solution is given by

$\phi\left[(1+x)^2 - \left(1+y^2\right), (2+x+y)/z\right] = 0$, ϕ being an arbitrary function.

Ex. 5. Find the general integral of $xzp + yzq = xy$.

Solution. Here the Lagrange's auxiliary equations are $(dx)/xz = (dy)/yz = (dz)/xy$ ....(1)

• From the first two fractions of (1), $(1/x)dx = (1/y)dy$.

Integrating, $\log x = \log y + \log c_1$ or $x/y = c_1$. ....(2)

• Choosing $1/x, 1/y, 0$ as multipliers, each fraction of (1) $= \dfrac{(1/x)dx + (1/y)dy}{(1/x)xz + (1/y)yz} = \dfrac{ydx + xdy}{2xyz}$ ....(3)

Combining the last fraction of (1) with fraction (3), we have

$\dfrac{ydx + xdy}{2xyz} = \dfrac{dz}{xy}$ or $ydx + xdy = 2zdz$ or $d(xy) = 2zdz$ or $d(xy) - 2zdz = 0$

Integrating, $xy - z^2 = c_2, c_2$ being an arbitrary constant. ....(4)

From (2) and (4) solution is $\phi(x/y, xy - z^2) = 0$, ϕ being an arbitrary function.

Ex. 6. Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Solution. Here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$. ....(1)

• Choosing 1, –1, 0 and 0, 1, –1 as multipliers in turn, each fraction of (1)

$$= \frac{dx - dy}{x^2 - y^2 + z(x - y)} = \frac{dy - dz}{(y - z)(y + z + x)}$$

so that $\dfrac{dx - dy}{(x - y)(x + y + z)} = \dfrac{dy - dz}{(y - z)(y + z + x)}$ or $\dfrac{d(x - y)}{x - y} - \dfrac{d(y - z)}{y - z} = 0$.

Integrating, $\log(x - y) - \log(y - z) = \log c_2$ or $(x - y)/(y - z) = c_1$. ....(2)

• Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}. \quad ....(3)$$

• Again, choosing 1, 1, 1 as multipliers, each fraction of (1) $= \dfrac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$ ....(4)

From (3) and (4), $\dfrac{xdx+ydy+zdz}{x+y+z}=dx+dy+dz$

or $2(x+y+z)d(x+y+z)-(2xdx+2ydy+2zdz)=0$.

Integrating, $(x+y+z)^2-(x^2+y^2+z^2)=2c_2$

or $(x^2+y^2+z^2+2xy+2yz+2zx)-(x^2+y^2+z^2)=2c_2$

or $xy+yz+zx=c_2, c_2$ being an arbitrary constant.

From (2) and (5), the required general solution is given by

$\phi[xy+yz+zx,(x-y)/(y-z)]=0$, ϕ being an arbitrary function.

Ex. 7. Solve $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$.

Solution. Here Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{z(x-y)} \quad ....(1)$$

• Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$=\frac{dx-dy}{(x^2-y^2-yz)-(x^2-y^2-zx)}=\frac{dx-dy}{z(x-y)} \quad .....(2)$$

• Choosing $x,-y,0$ as multipliers each fraction of (1)

$$=\frac{xdx-ydy}{x(x^2-y^2-yz)-y(x^2-y^2-zx)}=\frac{xdx-ydy}{(x-y)(x^2-y^2)}. \quad ....(3)$$

From (1), (2), (3) we have

$$\frac{dz}{z(x-y)}=\frac{dx-dy}{z(x-y)}=\frac{xdx-ydy}{(x-y)(x^2-y^2)} \text{ or } \frac{dz}{z}=\frac{dx-dy}{z}=\frac{2xdx-2ydy}{2(x^2-y^2)} \quad ....(4)$$

• Taking the first two fractions of (4), we have

$dz=dx-dy$ so that $z-x+y=c_1$ ....(5)

• Again , taking the first and third fractions of (4),

$d(x^2-y^2)/(x^2-y^2)-(2/z)dz=0$

Integrating, $\log\left(x^2-y^2\right)-2\log z=c_2$ or $\left(x^2-y^2\right)/z^2=c_2$. ....(6)

From (5) and (6), solution is $\phi\left(z-x+y,\left(x^2+y^2\right)/z^2\right)=0$, ϕ being an arbitrary function.

Ex. 8. Solve $\left(x^2+y^2+yz\right)p+\left(x^2+y^2-xz\right)q=z\left(x+y\right)$.

Solution. Here the auxiliary equations are $\dfrac{dx}{x^2+y^2+yz}=\dfrac{dy}{x^2+y^2-xz}=\dfrac{dz}{z\left(x+y\right)}$. ....(1)

• Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$=\frac{dx-dy}{\left(x^2+y^2+yz\right)-\left(x^2+y^2-xz\right)}=\frac{dx-dy}{z\left(x+y\right)}. \quad ....(2)$$

• Choosing $x, y, 0$ as multipliers, each fraction of (1)

$$=\frac{xdx+ydy}{x\left(x^2+y^2+yz\right)+y\left(x^2+y^2-xz\right)}=\frac{xdx+ydy}{\left(x+y\right)\left(x^2+y^2\right)}. \quad ....(3)$$

From (1), (2) and (3), we have

$$\frac{dz}{z\left(x+y\right)}=\frac{dx-dy}{z\left(x+y\right)}=\frac{xdx+ydy}{\left(x+y\right)\left(x^2+y^2\right)} \text{ or } \frac{dz}{z}=\frac{dx-dy}{z}=\frac{xdx+ydy}{x^2+y^2}. \quad ....(4)$$

• Taking the first two fractions of (4), we have

$dz=dx-dy$ or $dz-dx+dy=0$.

Integrating, $z-x+y=c_1, c_1$ being an arbitrary constant. ....(5)

• Taking the first and third fractions of (4), we have

$$\frac{2xdx+2ydy}{x^2+y^2}=2\frac{dz}{z} \text{ or } \frac{d\left(x^2+y^2\right)}{x^2+y^2}-2\frac{dz}{z}=0.$$

Integrating, $\log\left(x^2+y^2\right)-2\log z=\log c_2$ or $\left(x^2+y^2\right)/z^2=c_2$. ....(6)

From (5) and (6), solution is $\phi\left(z-x+y,\left(x^2+y^2\right)/z^2\right)=0$, ϕ being an arbitrary function.

Ex. 9. Solve $\cos\left(x+y\right)p+\sin\left(x+y\right)q=z$.

Solution. Here the Lagrange's auxiliary equations are $\dfrac{dx}{\cos\left(x+y\right)}=\dfrac{dy}{\sin\left(x+y\right)}=\dfrac{dz}{z}$. ....(1)

• Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$=\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \quad ....(2)$$

• Choosing 1, –1, 0 as multipliers, each fraction of (1) $=\dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ....(3)

From (1), (2) and (3), $\dfrac{dz}{z}=\dfrac{d(x+y)}{\cos(x+y)+\sin(x+y)}=\dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ....(4)

• Taking the first two fractions of (4), $\dfrac{dz}{z}=\dfrac{d(x+y)}{\cos(x+y)+\sin(x+y)}$. ....(5)

Putting $x+y=t$ so that $d(x+y)=dt$, (5) reduces to

$$\frac{dz}{z}=\frac{dt}{\cos t+\sin t}=\frac{dt}{\sqrt{2}\left\{\left(1/\sqrt{2}\right)\cos t+\left(1/\sqrt{2}\right)\sin t\right\}}=\frac{dt}{\sqrt{2}\left\{\sin(\pi/4)\cos t+\cos(\pi/4)\sin t\right\}}=\frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, $\left(\sqrt{2}/z\right)dz=\operatorname{cosec}(t+\pi/4)\,dt$.

Integrating, $\sqrt{2}\log z=\log\tan\dfrac{1}{2}\left(t+\dfrac{\pi}{4}\right)+\log c_1$, or $z^{\sqrt{2}}=c_1\tan\left(\dfrac{t}{2}+\dfrac{\pi}{8}\right)$

or $z^{\sqrt{2}}\cot\left(\dfrac{x+y}{2}+\dfrac{\pi}{8}\right)=c_1$ as $t=x+y$ ....(6)

• Taking the last two fraction of (4), $dx-dy=\dfrac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)$. ....(7)

On R.H.S. of (7), putting $x+y=t$, so that $d(x+y)=dt$, (7) reduces to

$dx-dy=\dfrac{\cos t-\sin t}{\cos t+\sin t}dt$ so that $x-y=\log(\sin t+\cos t)-\log c_2$

or $(\sin t+\cos t)/c_2=e^{x-y}$ or $e^{-(x-y)}(\sin t+\cos t)=c_2$

or $e^{y-x}\left[\sin(x+y)+\cos(x+y)\right]=c_2$, as $t=x+y$. ....(8)

From (6) and (8), the required general solution is

$\phi\left[z^{\sqrt{2}}\cot\left(\dfrac{x+y}{2}+\dfrac{\pi}{8}\right), e^{y-x}\left\{\sin(x+y)+\cos(x+y)\right\}\right]=0$, where ϕ is an arbitrary function.

Ex. 10. Solve $\cos(x+y)p+\sin(x+y)q=z+(1/z)$.

Ans. $\phi\left[\left(z^2+1\right)^{1/\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right), e^{y-x}\{\cos(x+y)+\sin(x+y)\}\right]=0$

Ex. 11. Solve $xp+yq=z-a\sqrt{\left(x^2+y^2+z^2\right)}$.

Solution.

Here the Lagrange's auxiliary equations are $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{\left(x^2+y^2+z^2\right)}}$. ....(1)

• Taking the first two fractions of (1), we have

$(1/x)dx=(1/y)dy$ or $(1/x)dx-(1/y)dy=0$.

Integrating, $\log x-\log y=\log c_1$ or $x/y=c_1$. ....(2)

• Choosing x, y, z as multipliers, each fraction of (1) $=\frac{xdx+ydy+zdz}{x^2+y^2+z^2-az\sqrt{\left(x^2+y^2+z^2\right)}}$ ....(3)

• Combining first and third fractions of (1) with fraction (3), we get

$$\frac{dx}{x}=\frac{dz}{z-a\sqrt{\left(x^2+y^2+z^2\right)}}=\frac{xdx+ydy+zdz}{x^2+y^2+z^2-az\sqrt{\left(x^2+y^2+z^2\right)}}. \quad ....(4)$$

• Putting $x^2+y^2+z^2=t^2$ so that $xdx+ydy+zdz=tdt$, (4) gives

$$\frac{dx}{x}=\frac{dz}{z-at}=\frac{tdt}{t^2-azt} \text{ or } \frac{dx}{x}=\frac{dz}{z-at}=\frac{dt}{t-az}. \quad ....(5)$$

• Choosing 0, 1, 1 as multipliers, each fraction of (5) $\frac{dz+dt}{(z+t)-a(t+z)}=\frac{d(z+t)}{(1-a)(z+t)}$. ....(6)

• Combining the first fraction of (5) with fraction (6), we get

$$\frac{dx}{x}=\frac{d(z+t)}{(1-a)(z+t)} \text{ or } (1-a)\frac{dx}{x}-\frac{d(z+t)}{z+t}=0.$$

Integrating, $(1-a)\log x-\log(z+t)=\log c_2, c_2$ being an arbitrary constant.

or $\frac{x^{a-1}}{z+t}=c_2$ or $\frac{x^{a-1}}{z+\sqrt{\left(x^2+y^2+z^2\right)}}=c_2$, as $t=\left(x^2+y^2+z^2\right)^{1/2}$ ....(7)

From (2) and (7), the required general solution is

$\phi\left[x^{a-1}\big/\left\{z+\sqrt{(x^2+y^2+z^2)}, x/y\right\}\right]=0$, ϕ being an arbitrary function.

Ex. 12. Solve $\left(x^3+3xy^2\right)p+\left(y^3+3x^2y\right)q=2z\left(x^2+y^2\right)$. **[I.A.S. 1993]**

Solution. Here subsidiary equations are $\dfrac{dx}{x^3+3xy^2}=\dfrac{dy}{y^3+3x^2y}=\dfrac{dz}{2z\left(x^2+y^2\right)}$. ....(1)

- Choosing 1, 1, 0 as multipliers, each fraction of (1) $=\dfrac{dx+dy}{x^3+3xy^2+3x^2y+y^3}=\dfrac{d(x+y)}{(x+y)^3}$. ....(2)

- Choosing 1, –1, 0 as multipliers, each fraction of (1) $=\dfrac{dx-dy}{x^3+3xy^2-y^3-3x^2y}=\dfrac{d(x-y)}{(x-y)^3}$ ....(3)

From (2) and (3), $(x+y)^{-3}d(x+y)=(x-y)^{-3}d(x-y)$

or $u^{-3}du-v^{-3}dv=0$, on putting $u=x+y$ and $v=x-y$.

Integrating, $u^{-2}/(-2)-v^{-2}/(-2)=c_1/2$ or $v^{-2}-u^{-2}=c_1$

or $(x-y)^{-2}-(x+y)^{-2}=c_1$, as $u=x+y$ and $v=x-y$. ....(4)

- Choosing $1/x, 1/y, 0$ as multipliers, each fraction of (1)

$$=\frac{(1/x)dx+(1/y)dy}{(1/x)\times\left(x^3+3xy^2\right)+(1/y)\times\left(y^3+3x^2y\right)}=\frac{(1/x)dx+(1/y)dy}{4\left(x^2+y^2\right)}. \qquad ....(5)$$

- Combining the last fraction of (1) with fraction (5), we have

$\dfrac{dz}{2z\left(x^2+y^2\right)}=\dfrac{(1/x)dx+(1/y)dy}{4\left(x^2+y^2\right)}$ or $\dfrac{dx}{x}+\dfrac{dy}{y}-2\dfrac{dz}{z}=0$.

Integrating, $\log x+\log y-2\log z=\log c_2$ or $(xy)/z^2=c_2$. ....(6)

From (4) and (6), the required general solution is given by

$\phi\left[(x-y)^{-2}-(x+y)^{-2}, (xy)/z^2\right]=0$, ϕ being an arbitrary function.

Ex. 13. Solve $p+q=x+y+z$.

Solution. Here Lagrange's auxiliary equations are $\dfrac{dx}{1}=\dfrac{dy}{1}=\dfrac{dz}{x+y+z}$. ....(1)

- Taking the first two fractions of (1), $dx-dy=0$ so that $x-y=c_1$. ....(2)

- Choosing 1, 1, 1 as multipliers, each fraction of (1) $=\dfrac{dx+dy+dz}{1+1+(x+y+z)}=\dfrac{d(2+x+y+z)}{2+x+y+z}$ ..(3)

Combining the first fraction of (1) with fraction (3), $d(2+x+y+z)/(2+x+y+z)=dx$.

Integrating, $\log(2+x+y+z)-\log c_2=x$ or $(2+x+y+z)/c_2=e^x$

or $e^{-x}(2+x+y+z)=c_2, c_2$ being arbitrary function. ....(4)

From (2) and (4), the required general solution is

$\phi\left[x-y, e^{-x}(2+x+y+z)\right]=0$, ϕ being an arbitrary function.

Ex. 14. Solve $(2x^2+y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z^2-yz-2zx-xy)q=x^2+y^2$ $+2z^2-yz-zx-2xy$. **[I.A.S. 1992]**

Solution. Here Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy} \quad ...(1)$$

- Choosing 1, –1, 0 ; 0, 1, –1 and –1, 0, 1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}=\frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\therefore \quad \frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)}. \quad ....(2)$$

- Taking the first two fractions of (2), we have

$(dx-dy)/(x-y)-(dy-dz)/(y-z)=0$.

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ or $(x-y)/(y-z)=c_1$. ....(3)

- Taking the last two fractions of (2), $(dy-dz)/(y-z)-(dz-dx)/(z-x)=0$.

Integrating, $\log(y-z)-\log(z-x)=\log c_2$ or $(y-z)/(z-x)=c_2$. ....(4)

From (3) and (4), the required general solution is

$\phi\left[(x-y)/(y-z),(y-z)/(z-x)\right]=0$, ϕ being an arbitrary function.

Ex. 15. Find the general solution of the partial differential equation $px(x+y)-qy(x+y)+(x-y)(2x+2y+z)=0$.

Solution. Given $x(x+y)p-y(x+y)q=-(x-y)(2x+2y+z)$. ....(1)

Lagrange's auxiliary equations are $\dfrac{dx}{x(x+y)}=\dfrac{dy}{-y(x+y)}=\dfrac{dz}{-(x-y)(2x+2y+z)}$. ....(2)

• Taking the first two fractions, $(1/x)dx=-(1/y)dy$ or $(1/x)dx+(1/y)dy=0$.

Integrating, $\log x+\log y=\log c_1$ or $xy=c_1$. ....(3)

• Again, each fraction of (2) $=\dfrac{dx+dy}{x(x+y)-y(x+y)}=\dfrac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$

$$=\frac{dx+dy}{(x-y)(x+y)}=\frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

Thus, $\dfrac{dx+dy}{(x+y)}=\dfrac{dx+dy+dz}{x+y-(2x+2y+z)}=-\dfrac{dx+dy+dz}{x+y+z}$

Thus, $\dfrac{dx+dy}{x+y}+\dfrac{dx+dy+dz}{x+y+z}=0$, so that $\log(x+y)+\log(x+y+z)=\log c_2$

or $(x+y)(x+y+z)=c_2, c_2$ being an arbitrary constant. ....(4)

From (3) and (4), solution is $\phi\left[xy,(x+y)(x+y+z)\right]=0$, ϕ being an arbitrary function.

<u>Assignment: Questions</u>

Q. 1. Solve $(x^2+2y^2)p-xyq=xz$.

Q. 2. Solve $xzp+yzq=xy$.

Q. 3. Solve $py+qx=xyz^2(x^2-y^2)$.

Q. 4. Solve $xp-yq=xy$.

Q. 5. Solve $p+3q=z+\cot(y-3x)$.

Q. 6. Solve $px(z-2y^2)=(z-qy)(z-y^2-2x^3)$.

Q. 7. Solve $(x-y)p+(x+y)q=2xz$.

Q.8. Solve $y^2p+x^2q=x^2y^2z^2$.

Q 9. Solve $(3x+y-z)p+(x+y-z)q=2(z-y)$.

Q. 10. Solve $x(x^2+3y^2)p-y(3x^2+y^2)q=2z(y^2-x^2)$.

Q. 11. Solve $(y-z)p+(z-x)q=x-y$.

Q. 12. Solve the general solution of the equation $(y+zx)p-(x+yz)q+y^2-x^2=0$.

Q. 13. Solve $x(y-z)p+y(z-x)q=z(x-y)$, i.e.,
$\{(y-z)/(yz)\}p+\{(z-x)/(zx)\}q=(x-y)/(xy)$. **[IAS 2005]**

Q. 14. Solve $2y(z-3)p+(2x-z)q=y(2x-3)$.

Q. 15. Solve $x^2(\partial z/\partial x)+y^2(\partial z/\partial y)=(x+y)z$.

Q. 16. Solve $z(x+2y)p-z(y+2x)q=y^2-x^2$.

Q. 17. Solve $\{my(x+y)-nz^2\}(\partial z/\partial x)-\{lx(x+y)-nz^2\}(\partial z/\partial y)=(lx-my)z$. **[I.A.S. 2001]**

Q. 18. Solve $px(z-2y^2)=(z-qy)(z-y^2-2x^2)$

Q. 19. Solve $px(z-2y^2)=(z-qy)(z-y^2-2x^3)$. **[I.A.S. 2006]**

Q. 20. Solve $x(z+2a)p+(xz+2yz+2ay)q=z(z+a)$.

Q. 21. Solve $2x(y+z^2)p+y(2y+z^2)q=z^3$.

Q. 22. $xp+zq+y=0$

Q. 23. Find the general solution of the differential equation $x^2(\partial z/\partial x)+y^2(\partial z/\partial y)=(x+y)z$.

Answers

Solution.1 The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{x^2+2y^2}=\frac{dy}{-xy}=\frac{dz}{xz} \quad ....(1)$$

• Taking the last two fractions of (2) and re–writing, we get

$(1/y)\,dy + (1/z)\,dz = 0$ so that $\log y + \log z = \log c_1$ or $yz = c_1$ ....(2)

• Taking the first two fractions of (1), we have

$$\frac{dx}{dy} = \frac{x^2 + 2y^2}{-xy} \text{ or } 2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right)x^2 = -4y \quad ....(3)$$

Putting $x^2 = v$ and $2x(dx/dy) = dv/dx$, (3) yields

$dv/dx + (2/y)v = -4y$, which is a linear equation.

Its integrating factor $= e^{\int (2/y)dy} - e^{2\log y} = y^2$ and hence its solution is

$$yv^2 = \int \{(-4y)\,xy^2\}dy + c_2 \text{ or } y^2x^2 + y^4 = c_2 \quad ....(4)$$

From (2) and (4), the required solution is $\phi\left(yz, y^2x^2 + y^4\right) = 0$, ϕ being an arbitrary function.

Solution.2 Given $xzp + yzq = xy$. ....(1)

The Lagrange's subsidiary equations for (1) are $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{xy}$. ....(2)

Taking the first two fractions of (2), $(1/x)\,dx - (1/y)\,dy = 0$ ....(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ....(4)

From (4), $x = c_1 y$. Hence, from second and third fractions of (2), we get

$$(1/yz)\,dy = \left(1/c_1y^2\right)dz \text{ or } 2c_1 y\,dy - 2z\,dz = 0. \quad ....(5)$$

Integrating (5), $c_1y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ....(6)

From (4) and (6), the required solution is $\phi\left(xy - z^2, x/y\right) = 0$, ϕ being an arbitrary function.

Solution.3 Given $py + qx = xyz^2\left(x^2 - y^2\right)$. ....(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2\left(x^2 - y^2\right)}$. ....(2)

• Taking the first two fractions of (2), $2xdx - 2ydy = 0$. ....(3)

Integrating. $x^2 - y^2 = c_1, c_1$ being an arbitrary constant. ....(4)

Using (4), the last two fractions of (2) give

$(dy)/x=(dz)/(xyz^2c_1)$ or $2c_1y\,dy-2z^{-2}dz=0$ . ....(5)

Integrating (5), $c_1y^2+(2/z)=c_2, c_2$ being an arbitrary constant.

or $y^2(x^2-y^2)+(2/z)=c_2$, using (4). ....(6)

From (4) and (6), the required general solution is

$y^2(x^2-y^2)+(2/z)=\phi(x^2-y^2)$, where ϕ is an arbitrary function.

Solution.4 The Lagrange's auxiliary equations for the given equation are

$(dx)/x=(dy)/(-y)=(dz)/(xy)$ ....(1)

Taking the first two fractions of (1), $(1/x)dx+(1/y)dy=0$

Integrating, $\log x+\log y=c_1$ so that $xy=c_1$ ....(2)

Using (2), (1) yields $(1/x)dx=(1/c_1)dz$ so that $\log x-\log c_2=z/c_1$

or $\log(x/c_2)=z/c_1$ or $\log(x/c_2)=z/(xy)$, by (2)

Thus, $x/c_2=e^{z/(xy)}$ or $xe^{-z/(xy)}=c_2, c_2$ being an arbitrary constant. ....(3)

From (2) and (3), the required solution is $xe^{-z/(xy)}=\phi(xy)$, ϕ being an arbitrary function.

Solution.5 The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{z+\cot(y-3x)}$$ ....(1)

Taking the first two fractions of (1), $dy-3dx=0$ so that $y-3x=c_1$ ....(2)

Taking the first and last fraction of (1), we have

$$dx=\frac{dz}{z+\cot(y-3x)}$$ or $$dx=\frac{dz}{z+\cot c_1}$$, using (2)

Integrating, $x=\log|z+\cot c_1|+c_2, c_1$ and c_2 being an arbitrary constants.

or $x-\log|z+\cot(y-3x)|=c_2$, using (2) ....(3)

From (2) and (3), the required general solution is

$x-\log|z+\cot(y-3x)|=\phi(y-3x)$, ϕ being an arbitrary function.

Solution.6 Re–writing the given equation, we have

$$x\left(z-2y^2\right)p+y\left(z-y^2-2x^3\right)q=z\left(z-y^2-2x^3\right) \quad ....(1)$$

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{x\left(z-2y^2\right)}=\frac{dy}{y\left(z-y^2-2x^3\right)}=\frac{dz}{z\left(z-y^2-2x^3\right)} \quad ....(2)$$

Taking the last two fraction, we get $(1/z)dz=(1/y)dy$

Integrating, $\log z=\log y+\log a$ or $z/y=a$ ....(3)

where a is an arbitrary constant. Using (3), (2) yields

$$\frac{dx}{x\left(ay-2y^2\right)}=\frac{dy}{y\left(ay-y^2-2x^3\right)} \text{ so that } \left(ay-y^2-2x^3\right)dx+x\left(2y-a\right)dy=0 \quad ....(4)$$

Comparing (4) with $Mdx+Ndy=0$, here $M=ay-y^2-2x^3$ and $N=x(2y-a)$. Then $\partial M/\partial y=a-2y$ and $\partial N/\partial x=2y-a$. Now, we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{x(2y-a)}\times 2(a-2y)=-\frac{2}{x}$$, which is a function of x alone.

Hence, by usual rule, integrating factor of (1) $=e^{\int(-2/x)dx}=e^{-2\log x}=e^{x^{-2}}=x^{-2}$

Multiplying (4) by x^{-2}, we get exact equation $\left(ayx^{-2}y^2x^{-2}-2x\right)dx+x^{-1}(2y-a)dy=0$

By the usual rule of solving an exact equation, its solution is

$$\int\left\{\left(ay-y^2\right)x^{-2}-2x\right\}dx+\int x^{-1}(2y-a)dy=b$$

(Treating y as constant) (Integrating terms free from x)

or $\left(ay-y^2\right)\times(-1/x)-x^2=b$ or $\left(y^2-ax\right)/x-x^2=b$

or $\left(y^2-ax-x^3\right)/x=b$, where b is an arbitrary constant. ....(5)

From (3) and (5), required solution is $\left(y^2-ax-x^3\right)/x=\phi(z/y)$, ϕ being an arbitrary function.

Solution.7 Here the Lagrange's subsidiary equations are $\dfrac{dx}{x-y}=\dfrac{dy}{x+y}=\dfrac{dz}{2xz}$. ....(1)

Taking the first two fractions of (1), $\frac{dy}{dx}=\frac{x+y}{x-y}=\frac{1+(y/x)}{1-(y/x)}$. ....(2)

Let $y/x=v$ i.e., $y=xv$. ....(3)

From (3), $(dy/dx)=v+x(dv/dx)$. ....(4)

Using (3) and (4), (2) gives $v+x\frac{dv}{dx}=\frac{1+v}{1-v}$ or $x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v-v(1-v)}{1-v}=\frac{1+v^2}{1-v}$

or $\frac{1-v}{1+v^2}dv=\frac{dx}{x}$ or $\left(\frac{2}{1+v^2}-\frac{2v}{1+v^2}\right)dv=\frac{2dx}{x}$

Integrating, $2\tan^{-1}v-\log(1+v^2)=2\log x-\log c_1$

or $\log x^2-\log(1+v^2)-\log c_1=2\tan^{-1}v$

or $\log\{x^2(1+v^2)/c_1\}=2\tan^{-1}v$ or $x^2(1+v^2)=c_1e^{2\tan^{-1}v}$

or $x^2\left[1+(y^2/x^2)\right]=c_1e^{2\tan^{-1}(y/x)}$, as $v=y/x$ by (3)

or $(x^2+y^2)e^{-2\tan^{-1}(y/x)}=c_1, c_1$ being an arbitrary constant.

Choosing $1,1,-1/z$ as multipliers, each fraction of (1)

$$=\frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)}=\frac{dx+dy-(1/z)dz}{0}$$

$\Rightarrow dx+dy-(1/z)dz=0$ so that $x+y-\log z=c_2$. ....(6)

From (5) and (6), the required general solution is

$\phi\left(x+y-\log z,(x^2+y^2)e^{-2\tan^{-1}(y/x)}\right)=0$, where ϕ is an arbitrary function.

Solution.8 Here Lagrange's auxiliary equations are $(dx)/y^2=(dy)/x^2=(dz)/x^2y^2z^2$. ....(1)

Taking the first two fractions of (1), we have

$3x^2dx-3y^2dy=0$ so that $x^3-y^3=c_1$. ....(2)

Choosing $x^2,y^2,-2/z^2$ as multipliers, each fraction of (1) $=\{x^2dx+y^2dy-(2/z^2)dz\}/0$

so that $3x^2dx+3y^2dy-(6/z^2)dz=0$.

Integrating, $x^3+y^3+(6/z)=c_2, c_2$ being an arbitrary constant. ....(3)

From (2) and (3), the required general solution is

$\phi\left[x^3-y^3, x^3+y^3+(6/z)\right]=0,\ \phi$ being an arbitrary function.

Solution. 9 Here Lagrange's auxiliary equations are $\dfrac{dx}{3x+y-z}=\dfrac{dy}{x+y-z}=\dfrac{dz}{2(z-y)}$ ....(1)

Choosing $1,-3,1$ as multipliers, each ratio of (1) $=\{dx-3dy-dz\}/0$

so that $dx-3dy-dz=0$.

Integrating, $x-3y-z=c_1, c_1$ being an arbitrary constant. ....(2)

From (2), $z=c_1-x+3y$. ....(3)

Substituting the above value of z, the first two fractions of (2) reduce to

$$\frac{dx}{3x+y-(c_1-x+3y)}=\frac{dy}{x+y-(c_1-x+3y)} \text{ or } \frac{dx}{2x+4y+c_1}=\frac{dy}{4y+c_1}. \quad ....(3)$$

Let $u=4y+c_1$ so that $dy=(1/4)\times du$. ....(4)

Then, (3) $\Rightarrow \dfrac{dx}{2x+u}=\dfrac{(1/4)du}{u}$ or $\dfrac{dx}{du}=\dfrac{1}{4}\dfrac{2x+u}{u}$ or $\dfrac{dx}{du}-\dfrac{1}{2u}x=\dfrac{1}{4}$, which is linear. ....(5)

Integrating factor of (5) $=e^{-\int(1/2u)du}=e^{-(1/2)\log u}=e^{\log(u)^{-1/2}}=u^{-1/2}=1/\sqrt{u}$.

Hence solution of (5) is $x\times\dfrac{1}{\sqrt{u}}=\displaystyle\int\frac{1}{4}\frac{1}{\sqrt{u}}du+c=\frac{1}{2}\sqrt{u}+c_2$

or $\dfrac{2x-u}{\sqrt{u}}=c_2$ or $\dfrac{2x-(4y+c_1)}{\sqrt{4y+c_1}}=c_2$, by (4)

or $\dfrac{2x-4y-(x-3y-z)}{\sqrt{4y+x-3y-z}}=c_2$, using (2) or $\dfrac{x-y+z}{\sqrt{x+y-z}}=c_2$ ....(6)

From (2) and (6), the required general solution is

$\phi\left(x-3y-z, (x-y+z)/\sqrt{x+y-z}\right)=0,\ \phi$ being an arbitrary function.

Solution.10

Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x\left(x^2+3y^2\right)}=\frac{dy}{-y\left(3x^2+y^2\right)}=\frac{dz}{2z\left(y^2-x^2\right)}. \quad \text{....(1)}$$

Choosing $1/x, 1/y, -1/z$ as multipliers, each fraction of (1)

$=\frac{(1/x)dx+(1/y)dy-(1/z)dz}{0}$ so that $\frac{1}{x}dx+\frac{1}{y}dy-\frac{1}{z}dz=0$.

Integrating, $\log x+\log y-\log z=\log c_1$ so that $(xy)/z=c_1$. ....(2)

Taking the first two ratios of (1), $\frac{dy}{dx}=-\frac{y\left(3x^2+y^2\right)}{x\left(x^2+3y^2\right)}=-\left(\frac{y}{x}\right)\frac{3+(y/x)^2}{1+3(y/x)^2}$. ....(3)

Put $y/x=v$ or $y=xv$ so that $(dy/dx)=v+x(dv/dx)$. ....(4)

Using (4), (3) reduces to $v+x\frac{dv}{dx}=-v\frac{3+v^2}{1+3v^2}$ or $x\frac{dv}{dx}=-v\left[\frac{3+v^2}{1+3v^2}+1\right]$

or $x\frac{dv}{dx}=-\frac{4\left(1+v^2\right)v}{1+3v^2}$ or $4\frac{dx}{x}+\frac{1+3v^2}{v\left(1+v^2\right)}dv=0$

or $4\frac{dx}{x}+\left(\frac{1}{v}+\frac{2v}{1+v^2}\right)dv$, on resolving into partial fractions

Integrating, $4\log x+\log v+\log\left(1+v^2\right)$ or $x^4v\left(1+v^2\right)=c_2'$

$x^4(y/x)\left[1+(y/x)^2\right]=c_2'$ or $xy\left(x^2+y^2\right)=c_2'$ or $c_1z\left(x^2+y^2\right)=c_2'$, by (2)

or $z\left(x^2+y^2\right)=c_2'/c_1$ or $z\left(x^2+y^2\right)=c_2$, where $c_2=c_2'/c_1$. ....(5)

$\therefore$ From (2) and (5) solution is $\phi\left(z\left(x^2+y^2\right), xy/z\right)=0$, ϕ being an arbitrary function.

Solution.11 Here the Lagrange's auxiliary equations are $\frac{dx}{y-z}=\frac{dy}{z-x}\frac{dz}{x-y}$. ....(1)

Choosing 1, 1, 1 as multipliers, each fraction of (1) $=\frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)}=\frac{dx+dy+dz}{0}$.

$\therefore\ dx+dy+dz=0$ so that $x+y+z=c_1$. ....(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore \;\; 2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = c_2$. ....(3)

$\therefore$ From (2) and (3) solution is $\phi\left(x+y+z, x^2+y^2+z^2\right) = 0$, ϕ being an arbitrary of function.

Solution.12 Given $(y+zx)p - (x+yz)q = x^2 - y^2$. ....(1)

Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+zx} = \dfrac{dy}{-(x+yz)} = \dfrac{dz}{x^2-y^2}$ .....(2)

Choosing $x, y, -z$ as multipliers, each fraction of (2)

$$= \frac{xdx + ydy - zdz}{x(y+zx) - y(x+yz) - z(x^2-y^2)} = \frac{xdx + ydy - zdz}{0} \;\; \therefore \;\; ydx + xdx + dz = 0 \text{ or } d(xy) + dz = 0.$$

Integrating, $xy + z = c_2, c_2$ being an arbitrary constant. ....(4)

$\therefore$ The required solution is $\phi\left(x^2+y^2-z^2, xy+z\right) = 0$, ϕ being an arbitrary function.

Solution.13 Given $x(y-z)p + y(z-x)q = z(x-y)$ ....(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ ....(2)

Choosing $1/x, 1/y, 1/z$ as multipliers each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{(y-z) + (z-x) + (x-y)} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\therefore \;\; \log(xyz) = c_1$ or $xyz = c_1$ ....(3)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(xy - xz) + (yz - yx) + (zx - zy)} = \frac{dx + dy + dz}{0}$$

$\Rightarrow dx + dy + dz = 0$ so that $x + y + z = c_2$ ....(4)

From (3) and (4), solution is $\phi(x+y+z, xyz) = 0$, ϕ being an arbitrary function.

Solution.14 The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{2y(z-3)}=\frac{dy}{2x-z}=\frac{dz}{y(2x-3)} \quad ....(1)$$

Taking the first and third fractions, $(2x-3)dx=2(z-3)dz$.

Integrating, $x^2-3x=z^2-6z+C_1$ or $x^2-3x-z^2+6z=C_1$ ....(2)

Choosing $1, 2y, -2$ as multipliers, each fraction of (1)

$$=\frac{dx+2ydy-2dz}{2y(z-3)+2y(2x-z)-2y(2x-3)}=\frac{dx+2ydy-2dz}{0}$$

$\therefore\ dx+2ydy-2dz=0$ so that $x+y^2-2z=C_2$ ....(3)

From (2) and (3), solution is $\phi\left(x^2-3x-z^2+6z, x+y^2-2z\right)=0$, ϕ being an arbitrary function.

Solution15. Re-writing the given equation $x^2p+y^2q=(x+y)z$ ....(1)

The Lagrange's auxiliary equations for (1) are $\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z}$ ....(2)

Taking the first two fractions of (2), $\left(1/x^2\right)dx-\left(1/y^2\right)dy=0$.

Integrating, $-(1/x)+(1/y)=C_1$ or $1/y-1/x=C_1$ ....(3)

Choosing $1/x, 1/y, -1/z$ as multipliers, each fraction of (2)

$$=\frac{(1/x)dx+(1/y)dy-(1/z)dz}{x+y-(x+y)}=\frac{(1/x)dx+(1/y)dy-(1/z)dz}{0}$$

$\therefore\ (1/x)dx+(1/y)dy-(1/z)dz=0$ so that $xy/z=C_2$ ....(4)

From (3) and (4), solution is $\Phi(1/y-1/x, xy/z)=0$, Φ being an arbitrary function.

Solution.16 The Lagrange's subsidiary equations are $\frac{dx}{z(x+2y)}=\frac{dy}{-z(y+2x)}=\frac{dz}{y^2-x^2}$ ....(1)

Taking the first two fraction of (1), we have

$(y+2x)dx+(x+2y)dy=0$ or $2xdx+2ydy+d(xy)=0$

Integrating, $x^2+y^2+xy=C_1, C_1$ being an arbitrary constant. ....(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{(x^2 z + 2xyz) - (y^2 z + 2xyz) + (zy^2 - zx^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = C_2$ ....(3)

From (2) and (3), solution is $\phi(x^2 + y^2 + z^2, x^2 + y^2 + xy) = 0$, ϕ being an arbitrary function.

Solution.17 Re-writing the given equation, $\{my(x+y) - nz^2\}p - \{lx(x+y) - nz^2\}q = (lx - my)z$ ....(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{my(x+y) - nz^2} = \dfrac{dy}{-lx(x+y) + nz^2} = \dfrac{dz}{(lx - my)z}$ ....(2)

Each fraction of (2) $= \dfrac{dx + dy}{(my - lx)(x+y)} = \dfrac{dz}{-(my - lx)z}$ so that $\dfrac{d(x+y)}{x+y} = -\dfrac{dz}{z}$

Integrating, $\log(x+y) = -\log z + \log C_1$ or $(x+y)z = C_1$ ....(3)

Taking lx, my, nz as multipliers, each fraction of (2)

$$= \frac{lxdx + mydy + nzdz}{lxmy(x+y) - lxnz^2 - mylx(x+y) + mynz^2 + nz^2(lx - my)} = \frac{lx\,dx + my\,dy + nz\,dz}{0}$$

$\therefore\ 2lx\,dx + 2my\,dy + 2nz\,dz = 0$ so that $lx^2 + my^2 + nz^2 = C_2$ ....(4)

From (3) and (4), solution is $\Phi(xz + yz, lx^2 + my^2 + nz^2) = 0$, Φ being an arbitrary function.

Solution.18 the given equation $x(z - 2y^2)p + y(z - y^2 - 2x^2)q = z(z - y^2 - 2x^2)$ ....(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(z - 2y^2)} = \dfrac{dy}{y(z - y^2 - 2x^2)} = \dfrac{dz}{z(z - y^2 - 2x^2)}$ ....(2)

Taking the last two fractions, $(1/y)dy - (1/z)dz = 0$ so that $y/z = C_1$ ....(3)

Taking $0, -2y, 1$ as multipliers, each fraction of (2)

$$= \frac{-2ydy + dz}{-2y^2(z - y^2 - 2x^2) + z(z - y^2 - 2x^2)} = \frac{d(z - y^2)}{(z - 2y^2)(z - y^2 - 2x^2)} \qquad ....(4)$$

Combining fraction (4) with first fraction of (2), we get

$$\frac{dx}{x\left(z-2y^2\right)}=\frac{d\left(z-y^2\right)}{\left(z-2y^2\right)\left(z-y^2-2x^2\right)} \text{ or } \frac{d\left(z-y^2\right)}{dx}=\frac{z-y^2-2x^2}{x}$$

or $du/dx=\left(u-2x^2\right)/x$, taking $z-y^2=u$ ....(5)

or $(du/dx)-(1/x)u=-2x$ which is an ordinary linear differential equation

whose I.F. $=e^{-\int(1/x)dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=1/x$ and solution is

$u\cdot\frac{1}{x}=\int(-2x)\left(\frac{1}{x}\right)dx+C_2$ or $\frac{z-y^2}{x}=-2x+C_2$, using (5)

or $\left(z-y^2\right)/x+2x=C_2$ or $\left(z-y^2+2x^2\right)/x=C_2$ ....(6)

From (3) and (6), the required general solution of (1)

$\Phi\left(y/z,\left(z-y^2-2x^2\right)/x\right)=0,$ Φ being an arbitrary function.

Solution.19 Do like. 17. Ans. $\Phi\left(y/z,\left(z-y^2+x^3\right)/x\right)=0$

Solution.20 The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{x(z+2a)}=\frac{dy}{xz+2yz+2ay}=\frac{dz}{z(z+a)} \quad ....(1)$$

Each fraction of (1) $\frac{dx+dy}{2(x+y)(z+a)}=\frac{dz}{z(z+a)}$ or $\frac{d(x+y)}{x+y}=\frac{2}{z}dz$

Integrating, $\log(x+y)=2\log z+\log C_1$ or $(x+y)/z^2=C_1$ ....(2)

Taking the first and third ratios of (4), $\frac{dx}{x}=\frac{z+2a}{z(z+a)}dz$ or $\frac{dx}{x}=\left(\frac{2}{z}-\frac{1}{z+a}\right)dz$

Integrating, $\log x=2\log z-\log(z+a)+\log C_2$ or $x(z+a)/z^2=C_2$ ....(3)

From (2) and (3), solution is $\Phi\left\{(x+y)/z^2,x(z+a)/z^2\right\}=0.$ ϕ being an arbitrary function.

Solution.21 The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{2x\left(y+z^2\right)}=\frac{dy}{y\left(2y+z^2\right)}=\frac{dz}{z^3} \quad ....(1)$$

Each fraction of (1) $= \dfrac{dx}{2x(y+z^2)} = \dfrac{z\,dy + y\,dz}{2yz(y+z^2)} = \dfrac{d(yz)}{2yz(y+z^2)}$

$\therefore\ (1/x)dx + (1/yz)d(yz) = 0$ so that $x/(yz) = C_1$ ....(2)

From the last two fractions of (1), $\dfrac{dy}{dz} = \dfrac{y(2y+z^2)}{z^3} = \dfrac{2y^2}{z^3} + \dfrac{y}{z}$ or $y^{-2}\dfrac{dy}{dz} - \dfrac{1}{z}y^{-1} = \dfrac{2}{z^3}$ ....(3)

Putting $-y^{-1} = u$ and $(1/y^2)\times(dy/dz) = du/dz$ in (3), we get

$(du/dz) + (1/z)u = 2/z^3$, which is an ordinary linear equation.

Its I.F. $= e^{\int (1/z)dz} = e^{\log z} = z$ and solution is $uz = \int (2/z^3)\,z\,dz - C_2 = -2z^{-1} - C_2$

or $-y^{-1}z - 2z^{-1} = -C_2$ or $z/y - 2/z = C_2$ ....(4)

From (3) and (4), solution is $\Phi(x/yz, z/y - 2/z) = 0,\ \phi$ being arbitrary function.

Solution.22 Given equation is $xp + zq = -y$

Its Lagrange's auxiliary equation are $\dfrac{dx}{x} = \dfrac{dy}{z} = \dfrac{dz}{-y}$ ....(1)

Taking the last two fractions of (2), $2ydy + 2zdz = 0$ so that $y^2 + z^2 = C_1$ ....(2)

Choosing $0, z, -y$ as multipliers, each fraction of (1)

$$= \frac{zdy - ydz}{z^2 + y^2} = \frac{(1/z)dy - (y/z^2)dz}{1 + (y/z)^2} = \frac{d(y/z)}{1 + (y/z)^2} \quad \text{....(3)}$$

Combining the first faction of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{d(y/z)}{1 + (y/z)^2} \quad \text{or} \quad \frac{dx}{x} - d\left(\tan^{-1}\frac{y}{z}\right) = 0$$

Integrating, $\log|x| - \tan^{-1}(y/z) = C_2, c_2$ being an arbitrary constant. ....(4)

From (2) and (4), the required general solution is $\log|x| - \tan^{-1}(y/z) = \phi(y^2 + z^2)$

Solution.23 Let $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Then, the given equation takes the form

$x^2 p = y^2 q = z(x+y)$ ....(1)

The Lagrange's auxiliary equations for (1) are

$$(dx)/x^2 = (dy)/y^2 = (dz)/z(x+y) \qquad \text{....(2)}$$

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0$

Integrating, $-(1/x)+(1/y) = c_1$ or $(x-y)/xy = c_1$ ....(3)

Choosing 1, –1, 0 as multipliers, each fraction of (2) $= \dfrac{dx-dy}{x^2-y^2}$ ....(4)

Combining the last fraction of (2) with fraction (4), we have

$$\frac{dx-dy}{(x-y)(x+y)} = \frac{dz}{z(x+y)} \text{ or } \frac{dx-dy}{x-y} - \frac{dz}{z} = 0$$

Integrating, $\log(x-y) - \log z = \sin c^2$ or $(x-y)/z = c^2$ ....(5)

From (5), $x - y = c_2 z$ ....(6)

using (6), (3) becomes $(c_2 z)/xy = a$ or $(xy)/z = c_2/c_1 = c_3$ say ....(7)

From (5) and (7), the required solution is $\phi((x,y)/z,(x-y)/z) = 0$.

Examples based on article 2

Ex. 1. Solve $x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 + x_1 x_2 x_3 = 0$.

Sol. Re-writing the given equation in standard form, we have

$x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 = -x_1 x_2 x_3$ ...(2)

The auxiliary equations for (2) are $\dfrac{dx_1}{x_2x_3} = \dfrac{dx_2}{x_3x_1} = \dfrac{dx_3}{x_1x_2} = \dfrac{dz}{-x_1x_2x_3}$. ...(3)

Taking the first and the fourth fractions of (3), $x_1\, dx_1 + dz = 0$ so that $x_1^2 + 2z = c_1$. ...(4)

Taking 1st and 2nd fractions of (3), $x_1\, dx_1 = x_2\, dx_2$ so that $x_1^2 - x_2^2 = c_2$. ...(5)

Finally, 2nd and 3rd fractions of (3) give $x_2\, dx_2 = x_3\, dx_3$ so that $x_2^2 - x_3^2 = c_3$. ...(6)

Hence the required general integral is

$$\phi\left(x_1^2 + 2z, x_1^2 - x_2^2, x_2^2 - x_3^2\right) = 0,$$ φ being an arbitrary function.

Ex. 2. Solve $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=az+\frac{xz}{t}$

Sol. Here auxiliary equations for the given equation are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{az+xy/t} \qquad \text{...(1)}$$

From the first two fractions of (1), $(1/x)dx-(1/y)dy=0$ so that $x/y=C_1$. ...(2)
From the first and third fractions of (1), $(1/x)dx-(1/t)dt=0$ so that $x/t=C_2$. ...(3)
Dividing (3) by (2), we have $y/t=C_2/C_1$. ...(4)

Taking the first and third fractions of (1) and using (4), we get

$$\frac{dx}{x}=\frac{dz}{az+\left(\frac{C_2}{C_1}\right)x} \quad \Rightarrow \quad \frac{dz}{dx}-\left(\frac{a}{x}\right)z=\left(\frac{C_2}{C_1}\right),$$ which is linear. ...(5)

I.F. of (5) = $e^{-\int(a/x)dx}=e^{-a\log x}=e^{\log x^{-a}}=x^{-a}$ and so solution of (5) is given by

$$zx^{-a}=C_3+\frac{C_2}{C_1}\int x^{-a}dx=C_3+\frac{C_2}{C_1}\frac{x^{1-a}}{1-a}=C_3+\frac{y}{t}\frac{x^{1-a}}{1-a},$$ using (4)

$\therefore\ zx^{-a}-\frac{y}{t}\frac{x^{1-a}}{1-a}=C_3, C_3$ being an arbitrary constant. ...(6)

From (2), (3) and (6), the required general solution is

$$\phi\left(\frac{x}{y},\frac{x}{t},zx^{-a}-\frac{y}{t}\frac{x^{1-a}}{1-a}\right)=0,$$ φ being an arbitrary function.

Ex. 3. Solve $x(\partial u/\partial x)+y(\partial u/\partial y)+z(\partial u/\partial z)=xyz$.

Sol. Here the auxiliary equations for the given equation are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{xyz} \qquad \text{...(1)}$$

Taking the first two fractions of (1), $(1/x)dx-(1/y)dy=0$.

Integrating it, $\log x-\log y=\log C_1$ or $x/z=C_1$. ...(2)

Taking the first and third fractions of (1), $(1/x)dx-(1/z)dz=0$

Integrating it, $\log x - \log z = \log C_2$ or $x/y = C_2$ ...(3)

Choosing yz, zx, xy as multipliers, each fraction of (1) $= \dfrac{yzdx + zxdy + xydz}{xyz + xyz + xyz} = \dfrac{d(xyz)}{3xyz}$. ...(4)

Combining the fourth fraction of (1) with fraction (4), we get

$\dfrac{du}{xyz} = \dfrac{d(xyz)}{3xyz}$ or $d(xyz) - 3du = 0$ so that $xyz - 3u = C_3$. ...(5)

From (2), (3) and (5), the required general solution is

$\varphi(x/y, x/z, xyz - 3u) = 0$, φ being an arbitrary function.

Ex. 4. Solve $(y + z + w)(\partial w/\partial x) + (z + x + w)(\partial w/\partial y) + (x + y + w)(\partial w/\partial z) = (x + y + z)$.

Sol. Here the auxiliary equations of the given equation are

$$\frac{dx}{y+z+w} = \frac{dy}{z+x+w} = \frac{dz}{x+y+w} = \frac{dw}{x+y+z}. \qquad ...(1)$$

Each fraction of $(1) = \dfrac{dw-dx}{-(w-x)} = \dfrac{dw-dy}{-(w-y)} = \dfrac{dw-dz}{-(w-z)} = \dfrac{dw+dx+dy+dz}{3(w+x+y+z)}$. ...(2)

Taking the first and the fourth fractions of (2), $\dfrac{dw+dx+dy+dz}{3(w+x+y+z)} + \dfrac{dw-dx}{w-x} = 0$.

Integrating, $(1/3) \times \log(w + x + y + z) + \log(w - x) = \log C_1$

or $(w + x + y + z)^{1/3}(w - x) = C_1$ ...(3)

Similarly, $(w + x + y + z)^{1/3}(w - y) = C_2$ ...(4)

and $(w + x + y + z)^{1/3}(x - z) = C_3$ ...(5)

From (3), (4) and (5), the required general solution is

$\varphi[(w + x + y + z)^{1/3}(w - x), (w + x + y + z)^{1/3}(w - y), (w + x + y + z)^{1/3}(w - z)] = 0$,

where φ is an arbitrary function.

Ex. 5. Prove that if $x_1^3 + x_2^3 + x_3^3 = 1$ when $z = 0$, the solution of the equation $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ can be given in the form $s^3\{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3\}^4 = (x_1 + x_2 + x_3 - 3z)^3$, *where* $s = x_1 + x_2 + x_3 + z$ *and* $p_i = \partial z/\partial x_i$, $i = 1,2,3$.

Sol. Given $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ ...(1)

where $s = x_1 + x_2 + x_3 + z$ ...(2)

The auxiliary equations for (2) are $\frac{dx_1}{s-x_1}=\frac{dx_2}{s-x_2}=\frac{dx_3}{s-x_3}=\frac{dz}{s-z}$

or $\frac{dx_1}{x_2+x_3+z}=\frac{dx_2}{x_3+x_1+z}=\frac{dx_3}{x_1+x_2+z}=\frac{dz}{x_1+x_2+x_3}$, using (2) ...(3)

Each fraction of $(3)=\frac{dx_1+dx_2+dx_3-3dz}{2(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)}=\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)}$ ...(4)

Again, each fraction of $(3)=\frac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)}=\frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)}$ ...(5)

Then, (4) and (5) give $\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)}=\frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)}$ ...(5)

or $\frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z}+3\frac{d(x_1+x_2+x_3-3z)}{x_1+x_2+x_3-3z}=0$

Integrating, log (x_1+x_2+x_3+z)+3log (x_1+x_2+x_3-3z)=log *a*

or $(x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3=a$, where *a* is an arbitrary constant. ...(6)

Given that $x_1^3+x_2^3+x_3^3=1$ when $z=0$ ...(7)

Hence (6) gives $a=(x_1+x_2+x_3)^4$. Then (6) reduces to

$(x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3=(x_1+x_2+x_3)^4$ ...(8)

Now, each fraction of (3) $=\frac{dx_1-dz}{-(x_1-z)}=\frac{3(x_1-z)^2\,d(x_1-z)}{-3(x_1-z)^3}=\frac{d(x_1-z)^3}{3(x_1-z)^3}$ ...(9)

By symmetry, each fraction of (3) is also $=\frac{d(x_2-z)^3}{-3(x_2-z)^3}=\frac{d(x_3-z)^3}{-3(x_3-z)^3}$ ...(10)

Using (9) and (10), we find that each fraction of (3)

$$=\frac{d(x_1-z)^3}{-3(x_1-z)^3}=\frac{d(x_2-z)^3}{-3(x_2-z)^3}=\frac{d(x_3-z)^3}{-3(x_3-z)^3}=\frac{d\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]}{-3\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]} \quad ...(11)$$

Then, from (4) and (11), we have

$$\frac{3d\left(x_1 + x_2 + x_3 - 3z\right)}{\left(x_1 + x_2 + x_3 - 3z\right)} = \frac{d\left[\left(x_1 - z\right)^3 + \left(x_2 - z\right)^3 + \left(x_3 - z\right)^3\right]}{\left[\left(x_1 - z\right)^3 + \left(x_2 - z\right)^3 + \left(x_3 - z\right)^3\right]}$$

Integrating it, 3log $(x_1 + x_2 + x_3 - 3z)$ + log b = log $\{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3\}$

$(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3 = b(x_1 + x_2 + x_3 - 3z)^3$ where b is an arbitrary constant ...(12)

Putting $z = 0$,(12) gives $\quad x_1^3 + x_2^3 + x_3^3 = b\left(x_1 + x_2 + x_3\right)^3$

$1 = b(x_1 + x_2 + x_3)^3$, using (7) so that $b = 1/(x_1 + x_2 + x_3)^3$

$\therefore$ (12) $\Rightarrow (x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3 = (x_1 + x_2 + x_3 - 3z)^3/(x_1 + x_2 + x_3)^3$ ....(13)

Raising both sides of (8) to power 3 , we have

$(x_1 + x_2 + x_3 + z)^3 (x_1 + x_2 + x_3 - 3z)^9 = (x_1 + x_2 + x_3)^{12}$ ...(14)

Raising both sides of (13) to power 4 , we have

$(x_1 - z)^3 + (x2 - z)^3 + (x_3 - z)^3 \}^4 = (x_1 + x_2 + x_3 - 3z)^{12}/(x_1 + x_2 + x_3)^{12}$ ...(15)

Multiplying the corresponding sides of (14) and (15), we have

$(x_1 + x_2 + x_3 + z)^3 \{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3 \}^4 = (x_1 + x_2 + x_3 - 3z)^3$

$s^3 \{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3 \}^4 = (x_1 + x_2 + x_3 - 3z)^3$,using (2)

Integral surfaces finding examples

Ex. 1. Find the integral surface of the linear partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0$, $z = 1$.

Sol. Given $\quad x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z.$ ...(1)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y\left(x^2+z\right)} = \dfrac{dz}{\left(x^2-y^2\right)z}.$ ...(2)

Two independent solutions by Lagrange's method (from past examples solutions)

$xyz = c_1$ and $x^2 + y^2 - 2z = c_2.$ ...(3)

Taking t as parameter, the given equation of the straight line $x + y = 0$, $z = 1$ can be put in parametric form $\quad x = t, \quad y = -t, \quad z = 1.$ ...(4)

Using (4), (3) may be re–written as $\quad -t^2 = c_1$ and $2t^2 - 2 = c_2.$ ... (5)

Eliminating t from the equations of (5), we have

$2(-c_1) - 2 = c_2$ or $2c_1 + c_2 + 2 = 0.$ ... (6)

Putting values of c_1 and c_2 from (3) in (6), the desired integral surface is

$2xyz + x^2 + y^2 - 2z + 2 = 0.$

Ex. 2. Find the equation of the integral surface of the differential equation $2y(z - 3)p + (2x - z)q = y(2x - 3)$, which pass through the circle $z = 0$, $x^2 + y^2 = 2x$.

Sol. Given equation is $2y(z - 3)p + (2x - z)q = y(2x - 3).$ ...(1)

Given circle is $x^2 + y^2 = 2x,$ $z = 0.$ ...(2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}.$ ...(3)

Taking the first and third fractions of (3), $(2x - 3)dx - 2(z - 3)dz = 0.$

Integrating, $x^2 - 3x - z^2 + 6z = c_1$, c_1 being an arbitrary constant. ...(4)

Choosing $1/2$, y, -1 as multipliers, each fraction of (3)

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

Hence $(1/2)dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0.$

Integrating, $x + y2 - 2z = c_2$, c2 being an arbitrary constant. ...(5)

Now, the parametric equations of given circle (2) are $x = t$, $y = (2t - t^2)^{1/2}$, $z = 0.$ ...(6)

Substituting these values in (4) and (5), we have

$t^2 - 3t = c_1$ and $3t - t^2 = c_2.$ ...(7)

Eliminating t from the above equations (7), we have $c_1 + c_2 = 0.$ ...(8)

Substituting the values of c_1 and c_2 from (4) and (5) in (8), the desired integral surface is

$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0 \Rightarrow x^2 + y^2 - z^2 - 2x + 4z = 0.$

Ex. 3. Find the integral surface of the partial differential equation $(x - y)p + (y - x - z)q = z$ through the circle $z = 1$, $x^2 + y^2 = 1$.

Sol. Given $(x - y)p + (y - x - z)q = z.$ ...(1)

Lagrange's auxiliary equations for (1) are $\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}.$ ...(2)

Choosing 1, 1, 1 as multipliers, each fraction on (2) = $(dx + dy + dz)/0$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_1.$ ...(3)

Taking the last two fractions of (2) and using (3) we get

$$\frac{dy}{y-(c_1-y)} = \frac{dz}{z} \quad \text{or} \quad \frac{2dy}{2y-c_1} - \frac{2dz}{z} = 0.$$

Integrating it, $\log(2y - c_1) - 2\log z = \log c_2$ or $(2y - c_1)/z^2 = c_2$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2.$ ....(4)

The given curve is given by $z = 1$ and $x^2 + y^2 = 1$. ...(5)

Putting $z = 1$ in (3) and (4), we get $x + y = c_1 - 1$ and $y - x = c_2 + 1$. ...(6)

But $2(x^2 + y^2) = (x + y)^2 + (y - x)^2$. ...(7)

Using (5) and (6), (7) becomes

$2 = (c_1 - 1)^2 + (c_2 + 1)^2$ or $c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$. ...(8)

Putting the values of c_1 and c_2 from (3) and (4) in (8), required integral surface is

$$(x + y + z)^2 + (y - x - z)^2/z^4 - 2(x + y + z) + 2(y - x - z)/z^2 = 0$$

i.e. $z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0$.

Ex. 4. Find the equation of the integral surface of the differential equation $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ which passes through the line $x = 1$, $y = 0$.

Sol. Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. ...(1)

By past examples,

$(x - y)/(y - z) = c_1$ ...(2)

and $xy + yz + zx = c_2$. ...(3)

The given curve is represented by $x = 1$ and $y = 0$. ...(4)

Using (4) in (2) and (3), we obtain $-1/z = c_1$ and $z = c_2$

so that $(-1/z) \times z = c_1c_2$ *or* $c_1c_2 + 1 = 0$. ...(5)

Putting the values of c_1 & c_2 from (2) and (3) in (5), the required integral surface is

$[(x - y)/(y - z)]\,(xy + yz + zx) + 1 = 0$ or $(x - y)\,(xy + yz + zx) + y - z = 0$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y2 + z2 = 1$, $x + z = 2$.

Sol. Given $4yzp + q = -2y$. ...(1)

Given curve is given by $y^2 + z^2 = 1$, and $x + z = 2$. ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}$. ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1$. ...(4)

Taking the last two fractions of (3),

$dz + 2ydy = 0$ so that $z + y^2 = c_2$. ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $1 + 2 = c_1 + c_2$, using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by

$3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0$.

Ex. 6. Find the general integral of the partial differential equation $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ and also the particular integral which passes through the line $x = 1$, $y = 0$.

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$. ...(1)

Given line is given by $x = 1$ and $y = 0$. ...(2)

Lagrange's auxiliary equations of (1) are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2x-2yz}. \quad ...(3)$$

Taking z, 1, x as multipliers, each fraction of (3) = $(zdx + dy + x\,dz)/0$

so that $zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1$. ...(4)

Again, taking x, y, 1/2 as multipliers, each fraction of (3) = $\{xdx + ydy + (1/2)dz\}/0$

so that $x\,dx + ydy + (1/2) \times dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2$. ...(5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x = 1$ and $y = 0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the eqution of the required surface is given by

$1 + xz + y = x^2 + y^2 + z$ or $x^2 + y^2 + z - xz - y = 1$.

Ex. 7. *Find the integral surface of* $x^2p + y^2q + z^2 = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ *which passes through the hyperbola* $xy = x + y$, $z = 1$.

Sol. Given $x^2p + y^2q + z^2 = 0$ or $x^2p + y^2q = -z^2$. ...(1)

Given curve is given by $xy = x + y$ and $z = 1$. ...(2)

Here Lagrange's auxiliary equations for (1) are $(dx)/x^2 = (dy)/y^2 = (dz)/(-z^2)$. ...(3)

Taking the first and third fractions of (1), $x^{-2}dx + z^{-2}dz = 0$.

Integrating, $-(1/x) - (1/z) = -c_1$ or $1/x + 1/z = c_1$. ...(4)

Taking the second and third fractions of (1), $y^{-2}dy + z^{-2}dz = 0$.

Integrating, $-(1/y) - (1/z) = -c_2$ or $1/y + 1/z = c_2$. ...(5)

Adding (4) and (5), $\frac{1}{x} + \frac{1}{y} = c_1 + c_2$ or $\frac{x+y}{xy} + \frac{2}{z} = c_1 + c_2$

or $(xy)/(xy) + 2 = c_1 + c_2$, using (2) or $c_1 + c_2 = 3$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$1/x + 1/z + 1/y + 1/z = 3$ or $yz + 2xy + xz = 3xyz$.

Ex. 8. *Find the integral surface of the linear first order partial differential equation* $yp + xq = z - 1$ *which passes through the curve* $z = x^2 + y^2 + z$, $y = 2x$

Sol. Given equation is $yp + xq = z - 1$ ... (1)

and the given curve is given by $z = x^2 + y^2 + 1$ and $y = 2x$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$ ...(3)

Taking the first two fractions, $2ydy - 2xdx = 0$

Integrating, it, $y^2 - x^2 = C_1$, C_1 being an arbitrary constant ... (4)

Taking the first and the last fractions of (3) and using (4), we get

$\frac{dx}{(x^2+C_1)^{1/2}} = \frac{dz}{z-1}$ so that $\log(z-1) - \log\{x + (x^2 + C_1)^{1/2}\} = \log C_2$

$\log(z-1) - \log(x+y) = \log C_2$, by (4) or $(z-1)/(x+y) = C_2$ ... (5)

The parametric form of the given curve (2) is

$x = t, y = 2t, z = 5t^2 + 1$ ... (6)

Substituting these values in (4) and (5), we get $3t^2 = C_1$ and $5t/3 = C_2$ ... (7)

Eliminating t from the above equations (7), we get

$5\sqrt{C_1}/3\sqrt{3} = C_2$ ...(8)

Substituting the values of C_1 and C_2 from (4) and (5) in (8), the required surface is given by

$5(y^2 - x^2)^{1/2}/3\sqrt{3}\,(z-1)/(x+y)$.

Ex. 10. Find the integral surface of the partial differential equation $(x-y)y^2p + (y-x)x^2q = (x^2 + y^2)z$ passing through the curve $xz = a^3$, $y = 0$.

Sol. Given equation is $(x+y)y^2\,p\ \&\ (y+x)x^2q = (x^2\ \&\ y^2)z$ ... (1)

and the given curve is given by

$xz = a^3$ and $y = 0$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$ ...(3)

Each fraction of (3) = $\frac{dx-dy}{(x-y)(y^2+x^2)} = \frac{dz}{(x^2+y^2)z}$ so that $\frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$

Integrating it, $(x-y)/z = C_1$, C_1 being an arbitrary constant ... (4)

Taking the first two fractions, $3x^2dx + 3y^2dy = 0$

Integrating it, $x^3 + y^3 = C_2$, C^2 being an arbitrary constant. ... (5)

The parametric form of the given curve (2) is

$z = t$, $x = a^3/t$, $y = 0$ ... (6)

Substituting these values in (4) and (5), we get

$a^3/t^2 = C_1$ so that $t^2 = a^3/C_1$ ...(7)

and $(a^3/t)^3 = C_2$ so that $t^3 = a^9/C_2$ ... (8)

Squaring both sides of (8), $t^6 = a^{18} / C_2^2$ or $(t^2)^3 = a^{18} / C_2^2$

or $(a^3 / C_1)^3 = a^{18} / C_2^2$, since $t^2 = a^3 / C_1$, by (7)

or $a^9 / C_1^3 = a^{18} / C_2^2$, or $C_2^2 = a^9 \, C_1^3$ ...(9)

Substituting the values of C_1 and C_2 from (4) and (5) in (9), the required integral surface of (1) is given by

$(x^3 + y^3)^2 = a^9 (x - y)^3 / z^3$ or $z^3 (x^3 + y^3)^2 = a^9 (x - y)^3$.

Orthogonal surfaces finding examples

Ex. 1. Find the surface which intersects the surfaces of the system z(x + y) = c(3z + 1) orthogonally and which passes through the circle $x^2 + y^2 = 1$, $z = 1$.

Sol. The given system of surfaces is $f(x, y, z) \equiv \{z(x + y\}/(3z + 1) = C$. ...(1)

$$\therefore \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \frac{\partial f}{\partial z} = (x+y)\frac{(3z+1) - z \times 3}{(3z+1)^2} = \frac{x+y}{(3z+1)^2}.$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \quad \text{or} \quad \frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $z(3z + 1)p + z(3z + 1)q = x + y$. ...(2)

Lagrange's auxiliary equations for (2) are $\dfrac{dx}{z(3z+1)} = \dfrac{dy}{z(3z+1)} = \dfrac{dz}{x+y}$ ...(3)

Taking the first two fractions of (3), we get $dx - dy = 0$ so that $x - y = C_1$. ...(4)

Choosing x, y, $-z(3z + 1)$ as multipliers, each fraction of (3) = $[xdx + ydy - z(3z+1)dz]/0$

$\therefore \quad xdx + ydy - 3z^2dz - zdz = 0 \quad or\ 2xdx + 2ydy - 6z^2\, dz - 2zdz = 0$

Integrating, $x^2 + y^2 - 2z^3 - z^2 = C_2$, C_2 being an arbitrary constant . ...(5)

Hence any surface which is orthogonal to (I) has equation of the form

$(x^2 + y^2 - 2z^3 - z^2 = \varphi(x - y)$, φ being an arbitrary function ...(6)

In order to get the desired surface passing through the circle $x^2 + y^2 = 1$, $z=1$ we must choose $\varphi(x - y) = -2$. Thus, the required particular surface is $x^2 + y^2 - 2z^3 - z^2 = -2$.

Ex. 2. Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by $x(x^2 + y^2 + z^2) = C_1 y^2$.

Sol. Given family of surfaces is $x(x^2 + y^2 + z^2)/y^2 = C_1$

Let $f(x,y,z) = x(x^2 + y^2 + z^2)/y^2 = C_1$ ...(1)

Then the surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z} \quad \text{or} \quad \frac{dx}{(3x^2+y^2+z^2)/y^2} = \frac{dy}{-2x(x^2+z^2)/y^3} = \frac{dz}{2x/y^2 z}$$

$$\frac{dx}{y(3x^2+y^2+z^2)} = \frac{dy}{-2x(x^2+z^2)} = \frac{dz}{2xyz}$$

Taking x, y, z as multipliers, each fraction of (2)

$$= \frac{xdx+ydy+zdz}{xy(3x^2+y^2+z^2)-2xy(x^2+z^2)+2xyz} = \frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} \quad \text{...(3)}$$

Combining this fraction (3) with the last fraction of (2), we get

$$\frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} = \frac{dz}{2xyz} \quad \text{or} \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$$

Integrating, $\log(x^2 + y^2 + z^2) = \log z + \log C_2$ or $(x^2 + y^2 + z^2)/z = C_2$ ...(4)

Taking $4x, 2y, 0$ as multipliers, each fraction of (2)

$$= \frac{4xdx+2ydy}{4xy(3x^2+y^2+z^2)-4xy(x^2+y^2)} = \frac{4xdx+2ydy}{4xy(2x^2+y^2)} \quad \text{...(5)}$$

Combining this fraction (5) with the last fraction of (2), we get

$$\frac{4xdx+2ydy}{4xy(2x^2+y^2)} = \frac{dz}{2xyz} \quad \text{or} \quad \frac{4xdx+2ydy}{2x^2+y^2} = \frac{2dz}{z}$$

Integrating, $\log(2x^2 + y^2) = 2\log z + \log C_3$ or $(2x^2 + y^2)/y^2 = C_3$ ...(6)

From (4) and (5), the required general equation of the surfaces which are orthogonal to the given family of surfaces (1) is of the form $(x^2 + y^2 + z^2)/z = \varphi\{(2x^2 + y^2)/z^2\}$, *i.e.*,

or $x^2 + y^2 + z^2 = z\varphi\{(2x^2 + y^2)/z^2\}$, where φ is an arbitrary function.

Ex. 3. Find the surface which is orthogonal to the one parameter system $z = cxy(x^2 + y^2)$ which passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$

Sol. The given system of surfaces is $f(x,y,z) = z/(x^3 y + xy^3) = C$ ...(1)

$$\frac{\partial f}{\partial x}=-\frac{z\left(3x^2y+y^3\right)}{\left(x^3y+xy^3\right)^2},\quad \frac{\partial f}{\partial y}=-\frac{z\left(3y^2x+x^3\right)}{\left(x^3y+xy^3\right)^2},\ \frac{\partial f}{\partial z}=\frac{1}{x^3y+xy^3}$$

The required orthogonal surface is solution of $p(\partial f/\partial x) + q(\partial f/\partial y) = \partial f/\partial z$

or $\quad -\frac{z\left(3x^2y+y^3\right)}{\left(x^3y+xy^3\right)^2}p-\frac{z\left(3y^2x+x^3\right)}{\left(x^3y+xy^3\right)^2}q=\frac{1}{x^3y+xy^3}$

or $\quad \{(3x^2 + y^2)/x\}p + \{(3y^2 + x^2)/y\}q = -(x^2+y^2)/z \qquad$ ...(2)

Lagrange's auxiliary equations for (2) are $\frac{dx}{\left(3x^2+y^2\right)/x}=\frac{dy}{\left(3y^2+x^2\right)/y}=\frac{dz}{-\left(x^2+y^2\right)/z}$ ...(3)

Taking the first two fractions of (3), $2xdx - 2ydy = 0$ so that $x^2 - y^2 = C_1$

Choosing $x,y,4z$ as multipliers, each fraction of (3) = $(xdx + ydy + 4zdz)/0$

$\therefore\ 2xdx + 2ydy + 8zdz = 0$ so that $x^2 + y^2 + 4z^2 = C_2$

Hence any surface which is orthogonal to (1) is of the form

$x^2 + y^2 + 4z^2 = \Phi(x^2 - y^2),\Phi$ being an arbitrary function. ...(4)

For the particular surface passing through the hyperbola $x^2 - y^2 = a^2, z = 0$ we must take $\Phi(x^2 - y^2) = a^4 (x^2 + y^2)/(x^2 - y^2)^2$. Hence, the required surface is given by

$(x^2 + y^2 + 4z^2)^2 (x^2 - y^2)^2 = a^4 (x^2 + y^2)$

Ex. 4. Find the family orthogonal to $\varphi[z(x + y)^2, x^2 - y^2] = 0$.
Sol. Given $\qquad \varphi[z(x + y)^2, x^2 - y^2] = 0 \qquad$ ...(1)

Let $\qquad u = z(x + y)^2$ and $v = x^2 - y^2$ ...(2)

Then (1) becomes $\qquad \varphi(u, v) = 0 \qquad$ ...(3)

From (2), $\qquad (\partial u/\partial x) = 2z(x + y), \quad (\partial u/\partial y)=2z(x + y), \quad (\partial u/\partial z)=(x + y)^2,$

$\qquad (\partial v/\partial x) = 2x, \qquad (\partial v/\partial y) = -2y, \quad (\partial v/\partial z) = 0$

The PDE represented after eliminating ϕ

$py(x + y) + qx(x + y) = -2z(x + y)$ or $py + qx = -2z$ ...(8)

which is differential equation of the family of surfaces given by (1). So the differential equation of the family of surfaces orthogonal to (8) is given by

$ydx + xdy - 2zdz = 0$ or $d(xy) - 2zdz = 0.$ ...(9)

Integrating (9), $xy - z^2 = C$,

which is the desired family of orthogonal surfaces, C being parameter

Ex. 5. Find the family of surfaces orthogonal to the family of surfaces given by the differential equation $(y + z)p + (z + x)q = x + y$.

Sol. Let $P = y + z$, $Q = z + x$ and $R = x + y$. ...(1)

Then, the given differential equation can be written as $Pp + Qq = R$. ...(2)

Now, the differential equation of the family of surfaces orthogonal to the given family is

$Pdx + Qdy + Rdz = 0$ or $(y + z)dx + (z + x)dy + (x + y)dz = 0$

or $(ydx + xdy) + (ydz + zdy) + (zdx + xdz) = 0.$

Integrating, $y + yz + zx = C$, which is the required family of surfaces, C being a parameter.

PREVIOUS QUESTIONS ANALYSIS

Q1.1. Find the integral surface of the following quasi-linear equation

$(y-\phi)\frac{\partial\phi}{\partial x}+(\phi-x)\frac{\partial\phi}{\partial y}=x-y$. Which passes through the curve $\phi=0$, $xy = 1$ and through the circle $x+y+\phi=0, x^2+y^2+\phi^2=a^2$. **[6a UPSC CSE 2024]**

Refer examples Integral Surfaces finding problems

Q1.2. Find the integral surface of the partial differential equitation:

$(x-y)y^2\frac{\partial z}{\partial x}+(y-x)x^2\frac{\partial z}{\partial y}=(x^2+y^2)z$ that contains the curve: $xz=a^3$, $y=0$ on it.

Refer example 7, page no. 45 type 3.

• Taking the first two fractions of • Choosing 1, –1, 0 as multipliers,

solution is $\phi(x^3+y^3,(x-y)/z)=0$, ϕ being an arbitrary function.

We have $x^3+y^3=3c_1..(1)\ \&\ c_2=(x-y)/z...(2)$. So on taking $x=t, z=a^3/t, y=0$, we get from (1) and (2); $t^3=3c_1\ \&\ t^2=a^3c_2 \Rightarrow 9c_1^{\ 2}=a^6c_2^{\ 3} \Rightarrow 9(x^3+y^3)^2=a^6\left(\frac{x-y}{z}\right)^3$ is required integral surface. **[6a UPSC CSE 2020]**

Q2. Find the general solution of the partial differential equation $p\tan x+q\tan y=\tan z$ where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$. **[(5e) 2020 IFoS]** **Refer Type-1 Problems Ex. 1. Page no. 6**

Q3. Find the general solution of the partial differential equation:

$(y^3x-2x^4)p+(2y^4-x^3y)q=9z(x^3-y^3)$, where $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$, and find its integral surface that passes through the curve: $x=t, y=t^2, z=1$. **[6a UPSC CSE 2018]**

Refer Ex 9 page 13; solution is $\phi\left(xyz^{1/3}, y/x^2+x/y^2\right)=0,\ \phi$ being an arbitrary function.

$xyz^{1/3}=c_1 \Rightarrow t.t^2.1=c_1\ \&\ y/x^2+x/y^2=c_2 \Rightarrow t+\frac{1}{t}=c_2$. So on eliminating the parameter t from two equations we get $(c_1)^{1/3}+\frac{1}{(c_1)^{1/3}}=c_2 \Rightarrow \left(xyz^{1/3}\right)^{2/3}+1=\left(\frac{y}{x^2}+\frac{x}{y^2}\right)\left(xyz^{1/3}\right)^{1/3}$.

Q4. Solve $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$, where $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$. If the solution of the above equation represents a sphere, what will be the coordinates of its centre?

Refer Ex 8 Type 2 page no. 12. solution is $\phi\left(y^2-z^2-2yz, x^2+y^2+z^2\right)=0,\ \phi$ being an arbitrary function.From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at $(0,0,0)$, i.e., origin. **[(7a) UPSC CSE 2018]**

Q5. Solve the partial differential equation: $(x-y)\frac{\partial z}{\partial x}+(x+y)\frac{\partial z}{\partial y}=2xz$. **[(6a) UPSC CSE 2017]**

Q6. Find the general integral of the partial differential equation $(y+zx)p-(x+yz)q=x^2-y^2$.

Refer Solution 12 of assignment on page no. 31 **[5e UPSC CSE 2016]**

Q7. Find the general solution of the partial differential equation

$xy^2\frac{\partial z}{\partial x}+y^3\frac{\partial z}{\partial y}=(zxy^2-4x^3)$. **[(5b) UPSC CSE 2016]**

Q8. Solve the partial differential equation $\left(y^2+z^2-x^2\right)p-2xyq+2xz=0$ where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$. **[5a UPSC CSE 2015]** **Refer Example 3 Type 3 on page no 16.**

Q9. Solve for the general solution $p\cos(x+y)+q\sin(x+y)=z$, where $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$.

Refer Example 9 Type 3 on page no 19. **[6a UPSC CSE 2015]**

Q10. Solve the PDE: $xu_x+yu_y+zu_z=xyz$. **[(6a) UPSC CSE 2013]**

Hint: Follow the method for 3 independent variable s, Article 2

Q11. Solve the partial differential equation $px+qy=3z$. **[6a UPSC CSE 2012]**

Q12. Solve $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy$ using Lagrange's Method. **[(8a) UPSC CSE 2012]**

Refer Example 6 Type 3 on page no 17.

Q13. Solve the PDE $(x+2z)\frac{\partial z}{\partial x}+(4zx-y)\frac{\partial z}{\partial y}=2x^2+y$. **[5b UPSC CSE 2011]**

Refer Example 7 Type 2 on page no 12.

Q14. Find the general solution of $x\left(y^2+z\right)p+y\left(x^2+z\right)q=z\left(x^2-y^2\right)$. **[(5a) 2010 IFoS]**

Refer Example 6 Type 2 on page no 11.

ORTHOGONAL SURFACES

For answers, follow examples based on finding orthogonal surfaces from page no. 46

Hint: Let $z=\phi(x,y)$ ....(1) be the equation of the given surface

Let $F(x,y,z)=\phi(x,y)-z$. ....(2)

From (1) and (2), $\frac{\partial F}{\partial x}=\frac{\partial \phi}{\partial x}=\frac{\partial z}{\partial x}=p$, $\frac{\partial F}{\partial y}=\frac{\partial \phi}{\partial y}=\frac{\partial z}{\partial y}=q$, $\frac{\partial F}{\partial z}=-1$

Since ∇F is normal to the surface $F(x,y,z)=0$, $\partial F/\partial x, \partial F/\partial y, \partial F/\partial z$ i.e. $p, q-1$ are direction ratios of the normal to $F(x,y,z)=0$ or $\phi(x,y)$.

Let required surfaces are given by $\psi(x,y,z)=0$. So we have

$\frac{\partial \psi}{\partial x} p+\frac{\partial \psi}{\partial y} q+\frac{\partial \psi}{\partial z}.(-1)=0$. Now by taking ϕ as given surfaces and find it's derivatives and use in this equation then by solving this resulting PDE we get the required surfaces.

Q15. Find the system of equations for obtaining the general equation of surfaces orthogonal to the family given by $x\left(x^2+y^2+z^2\right)=Cy^2$, where C is a parameter. **[6a IFoS 2022]**

Refer example 2 page no. 46

Q16. Find the equations of the system of curves on the cylinder $2y=x^2$ orthogonal to its intersection with the hyperboloids of the one-parameter system $xy=z+c$. **[(7c) 2019 IFoS]**

Q17. Find the surface which is orthogonal to the family of surfaces $z(x+y)=c(3z+1)$ and which passes through the circle $x^2+y^2=1, z=1$. **[(6b) 2017 IFoS]**

Q18. Find the general equation of surfaces orthogonal to the family of spheres given by $x^2+y^2+z^2=cz$. **[5a UPSC CSE 2016]**

Q19. Find the surface which intersects the surfaces of the system $z(x+y)=C(3z+1)$, (C being a constant) orthogonally and which passes through the circle $x^2+y^2=1, z=1$. **[6b UPSC CSE 2013]**

Chapter: First Order Non-linear , PDEs

The PDE: $f(x, y, z, p, q) = 0$ represents a non-linear PDE of order 1

E.g. (1) $x^2 - y + zp^4 + pq = 0$ (2) $z^2 - p^3 + pq + q^2 = 0$ (3) $x + y + z + pq = 0$

Here z is dependent on two independent variable x & y

$$\therefore dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \;\Rightarrow\; dz = pdx + qdy$$

Observe : suppose if someone asked us to solve a PDE $f(x, y, z, p, q) = 0$ ....(1)

Step (1) Let if somehow, we get $p = \phi(x, y, z)$ & $q = \psi(x, y, z)$

Step (2) : Now, by using p & q from step (1) in $dz = pdx + qdy$ and then solving this ordinary differential equation, we get $\boxed{z = \zeta(x, y, c_1, c_2)}$ ....(2) c_1 & c_2 are arbitrary constants.

Hence (2) is known as complete solution of (1).

Charpit's method

How to solve the PDE: $f(x, y, z, p, q) = 0$...(1)

Step (i):- To find p & q from (1), we solve Charpit's auxiliary equations; a system of differential equations which are given by

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \text{ ...(2)}$$

Now, by taking combinations of two fractions from (2) (like Lagrange's method) we try to get p & q .

Notice: here we'll get two constants of integration c_1 & c_2, while solving differential equations from (2)

Step (ii):- Now , by using p & q from step (i) in $dz = pdx + qdy$

And solving this differential equation, we get the complete integral or complete solution of (1).

Till Now, we've focused: on the general procedure to solve PDE $f(x, y, z, p, q) = 0$ ......(1)

Now we categories (1) into some special forms: If (1) is of the form

(i) $\boxed{f(p,q)=0}$

i.e., given PDE is a

function of $p \,\&\, q$ only

By general procedure

$p=a,\ q=b$ and

$z=ax + by + c$

(ii) $\boxed{z=px+qy+g(p,q)}$

known *n* as Clairaut's form

$f\left(x,y,z,p,q\right) = z-pz-qy-g(p,q)$

E.g. $z = px + qy + p^2 + q^2$, is

a PDE in Clairaut's from

(iii) $\boxed{f(p,q,z)}=0$

singular solution etc.

Now, we've interest in:-

Some given PDE can be reduced into these special forms!!

E.g. Solver $z^2p^2y+6zpxy+2zqx^2+4x^2y = 0$

Way 1:- By general procedure: always free to use, no issue, but may be lengthy.

Way 2:- Let's learn to reduce given PDE in some special form (if possible, as this is not rule; it just depends, on practicing examples.

☐ (1) can be managed as $\frac{z^2}{x^2y}p^2y+\frac{6zpxy}{x^2y}+\frac{2zqx^2}{x^2y}+4=0$

$$\Rightarrow \quad \left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2+6\left(\frac{z\partial z}{x\partial x}\right)^2+2\left(\frac{z}{R}\frac{\partial z}{\partial y}\right)+4=0 \qquad .....(2)$$

Procedure :- $x\partial x=\partial X$, $y\partial y=\partial Y$, $z\partial z=\partial Z$

$$\frac{x^2}{2}=X\ ,\qquad \frac{y^2}{2}=Y\ ,\qquad \frac{z^2}{2}=Z \qquad .....(3)$$

i.e., $\frac{z}{x}\frac{\partial z}{\partial x}=\frac{\partial Z}{\partial x}=P$, $\frac{z}{y}\frac{\partial z}{\partial y}=\frac{\partial Z}{\partial y}=Q$

On using (3) in (2), we get $P^2 + 6P + 2Q+ 4 = 0 \Rightarrow f\left(P,\ Q\right) = O$ .....(4)

$\therefore$ Complete integral is, $Z = aX + bY + C \Rightarrow \frac{z^2}{2}=a\frac{x^2}{2}+b\frac{y^2}{2}+C$

Next Category of problems.

Solving PDE $f\left(x,\ y,\ z,\ p,\ q\right) = 0 \ \ldots(1)\ with\ cond..x=f\left(\lambda\right), y=g\left(\lambda\right), z=h\left(\lambda\right)$

Finding integral surface given by (1) satisfying condition (2)

Working rule for solving Cauchy's problems

- Let the given PDE is $f(x, y, z, p, q) = 0 \ldots (1)$

Suppose we wish to find the integral surface of (1) which passes through a given curve with parametric equation

$x = f_1(\lambda),\ y = f_2(\lambda),\ z = f_3(\lambda),\ \lambda$ being the parameter ...(2)

- then in the solution $x = x\left(p_0, q_0, x_0, y_0, t_0, t\right)$ of the characteristic equations of (1)

$$dx/dt = \partial f/\partial p,$$
$$dy/dt = \partial f/\partial q,$$
$$dz/dt = p\left(\partial f/\partial p\right) + q\left(\partial f/\partial q\right),$$
$$dp/dt = -\left(\partial f/\partial x\right) - p\left(\partial f/\partial z\right),$$
$$dq/dt = -\left(\partial f/\partial y\right) - q\left(\partial f/\partial z\right)$$

, we shall assume that $x_0 = f_1(\lambda),\quad y_0 = f_2(\lambda),\quad z_0 = f_3(\lambda)$ are the initial values of x, y, z respectively.

- Then the corresponding initial values of p_0, q_0 can be obtained by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_3'(\lambda) \quad and \quad f\left\{f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0\right\} = 0$

[using in $dz = pdx + qdy \quad and \quad f(x, y, z, p, q) = 0$]

When the above values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 is substituted in characteristic equations of (1) , we shall be able to express x, y, z involving the two parameters *t* and λ of the form

$x = \phi_1(t, \lambda),\quad y = \phi_2(t, \lambda) \quad and \quad z = \phi_3(t, \lambda) \ldots (3)$

which are known as characteristics of (1)

Finally, by eliminating λ and *t* from (4), we arrive at a relation of the form *G*(*x*, *y*, *z*) = 0, which is the required equation of the integral surface of (1) passing through the given curve (2).

Ctegory-1 Examples

Ex. 1. Find a complete integral of $q = 3p^2$.

Sol. Here given equation is $f(x, y, z, p, q) = 3p^2 - q = 0$ ....(1)

- $\therefore$ Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$ ...(2)

$\dfrac{dp}{0 + p.0} = \dfrac{dp}{0 + q.0} = \dfrac{dz}{-6p^2 + q} = \dfrac{dx}{-6p} = \dfrac{dy}{1}$, using (1) in (2)

- Taking the first fraction, $dp = 0$ so that $p = a$. ....(3)

Substituting this value of p in (1), we get $q = 3a^2$. ....(4)

- Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + 3a^2dy$ so that $z = ax + 3a^2y + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 2. Find a complete integral of $px + qy = pq$.

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0$. ..... (1)

- Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

$\dfrac{dp}{-(x-q)} = \dfrac{dy}{-(y-q)} = \dfrac{dz}{-p(x-q) - q(y-p)} = \dfrac{dp}{p + p.0} = \dfrac{dq}{q + q.0}$

- Taking the last two fractions of (2), (1/p)dp = (1/q)dq.

Integrating, $\log p = \log q + \log a$ or $p = aq$. ..... (3)

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y$, as $q \neq 0$ ..... (4)

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y$ ..... (5)

- Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]dy$ or $adz = (ax + y)(adx + dy)$

$adz = (ax + y)d(ax + y) = udu$ where $u = ax + y$.

Integrating, $az = u^2/2 + b = (ax + y)^2/2 + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 3. Find a complete integral of $yzp^2 - q = 0$.

Sol. Here $f(x, y, z, p, q) = yzp^2 - q = 0$...(1)

• Charpit's uxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

$$\frac{dp}{0 + p(yp^2)} = \frac{dq}{zp^2 + q(yp^2)} = \frac{dz}{-2yzp^2 + q} = \frac{dx}{-2yzp} = \frac{dy}{1}, \ldots(2)$$

• Taking the first and fifth fractions, $(1/yp^3)\, dp = dy$

or p^{-3} dp = ydy or $-2p^{-3}$ dp = –2ydy.

Integrating, $p^{-2} = a^2 - y^2$ so that $p = 1/(a^2 - y^2)^{1/2}$.....(3)

Using (3), (1) ⇒ $q = yzp^2$ ⇒ $q = yz/(a^2 - y^2)$......(4)

• ∴ dz = pdx + qdy $= \dfrac{dx}{(a^2 - y^2)^{1/2}} + \dfrac{yzdy}{(a^2 - y^2)}$

or $(a^2 - y^2)^{1/2}$ dz – $\dfrac{yzdy}{(a^2 - y^2)^{1/2}}$ = dx or $d[z(a^2 - y^2)^{1/2}]$ = dx.

Integrating, $z(a^2 - y^2)^{1/2} = x + b$ or $z^2(a^2 - y^2) = (x + b)^2$, *a*, *b* being arbitrary constants.

Ex.4(a) Find a complete integral of $(p^2 + q^2)x = pz$.

(b). Find the compete integral of the partial differential equation $(p^2 + q^2)x = pz$ and deduce the solution which passes through the curve x = 0, $z^2 = 4y$.

Sol. Let f(x, y, q, p, q) = $(p^2 + q^2)x - pz = 0$. ....(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-fp} = \dfrac{dy}{-f_q}$

giving $dp/q^2 = dq/(-pq)$, by (1) or 2pdp + 2qdq = 0.

Integrating, $p^2 + q^2 = a^2$, where a is an arbitrary constant.......(2)

Solving (1) and (2), $p = a^2x/q$ and $q = (a/z) \times \sqrt{(z^2 - a^2x^2)}$ .... (3)

∴ dz = pdx + qdy = $\dfrac{a^2xdx}{z} + \dfrac{a\sqrt{(z^2 - a^2x^2)}dy}{z}$ or $\dfrac{zdz - a^2xdx}{\sqrt{(z^2 - a^2x^2)}}$ = ady.

Putting $z^2 - a^2x^2 = t$ so that $2(zdz - a^2\, xdx) = dt$, we get

$(1/2\sqrt{t}\,)$dt = adyor $(1/2)\times t^{-1/2}$ = ady.

Integrating, $t^{1/2} = ay + b$ or $\sqrt{(z^2 - a^2x^2)} = ay + b$, as $t = \sqrt{(z^2 - a^2x^2)}$

or $z^2 - a^2x^2 = (ay + b)^2$ or $z^2 = a^2x^2 + (ay + b)^2$ ...... (4)

(b) Proceeding as in part (a), (4) is the complete integral.

The parametric equations of the given curve x = 0, $z^2 = 4y$ are given by

x = 0, y = t^2, z = 2t ...... (5)

Therefore, from (4), we have $4t^2 = (at^2 + b)^2$ or $a^2t^4 + 2(ab - 2)t^2 + b^2 = 0$ ...... (6)

Equation (6) has equal roots if its discriminant = 0, i.e., if

$4(ab - 2)^2 - 4a^2b^2 = 0$ or $a^2b^2 = 1$ so that $b = 1/a$

Hence from (4), the appropriate one parameter sub-system is given by

$z^2 = a^2x^2 + (ay + 1/a)^2$ or $a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0$,

which is a quadratic equation in parameter 'a'. Therefore, this has for its envelope surface

$(2y - z^2)^2 - 4(x^2 + y^2) = 0$ or $\left(2y - z^2\right)^2 = 4\left(x^2 + y^2\right)$ ...... (7)

The desired solution is given by the function z defined by equation (7).

Ex. 5. Find a complete, singular and general integrals *of* $(p^2 + q^2)y = qz$

Sol. Here the given equation is $f(x, y, z, p, q) = \left(p^2 + q^2\right)y - qz = 0.$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

$\dfrac{dp}{-pq} = \dfrac{dq}{p^2} = \dfrac{dz}{-2p^2y + qz - 2q^2y} = \dfrac{dx}{-2py} = \dfrac{dy}{-2qy + z}$, by (1) ...... (2)

Taking the first two fractions, we get $2pdp + 2qdq = 0$ so that $p^2 + q^2 = a$ ...... (3)

Using (3), (1) gives $a^2y = qz$ or $q = a^2y/z.$

Putting this value of q in (3), we get

$$p = \sqrt{\left(a^2 - q^2\right)} = \sqrt{a^2 - \left(a^4y^2/z^2\right)} = \frac{a}{z}\sqrt{\left(z^2 - a^2y^2\right)}.$$

Now putting these values of *p* and q in $dz = pdx + qdy$, we have

$$dz = \frac{a}{z}\sqrt{\left(z^2 - a^2y^2\right)}dx + \frac{a^2ydy}{z}dy \text{ or } \frac{zdz - a^2ydy}{\sqrt{\left(z^2 - a^2y^2\right)}} = adx.$$

Integrating, $\left(z^2 - a^2y^2\right)^{1/2} = ax + b$ or $z^2 - a^2y^2 = (ax + b)^2$ ...... (4)

Which is a required complete integral, a, b being arbitrary constants.

Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$0 = 2ay^2 + 2(ax + b)x$ ...... (5) And $0 = 2(ax + b)$ .... (6)

Eliminating a and b between (4), (5) and (6), we get z = 0 which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$z^2 - a^2y^2 = [ax + \phi(a)]^2.$ ...... (7)

Differentiating (7) partially w.r.t. a, $-2ay^2 = 2[ax+\phi(a)].[x+\phi'(a)]$. ...... (8)

General integral is obtained by eliminating a from (7) and (8).

Ex. 6. Find a complete integral of $p\left(1+q^2\right)+(b-z)q=0$

Sol. Here given equation is $f(x,y,z,p,q) \equiv p\left(1+q^2\right)+(b-z)q=0$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

$\dfrac{dp}{pq} = \dfrac{dq}{p^2} = \dfrac{dz}{-p\left(1+q^2\right)-(b-z)q} = \dfrac{dx}{-\left(q^2+1\right)} = \dfrac{dy}{-2pq-(b-z)}$, by (1)

First two fractions give $(1/p)dp = (1/q)dq$ so that $q = pc$.

Putting $q = pc$ in (1), we have $p = \sqrt{[c(z-b)-1]}/c$.

∴ q = pc gives $q = \sqrt{[c(z-b)-1]}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = \sqrt{[c(z-b)-1]}\left(\dfrac{dx}{c}+dy\right)$ or $\dfrac{cdz}{\sqrt{[c(z-b)-1]}} = dx + cdy$

Integrating, $2\sqrt{[c(z-b)-1]} = x+cy+a$ or $4\{c(z-b)-1\} = (x+cy+a)^2$

Which is a complete integral, a and c being arbitrary constants.

Ex.7. Find a complete and singular integrals *of* $2xz - px^2 - 2qxy + pq = 0$

Sol. Here given equation is $f(x,y,z,p,q) = 2xz - px - 2qxy + pq = 0$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

$\dfrac{dp}{2z-2qy} = \dfrac{dq}{0} = \dfrac{dx}{x^2-q} = \dfrac{dy}{2xy-p} = \dfrac{dz}{px^2+2xyq-2pq}$, by (1)

The second fraction gives dq = 0 so that q = a

Putting = a in (1), we get $p = 2x(z-ay)/\left(x^2-a\right)$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$dz = \dfrac{2x(z-ay)}{x^2-a}dx + ady$ or $\dfrac{dz-ady}{z-ay} = \dfrac{2xdx}{x^2-a}$.

Integrating, $\log(z-ay) = \log\left(x^2-a\right) + \log b$

$z-ay=b\left(x^2-a\right)$ or $z=ay+b\left(x^2-a\right)$ ...... (2)

Which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get

$0 = y - b$ and $0=x^2-a$. ...... (3)

Solving (3) for a and b, $a = x^2$ and $b = y$ ...... (4)

Substituting the values of a and b given by (4) in (2), we get $z = x^2y$, which is the required singular integral.

Ex.8 Find a complete integral of $pxy + pq + qy = yz$.

Sol. Given $f(x,y,z,p,q)\equiv pxy+pq+qy-yz=0$ ...... (1)

Charpit's auxiliary equation are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+qf_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

$$\frac{dp}{0}=\frac{dq}{(px+q)+qy}=\frac{dz}{-p(xy+q)-q(p+y)}=\frac{dx}{-(xy+q)}=\frac{dy}{-(p+y)},\text{ by (1)}$$

The first fraction gives $dp = 0$ so that $p = a$.

Putting $p = a$ in (1) gives $axy + aq + qy = yz$ so that $q = y(z - ax)/(a + y)$.

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz=adx+\frac{y(z-ax)}{a+y}dy \text{ or } \frac{dz-adx}{z-ax}=\frac{ydy}{a+y}=\left(1-\frac{a}{a+y}\right)dy$$

Integrating, $\log(z - ax) = y - a\log(a + y) + \log b$, a, b, being arbitrary constants.

Or $\log(z-ax)+\log(a+y)^a-\log b=y$ or $(z-ax)(y+a)^a=be^y$

Ex. 9 *Find a complete integral* $p^2+q^2-2px-2qy+1=0.$

Sol. Given $f(x,y,z,p,q)\equiv p^2+q^2-2px-2qy+1=0.$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+qf_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

$$\frac{dp}{-2p}=\frac{dq}{-2q}=\frac{dz}{-p(2p-2x)-q(2q-2y)}=\frac{dx}{-(2p-2y)}=\frac{dy}{-(2q-2y)},\text{ by (1)}$$

The first two fractions give $(1/p)dp = (1/q)dq$ so that $p = aq$.

Putting $p = aq$ in (1), $a^2q^2+q^2-2aqx-2qy+1=0$ or $\left(a^2+1\right)q^2-2(ax-y)q+1=0.$

$$\Rightarrow q=\frac{2(ax+y)\pm\sqrt{\left\{4(ax+y)^2-4\left(a^2+1\right)\right\}}}{2\left(a^2+1\right)},\quad p=aq=a\frac{2(ax+y)\pm\sqrt{\left\{4(ax+y)^2-4\left(a^2+1\right)\right\}}}{2\left(a^2+1\right)}.$$

Putting these values of *p* and *q* in *dz = p dx + y dy*, we get

$$dz = \frac{(ax+y) \pm \sqrt{\left\{(ax+y)^2 - \left(a^2+1\right)\right\}}}{\left(a^2+1\right)}(adx+dy). \text{...... (2)}$$

Put *ax + y = v* so that a *dx + dy = dv*. Then (2) gives

$$\left(a^2+1\right)dz = \left[v \pm \sqrt{\left\{v^2 - \left(a^2+1\right)\right\}}\right]dv.$$

Integrating, $\left(a^2+1\right)z = v^2/2 \pm \left[(v/2) \times \sqrt{\left\{v^2 - \left(a^2+1\right)\right\}}\right.$

$$\left. -(1/2) \times \left(a^2+1\right)\log\left(v + \sqrt{\left\{v^2 - \left(a^2+1\right)\right\}}\right)\right] + b$$

is the complete integral, where *v = ax + b* and *a*, *b* are arbitrary constants.

Ex.10. Find a complete integral of $p^2x + q^2y = z$

Sol. Given equation is $f(x,y,z,p,q) = p^2x + q^2y - z = 0$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} - \dfrac{dy}{-f_q}$

Or $\dfrac{dp}{-p+p^2} = \dfrac{dq}{-q+q^2} = \dfrac{dz}{-2\left(p^2x+q^2y\right)} = \dfrac{dx}{-2px} = \dfrac{dy}{-2qy}$, by (1) ...... (2)

Now, each fraction in (2) $= \dfrac{2pxdp + p^2dx}{2px\left(-p+p^2\right) + p^2(-2px)} = \dfrac{2qydq + q^2dy}{2qy\left(-q+q^2\right) + q^2(-2qy)}$

Or $\dfrac{d\left(p^2x\right)}{-2p^2x} = \dfrac{d\left(q^2y\right)}{-2qy}$ i.e., $\dfrac{d\left(p^2x\right)}{p^2x} = \dfrac{d\left(q^2y\right)}{q^2y}$

Integrating it, $\log\left(p^2x\right) = \log\left(q^2y\right) + \log a$ or $p^2x = q^2ya.$ ...... (3)

Form (1) and (3), $aq^2y + q^2y = z$ or $q = [z/(1+a)]^{1/2}$ ...... (4)

Form (3) and (4), $p = q\left(\dfrac{ya}{x}\right)^{1/2} = \left\{\dfrac{za}{(1+a)x}\right\}^{1/2}.$

Putting the above values of p and q in dz = p dx + q dy, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2}dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2}dy \text{ or } (1+a)^{1/2}z^{-1/2}dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $(1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b, a, b$ being arbitrary constants.

Ex. 11. Find a complete integral of $2(z + px + qy) = yp^2$

Sol. Given equation is $f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-py_p - qy_y} = \dfrac{dp}{-f_q} = \dfrac{dq}{-f_q}$

Or $\dfrac{dp}{2p + 2p} = \dfrac{dp}{2q - p^2 + 2q} = \dfrac{dz}{-p(2x - 2yp) - q \times 2y} = \dfrac{dx}{-(2x - 2yp)} = \dfrac{dy}{-2y}, by(1)$

Taking the first and the last fractions, $\dfrac{dp}{4p} = \dfrac{dy}{-2y}$ or $\dfrac{dp}{p} + 2\dfrac{dy}{y} = 0.$

Integrating, $\log p + 2\log y = \log a$ or $py^2 = a.$ ...(2)

Solving (1) and (2) for *p* and *q*, $p = \dfrac{a}{y^2}$ and $q = -\dfrac{z}{y} - \dfrac{ax}{y^3} + \dfrac{a^2}{2y^4}.$

$\therefore dz = p\,dx + q\,dy = \dfrac{a}{y^2}dx + \left[-\dfrac{z}{y} - \dfrac{ax}{y^3} + \dfrac{a^2}{2y^4}\right]d$ Multiplying both sides by *y* and re–arranging, we get

$(ydz + zdy) - a\left(\dfrac{ydx - xdy}{y^2}\right) - \dfrac{a^2}{2y^3}dy = 0$ or $d(yz) - ad\left(\dfrac{x}{y}\right) - \dfrac{a^2}{2}y^{-3}dy = 0.$

Integrating, *yz* – *a*(*x*/*y*) + (*a*2/4*y*2) = *b*, *a*, *b* being arbitrary constants. ... (3)

Ex. 12. Find a complete integral of $z^2 = pqxy.$ ...(1)

Charpits's auxiliary equations are $\dfrac{dp}{f_x + pf_y} = \dfrac{dp}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_q} = \dfrac{dy}{-f_q}$

$\dfrac{dp}{-pqy + 2pz} = \dfrac{dp}{-pqy + 2qz} = \dfrac{dz}{-p(-qxy) - q(-pxy)} = \dfrac{dx}{qxy} = \dfrac{dy}{pxy}, by(1)$ ...(2)

Each fraction of (2) = $\dfrac{x\,dp + p\,dx}{x(-pqy + 2pz) + pqxy} = \dfrac{y\,dp + q\,dy}{y(-pqx + 2qz) + pqxy}$

$\dfrac{xdp + pdx}{2pxz} = \dfrac{ydq + qdy}{2qyz}$ or $\dfrac{d(xp)}{xp} = \dfrac{d(yq)}{yq.}$

Integrating, $\log(xp) = \log(yq)\log a^2$ or $xp = a^2 yq$ ...(3)

Solving (1) and (2) for p and q, $p = (az)/z$ and $q = z/(ay).$

$\therefore dz = p\,dx + q\,dy = (az/x)\,dx + (z/ay)$ and $(1/z)dz = (a/x)\,dx + (1/ay)\,dy.$

Integrating, $\log z = a\log x + (1/a)\log y + \log b$ or $z = x^a y^{1/b} b.$

Ex.13. Using Charpit's method, find the complete integrals of $pq = px + qy$

Sol. Here given equation is $f(x,y,z,p,q)=pq-px-qx=0$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_z+p\,f_z}=\dfrac{dp}{f_y+q\,f_z}=\dfrac{dz}{-pf_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

$$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-p(q-x)-p(p-y)}=\frac{dx}{-(q-x)}=\frac{dy}{-(p-y),}by(1)\;...(2)$$

To find first complete integral . Taking the first two fractions of (2), we get

$(1/p)dp=(1/q)dq$ so that $\log p=\log q+\log a$ or $p=aq.$ ...(3)

Using (3), (1) $\Rightarrow$ $aq^2=q(ax+y)$ $\Rightarrow$ $q=(ax+y)/a$ ...(4)

Here, From (3), we have $p=ax+y$ ...(5)

$\therefore dz=p\,dx+q\,dy=(ax+y)dx+[(ax+y)/a]dy=(1/a)(ax+y)(a\,dx+y).$

Putting $ax+y=t$ so that $z=(1/2a)\times t^2+b$ or $z=(1/2a)\times(ax+y)^2+b,$ as $t=ax+y$

To find second complete integral. Taking the second and the fourth ratios in (2) , we get

$dx/(q-x)=dq/q$ or $q\,dx+x\,dq=q\,dp.$

Integrating, $qx=q^2/2+a/2$ or $q^2-2xq+a=0.$

$\therefore\; q=[2x\pm 2(x^2-a)^{\frac{1}{2}}]/2$ so that $q=x+(x^2-a)^{1/2}$ ...(6)

Using(6), (1) $\Rightarrow$ $p[x+(x^2-a)^{1/2}]-px-y[x+(x^2-a)^{1/2}]=0$

So that $p=\{1+x/(x^2-a)^{1/2}\}y.$ ...(7)

$\therefore$ $dz=p\,dx+q\,dy=\{1+x/(x^2-a)^{1/2}\}y dx+[x+(x^2-a)^{1/2}]dy$

$dz=(y\,dx+x\,dy)+\left[\dfrac{xy\,dy}{(x^2-a)^{1/2}}+(x^2-a)^{1/2}dy\right]$ or $dz=d(xy)+d[y(x^2-a)^{1/2}]$

Integrating, $z=xy+y(x^2-a)^{1/2}+b,a,$ being arbitrary constants.

To find third complete integral. Taking the first and the fifth rations of (2) and proceeding as above third complete integral is $z=xy+x(y^2-a)^{1/2}+b$

Ex.14. Find complete integral of $xp+3yq=2(z-x^2q^2).$

Sol. Given equation is $f(x,y,z,p,q)=xp+3yq-2z+2x^2q^2=0.$...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_z+pf_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

$$\frac{dp}{-p+4xq^2}=\frac{dq}{q}=\frac{dz}{-px-q(3y+4x^2q)}=\frac{dx}{-x}=\frac{dy}{-3y-4x^2q},by(1)\qquad ...(2)$$

$$(2) \Rightarrow \frac{dq}{q} = \frac{dx}{-x} \Rightarrow \log q = \log a - \log x \Rightarrow qx = a \Rightarrow q = \frac{a}{x} \qquad ...(3)$$

Using (3),(1) $\Rightarrow xp + 3y(a/x) - 2z + 2x^2(a^2/x^2) = 0 \qquad \Rightarrow p = \frac{2(z-a^2)}{x} - \frac{3ay}{x^2}$ ...(4)

$$\therefore \qquad dz = p\,dx + q\,dy = \left\{\frac{2(z-a^2)}{x} - \frac{3ay}{x^2}\right\}dx + \frac{a}{x}dy$$

$$x^2 dz = 2x(z-a^2)dx - 3ay\,dx + ax\,dy \quad or \quad x^2 dz - 2(z-a)dx = -3ay\,dy + ax\,dy$$

$$\frac{x^2 dz - 2x(z-a^2)dx}{x^4} = \frac{3ay\,dx}{x^4} + \frac{a\,dy}{x^3} \text{ or } d\left(\frac{z-a^2}{x^2}\right) = d\left(\frac{ay}{x^3}\right)$$

Integrating, $(z-a^2)/x^2 = (ay)/x^3 + b$ or $z = a(a + y/x) + bx^2$

Ex. 15. find complete integral of $p^2 + q^2 - 2pq\tan h\,2y = \sec h^2 2y$.

Sol. $f(x,y,z,p,q) = p^2 + q^2 - 2pq\tanh 2y - \sec h^2\,2y = 0$ ...(1)

Charpit's auxiliary equations are $\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$

$$\frac{dp}{0} = \frac{dp}{-4pq\sec h^2\,2y + 4\sec h^2\,2y\tan 2y} = ..., by\ (1)$$

Then, first fraction $\Rightarrow \quad dp = 0 \Rightarrow p =$ constant = a, say ...(2)

Using (2), (1) $\Rightarrow \quad q^2 - (2a\tanh 2y)q + a^2 - \sec h^2\,2y = 0$

$$\Rightarrow \qquad q = \left[2a\tanh 2y \pm 2\sqrt{\left(a^2\tanh^2 2y - a^2 + \sec h^2\,2y\right)}\right]/2$$

$$\Rightarrow \qquad q = a\tanh 2y + \sqrt{\left(1-a^2\right)}, \sec h\,2y.$$

$\left[Note\,that\,\sec h^2 2y = 1 - \tanh^2 2y\right]$

Using (2) and (3), $dz = p\,dx + \{a\tanh 2y + \sqrt{\left(1-a^2\right)}\sec h\,2y\}\,dy$

Integrating, $z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{\left(1+a^2\right)}\int\frac{2\,dy}{e^{2y} + e^{-2y}}$

$$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{\left(1-a^2\right)}\int\frac{2e^{2y}\,dy}{1 + \left(e^{2y}\right)^2}$$

$$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{\left(1+a^2\right)}\tan^{-1}\left(e^{2y}\right)$$

$$\left[\therefore on\ putting\ e^{2y}=t \quad and \quad 2e^{2y}dy=dt, \int \frac{2e^{2y}dy}{1+\left(e^{2y}\right)^2}=\int \frac{dt}{1+t^2}=\tan^{-1}e^{2y}\right]$$

Ex.16. Find complete integral of $xp-yq=xq\,f\left(z-px-qy\right)$.

Sol. $f\left(x,y,z,p,q\right)=xp-yq-xq\,f\left(z-px-qy\right)=0 \qquad ...(1)$

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x+p\left(\partial F/\partial z\right)}=\frac{dq}{\partial F/\partial y+q\left(\partial F/\partial z\right)}=\frac{dz}{-p(\partial F/\partial p)-q\left(\partial F/\partial q\right)}=\frac{dx}{-\left(\partial F/\partial p\right)}=\frac{dy}{-\left(\partial F/\partial q\right)}=$$

$$\frac{dp}{p-qf+zpqf'-pqxf'}=\frac{dq}{-q+xq^2f'-xq^2f'}=.......by(2) \qquad ...(3)$$

Each ratio of $=\dfrac{x\,dp+y\,dq}{xp-yq-qxf}=\dfrac{x\,dp+y\,dq}{0}$, by (2)

$\Rightarrow \qquad x\,dp+y\,dq=0 \quad \Rightarrow \qquad x\,dp+\ y\,dq+p\,dx+q\,dy=p\,dx+q\,dy$

$\Rightarrow dz-d\left(xp\right)-d\left(yq\right)=0\ as\ dz=pdx+qdy$

Integrating, $\qquad z-xp-yq=$ constant = a, say $\qquad ...(4)$

$\therefore \qquad xp+yq=z-a \qquad ...(5)$

Using (4), (1) become $\quad xp-yq=x\,q\,f\left(a\right) \qquad ...(6)$

Subtracting (6) from (5), $2yq=z-a-xqf\left(a\right) \quad \Rightarrow \qquad q=\left(z-a\right)/\{2y+xf\left(a\right)\} \qquad ...(7)$

Using (7), (5) $\Rightarrow \qquad p=\dfrac{\left(z-a\right)\{y+xf\left(a\right)\}}{x\{2y+xf\left(a\right)\}} \qquad ...(8)$

Using (7) and (8), $\qquad dz=p\,dx+q\,dy$ reduces to

$$dz=\left(z-a\right)\left[\frac{\{y+xf\left(a\right)\}dx}{x\{2y+xf\left(a\right)\}}+\frac{dy}{2y+xf\left(a\right)}\right]$$

$$\frac{2dz}{z-a}=\frac{2y\,dx+2xf\left(a\right)dx+2x\,dy}{x\{2y+xf\left(a\right)\}}=\frac{2d\left(xy\right)+2xf\left(a\right)dx}{2xy+x^2f\left(a\right)}$$

Integrating, 2 log $\left(z-a\right)=\log\{2xy+x^2f\left(a\right)\}+\log b \quad or \qquad \left(z-a\right)^2=b\{2xy+x^2f\left(a\right)\}$.

Ex.17. Find a complete integral of $px+qy=z\left(1+pq\right)^{1/2}$

Sol. $f\left(x,y,z,p,q\right)=px+qy-z\left(1+pq\right)^{1/2}=0 \qquad$ **....**(1)

Charpit's auxiliary equation are $\dfrac{dp}{f_x + p\, f_z} = \dfrac{dq}{f_y + q\, f_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

$\dfrac{dp}{p - p(1+pq)^{1/2}} = \dfrac{dp}{q - q(1+pq)^{1/2}} = \ldots\ldots$ so that $\dfrac{dp}{p} = \dfrac{dq}{q}$, by (1)

$\Rightarrow \quad \log p = \log a + \log q \;\Rightarrow\quad p = aq. \qquad \ldots(2)$

Using (2), (1) $\Rightarrow q(ax+y) = z(1+aq^2)^{1/2}$ or $q^2\left[(ax+y^2) - az^2\right] = z^2$

$\therefore q = \dfrac{z}{\left[(ax+y)^2 - az^2\right]^{1/2}}$ and $\dfrac{dz}{z} = \dfrac{a\,dx + dy}{\sqrt{\{(ax+y)^2 - az^2\}}}$ ...(3)

Let $ax + y = \sqrt{a}u$ so that $a\,dx + dy = \sqrt{a}du.$

$\therefore (3) \Rightarrow \dfrac{dz}{z} = \dfrac{\sqrt{a}du}{\sqrt{(au^2 - az^2)}}$ or $\dfrac{du}{dz} = \sqrt{\dfrac{(u^2 - z^2)}{z}} = \sqrt{\left\{\left(\dfrac{u}{z}\right)^2 - 1\right\}}, \qquad \ldots(4)$

which is linear homogeneus equation. To solve it, we put

$\dfrac{u}{z} = v$ or $u = vz$ so that $\dfrac{du}{dz} = v + z\dfrac{du}{dz}$

$\therefore (4)\, yields \quad v + z\dfrac{du}{dz} = (v^2 - 1)^{1/2}$ or $\dfrac{dz}{z} = \dfrac{du}{(v^2-1)^{1/2} - v}$

Or $\qquad (1/z)\,dz = -\left[(v^2-1)^{1/2} + v\right]du$ on rationalization.

Integrating, log z $= -\left[\dfrac{v}{2}(v^2-1)^{1/2} - \dfrac{1}{2}\log\{v + (v^2-1)^{1/2}\right] - \dfrac{v^2}{2} + b,$ where, $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Ex.18. Find complete integral of $(x^2 - y^2)\,pq - xy(p^2 - q^2) = 1$

Sol. Let $f(x,y,z,p,q) = (x^2 - y^2)\,pq - xy(p^2 - q^2) - 1 = 0$

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\, f_z} = \dfrac{dq}{f_y + q\, f_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

$\dfrac{dp}{2pqx - z(p^2 - q^2)} = \dfrac{dq}{-2pqy - x(p^2 - q^2)} = \dfrac{dx}{-(x^2 - y^2)y + 2pxy} = \dfrac{dy}{-(x^2-y^2)p - 2pxy}, by\,(1)$ Using

- x,y, p,q as multipliers, each fraction $= \dfrac{x\,dp + y\,dp + p\,dx + q\,dy}{0} = \dfrac{d(xp) + d(yq)}{0}$

$$\frac{a-qy}{x}\{\left(x^2-y^2\right)q-\left(a-qy\right)y\}+xyq^2-1=0 \text{ or } \{\left(a-qy\right)/x\}\left(x^2q-ay\right)+xyq^2-1=0$$

Or $\left(a-qy\right)\left(x^2q-ay\right)+x^2yq^2-x=0$ or $aq\left(x^2+y^2\right)=a^2y+x$

$$\therefore \qquad q=\frac{a^2y+x}{a\left(x^2+y^2\right)} \text{ and } p=\frac{1}{x}\left[a-\frac{\left(a^2y+x\right)y}{a\left(x^2+y^2\right)}\right]=\frac{a^2x-y}{a\left(x^2+y^2\right)}$$

Substituting these values in $dz=p\,dx+q\,dy$, we have

$$dz=\frac{\left(a^2x-y\right)dx+\left(a^2y+x\right)dy}{a\left(x^2+y^2\right)}=a\frac{x\,dx+y\,dy}{x^2+y^2}+\frac{x\,dy-y\,dy}{a\left(x^2+y^2\right)}$$

Integrating, $z=\left(a/2\right)\times\log\left(x^2+y^2\right)+\left(1/a\right)\times\tan^{-1}\left(y/x\right)+b$.

Ex.19. Find a complete integral of $2\left(pq+yp+qx\right)+x^2+y^2=0$

Sol. Given equation is $f\left(x,y,z,p,q\right)=2\left(pq+yp+qx\right)+x^2+y^2=0$ ...(1)

$$\frac{dp}{\partial F/\partial x+p\left(\partial F/\partial z\right)}=\frac{dq}{\partial F/\partial y+q\left(\partial F/\partial z\right)}=\frac{dz}{-p(\partial F/\partial p)-q\left(\partial F/\partial q\right)}=\frac{dx}{-\left(\partial F/\partial p\right)}=\frac{dy}{-\left(\partial F/\partial q\right)}=$$

$$\frac{dp}{2q+2x}=\frac{dq}{2p+2y}=\frac{dz}{-p\left(2q+2y\right)-q\left(2p+2x\right)}=\frac{dx}{-\left(2q+2y\right)}=\frac{dy}{-\left(2p+2x\right)}, \text{ by (1)}$$

Each of these above fractions $=\dfrac{dp+dq+dx+dy}{\left(2q+2x\right)+\left(2p+2y\right)-\left(2q+2y\right)-\left(2p+2x\right)}=\left(dp+dq+dx+dy\right)/0$

$\Rightarrow \qquad dp+dq+dx+dy=0$ so that $\left(p+x\right)+\left(q+y\right)=a$ ...(2)

Re-writing (1), $2\left(p+x\right)\left(q+y\right)+\left(x-y\right)^2=$ or $\left(p+x\right)\left(q+y\right)=-\left(x-y\right)^2/2.$ ...(3)

Now, $\left(p+x\right)-\left(q+y\right)=\sqrt{\{\left(p+x\right)^2+\left(q+y\right)\}^2-4\left(p+x\right)\left(q+x\right)}$

$\left(p+x\right)-\left(q+y\right)=\sqrt{a^2+2\left(x-y\right)^2}$, using (2) and (3) ...(4)

Adding (2) and (4), $2\left(p+x\right)=a+\sqrt{a^2+2\left(z-y\right)^2}$

Subtracting (4) from (2), $2\left(q+y\right)=a-\sqrt{a^2+2\left(x-y\right)^2}$

These give $p=-x+\dfrac{a}{2}+\dfrac{1}{2}\sqrt{a^2+2\left(x-y\right)^2}$ $q=-y+\dfrac{a}{2}-\dfrac{1}{2}\sqrt{a^2+2\left(x-y\right)^2}$

Substituting the above values of p and q, $dz=p\,dx+q\,dy$ becomes d

$$dz = -\left(x\,dx + y\,dy\right) + \left(a/2\right)\times\left(dx+dy\right) + \left(1/2\right)\times\sqrt{a^2 + 2\left(x-y\right)^2}\left(dx-dy\right)$$

$$dz = -\frac{1}{2}d\left(x^2 - y^2\right) + \frac{a}{2}d\left(x+y\right) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2} + \left(x-y\right)^2}\,d\left(x-y\right) \quad ...(5)$$

Put $x - y = t \;\; d\left(x-y\right) = dt$ Then (5) becomes

$$dz = -\left(1/2\right)\times d\left(x^2 + y^2\right) + \left(a/2\right)\times d\left(x+y\right) + \left(1/\sqrt{2}\right)\times\sqrt{\left(a/\sqrt{2}\right)^2 + t^2}\,dt.$$

$$\therefore z = -\frac{x^2+y^2}{2} + a\frac{x+y}{2} + \frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{\left(a\sqrt{2}\right)^2 + t^2} + \frac{\left(a/\sqrt{2}\right)^2}{2}\log\left\{t + \sqrt{\left(a\sqrt{2}\right)^2 + t^2}\right\}\right] + b$$

Putting the value of t, the required complete integral is

$$z = -\frac{x^2+y^2}{2} + \frac{a\left(x+y\right)}{2} + \frac{1}{2\sqrt{2}}\left[\left(x-y\right)\sqrt{\frac{a^2}{2} + \left(x-y\right)^2} + \frac{a^2}{2}\log\left\{x - y + \sqrt{\frac{a^2}{2} + \left(x-y\right)^2}\right\}\right] + b\,.$$

Ex. 20. Use Charpit's method to find the complete integral of $2x\{z^2\left(\partial z/\partial y\right)^2 + 1\} = z\left(\partial z/\partial x\right)$

Sol. Given $2x\left(\partial Z/\partial y\right)^2 + 2x - \left(\partial Z/\partial x\right) = 0$ ...(1)

Let $zdz = dz$ so that $z^2 = 2z$ ...(2)

Then (1) becomes $2x\left(\partial Z/\partial y\right)^2 + 2x - \left(\partial Z/\partial x\right) = 0$ or $2xQ^2 + 2x - p = 0$

Where $p = \partial Z/\partial x$ and $Q = \partial Z/\partial y$ ...(3)

Let $f\left(x, y, Z, P, Q\right) = 2xQ^2 + 2x - p = 0$ ...(4)

Charpit's auxiliary equations are $\dfrac{dP}{f_x + p\,f_Z} = \dfrac{dQ}{f_y + Q\,f_Z} = \dfrac{dZ}{-pf_P - Qf_Q} = \dfrac{dx}{-f_P} = \dfrac{dy}{-f_Q}$

Giving $\dfrac{dP}{2Q^2+2} = \dfrac{dQ}{0} =$ by (4) so that $dQ = 0.$

Integrating, $Q = a, a$ being an arbitrary constant

Using $Q = a$, (4) gives $P = 2x\left(a^2+1\right) \;\; Q = a \quad (6)$

$dZ = Pdx + Q\,dy = 2x\left(a^2+1\right)dx + a\,dy$, by (5) and (6)

Integrating, $Z = x^2\left(a^2+1\right) + ay + b/2$, or $z^2/2 = x^2\left(a^2+1\right) + ay + b/2$, Using (2)

$z^2 = 2x^2\left(a^2+1\right) + 2ay + b$, which is complete integral of (1)

Ex. 21. Solve by Charpit's method the partial differential equation.

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0.$$

Sol. Let $f(x,y,z,p,q)=p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$ ...(2)

From (1) $f_x=p^2(2x-1)+2pqy-2pz,\ \ f_y=2pqx+q^2(2y-1)-2pz,$

$f_z=-2px-2qy+2z,\ \ f_p=2px(x-1)+2qxy-2xz\ \ \ f_q=2pxy+2qy(y-1)-2yz$

And so $f_x+p\,f_z=-p^2\ \ \ f_y+q\,f_z=-p^2$ Then (2) becomes

$$\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{dz}{-p\{2px(x-1)+2qxy-2xz\}-q\{2pxy+2qy(y-1)-2yz\}}$$

$$=\frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)}\ \text{...(3)}$$

Each fraction of (3) $=\dfrac{(1/p)dp}{-p}=\dfrac{(1/q)dq}{-q}=\dfrac{(1/p)dp-(1/q)dq}{-p+q}$ ...(4)

Also, each fraction of (3) $=\dfrac{(1/x)dx-(1/y)dy}{-2px+2p-2qy+2z+2px+2qy-2q-2z}$ ....(5)

$\therefore\ (4)\,and\,(5)\Rightarrow\dfrac{(1/p)dp-(1/q)dq}{-(p-q)}=\dfrac{(1/x)dx-(1/y)dy}{2(p-q)}$

Or $(1/2)\times\{(1/x)dx-(1/y)dy\}=(1/q)dq-(1/p)dp$

Integrating, $(1/2)\times\{\log x-\log y\}=\log q-\log p+\log a$ or $(x/y)^{1/2}=aq/p$

Or $p=(ay^{1/2}q)/x^{1/2}$, a being and arbitrary constant....(5)

Re-writing (1), $(px+qy-z)^2=p^2x+q^2y$ or $px+qy-z=\pm(p^2x+qy^2)^{1/2}$ ...(6)

Taking + ive sign in (7), $px=qy-z=(p^2x+q^2y)^{1/2}$ ....(7)

Substituting the value of p given by (6) in (8), $aqy^{1/2}x^{1/2}+qy-z=(a^2q^2y+q^2y)^{1/2}$

Or $q\{y+a(xy)^{1/2}-(1+a^2)^{1/2}y^{1/2}\}=z$ so that $q=z/y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}$ ...(9)

Then (6) gives $p=az/x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}$ ...(10)

Putting these values of p and q in $dz=p\,dx+q\,dy+q\,dy$, we get

$$dz = \frac{az\,dx}{x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}} = \frac{z\,dy}{y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

$$\frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

Integrating, $\log z = 2\log\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2} + \log b$

Or $z = b\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}^{2,}$ a and b being arbitrary constants.

Ex.22. Find the complete integral of $(p+q)(px+qy)=1$

Sol. Let $f(x,y,z,p,q)=(p+q)(px+qy)-1=0$ ...(1)

Charpit's auxiliary equations $\dfrac{dp}{f_x+p\,f_z} = \dfrac{dq}{f_y+q\,f_z} = \dfrac{dz}{-pf_p-qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

Give $\dfrac{dp}{p(p+q)} = \dfrac{dq}{q(p+q)} =$ so that $\dfrac{dp}{p} = \dfrac{dq}{q}$ using (1)

Integrating, $p = aq, a$ being an arbitrary constant ...(2)

Putting $p = aq$ in (2) gives $(aq+q)(aqx+qy)-1=0$ or $q^2(1+a)(ax+y)=1$...(3)

$\therefore$ From (2) and (3), $q = 1/(1+a)^{1/2}(ax+y)^{1/2}$ $p = a/(1+a)^{1/2}(ax+y)^{1/2}$

Putting these values of p and q in $dz = p\,dx + q\,dy$ we get

$$dz = \frac{a\,dx}{(1+a)^{1/2}(ax+y)^{1/2}} = \frac{dy}{(1+a)^{1/2}(ax+y)^{1/2}} = \frac{d(ax+y)}{(1+a)^{1/2}(ax+y)^{1/2}}$$

Integrating, $z(1+a)^{1/2} = 2(ax+y)^{1/2} + b, a, b$ being arbitrary constants.

Category-2 Examples: Special Case-I: Form $f(p,q)=0$

Ex. 1. *Solve (a)* $p^2 + q^2 = m^2$*, where m is a constant (b)* $p^2 + q^2 = 1$

Sol. (a) Given that $p^2 + q^2 = m^2$....(1)

Since (1) is of the form *f*(*b*, *q*) = 0, its solution is $z = ax + by + c$,...(2)

where *a2 + b2 = m2* or $b = (m^2 - a^2)^{1/2}$, on putting a for *p* and *b* for *b* in (1).

$\therefore$ From (2), the complete integral is $z = ax + y(m^2 - a^2)^{1/2} + c$,...(3)

which contains two arbitrary constants *a* and *c*.

For singular solution, differentiating (3) partially with respect to *a* and *c*, we get $0 = x - ay/(m^2 - a^2)^{1/2}$ and $0 = 1$. But $0 = 1$ is absurd.

Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get $z = ax + y(m^2 - a^2)^{1/2} + \phi(a)$....(4)

Differentiating (4) partially wit respect to 'a', we get $0 = a - ay/(m^2 - a^2)^{1/2} + \phi'(a)$....(5)

Eliminating a from (4) and (5), we get the required general solution.

(*b*) **Hint.** taking $m = 1$

Ex. 2. Find the complete integral *of* $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0$.

Sol. The given equation can be rewritten as

$$z^2 y\ (\partial z/\partial x)^2 + 6zxy(\partial z/\partial x) + 2zx^2(\partial z/\partial y) + 4x^2 y = 0$$

or $\left(\dfrac{z\partial z}{x\partial x}\right)^2 + 6\left(\dfrac{z\partial z}{x\partial x}\right) + 2\left(\dfrac{z\partial z}{y\partial y}\right) + 4 = 0$, dividing by x^2y...(1)

Put $x\,dx = dX$, $y\,dy = dy$ and $z\,dz = dZ$....(2)

So that $x^2/2 = X$, $y^2/2 = Y$ and $z^2/2 = Z$....(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + 6\,(\partial Z/\partial X) + 2(\partial Z/dY) + 4 = 0$

or $P^2 + 6P + 2Q + 4 = 0$, Where $P = \partial Z/\partial X.$ $Q = \partial Z/\partial Y$ ...(4)

Equation (4) is of the form $f(P, Q) = 0$. Note that now we have *P, Q, X, Y, Z* in place of *p, q, x, y, z* in usual equations. Accordingly, solution of (4) is $Z = aX + bY + c$, ...(5)

where $a^2 + 6a + 2b + 4 = 0$ or $b = -(a^2 + 6a + 4)/2$, on putting a for *P* and b for *Q* in (4).

So, from (5), the required complete integral is

$Z = aX - \{(a^2 + 6a + 4)/2\}Y + c$, where *a* and *c* are arbitrary constants.

or $z^2/2 = a(x^2/2) - (a^2 + 6a + 4) \times (y^2/4) + c$, using (3)

or $z^2 = ax^2 - (2 + 3a + a^2/2)y^2 + c'$, where $c' = 2c$.

Ex. 3. Find the complete integral of (i) $x^2p^2 + y^2q^2 = z$ (ii) $p^2x + q^2y = z$

Sol. (i) The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \text{ or } \left(\frac{x\partial z}{\sqrt{z}\partial x}\right)^2 + \left(\frac{y\partial z}{\sqrt{z}\partial y}\right)^2 = 1 \text{...(1)}$$

Pur $(1/x)dx = dX$, $(1/y)dy = dY$ and $\left(1/\sqrt{z}\right)dz = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $2\sqrt{z} = Z$....(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$,...(4)

Where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f\left(P, Q\right) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c$, ...(5)

Where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$ on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $2\sqrt{z} = a\log x + \log y \cdot \sqrt{1-a^2} + c,$ by (3)

or $\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z},$ taking $c = -\log c'$

or $\log\{x^a y^{\sqrt{1-a2}} / c'\} = 2\sqrt{z}$ or $xay^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$

where *a* and *c′* are two arbitrary constants.

(ii) The given equation can be re-written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \text{ or } \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1 \quad ...(1)$$

Put $(1/\sqrt{x})dx = dX$, $(1/\sqrt{y})dy = dY$ and $(1/\sqrt{z})dz = dZ$ ...(2)

so that $2\sqrt{x} = X$, $2\sqrt{y} = Y$ and $2\sqrt{z} = Z$ ....(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$,...(4)

Where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P,Q) = 0$.

$\therefore$ solution of (4) is $z = a\,X + b\,Y + c$, ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, putting *a* for *P* and *b* for *Q* in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1-a^2} + c,$ by (3)

where *a* and *c* are two arbitrary constants.

Ex.4. Find the complete integral of $\left(1-x^2\right)yp^2 + x^2q = 0.$

Sol. The given equation can be rewritten as

$$\frac{1-x^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{1}{y}\frac{\partial z}{\partial y} = 0 \text{ or } \left(\frac{\left(1-x^2\right)^{1/2}}{x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{y}\frac{\partial z}{\partial y}\right) = 0. ...(1)$$

Put $\left\{x/\left(1-x^2\right)^{1/2}\right\}dx = dX$ and $y\,dy = dY$ ...(2)

so that $X = \int \frac{xdx}{\left(1-x^2\right)^{1/2}} = -\frac{1}{2}\int\left(1-x^2\right)^{-1/2}(-2x)dx = -\left(1-x^2\right)^{1/2}$ and $Y = \frac{y^2}{2}$ ...(3)

Using (2), (1) becomes $(\partial z/\partial X)^2+(\partial z/\partial Y)=0$ or $P^2+Q=0,$ ...(4)

where $P=\partial z/\partial X$ and $Q=\partial z/\partial Y$. Note carefully that here the old variable *z* remains unchanged

even after transformation (2). Here (4) is of the form $f(P, Q) = 0$.

∴ Solution of (4) is $z = aX + bY + c$, ...(5)

where $a^2 + b = 0$ or $b = -a^2$, on putting *a* for *P* and *b* for *Q* in (4),

∴ from (5), the required complete integral is.

$z = aX - a^2Y + c$ or $z = -a(1 - x^2)^{1/2} - (a^2 y^2)/2 + c$, by (3).

Ex. 5. Find the complete integral of $(y-x)\ (qy-px)\ =\ (p-q)^2$

Sol. Let *X* and *Y* be two new variables such that

$X = x + y$ and $Y = xy$. ...(1)

Given equation is $(y - x)(qy - px) = (p - q)^2$. ...(2)

Now, $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}=\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}$ ...(3) [∵ from (1), $\partial X/\partial x = 1$ and $\partial Y/\partial x = y$]

and $p=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}=\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}$. ...(4) [∵ from (1), $\partial X/\partial y = 1$ and $\partial Y/\partial y = x$]

Substituting the above values of *p* and *q* in (2), we have

$$(y-x)\left[y\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial y}\right)-x\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)\right]=\left[\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)-\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}\right)\right]^2$$

or $(y-x)^2\frac{\partial z}{\partial X}=(y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$ or $\frac{\partial z}{\partial X}=\left(\frac{\partial z}{\partial Y}\right)^2$ or $P = Q^2$, ...(5)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

∴Solution of (4) is $z = aX + bY + c$,...(6)

where $a = b^2$, on putting *a* for *P* and *b* for *Q* in (5).

∴ from (6), the required complete integral is

$z = b^2X + bY + c$ or $z = b2(x + y) + bxy + c$, by (1).

<u>**Category-2 Examples: Special Case-II:**</u> **Form Clairaut's**

Ex. 1. Solve $z = px+qy+c\sqrt{(1+p^2+q^2)}$

Sol. The complete integral of the given equation is

$z = ax + by + c\sqrt{(1+a^2+b^2)}$, *a, b* being arbitrary constants....(1)

Singular Integral. Differentiating (1) partially w.r.t. *a* and *b*, we get

$$0 = x + ac / \sqrt{(1 + a^2 + b^2)} \quad ...(2)$$

$$0 = y + bc / \sqrt{(1 + a^2 + b^2)} \quad ...(3)$$

□From (2) and (3),

$$x^2 + y^2 = (a^2c^2 + b^2c^2)/(1 + a^2 + b^2).$$

$$□ c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2 + b^2c^2}{1 + a^2 + b^2} = \frac{c^2}{1 + a^2 + b^2}$$

so that $1 + a^2 + b^2 = c^2 / \left(c^2 - x^2 - y^2\right)$. ...(4)

From (2), $a = -\frac{x\sqrt{(1 + a^2 + b^2)}}{c} = -\frac{x}{\sqrt{(c^2 - x^2 - y^2)}}$, by (4)

Similarly from (3) and (4), we obtain $b = -y / \sqrt{c^2 - x^2 - y^2}$.

Putting these values of *a* and *b* in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{(c^2 - x^2 - y^2)}} - \frac{y^2}{\sqrt{(c^2 - x^2 - y^2)}} + \frac{c^2}{\sqrt{(c^2 - x^2 - y^2)}} = \frac{c^2 - x^2 - y^2}{\sqrt{(c^2 - x^2 - y^2)}}$$

Or $z = \left(c^2 - x^2 - y^2\right)^{1/2}$ or $z^2 = c^2 - x^2 - y^2$ or $x^2 + y^2 + z^2 = c^2$ ...(5)

We can easily verify that (1) is satisfied by (5).

General Integral. Take *b* = ϕ(*a*), where ϕ is an arbitrary function.

Then, (1) yields $z = ax + y\phi(a) + c[1 + a^2 + \{\phi(a)\}^2]^{1/2}$ ...(6)

Differentiating both sides of (6) partially w.r.t. '*a*', we get

$$0 = x + y\phi'(a) + (c/2) \times [1 + a^2 + \{\phi(a)\}^2]^{-1/2} \times [2a + 2\phi(a)\phi'(a)]. \quad ...(7)$$

Eliminating *a* from (6) and (7), we get the general integral.

Ex. 2. Find the complete and singular integrals of the following equations:

(*i*) *z = px + qy + log (pq)* (*ii*) $z = px + qy - 2\sqrt{pq}$

Sol. (i) The complete integral is z = ax + by + log (ab)

Or *z = ax + by +* log *a* + log *b, a, b* being arbitrary constants .....(1)

Differentiating (1) partially with respect to *a* and *b*, we get

0 = *x* +(1/*a*) and 0 = *y* + (1/b) so that *a* = -1/*x* and *b* = -1/*y.* ...(2)

Eliminating *a* and *b* from (1) and (2), the required singular integral is

$z = -1 - 1 + \log(1/xy)$ or $z = -2 - \log(xy)$

(*ii*) The complete integral is $z = ax + by - 2\sqrt{ab}$ ...(1)

Differentiating (1) partially with respect to *a* and *b*, we get

$0 = x - \dfrac{2b}{2\sqrt{ab}}$ and $0 = y - \dfrac{2a}{2\sqrt{ab}}$ so that $x = \sqrt{\dfrac{b}{a}}$ and $y = \sqrt{\dfrac{a}{b}}$ ....(2)

Now, using (1) $x - z = x(ax + by - 2\sqrt{ab} = \sqrt{\dfrac{b}{a}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab},$ using (2)

□$x - z = \sqrt{(b/a)}.$ ...(3)

Similarly, using (1) $y - z = y - \left(ax + by - 2\sqrt{ab}\right), = \sqrt{\dfrac{a}{b}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$

□$y - z = \sqrt{(a/b)}.$ ...(4)

From (3) and (4), $(x - z)(y - z) = 1$,

which is singular integral as it satisfies the given equation.

Ex. 3. Prove that the complete integral of z = px + qy – 2p – 3q represents all possible planes through the point (2, 3, 0). Also find the envelope of all planes represented by the complete integral (i.e., find the singular integral).

Sol. Given that $z = px + qy - 2p - 3q$,...(1)

which is of the form $z = px + qy + f(p, q)$ and so its complete integral is

$z = ax + by - 2a - 3b$, *a, b* being arbitrary constants...(2)

Since (2) is a linear equation in *x, y, z,* it follows that (2) represents planes for various values of *a* and *b.* Again putting x = 2, *y* = 3, *z* = 0 in (2), we have

$0 = 2a + 3b - 2a - 3b$ *i.e.,* $0 = 0$,

showing that coordinates of the point (2, 3, 0) satisfy (2). Hence the complete integral (2) of (1) represents all possible planes passing through the point (2, 3, 0).

Differentiating (2) partially with respect to *a* and *b*, we get

$0 = x - 2$ and $0 = y - 3$ so that $x = 2$ and $y = 3$.

Substituting these values in (2), we get $z = 0$ as the required envelope (*i.e.,* singular integral).

Ex. 4. Prove that the complete integral of z = px + qy + [pq/(pq – p – q)] represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.

Sol. Since the given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$z = ax + by + [ab/(ab - a - b)]$, *a* and *b* being arbitrary constants. ...(1)

Since (2) is a linear equation in *x, y, z,* it follows that (1) represents planes for various values of *a* and *b*. We now rewrite (1) in the intercept form of a plane as follows:

ax + *by* – *z* = *ab*/(*a* + *b* – *ab*)

or $\frac{x}{[b/(a+b-ab)]}+\frac{y}{[a/(a+b-ab)]}+\frac{z}{[-ab/(a+b-ab)]}=1.$

□The algebraic sum of the intercepts on three coordinate axes

$=\frac{b}{a+b-ab}+\frac{a}{a+b-ab}+\frac{(-ab)}{a+b-ab}=\frac{b+a-ab}{a+b-ab}=1,$ as required.

Ex. 5. *Show that the complete integral of the equation* $z=px+qy+\left(p^2+q^2+1\right)^{1/2}$ *represents all planes at unit distance from the origin.*

Sol. Given equation is of the form *z* = *px* + *qy* + *f*(*p*, *q*), so its complete integral is

$z = ax + by + \left(a^2 + b^2 + 1\right)^{1/2}$, a, b being an arbitrary constant.

$ax + by - z + \left(a^2 + b^2 + 1\right)^{1/2} = 0.$ ...(1)

Since (2) is a linear equation in *x, y, z,* it follows that (1) represents planes for various values of *a* and *b*.

The perpendicular distance of (1) from origin (0, 0, 0)

$=\frac{a.0+b.0-0+\sqrt{a^2+b^2+1}}{\sqrt{\{a^2+b^2+(-1)^2\}}}=\frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}}=1,$ as required

Ex. 6. *Find the complete integral of the following equations:*

(i) $(p+q)(z-px-qy)=1$ (*ii*) $pqz=p^2\left(xq+p^2\right)+q^2\left(yp+q^2\right)$

Sol. (*i*) Re–writing the given equation in the standard form *z* = *px*+*qy* + *f* (*p*,*q*) we get

z – *px* – *qy* = *1* / (*p* + *q*) or *z* = *px* + *qy* *1* / (*p* + *q*)

□ Its complete integral is *z* = *ax* + *by* + 1 / (*a* + *b*), where a and b are arbitrary constants.

(*ii*) Dividing both sides of the given equation *by pq,* $z=px+qy+\left(p^4+q^4\right)/pq,$

Its complete integral is $z=ax+by+\left(a^4+b^4\right)/ab$, *a, b* being arbitrary constants

Ex. 7. (*a*) *Find the complete integral the equation* $2(y+zq)=q(xp+yq).$

Sol. Re–writing the given equation, we have

$2zq=xpq+yq^2$ or $z=(1/2)px+(1/2)qy-(y/q)$

Or $z = x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) - \frac{1}{2}\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)^{-1}$ ...(1)

Putting 2*x dx* =*d X* and 2*y dy*= *dY* so that $x^2 = X$ and $y^2 = Y$, (1) gives

$z = X(\partial z/\partial X) + Y(\partial z/\partial Y) - 1/\{2(\partial z/\partial Y)\}$ or *z = PX + QY* – (1/2Q),

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. The above equation is of the form $z = PX + Qy + f(P, Q)$ and hence its complete integral is

$z = aX + bY - (-1/2b)$ or $z = ax^2 + by^2 - (1/2b)$ *a* and *b* being arbitrary constants.

Ex. 7. (*b*) *Find the complete integral of* $2q(z - px - qy) = 1 + q^2$.

Sol. Re–writing the given equation in the form $z = px + qy + (1+q^2)/2q$, we have

$z - px - qy = (1+q^2)/2q$ or $z = px + qy + (1+q^2)/2q,$

Its complete integral is $z = ax + by + (1+q^2)/2b,$ *a* and *b* being arbitrary constants.

<u>Category-2 Examples: Special Case-III:</u> Form $f(p, q, z) = 0$

Ex. 1. *Find a complete integral of* $9(p^2 z + q^2) = 4$.

Sol. Given equation is $9(p^2 z + q^2) = 4$, ...(1)

which is of the form $f(p, q, z) = 0$. Let *u = x + ay,* where *a* is an arbitrary constant. Now, replacing *p* and *q* by *dz/du* and *a*(*dz/du*) respectively in (1), we get

$$9\left[z\left(\frac{dz}{du}^2\right) + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \text{ or } \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z + a^2)}.$$

$du = \pm (3/2) \times (z + a^2)^{1/2} dz$, separating variables *u* and *z.*

Integrating, $u + b = \pm (3/2) \times \left[(z + a^2)^{3/2}/(3/2)\right]$ or $u + b = \pm (z + a^2)^{3/2}$

Or $(u + b)^2 = (z + a^2)^3$ or $(x + ay + b)^2 = (z + a^2)^3$, as *u = x + ay*

which is a complete integral containing two arbitrary constants *a* and *b.*

Ex. 2. *Find a complete integral of* $p^2 = qz$.

Sol. Given equation is $p^2 = qz$, ...(1)

which is of the form *f*(*p, q, z*) = 0. Let *u* = *x* + *ay*, where *a* is an arbitrary constant. Now, replacing *p* and *q* by *dz/du* and *a*(*dz/du*) respectively in (1), we get

$$\left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z \text{ or } \frac{dz}{du} = az \text{ or } \frac{dz}{z} a\, du.$$

Integrating, log *z* – *log b* = *au* or $z = be^{au}$ or $z = be^{a(x + ay)}$,

which is a complete integral containing two arbitrary constants *a* and *b*.

Ex.3. *Find complete integrals of the following partial differential equations.*

(*i*) *p*(*z* + *p*) + *q* = 0 (*ii*) *p*(1 + *q*) = *qz*.

Sol. (*i*) The given equation is of the form *f*(*p*, *q*, *z*) = 0. Let *u* = *x* + *ay*, *a* being an arbitrary constant. Replacing *p* by *dz*/*du* and *q* by *a*(*dz*/*du*) in the given equation, we get

$$\frac{dz}{du}\left(z + \frac{dz}{du}\right) + \frac{dz}{du} = 0 \text{ or } \frac{dz}{du} = -(z-a) \text{ or } \frac{dz}{z+a} = -du \text{ or } \frac{dz}{z+a} = du.$$

Integrating, log (*z* + *a*) – log *b* = –*u* or $z + a = be^{-u}$ or $z + a = be^{-(x + ay)}$.

(*ii*) **Ans.** *az* – 1 = *bex* + *ay*.

Ex.4. *Find a complete integral of* $p^3 + q^3 - 3pqz = 0$.

Sol. The given equation is of the form *f*(*p*, *q*, *z*) = 0. Let *u* = *x* + *ay*, *a* being an arbitrary constant. Replacing *p* by *dz*/*du* and *q* by *a*(*dz*/*du*) in the given equation,

$$\left(\frac{dz}{dy}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 - 3az\left(\frac{dz}{du}\right)^2 = 0\ 9 \text{ or } (1+a^3)\frac{dz}{du} = 3az \text{ or } \frac{1+a^2}{a}dz = 3au.$$

Integrating $(1+a^3)\log z = 3au + b$ or $(1+a^3)\log z = 3a(x+ay)+b.$

Ex. 5. *Find complete and singular integrals of* $z^2(p^2z^2 + q^2) = 1$.

Sol. The given equation is of the form *f*(*p*, *q*, *z*) = 0. Let *u* = *x* + *ay*, *a* being an arbitrary constant. Replacing *p* by *dz*/*du* and *q* by *a*(*dz*/*du*) in the given equation, we have

$$z^2\left[z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1 \text{ or } z^2(z^2+a^2)\left(\frac{dz}{du}\right)^2 = 1$$

$$du = \pm z(z^2+a^2)^{1/2}dz = \pm(1/2)\times(z^2+a^2)^{1/2}(2zdz)$$

Integrating, $u + b = \pm(1/2)\times[(z^2+a^3)^{3/2}/(3/2)]$

or $9(u+b)^2 = (z^2+a^2)^3$ or $9(x+ay+b)^2 = (z^2+a^3)^3$, ...(1)

which is a complete integral containing two arbitrary constants *a* and *b*.

Singular Integral. Differentiating (1) partially, w.r.t. '*a*' and '*b*', we get

$18(x + ay + b)y = 3(z^2 + a^2) \times 2a$ ...(2)

and 18(*x* + *ay* + *b*) = 0. ...(3)

From (2) and (3), *x* + *ay* + *b* = 0 and *a* = 0. Putting these values in (1), we get *z* = 0, which is free from a and b. Again, from z = 0, we get $p = \partial z / \partial x$ = 0 and $q = \partial z / \partial y = 0$ These values *i.e.*,

z = 0, p = 0 and q = 0 do not satisfy the given equation. Hence z = 0 is not a singular solution of the given equation.

Ex. 6 . (*i*) *Find a complete integral of* $z^2(p^2 + q^2 + 1) = k^2$.

(*ii*) *Find a complete and singular integral of* $z^2(p^2 + q^2 + 1) = 1$.

Sol. (*i*) The given equation is of the form $f(p, q\, z) = 0$. Let $u = x + ay$ where a is an arbitrary constant. Replacing p by (dz/du) and q by $a(dz/du)$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right]k^2 \text{ or } (1+a^2)\left(\frac{dz}{du}\right)^2 = \frac{k^2 - z^2}{z^2}$$

$$\text{Or } \pm(1+a^2)^{1/2}\frac{z}{(k^2-z^2)^{1/2}}dz = du \text{ or } \pm\frac{1}{2}(1+a^2)(k^2-z^2)^{1/2}(-2zdz) = du$$

Integrating, $\pm(1+a^2)^{1/2}(k^2-z^2)^{1/2} = u+b$ or $(1+a^2)(k^2-z^2) = (u+b)^2$ $(1+a^2)(1-z^2) = (x+ay+b)^2$

(*ii*) Here k = 1. Proceed as in part (*i*) and get complete integral

$(1 + a2)(1 – z2) = (x + ay + b)2$. ...(1)

Differentiating (1) partially w.r.t. a and b, we get

$2a(1 – z^2) = 2(x + ay + b) \times y$ ...(2)

and $0 = 2(x + ay + b)$. ...(3)

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With these values (1) reduces to $z^2 = 1$, which is free from a and b. Again, from $z^2 = 1$, $p = \partial z / \partial x$ = 0 and $q = \partial z / \partial y = 0$. Now, p = 0, q =0 and $z^2 = 1$, satisfy the given equation and hence singular integral of the given equation is $z2 = 1$.

<u>Category-2 Examples: Special Case-IV:</u> Form $f_1(x, p) = f_2(y, q)$

Ex. 1. *Find a complete integral of* $x(1 + y)p = y(1 + x)q$.

Sol. Separating p and x from q and y, the given equation reduces to

$(xp)/(1 + x) = (yq)/(1 + y)$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \text{ and } \frac{yq}{1+y} = a \text{ so that } p = a\left(\frac{1+x}{x}\right) \text{ and } q = a\left(\frac{1+y}{y}\right)$$

Putting these value of p and q in $dz = p\, dx = q$ dy, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}dy \text{ or } dz = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy.$$

Integrating,z = a (log *x* + *x*) + a (log *y* + *y*) + b = a (log *xy* + *x* + *y*) + *b*,

Ex. 2. *Find a complete integral of* $p - 3x^2 = q^2 - y$.

Sol. Equating each side to an arbitrary constant *a*, we get

$p - 3x^2 = a$ and $q^2 - y = a$ so that $p = a + 3x^2$ and $q = (a+y)^{1/2}$

Putting these values of *p* and *q* in *dz* = *pdx* + *qdy*, we get

$dz = (a + 3x^2)dx + (a + y)^{1/2}\, dy$ so that $z = ax + x^3 + (2/3) \times (a + y)^{3/2} + b$.

Ex. 3. *Find a complete integral of* $z^2(p^2 + q^2) = x^2 + y^2$, *i.e.*, $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2] = x^2 + y^2$.

Sol. Given $z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$ or $\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2$ ...(1)

Let *z dz* = *dz* so that $z^2/2 = Z$

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y) = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

Where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$ Separating *P* and *x* from *Q* and *y*, we get

$P^2 - x^2 = y^2 - Q^2$.

Equating each side of the above equation to an arbitrary constant *a*2, we get

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$.

Putting these values of *P* and *Q* in *dZ* = *P dx* + *Q dy*, we have

$dZ = (a^2 + x^2)^{1/2}\, dx + (y^2 - a^2)^{1/2}\, dy$.

Integrating, $Z = (x/2) \times (a^2 + x^2)1/2 + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\} + (y/2) \times (y^2 - a^2)^{1/2} - (a^2/2) \times \log\{y + (y^2 - a^2)^{1/2}\} + (b/2)$

$z^2 = x^2 (a^2 + x^2)1/2 + a^2 \log[x + (a^2 + x^2)1/2\} + y(y^2 + a^2)^{1/2} - a^2\log\{y + (y^2 + a^2)^{1/2}\} + b$

$[From\ (2), Z = z^2/2]$

Ex. 4. *Find a complete integral of* $z(p^2 - q^2) = x - y$.

Sol. Re-writing the given equation,

$(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y$ ...(1)

Let $\sqrt{z}\, dz = dZ$ so that (2/3) × $z^{3/2}$ = Z ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 - (\partial Z/\partial x)^2 = x - y$ or $P^2 - Q^2 = x - y$,

Where P $= \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$P^2 - x = Q^2 - y$ ...(3)

Equating each side to an arbitrary constant *a*, we get

$P^2 - x = a$ and $Q^2 - y = a$ so that $P = (x + a)^{1/2}$ and $Q = (y + a)^{1/2}$

Putting these values of *P* and *Q* in $dZ = P\,dx + Q\,dy$, $dZ = (x + a)^{1/2}dx + (y + a)^{1/2}\,dy$.

Integrating, $Z = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$

or $(2/3) \times z^{3/2} = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$, as $Z = (2/3) \times z^{3/2}$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$, *a*, *b* being arbitrary constants.

Ex.5. *Find the complete integral of* $\left(1-x^2\right)yp^2 + x^2q = 0$

Sol. Re-writing, we have $\left(x^2-1\right)\dfrac{p^2}{x^2} = \dfrac{q}{y} = a^2$, say

$\therefore p = \dfrac{ax}{\left(x^2-1\right)^{\frac{1}{2}}}$ and $q = a^2y$. so that $q = a^2y$. Hence $d\,z = pdx + qdy$ becomes

$dz = ax\left(x^2-1\right)^{-\frac{1}{2}}dx + a^2ydy$ so that $z = a\left(x^2-1\right)^{\frac{1}{2}} + \dfrac{\left(a^2y^2\right)}{2+b}$

Ex. 16. *Find the the complete integral of p + q – 2px – 2qy + 1 = 0.*

Sol. Re-writing, *p – 2px = 2qy – q – 1 = a,* say

$\therefore p = \dfrac{a}{1-2x}, \quad q = \dfrac{a+1}{2y-1}$ and so $dz = pdx + qdy = \dfrac{adx}{1-2x} + \dfrac{(a+1)dy}{2y-1}$

Integrating $z = -\left(\dfrac{a}{2}\right)\times\log|1-2x| + \left(\dfrac{1}{2}\right)\times(a+1)\log|2y+1| + b$

CAUCHY CHARACTERSTIC METHOD EXAMPLES

Ex. 1. Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0$, $y^2 = z$.

Sol. Given equation is $pq = z,$ ...(1)

We are to find its integral surface which passes through the given parabola given by

$x = 0, \quad and \quad y^2 = z$ ...(2)

Re-writing (2) in parametric form, we have

$x = 0, \ y = \lambda, \ z = \lambda^2,$ λ being a parameter ...(3)

Let the initial values x_0, y_0, z_0, p_0, q_0 *of* x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = 0, \quad y_0 = y_0(\lambda) = \lambda, \quad z_0 = z_0(\lambda) = \lambda^2$ ...(4A)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values (x_0, y_0, z_0, p, q_0) satisfy (1), we have

$p_0 q_0 = z_0$, *or* $p_0 q_0 = \lambda^2$, by (4A) ...(5)

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $2\lambda = p_0 \times 0 + q_0 \times 1$ *or* $q_0 = 2\lambda$, by (4A) ...(6)

Solving (5) and (6), $p_0 = \lambda/2$ *and* $q_0 = 2\lambda$ ...(4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0 = 0, y_0 = \lambda, z_0 = \lambda^2, p_0 = \lambda/2, q_0 = 2\lambda$ *when* $t = t_0 = 0$ ...(7)

Re-writing (1), let $f(x, y, z, p, q) = pq - z = 0$ ...(8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = q \quad \ldots(9)$$

$$dy/dt = \partial f/\partial q = p \quad \ldots(10)$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = 2pq \quad \ldots(11)$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p \quad \ldots(12)$$

and $$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = q) \quad \ldots(13)$$

From (9) *and* (13), $(dx/dt) - (dq/dt) = 0$, *so that* $x - q = C_1$ ...(14)

where C_1 is an arbitrary constant. Using initial values (7), (14) gives

$x_0 - q_0 = C_1$ or $0 - 2\lambda = C_1$ or $C_1 = -2\lambda$, Then (14) becomes

$x - q = -2\lambda$ or $x = q - 2\lambda$, ...(15)

From (10) and (12), $(dy/dt) - (dp/dt) = 0$ so that $y - p = C_2$, ...(16)

where C_2 is an arbitrary constant. Using initial values (7), (16) gives

$y_0 - p_0 = C_2$ or $\lambda - (\lambda/2) = C_2$ or $C_2 = \lambda/2$. Then (16) becomes

$y - p = \lambda/2$ or $y = p + (\lambda/2)$ ...(17)

From (12), $(1/p)dp = dt$ so that $\log p - \log C_3 = t$ or $p = C_3 e^t$ ...(18)

Using initial values (7), (18) gives $p_0 = C_3 e^0$ or $\lambda/2 = C_3$

Hence (18) reduces to $p = (\lambda/2)\times e^t$ ...(19)

From (13), $(1/q)dq = dt$ so that $\log q - \log C_4 = t$ or $q = C_4 e^t$ ...(20)

Using initial values (7), (20) gives $q_0 = C_4 e^0$ or $2\lambda = C_4$

Hence (20) reduces to $q = 2\lambda e^t$ ...(21)

Using (21), (15) becomes $x = 2\lambda e^t - 2\lambda$ or $x = 2\lambda(e^t - 1)$ ...(22)

Using (19), (17) becomes $y = (\lambda/2)e^t + \lambda/2$ or $y = (\lambda/2)\times(e^t + 1)$ ...(23)

Substituting values of p and q from (19) and (21) in (11), we get

$$dz/dt = 2\{(\lambda/2)\times e^t\}\times\{2\lambda e^t\} \quad \text{or} \quad dz = 2\lambda^2 e^{2t}\, dt$$

Integrating, $z = \lambda^2 e^{2t} + C_5$, C_5 being arbitrary constant ...(24)

Using initial values (7), (24) *gives* $z_0 = \lambda^2 e^0 + C_5$ or $\lambda^2 = \lambda^2 + C_5$ or $C_5 = 0$

Then, (24) gives $z = \lambda^2 e^{2t}$ or $z = \lambda^2 (e^t)^2$ ...(25)

The required characteristics of (1)are given by (22), (23) and (25)

To find the required integral surface of (1), we now proceed to eliminate two parameters t and λ from three equations (22), (23)and (25). Solving (22) and (23) for e^t and λ, we have

$e^t = (x + 4y)/(4y - x)$ and $\lambda = (4y - x)/4$

Substituting these values of e^t and λ in (25), we have

$z = \{(4y - x)^2/16\}\times\{(x + 4y)/(4y - x)\}^2$ or $16z = (4y + x)^2$

which is the required integral surface of (1) passing through (2).

Ex. 2. Find the solution of the equation $z = (p^2 + q^2)/2 + (p - x)(q - y)$ which passes through the x – axis.

Sol. Given equation is $z = (p^2 + q^2)/2 + (p - x)(q - y)$ ...(1)

We are to find its integral surface which passes through x – axis which is given by equations

$y = 0$ and $z = 0$ ...(2)

Re-writing (2) in parametric form, $x = \lambda$, $y = 0$, $z = 0$, λ being the parameter ...(3)

Let the initial values x_0, y_0, z_0, p_0, q_0 *of* x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = \lambda$, $y_0 = y_0(\lambda) = 0$, $z = z_0(\lambda) = 0$ ...(4A)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$z_0 = (p_0^2 + q_0^2)/2 + (p_0 - x_0)(q_0 - x_0)$ or $0 = (p_0^2 + q_0^2)/2 + q_0(p_0 - \lambda)$, by (4A)

or $p_0^2 + q_0^2 + 2q_0 p_0 - 2q_0 \lambda = 0$ ...(5)

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

so that $0 = p_0\times1 + q_0\times0$ or $p_0 = 0$, by (4A) ...(6)

Solving (5) and (6), $p_0 = 0$ and $q_0 = 2\lambda$ ...(4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0 = \lambda$, $y_0 = 0$, $z_0 = 0$, $p_0 = 0$, $q_0 = 2\lambda$ when $t = t_0 = 0$ ...(7)

Let $f(x, y, z, p, q) = (p^2 + q^2)/2 + pq - py - qx + xy - z = 0$ ...(8)

The usual characteristic equations of (8) are given by

$dx/dt = \partial f/\partial p = p + q - y$ ...(9)

$dy/dt = \partial f/\partial q = q + p - x$ ...(10)

$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(p + q - y) + q(q + p - x)$, ...(11)

$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p + q - y$ ...(12)

and $dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = p + q - x$ ...(13)

From (9) and (12), $(dx/dt) - (dp/dt) = 0$ so that $x - p = C_1$ ...(14)

where C_1 is an arbitrary constant. Using initial conditions (7), (14) gives $\lambda - 0 = C_1$ or $C_1 = \lambda$.

Hence (14) reduces to $x - p = \lambda$ *or* $x = p + \lambda$ ...(15)

From (10) and (13), $(dy/dt) - (dq/dt) = 0$ so that $y - q = C_2$, ...(16)
where C_2 is an arbitrary constant.
Using initial conditions (7), (16) gives $0 - 2\lambda = C_2$ or $C_2 = -2\lambda$
Hence (16) reduces to $y - q = -2\lambda$ or $y = q - 2\lambda$ ...(17)

$$\therefore \frac{d(p+q-x)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p + q - y + p + q - x - (p + q - y), \text{ using (9), (12) and (13)}$$

$$\text{or} \quad \frac{d(p+q-x)}{dt} = p + q - x \quad \text{or} \quad \frac{d(p+q-x)}{p+q-x} = dt$$

Integrating, $\log(p + q - x) - \log C_3 = t$ or $p + q - x = C_3 e^t$, ...(18)
where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0 + 2\lambda - \lambda = C_3$ or $C_3 = \lambda$.
Hence (18) reduces to $p + q - x = \lambda e^t$ ...(19)

$$\text{Now,} \quad \frac{d(p+q-y)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p + q - y + p + q - x - (q + p - x), \text{ by (10), (12) and (13)}$$

$$\text{or} \quad \frac{d(p+q-y)}{dt} = p + q - y \quad \text{or} \quad \frac{d(p+q-y)}{b+q-y} = dt$$

Integrating, $\log(p + q - y) - \log C_4 = t$ or $p + q - y - C_4 e^t$ ...(20)
where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0 + 2\lambda - 0 = C_4$ or $C_4 = 2\lambda$.
Hence (20) reduces to $p + q - y = 2\lambda e^t$ ...(21)
From (9) and (21), $dx/dt = 2\lambda e^t$ so that $x = 2\lambda e^t + C_5$ ...(22)
where C_5 is an arbitary constant. Using initial conditions (7), (22) gives $\lambda = 2\lambda + C_5$ or $C_5 = -\lambda$.
Hence (22) reduces to $x = 2\lambda e^t - \lambda$ or $x = \lambda(2e^t - 1)$ ...(23)
From (10) and (19), $dy/dt = \lambda e^t$ so that $y = \lambda e^t + C_6$ ...(24)
where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives $0 = \lambda + C_6$ or $C_6 = -\lambda$.
Hence (24) reduces to $y = \lambda e^t - \lambda$ or $y = \lambda(e^t - 1)$ ...(25)
Substituting value of y from (17) in (12), we get

$dp/dt = p + q - (q - 2\lambda)$ or $(dp/dt) - p = 2\lambda$, ...(26)

which is a linear equation whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and solution is

$pe^{-t} = \int(2\lambda)e^{-t}\,dt + C_7 = -2\lambda e^{-t} + C_3$ or $p = -2\lambda + C_3 e^t$ ...(27)

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0 = -2\lambda + C_7$ or $C_7 = 2\lambda$.
Hence (27) reduces to $p = -2\lambda + 2\lambda e^t$ or $p = 2\lambda(e^t - 1)$ ...(28)
Substituting value of x from (15) in (13), we get

$dq/dt = p + q - (p + \lambda)$ or $dq/dt - q = -\lambda$ ...(29)

which is a linear equation whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and solution is

$qe^{-t} = \int(-\lambda)e^{-t}\,dt + C_8 = \lambda e^{-t} + C_8$ *or* $q = \lambda + C_8 e^t$ ...(30)

where C_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + C_8$ or $C_8 = \lambda$.
Hence (30) reduces to $q = \lambda + \lambda e^t$ or $q = \lambda(1 + e^t)$ ...(31)
Substitutions the values of $p + q - x$ and $p + q - y$ from (13) and (24) respectively in (1) gives

$$dz/dt = p(2\lambda e^t) + q(\lambda e^t) = 2\lambda(e^t - 1)(2\lambda e^t) + \lambda(1 + e^t)(\lambda e^t)$$

[on putting values of p and q with help of (28) and (31)]

& $dz/dt = 5\lambda^2 e^{2t} - 3\lambda^2 e^t$ or $dz = (5\lambda^2 e^{2t} - 3\lambda^2 e^t)dt$

Integrating, $z = (5/2) \times \lambda^2 e^{2t} - 3\lambda^2 e^t + C_9$ ...(32)

where C_9 is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives
$0 = (5/2) \times \lambda^2 - 3\lambda^2 + C_9$ or $C_9 = 3\lambda^2 - (5/2)\lambda^2$. Hence (32) reduces to
$z = (5/2) \times \lambda^2 (e^{2t} - 1) - 3\lambda^2 (e^t - 1)$ ...(33)
Solving (23) and (25) for λ and e^t, $\lambda = x - 2y$ and $e^t = (x - y)/(x - 2y)$ ...(34)
Eliminating λ and e^t from (33) and (34), we have

$$(p^2 + q^2)x = pz$$

$$z = \frac{5}{2}(x-2y)^2\left\{\left(\frac{x-y}{x-2y}\right)-1\right\}-3(x-2y)^2\left(\frac{x-y}{x-2y}-1\right)$$

or $z = (5/2)\times\{(x - y)^2 - (x - 2y)^2\} - 3\{(x - 2y)(x - y) - (x - 2y)^2\}$

$z = (y/2)\times(4x - 3y)$, on simplification.

Ex. 3. Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0$, $y = 0$.

Sol. Do yourself, the required characteristics are $x = 2\lambda(2 - e^{-t})$, $y = 2\sqrt{2}\,\lambda(e^{-t} - 1)$, $z = -\lambda^2 e^{-2t}$, λ being parameter. Solution is $4z + (x + y\sqrt{2})^2 = 0$.

JACOBI'S METHOD

Ex.1. *Find a complete integral of* $p_1{}^3 + p_2{}^2 + p_3 = 1.$

Sol. Let the given equation be rewritten as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0.$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $\dfrac{dp_1}{0} = \dfrac{dx_1}{-3p_1^2} = \dfrac{dp_2}{0} = \dfrac{dx_2}{-2p_2} = \dfrac{dp_3}{0} = \dfrac{dx_3}{-1}$, using...(1)

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$.

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$. ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(3)

Now, $(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$

or *(F_1,F_2) = (0)(0) – (1)(0) + (0)(1) – (0)(0) + (0)(0) – (0)(0) = 0*, by (3) and (4).

Thus, we have verified that for relations (2) and (3), *$(F_1,F_2) = 0$*. Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $p_1 = a_1$, $p_2 = a_2$, $p_3 = 1 - a_1{}^3 - a_2{}^2$.

Putting these values in *$dz = p_1dx_1 + p_2dx_2 + p_3dx_3$,* we have

$dz = a_1dx_1 + a_2dx_2 + \left(1 - a_1{}^3 - a_2{}^2\right)dx_3.$

Integrating, $z = a_1x_1 + a_2x_2 + \left(1 - a_1{}^3 - a_2{}^2\right)x_3 + a_3$,

which is a complete integral of given equation containing three arbitrary constants *a_1, a_2*, and *a_3*. $a_1, a_2,$

Ex.2. Find a complete integral of $x_3{}^2p_1{}^2p_2{}^2p_3{}^2 + p_1{}^2p_2{}^2 - p_3{}^2 = 0$.

Sol. Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = x_3^2p_1^2p_2^2p_3^2 + p_1^2p_2^2 - p_3^2 = 0$. ...(1)

$\therefore$ Jacobi's auxiliary equations are $\dfrac{dp_1}{\frac{\partial f}{\partial x_1}} = \dfrac{dx_1}{\frac{-\partial f}{\partial p_1}} = \dfrac{dp_2}{\frac{\partial f}{\partial x_2}} = \dfrac{dx_2}{\frac{-\partial f}{\partial p_2}} = \dfrac{dp_3}{\frac{\partial f}{\partial x_3}} = \dfrac{dx_3}{\frac{-\partial f}{\partial p_3}}$

or $\dfrac{dp_1}{0} = \dfrac{dx_1}{-\left(2p_1x_3^2p_2^2p_3^2 + 2p_1p_2^2\right)} = \dfrac{dp_2}{0} = \dfrac{dx_2}{-\left(2p_2x_3^2p_1^2p_3^2 + 2p_2p_1^2\right)} = \ldots$, by (1)

$\therefore$ From first and third fractions, *$dp_1=0$* and *$dp_2=0$* so that *$p_1=a_1$* and *$p_2=a_2$*.

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$, ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ....(3)

As in Ex. 1, verify that for relations (2) and (3), *$(F_1,F_2) = 0$*

Hence (2) and (3) may be taken as the additional equations.

Solving (1), (2) and (3) for *p_1,p_2,p_3*, we have $p_1 = a_1, p_2 = a_2, p_3 = \pm a_1a_2 / \sqrt{\left(1 - a_1^2a_2^2x_3^2\right)}$.

Putting these values in *$dz = p_1dx_1 + p_2dx_2 + p_3dx_3$*, we get\

$dz = a_1dx_1 + a_2dx_2 \pm \left\{a_1a_2 / \sqrt{\left(1 - a_1^2a_2^2x_3^2\right)}\right\}dx_3$, whose integration gives

$z = a_1x_1 + a_2x_2 \pm \sin^{-1}(a_1a_2x_3) + a_3, a_1, a_2, a_3$ being arbitrary constants.

ASSIGNMENT 1: QUESTIONS

Q1. Find a complete integral of z = pz + qy + p^2 + q^2.

Q2. Find a complete integral of $p^2 - y^2 q = y^2 - x^2$

Q3. Find the complete integrals of following equations: (i) $q = (z + px)^2$ (ii) $p = (z + qy)^2$

Q5. Find a complete integrals of the following partial differential equations:

(i) q = px + p^2. (ii) q = – px + p^2.

Q6. Find a complete integral of $p^2 + q^2 - 2px - 2qy + 2xy = 0$

Q8. Find the complete integral of the following partial differential equations.

(a) $px^5 - 4q^2x^2 + 6x^2z - 2$ (b) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

Q9. Find the complete integral of $(p + y)^2 + (q + x)^2 = 1$

Q10. Find the complete integral of 2 (y + zq) = q(xp + yq)

Solutions.

Sol 1. Let f(x, y, z, p, q) = z – px – qy – p^2 + q^2 = 0 ....(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$ ....(2)

From (1), $f_x = -p$, $f_y = -q$, $f_z = 0$, $f_p = -x - 2p$ and $f_q = -y - 2q$ ....(3)

Using (3), (2) reduces to

$\dfrac{dp}{0} = \dfrac{dq}{0} = \dfrac{dq}{p(x+2p)+q(y+2p)} = \dfrac{dx}{x+2p} = \dfrac{dy}{y+2q}$ ....(4)

Taking the first fraction of (4), dp = 0 so that p = a ....(5)

Taking the second fraction of (4), dq = 0 so that q = b ....(6)

Putting p = a and q = b in (1), the required complete integral is

z = ax + by + a^2 + b^2, a, b being arbitrary constants.

Sol 2. Here given equation is $f(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0$ ..... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

Or $\dfrac{dp}{p\left(4p^2z^3 + 2zq^2\right)} = \dfrac{dq}{q\left(4p^2z^3 + 2zq^2\right)} = \dfrac{dz}{-2p^2z^4 - 2q^2z^2} = \dfrac{dx}{-2pz^4} = \dfrac{dy}{-2qz^2}$, by (1) ..... (2)

Taking the first two fractions, (1/p)dp = (1/q)dq so that p = aq.

Solving (1) and (2) for p and q, $p = \dfrac{a}{z\left(a^2z^2+1\right)^{1/2}}$, $q = \dfrac{1}{z\left(a^2z^2+1\right)^{1/2}}$

$\therefore\ dz = pdx + qdy = (adx + dy)/z\left(a^2z^2+1\right)^{1/2}$ or $adx + dy = z\left(a^2z^2+1\right)^{1/2} dz$

Integrating, $ax + y = \int\left(a^2z^2+1\right)^{1/2} \cdot zdz$ ..... (3)

Putting $a^2z^2+1=t^2$ so that $2a^2zdz = 2tdt$, (3) becomes

$ax + y = \int\left(1/a^2\right)t\cdot tdt$ or $ax+y+b=\left(1/3a^2\right)t^3$, where $t=\left(a^2z^2+1\right)^{1/2}$

Or $ax+y+b=\left(1/3a^2\right)\times\left(a^2z^2+1\right)^{3/2}$ or $9a^4(ax+y+b)^2=\left(a^2z^2+1\right)^3$,

which is a complete integral, a and b being arbitrary constants.

Sol 3. (i). Here given equations is $f(x,y,z,p,q)=(z+px)^2-q=0$ ..... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+qf_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dq}{-f_q}$

Or $\dfrac{dp}{2p(z+px)+2p(z+px)}=\dfrac{dq}{2q(z+px)}=\dfrac{dz}{-2px(z+px)+q}=\dfrac{dx}{-2x(z+px)}=\dfrac{dy}{0}$, by (1)

Taking the second and fourth fractions, $(1/q)dq=-(1/x)dx.$

Integrating, $\log q=\log a-\log x$ so that $q=a/x.$ ..... (2)

Substituting the above value of q in (1), we have

$(z+px)^2=a/x$ or $px=\sqrt{a}/\sqrt{x}-z$ or $p=\sqrt{a}/x\sqrt{x}-z/x.$ ..... (3)

$\therefore\ dz=pdx+qdy=\left(\dfrac{\sqrt{a}}{x\sqrt{x}}-\dfrac{z}{x}\right)dx+\dfrac{a}{x}dy$, by (2) and (3)

Or $xdz=\sqrt{a}x^{-1/2}dx-zdx+ady$ or $xdz+zdx=\sqrt{a}x^{-1/2}dx+ady$

Or $d(xz)=\sqrt{a}x^{-1/2}dx+ady$

Integrating, $xz=2\sqrt{a}\sqrt{x}+ay+b, a, b$ being arbitrary constants

(ii) **Ans.** $yz=ax+\sqrt{ay}+b.$

Sol 5. (i) Here given equation is $f(x,y,z,p,q)\equiv q-px-p^2=0$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+qf_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

Or $\dfrac{dp}{-p}=\dfrac{dq}{0}=\dfrac{dz}{-p(-x-2p)-q}=\dfrac{dx}{-(-x-2p)}=\dfrac{dy}{-1}$, by (1)

The 2nd fraction gives dq = 0 so that q = a.

Putting q = a in (1) gives $p^2 + px - a = 0$ so that $p = (1/2) \times \left[-x \pm \left(x^2 + 4a \right)^{1/2} \right]$

Putting these values of p and q in dz = pdx + qdy, we get

$$dz = -(x/2) \times dx \pm (1/2) \times \left(x^2 + 4a \right)^{1/2} dx + ady.$$

Integrating, the required complete integral is

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\left(x^2 + 4a \right)^{1/2} + 2a \log\left\{ x + \left(x^2 + 4a \right)^{1/2} \right\} \right] + ay + b$$

Part (ii). Proceed like part (i) yourself. Complete integral is

$$z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\left(x^2 + 4a \right)^{1/2} + 2a \log\left\{ x + \left(x^2 + 4a \right)^{1/2} \right\} \right] + ay + b$$

Sol. Given equation is $f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0.$ ...... (1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

Or $\dfrac{dp}{-2p + 2y} = \dfrac{dq}{-2q + 2x} = \dfrac{dx}{2x - 2p} = \dfrac{dy}{2y - 2q}$, by (1)

which gives $\dfrac{dp + dq}{2(x + y - p - q)} = \dfrac{dx + dy}{2(x + y - p - q)}$

Or dp + dq = dx + dy i.e., dp – dx + dq – dy = 0.

Integrating, (p – x) + (q – y) = a ...... (2)

Re–writing (1), $(p - x)^2 + (q - y)^2 = (x - y)^2.$ ...... (3)

Putting the value of (q – y) from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2$ or $2(p - x)^2 - 2a(p - x) + \left\{ a^2 - (x - y)^2 \right\} = 0.$

$$\therefore\ p - x = \frac{2a \pm \sqrt{\left[4a^2 - 4.2 \cdot \left\{ a^2 - (x - y)^2 \right\} \right]}}{4} \Rightarrow p = x + \frac{1}{2}\left[a \pm \sqrt{\left\{ 2(x - y)^2 - a^2 \right\}} \right]$$

∴ (2) gives = a + y – p + x or $q = y + (1/2) \times \left[a \mp \sqrt{\left\{ 2(x - y)^2 - a^2 \right\}} \right].$

Putting these value of p and q in dz = p dx + q dy, we get

$$dz = xdx + ydy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\left\{ 2(x - y)^2 - a^2 \right\}}(dx - dy)$$

Or $dz = xdx + ydy + \dfrac{a}{2}(dx + dy) \pm \dfrac{1}{\sqrt{2}}\sqrt{(x - y)^2 - a^2/2} \cdot (dx - dy)$

Integrating, the desired complete integral is

$$z=\frac{x^2+y^2}{2}+\frac{a(x+y)}{2}\pm\frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2-a^2/2}-\frac{a^2}{4}\log\left[(x-y)+\sqrt{(x-y)^2-a^2/2}\right]\right)$$

Sol 8. (a) Let $f(x,y,z,p,q)=px^5-4q^2x^2+6x^2z-2=0$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_y}=\dfrac{dz}{-pf_p-q\,f_q}=\dfrac{dz}{-f_p}=\dfrac{dy}{-f_q}$

Or $\dfrac{dp}{5px^4-8q^2x+12xz+6px^2}=\dfrac{dq}{6qx^2}=\dfrac{dz}{-px^5+8q^2x^2}=\dfrac{dx}{-x^5}=\dfrac{dy}{8qx^2}$, by (1)

Taking the second and the last fractions, $4dq=3dy$

Integrating, $4q=3y+3a$ or $q=3(y+a)/4$ ...(2)

Using (2), (1) given $p=\{(9/4)\times(y+a)^2-6x^2z+2\}/x^5$ ...(5)

$$dz=(9/4x^3)(y+a)^2\,dx-(6z/x^3)dx+(2/x^5)dx+(3/4)(y+a)dy$$

$(6z/x^3)dx+dz=\{(9/4x^3)(y+a)^2\,dx+(3/4)(y+a)dy\}+(2/x^5)dx$ ...(4)

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial z}=\dfrac{\partial N}{\partial x}\right)=\dfrac{6}{x^3}$, which is function x alone and so I.F. $=e^{\int(6/x^3)dx}=e^{-3/x^2}$

Multiplying both sides of (4) by I.F. e^{-3/x^2} we, get

$$(6z/x^3)e^{-3/x^2}dx+e^{-3/x^2}dz=(3/8)\times\{(6/x^3)(y+a)^2e^{-3/x^2}dx+2(y+a)e^{-3/x^2}dy\}+(2/x^5)e^{-3/x^2}dx$$

$$d\left(ze^{-3/x^2}\right)=(3/8)\times d\{(y+a)^2e^{-3/x^2}\}+(2/x^5)\times e^{-3/x^2}dx$$

Integrating, $ze^{-3/x^2}=(3/8)\times(y+a)^2e^{-3/x^2}+2\int(1/x^2)e^{-3/x^2}(1/x^3)dx$

Or $ze^{-3/x^2}=(3/8)\times(y+a)^2e^{-3/x^2}-(1/9)\times\int ue^u\,du$, putting $(-3/x^2)=$ u so that $(6/x^3)dx=du$

Or $ze^{-3/x^2}=(3/8)\times(y+a)^2e^{-3/x^2}-(1/9)\times(ue^u-e^u)+b$

Or $ze^{-3/x^2}=(3/8)\times(y+a)^2e^{-3/x^2}-(1/9)\times(-3/x^2)e^{-3/x^2}+(1/9)\times e^{-3/x^2}+b$

Or $z=(3/8)\times(y+a)^2+(1/3x^2)+(1/9)+be^{-3/x^2}$, a, b, being arbitrary constants.

(b) **Ans.** $z=(2/3)\times(y+a)^{3/2}+(1/3x^2)+(1/9)+be^{-3/x^2}$

Sol 9. Let f (x,y, z, p, q) = $(p+y)^2+(q+x)^2-1=0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+qf_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

$$\frac{dp}{2(q+x)} = \frac{dq}{2(p+y)} = \frac{dz}{-2(p^2+q^2+py+qx)} = \frac{dx}{-2(p+y)} = \frac{dy}{-2(q+x)}, \text{ by (1)}$$

Taking the first and the last fractions, dp + dy = 0 so that p + y = a ... (2)

Using (2), (1) gives $a^2 + (q + x)^2 - 1 = 0$ or $q + x = (1-a^2)^{1/2}$ ... (3)

Using (2) and (3) in dz = pdx + qdy , we get

$dz = (a - y)dx + \{(1-a^2)^{1/2} - x\}dy = adx - 1(1-a^2)^{1/2} dy - (ydx + xdy)$

Integrating, $z = ax - (1-a^2)^{1/2} y - xy + b$, a, b being arbitrary constants.

Sol 10. Let $f(x, y, z, p, q) = 2y + 2zq - xpq - yq^2 = 0$ ... (1)

Charpit's auxiliary equations are $\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$ ...(2)

$$\frac{dp}{-pq+2pq} = \frac{dq}{2-q^2+2q^2(p+y)} = \frac{dz}{2pqx+2py-2qz} = \frac{dx}{qx} = \frac{dy}{xp+2yp-2z}, \text{ by (1)}$$

Taking the first and fourth fractions, (1/pq)dp = (1/qx)dx or (1/p)dp = (1/x)dx

Integrating, log p = log a + log x or p = ax, ... (3)

where a is an arbitrary constant. Substituting the value of p given by (3) in (1), we have

$2y + 2zq - ax^2q - yq^2 = 0$ or $yq^2 + q(ax^2 - 2z) - 2y = 0$.

$\Rightarrow$ $q = [-(ax^2-2x) \pm \{(ax^2-2z)^2+8y^2\}^{1/2}]/(2/y)$... (4)

Substituting the values of p and q given by (3) and (4) in dz = p dx + q dy, we obtain

$$dz = axdx + (1/2y) \times [2z - ax^2 \pm \{(2z - ax^2)2 + 8y^2\}^{1/2}]by$$

$$\frac{2dz - 2axdx}{(2z-ax^2) \pm \{(2x-ax^2)2+by^2\}^{1/2}} = \frac{dy}{y} \text{ ...(5)}$$

Putting $2z - ax^2 = u$ and $2dz - 2\,ax\,dx = du$, (5) yields

$$\frac{du}{u \pm (u^2+8y^2)^{1/2}} = \frac{dy}{y} \text{ or } \frac{du}{dy} = \frac{u}{y} \pm \left\{\left(\frac{u}{y}\right)^2 + 8\right\}^{1/2} \text{ ...(6)}$$

which is linear homogeneous differential equation. To solve it, we put u/y = v, i.e., u = yv so that du/dy = v + y(dv/dy) and so (6) reduces to

$$v + y\frac{dv}{dy} = v \pm (v^2+8)^{1/2} \text{ or } \frac{dv}{(v^2+8)^{1/2}} = \frac{dy}{y},$$

taking positive sign. Integrating it, we have

$\log\{v + (v^2 + 8)^{1/2}\}/ \log y + \log b$ or $v + (v^2 + 8)1/2 = by$

or $u / y + \{(u / y)^2 + 8\}^{1/2} = by$ or $u + (u2 + 8y^2)^{1/2} = by^2$

or $2z - ax^2 + \{(2z - ax^2)^2 + 8y^2\}^{1/2} = by^2$, as $u = 2z - ax^2$; a, b being arbitrary constants

PREVIOUS YEARS QUESTIONS

Q1. Find a complete integral of the partial differential equation

$\left(p^2+q^2\right)x = pz;\ p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$ using Charpit's method and hence deduce the solution which passes through the curve $x=0, z^2=4y$. **[8c IFoS 2022]**

Refer example 4a, 4b, category I

Q2. Solve the following by Charpit's method: $pxy+pq+qy=yz,\ p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$. **[6a IFoS 2021]**

Refer example 8, category I

Q3. Solve the following differential equation:

$\left(y^2+z^2-x^2\right)p-2xyq+2xz=0,\ p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$. **[7c IFoS 2021]**

Q4. Find a complete integral of the partial differential equation $p=(z+qy)^2$ by using Charpit's method. **Refer Solution 3, Assignment-1** **[8a UPSC CSE 2021]**

Q5. Find the general solution and singular solution of the partial differential equation $6yz-6pxy-3qy^2+pq=0$. **[(6a) 2020 IFoS]**

Take help of Refer example 7, category I

Q6. Find a complete integral of the equation by Charpit's method $p^2x+q^2y=z$. Here $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$. **[(5e) 2019 IFoS]**

Refer example 10, category I

Q7. Find the complete integral of the partial differential equation $\left(p^2+q^2\right)x=zp$ and deduce the solution which passes through the curve $x=0, z^2=4y$. Here $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$. **[(6a) 2018 IFoS]**

Refer example 4a, 4b, category I

Q8. Find a complete integral of the partial differential equation $2(pq+yp+qx)+x^2+y^2=0$.

Refer example 21, category I **[6a UPSC CSE 2017]**

Q9. Find complete integral of $xp-yq=xqf(z-pz-qy)$ where $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$.

Refer example 18, category I **[(6c) UPSC CSE 2017]**

Q10. Find the solution of the equation $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=1$ the passes through the circle $x^2+y^2=1, u=1$. **[(7c) UPSC CSE 2013]**

To find complete solution Refer **Category-2 Examples: Special Case-I: Form** $f(p,q)=0$

Then, follow the same procedure as we did for example 4 in Category I

CAUCHY'S CHARACTERISTIC METHOD

Q1. Find the solution of the partial differential equation

$$z=\frac{1}{2}\left(p^2+q^2\right)+(p-x)(q-y);\, p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$$

which passes through the *x*-axis, using Cauchy's method of characteristics.

[7c IFoS 2022], [7a UPSC CSE 2020]

Q2. Solve the first order quasilinear partial differential equation by the method of characteristics: $x\frac{\partial u}{\partial x}+(u-x-y)\frac{\partial u}{\partial y}=x+2y$ in $x>0, -\infty<y<\infty$ with $u=1+y$ on $x=1$.

[6a UPSC CSE 2019]

Q3. Determine the characteristics of the equation $z=p^2-q^2$, and find the integral surface which passes through the parabola $4z+x^2=0,\ y=0$. **[6a UPSC CSE 2016]**

Q4. Solve the following partial differential equation $zp+yq=x$

$x_0(s)=x, y_0(s)=1, z_0(s)=2s$ by the method of characteristics. **[6a UPSC CSE 2010]**

For answers refer Examples based on CAUCHY'S CHARACTERISTIC METHOD

HIGHER ORDER PDEs WITH CONSTANT COEFFICIENTS

General form of higher order PDE

$$a_0 \frac{\partial^n z}{\partial x^n} + \beta_0 \frac{\partial^n z}{\partial y^n} + \alpha_1 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \beta_1 \frac{\partial^{n-1} z}{\partial y^{n-1}} + \ldots\ldots\ldots\ldots = f(x, y) \qquad \ldots\ldots(1)$$

E.g. **(1)** $2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial^2 z}{\partial x \partial y} + 2x^3 y \frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x^2} = e^{x+y}$

(2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + x\cos y \frac{\partial^2 z}{\partial y^2} = x^2 + y$

(3) $2\frac{\partial^3 z}{\partial x^3} + 5\frac{\partial^3 z}{\partial x^2 \partial y} + 6\frac{\partial^3 z}{\partial y^3} = x + y$

OBSERVE!

(1) $\left(2D^2 + xDD' + 2x^3 yD'D^2\right)z = e^{x+y}$

Terms of degree 2 and Terms of degree 3. Terms are not of same degree (as far as partial derivative are concerned)

(2) $\left(D^2 + DD' + x\cos yD'^2\right)z = x^2 + y$; Terms are of degree 2 (derivatives)

(3) $\left(2D^3 + 5D^2D' + 6D'^3\right)z = x + y$; Terms are of degree 3 (derivative)

Representations: In general, $\frac{\partial}{\partial x} = D,\qquad \frac{\partial}{\partial y} = D'$

- **<u>Homogeneous PDEs</u>** $\left(F\left(D^m, D'^m\right)\right)z = f(x, y)$ **Each term of same degree *m(derivatives)***
  - With constant coefficient
  - with variable coefficient
    - Special form (Cauchy Euler's type)
    - $Rr + Ss + Tt = 0$

- **<u>Non – Homogenous PDEs</u>** $\left(F(D, D')\right)z = f(x, y)$; **Terms need not of same degree(derivatives)**
  - With constant coefficient
  - with variable coefficient

Part (1): we will study: Solving Homogenous PDEs

Part (2): We will study: Solving non-homogenous PDEs

When F(D, D') can be factored into linear factors of the form $\left((\mathrm{D}-m_1\mathrm{D}'-\mathrm{K}_1)(\mathrm{D}-m_2\mathrm{D}'-\mathrm{K}_2)\ldots\right)z = f(x,y)$

When *f*(D, D') cannot be factored into linear factors

<u>Part (1): Solving homogenous PDEs:</u> $\left(\mathrm{F}\left(\mathrm{D}^m,\mathrm{D}'^m\right)\right)z = f(x,y)$ .....(1)

Solution of (1) is given by z = C.F + P.I

Just to observe! Let's visualize through examples:

E.g.(1) Solving $\left(\mathrm{D}^2-5\mathrm{DD}'+6\mathrm{D}'^2\right)z = x+y$

To find CF of above equation:

$\left(\mathrm{D}^2-5\mathrm{DD}'+6\mathrm{D}'^2\right)z=0 \Rightarrow \left((\mathrm{D}-2\mathrm{D}')(\mathrm{D}-3\mathrm{D}')\right)z=0$ .....(i)

This is how we may Think!!

Let $(D-2D')z = v$ .....(2)

$\therefore$ We have, $(\mathrm{D}-3\mathrm{D}')v = 0 \Rightarrow Dv - 3D'v = 0 \Rightarrow \frac{\partial v}{\partial x} - 3\frac{\partial v}{\partial y} = 0$

By Lagrange's method; $\frac{dx}{1} = \frac{dy}{-3} = \frac{dv}{0}$ $\therefore$ From 1st two fractions, $y = -3x + c_1$ $\therefore$ $u(x,y,z) = y + 3x$

Also, $dv = 0 \Rightarrow v = c_2$

$(\mathrm{D}-2\mathrm{D}')z = c_2 \Rightarrow \frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = c_2$

So, Lagrange's; $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{c_2}$. Solving again and then using these in $\left((\mathrm{D}-2\mathrm{D}')(\mathrm{D}-3\mathrm{D}')\right)z = 0$

<u>Rule to find CF of (F(D^m, D$'^m$)) z = 0</u>

- Manage (F(D^m, D$'^m$)) z = 0 as,

  $(D-m_1D')(D-m_2D')(D-m_3D')\ldots\ldots(D-m_nD') = 0$ .....(1)

Now, Let's take $(D - m_n D')z = 0 \Rightarrow \frac{\partial z}{\partial x} - m_n \frac{\partial z}{\partial y} = 0 \Rightarrow 1.p \; - m_n q = \; 0 \;\therefore$

$P = 1,\; Q = -m_n,\; R = 0$

$\therefore$ Lagrange's auxiliary equation are,

$$\underbrace{\frac{dx}{1} = \frac{dy}{-m_n}}_{\Downarrow} = \underbrace{\frac{dz}{0}}_{\Downarrow}$$

$$y = -m_n x + c_1 \qquad z = c_2$$

$$u(x,y) = y + m_n x \;,\; v(x,y) = z$$

$\therefore$ Solution of (2) is given by $z = \phi_n(y + m_n x)$. In general, we have; $z = \phi_n(y + m_n x)$, then

Exam Point- C.F of (1) is given by

$\boxed{z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \ldots.. + \phi_n(y + m_n x)}$; where $\phi_1, \phi_2, \ldots.\phi_n$ are arbitrary functions.

Procedure:

- Replace D by *m*, D' by 1 in differential equation (1).

Let's say, we get $(m - m_1)(m - m_2)(m - m_3)\ldots.\;(m - m_n) = 0$; (Known as auxiliary equation of (1))

$$\therefore\; m = m_1, m_2, \ldots\ldots, m_n$$

<u>Case (1):</u> When $m_1, m_2, \ldots\ldots, m_n$ all are distinct.

C.F = $\phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \ldots\ldots + \phi_n(y + m_n x)$

<u>Case (2):</u> If auxiliary equation has two repeated roots & rest are distinct.

Let if $m_1 = m_2 = \alpha$, $m_3, m_4, \ldots\ldots., m_n$ are distinct

Then C.F = $\phi_1(y + \alpha x) + x\phi_2(y + \alpha x) + \phi_3(y + m_3 x) + \ldots\ldots + \phi_n(y + m_n x)$

<u>**Finding P.I of**</u> $\left(\mathrm{F}\left(\mathrm{D}^m, \mathrm{D}'^m\right)\right) z = f(x,y)$

<u>Case (1)</u>: when $(\mathrm{F}(\mathrm{D},\mathrm{D}'))$ is <u>homogenous of degree m</u> & $f(x,y) = \phi(ax + by)$

Exam point: $\boxed{P.I \;= \frac{1}{F(\mathrm{D},\mathrm{D}')}\phi^{(m)}(ax + by) = \frac{1}{F(a,b)}\phi(ax + by)}$

To remember: we may think like, let $ax + by = v$ $\therefore$ We have,

$\frac{1}{F(D,D')}\phi^{(m)}(v) = \frac{1}{F(a,b)}\phi(v)$ On integrating m times w.r.t. v we get

$$\boxed{\frac{1}{F(D,D')}\phi(v) = \frac{1}{F(a,b)}\underbrace{\int\int\ldots\int}_{m}\phi(v)dv^m}$$

Note: If F(a, b) = 0; then $\boxed{\frac{1}{(bD - aD')}\phi(ax+by) = \frac{x^m}{b^m m!}\phi(ax+by)}$

<u>Case (2)</u> when $f(x, y) = x^m y^n$; P.I = $\frac{1}{F(D,D')}x^m y^n$

Exam point: Expand F(D, D') using binomial in powers of D or D' according as $m < n$ or $n < m$

E.g. $F(D,D') = D^2 - 5DD' + 6D'^2$, $f(x,y) = xy^2$

PI = $\frac{1}{(D^2 - 5DD' + 6D'^2)}xy^2 = \frac{1}{D'}\left\{\frac{1}{D-3D'} - \frac{1}{D-2D'}\right\}xy^2 = \frac{1}{D'}\left\{\frac{1}{(D-3D')}xy^2 - \frac{1}{(D-2D')}xy^2\right\}$

$= \frac{1}{D'}\left\{\frac{1}{D\left(1-\frac{3D'}{D}\right)}xy^2 - \frac{1}{D\left(1-\frac{2D'}{D}\right)}xy^2\right\} = \frac{1}{DD'}\left\{\left(1-\frac{3D'}{D}\right)^{-1}xy^2 - \left(1-\frac{2D'}{D}\right)^{-1}xy^2\right\}$

$= \frac{1}{DD'}\left\{\left(1+\frac{3D'}{D}+\frac{9D'^2}{D^2}+\frac{27D'^3}{D^3}+\ldots\right)xy^2 - \left(1+\frac{2D'}{D}+\frac{4D'^2}{D^2}+\ldots\right)xy^2\right\} = \ldots$

Case (IV): A general method to find PI of $(F(D,D'))z = f(x,y)$

If more than one factors, then use partial fraction and then

$\frac{1}{(D-mD')}f(x,y) = \int f(x, c-mx)dx$, where $c = y + mx$

E.g. **Finding** $\frac{1}{(D-5D')}\sin(x+y)$

Way 1: $\sin(x+y) = f(ax+by)$; $a = 1, b = 1$ & proceeding by case (1)

Way 2: By general method,

P.I = $\frac{1}{(D-5D')}\sin(x+y)=\int \sin(x+c-mx)dx=\int \sin(x+c-5x)dx=\int \sin(c+4x)dx$

$$=\frac{-\cos(c-4x)}{-4}=\frac{1}{4}\cos(y+5x-4x)=\frac{1}{4}\cos(x+y)$$

Working Procedure to solve: Homogenous Linear $F(D,D')z=f(x,y)$; where

$$F(D,D')=(D-m_1D')(D-m_2D')......(D-m_nD')$$

Finding C.F of (F(D, D')) *z* = 0 .....(1)

Write auxiliary equation by replacing D by m & D' by 1 in (1), we get auxiliary equation:

$$(m-m_1)(m-m_2).....(m-m_n)=0 \quad \therefore \; m=m_1,m_2,......m_n$$

Case (1):- Let if $m_1,m_2,......m_n$ are distinct then C.F = $\phi_1(y+m_1x)+\phi_2(y+m_2x)+.....+\phi_n(y+m_nx)$

Where $\phi_1,\phi_2,.....,\phi_n$ are arbitrary functions.

Case (2):- Let if $m=m_1,m_2,.......m_r=\alpha$ i.e., r-repeated roots of m; then

C.F = $\phi(y+\alpha x)+x\phi(y+\alpha n)+x^2\phi(y+\alpha x)+.....+x^{r-1}\phi(y+\alpha x)+\phi_{r+1}(y+m_rx)+.....+\phi_n(y+m_nx)$

Finding PI

Case (1): $\frac{1}{F(D^m,D'^m)}\phi(v)=\frac{1}{F(a,b)}\underbrace{\iiint}_{m\text{ times}}\phi(v)dv$; $v=ax+by$

Case (2): $\frac{1}{(bD-aD')^m}\phi(ax+by)=\frac{x^m}{b^m m!}\phi(ax+by)$; If F($a$, b) = 0

Case (3): When $f(x,y)=x^my^n$ (use binomial to expand F(D, D'))

Case (4): General method $\frac{1}{D-mD'}f(x,y)=\int f(x,c-mx)dx;\qquad y=mx+c$

Non-homogenous PDEs with constant coefficient

Finding C.F & PI = $F(D,D')z=f(x,y)$: Now $F(D,D')$ need not be homogenous.

Finding C.F when $F(D,D')$ can be factorized as

- $\left(\left(a_1D-b_1D'-c_1\right)\left(a_2D-b_2D'-c_2\right)\left(a_nD-b_nD'-c_z\right)\right)z=0$

  i.e. $\left(\left(D-m_1D'-K_1\right)\right)\left(\left(D-m_2D'-K_2\right)\right)......\left(D-m_nD'-K_n\right)z=0$

$\because\ \left(D-mD'-K\right)z=0\Rightarrow \frac{\partial z}{\partial x}-m\frac{\partial z}{\partial y}-kz=0\Rightarrow\ Pp-mq=kz$

Lagrange's method: $\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{kz}\ give: ze^{-kx}=\phi\left(y+mx\right)\Rightarrow z=e^{kx}\phi\left(y+mx\right)$

Exam point: To find C.F of F(D, D')

- **Factorize** $\mathrm{F}(\mathrm{D},\ \mathrm{D}')$ **as** $\left(\left(D-m_1D'-K_1\right)\left(D-m_2D'-K_2\right)..........\left(D-m_nD'-K_n\right)\right)z=0$

Then C.F = $e^{k_1x}\phi_1\left(y+m_1x\right)+e^{k_2x}\phi_2\left(y+m_2x\right)+......+e^{k_nx}\phi_n\left(y+m_nx\right)$

Case (2): When $\mathrm{F}(\mathrm{D},\ \mathrm{D}')$ cannot be factorized as above : In that case we use **TRIAL METHOD**.

Step (1): Let $z=Ae^{hx+ky}$; where *h*, *k* are constants to be chosen. So, for given PDE

We get (*h*, *k*) ← $\begin{cases} z=e^{hx+ky} \\ \frac{\partial z}{\partial x}=he^{hx+ky} \\ \frac{\partial^2 z}{\partial x^2}=h^2e^{hx+ky} \end{cases}$

Step (2): Required solution is given by $z=\sum_{i=1}^{n}Ae^{h_ix+k_iy}$

<u>Finding PI of</u> $\mathrm{F}(\mathrm{D},\ \mathrm{D}')$ ***z* = *f*(*x*, *y*)**

Case (1): When $f\left(x,y\right)=e^{ax+by}$, Then P.I = $\frac{1}{F\left(D,D'\right)}e^{ax+by}=\frac{e^{ax+by}}{F\left(a,b\right)}$

Case (2): When $f\left(x,y\right)=x^xy^n$

Then P.I = $\frac{1}{F\left(D,D'\right)}x^my^n$; expand F(D, D') in powers of D or D' by binomial expansion.

Case (3): When $f\left(x,y\right)=\sin\left(ax+by\right)$ or $\cos\left(ax+by\right)$

Then for P.I: Replace D^2 by $-a^2$ & D'^2 by $-b^2$ & DD'^2 by ab

Case (4): When $f\left(x,y\right)=e^{ax+by}v\left(x,y\right)$

Then P.I = $\frac{1}{F(D,D')}e^{ax+by}v(x,y)$ $=e^{ax+by}\frac{1}{F(D+a,D'+b)}v(x,y)$;*v*(*x*, *y*) is any of above three cases.

Note: if in case (1) $\frac{1}{F(D,D')}e^{ax+by};F(a,b)=0$ then we do by:

$$\frac{1}{F(D,D')}e^{ax+by}=\frac{1}{F(D,D')}e^{ax+by}.1=e^{ax+by}\frac{1}{F(D+a,D'+b)}.1$$

Similar as we have done in ODEs **Recalling from ODEs**

PI = $\frac{1}{F(D)}e^{ax}=\frac{e^{ax}}{F(a)};$

PI = $\frac{1}{F(D)}\left(x^m+ax^{m-1}+......+\right)$; Binomial expansion

PI = $\frac{1}{F(D)}\cos ax$ or sin*ax*; Replacing $D^2 \to -a^2$

PI = $\frac{1}{F(D)}e^{ax}v(x)$ $=e^{ax}\frac{1}{F(D+a)}v(x)$

To Solve PDEs of the form $\boxed{x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0}$

The similar process as ODE, is followed.

We reduce given PDE by substituting: $x=e^u$, $y=e^v$

We get,

$x^2\frac{\partial^2 z}{\partial x^2}=D(D-1)$, where $D=\frac{\partial}{\partial u}$

$x^2\frac{\partial^2 z}{\partial x^2}=D'(D'-1)$, where $D'=\frac{\partial}{\partial v}$

$xy\frac{\partial z}{\partial x\partial y}=DD'$

<u>EXAMPLES & PYQs : Linear Higher order PDEs with constant coefficients</u>

CATEGORY-1

Ex. 1. Solve the following partial differential equations:

(a) $\left(D^2 - 3DD' + 2D'^2\right)z = e^{2x-y} + e^{x+y} + \cos(x+2y)$.

(b) $\left(D^2 - 3DD' + 2D'^2\right)z = e^{2x-y} + \cos(x+2y)$

(c) $\left(D^3 - 4D^2D' + 5DD'^2 - 2D'^3\right)z = e^{y+2x} + (y+x)^{1/2}$.

(d) $\left(D_x^3 - 7D_xD_y^2 - 6D_y^3\right)z = \sin(x+2y) + e^{3x+y}$.

Sol. (a) Here auxiliary equation is $m^2 - 3m + 2 = 0$ so that $m = 1, 2$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary function. ...(1)

• Now, P.I. corresponds to e^{2x-y}

$$= \frac{1}{D^2 - 3DD' + 2D'^2} e^{2x-y} = \frac{1}{2^2 - 3\times 2\times(-1) + 2\times(-1)^2} \iint e^v dv dv \text{, where } v = 2x - y$$

$$= (1/12)\times \int e^v dv = (1/12)\times e^v = (1/12)\times e^{2x-y} \quad \text{...(2)}$$

• P.I. corresponding to $e^{x+y} = \dfrac{1}{D^2 - 3DD' + 2D'^2} e^{x+y} = \dfrac{1}{D - D'}\left\{\dfrac{1}{D - 2D'} e^{x+y}\right\}$

$$= \frac{1}{D - D'}\left\{\frac{1}{1 - (2\times 1)} \int e^v dv\right\} \text{, where } v = x + y$$

$$= -\frac{1}{D - D'} e^v = -\frac{1}{(D - D')^1} e^{x+y} = -\frac{x}{1^1 \times 1!} e^{x+y} = -xe^{x+y}$$

• P.I. corresponding to $\cos(x+2y)$

$$= \frac{1}{D^2 - 3DD' + 2D'^2} \cos(x+2y) = \frac{1}{1^2 - (3\times 1\times 2) + (2\times 2^2)} \iint \cos v\, dv dv \text{, where}$$

$v = x + 2y$

$$= (1/3)\times \int \sin v\, dv = -(1/3)\times \cos v = -(1/3)\times \cos(x+2y)$$

From (1), (2), (3) and (4), the required solution is z = C.F. + P.I.

$$z=\phi_1(y+x)+\phi_2(y+2x)+(1/12)\times e^{2x-y}-xe^{x+y}-(1/3)\times\cos(x+2y)$$

(b) This problem is same as part (a) except that the term e^{x+y} is missing on R.H.S. So, now you need not compute P.I. corresponding to e^{x+y}. Therefore, the solution will take the form

$$y=\phi_1(y+x)+\phi_2(y+2x)+(1/12)\times e^{2x-y}-(1/3)\times\cos(x+2y)$$

(c) Here auxiliary equation is $m^3-4m^2+5m-2=0$ giving $m=1,1,2$.

$\therefore$ C.F. $=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y+2x).,\phi_1,\phi_2,\phi_3$ being arbitrary function ...(1)

• P.I. corresponding to e^{y+2x}

$$=\frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}e^{y+2x}=\frac{1}{D-2D'}\left\{\frac{1}{(D-D')^2}e^{y+2x}\right\}$$

$$=\frac{1}{D-2D'}\frac{1}{(2-1)^2}\int\int e^v dvdv, \text{ where } v=y+x,$$

$$=\frac{1}{D-2D'}\int e^v dv=\frac{1}{D-2D'}e^v=\frac{1}{(1\times D-2\times D')^1}e^{y+2x}=\frac{x}{1\times 1!}e^{y+x}=xe^{y+x} \quad ...(2)$$

• P.I. corresponding to $(y+x)^{1/2}$

$$=\frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}(y+x)^{1/2}=\frac{1}{(D-D')^2}.\left\{\frac{1}{D-2D'}(y+x)^{1/2}\right\}$$

$$=\frac{1}{(D-D')^2}\times\frac{1}{1-(2\times 1)}\int v^{1/2}dv, \text{ where } v=y+x$$

$$=-\frac{1}{D-D'}\times\frac{2}{3}v^{3/2}=-\frac{2}{3}\frac{1}{(D-D')^2}(y+x)^{3/2}=-\frac{2}{3}\times\frac{x^2}{1^2\times 2!}(y+x)^{3/2}$$

$$=-(x^2/3)\times(y+x)^{3/2}, \quad ...(3)$$

From (1), (2) and (3), the required general solution is

$$z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y+2x)+xe^{y+x}-(x^2/3)\times(y+x)^{3/2}$$

(d) Here note that D_x and D_y stand for D and D' respectively.

$\therefore$ Auxiliary equation is $m^3-7m-6=0$ so that $m=-1,-2,3$.

$\therefore$ C.F. $= \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x), \phi_1, \phi_2, \phi_3$ being arbitrary functions.

- P.I. corresponding to $\sin(x+2y)$

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}\sin(x+2y) = \frac{1}{1^3-(7\times1\times2^2)-(6\times2^3)}\iiint \sin v\,dv\,dv\,dv, \text{ where}$$

$v = x+2y$

$$= -(1/75)\times\iint(-\cos v)\,dvdv = -(1/75)\times\int(-\sin v)\,dv = -(1/75)\times\cos v = -(1/75)\times\cos(x+2y)$$

- P.I. corresponding to e^{3x+y}

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}e^{3x+y} = \frac{1}{D_x - 3D_y}\left[\frac{1}{(D_x+D_y)(D_x+2D_y)}e^{3x+y}\right]$$

$$= \frac{1}{D_x-3D_y}\cdot\frac{1}{(3+1)(3+2)}\iint e^v\,dvdv, \text{ where } v = 3x+y$$

$$= \frac{1}{20}\frac{1}{D_x-3D_y}\int e^v\,dv = \frac{1}{20}\frac{1}{D_x-3D_y}e^v = \frac{1}{20}\frac{1}{(D_x-3D_y)^1}e^{3x+y}$$

$$= \frac{1}{20}\times\frac{x}{1^1\times1!}e^{3x+y} = \frac{x}{20}e^{3x+y}$$

Hence the required general solution is z = C.F. + P.I.

$$z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)-(1/75)\times\cos(x+2y)+(1/20)\times xe^{3x+y}$$

Ex.2. Solve (a) $(D^2 - 6DD' + 9D'^2)z = \tan(y+3x)$ (b) $(D^2 - 6DD' + 9D'^2)z = 6x+2y$

Sol. (a) Here auxiliary equation is $(m-3)^2 = 0$ so that $m = 3, 3$.

$\therefore$ C.F. $= \phi_1(y+3x)+x\phi_2(y+3x), \phi_1, \phi_2$ being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-3D')^2}\tan(y+3x) = \frac{x^2}{1^2\times2!}\tan(y+3x) = \frac{x^2}{2}\tan(y+3x)$$

$\therefore$ The required solution is

$$z = \phi_1(y+3x)+x\phi_2(y+3x)+(x^2/2)\times\tan(y+3x).$$

(b) Re-writing the given equation reduces to $(D-3D')^2 z = 2(3x+y)$

$\therefore$ C.F. $=\phi_1(y+3x)+x\phi_1(y+3x), \phi_1, \phi_2$ being arbitrary constants.

Now, P.I. $=\dfrac{1}{(D-3D')^2}2(3x+y)=2\dfrac{x^2}{1^2\times 2!}(3x+y)=x^2(3x+y)$

$\therefore$ The required solution is $z=\phi_1(y+3x)+x\phi_2(y+3x)+3x^3+x^2y$.

Ex.3. Solve

(a) $(D-3D')^2(D+3D')z=e^{3x+y}$

(b) $(D-2D')(D+D')^2 z=\cos(2x+y)$

Sol. (a) C.F. $=\phi_1(y+3x)+x\phi_2(y+3x)+\phi_3(y-3x)$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

P.I. $=\dfrac{1}{(D-3D')^2}\dfrac{1}{D+3D'}e^{3x+y}=\dfrac{1}{(D-3D')^2}\dfrac{1}{3+(3\times 1)}\int e^v dv$, where $v = 3x + y$

$=\dfrac{1}{6}\dfrac{1}{(D-3D')^2}e^y=\dfrac{1}{6}\dfrac{1}{(D-3D')^2}e^{3x+y}=\dfrac{1}{6}\dfrac{x}{1\times 2!}e^{3x+y}$

The required solution is $z=\phi_1(y+3x)+x\phi_2(y+3x)+\phi_3(y-3x)+(x^2/12)\times e^{3x+y}$

(b) **Ans.** $z=\phi_1(y+2x)+\phi_2(y-x)+x\phi_3(y-x)-(x/9)\times\cos(2x+y)$

Ex.4. Solve

(a) $r+s-2t=e^{x+y}$

(b) $(D^3-7DD'^2-6D'^3)y=\sin(x+2y)$

(c) $(D^3-3DD'^2+2D'^3)y=(x-2y)^{1/2}$

Sol. (a) Re-writing given equation becomes $(\partial^2 z/\partial x^2)+(\partial^2 z/\partial x\partial y)-2(\partial^2 z/\partial y^2)=e^{x+y}$

or $(D^2+DD'-2D'^2)z=e^{x+y}$ or $(D-D')(D+2D')z=e^{x+y}$

Its C.F. $=\phi(y+x)+\phi_2(y-2x), \phi_1, \phi_2$ being arbitrary functions

P.I. $=\dfrac{1}{D-D'}\dfrac{1}{D+2D'}e^{x+y}=\dfrac{1}{D-D'}\dfrac{1}{1+(2\times 1)}\int e^v dv$, where $v = x + y$

$$= \frac{1}{3}\frac{1}{\mathrm{D}-\mathrm{D}'}e^{v} = \frac{1}{3}\frac{1}{\mathrm{D}-\mathrm{D}'}e^{x+y} = \frac{1}{3}\frac{x}{1!}e^{x+y}$$

$\therefore$ The required solution is $z = \phi_1(y+x) + \phi_2(y-2x) + (x/3)\times e^{x+y}$

(b) Here the auxiliary equation is $m^3 - 7m - 6 = 0$ giving $m = -1, -2, 3$.

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x), \phi_1, \phi_2, \phi_3$ are arbitrary functions.

$$\text{P.I.} = \frac{1}{\mathrm{D}^3 - 7\mathrm{D}\mathrm{D}'^2 - 6\mathrm{D}'^3}\sin(x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (16\times 2^3)}\iiint \sin v (dv)^3,$$

where $v = x+2y$

$$= -\frac{1}{75}\iint(-\cos v)\,dv dv = -\frac{1}{75}\int(-\sin v)\,dv = -\frac{1}{75}\cos v = -\frac{1}{75}\cos(x+2y)$$

$\therefore$ Required solution $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - (1/75)\times\cos(x+2y)$

(c) The auxiliary equation is $m^3 - 3m + 2 = 0$ giving $m = 1, 1, 2$.

$\therefore$ C.F. $= \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x), \phi_1, \phi_2, \phi_3$ are arbitrary functions

$$\text{P.I.} = \frac{1}{\mathrm{D}^3 - 3\mathrm{D}\mathrm{D}'^2 + 2\mathrm{D}'^3}(x-2y)^{1/2} = \frac{1}{1^3 - 3\times 1\times(-2)^2 + 2\times(-2)^3}\iiint v^{1/2}(dv)^3, \text{ where } v = x-2y$$

$$= -\frac{1}{27}\iint\frac{v^{3/2}}{(3/2)}dv dv = -\frac{1}{27}\int\frac{v^{5/2}}{(3/2)\times(5/2)}dv = -\frac{1}{27}\frac{v^{7/2}}{(3/2)\times(5/2)\times(7/2)}$$

$$= -(8/2835)\times v^{7/2} = -(8/2835)\times(x-2y)^{7/2}$$

General solution is $z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x) - (8/2835)\times(x-2y)^{7/2}$.

Ex.5. Solve $(\mathrm{D}^2 - 3\mathrm{D}\mathrm{D}' + 2\mathrm{D}'^2)z = \cos(x+2y)$

Sol. The auxiliary equation $m^2 - 3m + 2 = 0$ gives $m = 1, 2$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x), \phi_1, \phi_2$ being arbitrary functions

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 - 3\mathrm{D}\mathrm{D}' + 2\mathrm{D}'^2}\cos(x+2y) = \frac{1}{1^2 - 3\cdot 1\cdot 2 + 2\cdot 2^2}\iint\cos v(dv)^2, \text{ where } v = x+2y$$

$$= (1/3)\times\int\sin v dv = -(1/3)\times\cos v = -(1/3)\times\cos(x+2y)$$

$\therefore$ Solution is $z = \phi_1(y+x) + \phi_2(y+2x) - (1/3) \times \cos(x+2y)$

Ex.6. Solve $\left(D^2 - DD' - 2D'^2\right) z = 2x + 3y + e^{3x+4y}$.

Sol. The auxiliary equation $m^2 - m - 2 = 0$ giving $m = 2, -1$.

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-x), \phi_1, \phi_2$ being arbitrary functions

P.I. corresponding to $(2x+3y)$

$$= \frac{1}{D^2 - DD' - 2D'^2}(2x+3y) = \frac{1}{2^2 - (2\times3) - (2\times3^2)} \iint v (dv)^2 \text{, where } v = 2x + 3y$$

$$= -\frac{1}{20}\int \frac{v^2}{2} dv = -\frac{1}{20}\left(\frac{v^3}{2\times3}\right) = -\frac{1}{60}(2x+3y)^3$$

P.I. corresponding to e^{3x+4y}

$$= \frac{1}{D^2 - DD' - 2D'^2} e^{3x+4y} = \frac{1}{3^2 - (3\times4) - (2\times4^2)} \iint e^v (dv)^2 \text{, where } v = 3x + 4y$$

$$= -(1/35) \times e^v = -(1/35) \times e^{3x+4y}.$$

$\therefore$ General solution is $z = \phi_1(y+2x) + \phi_2(y-x) - (1/60)\times(2x+3y)^3 - (1/35)\times e^{3x+4y}$.

Ex. 7. Solve $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \cos ny$.

Sol. Given equation can be written as $\left(D^2 + D'^2\right) z = \cos mx \cos ny$.

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = \pm i$.

$\therefore$ C.F. $= \phi_1(y+ix) + \phi_2(y-ix), \phi_1$ and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + D'^2}\cos mx \cos ny = \frac{1}{D^2 + D'^2}\frac{\cos(mx+ny) + \cos(mx-ny)}{2}$$

$$= \frac{1}{2}\frac{1}{D^2 + D'^2}\cos(mx+ny) + \frac{1}{2}\frac{1}{D^2 + D'^2}\cos(mx-ny)$$

$$= \frac{1}{2}\frac{1}{m^2 + n^2}\iint \cos v \, dv \, dv + \frac{1}{2}\frac{1}{m^2 + (-n)^2}\iint \cos u \, du \, du$$

where $v = mx + ny$ and $u = mx - ny$

$$= \frac{1}{2}\frac{1}{m^2+n^2}\int \sin v\,dv + \frac{1}{2}\frac{1}{m^2+n^2}\int \sin u\,du = \frac{1}{2}\frac{1}{m^2+n^2}\left[-\cos v - \cos u\right]$$

$$= -\frac{1}{2\left(m^2+n^2\right)}\left[\cos\left(mx+ny\right)+\cos\left(mx-ny\right)\right],\text{ as } v = mx+ny,\ u = mx-ny$$

$$= -\frac{1}{2\left(m^2+n^2\right)}\times 2\cos mx\cos ny = -\left(m^2+n^2\right)^{-1}\cos mx\cos ny$$

Hence the required general solution is $z = \phi_1\left(y+ix\right)+\phi_2\left(y-ix\right)-\left(m^2+n^2\right)^{-1}\cos mx\cos ny$

Ex.7.(b) Solve $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \sin ny$

Sol. Ans. $z = \phi_1\left(y+ix\right)+\phi_2\left(y-ix\right)+\left(\sin mx+\sin ny\right)/\left(m^2+n^2\right)$

Ex. 8. Solve the following partial differential equations:

(a) $\left(D^2 - 2DD' + D'^2\right)z = \tan\left(y+x\right)$ or $\left(D-D'\right)^2 z = \tan\left(y+x\right)$

(b) $\left(D^2 - 2aDD' + a^2D'^2\right)z = f\left(y+ax\right)$ or $\left(D-aD'\right)^2 z = f\left(y+ax\right)$.

(c) $4r - 4s + t = 16\log\left(x+2y\right)$.

Sol. (a) Ans. $z = \phi_1\left(y+x\right)+x\phi_2\left(y+x\right)+\left(x^2/2\right)\times\tan\left(y+x\right)$.

(b) Ans. $z = \phi_1\left(y+x\right)+x\phi_2\left(y+x\right)+\left(x^2/2\right)\times f\left(y+ax\right)$.

(c) Ans. $z = \phi_1\left(2y+x\right)+x\phi_2\left(2y+x\right)+2x^2\log\left(x+2y\right)$

(d) Ans. $z = \phi_1\left(y+x\right)+\phi_2\left(y-x\right)+\left(x/4\right)\times\left(x-y\right)^2$

(e) Ans. $z = \phi_1\left(y-x\right)+\phi_2\left(2y+3x\right)+xe^{x-y}$.

(f) Hint: P.I. $= \dfrac{1}{D^2+5DD'+6D'^2}(y-2x)^{-1} = \dfrac{1}{D+2D'}\left[\dfrac{1}{D+3D'}(y-2x)^{-1}\right]$

$$= \frac{1}{D+2D'}\times\frac{1}{-2+\left(3\times 1\right)}\int v^{-1}dv,\text{ where } v = y-2x$$

$$=\frac{1}{D+2D'}\log v=\frac{1}{D+2D'}\log(y-2x)=\frac{1}{\left[1\times D-(-2)\times D'\right]^1}\log(y-2x)$$

$$=\frac{x}{1^1\times 1!}\log(y-2x)$$

$\therefore$ The general solution is $z=\phi_1(y-2x)+\phi_2(y-3x)+x\log(y-2x)$.

Ex.9. Solve $\partial^3 z/\partial x^2\partial y-2\left(\partial^3 z/\partial x\partial y^2\right)+\partial^3 z/\partial y^3=1/x^2$

Sol. Let $D=\partial/\partial x$ and $D'=\partial/\partial y$. Then the given equation becomes

$\left(D^{'}D'=2DD'^2+D'^3\right)z=1/x^2$ or $(D-D')^2D'z=1/x^2$ .....(1)

Corresponding to repeated factor $(D-D')^2$, the part of C.F. is $\phi_1(y+x)+x\phi_2(y+x)$. Again corresponding to factor *D'*, the part of C.F. is $f_1(x)$.

$\therefore$ C.F. of (1) = $\phi_1(y+x)+x\phi_2(y+x)+\phi_3(x)$

P.I = $\frac{1}{(D-D')^2D'}\frac{1}{x^2}=\frac{1}{(D-D')^2}\int\frac{1}{x^2}dy=\frac{1}{(D-D')^2}\frac{y}{x^2}=\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}\frac{y}{x^2}$

$$=\frac{1}{D^2}\left(1+\frac{2D'}{D}+\frac{3D'^2}{D^2}+....\right)\frac{y}{x^2}=\frac{1}{D^2}\left\{\frac{y}{x^2}+\frac{2}{D}\left(\frac{1}{x^2}\right)\right\}=y\frac{1}{D^2}\frac{1}{x^2}+\frac{2}{D^3}\frac{1}{x^2}$$

$$=y\frac{1}{D}\left(-\frac{1}{x}\right)=-y\log x$$

where we have omitted a function of *x* as it can be included in the term $\phi_3(x)$ of C.F.

$\therefore$ Required general solution is $z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3-y\log x$

Where ϕ_1,ϕ_2 and ϕ_3 are arbitrary function .

Ex.10- Solve $\left(D^3-7DD'-6D'^3\right)z=x^2+xy^2+y^3+c(x-y)$

Sol. Given $\left(D^3-7DD'^2-6D'^3\right)z=x^2+xy^2+y^3+\cos(x-y)$ .....(1)

Here auxiliary equation is $m^3-7m-6=0$ so that $m=-1,-2,3.$

$\therefore$ C.F. = $\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x).\phi_1,\phi_2,\phi_3$ being arbitrary function

P.I. Corresponding to $(x^2+xy^2+y^3)$

$$=\frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3)=\frac{1}{D^3}\left\{1-\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)\right\}^{-1}(x^2+xy^2+y^3)$$

$$=\frac{1}{D^3}\left\{1+\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)+....\right\}(x^2+xy^2+y^3)=\frac{1}{D^3}(x^2+xy^2+y^3)+\frac{7}{D^5}(2x+6y)+\frac{36}{D^6}1$$

$$=\left(x^5/60+x^4y^2/24+x^3y^3/(6)+7\left(x^6/360+x^5y/20\right)+36\times\left(x^6/720\right)\right)$$

$$=5x^6/72+x^5/60+7x^5y/20+x^4y^2/24+x^3y^3/6$$

P.I. Corresponding to cos(*x* - *y*)

$$=\frac{1}{D^3-7DD'^2-6D'^3}\cos(x-y)=\frac{1}{(D+D')}\frac{1}{\left(D^2-DD'-6D'^2\right)}\cos(x-y)$$

$$=\frac{1}{D+D'}\frac{1}{1^2-4\times1\times(-1)-6\times(-1)^2}\int\int\cos v\,dv\,dv$$, where $v = x - y$

$$=\frac{1}{DD'D''}\left\{1-\frac{D^3D'^3+D''^3}{3DD'D''}+......\right\}xyz=\frac{1}{DD'D''}xyz=\frac{x^2y^2z^2}{8}$$

$$=-\frac{1}{4}\frac{x}{(-1)^1\times1!}\cos(x-y)$$

$$=(x/4)\times\cos(x-y)$$

Hence the required general solution is $z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

$$+(5/72)\times x^6+x^5/60+(7/20)\times x^5y+(1/24)\times x^4y^2+(1/6)\times\left(x^3y^3\right)+(x/4)\times\cos(x-y)$$

Ex.11. Solved $\left(D^2+DD'-6D'^2\right)z=x^2\sin(x+y)$.

Sol. Re-writing the given equation is $(D+3D')(D-2D')z=x^2\sin(x+y)$ ....(1)

$\therefore$ C.F. = $\phi_1(y-3x)+\phi_2(y+2x), \phi_1, \phi_2$ being arbitrary functions.

P.I. = $\dfrac{1}{(D+3D')(D-2D')}x^2\sin(x+y) = \dfrac{1}{D+3D'}.\left\{\dfrac{1}{D-2D'}, x^2\sin(x+y)\right\}$

$= \dfrac{1}{D+3D'}\int x^2\sin(x+c-2x)dx = \dfrac{1}{D+3D'}\int x^2\sin(c-x)dx$, where $c = y+2x$

$= \dfrac{1}{D+3D'}\left[x^2\cos(c-x) - \int 2x\cos(c-x)dx\right]$, integrating by parts

$= \dfrac{1}{D+3D'}\left[x^2\cos(c-x) - \left\{-2x\sin(c-x) + \int 2\sin(c-x)dx\right\}\right]$, integrating by parts

$= \dfrac{1}{D+3D'}\left[x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)\right]$

$= \dfrac{1}{D+3D'}\left[(x^2-2)\cos(x+y) + 2x\sin(x+y)\right]$, as $c = y+2x$

$= \int\left[(x^2-2)\cos(x+c'+3x) + 2x\sin(x+c'+3x)\right]dx$, where $c' = y-3x$

$= \int(x^2-2)\cos(4x+c')dx + 2\int x\sin(4x+c')dx$

$= (x^2-2)\dfrac{\sin(4x+c')}{4} - \int 2x\dfrac{\sin(4x+c')}{4}dx + 2\int x\sin(4x+c')dx$

[Integrating by part 1st integral and keeping the second integral unchanged]

$= \dfrac{1}{4}(x^2-2)\sin(4x+c') + \dfrac{3}{2}\int x\sin(4x+c')dx$

$= \dfrac{x^2-2}{4}\sin(4x+c') + \dfrac{3}{2}\left[-\dfrac{x\cos(4x+c')}{4} + \int\dfrac{\cos(4x+c')}{4}dx\right]$

$= \dfrac{x^2-2}{4}\sin(4x+c') - \dfrac{3}{8}x\cos(4x+c') + \dfrac{3}{32}\sin(4x+c')$

$= \dfrac{1}{4}(x^2-2)\sin(4x+y-3x) - \dfrac{3}{8}x\cos(4x+y-3x) + \dfrac{3}{32}\sin(4x+y-3x)$, as $c' = y-3x$

$= (x^2/4 - 13/32)\sin(x+y) - (3x/8)\times\cos(x+y)$,

The solution $z = \phi_1(y-3x) + \phi_2(y+2x) + \left[(x^2/4) - (13/32)\right]\sin(x+y) - (3x/8)\times\cos(x+y)$.

Ex. 12. Solve $\left(D^3+D^2D'-DD'^2-D'^3\right)z=e^y\cos 2x$

Sol. Here $D^3+D^2D'-DD'^2-D'^3=D^2\left(D+D'\right)-D'^2\left(D+D'\right)=\left(D+D'\right)^2\left(D-D'\right)$

So the given equation reduces to $\left(D-D'\right)\left(D+D'\right)^2 z=e^y\cos 2x$

$\therefore$ C.F. $=\phi_1\left(y+x\right)+\phi_2\left(y-x\right)+x\phi_3\left(y-x\right),\phi_1,\phi_2$ being arbitrary functions

P.I. $=\dfrac{1}{\left(D-D'\right)\left(D+D'\right)}\dfrac{1}{D+D'}e^y\cos 2x=\dfrac{1}{\left(D-D'\right)\left(D+D'\right)}\int e^{a+x}\cos 2x dx$, where $y-x=a$

$$=\frac{1}{\left(D-D'\right)\left(D+D'\right)}e^a\int e^x\cos 2x dx=\frac{1}{\left(D-D'\right)\left(D+D'\right)}e^{y-x}\frac{1}{1^2+2^2}e^x(\cos 2x+2\sin 2x)^*$$

$$=\frac{1}{5}\frac{1}{\left(D-D'\right)\left(D+D'\right)}e^y\left(\cos 2x+2\sin 2x\right)=\frac{1}{5}\frac{1}{D-D'}\int e^{x+a}\left(\cos 2x+2\sin 2x\right)dx,$$

where $y-x=a$

$$=\frac{1}{5}\frac{1}{D-D'}e^a\left\{\int e^x\cos 2x dx+2\int e^x\sin 2x dx\right\}$$

$$=\frac{1}{5}\frac{1}{D-D'}e^{y-x}\left\{\frac{e^x}{1^2+2^2}\left(\cos 2x+2\sin 2x\right)+\frac{2e^x}{1^2+2^2}\left(\sin 2x-2\cos 2x\right)\right\}$$

$$=\frac{1}{25}\frac{1}{D-D'}e^y\left(4\sin 2x-3\cos 2x\right)=\frac{1}{25}\int e^{b-x}\left(4\sin 2x-3\cos 2x\right)dx, \text{ where } b=y+x$$

$$=\frac{1}{25}e^b\left\{4\int e^{-x}\sin 2x dx-3\int e^{-x}\cos 2x dx\right\}$$

$$=\frac{1}{25}e^{y+x}\left\{\frac{4e^{-x}}{1^2+2^2}(-\sin 2x-2\cos 2x)-\frac{3e^{-x}}{1^2+2^2}(-\cos 2x+2\sin 2x)\right\}$$

$$=-(1/25)\times e^y\left(\cos 2x+2\sin 2x\right)$$

$\therefore$ Required solution is

$$z=\phi_1\left(y+x\right)+\phi_2\left(y-x\right)+x\phi_3\left(y-x\right)-\left(e^y/25\right)\times\left(\cos 2x+2\sin 2x\right)$$

Ex. 13. Find a surface passing through the two lines $z = x = 0$, $z - 1 = x - y = 0$ satisfying $r - 4s + 4t = 0$.

Sol. The given equation may be written as $\partial^2 z/\partial x^2 - 4(\partial^2 z/\partial x\partial y) + 4(\partial^2 z/\partial y^2) = 0$ or

$$(D^2 - 4DD' + 4D'^2)z = 0 \quad \text{or} \quad (D - 2D')^2 z = 0$$

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Its solution is $z = \varphi_1 (y + 2x) + x\varphi_2 (y + 2x)$. φ_1, φ_2 being arbitrary functions…(1)

Since (1) passes through $z = x = 0$, we have $0 = \varphi_1 (y)$ which gives $\varphi_1 (y + 2x) = 0$.

∴ (1) becomes

$$z = x\varphi_2 (y + 2x) \qquad \text{…(2)}$$

Since (2) passes through $z - 1 = x - y = 0$, *i.e.* $z = 1$ *and* $y = x$, we get

$$1 = x\varphi_2 (3x) \text{ or } \varphi_2 (3x) = 3/(3x) \text{ so that } \varphi_2 (y + 2x) = 3/(y + 2x)$$

∴ from (2), we have $3x = z(y + 2x)$, which is the required surface.

Ex. 14. Find the surface satisfying the equation $r + t - 2s = 0$ and the conditions that $bz = y^2$ when x = 0 and $az = x^2$ when $y = 0$.

Sol. Re – writing the given equation,

$$\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 - 2(\partial^2 z/\partial x\partial y) = 0$$

$$\text{or } (D - D')^2 z = 0 \quad \text{Or } (D^2 - 2DD' + D'^2) = 0$$

Its solution is $z = \varphi_1 (y + x) + x\varphi_2 (y + x)$. φ_1, φ_2 being arbitrary functions…(1)

Since $z = y^2/b$ when $x = 0$, (1) gives $y^2/b = \varphi_1 (y)$, $\Rightarrow \varphi_1 (y + x) = (y + x)^2/b$…(2)

Again since $z = x^2/a$ when $y = 0$, (1) gives $x^2\, a = x\varphi_2 (x) + \varphi_1 (x)$….(3)

Since from (2), $\varphi_2 (x) = x^2/b$, (3) becomes $\dfrac{x^2}{a} = x\varphi_2 (x) + \dfrac{x^2}{b}$ i.e. $\varphi_2 (x) = \dfrac{b-a}{ab}x$ which gives

$$\varphi_2 (y + x) = \frac{b-a}{ab}(y + x) \qquad \text{…(4)}$$

Using (2) and (4) in (1), the required surface is

1. $z = \dfrac{b-a}{ab}x(y+x) + \dfrac{(y+x)^2}{b} = \dfrac{y+x}{b}\left(\dfrac{b-a}{a}x + y + x\right)$ or $z = (y+x)\left(\dfrac{x}{a} + \dfrac{y}{b}\right)$

Ex.5. Find a surface satisfying the equation $D^2 z = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$.

Ans. $z = x^3 + y^3 + (x + y + 1)^2$

CATEGORY-2

Ex. 1. Solve $(a)\left(D^2 - D'\right)z = 2y - x^2$. (b) $\left(2D^2 - D'^2 + D\right)z = x^2 - y$.

Sol. (a) Here $D^2 - D'$ cannot be resolved into linear factors in D and D'. Hence to find C.F., we consider the equation $\left(D^2 - D'\right)z = 0$ ……(1)

Let a trial solution of (1) be $z = Ae^{hx+ky}$. ……(2)

So $D^2 z = Ah^2 e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) gives

$A\left(h^2-k\right)e^{hx+ky}=0$ or $h^2-k=0$ so that $k=h^2$

$\therefore$ C.F. $=\Sigma Ae^{hx+ky}=\Sigma Ae^{hx+h^2y}, A, h$ being arbitrary constants.

Now, P.I. $=\dfrac{1}{D^2-D'}\left(2y-x^2\right)=\dfrac{1}{D^2\left(1-D'/D^2\right)}\left(2y-x^2\right)=\dfrac{1}{D^2}\left(1-\dfrac{D'}{D^2}\right)^{-1}\left(2y-x^2\right)$

$$=\frac{1}{D^2}\left(1+\frac{D'}{D^2}+\ldots\right)\left(2y-x^2\right)=\frac{1}{D^2}\left\{\left(2y-x^2\right)+\frac{1}{D^2}D'\left(2y-x^2\right)\right\}$$

$$=\frac{1}{D^2}\left(2y-x^2+\frac{1}{D^2}2\right)$$

$$=\frac{1}{D^2}\left(2y-x^2+2\times\frac{x^2}{2}\right)=\frac{1}{D^2}(2y)=2y\times\frac{x^2}{2}=x^2y.$$

General solution is $z=\Sigma Ae^{hx+h^2y}+x^2y, A$ and h being arbitrary constants.

(b) Ans. $z=\Sigma Ae^{hx+ky}-(1/2)\times x^2y^2+(1/6)\times y^3-(1/12)\times xy^4-(1/6)\times y^4-(1/360)\times y^6$

where h and k are connected by the relation $2h^2-k^2+h=0$

Ex. 2. Solve $\left(3D^2-2D'^2+D-1\right)z=4e^{x+y}\cos(x+y)$.

Sol. Since $\left(3D^2-2D'^2+D-1\right)$ cannot be resolved into linear factors in D and D', hence C.F.

$=\Sigma Ae^{hx+ky}$, where A, h are arbitrary constants connected by $3h^2-2k^2+h-1=0$.

P.I. $=\dfrac{1}{3D^2-2D'^2+D-1}4e^{x+y}\cos(x+y)=4e^{x+y}\dfrac{1}{3(D+1)^2-2\left(D'+1\right)^2+\left(D+1\right)-1}\cos(x+y)$

$$=4e^{x+y}\frac{1}{3D^2+7D-2D'^2-4D'+1}\cos(x+y)$$

$$=4e^{x+y}\frac{1}{3\left(-1^2\right)+7D-2\left(-1^2\right)-4D'+1}\cos(x+y)$$

$$=4e^{x+y}\frac{1}{7D-4D'}\cos(x+y)=4e^{x+y}\left(7D+4D'\right)\frac{1}{49D^2-16D'^2}\cos(x+y).$$

$$= 4e^{x+y}\frac{7D+4D'}{49(-1^2)-16(-1^2)}\cos(x+y)$$

$$= -(4/33)\times e^{x+y}(7D+4D')\cos(x+y)$$

$$= -(4/33)e^{x+y}\times\left[7D\cos(x+y)+4D'\cos(x+y)\right]$$

$$= -(4/33)\times e^{x+y}\left[-7\sin(x+y)-4\sin(x+y)\right]=(4/3)\times e^{x+y}\sin(x+y).$$

Hence general solution is $z=\sum Ae^{hx+ky}+(4/3)\times e^{x+y}\sin(x+y)$.

Ex. 3. Solve $(D-3D'-2)^2 z = 2e^{2x}\sin(y+3x)$

Sol. Here C.F. $= e^{2x}\left[\phi_1(y+3x)+x\phi_2(y+3x)\right], \phi_1,\phi_2$ being arbitrary functions

P.I. $= \frac{1}{(D-3D'-2)^2}2e^{2x+0.y}\sin(y+3x) = 2e^{2x+0.y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\sin(y+3x)$

$$= 2e^{2x}\frac{1}{(D'-3D')^2}\sin(y+3x) = 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x),$$

$\therefore$ Required solution is $z=e^{2x}\left[\phi_1(y+3x)+x\phi_2(y+3x)\right]+x^2e^{2x}\sin(y+3x)$.

Ex. 4. Solve $\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}-2\frac{\partial z}{\partial y}=e^{x+y}$.

Sol. Given $\left(D^2-4DD'+4D'^2+D-2D'\right)z=e^{x+y}$ or $(D-2D')(D-2D'+1)z=e^{x+y}$.

$\therefore$ C.F. $=\phi_1(y+2x)+e^{-x}\phi_2(y+2x), \phi_1,\phi_2$ being arbitrary functions

P.I. $= \frac{1}{(D-2D'+1)}\left[\frac{1}{D-2D'}e^{x+y}\right] = \frac{1}{D-2D'-1}\frac{1}{1-2}e^{x+y}$,

$$= -e^{x+y}\frac{1}{(D+1)-2(D'+1)+1}.1,$$

$$= -e^{x+y}\frac{1}{D-2D'}1 = -e^{x+y}\frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1}1 = -e^{x+y}\frac{1}{D}1 = -xe^{x+y}$$

$\therefore$ The required solution is $z=\phi_1(y+2x)+e^{-x}\phi_2(y+2x)-xe^{x+y}$.

Ex. 5. Solve $(D-3D'-2)^2 z=2e^{2x}\tan(y+3x)$.

Sol. Here C.F. $=e^{2x}\{\phi_1(y+3x)+x\phi_2(y+3x)\}, \phi_1$ and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}2e^{2x}\tan(y+3x)=2\frac{1}{(D-3D'-2)^2}e^{2x+0\cdot y}\tan(y+3x)$$

$$2e^{2x+0\cdot y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\tan(y+3x)=2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$$=2e^{2x}\times\left(x^2/2!\right)\tan(y+3x),$$

$\therefore$ General solution is $z=e^{2x}\{\phi_1(y+3x)+x\phi_2(y+3x)\}+x^2e^{2x}\tan(y+3x)$

Ex. 6. Solve $r-3s+2t-p+2q=(2+4x)e^{-y}$

Sol. Re-writing the given equation $\left(D^2-3DD'+2D'^2-D+2D'\right)z=(2+4x)e^{-y}$

or $(D-2D')(D-D'-1)z=(2+4x)e^{-y}$

$\therefore$ C.F. $=\phi_1(y+2x)+e^x\phi_2(y+x)$, where ϕ_1,ϕ_2 are arbitrary functions.

P.I.

$$=\frac{1}{(D-2D')(D-D'-1)}2e^{0\cdot x-y}(1+2x)=2e^{0\cdot x-y}\frac{1}{\{D+0-2(D'-1)\}\{D+0-(D'-1)-1\}}(1+2x)$$

$$=2e^{-y}\frac{1}{(D-2D'+2)(D-D')}(1+2x)=2e^{-y}\frac{1}{2D}\left(1+\frac{D-2D'}{2}\right)^{-1}\left(1-\frac{D'}{D}\right)^{-1}(1+2x)$$

$$=e^{-y}\frac{1}{D}\left\{1-\frac{1}{2}(D-2D')+\ldots\right\}\left(1+\frac{D'}{D}+\ldots\right)(1+2x)$$

$$=e^{-y}\frac{1}{D}\left(1-\frac{D}{2}+\ldots.\right)(1+2x)$$

$$= e^{-y}(1/D)(1+2x-1) = x^2 e^{-y}$$

$\therefore$ The required solution is $z = \phi_1(y+2x) + e^x \phi_2(y+x) + x^2 e^{-y}$.

Ex. 7. Solve (a) $\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = e^{2x+3y} + xy + \sin(2x+y)$.

(b) $\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = e^{2x+3y} + xy$.

(c) $\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right)z = xy + \sin(2x+y)$.

Sol. (a) The given equation can be rewritten as

$$(D+D')(D-2D'+2)z = e^{2x+3y} + xy + \sin(2x+y) \qquad .....(1)$$

$\therefore$ C.F. $= \phi_1(y-x) + e^{-2x}\phi_2(y+2x), \phi_1, \phi_2$ being arbitrary functions.

P.I. corresponding to e^{2x+3y}

$$= \frac{1}{(D+D')(D-2D'+2)} e^{2x+3y} = \frac{1}{(2+3)(2-6+2)} e^{2x+3y} = -\frac{1}{10} e^{2x+3y}$$

P.I. corresponding to xy

$$= \frac{1}{(D+D')(D-2D'+2)} xy = \frac{1}{D(1+D'/D) \times 2\{1+(D/2-D')\}} xy$$

$$= \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left\{1+\left(\frac{D}{2}-D'\right)\right\}^{-1} xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\dots\right)\left\{1-\left(\frac{D}{2}-D'\right)+\left(\frac{D}{2}-D'\right)^2+\dots\right\} xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\dots\right)\left(1-\frac{D}{2}+D'-DD'+\dots\right) xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\dots\right)\left(xy-\frac{y}{2}+x-1\right)$$

$$=\frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{1}{D}\left(x-\frac{1}{2}\right)\right]=\frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{x^2}{2}+\frac{x}{2}\right]$$

$$=\frac{1}{2}\left[\frac{x^2y}{2}-\frac{xy}{2}+\frac{x^2}{2}-x-\frac{x^3}{6}+\frac{x^2}{4}\right]=\frac{x^2y}{4}+\frac{3x^2}{8}-\frac{xy}{4}-\frac{x}{2}-\frac{x^3}{12}.$$

P.I. corresponding to $\sin(2x+y)$

$$=\frac{1}{D^2-DD'-2D'^2+2D+2D'}\sin(2x+y)$$

$$=\frac{1}{-2^2+(2\times1)-2\times\left(-1^2\right)+2D+2D'}\sin(2x+y)$$

$$=\frac{1}{2(D+D')}\sin(2x+y)=\frac{1}{2}(D-D')\frac{1}{\left(D^2-D'^2\right)}\sin(2x+y)$$

$$=\frac{1}{2}\frac{1}{-2^2-\left(-1^2\right)}(D-D')\sin(2x+y)$$

$$=-(1/6)\times(D-D')\sin(2x+y)=-(1/6)\times\left[D\sin(2x+y)-D'\sin(2x+y)\right]$$

$$=-(1/6)\times\left[2\cos(2x+y)-\cos(2x+y)\right]=-(1/6)\times\cos(2x+y)$$

The required solution is $z=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)-(1/10)\times e^{2x+3y}+(1/4)\times x^2y$

$+(3/8)\times x^2-(1/4)\times xy-(x/2)-\left(x^3/12\right)-(1/6)\times\cos(2x+y)$

(b) As in part (a), $C.F.=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)$

P.I. corresponding to $e^{2x+3y}=-(1/10)\times e^{2x+3y}$

P.I. corresponding to $xy=(1/4)\times x^2y+(3/8)\times x^2-(1/4)\times xy-(1/2)\times x-(1/12)\times x^3$.

$\therefore$ The required general solution is $z=$ C.F. + P.I., i.e.

$$z=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)-(1/10)\times e^{2x+3y}+(1/4)\times x^2y+(3/8)\times x^2$$

$-(1/4)\times xy-(x/2)-\left(x^{3}/12\right)$

(c) As in part (a), C.F. $=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)$

P.I. corresponding to $xy=(1/4)\times x^2+(3/8)\times x^2-(1/4)\times xy-(x/2)-\left(x^3/12\right)$

and P.I. corresponding to $sin(2x+y)=-(1/6)\times cos(2x+y)$.

$\therefore$ The required solution is $z=$ C.F. + P.I., i.e., $z=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)$

$+(1/4)\times x^2+(3/8)\times x^2-(1/4)\times xy-(x/2)-\left(x^3/12\right)\times-(1/6)\times\cos(2x+y)$.

Ex. 8. Find a particular integral of the differential equation:

$\left(D^2-D'\right)z=e^{x+y}+5\cos(x+2y)$.

Sol. P.I. corresponding to e^{x+y}

$$=\frac{1}{D^2-D'}e^{x+y}=e^{x+y}\frac{1}{(D+1)^2-(D'+1)}1=e^{x+y}\frac{1}{D^2+2D-D'}1$$

$$=e^{x+y}\frac{1}{2D}\left[1+\left(\frac{D}{2}-\frac{D'}{2D}\right)\right]^{-1}1=e^{x+y}\frac{1}{2D}\{1+\ldots\}1=\frac{1}{2}xe^{x+y}$$

P.I. corresponding to $5\cos(x+2y)$

$$=5\frac{1}{D^2-D'}\cos(x+2y)=5\frac{1}{-1^2-D'}\cos(x+2y)=-\frac{5}{D'+1}\cos(x+2y)$$

$$=-5(D'-1)\frac{1}{D'^2-1}\cos(x+2y)=-5\frac{1}{-2^2-1}(D'-1)\cos(x+2y)$$

$$=(D'-1)\cos(x+2y)=D'\cos(x+2y)-\cos(x+2y)=-2\sin(x+2y)-\cos(x+2y)$$

$\therefore$ Required P.I. $=(x/2)\times e^{x+y}-2\sin(x+2y)-\cos(x+2y)$.

Ex. 9. Solve $\left(D^2-D'^2-3D+3D'\right)z=xy+e^{x+2y}$.

Sol. The given equation can be re-written as $(D-D')(D+D'-3)z=xy+e^{x+2y}$.

$\therefore$ C.F. $=\phi_1(y+x)+e^{3x}\phi_2(y-x), \phi_1, \phi_2$ being arbitrary functions.

P.I. corresponding to $xy=\dfrac{1}{(D-D')(D+D'-3)}xy=-\dfrac{1}{3D}\left(1-\dfrac{D'}{D}\right)^{-1}\left(1-\dfrac{D+D'}{3}\right)^{-1}xy$

$$=-\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left[1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\ldots\right]xy$$

$$=-\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left(1+\frac{D+D'}{3}+\frac{2DD'}{9}+\ldots\right)xy$$

$$=-\frac{1}{3D}\left(1+\frac{D}{3}+\frac{D'}{3}+\frac{D'}{D}+\frac{D'}{3}+\frac{2DD'}{9}+\ldots\right)xy$$

$$=-\frac{1}{3D}\left(xy+\frac{y}{3}+\frac{2x}{3}+\frac{1}{D}x+\frac{2}{9}\right)=-\frac{1}{9}\left(\frac{x^2y}{2}+\frac{xy}{2}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right)$$

P.I. corresponding to e^{x+2y}

$$=\frac{1}{(D+D'-3)}\frac{1}{D-D'}e^{x+2y}=\frac{1}{D+D'-3}\frac{1}{(1-2)}e^{x+2y}$$

$$=-\frac{1}{D+D'-3}e^{1\cdot x+2y}1=-e^{1\cdot x+2y}\frac{1}{(D+1)+(D'+2)-3}1=-e^{x+2y}\frac{1}{D+D'}1$$

$$=-e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}1=-e^{x+2y}\frac{1}{D}(1+\ldots)1=-xe^{x+2y}.$$

Hence the required general solution is $z=C.F+$ P.I., i.e

$$z=\phi_1(y+x)+e^{3x}\phi_2(y-x)-\left(x^2y/6\right)\ -(xy/6)-\left(x^2/9\right)-\left(x^3/18\right)-(2x/27)-xe^{x+2y}$$

Ex. 10. Solve $(D-D'-1)(D-D'-2)z=e^{2x-y}+x$.

Sol. Here C.F. $=e^{x}\phi_1(y+x)+e^{2x}\phi_2(y+x), \phi_1, \phi_2$ being arbitrary functions

Now, P.I. corresponding to e^{2x-y}

$$=\frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y}$$

$$=\frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}}e^{2x-y}=\frac{1}{2}e^{2x-y}$$

and P.I. corresponding to x

$$=\frac{1}{(D-D'-1)(D-D'-2)}x=\frac{1}{2\{1-(D-D')\}\{1-(D-D')/2\}}x$$

$$=\frac{1}{2}\left[1-(D-D')\right]^{-1}\left\{1-\frac{D-D'}{2}\right\}^{-1}x=\frac{1}{2}\left[1+(D-D')+\ldots\left\{1+\frac{D-D'}{2}+\ldots\right\}x\right.$$

$$=\frac{1}{2}\left\{1+(D-D')+\frac{D-D'}{2}+\ldots\right\}x=\frac{1}{2}\left\{1+\frac{3}{2}D+\ldots\right\}x=\frac{1}{2}\left(x+\frac{3}{2}\right).$$

$\therefore$ General solution is $z=e^{x}\phi_1(y+x)+e^{2x}\phi_2(y+x)+(1/2)\times e^{2x-y}+x/2+3/4$.

Ex. 11. Solve (a) $\left(D^2-DD'-2D\right)z=\sin(3x+4y)-e^{2x+y}$.

(b) $\left(D^2-DD'-2D\right)z=\sin(3x+4y)+x^2y$

Sol. (a) The given equation can be re-written as $D(D-D'-2)z=\sin(3x+4y)-e^{2x+y}$.

$\therefore$ C.F. $=\phi_1(y)+e^{2x}\phi_2(y+x),\phi_1,\phi_2$ being arbitrary functions.

P.I. corresponding to $\sin(3x+4y)$

$$=\frac{1}{D^2-DD'-2D}\sin(3x+4y)=\frac{1}{-3^2+(3\times 4)-2D}\sin(3x+4y)$$

$$=\frac{1}{3-2D}\sin(3x+4y)=(3+2D)\frac{1}{9-4D^2}\sin(3x+4y)=\frac{3+2D}{9-4\left(-3^2\right)}\sin(3x+4y)$$

$$=(1/45)\times\left[3\sin(3x+4y)+2D\sin(3x+4y)\right]$$

$$=(1/45)\times\left[3\sin(3x+4y)+6\cos(3x+4y)\right]$$

and P.I. corresponding to $\left(-e^{2x+y}\right)$

$$= -\frac{1}{D(D-D'-2)}e^{2x+y} = -\frac{1}{2(2-1-2)}e^{2x+y} = \frac{1}{2}e^{2x+y}$$

Hence the required general solution is $z = C.F. + P.I.,\ \ i.e.$

$$z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)] + (1/2)\times e^{2x+y}$$

(b) As in part (a), C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x), \phi_1, \phi_2$ being arbitrary functions.

P.I. corresponding to $\sin(3x+4y) = (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)]$.

P.I. corresponding to $x^2y = \frac{1}{D(D-D'-2)}x^2y = -\frac{1}{2D}\left\{1 - \left(\frac{D-D'}{2}\right)\right\}^{-1} x^2y$

$$= -\frac{1}{2D}\left\{1 + \frac{D-D'}{2} + \left(\frac{D-D'}{2}\right)^2 + \left(\frac{D-D'}{2}\right)^3 + \ldots\right\}x^2y$$

$$= -\frac{1}{2D}\left(1 + \frac{D}{2} - \frac{D'}{2} + \frac{D^2}{4} - \frac{DD'}{2} - \frac{3D^2D'}{8} + \ldots\right)x^2y$$

$$= -\frac{1}{2D}\left(x^2y + xy - \frac{x^2}{2} + \frac{y}{2} - x - \frac{3}{4}\right)$$

$$= -\frac{1}{2}\left(\frac{x^3y}{3} + \frac{x^2y}{2} - \frac{x^3}{6} + \frac{xy}{2} - \frac{x^2}{2} - \frac{3x}{4}\right)$$

Hence the solution is $z =$ C.F. + P.I., i.e.

$$z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)]$$

$$-(1/6)\times x^3y - (1/4)\times x^2y + (1/12)\times x^3 - (1/4)\times xy - \left(x^2/4\right) + 3x/8$$

Ex. 12. Solve $\left(\partial^2 z/\partial x^2\right) - \left(\partial^2 z/\partial y^2\right) + (\partial z/\partial x) + 3(\partial z/\partial y) - 2z = e^{x-y} - x^2y$.

Sol. The given equation can be re-written as $\left(D^2 - D'^2 + D + 3D' - 2\right)z = e^{x-y} - x^2y$

or $\{(D-D')(D+D') + 2(D+D') - (D-D'+2)\}z = e^{x-y} - x^2y$

or $\{(D+D')(D-D'+2)-(D-D'+2)\}z=e^{x-y}-x^2y$

or $(D-D'+2)(D+D'-1)z=e^{x-y}-x^2y$.

$\therefore$ C.F. $=e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x), \phi_1, \phi_2$ being arbitrary functions.

P.I. corresponding to e^{x-y}

$$=\frac{1}{(D-D'+2)(D+D'-1)}e^{x-y}=\frac{1}{\{1-(-1)+2\}(1-1-1)}e^{x-y}=-\frac{1}{4}e^{x-y}$$

and P.I. corresponding to $(-x^2y)$

$$=\frac{1}{(D-D'+2)(D+D'-1)}(-x^2y)=\frac{1}{2}\left\{1+\frac{D-D'}{2}\right\}^{-1}\{1-(D+D')\}^{-1}x^2y$$

$$=\frac{1}{2}\left\{1-\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2-\left(\frac{D-D'}{2}\right)^3+\ldots\right\}\times\{1+(D+D')+(D+D')^2+(D+D')^3+\ldots\}x^2y$$

$$=\frac{1}{2}\left(1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}+\frac{3D^2D'}{8}+\ldots\right)\times(1+D+D'+D^2+2DD'+3D^2D'+\ldots)x^2y$$

$$=(1/2)\times\left[1+(1/2)\times D+(3/2)\times D'+(3/4)\times D^2+(3/2)\times DD'+(21/8)\times D^2D'+\cdots\right]x^2y$$

$$=(1/2)\times\left[x^2y+xy+(3x^2/2)+(3y/2)+3x+21/4\right].$$

Hence general solution is $z=$ C.F. + P.I., i.e. $z=e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x)$

$$-(1/4)\times e^{x-y}+(1/2)\times x^2y+(1/2)\times xy+(3/4)\times x^2+(3/4)\times y+(3/2)\times x+21/8$$

Ex. 13. Solve $(D + D')(D + D' - 2)z = \sin(x + 2y)$

Sol. Here C.F. = $\varphi_1(y-x) + e^{2}x\,\varphi_2(y-x)$, φ_1, φ_2 being arbitrary function

P.I. = $\dfrac{1}{D+D'-2}\left\{\dfrac{1}{D+D'}\sin(x+2y)\right\} = \dfrac{1}{D+D'-2}\int \sin(3x+2d)\,dx$, where, $y - x = d$,

$$=\frac{1}{D+D'-2}\left\{-\frac{\cos(3x+2d)}{3}\right\} = -\frac{1}{3}\frac{1}{D+D'-2}\cos(2x+y)$$

$= -\frac{1}{3}\, e^{2x} \int e^{-2x} \cos(3x + 2d)\,dx$, where $y - x = d$

$= -\frac{1}{3} e^{2x} \frac{1}{(-2)^2 + 3^2}\, e^{-2x}\{-2\cos(3x + 2d) + 3\sin(3x + 2d)\}$

$= \frac{2}{39}\cos(x + 2y) - \frac{1}{13}\sin(x + 2y)$

$\therefore$ Solution is $z = \varphi_1(y - x) + e^{2x}\varphi_2(y - x) + (2/39)\times\cos(x + 2y) - (1/13)\times\sin(x + 2y)$

Ex. 14. Solve $(D^3 - DD'^2 - D^2 + DD')z = (x + 2)/x^3$

Sol. Re-writing, the given equation

$$D(D - D')(D + D' - 1)z = (x + 2)/x^3$$

Its C.F. $= \varphi_1(y) + \varphi_2(y + x) + e^x\varphi_3(y - x)$, $\varphi_1, \varphi_2, \varphi_3$ being arbitrary functions

P.I. $= \frac{1}{(D - D')(D + D' - 1)}\frac{1}{D}\left(\frac{1}{x^2} + \frac{2}{x^3}\right) = \frac{1}{(D + D' - 1)(D - D)}\left(-\frac{1}{x} - \frac{1}{x^2}\right)$

$= \frac{1}{D + D' - 1}\int\left(-\frac{1}{x} - \frac{1}{x^2}\right)dx = \frac{1}{D + D' - 1}\left(-\log x + \frac{1}{x}\right) = e^x\int e^{-x}\left(-\log x + \frac{1}{x}\right)dx$

1. $= -e^x\int e^{-x}\log x\,dx + e^x\int e^{-x}\frac{1}{x}dx = -e^x\left[(-e^{-x})\log x - \int(-e^{-x})\frac{1}{x}dx\right] + e^x\int e^{-x}\frac{1}{x} = \log x$

[on integration by parts first integral only]

$\therefore$ General solution is $z = \varphi_1(y) + \varphi_2(y + x) + ex\,\varphi_3(y - x) + \log x$.

Ex. 15. Solve $(D^2 + DD' + D' - 1)z = 4\sinh x$.

Sol. Re-writing, the given equation

$$(D + 1)(D + D' - 1)z = 2(e^x - e^{-x})$$

C.F. $= e^{-x}\varphi_1(y) + e^x\varphi_2(y - x)$, where φ_1, φ_2 are arbitrary functions

P.I. $= \frac{1}{(D + 1)(D + D' - 1)}2(e^x - e^{-x}) = \frac{1}{(D + 1)}2e^x\int e^{-x}e^x - e^{-x}\,dx$

$= \frac{1}{(D + 1)}2e^x\left(x + \frac{1}{2}e^{-2x}\right) = \frac{1}{D + 1}(2xe^x + e^{-x} = e^{-x}\int e^x(2xe^x + e^{-x})dx$

$= 2e^{-x}\int xe^{2x}\,dx + e^{-x}x = 2e^{-x}[x\times(e^{2x}/2) - \int 1\cdot(e^{2x}/2)dx] + xe(-x)$

$= xe^x - e^{-x}\int e^{2x}\,dx + xe^{-x} = xe^x - e^{-x}\times(1/2)\times e^{2x} + xe^{-x} = (x - 1/2)e^x + xe^{-x}$

General solution is $z = e^{-x}\varphi_1(y) + e^x\varphi_2(y - x) + (x - 1/2)e^x + xe^{-x}$

Ex. 16. Solve $(D^2 - DD' + D' - 1)z = 1 + xy + ey + \cos(x + 2y)$

Sol. Re-writing the given equation $(D - 1)(D - D' + 1)z = 1 + xy + ey + \cos(x + 2y)$

C.F. $= e^x\varphi_1(y) + e^{-x}\varphi_2(y + x)$, φ_1, φ_2 being arbitrary functions.

P.I. $= \dfrac{1}{(D-1)(D-D'+1)}\{1 + xy + e^y + \cos(x + 2y)\}$

$= \dfrac{1}{D-1} e^{-x} \int e^x\{1 + x(d - x) + e^{d-x} + \cos(2d - x)\}dx$, where $d = y + x$

$= \dfrac{1}{D-1} e^{-x} \{\int(1 + dx - x^2) e^x\, dx + \int e^a\, dx + \int e^x \cos(x - 2d)dx\}$

$= \dfrac{1}{D-1} e^{-x}\left[\left(1 + dx - x^2\right)e^x - (d-2x)e^x + (-2)e^x + e^d x + \dfrac{e^x}{1^2+1^2}\left\{\cos(x-2d) + \sin(x-2d)\right\}\right]$

$= \dfrac{1}{D-1}$ $[(1 + dx - x^2 - d + 2x - 2) + e^{d-x} x + (1/2) \times \{\cos(x - 2d) + \sin(x - 2d)\}]$

$= \dfrac{1}{D-1}$ $[-1 + x(y + x) - x^2 - (y + x) + 2x + e^y x + (1/2) \times \{\cos(-2y - x) + \sin(-2y - x)\}]$

$= \dfrac{1}{D-1}\{xy - y + x - 1 + xe^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin(2y + x)\}$

$= \dfrac{1}{D-1}\{(x - 1)(y + 1) + xe^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin(2y + x)\}$

$= e^x \int e^{-x} \{(x - 1)(k + 1) + xe^k + (1/2) \times \cos(2k + x) - (1/2) \times \sin(2k + x)\}$, where $y = k$

$= e^x\left[(k+1)\int e^{-x}(x-1)dx + e^k \int xe^{-x}dx + \dfrac{1}{2}\int e^{-x}\cos(2k + x)\right]dx$

$- \dfrac{1}{2} \int e^{-x} \sin(2k + x)dx]$

$= e^x (k + 1)\{(-e^{-x})(x + 1) - (e^{-x})(1)\} + e^x e^k \{(-e^{-x})(x) - (e^{-x})(1)\}]$

$+ \dfrac{e^x}{2}\dfrac{e^{-x}}{(-1)^2+1^2}\{-\cos(2k + x) + \sin(2k + x)\} - \dfrac{e^x}{2}\dfrac{e^{-x}}{(-1)^2+1^2}\{-\sin(2k + x) - \cos(2k + x)]$

$= -(k + 1)x - e^k (x + 1) + (1/2) \times \sin(2k + x)] = -(y + 1)x - e^y (x + 1) + (1/2)\sin(2y + x)$

$\therefore$ solution is $z = e^x \varphi_1(y) + e^{-x} \varphi_2(y + x) - x(y + 1) - (x + 1)e^y + (1/2) \times \sin(2y + x))$

Ex 17. Solve $(D^2 - DD' - 2D'2 + 2D + 2D')z = xy + \sin(2x + y)$

Ans. $z = \varphi_1(y - x) + e^{-2x} \varphi_2(y + 2x) + (x/24) \times (6xy - 6y + 9x - 2x^2 - 12) - (1/6) \times \cos(2x + y)$.

Ex. 18. Find a surface satisfying $r + s = 0$, i.e., $(D^2 + DD')z = 0$ and touching the elliptic paraboloid $z = 4x^2 + y^2$ along its section by the plane $y = 2x + 1$.

Sol. Given $(D^2 + DD')z = 0$ or $D(D + D')=0$

$\therefore$ Solution of (1) is $z =$ C.F. $= \varphi_1(y) + \varphi_2(y - x)$ ...(1)

where φ_1 and φ_2 are arbitrary functions.

Since (2) touches the curve given by $z = 4x^2 + y^2$

and $y = 2x + 1$ ...(3)

values of $p(=\partial z/\partial x)$ and $q(=\partial z/\partial y)$ obtained from (2) and (3) must be equal for any point on (4).

$\therefore -\varphi_2'(y-x) = 8x$ for $y = 2x+1$ or $\varphi_2'(x+1) = -8x$.

and $\varphi_1'(y) + \varphi_2'(y-x) = 2y$ for $y = 2x+1$ or $\varphi_1'(2x+1) + \varphi_2'(x+1) = 4x+2$

From (5), $\varphi_2'(x) = 8 - 8x$ ...(5)

Integrating it, $\varphi_2(x) = 8x - 4x^2 + c_1, c_1$ being an arbitrary constant

Subtracting (5) from (6), $\varphi_1'(2x+1) = 12x + 2 = 6(2x+1) - 4$

so that $\varphi_1'(x) = 6x - 4$ ...(7)

Integrating it, $\varphi_1(x) = 3x^2 - 4x + c_2$, c_2 being an arbitrary constant

From (8),

$\varphi_1(y) = 3y^2 - 4y + c_2$ ...(8)

and from (7),

$\varphi_2(y-x) = 8(y-x) - 4(y-x)^2 + c_1$

Putting the above values of $\varphi_1(y)$ and $\varphi_2(y-x)$ in (2), we get

$z = 3y^2 - 4y + c_2 + 8(y-x) - 4(y-x)^2 + c_1$ ...(9)

$z = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3$, where $c_3 = c_1 + c_2$

Equating the values of z from (3) and (9), we get

$4x^2 + y^2 = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3$, where $y = 2x+1$.

$\therefore c_3 = 8x^2 + 2y^2 - 4y + 8x - 8xy = 8x^2 + 2(2x+1)^2 - 4(2x+1) + 8x - 8x(2x+1) = -2$

Hence, from (9), the required surface is $4x^2 - 8xy + y^2 - 4y + z + 2 = 0$

CATEGORY-3

Ex.1. Solve $x^2\left(\partial^2 z/\partial x^2\right) + 2xy\left(\partial^2 z/\partial x\partial y\right) - x\left(\partial z/\partial x\right) = x^3/y^2$

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. .....(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$ $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$

Then the given equation $\left(x^2D^2 + 2xyDD' - xD\right)z = x^3y^{-2}$ becomes

$$\left[D_1(D_1-1) + 2D_1D_1' - D_1\right]z = \left(e^u\right)^3\left(e^v\right)^{-2}$$

or $\left(D_1^2 + 2D_1D_1' - 2D_1\right)z = e^{3u-2v}$ or $D_1\left(D_1 + 2D_1' - 2\right)z = e^{3u-2v}$

$\therefore$ C.F. = $\phi_1(v) + e^{2u}\phi_2(v-2u) = \phi_1(v) + \left(e^u\right)^2\phi_2(v-2u) = \phi_1(\log y) + x^2\phi_2(\log - 2\log x)$,

using (1)

$= \phi_1(\log y) + x^2\phi_2\left(\log\left(y/x^2\right)\right) = f_1(y) + x^2 f_2\left(y/x^2\right)$, where f_1 and f_2 are arbitrary functions

Now, P.I. = $\dfrac{1}{D_1\left(D_1+2D_1'-2\right)}e^{3u-2v} = \dfrac{1}{3(3-4-2)}\left(e^u\right)^3\left(e^v\right)^{-2} = \dfrac{x^3y^{-2}}{9} = -\dfrac{x^3}{9y^2}$

Hence the required general solution is z = C.F. + P.I. or $z = f_1(y) + x^2f_2(y/x^2) - (x^3/9y^2)$.

Ex. 2. Solve $x^2r - 3xys + 2y^2t + px + 2qy = x + 2y$.

Sol. The given equation can be re-written as

$$x^2\left(\partial^2 z/\partial x^2\right) - 3xy\left(\partial^2 z/\partial x\partial y\right) + 2y^2\left(\partial^2 z/\partial y^2\right) + x(\partial z/\partial y) + 2y(\partial z/\partial y) = x + 2y$$

or $\left(x^2D^2 - 3xyDD' + 2y^2D'^2 + xD + 2yD'\right)z = x + 2y.$ .....(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$

$\therefore$ (1) becomes $\left[D_1(D_1-1) - 3D_1D_1' + 2D_1'\left(D_1'-1\right) + D_1 + 2D_1'\right]z = e^u + 2e^v$

or $\left(D_1^2 - 3D_1D_1' + 2D_1'^2\right)z = e^u + 2e^v$ or $\left(D_1 - D_1'\right)\left(D_1 - 2D_1'\right)z = e^u + 2e^v$.

$\therefore$ C.F. = $\phi_1(v+u) + \phi_2(v+2u) = \phi_1(\log y + \log x) + \phi_2(\log y + 2\log x)$

or C.F. = $\phi_1\log(xy) + \phi_2\log\left(x^2y\right) = f_1(xy) + f_2\left(x^2y\right)$, where f_1 and f_2 are arbitrary functions.

$$\frac{1}{\left(D_1-D_1'\right)\left(D_1-2D_1'\right)}\left(e^v + 2e^v\right) = \frac{1}{\left(D_1-D_1'\right)\left(D_1-2D_1'\right)}e^{1-u+0.v} + 2\frac{1}{\left(D_1-D_1'\right)\left(D_1-2D_1'\right)}e^{0.4+1.v}$$

$= \dfrac{1}{(1-0)(1-0)}e^u + 2\dfrac{1}{(0-1)(0-2)}e^v = x + y$, using (2)

Hence the required general solution is z = C.F. + P.I. or $z = f_1(xy) + f_2\left(x^2y\right) + x + y$

Ex.3. Find the general solution of $x^2\left(\partial^2 z/\partial x^2\right) + 2xy\left(\partial^2 z/\partial x\partial y\right) + y^2\left(\partial^2 z/\partial y^2\right) = nz$

$$= n\{x(\partial z/\partial x) + y(\partial z/\partial y)\} + x^2 + y^2 + x^3$$

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$

Then, the given equation reduces to

$[x^2D^2 + 2xyDD' + y^2D'^2 - n(xD + yD') + n]z = x^2 + y^2 + x^3$

or $[x^2D^2 + 2xyDD' + y^2D'^2 - n(xD + yD\%) + n]z = x2 + y2 + x3$

or $$\left\{(D_1 + D_1')^2 - (D_1 + D_1') - n(D_1 + D_1' - 1)\right\}z = e^{2u} + e^{2v} + e^{3u}$$

or $$\left\{(D_1 + D_1')(D_1 + D_1' - 1) - n(D_1 + D_1' - 1)\right\}z = e^{2u} + e^{2v} + e^{3u}$$

or $$(D_1 + D_1' - 1)(D_1 + D_1' - n)z = e^{2u} + e^{2v} + e^{3u}$$

$\therefore$ C.F. = $e^u\phi_1(v-u) + e^{nu}\phi_2(v-u) = e^u\phi_1(v-u) + (e^u)^n\phi_2(v-u)$

$x = \phi_1(\log y - \log x) + x^n\phi_2(\log y - \log x) = x\phi_1\log(y/x) + x^n\phi_2\log(y/x)$, using (1)

$= xf_1(y/x) + x^nf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Also, P.I. = $$\frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}\left(e^{2u} + e^{2v} + e^{3u}\right) = \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}e^{2u+0.v}$$

$$+\frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}e^{0.u+2v} + \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}e^{3u+0.v}$$

$$= \frac{(e^u)^2}{(2+0-1)(2+0-n)} + \frac{(e^v)^2}{(0+2-1)(0+2-n)} + \frac{(e^u)^3}{(3+0-1)(3+0-n)} = \frac{x^2+y^2}{2-n} + \frac{x^3}{2(3-n)}$$

Hence general solution is $z = xf_1(y/x) + x^nf_2(y/x) + (x^2+y^2)/(2-n) + x^3/2(3-n)$

Ex.4. Solve $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) = xy$ or $(x^2D^2 - y^2D'^2)z = xy$

Sol. Let $x = eu$, $y = ev$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$

Then the given equation $\left(x^2D^2 - y^2D'^2\right)z = xy$ becomes

$$\left[D_1(D_1-1) - D_1'\left(D_1'-1\right)\right]z = e^u e^v \quad \text{or} \quad \left(D_1^2 - D_1'^2 - D_1 + D_1'\right)z = e^{u+v}$$

or $\left[\left(D_1 - D_1'\right)\left(D_1 + D_1'\right) - \left(D_1 - D_1'\right)\right]z = e^{u+v}$ or $\left(D_1 - D_1'\right)\left(D_1 + D_1' - 1\right)z = e^{u+v}$

or C.F. = $\phi_1 \log(xy) + x\phi_2 \log(y/x) = f_1(xy) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Also, P.I. = $\dfrac{1}{\left(D_1 - D_1'\right)\left(D_1 + D_1' - 1\right)} e^{u+v} = \dfrac{1}{D_1 - D_1'} \dfrac{1}{(1+1-1)} e^{u+v} = \dfrac{u}{1!} e^{u+v} = ue^u e^v = xy \log x$

Hence the required general solution is z = C.F. + P.I. or $z = f_1(xy) + xf_2(y/x) + xy\log x$

Ex. 5. Solve $yt - q = xy$.

Sol. The given equation can be rewritten as

$$y\left(\partial^2 z/\partial y^2\right) - (\partial z/\partial y) = xy$$

$$y^2\left(\partial^2 z/\partial y^2\right) - y\left(\partial z/\partial y\right) = xy^2$$

or

Let $x = e^u$, $y = e^v$ so that $u = logx,\ v = logy$.

Also, let $D \equiv \partial/\partial x,\ D' \equiv \partial/\partial y,\ D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then (1) becomes $\left[D_1'(D_1'-1) - D_1'\right]z = e^u e^{2v}$

or $D_1'\left(D_1' - 2\right)z = e^{u+2v}$

$\therefore$ C.F. = $\phi_1(u) + e^{2v}\phi_2(u) = \phi_1(logx) + y^2\phi_2(logx),\ by(2)$

$= f_1(x) + y^2 f_2(x),\ f_1.$ and f_2 being arbitrary function

Also, P.I. $= \dfrac{1}{\left(D_1' - 2\right)D_1'} e^{u+2v} = \dfrac{1}{D_1 - 2} \dfrac{1}{2} e^{u+2v} = \dfrac{1}{2} e^{u+2v} \dfrac{1}{D' + 2 + 2} \cdot 1$

$$=\frac{1}{2}e^{u+2v}\frac{1}{D'}1=\frac{1}{2}e^{u}\times\left(e^{v}\right)^{2}\times v=\frac{xy^{2}}{2}logy.$$

Hence the required general solution is $z=$ C.F. + P.I. or

$$z=f_1(x)+y^2 f_2(x)+\left(\frac{1}{2}\right)\times xy^2\log y\ .$$

Ex. 6. Solve $\left(x^2D^2-xyDD'-2y^2D'^2+xD-2yD'\right)z=log\left(y/x\right)-(1/2).$

Sol. Let $x=e^u$, $y=e^v$ so that $u=logx$,

$v=logy...(1)$ Also, let $D\equiv\partial/\partial x,\ D'\equiv\partial/\partial y,\ D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$.

Then the given equation reduces to

$$\left[D_1(D_1-1)-D_1D_1'-2D_1'\left(D_1'-1\right)+D_1-2D_1'\right]z=logy-logx-(1/2)$$

or $\left(D_1^2-D_1D_1'-2D_1'^2\right)z=v-u-(1/2)$ *or* $\left(D_1-2D_1'\right)\left(D_1+D_1'\right)=v-u-(1/2).$

$\therefore$ C.F. $=\phi_1(v+2u)+\phi_2(v-u)=\phi_1(logy+2logx)+\phi_2(logy-logx)$

C.F. $=\phi_1\left(log\left(yx^2\right)\right)+\phi_2\left(log\left(y/x\right)\right)=f_1\left(yx^2\right)+f_2\left(y/x\right),$

where f_1 *and* f_2 *are arbitrary functions.*

or $\quad P.I.=\dfrac{1}{D_1^2-D_1D_1'-2D_1'^2}\left(v-u-\dfrac{1}{2}\right)$

$$=\frac{1}{D_1^2\left(1-D_1'/D_1-2D_1'^2/D_1^2\right)}\left(v-u-\frac{1}{2}\right)$$

$$=\frac{1}{D_1^2}\left\{1-\left(\frac{D_1'}{D_1}+\frac{2D_1'^2}{D_1^2}\right)\right\}^{-1}\left(v-u-\frac{1}{2}\right)=\frac{1}{D_1^2}\left(1+\frac{D_1'}{D_1}+\ldots\right)\left(v-u-\frac{1}{2}\right)$$

$$=\frac{1}{D_1^2}\left\{v-u-\frac{1}{2}+\frac{1}{D_1}D_1'\left(v-u-\frac{1}{2}\right)\right\}=\frac{1}{D_1^2}\left(v-u-\frac{1}{2}+\frac{1}{D_1}\cdot 1\right)=\frac{1}{D_1^2}\left(v-u-\frac{1}{2}+u\right)$$

$$= \frac{1}{D_1^2}\left(v - \frac{1}{2}\right) = \left(v - \frac{1}{2}\right)\frac{u^2}{2} = \frac{1}{2}u^2 v - \frac{1}{4}u^2 = \frac{(logx)^2 logy}{2} - \frac{(logx)^2}{4}, \text{ by (1)}$$

$\therefore$ Required solution is

$$z = f_1\left(yx^2\right) + f_2\left(y/x\right) + (1/2)\times(logx)^2 logy - (1/4)\times(logx)^2.$$

Ex. 7. Solve $\left(x^2D^2 - 4y^2D'^2 - 4yD' - 1\right)z = x^2y^2\log y$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$.

Also, let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$\left[D_1(D_1 - 1) - 4D_1'(D_1' - 1) - 4D_1' - 1\right]z = e^{2u}e^{2v}v$$

or $\left(D_1^2 - D_1 - 4D_1'^2 - 1\right)z = e^{2u+2v}v \# (2)$

Here $\left(D^2{}_1 - D_1 - 4D_1'^2 - 1\right)$ cannot be resolved into linear factors in D_1 and D_1'. To find C.F. corresponding to it, we consider the equation.

$$\left(D_1^2 - D_1 - 4D_1'^2 - 1\right)z = 0 \qquad .....(3)$$

$$z = Ae^{hu+kv} \qquad .....(4)$$

Let a trial solution of (3) be

$$D_1 z = Ahe^{hu+kv}, \text{ and } D_1'^2 z = Ak^2 e^{hu+kv}$$

Then, $(3) \Rightarrow A\left(h^2 - h - 4k^2 - 1\right)e^{hu+hv} = 0 \Rightarrow h^2 - h - 4k^2 - 1 = 0.$ . ...(5)

$$\therefore \text{ C.F. of } (2) = \Sigma Ae^{hu+kv} = \Sigma A\left(e^u\right)^h\left(e^v\right)^k = \Sigma Ax^h y^k$$

P.I. of $(2) = \dfrac{1}{D_1^2 - D_1 - 4D_1'^2 - 1}e^{2u+2v}v = e^{2u+2v}\dfrac{1}{(D_1+2)^2 - (D_1+2) - 4\left(D_1'+2\right)^2 - 1}v$

$$= e^{2u+2v}\frac{1}{D_1^2 + 3D_1 - 4D_1'^2 - 16D_1' - 15}v$$

$$= e^{2u+2v} \frac{1}{(-15)\times\left[1+(16/15)\times D_1^{'}+(4/15)\times D_1^{'2}-(1/5)\times D_1-(1/15)\times D_1^2\right]} v$$

$$= e^{2u+2v} \frac{1}{(-15)}\left[1+\left\{\frac{16}{15}D_1^{'}+\frac{4}{15}D_1^{'2}-\frac{1}{5}D_1-\frac{1}{15}D_1^2\right\}\right]^{-1} v$$

$$= \frac{e^{2u+2v}}{(-15)}\left(1-\frac{16}{15}D_1^{'}+\dots\right)v = \frac{e^{2u+2v}}{(-15)}\left(v-\frac{16}{15}\right) = \frac{(e^u)^2\times(e^v)^2(16-15v)}{225}$$

$= (1/225)\times x^2y^2(16-15\log y)$, using (1).

The required general solution is $z = \Sigma Ax^h y^k + (1/225)\times x^2 y^2 (16-15\log y)$

where $h^2-h-4k^2-1=0$, and A, h and k are arbitrary constants.

Ex.8. Solve $\dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial x^2}-\dfrac{1}{x^3}\dfrac{\partial z}{\partial x}=\dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial y^2}-\dfrac{1}{y^3}\dfrac{\partial z}{\partial y}$.

Sol. Let $x^2/2=u,\ y^2/2=v$ so that $dx/du=1/x,\ dy/dv=1/y$ ....(1)

Now, $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{dx}{du}=\frac{1}{x}\frac{\partial z}{\partial x}, \text{ using } (1) \qquad ...(2)$$

and $$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\frac{dx}{du}$$

$$=\left(\frac{1}{x}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^2}\frac{\partial z}{\partial x}\right)\frac{1}{x}, \text{using } (1)$$

$$\therefore \frac{\partial^2 z}{\partial u^2}=\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x} \qquad .....(3)$$

Similarly, $$\frac{\partial^2 z}{\partial v^2}=\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y} \qquad .....(4)$$

Using (3) and (4), the given equation reduces to

$$\partial^2 z/\partial u^2=\partial^2 z/\partial v^2 \quad \text{or} \quad \left(D_1^2-D_1^{'2}\right)z=0$$

or $$\left(D_1-D_1^{'}\right)\left(D_1+D_1^{'}\right)z=0 \qquad .....(5)$$

where $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$. Hence solution of (5) is

$$z=\phi_1(v+u)+\phi_2(v-u)=\phi_1\left\{(1/2)\times\left(x^2+y^2\right)\right\}+\phi_2\left\{(1/2)\times\left(y^2-x^2\right)\right\}$$

or $z=f_1\left(y^2+x^2\right)+f_2\left(y^2-x^2\right)$, f_1, f_2 being arbitrary functions.

Ex. 9. Find a surface satisfying equation $2x^2r-5xys+2y^2t+2(px+qy)=0$ and touching the hyperbolic paraboloid $z=x^2-y^2$ along its section by the plane $y=1$.

Sol. Re-writing given equation, $2x^2\dfrac{\partial^2 z}{\partial x^2}-5xy\dfrac{\partial^2 z}{\partial x\partial y}+2y^2\dfrac{\partial^2 z}{\partial y^2}+2\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=0$.

Or $\left\{2x^2D^2-5xyDD'+2y^2D'^2+2(xD+yD')\right\}z=0$ ....(1)

Put $x=e^u$, $y=e^v$ so that $u=\log x$ and $v=\log y$.

If $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$, then (1) reduces to

$$\left[2D_1(D_1-1)-5D_1D_1'+2D_1'(D_1'-1)+2(D_1+D_1')\right]z=0$$

or $\left(2D_1^2-5D_1D_1'+2D_1'^2\right)=0$ or $(2D_1-D_1')(D_1-2D_1')=0$.

$\therefore$ solution is $z=$ C.F. $=\phi_1(2v+u)+\phi_2(u+2v)$, ϕ_1,ϕ_2 being arbitrary function

The given surface is $z=x^2-y^2$.

Now (2) and (3) are to touch each other along the section by the plane

$y=1\#(4)$

Therefore the values of p and q for (2) and (3) must be equal at $y=1$. Equating values of p and q from (2) and (3), we get

$$y^2f_1'\left(y^2x\right)+2xyf_2'\left(x^2y\right)=2x \quad .....(5)$$

and $$2xyf_1'\left(y^2x\right)+x^2f_2'\left(x^2y\right)=-2y \quad ....(6)$$

Putting $y=1$, (5) and (6) reduce to

$f_1'(x)+2xf_2'\left(x^2\right)=2x$ and $2xf_1'(x)+x^2f_2'\left(x^2\right)=-2$.

Solving these, $f_1'(x)=-(2/3)\times x-(4/3)\times x^{-1}$ .....(7)

and $f_2'\left(x^2\right)=(2/3)\times x^{-2}+(4/3)$ .....(8)

Integrating (7), $f_1(x) = -(1/3)\times x^2 - (4/3)\times log x + c_1$

Which gives $f_1(y^2x) = -(1/3)\times y^4x^2 - (4/3)\times log(y^2x) + c_1$ ....(9)

Writing X for x^2 in (8). $f_2'(X) = (2/3)\times(1/X) + (4/3)$

Integrating it, $f_2(X) = (2/3)\times \log X + (4/3)\times X + c_2$

Putting the values of $f_1(y^2x)$ and $f_2(yx^2)$ from (9) and (10) in (2) and writing $c_1 + c_2 = c/3$, the complete solution is

$$z = -(1/3)\times y^4x^2 - (4/3)\times \log(y^2x) + (2/3)\times \log(yx^2) + (4/3)\times(yx^2) + c/3$$

or $\quad 3z = -y^4x^2 - 4(\log x + 2\log y) + 2(\log y + 2\log x) + 4yx^2 + c$

or $\quad 3z = -y^4x^2 - 6\log y + 4yx^2 + c$

Now equating values of z from (3) and (11) and putting $y = 1$, we have

$$x^2 - 1 = (1/3)\left[-x^2 - 6\log 1 + 4x^2 + c\right], \text{ giving } c = -3.$$

So the required surface is $3z = 4yx^2 - y^4x^2 - 6\log y - 3$

PREVIOUS YEARS QUESTIONS

Q1. Find the general solution of the partial differential equation

$(D^2 + DD' - 6D'^2)z = x^2\sin(x+y)$ where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$. **[7a UPSC CSE 2022]**

Q2. Solve the partial differential equation: $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{2x+y} + \sin(x-2y)$;

$D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$. **[5d UPSC CSE 2020]**

Q3. Find the solution of the following differential equation:

$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} = ye^x$. **[(7c) 2020 IFoS]**

Q4. Find the solution of the equation: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$. **[(5a) 2019 IFoS]**

Q5. Solve the partial differential equation:

$\left(2D^2 - 5DD' + 2D'^2\right)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$ where $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$.

[7a UPSC CSE 2018]

Q6. Find a real function V of *x* and *y*, satisfying $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi\left(x^2 + y^2\right)$ and reducing to zero, when $y = 0$. **[(8a) 2018 IFoS]**

Q7. Solve $\left(D^2 - 2DD' + D'^2\right)z = e^{x+2y} + x^3 + \sin 2x$ where $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}, D^2 \equiv \frac{\partial^2}{\partial x^2}, D'^2 \equiv \frac{\partial^2}{\partial y^2}$. **[5a UPSC CSE 2017]**

Q8. Solve the partial differential equation

$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$. **[7a UPSC CSE 2016]**

Q9. Find the particular integral of $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 2x\cos y$. **[(6b) 2016 IFoS]**

Q10. Solve $\left(D^2 + DD' - 2D'^2\right)u = e^{x+y}$, where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$. **[5b UPSC CSE 2015]**

Q11. Find the solution of the equation $u_{xx} - 3u_{xy} + u_{yy} = \sin(x - 2y)$. **[(5d) 2015 IFoS]**

Q12. Solve the partial differential equation $\left(2D^2 - 5DD' + 2D'^2\right)z = 24(y-x)$.

[5a UPSC CSE 2014]

Q13. Solve $\left(D^2 + DD' - 6D'^2\right)z = x^2 \sin(x+y)$ where D and D' denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

[6a UPSC CSE 2013]

Q14. Solve the partial differential equitation $(D - 2D')(D - D')^2 z = e^{x+y}$. **[5a UPSC CSE 2012]**

Q15. Solve $\left(D^3 D'^2 + D^2 D'^3\right)z = 0$ where D stands for $\frac{\partial}{\partial x}$ and D' stands for $\frac{\partial}{\partial y}$. **[(5b) 2012 IFoS]**

Q16. Find the complementary function and particular integral of the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$.

[(7b) 2011 IFoS]

Q17. Solve the PDE $\left(D^2 - D'\right)\left(D - 2D'\right)Z = e^{2x+y} + xy$. **[5a UPSC CSE 2010]**

Q18. Find the surface satisfying the PDE $\left(D^2 - 2DD' + D'^2\right)Z = 0$ and the conditions that $bZ = y^2$ when $x = 0$ and $aZ = x^2$ when $y = 0$. **[5b UPSC CSE 2010]**

NON-HOMOGENEOUS

Q19. Find the general solution of the partial differential equation

$\left(D^2 - D'^2 - 3D + 3D'\right)z = xy + e^{x+2y}$ where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$. **[7a UPSC CSE 2021]**

Q20. Solve: $\left(D - 3D' - 2\right)^2 z = 2e^{2x}\cot\left(y + 3x\right)$. **[(7a) 2014 IFoS]**

Q21. Solve the PDE $\left(D^2 - D'^2 + D + 3D' - 2\right)z = e^{(x+y)} - x^2 y$. **[5a UPSC CSE 2011]**

Q22. Find the general solution of $\left(D - D' - 1\right)\left(D - D' - 2\right)z = e^{2x-y} + \sin\left(3x + 2y\right)$.

[(7a) 2010 IFoS]

Partial Differential Equations of order Two with variable coefficients

Ex 1. Solve the following partial differential equations:

(i) $r = 6x$ (ii) $ar = xy$ (iii) $r = \sin(xy)$

Solution.

(i) Given equation can be written as $\partial z^2/\partial x^2 = 6x$ ....(1)

Integrating (1) with respect to 'x' $\partial z/\partial x = 3x^2 + \phi_1(y)$, ...(2)

where $\phi_1(y)$ is an arbitrary function of y.

Integrating (2) with respect to 'x', $z = x^3 + x\phi_1(y) + \phi_2(y)$,

where $\phi_2(y)$ is an arbitrary function of y.

(ii) Given equation can be written as $\partial z^2/\partial x^2 = (1/a) \times xy$ ....(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial x = (y/a) \times (x^2/2) + \phi_1(y)$ ....(2)

Integrating (2) w.r.t. 'x', $z = (y/6a) \times x^3 + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

(iii) Given equation can be written as $\partial^2 z/\partial x^2 = \sin(xy)$ ....(1)

Integrating (1) w.r.t. 'x' $\partial z/\partial x = -(1/y) \times \cos(xy) + \phi_1(y)$ ....(2)

Integrating (2) w.r.t. 'x' $z = -(1/y^2) \times \sin(xy) + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. Solve (i) $t - xq = x^2$ (ii) $yt - q = xy$

Solution.

(i) The given equation can be rewritten as $(\partial q/\partial y) - xq = x^2$, ...(1)

which is linear differential equation in variables q and y, regarding x as constant.

Integrating factor (I.F.) of (1) $= e^{\int(-x)dy} = e^{-xy}$ and solution of (1) is

$q(I.F.) = \int (x^2)(I.F.)dy + \phi_1(x)$ or $qe^{-xy} = \int x^2 e^{-xy} dy + \phi_1(x)$

or $qe^{-xy} = x^2 \times (-1/x) \times e^{-xy} + \phi_1(x)$ or $q = \partial z/\partial y = -x + e^{xy}\phi_1(x)$

Integrating it w.r.t. $'y'$, $z = -xy + (1/x) \times \phi_1(x) e^{xy} + \psi_2(x)$

or $z = -xy +_1 (x) e^{xy} + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$

It is the required solution, ψ_1, ψ_2 being arbitrary functions.

(ii) The given equation can be rewritten as $y(\partial q/\partial y) - q = xy$ or $(\partial q/\partial y) - (1/y) \times q = x$, which is differential equation linear in variables q and y, regarding x as constant.

I.F. of (1) $= e^{\int(-1/y)dy} = e^{-\log y} = 1/y$ and solution of (1) is

$q \times \frac{1}{y} = \int\left(x \times \frac{1}{y}\right) dy + \phi_1(x)$ or $\frac{q}{y} = x \log y + \phi_1(x)$

or $q = xy \log y + y\phi_1(x)$ or $\partial z/\partial y = xy \log y + y\phi_1(x)$

Integrating it, $z = x\left[(y^2/2) \times \log y - \int\{(y^2/2) \times (1/y)\} dy\right] + (y^2/2) \times \phi_1(x) + \phi_2(x)$

or $z = (1/2) \times xy^2 \log y - (1/4) \times xy^2 + (1/2) \times y^2\phi_1(x) + \phi_2(x), \phi_1, \phi_2$

or $z = (1/2) \times xy^2 \log y - (1/4) \times xy^2 + (1/2) \times y^2\phi_1(x) + \phi_2(x), \phi_1, \phi_2$ being arbitrary functions.

Classification of P.D.E. Reduction to Canonical or Normal Forms.

Let's consider a PDE: $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ .....(1)

Where $R,\ S,\ T$ some function of $x,\ y$

(1) If $S^2 - 4RT > 0$; Then PDE (1) is hyperbolic PDE

E.g. One-dimensional wave equation, $r - t = 0; R = 1,\ T = -1,\ S = 0;\ S^2 - 4RT = 0 - 4\times1\times(-1) > 0$

(2) If $S^2 - 4RT = 0$; PDE (1) is parabolic PDE

E.g. $r + 2s + t = 0$; $R = 1,\ T = 1,\ S = 2 \therefore\ S^2 - RT = 4 - 4 = 0$

(3) If $S^2 - 4RT < 0$; PDE (1) is Elliptic PDE,

E.g. $r + s + t = 0$; R = 1, S = 1, T = 1 $\therefore\ S^2 - 4RT = 1 - 4 < 0$

Part (1): Reducing $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ into canonical form: like

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

How!! By given PDE: we try to find

Step-I: *u* & *v* i.e., *z* = function of *x*,*y* $\xrightarrow{\text{To}}$ *z* = function of *u* & *v*

then How to find *u* & *v* ??

↓ for this

we write ⍰ characteristic equation for given PDE as $R\lambda^2 + S\lambda + T = 0$ ...(1)

(i.e., $S^2 - 4RT > 0$) Hyperbolic PDE	(i.e., $S^2 - 4RT = 0$) Parabolic PDE	i.e., $S^2 - 4RT < 0$ Elliptic PDE
• We get λ_1 & λ_2 : real & distinct roots of (1) • Now, we find *u* & *v* by solving: $\frac{dy}{dx} + \lambda_1 = 0$, $\frac{dy}{dx} + \lambda_2 = 0$ ↓ ↓ $f_1(x, y) = c_1$ $f_2(x, y) = c_2$ $\therefore$ We take, $u = f_1(x, y)$, $v = f_2(x, y)$	We get two equal roots $\lambda_1 = \lambda_2 = \lambda$ $\therefore$ solve, $\frac{dy}{dx} + \lambda = 0$ Get $f_1(x, y) = c_1$ Take $u = f_1(x, y)$ Now choose another function $f_2(x, y)$ which is linearly independent with $f_1(x, y)$; using jacobians to verify linear independence	No real root $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ Find u & v by: $\frac{dy}{dx} + \lambda_1 = 0$, $\frac{dy}{dx} + \lambda_2 = 0$

Step (II): We try to find p, q, r, s, t

$\because$ We have, *u* = function of *x* & *y*, *v* = function of *x* & *y*; *z* as some function of *u*, *v*

$\therefore$ *z* is a function of *u* & *v* ; *u* & *v* are function of *x*, *y*.

$$\therefore \; \partial z = \frac{\partial z}{\partial u}\partial u + \frac{\partial z}{\partial v}\partial v \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = p$$

E.g. if *u* = *x* + *y* , *v* = *x* - *y*

i.e., we have $p = \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}.\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial z}{\partial v} = 1.\frac{\partial z}{\partial u} + 1.\frac{\partial z}{\partial v}$

Note: from line; we get $\frac{\partial}{\partial x}$ terms of $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, u, v$ the finding $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial x}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

E.g. Let if *u* = *x* + *y*, *v* = *x* – *y*

$$\therefore \; p = \frac{\partial z}{\partial x} = 1.\frac{\partial z}{\partial u} + 1.\frac{\partial z}{\partial v} \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \Rightarrow \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)$$

$$\therefore \; r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial y^2}$$

Similarly, we get $q = \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial z}{\partial v} \therefore \; t = \frac{\partial^2 z}{\partial y^2} = ...$And *s* = $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)$ from above.

Ex. 1. Classify the following partial differential equations

(i) $2\left(\partial^2 u/\partial x^2\right) + 4\left(\partial^2 u/\partial x \partial y\right) + 3\left(\partial^2 u/\partial y^2\right) = 2$

(ii) $\partial^2 u/\partial x^2 + 4\left(\partial^2 u/\partial x \partial y\right) + 4\left(\partial^2 u/\partial y^2\right) = 0$

(iii) $xyr - \left(x^2 - y^2\right)s - xyt + py - qx = 2\left(x^2 - y^2\right)$

(iv) $x^2\left(y-1\right)r - x\left(y^2 - 1\right)s + y\left(y-1\right)t + xyp - q = 0$

(v) $x\left(xy-1\right)r - \left(x^2y^2 - 1\right)s + y\left(xy-1\right)t + xp + yq = 0$

(vi) $\left(x-y\right)\left(xr - xs - ys + yt\right) = \left(x+y\right)\left(p-q\right)$

Solution.

(i) Re-writing the given equation, we get $2r + 4s + 3t - 2 = 0$ ....(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, u, p, q) = 0$, we get $R = 2, S = 4$ and $T = 3$. So

$S^2 - 4RT = (4)^2 - (4 \times 2 \times 3) = -8 < 0$, showing that the given equation is elliptic at all points.

(ii) Re-writing the given equation, we get $r + 4s + 4t = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, u, p, q) = 0$, we get $R = 1, S = 4$ and $T = 4$. So

$S^2 - 4RT = (4)^2 - (4 \times 1 \times 4) = 0$, showing that the given equation is parabolic at all points.

(iii) Given $xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0$ ....(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0$, we get $R = xy, S = -(x^2 - y^2)$ and

$T = -xy$. So, here $S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 > 0$,

showing that the given equation is hyperbolic at all points.

(iv) Hyperbolic (v) Hyperbolic

(vi) Hyperbolic

Ex. 2. Classify $u_{xx} + u_{yy} = y_{zz}$

The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

The eigenvalues of A are given by $|A - \lambda I| = 0$, i.e.,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \text{ or } -(1+\lambda)(1-\lambda)^2 = 0$$

Hence $\lambda = -1, 1, 1$ showing that all the eigenvalues are non-zero and have the same sign except one. Hence the given equation is of hyperbolic type.

Ex.3. Classify $u_{xx} + u_{yy} + u_{zz} + u_{yz} + u_{zy} = 0$

Solution.

The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + u_{yz} + 0 \cdot u_{zx} + u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$, using properties of determinants

Since $|A| = 0$, the given equation is of parabolic type.

Ex.4. Find the characteristics of $y^2 r - x^2 t = 0$

Solution.

Given $y^2 r - x^2 t = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2, S = 0$ and $T = -x^2$.

Then $S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$ and hence (1) is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or $y^2\lambda^2 - x^2 = 0$ ....(2)

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots).

Corresponding characteristic equations are

$(dy/dx) + (x/y) = 0$ and $(dy/dx) - (x/y) = 0 \Rightarrow x\,dx + y\,dx = 0$ and $xdx - ydy = 0$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$, which are the required families of characteristics. Here these are families of circles and hyperbolas respectively.

Category-1:Hyperbolic PDE: canonical form

Ex.1. (i) Write canonical form of $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$

(ii) Reduce $3(\partial^2 z/\partial x^2) + 10(\partial^2 z/\partial x \partial y) + 3(\partial^2 z/\partial y^2) = 0$ to canonical form and hence solve it.

Solution.

(i) Re-writing the given equation, we get $r - t = 0$ ....(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1, S = 0$ and $T = -1$ so that $S^2 - 4RT = 4 > 0$, showing that (1) is hyperbolic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$

Hence $\lambda = 1, -1$. So $\lambda_1 = 1, \lambda_2 = -1$ [Real and distinct roots]

Then the characteristic equations $dy/dx + \lambda_1 = 0, dy/dx + \lambda_2 = 0$ reduces to $(dy/dx) + 1 = 0$ and $(dy/dx) - 1 = 0$

Integrating these, $y + x = c_1$ and $y - x = c_2$

In order to reduce (1) to its canonical form, we choose

$u = y + x$ and $v = y - x$ ....(2)

$$\therefore \; p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ....(3)$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ....(4)$$

From (3) and (4), $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ ....(5)

$$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

$$\text{or } r = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ....(6)$$

$$\text{and } t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

$$\text{or } t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ....(7)$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0 \text{ or } \frac{\partial^2 z}{\partial u \partial z} = 0$$

(ii) Ans. $\partial^2 z/\partial u \partial v = 0$; $z = f(y - 3x) + g(3y - x)$

Ex. 2. Reduce $\partial^2 z/\partial x^2 = (1 + y)^2 (\partial^2 z/\partial y^2)$ to canonical form

Solution.

Re-writing the given equation, $r-\left(1+y^2\right)t=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f\left(x,y,z,p,q\right)=0$, here $R=1, S=0$ and $T=-\left(1+y\right)^2$ so that $S^2-4RT=\left(1+y^2\right)>0$ for $y\neq-1$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2-\left(1+y\right)^2=0$ so that $\lambda=1+y,-\left(1+y\right)$. Hence the corresponding characteristic equations are given by

$\left(dy/dx\right)+\left(1+y\right)=0$ and $\left(dy/dx\right)-\left(1+y\right)=0$

Integrating these, $\log\left(1+y\right)+x=C_1$ and $\log\left(1+y\right)-x=C_2$

In order to reduce (1) to its canonical form, we choose

$u=\log\left(1+y\right)+x$ and $v=\log\left(1+y\right)-x$ ....(2)

$\therefore\ p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}$, using (2) ....(3)

and $q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}=\dfrac{1}{1+y}\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)$ ....(4)

From (3) $\partial/\partial x\equiv\partial/\partial u-\partial/\partial v$ ....(5)

$\therefore\ r=\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)=\left(\dfrac{\partial}{\partial u}-\dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}\right)$, using (3) and (5)

or $r=\partial^2 z/\partial u^2-2\left(\partial^2 z/\partial u\,\partial v\right)+\partial^2 z/\partial v^2$ ....(6)

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left\{\frac{1}{1+y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\right\}=-\frac{1}{\left(1+y\right)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\text{, by (4)}$$

$$=-\frac{1}{\left(1+y\right)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=-\frac{1}{\left(1+y\right)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\left[\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}\right)\frac{1}{y+1}+\left(\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}\right)\frac{1}{1+y}\right]\text{, by (2)}$$

or $t=\dfrac{1}{\left(1+y\right)^2}\left(\dfrac{\partial^2 z}{\partial u^2}+2\dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{\partial^2 z}{\partial v^2}-\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}\right)$ ....(7)

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)=0 \text{ or } 4\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

Ex. 3. Reduce the differential equation $t-s+p-q(1+1/x)+(z/x)=0$ to canonical form.

Solution.

Given $0\cdot r-s+t+p-q(1+1/x)+(z/x)=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=0, S=-1$ and $T=1$

Hence $S^2-4RT=1>0$, showing that the given equation is hyperbolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $-\lambda+1=0$ giving $\lambda=1$. Hence the corresponding characteristic equation $dy/dx+\lambda=0$ yields $dy/dx+1=0$ or $dx+dy=0$

Integrating it, $x+y=c$, c being an arbitrary constant

Choose $u=x+y$ and $v=x$, ...(2)

where we have chosen $v=x$ in such a manner that u and v are independent as verified below:

Jacobian of u and $v=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}=1\neq 0 \Rightarrow u$ and v are independent functions.

Now, $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, using (2) ....(3)

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}$, using (2) ....(4)

From (4), we have $\partial/\partial y\equiv\partial/\partial u$ ....(5)

$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$, using (3) and (5) ....(6)

and $t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$, using (5)

or $t=\partial^2 z/\partial u^2$

Using (2), (3), (4), (6) and (7), (1) reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}\right)+\frac{\partial^2 z}{\partial u^2}+\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\left(1+\frac{1}{v}\right)+\frac{z}{v}=0$$

or $\partial^2 z/\partial u \partial v - (\partial z/\partial v) + (1/v)\times(\partial z/\partial u) - (z/v) = 0$, which is the required canonical form.

Ex. 4. Reduce the equation $yr + (x+y)s + xt = 0$ to canonical form and hence find its general solution.

Solution.

Given $yr + (x+y)s + xt = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y, S = (x+y)$ and $T = x$ so that $S^2 - 4RT = (x+y)^2 - 4xy = (x-y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $y\lambda^2 + (x+y)\lambda + x = 0$ or $(y\lambda + x)(\lambda + 1) = 0$ so that $\lambda = -1, -x/y$. Then the corresponding characteristic equations are given by

$(dy/dx) - 1 = 0$ and $(dy/dx) - (x/y) = 0$

Integrating these, $y - x = c_1$ and $y^2/2 - x^2/2 = c_2$

In order to reduce (1) to its canonical form, we choose

$u = y - x$ and $v = y^2/2 - x^2/2$ ....(2)

Final required form is: $u^2 \dfrac{\partial^2 z}{\partial u \partial v} + u\dfrac{\partial z}{\partial v} = 0$, by (2) or $u\dfrac{\partial^2 z}{\partial v \partial v} + \dfrac{\partial z}{\partial v} = 0$, as $u \neq 0$ ....(3)

Solution of (3). Multiplying both sides of (8) by v, we get

$uv(\partial^2 z/\partial u \partial v) + v(\partial z/\partial v) = 0$ or $(uvDD' + vD')z = 0$ ....(4)

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial u$. To reduce (9) into linear equation with constant coefficients, we take new variables X and Y as follows.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$ ....(5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$ .Then (9) reduces to

$(D_1 D_1' + D_1')z = 0$ or $D_1'(D_1 + 1)z = 0$

Its general solution is $z = e^{-X}\phi_1(Y) + \phi_2(X) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$

or $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1(y^2 - x^2) + \psi_2(y-x)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex.5. Reduce the equation $r\left(2\sin x\right)s-\left(\cos^2 x\right)t-\left(\cos x\right)q=0$ to canonical form and hence solve it.

Sol. Given $r-\left(2\sin x\right)s-\left(\cos^2 x\right)t-\left(\cos x\right)q=0$ ....(1)

Comparing (1) with $Rs+Ss+f\left(x,y,z,p,q\right)=0$, here $R=1, s=-2\sin x$ and $T=-\cos^2 x$ so that $S^2-4RT=4\left(\sin^2 x+\cos^2 x\right)=4>0$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2-\left(2\sin x\right)\lambda-\cos^2 x=0$ so that $\lambda=\sin x+1, \sin x-1$. Hence the corresponding characteristic equations become

$dy/dx+\sin x+1=0$ and $dy/dx+\sin x-1=0$

Integrating these, $y-\cos x+x=c_1$ and $y-\cos x-x=c_2$

Choose $u=y-\cos x+x$ and $v=y-\cos x-x$ ....(2)

$\therefore \quad p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=\left(1+\sin x\right)\dfrac{\partial z}{\partial u}+\left(\sin x-1\right)\dfrac{\partial z}{\partial v}$, by (2) ....(3)

$q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial v}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$, using (2) ....(4)

From (4), we have $\partial/\partial y=\partial/\partial u+\partial/\partial v$ ....(5)

$\therefore \quad t=\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\left(\dfrac{\partial}{\partial u}+\dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)$, using (4) and (5)

or $t=\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)+\dfrac{\partial}{\partial v}\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial^2 z}{\partial u^2}+2\dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{\partial^2 z}{\partial v^2}$ ....(6)

Now, $s=\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)=\dfrac{\partial}{\partial y}\left\{\left(1+\sin x\right)\dfrac{\partial z}{\partial u}+\left(\sin x-1\right)\dfrac{\partial z}{\partial v}\right\}$, by (3)

$$=\left(\sin x+1\right)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\left(\sin x-1\right)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$

$$=\left(\sin x+1\right)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right\}+\left(\sin x-1\right)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$=\left(\sin x+1\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}\right)+\left(\sin x-1\right)\left(\frac{\partial^2 z}{\partial u\partial x}+\frac{\partial^2 z}{\partial v^2}\right)$$

or $s = \sin x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}$ ....(7)

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{(\sin x+1)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\} = \cos x\frac{\partial z}{\partial u} + (\sin x+1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \cos x\frac{\partial z}{\partial v} + (\sin x - 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \cos x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + (\sin x+1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\} + (\sin x - 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= \cos x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + (\sin x+1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u^2} + (\sin x - 1)\frac{\partial^2 z}{\partial v^2}\right\} + (\sin x - 1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u \partial v} + (\sin x - 1)\frac{\partial^2 z}{\partial v^2}\right\}$$

$\therefore \quad r = \cos x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + (1+\sin x)^2\frac{\partial^2 z}{\partial u^2}(\sin x - 1)^2\frac{\partial^2 z}{\partial v^2} - 2\cos^2 x\frac{\partial^2 z}{\partial u \partial v}$ ....(8)

Using (4) (6), (7) and (8) in (1), we get

$$\cos x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \left(1 + 2\sin x + \sin^2 x\right)\frac{\partial^2 x}{\partial u^2} + \left(\sin^2 x + 1 - 2\sin x\right)\frac{\partial^2 z}{\partial v^2} - 2\cos^2 x\frac{\partial^2 z}{\partial u \partial v}$$

$$-2\sin x\left\{\sin x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right\} - \cos^2 x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) - \cos x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = 0$$

or $\left(1 + 2\sin x + \sin^2 x - 2\sin^2 x - 2\sin x - \cos^2 x\right) \times \left(\partial^2 z/\partial u^2\right) + \left(\sin^2 x + 1 - 2\sin x - 2\sin^2 x\right.$

$\left. + 2\sin x - \cos^2 x\right) \times \left(\partial^2 z/\partial v^2\right) - \left(2\cos^2 x + 4\sin^2 x + 2\cos^2 x\right) \times \left(\partial^2 z/\partial u \partial v\right) = 0$

or $\partial^2 z/\partial u \partial v = 0$, on simplification. ....(9)

(9) is the required canonical form of (1).

Solution of (9). Integrating (9) w.r.t. $\partial z/\partial v = \phi(v)$, ϕ being an arbitrary function ....(10)

Integrating (10) w.r.t. 'v', $z = \int \phi(v)\,dv + F(u) = G(v) + F(u)$,

where $G(v) = \int \phi(v)\,dv$, F and G are arbitrary functions.

$\therefore \quad z = G(y - \cos x - x) + F(y - \cos x + x)$ is the required solution.

Ex. 6. Reduce the equation $(y-1)r - \left(y^2 - 1\right)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3$ to canonical form and hence solve it.

Solution

Given $\left(y-1\right)r-\left(y^2-1\right)s+y\left(y-1\right)t+p-q=2ye^{2x}\left(1-y\right)^3=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f\left(x,y,z,p,q\right)=0$, we get

$R=y-1,\ S=-\left(y^2-1\right)$ and $T=y\left(y-1\right)$ ....(2)

$\therefore$ The λ-quadratic $R\lambda^2-S\lambda+T=0$ gives

$\left(y-1\right)\lambda^2-\left(y^2-1\right)\lambda+y\left(y-1\right)=0\Rightarrow\lambda_1=1$ and $\lambda_2=y$ (real and distinct roots)

Hence characteristic equations $\left(dy/dx\right)+\lambda_1=0$ and $\left(dy/dx\right)+\lambda_2=0$ become

$\left(dy/dx\right)+1=0$ and $\left(dy/dx\right)+y=0$

Integrating these, $x+y=c_1$ and $ye^x=c_2$

To reduce (1) to canonical form, we change the independent variables x,y, to new independent variables u,v by taking

$u=x+y$ and $v=ye^x$ ....(3)

$\therefore\ p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}+ye^x\dfrac{\partial z}{\partial v}=\dfrac{\partial z}{\partial u}+v\dfrac{\partial z}{\partial v}$, by (3) ....(4)

$q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}=\dfrac{\partial z}{\partial u}+e^x\dfrac{\partial z}{\partial v}$, by (3) ....(5)

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{\partial}{\partial u}\right)=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial u\partial v}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v}\text{, by (4)}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+\left(e^x+v\right)\frac{\partial^2 z}{\partial u\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and $t=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial u}+e^x\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right)+e^x\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial v}\right)$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}$$

Substituting the above values in (1) and simplifying, we have

$$(1-y)^3 e^x \frac{\partial^2 z}{\partial u \partial v} = 2ye^{2x}(1-y)^3 \text{ or } \frac{\partial^2 z}{\partial u \partial v} = 2v \quad \text{....(6)}$$

which is the canonical form of (1).

Integrating (6) w.r.t. $'v'$, $\partial z/\partial u = v^2 + \phi(u), \phi(u)$ being an arbitrary function, ...(7)

Integrating (7) w.r.t. $'u'$, $z = uv^2 + \phi_1(u) + \phi_2(v)$, where $\phi_1(u) = \int \phi(u)\, du$

$\therefore$ Using (3) $z = (x+y)y^2e^{2x} + \phi_1(x+y) + \phi_2(ye^x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 7. Solve $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$

Solution

Given $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$R = x^2(y-1)$, $S = -x(y^2-1)$ and $T = y(y-1)$

$\therefore$ λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$x^2(y-1)\lambda^2 - x(y^2-1)\lambda + y(y-1) = 0 \Rightarrow \lambda_1 = y/x$ and $\lambda_2 = 1/x$ (real and distinct)

So characteristic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$ become

$(dy/dx) + (y/x) = 0$ and $(dy/dx) + (1/x) = 0$

Integrating these, $xy = c_1$ and $xe^y = c_2$ so for canonical form, we take

$u = xy$ and $v = xe^y$ ....(2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{....(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{....(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\left(\frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]=y^{2}\frac{\partial^{2}z}{\partial u^{2}}+2ye^{x}\frac{\partial^{2}z}{\partial u\partial v}+e^{2y}\frac{\partial^{2}z}{\partial y^{2}}$$

$$s=\frac{\partial^{2}z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+xe^{y}\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial u}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^{y}\frac{\partial z}{\partial v}+xe^{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial z}{\partial u}+e^{y}\frac{\partial z}{\partial v}+x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+xe^{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=\frac{\partial z}{\partial u}+e^{y}\frac{\partial z}{\partial v}+xy\frac{\partial^{2}z}{\partial u^{2}}+\left(yxe^{y}+e^{y}x\right)\frac{\partial^{2}z}{\partial u\partial v}+xe^{2y}\frac{\partial^{2}z}{\partial v^{2}}$$

and $t=\dfrac{\partial^{2}z}{\partial y^{2}}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(x\dfrac{\partial z}{\partial u}+xe^{y}\dfrac{\partial z}{\partial v}\right)=x\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right)+xe^{y}\dfrac{\partial z}{\partial v}+xe^{y}\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial v}\right)$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+xe^{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x^{2}\frac{\partial^{2}z}{\partial u^{2}}+2x^{2}e^{y}\frac{\partial^{2}z}{\partial u\partial v}+x^{2}e^{2y}\frac{\partial^{2}z}{\partial y^{2}}+xe^{y}\frac{\partial z}{\partial v}$$

Substituting the above values in (1) and simplifying, we get $\partial^{2}z/\partial u\partial v=0$, ...(5)

which is canonical form of (1).

Integrating (5) w.r.t. 'u' $\partial z/\partial v=\phi(v)$, $\phi(v)$ being an arbitrary function.

Integrating it w.r.t. 'v', $z=\phi_{1}(v)+\phi_{2}(u)$, where $\phi_{1}(v)=\int\phi(v)dv$

$\therefore\ z=\phi_{1}\left(xe^{y}\right)+\phi_{2}(xy)$, by (2). This is the required solution, ϕ_{1},ϕ_{2} being arbitrary functions.

Ex. 8. Solve (i) $xyr-\left(x^{2}-y^{2}\right)s-xyt+py-qx=2\left(x^{2}-y^{2}\right)$

(ii) $x(y-x)r-\left(y^{2}-x^{2}\right)s+y(y-x)t+(y+x)(p-x)=2x+2y+2$

Solution

(i) Given $xyr-\left(x^{2}-y^{2}\right)s-xyt+py-qx=2\left(x^{2}-y^{2}\right)=0$ ....(1)

Comparing (i) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have

$R=xy$, $S=-\left(x^{2}-y^{2}\right)$ and $T=-xy$

So λ-quadratic $R\lambda^{2}+S\lambda+T=0$ becomes $xy\lambda^{2}-\left(x^{2}-y^{2}\right)\lambda-xy=0$ giving $\lambda=-y/x,x/y$

$\therefore\ \dfrac{dy}{dx}+\lambda_1=0$ and $\therefore\ \dfrac{dy}{dx}+\lambda_2=0 \Rightarrow \dfrac{dy}{dx}-\dfrac{y}{x}=0$ and $\dfrac{dy}{dx}+\dfrac{y}{x}=0$

Integrating, $y/x=c_1$ and $x^2+y^2=c_2$. So, we take

$u=y/x$ and $v=x^2+y^2$ ....(2)

$\therefore$ Proceeding as usual, we obtain

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\left(-\frac{y}{x^2}\right)\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v},\quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{x}\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v}$$

$$r=\left(-\frac{y}{x^2}\right)^2\frac{\partial^2 z}{\partial u^2}+2\times(2x)\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial v\partial u}+4x^2\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^3}\frac{\partial z}{\partial u}+2\frac{\partial z}{\partial v}$$

$$s=\left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial u^2}+\left\{2y\left(-\frac{y}{x^2}\right)+2x\times\frac{1}{x}\right\}\frac{\partial^2 z}{\partial u\partial v}+4xy\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial u}$$

and $t=\left(\dfrac{1}{x}\right)^2\dfrac{\partial^2 z}{\partial u^2}+2\times\dfrac{1}{x}\times(2y)\dfrac{\partial^2 z}{\partial u\partial v}+4y^2\dfrac{\partial^2 z}{\partial v^2}+2\dfrac{\partial z}{\partial v}$

Substituting these in (1) we get

$\left(x^2+y^2\right)^2\dfrac{\partial^2 z}{\partial u\partial v}=\left(y^2-x^2\right)x^2$ or $\dfrac{\partial^2 z}{\partial u\partial v}=\dfrac{\left(y^2-x^2\right)x^2}{\left(x^2+y^2\right)^2}=\dfrac{u^2-1}{\left(u^2+1\right)^2}$, by (2) ....(3)

Integrating (3) w.r.t. 'u', we have

$\dfrac{\partial z}{\partial v}=\displaystyle\int\frac{u^2-1}{\left(u^2+1\right)^2}du+\phi(v)=\int\frac{du}{u^2+1}-2\int\frac{du}{\left(u^2+1\right)^2}+\phi(v)$ ....(4)

We have, $\displaystyle\int 1\cdot\frac{1}{u^2+1}du=u\times\frac{1}{u^2+1}-\int u\times\left(\frac{-2u}{\left(u^2+1\right)^2}\right)du$, integrating by parts

or $\displaystyle\int\frac{du}{u^2+1}=\frac{u}{u^2+1}+2\int\frac{\left(u^2+1\right)-1}{\left(u^2+1\right)^2}du=\frac{u}{u^2+1}+2\int\frac{du}{u^2+1}-2\int\frac{du}{\left(u^2+1\right)^2}$

Then, $\displaystyle\int\frac{du}{u^2+1}-2\int\frac{du}{\left(u^2+1\right)^2}=-\frac{u}{u^2+1}$ ....(4)

Using (5), (4) gives $\partial z/\partial v=-u/\left(u^2+1\right)+\phi(v),\phi(v)$ being an arbitrary function ....(5)

Integrating (5) w.r.t. $z=-(uv)/(u^2+v^2)+\phi_1(v)+\phi_2(u)$, where $\phi_1(v)=\int\phi(v)dv$

$\therefore$ Using (2), $z=-xy+\phi_1(x^2+y^2)+\phi_2(y/x), \phi_1, \phi_2$ being arbitrary functions.

(ii) Hint. Since $R=x(y-x)$, $S=-(y^2-x^2)$, $T=y(y-x)$, so here $\lambda_1=y/x$, $\lambda_2=1$

So we get $(dy/dx)+(y/x)=0$ and $(dy/dx)+1=0$ as characteristic equations

These give $xy=c_1$ and $x+y=c_2$. Hence take

$y=xy$ and $v=x+y$ ....(1)

As usual, $p=y\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$ and $q=x\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$,

$$r=y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},\ t=x^2\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},$$

$$x=xy\frac{\partial^2 z}{\partial u^2}+(x+y)\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial u}$$

$\therefore$ Given equation becomes $-(y-x)^3\dfrac{\partial^2 z}{\partial v\partial u}=2x+2y+2$ ....(2)

or $\dfrac{\partial^2 z}{\partial v\partial u}=-\dfrac{2(x+y+1)}{(y-x)^3}=-\dfrac{2(x+y+1)}{\left[(y+x)^2-4xy\right]^{3/2}}=\dfrac{2(v+1)}{(v^2-4u)^{3/2}}$, by (1)

Integrating (2) w.r.t. 'u', we get $\dfrac{\partial z}{\partial v}=\dfrac{v+1}{\sqrt{(v^2-4u)}}+\phi(v)$ ....(3)

Integrating, (3) w.r.t. v, $z=\sqrt{(v^2-4u)}+\log\left[v+\sqrt{(v^2-4u)}\right]+\phi_1(v)+\phi_2(u)$

or $z=x-y+\log(2x)+\phi_1(x+y)+\phi_2(xy), \phi_1, \phi_2$ being arbitrary functions.

Ex.9. Solve (i) $y(x+y)(r-s)-xp-yq-z=0$

(ii) $xys-x^2r-px-qy+z=-2xy^2y$

Solution

(i) Given $y(x+y)r-y(x+y)s-xp-yq-z=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0, R=y(x+y), S=-y(x+y), T=0$

So, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$y(x+y)\lambda^2 - y(x+y)\lambda = 0$, giving $\lambda = 0,1$. Thus $\lambda_1 = 1$ and $\lambda_2 = 0$ and so $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0 \Rightarrow \frac{dy}{dx} + 1 = 0$ and $\frac{dy}{dx} = 0$

Integrating these, $x + y = c_1$ and $y = c_2$

So we take $u = x + y$ and $v = y$ ....(2)

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}$, by (2) ....(3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, by (2) ....(4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}$, by (3) ....(5)

$s = \frac{\partial^2 z}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}$, using (3) and (4) ....(6)

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), we have

$$y(x+y)\left(-\frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0 \text{ or } uv\frac{\partial^2 z}{\partial v \partial u} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} + z = 0$$

or $\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{1}{uv}z = 0$ or $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0$ ....(7)

Let $\partial z/\partial v + (z/v) = w$ ....(8)

Then, the above equation (7) becomes $\partial w/\partial u + w/u = 0$

Integrating, $wu = \phi(v)$ or $w = (1/u) \times \phi(v)$

Substituting this value of w in (8), we have $\frac{\partial z}{\partial v} + \frac{1}{v}z = \frac{1}{u}\phi(v)$ ....(9)

I.F. of (9) $= e^{\int (1/v)dv} = v$ and solution of (9) is

$$zv = \frac{1}{u}\int \phi(v)dv + \phi_2(u) \text{ or } z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u), \text{ where } \phi_1(v) = \int \phi(v)\,dv$$

or $z = \dfrac{1}{y(x+y)}\phi_1(y) + \dfrac{1}{y}\phi_2(x+y)$, by (2) ; ϕ_1, ϕ_2 being arbitrary functions

(ii) Hint. Given $xys - x^2 r - px - qy + z = -2x^2 y$ ....(1)

Here, $R = -x^2, S = xy$, $T = 0$ and λ-quadratic is $-x^2\lambda^2 + xy\lambda = 0$

so that $\lambda_1 = y/x$ and $\lambda_2 = 0$. Hence, characteristic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = 0 \text{ and } \frac{dy}{dx} = 0$$

Integrating these, $xy = c_1$, $y = c_2$. So we take $u = xy$ and $v = y$ ....(2)

Then, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} y = v\dfrac{\partial z}{\partial u}$, by (2) ....(3)

$q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = x\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} = \dfrac{u}{v}\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$, by (2) ....(4)

$r = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = v\dfrac{\partial}{\partial u}\left(v\dfrac{\partial z}{\partial u}\right) = v^2\dfrac{\partial^2 z}{\partial u^2}$, by (3)

$s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = v\dfrac{\partial}{\partial u}\left(\dfrac{u}{v}\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right) = v\left(\dfrac{1}{v}\dfrac{\partial z}{\partial u} + \dfrac{u}{v}\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial u \partial v}\right)$, by (3) and (4)

Substituting these values in (1), we have

$$xy\left(\frac{\partial z}{\partial u} + u\frac{\partial^2 z}{\partial u^2} + v\frac{\partial^2 z}{\partial u \partial v}\right) - x^2v^2\frac{\partial^2 z}{\partial u^2} - v\frac{\partial z}{\partial u}x - y\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + z = -2x^2 y$$

or $u\dfrac{\partial z}{\partial u} + u^2\dfrac{\partial^2 z}{\partial u^2} + uv\dfrac{\partial^2 z}{\partial u \partial v} - u^2\dfrac{\partial^2 z}{\partial u^2} - u\dfrac{\partial z}{\partial u} - u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v} + z = -2\left(u^2/v^2\right)v$, by (2)

or $uv\dfrac{\partial^2 z}{\partial u \partial v} - u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v} + z = -\dfrac{2u^2}{v}$ or $\dfrac{\partial^2 z}{\partial u \partial v} - \dfrac{1}{v}\dfrac{\partial z}{\partial u} - \dfrac{1}{u}\dfrac{\partial z}{\partial v} + \dfrac{z}{uv} = -\dfrac{2u}{v^2}$

or $\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v} - \dfrac{z}{v}\right) - \dfrac{1}{u}\left(\dfrac{\partial z}{\partial v} - \dfrac{z}{v}\right) = -\dfrac{2u}{v^2}$ ....(5)

Let $\partial z/\partial v - z/v = w$ ....(6)

Then (5) becomes $\frac{\partial w}{\partial u}-\frac{1}{u}w=\frac{2u}{v^2}$, which is linear differential equation ....(7)

I.F. of (7) $=e^{-\int(1/u)du}=e^{-\log u}=e^{\log u^{-1}}=(1/u)$ and so its solution is

$$\frac{w}{u}=-\int\left(\frac{2u}{v^2}\times\frac{1}{u}\right)du=-\frac{2u}{v^2}+\phi(v) \text{ or } w=-\frac{2u^2}{v^2}+u\phi(v)$$

Substituting this value of w in (6), we get $\frac{\partial z}{\partial v}-\frac{1}{v}z=-\frac{2u^2}{v^2}+u\phi(u)$

Its I.F. $=e^{-\int(1/v)dv}=e^{-\log v}=e^{\log v^{-1}}=(1/v)$ and so its solution is

$$\frac{z}{v}=\int\frac{1}{v}\left[-\frac{2u^2}{v^2}+u\phi(v)\right]dv=\frac{u^2}{v^2}+u\psi(v)+\phi_2(u)$$

or $z=(u^2/v)+uv\psi(v)+v\phi_2(u)=(u^2/v)+u\phi_1(v)+v\phi_2(u)$ or $z=x^2y+xy\phi_1(y)+y\phi_2(xy)$, by (2)

Ex. 10. Solve $x^2r-y^2t+px-qy=x^2$

Solution

Given $x^2r-y^2t+px-qy=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R=x^2$, $S=0$ and $T=-y^2$ ....(2)

Now, the λ-quadratic $R\lambda^2+S\lambda+T=0$ and (2) give

$x^2\lambda^2-y^2=0$ so that $\lambda=\pm y/x$ (real and distinct roots)

Take $\lambda_1=y/x$ and $\lambda_2=-y/x$

Hence characteristic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$

become $(dy/dx)+(y/x)=0$ and $(dy/dx)-(y/x)=0$

or $(1/x)dx+(1/y)dy=0$ and $(1/x)dx-(1/y)dy=0$

Integrating, $\log x+\log y=\log c_1$ and $\log x-\log y=\log c_2$

or $xy=c_1$ and $x/y=c_2$

To reduce (1) to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u = xy \text{ and } v = x/y \qquad \text{....(3)}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}, \text{ using (3)} \qquad \text{....(4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}, \text{ using (3)} \qquad \text{....(5)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2} \times y + \frac{\partial^2 z}{\partial v \partial u} \times \frac{1}{y}\right) + \frac{1}{y}\left(\frac{\partial^2 z}{\partial v \partial u} \times y + \frac{\partial^2 z}{\partial v^2} \times \frac{1}{y}\right), \text{ using (3)}$$

$$\therefore \; r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} \qquad \text{....(6)}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \left[-\frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2} \times x + \frac{\partial^2 z}{\partial v \partial u} \times \left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u \partial v} \times x + \frac{\partial^2 z}{\partial v^2} \times \left(-\frac{x}{y^2}\right)\right]$$

$$\therefore \qquad t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} \qquad \text{.....(7)}$$

Substituting the values of r, t, p and q given by (6), (7) (3) and (4) in (1), we obtain

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial u^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)$$

$$+x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) - y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) - x^2 = 0$$

or $4x^2 \frac{\partial^2 z}{\partial u \partial v} = x^2$ so that $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}$ ....(8)

which is the canonical form of (1).

Now, integrating (8) w.r.t. 'u', $\partial z/\partial u = (u/4) + f(v)$ ....(9)

Integrating (9) w.r.t. 'v' $z = (uv)/4 + \int f(v)\,dx + \phi(u)$

or $z = (uv)/4 + \psi(v) + \phi(u)$, where $\psi(v) = \int f(v)\,dv$

or $z = x^2/4 + \psi(x/y) + \phi(xy)$, which is the required solution, ϕ, ψ being arbitrary functions.

Ex. 11. (a) Reduce $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) = 0$ to canonical form and hence solve it.

(b) Reduce $y^2(\partial^2 z/\partial x^2) - x^2(\partial^2 z/\partial y^2) = 0$ to canonical form.

Sol. (a) Re-writing the given equation, $x^2 r - y^2 t = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2, S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2y^2 > 0$ for $x \neq 0, y \neq 0$ and hence (1) is hyperbolic. The λ-quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x$, $-y/x$ and hence the corresponding characteristics equations become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

Integrating these, $xy = c_1$ and $x/y = c_2$

In order to reduce (1) to its canonical form, we choose $u = xy$ and $v = x/y$ ....(2)

Now, doing exactly as in solved Ex. 12, we get

$$r = y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \text{ and } t = x^2 \frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1), we get

$$x^2\left(y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2 \frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $4x^2 \frac{\partial^2 z}{\partial u \partial v} - \frac{2x}{y}\frac{\partial z}{\partial v} = 0$ or $2xy\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$

or $2u(\partial^2 z/\partial u \partial v) - (\partial z/\partial v) = 0$, using (2) ....(3)

This is the required canonical form of (1).

We now proceed to find solution of (1). Multiplying both sides of (3) by v, we get

$$2uv\frac{\partial^2 z}{\partial u \partial v} - v\frac{\partial z}{\partial v} = 0 \text{ or } \left(2uv\, DD' - vD'\right)z = 0 \qquad \text{....(4)}$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial u$. We now reduce (4) to a linear equation with constant coefficients by usual method (refer Art. 6.3 of chapter 6).

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $y = \log v$ ....(5)

Let $D_1 \equiv \partial/\partial X$ and $D_1^{'} \equiv \partial/\partial Y$. Then (4) reduces to

$$\left(2D_1D_1^{'} - D_1^{'}\right)z = 0 \text{ or } D_1^{'}\left(2D_1 - 1\right)z = 0$$

Its general solution is given by (use Art. 5.6 of chapter 5)

$$z = e^{X/2}\phi_1\left(Y\right) + \phi_2\left(X\right) = u^{1/2}\phi_1\left(\log v\right) + \phi_2\left(\log u\right) = u^{1/2}\psi_1\left(v\right) + \psi_2\left(u\right), \text{ using (5)}$$

$$= \left(xy\right)^{1/2}\psi_1\left(x/y\right) + \psi_2\left(xy\right) = x\left(y/x\right)^{1/2}\psi_1\left(x/y\right) + \psi_2\left(xy\right) = xf\left(x/y\right) + \psi_2\left(xy\right), \text{ using (2)}$$

where f and ψ_2 are arbitrary functions.

(b) Try yourself. Choose $u = \left(y^2 - x^2\right)/2$, $v = \left(y^2 + x^2\right)/2$

Ans. $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{2\left(u^2 - v^2\right)}\left(v\dfrac{\partial z}{\partial u} - u\dfrac{\partial z}{\partial v}\right)$

Ex. 12. Reduce the equation $x\left(xy-1\right)r - \left(x^2y^2-1\right)s + y\left(xy-1\right)t + \left(x-1\right)p + \left(y-1\right)q = 0$ to canonical form and hence solve it.

Sol. Comparing the given equation with $Rr + Ss + Tt + f\left(x, y, z, p, q\right) = 0$

here, $R = x\left(xy-1\right)$, $S = -\left(x^2y^2-1\right)$, $T = y\left(xy-1\right)$ ....(1)

Now, the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ and (1) give

$$x\left(xy-1\right)\lambda^2 - \left(x^2y^2-1\right)\lambda + y\left(xy-1\right) = 0 \text{ or } x\lambda^2 - \left(xy+1\right)\lambda + y = 0$$

or $\left(x\lambda - 1\right)\left(\lambda - y\right) = 0$ so that $\lambda = 1/x,\ y$. Take $\lambda_1 = 1/x$ and $\lambda_2 = y$

Hence characteristic equations $\left(dy/dx\right) + \lambda_1 = 0$ and $\left(dy/dx\right) + \lambda_2 = 0$

become $(dy/dx)+(1/x)=0$ and $(dy/dx)+y=0$

or $dy+(1/x)dx=0$ and $(1/y)dy+dx=0$ ....(2)

Integrating (2), $y+\log x=\log c_1$ and $\log y+x=\log c_2$

or $\log e^y+\log x=\log c_1$ and $\log y+\log e^x=\log c_2$

$xe^y=c_1$ and $ye^x=c_2$

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v, by taking

$u=xe^y$ and $v=ye^x$ ....(3)

$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}$, using (3) ....(4)

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$, using (3) ....(5)

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+ye^x\frac{\partial z}{\partial v}+ye^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+ye^x\frac{\partial z}{\partial v}+ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+ye^x\frac{\partial z}{\partial v}+ye^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$\therefore \quad r=e^{2y}\frac{\partial^2 z}{\partial u^2}+2ye^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y^2e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial z}{\partial u}+xe^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial z}{\partial v}+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\frac{\partial z}{\partial u}+xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\frac{\partial z}{\partial u}+xe^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$=xe^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+ye^{2x}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(xe^{y}\frac{\partial z}{\partial u}+e^{x}\frac{\partial z}{\partial v}\right)=xe^{y}\frac{\partial z}{\partial u}+xe^{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=xe^{y}\frac{\partial z}{\partial u}+xe^{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+e^{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=xe^{y}\frac{\partial z}{\partial u}+xe^{y}\left[\frac{\partial^2 z}{\partial u^2}xe^{y}+\frac{\partial^2 z}{\partial u\partial v}e^{x}\right]+e^{x}\left[\frac{\partial^2 z}{\partial u\partial v}xe^{y}+\frac{\partial^2 z}{\partial v^2}e^{x}\right],$$

$$\therefore \qquad t=x^2e^{2y}\frac{\partial^2 z}{\partial u^2}+2xe^{x+y}\frac{\partial^2 z}{\partial u\partial v}+xe^{y}\frac{\partial z}{\partial u}+e^{2x}\frac{\partial^2 z}{\partial v^2}$$

Putting the above values of r,s,t,p,q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u\partial v}=0 \text{ or } \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)=0 \qquad ....(6)$$

Integrating (6) w.r.t. $'v'$, $\partial z/\partial u=f(u)$, f being an arbitrary function. ....(7)

Integrating (7) w.r.t. $'u'$, $z=\int f(u)\,du+\psi(v)$ or $z=\phi(u)+\psi(v)$, where $\phi(u)=\int f(u)\,du$.

Using (3), the required solution is $z=\phi\left(xe^{y}\right)+\psi\left(ye^{x}\right)$, ϕ and ψ being arbitrary function.

Ex. 13. (a) Reduce the one-dimensional wave equation $\partial^2 z/\partial x^2=\left(1/c^2\right)\times\left(\partial^2 z/\partial t^2\right)$, $(c>0)$ to canonical form and hence find its general solution.

(b) Find the D'Alembert's solution of the Cauchy's problem: $\partial^2 z/\partial x^2=\left(1/c^2\right)\times\left(\partial^2 z/\partial t^2\right)$, $(c>0)$ satisfying $z(x,0)=f(x)$ and $z_t(x,0)=g(x)$ where $f(x)$ and $g(x)$ are given functions representing the initial displacement and initial velocity, respectively. Also, $z_t=\partial z/\partial t$.

Sol. (a) Given $\partial^2 z/\partial x^2-\left(1/c^2\right)\times\left(\partial^2 z/\partial t^2\right)=0, c>0$ ....(1)

To re-write (1), put $y=ct$, ...(2)

Then, (1) reduces to $\partial^2 z/\partial x^2-\left(\partial^2 z/\partial y^2\right)=0$ or $r-t=0$ ....(3)

Proceed now exactly as in solved Ex. 1 to reduce (3) to its canonical form

$$\frac{\partial^2 z}{\partial u\partial v}=0 \text{ or } \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=0 \qquad ....(4)$$

where $u = y + x$, $v = y - x$ or $u = ct + x$ and $v = ct - x$ ....(5)

Integrating (4) w.r.t. $'u'$, $\partial z/\partial v = f(v)$, where f is an arbitrary function ....(6)

Integrating (6) w.r.t. $'v'$, $z = \int f(v)dv + \psi(u) = F(v) + \psi(u)$, where $f(v) = \int f(v)dv$

or $z(x,t) = F(ct - x) + \psi(ct + x)$, using (5)

or $z(x,t) = \phi(x - ct) + \psi(x + ct)$, ...(7)

where we take $\phi(x - ct) = F(ct - x)$ and ϕ, ψ as arbitrary functions.

(7) is the required general solution of (1).

(b) We are to solve $\partial^2 z/\partial x^2 - (1/c^2) \times (\partial^2 z/\partial t^2) = 0$ ....(i)

subject to the conditions $z(x,0) = f(x)$ ....(ii)

and $(\partial z/\partial t)_{t=0} = g(x)$ ....(iii)

Proceed exactly as in part (a) and get solution of (i) as

$z(x,t) = \phi(x - ct) + \psi(x + ct)$ ....(iv)

Differentiating (iv) partially w.r.t. $'t'$, we get

$\partial z/\partial t = -c\phi'(x - ct) + c\psi'(x + ct)$ ....(v)

where dash denotes the derivative w.r.t. the argument. Putting $t = 0$ in (iv) and (v) and using (ii) and(iii) respectively, we get $\phi(x) + \psi(x) = f(x)$ ....(vi)

and $-c\phi'(x) + c\psi'(x) = g(x)$ ....(vii)

Integrating (vii), $-c\phi(x) + c\psi(x) = \int_a^x g(u)du$, ...(viii)

where a is an arbitrary constant. Solving (vi) and (viii) for $\phi(x)$ and $\psi(x)$, we have

$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(x)du$, and $\psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(u)du$

so that $\phi(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c}\int_a^{x-ct} g(u)du$ ....(ix)

and $\psi(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c}\int_a^{x+ct} g(u)du$ ....(x)

Using (ix) and (x) in (iv), we get the required so called D'Alembert's solution of the Cauchy problem (which represents the vibrations of an infinite string in the present problem)

$$z(x,t)=\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\left[\int_{x-ct}^{a} g(u)\,du+\int_{a}^{x+ct} g(u)\,dx\right]$$

or $z(x,t)=\frac{1}{2}\{f(x-ct)+f(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$ ....(xi)

Particular Case I. If in the above problem, we take $g(x)=0$ so that the initial velocity of the string is zero, then (xi) reduces to

$$z(x,t)=\{f(x-ct)+f(x+ct)\}/2$$

where $f(x-ct)$ represents a right travelling wave travelling with the speed c (along OX) and $f(x+ct)$ represents a left travelling wave travelling with the speed c.

Particular case II. If $f(x)=\sin x$ and $g(x)=\cos x$ in the above problem, then the corresponding solution (xi) reduces to

$$z(x,t)=\frac{1}{2}\{\sin(x-ct)+\sin(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct}\cos u\,du$$

or $z(x,t)=\sin x\cos ct+(1/2c)\times\{\sin(x+ct)-\sin(x-ct)\}$ or

$z(x,t)=\sin x\cos ct+(1/c)\times\cos x\sin ct$

Particular case III. If $f(x)=\sin x$ and $g(x)=x^2$, then (xi) gives

$z(x,t)=\sin x\cos ct+x^2t+(c^3t^3)/3$, on simplification.

<u>Category-2: Parabolic PDE: canonical form</u>

Ex. 1. Reduce the equation $\partial^2 z/\partial x^2+2(\partial^2 z/\partial x\partial y)+\partial^2 z/\partial y^2=0$ to canonical form and hence solve it.

Sol. Re-writing the given equation, we get $r+2s+t=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$ here $R=1, S=2, T=1$ so that $S^2-4RT=0$, showing that (1) is parabolic.

The λ-quadrate equation reduces to $\lambda^2+2\lambda+1=0$ so that $\lambda=-1,-1$ (equal roots).

The corresponding characteristic equation is $(dy/dx)-1=0$ or $dx-dy=0$

Integrating, $x - y = c, c$ being an arbitrary constant.

Choose $u = x - y$ and $v = x + y$, ...(2)

where we where we have chosen $v = x + y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1\cdot 1 + 1\cdot 1 = 2 \neq 0$$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, using (2) ....(3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, using (2) ....(4)

From (3) and (4), $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ ....(5)

$\therefore \; r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$, by (3) and (5)

$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$ ....(6)

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$, by (4) and (5)

$= -\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 v}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$ ....(7)

and $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$, by (4) and (5)

$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$ ....(8)

Using (6), (7) and (8) in (1), the required canonical form is

$\frac{\partial^2 z}{\partial v^2} = 0$ or $\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0$ ....(9)

To find the required solution. Integrating (9) partially w.r.t. 'v', we get

$\partial z/\partial v = \phi(u), \phi$ being an arbitrary function. ....(10)

Integrating (10) partially w.r.t 'v', $z=\int \phi(u)\,dv+\psi(u)=v\phi(u)+\psi(u)$

or $z=(x+y)\phi(x-y)+\psi(x-y)$, which is the desired solution, ϕ,ψ being arbitrary functions.

Ex. 2. Reduce the equation
$y^2\left(\partial^2 z/\partial x^2\right)-2xy\left(\partial^2 z/\partial x\partial y\right)+x^2\left(\partial^2 z/\partial x^2\right)=\left(y^2/x\right)\left(\partial z/\partial x\right)+\left(x^2/y\right)\left(\partial z/\partial y\right)$ to canonical form and hence solve it.

Sol. Re-writing the given equation, $y^2r-2xys+x^2t-\left(y^2/x\right)p-\left(x^2/y\right)q=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=y^2, S=-2xy, T=x^2$ so that $S^2-4RT=0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$y^2\lambda^2-2xy\lambda+x^2=0$ or $(y\lambda-x)^2=0$ so that $\lambda=x/y, x/y$.

The corresponding characteristic equation is $dy/dx+x/y=0$

or $x\,dx+y\,dy=0$ so that $x^2/2+y^2/2=C_1$

Choose $u=x^2/2+y^2/2$ and $v=x^2/2-y^2/2$, ...(2)

where we have chosen $v=x^2/2-y^2/2$ in such a manner that u and v are independent functions as verified below

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=-2xy\neq 0$$

Now, $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=x\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)$, using (2) ....(3)

$q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}=x\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)$, using (2) ....(4)

$r=\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial}{\partial x}\left\{x\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)\right\}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}+x\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)$, by (3)

$$=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}+x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}+x^2\left(\dfrac{\partial^2 z}{\partial u^2}+2\dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{\partial^2 z}{\partial v^2}\right)$, using (2) ....(5)

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\text{, by (4)}$$

$$=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}+y^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right) \quad \text{....(6)}$$

and $s=\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial x}\left\{y\left(\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}\right)\right\}=y\left\{\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}\right)\dfrac{\partial u}{\partial x}+\dfrac{\partial}{\partial v}\left(\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}\right)\dfrac{\partial v}{\partial x}\right\}$

or $s=xy\left(\partial^2 z/\partial u^2-\partial^2 z/\partial v^2\right)$ ....(7)

Using (3), (4), (5), (6) and (7) in (1) and simplifying, we get

$4x^2y^2\left(\partial^2 z/\partial v^2\right)=0$ so that $\partial^2 z/\partial v^2=0$ ....(8)

which is the required canonical form.

Integrating (8) partially w.r.t. 'v', $\partial z/\partial v=\phi(u)$, ϕ being arbitrary function. ....(9)

Integrating (9) partially w.r.t. 'v', $z=v\phi(u)+\psi(u)$, ψ being arbitrary function.

or $z=\left[\left(x^2-y^2\right)/2\right]\phi\left\{\left(x^2+y^2\right)/2\right\}+\psi\left\{\left(x^2+y^2\right)/2\right\}$, using (2)

or $z=\left(x^2-y^2\right)F\left(x^2+y^2\right)+G\left(x^2+y^2\right)$, F, G being arbitrary functions

Ex. 3. (i) Reduce $r+2xs+x^2t=0$ to canonical form

(ii) Reduce $r-6s+9t+2p+3q-z=0$ to canonical form

(iii) Reduce $r-2s+t+p-q=0$ to canonical form and hence solve it.

Sol. **Hint** (i) Given $r+2xs+x^2t=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f\left(x,y,z,p,q\right)=0$, here $R=1, S=2x$ and $T=x^2$ so that $S^2-4RT=0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$\lambda^2+2\lambda x+x^2=0$ or $\left(\lambda+x^2\right)=0$ so that $\lambda=-x,-x$

The corresponding characteristic equation is $\left(dy/dx\right)-x=0$ or $dy-x\,dx=0$

Integrating, $y-x^2/2=c_1, c_1$ being an arbitrary constant. ....(2)

Choose $u = y - x^2/2$ and $v = x$

where we have chosen $v = x$ in such a manner that u and v are independent functions.

we finally obtain $\partial^2 z/\partial v^2 = \partial z/\partial u$, which is required canonical form.

3. (ii) Hint. Here $\lambda = 3, u = y + 3x$. Choose $v = y$. The canonical form will be

$$\partial^2 z/\partial v^2 = z/9 - (\partial z/\partial u) + (1/3) \times (\partial z/\partial v)$$

3. (iii) Hints. Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v$ solution is $z = \phi(x+y) + e^y \psi(x+y), \phi, \psi$ being arbitrary functions. 31

Ex. 4. Reduce the following to canonical form and hence solve

(i) $x^2 r + 2xy\,s + y^2 t = 0$ (ii) $r - 4s + 4t = 0$ (iii) $x^2 r + 2xys + y^2 t + xyp + y^2 q = 0$

(iv) $2r - 4s + 2t + 3z = 0$

Sol. **Hint**(i) Given $x^2 r + 2xy\,s + y^2 t = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2, S = 2xy$ and $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$x^2\lambda^2 + 2xy\lambda + y^2 = 0$ or $(x\lambda + y)^2 = 0$ giving $\lambda = -y/x, -y/x$

The corresponding characteristic equation is $dy/dx - y/x = 0$

or $(1/y)dy - (1/x)dx = 0$ so that $\log y - \log x = c_1$ or $y/x = c_1$

Choose $u = y/x$ and $v = y$ ....(2)

where we have chosen $v = y$ in such a manner that u and v are independent functions

we finally get as the canonical form $\partial^2 z/\partial v^2 = 0$

Integrating (8) partially w.r.t. $'v'$, $\partial z/\partial v = \phi(u)$

Integrating, partially w.r.t $'v'$, $z = v\phi(u) + \psi(u)$

or $z = y\phi(y/x) + \psi(y/x)$, ϕ, ψ being arbitrary functions.

(ii) Hint. Here $\lambda = 2, u = y + 2x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = 0$ and solution is

$z = y\phi(y + 2x) + \psi(y + 2x)$.

(iii) Hint. Here $\lambda = -y/x, u = y/x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(\partial z/\partial v)$ and solution is $z = \phi(y/x) + e^{-y}\psi(y/x)$

(iv) Hint. Here $\lambda = 1, u = x + y$. Choose $v = y$. The canonical for $\partial^2 z/\partial v^2 = -(3z/2)$ and solution is $z = e^{(i\sqrt{3/2})y}\phi(y+x) + e^{-(i\sqrt{3/2})y}\psi(y+x)$

Category-3: Elliptic PDE: canonical form

Ex. 1. Reduce the following partial differential equations to canonical forms:

(i) $\partial^2 z/\partial x^2 + x^2(\partial^2 z/\partial y^2) = 0$ or $r + x^2 t = 0$ (ii) $y^2(\partial^2 z/\partial y^2) + \partial^2 z/\partial x^2 = 0$

Sol. (i) Re-writing the given equations, we get $r + x^2 t = 0$ ....(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1, S = 0, T = x^2$ so that

$S^2 - 4RT = -4x^2 < 0, x \neq 0$, showing that (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + x^2 = 0$ giving $\lambda = ix, -ix$

The corresponding characteristic equations are given by

$dy/dx + ix = 0$ and $dy/dx - ix = 0$

Integrating, $y + i(x^2/2) = c_1$ and $y - i(x^2/2) = c_2$

Choose $u = y + i(x^2/2) = \alpha + i\beta$ and $v = y - i(x^2/2) = \alpha - i\beta$

where $\alpha = y$ and $\beta = x^2/2$ ....(2)

are now two new independent variables.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial x} = x\dfrac{\partial z}{\partial \beta}$, by (2) ....(3)

$q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial y} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial y} = \dfrac{\partial z}{\partial \alpha}$, by (2) ....(4)

$r = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(x\dfrac{\partial z}{\partial \beta}\right) = \dfrac{\partial z}{\partial \beta} + x\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial \beta}\right)$, by (3)

$= \dfrac{\partial z}{\partial \beta} + x\left[\dfrac{\partial}{\partial \alpha}\left(\dfrac{\partial z}{\partial \beta}\right)\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial}{\partial \beta}\left(\dfrac{\partial z}{\partial \beta}\right)\dfrac{\partial \beta}{\partial x}\right] = \dfrac{\partial z}{\partial \beta} + x^2\dfrac{\partial^2 z}{\partial \beta^2}$ ....(5)

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial \alpha}\left(\dfrac{\partial z}{\partial \alpha}\right) = \dfrac{\partial^2 z}{\partial \alpha^2}$, by (4) ....(6)

Using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2}+x^2\frac{\partial^2 z}{\partial \alpha^2}=0 \text{ or } \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=-\frac{1}{2\beta}\frac{\partial z}{\partial \beta}\text{, as } \beta=\frac{x^2}{2}$$

(ii) Ans. $\partial^2 z/\partial\alpha^2+\partial^2 z/\partial\beta^2=-(1/2\alpha)\times(\partial z/\partial\alpha)$, where $\alpha=y^2/2, \beta=x$

Ex.2. Reduce $y^2\left(\partial^2 z/\partial x^2\right)+x^2\left(\partial^2 z/\partial y^2\right)=0$ to canonical form

Sol. Re-writing the given equation, we get $y^2 r+x^2 t=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=y^2, S=0, T=x^2$ so that $S^2-4RT=-4x^2y^2<0$ for $x\neq 0$, $y\neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$y^2\lambda^2+x^2=0$ or $\lambda^2=-x^2/y^2$ so that $\lambda=ix/y$, $-ix/y$

The corresponding characteristic equations are

$dy/dx+ix/y=0$ and $dy/dx-ix/y=0$

Integrating, $y^2+ix^2=C_1$ and $y^2-ix^2=C_2$

Choose $u=y^2+ix^2=\alpha+i\beta$ and $v=y^2-ix^2=\alpha-i\beta$

where $\alpha=y^2$ and $\beta=x^2$ ....(2)

are now two new independent variables

Now, $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}=2x\frac{\partial z}{\partial \beta}$, by (2) ....(3)

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=2y\frac{\partial z}{\partial \alpha}$, by (2) ....(4)

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right)=2\frac{\partial z}{\partial \beta}+2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)\text{, by (3)}$$

$$=2\frac{\partial z}{\partial \beta}+2x\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}=2\frac{\partial z}{\partial \beta}+4x^2\frac{\partial^2 z}{\partial \beta^2} \quad ....(5)$$

$$\text{and } t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right)=2\frac{\partial z}{\partial \alpha}+2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$=2\frac{\partial z}{\partial \alpha}+2y\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\}=2\frac{\partial z}{\partial \alpha}+4y^2\frac{\partial^2 z}{\partial \alpha^2} \quad ....(6)$$

Using (5) and (6) in (1), the required canonical form is

$$2y^2\frac{\partial z}{\partial \beta}+4x^2y^2\frac{\partial^2 z}{\partial \beta^2}+2x^2\frac{\partial z}{\partial \alpha}+4x^2y^2\frac{\partial^2 z}{\partial \alpha^2}=0 \text{ or } 2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}\right)+\alpha\frac{\partial z}{\partial \beta}+\beta\frac{\partial z}{\partial \alpha}=0$$

or $\dfrac{\partial^2 z}{\partial \alpha^2}+\dfrac{\partial^2 z}{\partial \beta^2}+\dfrac{1}{2}\left(\dfrac{1}{\alpha}\dfrac{\partial z}{\partial \alpha}+\dfrac{1}{\beta}\dfrac{\partial z}{\partial \beta}\right)=0$

Ex. 3. Reduce $\partial^2 z/\partial x^2+y^2\left(\partial^2 z/\partial y^2\right)=y$ to canonical form.

Sol. **Hint.** Re-writing the given equation, we get $r+y^2t-y=0$ ....(1)

Comparing (1) with $Rr+Ss+Tt+f\left(x,y,z,p,q\right)=0$, here $R=1, S=0$ and $T=y^2$ so that $S^2-4RT=-4y^2<0$ for $y\neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2+y^2=0 \Rightarrow \lambda=iy,-iy$

The corresponding characteristic equations are given by

$dy/dx+iy=0$ and $dy/dx-iy=0$

Integrating these, $\log y+ix=c_1$ and $\log y-ix=c_2$

Choose $u=\log y+ix=\alpha+i\beta$ and $v=\log y-ix=\alpha-i\beta$,

where $\alpha=\log y$ and $\beta=x$ ....(2)

are now two new independent variables.

The required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2}+\frac{\partial^2 z}{\partial \alpha^2}-\frac{\partial z}{\partial \alpha}-y=0 \text{ or } \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\frac{\partial z}{\partial \alpha}+e^{\alpha}.$$

PREVIOUS YEARS QUESTIONS

For Answers hints, see examples of respective categories.

Q1.1. Solve the partial differential equation $\dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}+\phi\right)+2x^2y\left(\dfrac{\partial \phi}{\partial x}+\phi\right)=0$

By transforming it to the canonical form. **[8a UPSC CSE 2024]**

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}+\phi\right)+2x^2y\left(\frac{\partial \phi}{\partial x}+\phi\right)=0 \Rightarrow \frac{\partial^2 \phi}{\partial x\partial y}+\frac{\partial \phi}{\partial y}+2x^2y\left(\frac{\partial \phi}{\partial x}+\phi\right)=0$$

$S=1, R=0, T=0; S^2-4RT=1>0; HEPERBOLIC$

Now **Refer Example-4 Category-1 Hyperbolic PDE**

Q1. Reduce the partial differential equation $\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial x}-\frac{\partial z}{dy}\left(1+\frac{1}{x}\right)+\frac{z}{x}=0$ to canonical form. **[8a UPSC CSE 2023]** **Refer Example-3 Category-1 Hyperbolic PDE**

Q2. Reduce the following partial differential equation to a canonical form and hence solve it: $yu_{xx}+(x+y)u_{xy}+xu_{yy}=0$. **[8a UPSC CSE 2022] Refer Example-4 Category-1 Hyperbolic PDE**

Q3. Reduce the following second order partial differential equation to canonical form and find the general solution: $\frac{\partial^2 u}{\partial x^2}-2x\frac{\partial^2 u}{\partial x \partial y}+x^2\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial y}+12x$. **[7c UPSC CSE 2019]**

Take help from Example 3(i) Category-2: parabolic PDE: canonical form

Q4. Reduce the equation $y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x \partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$. to canonical form and hence solve it. **[7a UPSC CSE 2017]**

Take help from Example 2 Category-2: parabolic PDE: canonical form

Q5. Reduce the second-order partial differential equation

$x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$ into canonical form. Hence, find its general solution.

[8a UPSC CSE 2015] **Take help from Example 4 Category-2: parabolic PDE: canonical form**

Q6. Reduce the equation $\frac{\partial^2 z}{\partial x^2}=x^2\frac{\partial^2 z}{\partial y^2}$ to canonical form. **[6a UPSC CSE 2014]**

Refer Example-6 Category-1 Hyperbolic PDE

Q7. Reduce the equation $y\frac{\partial^2 z}{\partial x^2}+(x+y)\frac{\partial^2 z}{\partial x \partial y}+x\frac{\partial^2 z}{\partial y^2}=0$ to its canonical form when $x \neq y$. **[5b UPSC CSE 2013]** **Refer Example-4 Category-1 Hyperbolic PDE**

Q8. Rewrite the hyperbolic equation $x^2u_{xx}-y^2u_{yy}=0\left(x>0, y>0\right)$ in canonical form.

Refer Example-11 Category-1 Hyperbolic PDE **[(6c) 2013 IFoS]**

Q9. Reduce the equation $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=0$ to its canonical form and solve. **[(5a) 2011 IFoS]**

Take help from Example 1 Category-2: parabolic PDE: canonical form

Q10. Reduce the following 2nd order partial differential equation into canonical form and find its general solution $xu_{xx} + 2x^2 u_{xy} - u_x = 0$. **[6b UPSC CSE 2010]**

Take help from Examples Category-1: Hyperbolic PDE: canonical form

CHAPTER: INITIAL BOUNDARY VALUE PROBLEMS(IBVPs)

Demand of exam UPSC CSE/IFoS
Example 1. Solve the following

initial boundary value problem $IBVP$ of heat conduction given by

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}; \qquad 0\le x\le l;\ \ t>0$$

$$B.C.:\ \ u(0,t)=0=u(l,t)\ ;\ t>0$$

$$I.C.:\ \ u(x,0)=f(x) \quad ;\ \ 0\le x\le l$$

Example 2. Sometimes the problem is stated like below and then we must know what is the PDE representing Heat equation and how to use given conditions of physics in terms of mathematical problem. For such questions, there are few selected equations, we need to remember like Heat equation, Wave equation, Laplace etc. About physics terminology, no need to worry much, simply try to read examples in this document, after few, you'll have idea what are those just few terms to be take care of.

(a) If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $f(x)$ in the bar, then find temperature at subsequent time t

The initial boundary value problem $IBVP$ of heat conduction is given by

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}; \qquad 0\le x\le l;\ \ t>0 \qquad .(1)$$

$$B.C.:\ \ u(0,t)=0=u(l,t)\ ;\ t>0 \qquad .(2)$$

$$I.C.:\ \ u(x,0)=f(x) \quad ;\ \ 0\le x\le l \qquad .(3)$$

(b) A uniform rod of length l whose surface is thermally insulated is initially at temperature α. At $t=0$, one end is suddenly cooled to $0^\circ C$ and subsequently maintained at this Temperature other end remain thermally insulated Find temperature distribution.

The $IBVP$ is given by $\dfrac{\partial u}{\partial t}=k\dfrac{\partial^2 u}{\partial x^2}; \quad 0\le x\le l\ ; \qquad t>0 \qquad (1)$

$$BC:\quad u(0,t)=0,\quad \frac{\partial u(t,t)}{\partial x}=0; \qquad t>0\ ..(2) \quad IC: u(x,0)=\alpha; 0\le x\le l\ ..(3)$$

Example 3. A thin rectangular homogeneous thermally conducting plate lies. In $x-y$ plane $0\le x=a, 0\le y\le b.$ The edge $y=0$ is held at the temperature $Tx(x-a)$ where T is Constant while the remaining edges are held at 0°. the other faces are insulated and no Internal sources and sinks are present find temperature distribution

Note: The given problem is of Dirichlet type and can be defined as

$$u_{xx}+u_{yy}=0 \qquad (1)$$

$$BC:\quad u(0,y)=0=u(a,y)=u(x,b),\quad u(x,0)=Tx(x-a) \qquad (2)$$

Example 4. The points of trisection of a string are pulled aside through h on opposite sides of the Position of equilibrium and the string is released from rest. Derive an expression for the string At any subsequent time and show that the middle point of the middle point of the string always Remains at rest

i.e. Given $IBVP$ can be defined as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $\quad 0 < x < 3l$

$BC: \quad u(0,t) = 0 = u(3l,t); \; IC: \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = f(x)$

Mentor's words: Now we'll prepare for this chapter in a Systemtic Way.

Step-I: Solving PDE by method of separation of variables.

Step-II: Combining the result of step-I with given boundary conditions: eigenvalues and eigen functions of the Boundary value Problem.

As we have the general solution of given problem addressed with PDE and Boundary conditions. By the Principle of Super position, we write this solution in infinite series summation form.

Step-III: Now on using the given Initial Condition in this solution, we get a Fourier Series. So at this third step basically, we try to find out Fourier Coefficients.

Let's explain now:

Step-I: Method of separation of variables

Example: Solve the following PDE using separation of variable method

(i) $\frac{\partial u}{\partial t} = 4\frac{\partial u}{\partial x}, \qquad u(x,0) = 8e^{-3x}$ **(ii)** $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

- Let's consider a PDE $\frac{\partial u}{\partial t} = 4\frac{\partial u}{\partial x}$ .....(1)

  Here u is the dependent variable on two independent variables x & t
- Let $u(x,t) = \mathrm{X}(x).\mathrm{T}(t)$ is solution of given PDE.

  i.e., we suppose u as product to two functions X & T; which are variable separated
- $\because \; u = \mathrm{X.T} \Rightarrow \frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = \mathrm{X.T'}$ .....(2)

  {$\because$ X is function of x alone $\therefore \frac{\partial}{\partial x}(X) = \frac{d}{dx}(X) = X'$} and $\mathrm{T'} = \frac{dT}{dt} = \frac{\partial}{\partial t}(\mathrm{T})$

  Now, using (2) in (1) we get $XT' = 4X'T \Rightarrow \frac{\mathrm{X'}}{\mathrm{X}} = \frac{\mathrm{T'}}{4\mathrm{T}}$

  Note that LHS & RHS both are only one variable functions, we can take them equal to some separation constant λ

  i.e., we have, $\frac{\mathrm{X'}}{\mathrm{X}} = \frac{\mathrm{T'}}{4\mathrm{T}} = \lambda$ .....(3)
- Now, we solve separately above ODEs'

  $\frac{\mathrm{X'}}{\mathrm{X}} = \lambda \Rightarrow \log \mathrm{X} = \lambda x + \log c_1 \Rightarrow X = c_1 e^{\lambda x}$ .....(4)

Similarly, $\frac{T'}{4T}=\lambda \Rightarrow \frac{1}{4}\log \mathrm{T}=\lambda t+c_2 \Rightarrow \log \mathrm{T}=4\lambda t+c_2 \Rightarrow T=De^{4\lambda t}$; D is integration constant

So, the complete solution of given PDE is, $u(x,t)=ce^{\lambda x}De^{4\lambda t} \Rightarrow u(x,t)=cDe^{(x+4t)\lambda}$ .....(5)

$\because$ Given condition is $u(x,0)=8e^{-3x}$ $\therefore$ (5) given $8.e^{-3x}=CDe^{\lambda x} \Rightarrow$ CD = 8 & λ = −3

Required solution of given problem is, $u(x,t)=8e^{-3(x+4t)}$

(II) $\therefore$ u is the dependent variable on two independent variables x & y i.e., $u(x, y)$

Let $u(x,y)$ = X. Y

$\therefore \frac{\partial u}{\partial x}=X'Y, \frac{\partial^2 u}{\partial x^2}=X''Y, \frac{\partial u}{\partial y}=XY'$

$\therefore$ Using these in given PDE, we have

$X''Y-2X'Y+XY'=0 \Rightarrow \frac{X''}{X}-\frac{2X'}{X}=\frac{-Y'}{Y}$

Let's take both LHS & RHS of above equation as some separation constant λ.

$\frac{X''}{X}-\frac{2X'}{X}=\lambda$, $\qquad \frac{-Y'}{Y}=\lambda$

↓Solving ODE $\qquad$ ⇓

$\frac{d^2X}{dx^2}-2\frac{dX}{dx}-\lambda X=0 \qquad y=c_3e^{-\lambda y}$ .....(2)

$m^2-2m-\lambda=0$

$m=\frac{2\pm\sqrt{4+4\lambda}}{2}$

$m=1+\sqrt{1+\lambda}, \qquad m=1-\sqrt{1+\lambda}$

$\because$ PI = 0

$\therefore$ X = C.F = $c_1e^{(1+\sqrt{1+\lambda})x}+c_2e^{(1-\sqrt{1+\lambda})x}$ .....(1)

$\therefore$ The complete solution of given PDE is, $u(x,y)=\left\{c_1e^{(1+\sqrt{1+\lambda})x}+c_2e^{(1-\sqrt{1+\lambda})x}\right\}c_3e^{-\lambda y}$

(III) $\frac{\partial u}{\partial x}=u+\frac{2\partial u}{\partial t}, \quad u(x,0)=6e^{-3x}$ .....(1)

Here $u(x,t)$ is of function of x & t.

Let $u(x,t)=\mathrm{X.T}$ and by following previous procedure $u(x,t)$ =CD. $e^{(1+\lambda)x}e^{\lambda\frac{t}{2}}$

Also, given $u(x,0)=6e^{-3x}$ $\therefore$ $6e^{-3x}=\mathrm{CD}e^{(1+\lambda)x}e^{\circ}$

$\therefore$ Required solution of given problem is $u(x,t)=6e^{-3x-t}$

Step-II: Eigenvalues & Eigen functions: for a given PDE with given boundary conditions & initial condition: Values of λ for which given PDE has non-terminal solution is called eigenvalue & this non-trivial solution is called reign function of given problem.

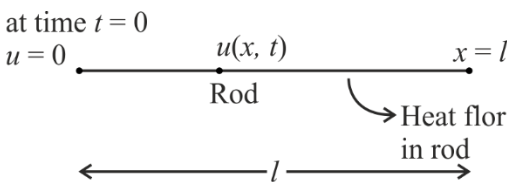
Let's say the given problem is to solve **following Initial Boundary Value Problem(IBVP)**

$\frac{\partial u}{\partial t}=k.\frac{\partial^2 u}{\partial x^2}, \qquad 0\le x\le l \qquad ; \; t>0$; t is time.

$BC: u(0,t) = 0 = u(l,t) = 0, t > 0$

$I.C: u(x,0) = f(x); \quad 0 \le x \le l$

Let's try to solve by variable separable method



$u(x,t)$ be the Temperature at a point x & time t

Let $u(x,t) = X(x).T(t)$

$\therefore \dfrac{\partial u}{\partial t} = XT', \qquad \dfrac{\partial^2 u}{\partial x^2} = X''T$

$\therefore$ From given PDE, $XT' = k.X''T \Rightarrow \dfrac{T'}{kT} = \dfrac{X''}{X} = \lambda$ .....(1) ; where λ is separation constant.

So, we have $\dfrac{X''}{X} = \lambda$ and $\dfrac{T}{kT} = \lambda$

$\Rightarrow X'' - \lambda X' = 0 \qquad \Rightarrow T' - k\lambda T = 0 \Rightarrow T = c.e^{\lambda kt}$

$\therefore$ Auxiliary equation $m^2 - \lambda = 0$

$m = \pm\sqrt{\lambda}$

$\therefore$ C.F = $c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$\therefore X = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$\therefore u(x,t) = \left(c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}\right) c e^{\lambda kt}$ .....(2)

- Now, using boundary condition

$\therefore u(0,t) = 0 \Rightarrow (c_1 + c_2)e^{\lambda kt} = 0 \qquad \because e^{\lambda kt} \neq 0$ for any for value of t

$\therefore$ We must have $c_1 + c_2 = 0$ .....(2)

$u(l,t) = 0 \Rightarrow \left(c_1 e^{\sqrt{\lambda}l} + c_2 e^{-\sqrt{\lambda}l}\right) e^{\lambda kt} = 0 \qquad \{\because e^{\lambda kt} \neq 0\}$

$\therefore \left(c_1 e^{\sqrt{\lambda}l} + c_2 e^{-\sqrt{\lambda}l}\right) = 0$ .....(3)

Note: But what happens, if we categories λ as +ve, –ve, zero.

Case (1) Let λ is positive : Let $\lambda = \mu^2$

$\therefore$ We have, $\dfrac{X''}{X} = \mu^2 \Rightarrow X'' - \mu^2 X = 0 \Rightarrow X = c_1 e^{\mu x} + c_2 e^{-\mu x}$

Now, using Boundary condition; we have,

$$\left.\begin{aligned} u(0,t) = 0 &\Rightarrow c_1 + c_2 = 0 \\ u(l,t) = 0 &\Rightarrow c_1 e^{\mu l} + c_2 e^{-\mu l} = 0 \end{aligned}\right\} \therefore \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

i.e., X = 0 $\therefore\ u(x,t) = X.T = 0.ce^{\mu^2 kT} = 0$ **(TRIVIAL SOLUTION)**

i.e., for positive λ ; we cannot have non-trivial solution

$\therefore$ given problem has no positive eigenvalue.

Case (II) Let λ is negative; Let $\lambda = -\mu^2$

$\therefore$ We have, $\frac{X''}{X} = -\mu^2$, $\frac{T''}{kT} = -\mu^2$

$X'' + \mu^2 X = 0$ $\log T = -\mu^2 kt + \log c$

$\therefore$ C.F = $c_1 \cos\mu x + c_2 \sin\mu x$ $T = ce^{-\mu^2 kT}$

$$\therefore\ u(x,t) = (c_1 \cos\mu x + c_2 \sin\mu x)ce^{-\mu^2 kT}$$

Now, using B.C,

$u(0,t) = 0 \Rightarrow (c_1 \cos 0 + c_2 \sin 0)ce^{-\mu^2 kt} = 0 \Rightarrow c_1 + c_2 \times 0 = 0 \Rightarrow c_1 = 0$ $\left\{\because ce^{-\mu^2 kt} \neq 0\right\}$

$u(l,t) = 0 \Rightarrow (c_1 \cos\mu l + c_2 \sin\mu l)ce^{-\mu^2 kt} = 0 \Rightarrow (0 + c_2 \sin\mu l)ce^{-\mu^2 kt} = 0$

$\Rightarrow c_2 \sin\mu l = 0$ $\left\{\because ce^{-\mu^2 kt}\right\}$

Either c_2 = 0 or $\sin\mu l = 0$

$\downarrow$ $\downarrow$

i.e., we have, $c_1 = 0, c_2 = 0$ $\mu l = n\pi$; $n \in \mathbf{Z}$

So, we have $u(x,0) = 0.T(t) = 0$ $\therefore \mu = \frac{n\pi}{l}$

(Trivial solution)

So, we don't consider this possibility of taking c_2 = 0

Therefore, for non-trivial solution; to get eigenvalue/eigen function); we take $\mu = \frac{n\pi}{l}$

$\therefore\ u(x,t) = A\left(\sin\frac{n\pi}{l}x\right).e^{-\frac{n^2\pi^2}{l^2}kt}$; where A is some arbitrary constant.

Here, $\mu = \frac{n\pi}{l}$ i.e., $\mu = \frac{0.\pi}{l}, 1\frac{\pi}{l}, \frac{2.\pi}{l}$ where each value is called an eigenvalue and for each eigenvalue, $u(x,t)$ is called an eigen function of given problem.

Case (III): When $\lambda = 0$ i.e., $\mu = 0$

$\therefore$ we have, $\frac{T'}{kT} = \frac{X''}{X} = 0$

i.e., $\frac{X''}{X} = 0 \Rightarrow X = Ax + B$, When A, B are arbitrary constants.

$\frac{T'}{KT} = 0$ $\Rightarrow$ T = C

$\therefore u(x,t)=(\mathrm{A}x+\mathrm{B}).\mathrm{C}$

Using boundary condition $u(0,t)=0 \Rightarrow (\mathrm{A}.0+\mathrm{B}).C=0 \Rightarrow$ B.C = 0

$u(l,t)=0 \Rightarrow (\mathrm{A}l+\mathrm{B})\mathrm{C}=0 \Rightarrow \mathrm{A}l.\mathrm{C}+\mathrm{B}.\mathrm{C}=0 \Rightarrow \mathrm{A}l.\mathrm{C}=0 \Rightarrow$ A = 0 or C = 0

i.e., In this case $u(x, t)$ = C; where C is some arbitrary constant

But if $C \neq 0$ then $u(x, t)$ will not satisfy boundary condition

$\therefore\ u(x,t)=0 \therefore$ **Trivial solution** $\therefore\ \lambda$ = 0 is not an eigenvalue.

Observation: If we have been given an IBVP (Initial Boundary value Problem) as $\dfrac{\partial u}{\partial t}=k\dfrac{\partial^2 u}{\partial x^2}$ with

B.C : $u(0,t)=0$, $u(l,t)=0$ then the solution is of the form $u(x,t) \Rightarrow \left(A\sin\dfrac{n\pi}{l}x\right)e^{-\frac{n^2\pi^2}{l^2}kt}$.

$\because$ equation is linear (given PDE), so we can apply the principle of superposition and so the solution of given problem is written as,

$$u(x,t)=\sum_{n=1}^{\infty}A_n\left(\sin\frac{n\pi}{l}x\right)e^{-\frac{n^2\pi^2}{l^2}kt} \qquad \text{......(4)}$$

Now, if we apply the given initial condition $u(x,\ 0)\ = f(x)$ on above $u(x,\ \mathrm{t})$

So, we get,

$f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\dfrac{n\pi}{l}x\right).e^0=\sum_{n=1}^{\infty}A_n\sin\left(\dfrac{n\pi}{l}x\right)$. Which is a Fourier series

Where, $A_n=\dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi}{l}xdx$ .....(5) ; Fourier coefficients

$\therefore$ (4) & (5) together give the required solution of given IBVP.

Step-III: Fourier Series: let us assume that $f(x)$ is a periodic function of period 2π, let $x\in[-\pi,\pi]$ and is integrable. Let us further assume that $f(x)$ can be represented by a trigonometric series

$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ....(1)then the Fourier coefficients are determined by

Hint: Integrating (1) w.r.t. x, we get a_0; Multiplying by $cosnx$ in (1) and then integrating, we get a_n

Multiplying by $\sin nx$ in (1) and then integrating, we get b_n

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

Exam point-1 for period 2L, i.e. $x\in[-L,L]$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

Exam point-2 for period L, i.e. $x\in[0,L]$

$$a_0=\frac{1}{L}\int_{0}^{L}f(x)dx,\quad a_n=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

So, how this will be used in our chapter's demand?
Ans. Let's say after step-II, For some given IBVP, for $0 < x < l$,

we found $u(x,t) = B_1 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}}$ (1)

Applying Initial Condition IC, we get

$$x(l-x) = B_1 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \qquad (2)$$

Which is half –range Fourier cosine series. Therefore

$$B_1 = \frac{1}{l}\int_0^l x(l-x)\,dx$$

$$B_1 = \frac{1}{l}\left(\frac{l^3}{2} - \frac{l^3}{3}\right) = \frac{l^2}{6}$$

And $A_n = \frac{2}{l}\int_0^l x(l-x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2l^2}{n^2\pi^2}\left[1+(-1)^n\right]$

$$A_n = \begin{cases} -\frac{4l^2}{n^2\pi^2} \; ; & n \text{ is even} \\ 0 \; ; & n \text{ is odd} \end{cases}$$

Using in Eq.(1) we get $u(x,t) = \frac{l^2}{6} + \sum_{n=1}^{\infty} -\frac{l^2}{n^2\pi^2}\cos\left(\frac{2n\pi x}{l}\right) e^{-\frac{4n^2\pi^2 kt}{l^2}}$

Mentor's advice: Although we may go for explaining different kinds of PDEs with different kind of Boundary conditions like above but that will not be a productive process to address the demand of exam UPSC CSE/IFoS. So, what should be the Right way to Prepare this topic!

In this document, all kinds of problems are categorically explained through good examples. Aspirants are suggested to try those examples and if they feel like why the solution is written in that particular format, then they may refer above discussion with given IBVP's scenario logically.

Example1: If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $f(x)$ in the bar, then find temperature at subsequent time t

Solution:- The initial boundary value problem $IBVP$ of heat conduction is given by

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}; \qquad 0\le x\le l;\ \ t>0 \qquad .(1)$$

$$B.C.:\ \ u(0,t)=0=u(l,t)\ ;\ t>0 \qquad .(2)$$

$$I.C.:\ \ u(x,0)=f(x)\quad ;\ \ 0\le x\le l \qquad .(3)$$

Let $u=X(x)T(t)$ be variable separable solution of Eq.(1).Then Eq.(1) $\Rightarrow \dfrac{T'}{kT}=\dfrac{X''}{X}$.(4)

Eq.(4) holds good if $\dfrac{T'}{kT}=\dfrac{X''}{X}=$ Constant. Since B.C. are periodic and homogeneous in x

Therefore periodic solution for Eq.(1) exists only when we consider

$$\frac{X''}{X}=-\lambda^2 \qquad .(5)$$

$$\Rightarrow \qquad X=C_1\cos\lambda x+C_2\sin\lambda x \qquad .(6)$$

Using Eq.(5) in Eq.(4),we get

$$\frac{T'}{kT}=-\lambda^2 \Rightarrow \qquad T=C_3e^{-\lambda^2kt} \qquad .(7)$$

Hence complete solution of Eq.(1) is $u=(A\cos\lambda x+B\sin\lambda x)e^{-\lambda^2kt}$.(8)

Applying B.C. Eq.(2), we get, $0=(A\cos 0+B\sin 0)e^{-\lambda^2kt}$

And $\quad 0=(A\cos\lambda l+B\sin\lambda l)e^{-\lambda^2kt}\Rightarrow A=0,\qquad A=\cos\lambda l+B\sin\lambda l=0$

i.e., $\qquad A=0,\qquad B\sin\lambda l=0$

For non-trivial solution ,we assume that $B\neq 0$ but $\sin\lambda l=0,\ i.e.,\ \lambda=\dfrac{n\pi}{l}; n\ is\ Integer$

Hence the solution is found in the form

$$u(x,t)=B\sin\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2}{l^2}Kt} \qquad .(9)$$

Since heat conduction equation is linear, therefore its most general solution is obtained by using **The principle of superposition**. Thus

$$u(x.t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2Kt}{l^2}} \qquad .(10)$$

Using $I.C.$, we get $f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\dfrac{n\pi x}{l}\right)$.(11)

Which is half range Fourier sine series. Therefore

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad .(12)$$

Eq.(10) along with Eq.(12) represents the required solution of given $IBVP$.

Example2:- A rod of length l with insulated sides is initially at uniform temperature u_0. Its ends are Suddenly cooled at $0^\circ C$ and are kept at that temperature. Find the temperature function $u(x,t)$.

Solution:- The $IBVP$ of heat conduction is given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}; \qquad 0 \le x \le l; \quad t > 0 \qquad .(1)$$

$$BC: \quad u(0,t) = u(l,t) = 0; \quad t > 0 \qquad .(2)$$

$$IC: \quad u(x,0) = u_0; \qquad 0 \le x \le l \qquad .(3)$$

Proceeding on the same line as in example 1, we get

$$u(x,t) = \sum B_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l}} \qquad .(10)$$

Using IC, we get $u_0 = \sum B_n \sin\left(\frac{n\pi x}{l}\right)$

Which is half-range Fourier sine series.

Therefore, $B_n = \frac{2}{l}\int_0^l u_0 \sin\left(\frac{n\pi x}{l}\right)dx = \frac{2u_0}{l}.\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)_0^l = \frac{2u_0}{n\pi}(1-\cos n\pi) = \frac{2u_0}{n\pi}\left(1-(-1)^n\right)$

$$B_n = \begin{cases} \frac{4u_0}{n\pi}; & n \text{ is odd} \\ 0 \;; & n \text{ is even} \end{cases} \qquad \text{Or} \quad B_{2n+1} = \frac{4u_0}{(2n+1)\pi}; \quad n = 0,1,2,\ldots \qquad .(12)$$

Using in Eq. (10) , we get $u(x,t) = \sum_{n=0}^{\infty} \frac{4u_0}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi x}{l}\right) e^{-\frac{(2n+1)^2\pi^2 kt}{l^2}}$.(13)

Note: Above two examples are in their explained and standard answer formats as to required to write an answer in exam. So now following examples are being given for practice and in solutions, direct hints or only crucial steps are given.

Example 3:- A homogeneous rod of length l has its ends kept at zero temperature and the Temperature is initially is $u(x,0) = \begin{cases} x & ; \; 0 \le x \le l/2 \\ l-x & ; \; l/2 \le x \le l \end{cases}$. Find temperature distribution $u(x,t)$.

Solution:- The $IBVP$ of heat conduction is given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}; \qquad 0 \le x \le l \qquad .(1)$$

$$BC: \quad u(0,t) = 0 = u(l,t); \quad t > 0 \qquad .(2)$$

$$IC: \quad u(x,0) = f(x) = \begin{cases} x & ; \; 0 \le x \le l/2 \\ l-x & ; \; l/2 \le x \le l \end{cases} \qquad .(3)$$

Proceeding on the same line as in example (1), we get $u(x,t) = \sum_{n=0}^{\infty} B_n \left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}}$.(10)

Applying IC, we get $f(x)=\sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$.(11)

Which is half-range Fourier sign series, Therefore, $B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$=\frac{2}{l}\left[\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx+\int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]=\frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)=\begin{cases}\frac{4l}{n^2\pi^2}(-1)^{\frac{n-1}{2}} & ; \quad n \text{ is odd}\\ 0 & ; \quad n \text{ is even}\end{cases}$$

$$\Rightarrow \qquad B_{2n+1}=\frac{4l(-1)^n}{\left((2n+1)\pi\right)^2}\ ; \quad B_{2n}=0\ ; \quad n=0,1,2,\ldots \qquad .(12)$$

Using in Eq.(10),we get $u(x,t)=\sum_{n=0}^{\infty}\frac{4l(-1)^n}{(2n+1)^2\pi^2}\sin\left(\frac{(2n+1)\pi x}{l}\right)e^{-\frac{(2n+1)^2\pi^2}{l^2}kt}$. (13)

Example4: Determine the solution of one dimensional heat equation $\frac{\partial u}{\partial t}=K\frac{\partial^2 u}{\partial x^2}; -a\le x\le a$, $B.C.$: $u(\pm a,t)=0$, with IC, $u(x,0)=x$; $t=0; -a<x<a$.

Solution. Proceeding as in example (1), we get $u(x,t)=(A\cos\lambda x+B\sin\lambda x)e^{-\lambda^2 Kt}$.(8)

Applying $B.C.$: $u(\pm a,t)=0$, we get

$0=A\cos\lambda a\pm B\sin(\lambda a)\Rightarrow A=0, B\neq 0$ but $\sin(\lambda a)=0$, *i.e.*, $\lambda=\left(\frac{n\pi}{a}\right)$

Hence the solution is found in the form $u(x,t)=B\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n^2\pi^2 Kt}{a^2}}$ (9)

Since heat conduction equation is linear, therefore its most general solution is obtained by Using principle of superposition .Thus $u(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n^2\pi^2 Kt}{a^2}}$ (10)

Applying IC, we get $x=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{a}\right); \quad -a\le x\le a$ (11)

Which is full range Fourier sine series, therefore $B_n=\frac{1}{a}\int_{-a}^{a}x\sin\left(\frac{n\pi x}{a}\right)dx$

$$B_n=\frac{1}{a}\int_0^a x\sin\left(\frac{n\pi x}{a}\right)dx=\frac{2a(-1)^{n+1}}{n\pi} \qquad (12)$$

*Note: for $-a<x<a$, Fourier coefficients are calculated like above.

Using in Eq.(10),we get, $u(x,t)=\sum_{n=1}^{\infty}\frac{2a(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n^2\pi^2 Kt}{a^2}}$ (13)

Exam point: <u>The Boundary Conditions are **Non-homogeneous and non-periodic/periodic,** therefore both solution corresponding to zero and Negative Value of separation constants: Examples</u>

Example5:- Solve the $IBVP$

(a) $$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2};\quad 0\le x\le 1;\qquad t>0$$

$$BC:\quad u(0,t)=2,\qquad u(1,t)=3$$

$$IC:\quad u(x,0)=x(1-x)$$

(b) $$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2};\quad 0\le x$$

$$BC:\quad u(0,t)=5=u(1,t)$$

$$BC:\quad u(x,0)=x$$

Solution.(a) We have

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2};\quad 0\le x\le 1;\qquad t>0 \tag{1}$$

$$BC:\quad u(0,t)=2,\qquad u(1,t)=3;\qquad t>0 \tag{2}$$

$$IC:\quad u(x,0)=x(1-x);\qquad \le 0x\le 1 \tag{3}$$

Let $u=X(x)T(t)$ be variables separable solution of Eq.(1). Then Eq.(1) gives

$$\frac{T'}{kT}=\frac{X''}{X} \tag{4}$$

Eq. (4) holds good if each side is equal to same separation constant. Since $B.C.$ are Non-homogeneous and non-periodic therefore both solution corresponding to zero and Negative Value of separation constants. Constitute the general solution.

Let $\frac{X''}{X}=0$ & $\frac{T'}{kT}=0$. Then On solving , we get $X=C_1x+C_2$, $T=C_3$

Hence $$u=(A_1x+B_1)$$

Applying BC, we get

$$2=A_1.0+B_1 \text{ and } 3=A_1(1)+B_1$$

$\Rightarrow$ $$B_1=2 \text{ and } A_1=1$$

Hence $$u=x+2 \tag{6}$$

Let $\frac{X''}{X}=-\lambda^2$. Then Eq.(4) gives $\frac{T'}{kT}=-\lambda^2$. On solving ,we get

$$X=(C_4\cos\lambda x+C_5\sin\lambda x),\quad T=C_6e^{-\lambda^2kt}$$

Hence $$u=(A_2\cos\lambda x+B_2\sin\lambda x)e^{-\lambda^2kt} \tag{7}$$

The general solution of given $IBVP$ is given by

$$u = x + 2 + \left(A_2 \cos\lambda x + B_2 \sin\lambda x\right)e^{-\lambda^2 kt} \qquad (8)$$

Applying BC in Eq. (8) , we get

And
$$\left.\begin{aligned} 2 &= 2 + \left(A_2 + B_2.0\right)e^{-\lambda^2 kt} \\ 3 &= 1 + 2 + \left(A_2 \cos\lambda + B_2 \sin\lambda\right) \end{aligned}\right] \qquad (9)$$

$\Rightarrow$ $\qquad A_2 = 0, A_2 \cos\lambda + B_2 \sin\lambda = 0$

For non-trivial solution

$$A_2 = 0,\ \ B_2 \neq 0,\ \sin\lambda = 0,\ \ i.e.,\ \lambda = n\pi;\ \ n \in I$$

Hence solution is found in the form

$$u = x + 2 + B_2 \sin\left(nx\pi\right)e^{-n^2\pi^2 kt} \qquad (10)$$

Since Eq. (10) is linear, therefore its most general solution is obtained by using principle Of superposition. Thus, $u = x + 2 + \sum_{n=1}^{\infty} B_n \sin\left(nx\pi\right)e^{-n^2 kt\pi^2}$ (11)

Applying IC, we get

$$x\left(1-x\right) = x + 2 + \sum_{n=1}^{\infty} B_n \sin\left(n\pi x\right)$$

$$-2 - x^2 = \sum_{n=1}^{\infty} B_n \sin\left(n\pi x\right) \qquad (12)$$

Which is half –range Fourier sine series. Therefore

$$B_n \frac{2}{1}\int_0^1 -\left(2 + x^2\right)\sin\left(n\pi x\right)dx$$

$$B_n = \frac{6\left(-1\right)^n}{n\pi} - \frac{4}{n\pi} - \frac{4}{\left(n\pi\right)^3}\left(\left(-1\right)^n - 1\right)$$

Using in Eq. (11) we get

$$u\left(x,t\right) = x + 2 + \sum_{n=1}^{\infty}\left(\frac{6\left(-1\right)^n}{n\pi} - \frac{4}{n\pi} - \frac{4\left(\left(-1\right)^n - 1\right)}{\left(n\pi\right)^3}\right)\sin\left(n\pi x\right)e^{-n^2\pi^2 kt} \qquad (13)$$

(b)Here $B.C.$ are non-homogeneous but periodic . therefore both solution corresponding to separation constants zero and $-ve$, constitute the general solution .

Let $\frac{X''}{X} = 0$. Then Eq. (4) gives $\frac{T'}{kT} = 0$. On solving, we get

$$X = C_1 x + C_2\,,\ \ T = C_3$$

Hence $\qquad u = \left(A_1 x + B_1\right) \qquad .(5)$

Applying BC, we get

$$5 = A_1.0 + B_1 \text{ and } \ \ 5 = A_1 + B_1$$

$\Rightarrow$ $\qquad B_1 = 5 \qquad$ and $\qquad A_1 = 0$

Hence $\qquad u = 5 \qquad .(6)$

Let $\frac{X''}{X}=-\lambda^2$. Then Eq.(4) gives $\frac{T'}{kT}=-\lambda^2$.

On solving, we get

$$X=\left(c_4\cos\lambda x+C_5\sin\lambda x\right).\quad T=C_6e^{-\lambda^2kt}$$

Hence $\quad u=\left(A_2\cos\lambda x+B_2\sin\lambda x\right)e^{-\lambda^2kt}$.(7)

The general solution of $IVBP$ is given by

$$u=5+\left(A_2\cos\lambda x+B_2\sin\lambda x\right)e^{-\lambda^2kt} \qquad .(8)$$

Applying BC in Eq.(8), we get

$$\left.\begin{aligned}&5=5+\left(A_2+B_2.0\right)e^{-\lambda^2kt}\\ &5=5+\left(A_2\cos\lambda+B_2\sin\lambda\right)e^{-\lambda^2kt}\end{aligned}\right] \qquad .(9)$$

and

$\Rightarrow \qquad A_2=0, B_2\sin(\lambda)=0$

For non – trivial solution, we set $\quad B_2\neq 0,\ \sin\lambda=0$

i.e., $\qquad \lambda=n\pi;\ n\in I$

Hence solution is found in the form

$$u(x,t)=5+B_2\sin(n\pi x)e^{-n^2\pi^2kt} \qquad .(10)$$

Now proceed, apply super position and then by taking initial conditions and then Fourier like part (a).

Exam point: <u>BC are of **Neumann type**, therefore both solution corresponding to zero and –ve separation constants: Examples</u>

***Example6*:- (a)** Solve the following $IBVP$

$$\frac{\partial u}{\partial t}=k\frac{\partial^2u}{\partial x^2};\quad 0\le x\le l;\quad t>0 \qquad (1)$$

$$BC:\quad u_x(0,t)=0=u_x(l,t) \qquad (2)$$

$$IC:\quad u(x,0)=\sin\left(\frac{x\pi}{l}\right) \qquad (3)$$

(b) Find the temperature function when both ends of a rod of length l are kept insulated And initial temperature is $x(l-x);\ t>0$

***Solution*. (a)** Let $u=X(x)T(t)$ be variable separable solution of given $IBVP$. then

$$\frac{X''}{X}=\frac{T'}{kT} \qquad (4)$$

Eq.(4) holds good if each side is equal to same separation constant . since BC are of Neumann type, therefore both solution corresponding to zero and –ve separation constants Constitute the general solution

Let $\frac{X''}{X}=0$. Then Eq.(4) gives $\frac{T'}{kT}=0$ on solving, we get $X=C_1x+C_2,\ T=C_3$.

Hence $\qquad u=\left(A_1x+B_1\right)$ (5)

Applying BC, we get $\qquad 0=A_1$

Hence $\qquad u=B_1$ (6)

Let $\dfrac{X''}{X}=-\lambda^2$. Then Eq. (4) gives $\dfrac{T'}{kT}=-\lambda^2$

On solving ,we get $\qquad X=\left(C_4\cos\lambda x+C_5\sin\lambda x\right), T=C_6e^{-\lambda^2kt}$

Hence $\qquad u=\left(A_2\cos\lambda x+B_2\sin\lambda x\right)e^{-\lambda^2kt}$ (7)

The general solution of $IBVP$ is given by

$$u(x,t)=B_1+\left(A_2\cos\lambda x+B_2\sin\lambda x\right)e^{-\lambda^2kt} \tag{8}$$

Applying BC, we get

$$0=B_2,\quad 0=\left(-A_2\lambda\sin(\lambda l)+B_2\lambda\cos\lambda l\right)$$

$$\Rightarrow \qquad B_2=0,\quad A_2\lambda\sin\lambda l=0 \tag{9}$$

For non – trivial solution , we set $\quad A_2\neq 0,\ \sin\lambda l=0$

i.e., $\qquad \lambda=\dfrac{n\pi}{l};\qquad n\in I.$

Hence solution is found in the form

$$u(x,t)=B_1+A_2\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2kt}{l^2}} \tag{10}$$

Since Eq.(1) is linear , therefore its most general solution of Eq.(1) is obtained by using principle of superposition .thus

$$u(x,t)=B_1+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2kt}{l^2}} \tag{11}$$

Applying IC, we get

$$\sin\left(\frac{x\pi}{l}\right)=B_1+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2kt}{l^2}} \tag{12}$$

Which is half -range Fourier cosine series. Therefore

$$B_1=\frac{1}{l}\int_0^l\sin\left(\frac{x\pi}{l}\right)dx=\frac{1}{l}\left(-\frac{l}{\pi}\cos\frac{x\pi}{l}\right)_0^l=\frac{2}{\pi}$$

And $\qquad A_n=\dfrac{2}{l}\displaystyle\int_0^l\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$

$$=\frac{1}{l}\int_0^l\left[\sin(n+1)\frac{\pi x}{l}-\sin\frac{(n-1)\pi x}{l}\right]dx$$

$$=-\frac{1}{l}\left[\frac{l}{(n+1)\pi}\cos\frac{(n+1)\pi x}{l}-\frac{l}{(n-1)\pi}\frac{\cos(n-1)\pi x}{l}\right]_0^l$$

$$=-\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n+1}-\frac{(-1)^{n-1}}{n-1}-\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$=\frac{-1}{\pi}\left[\frac{1+(-1)^n}{n+1}-\frac{1+(-1)^n}{n-1}\right]$$

$$A_n=\frac{2\left(1+(-1)^n\right)}{(n^2-1)\pi}=\begin{cases}0 & ; \quad n \text{ is odd}\\ \dfrac{4}{(n^2-1)\pi} & ; \quad n \text{ is even}\end{cases}$$

Using in Eq.(11),we get

$$u(x,t)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{4}{(4n^2-1)\pi}\cos\left(\frac{2n\pi x}{l}\right)e^{-\frac{4n^2\pi^2kt}{l^2}} \qquad (13)$$

(b) The $IBVP$ is given by

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}\ ; \quad 0\le x\le l;\ t>0 \qquad (1)$$

$$B.C.: \quad \frac{\partial u}{\partial x}(0,t)=0=\frac{\partial u}{\partial x}(l,t); \quad t>0 \qquad (2)$$

$$IC: \quad u(x,0)=x(l-x); \quad 0<x<l \qquad (3)$$

Proceeding on the same line as in above example we get

$$u(x,t)=B_1+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2kt}{l^2}} \qquad (11)$$

Applying IC, we get

$$x(l-x)=B_1+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

Which is half –range Fourier cosine series. Therefore

$$B_1=\frac{1}{l}\int_0^l x(l-x)\,dx$$

$$B_1=\frac{1}{l}\left(\frac{l^3}{2}-\frac{l^3}{3}\right)=\frac{l^2}{6}$$

And $\quad A_n=\dfrac{2}{l}\displaystyle\int_0^l x(l-x)\cos\left(\frac{n\pi x}{l}\right)dx=\frac{2l^2}{n^2\pi^2}\left[1+(-1)^n\right]$

$$A_n=\begin{cases}-\dfrac{4l^2}{n^2\pi^2}\ ; & n \text{ is even}\\ 0\ ; & n \text{ is odd}\end{cases}$$

Using in Eq.(11) we get $u(x,t)=\dfrac{l^2}{6}+\displaystyle\sum_{n=1}^{\infty}-\frac{l^2}{n^2\pi^2}\cos\left(\frac{2n\pi x}{l}\right)e^{-\frac{4n^2\pi^2kt}{l^2}}$

***Example7*:- (a)** A bar of length unity has its end at $x=0$ is insulated and its end at $x=1$ is kept at Temperature zero .Find an expression for the temperature $u(x,t)$ if

$$u(x,0)=\begin{cases}1 \quad ; & 0\le x\le l/2\\ 2(1-x); & 1/2\le x\le l\end{cases}$$

(b) A uniform rod of length l whose surface is thermally insulated is initially at temperature α. At $t=0$, one end is suddenly cooled to $0^{\circ}C$ and subsequently maintained at this Temperature other end remain thermally insulated Find temperature distribution

(c) Find the temperature $u(x,t)$ is a uniform bar of length l whose end $x=0$ is kept at zero temperature and other end $x=l$ is poorly insulated and radiates energy into the medium (i)At a rate proportional to the temperature (ii) at a constant rate and initial temperature is $f(x): 0\le x\le l.$

***Solution*:- Just hints at crucial steps while solving to reduce large number of pages.**

(a) the $IBVP$ is given by $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$; $0<x<1$..(1) $BC:\frac{\partial u}{\partial x}(0,t)=0, u(1,t)=0$ ..(2)

$$IC: \quad u(x,0)=\begin{cases}1 \quad ; & 0\le x\le \frac{1}{2}\\ 2(1-x); & \frac{1}{2}\le x\le 1\end{cases} \qquad (3)$$

- Let $u=X(x)T(t)$ be variables separable solution of Eq.(1) Then
- $u(x,t)=\sum_{n=1}^{\infty}A_n\cos\left(\frac{2n-1}{2}\right)\pi x e^{-\left(\frac{2n-1}{2}\right)^2\pi^2 kt}$ (10)

Applying IC, we get $u(x,0)=\sum_{n=1}^{\infty}A_2\cos\left(\frac{2n-1}{2}\right)\pi x$ (11)

Which is half range Fourier cosines .therefore , $A_n=\frac{2}{1}\int_0^1 u(x,0)\cos s\left(\frac{2n-1}{2}\right)\pi x\, dx$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{16\cos\left(\frac{2n-1}{4}\right)\pi}{(2n-1)^2\pi^2}e^{-\left(\frac{2n-1}{2}\right)^2\pi^2 kt} \qquad (12)$$

(b) The $IBVP$ is given by $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$; $0\le x\le l$; $t>0$ (1)

$BC:$ $u(0,t)=0,$ $\frac{\partial u(t,t)}{\partial x}=0;$ $t>0$ ..(2) $IC: u(x,0)=\alpha; 0\le x\le l$ ..(3)

- $u(x,t)=B_2\sin\frac{(2n-1)\pi x}{l}e^{-\left(\frac{(2n-1)}{l}\pi\right)^2 kt}$ ..(9) • $u(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{(2n-1)\pi x}{l}.e^{-\left(\frac{(2n-1)}{l}\pi\right)^2 kt}$ ..(10)
- Applying IC we get $\alpha=\sum_{n=1}^{\infty}B_n\sin\left(\frac{(2n-1)\pi x}{l}\right)$; $B_n=\frac{2}{1}\int_0^l\alpha\frac{\sin(2n-1)\pi x}{l}dx$
- Using in Eq. (10) we get $u(x,t)=\frac{4\alpha}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left(\frac{(2n-1)\pi x}{l}\right)e^{-\frac{(2n-1)^2\pi^2 kt}{l^2}}$ (12)

(c) **(i) The** $IBVP$ is given by $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$; $\quad 0 \le x \le l$

$BC: \quad u(0,t) = 0, \quad \dfrac{\partial u(l,t)}{\partial x} = -\alpha u(l,t); \quad t > 0; \quad IC: \quad u(x,0) = f(x): \qquad 0 \le x \le l$

• $u(x,t) = \sum_{n=1}^{\infty} B_2 \sin\left(\dfrac{\beta_n x}{l}\right) e^{\frac{-\beta_n^2 kt}{l^2}}$ (10)

• Applying IC, we get $f(x) = \sum_{n=1}^{\infty} B_2 \sin\left(\dfrac{\beta_n x}{l}\right)$; $B_n = \dfrac{2}{l}\int_0^l f(x)\sin\left(\dfrac{\beta_n x}{l}\right)dx$ (11)

Eq.(10) along with Eq.(11) represents the solution of $IBVP$.

(ii) $\dfrac{\partial v}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$. $BC: \quad u(0,t) = 0 \quad \dfrac{\partial u(l,t)}{\partial x} = -\alpha;\ \alpha > 0;\ IC: \quad u(x,0) = f(x)$

• $u(x,t) = -\alpha x + \sum_{n=1}^{\infty} B_2 \sin\left(\dfrac{2n-1}{2}\right)\dfrac{\pi}{l} x\, e^{-\left(\frac{2n-1}{2}\right)^2 \frac{\pi^2 kt}{l^2}}$ (11)

• $IC,\ f(x) = -\alpha x + \sum_{n=1}^{\infty} B_2 \sin\left(\dfrac{2n-1}{2}\right)\dfrac{\pi}{l} x$ ..(12); $B_n = \dfrac{2}{l}\int_0^l f(x)\sin\left(\dfrac{2n-1}{2}\dfrac{\pi}{l}x\right)dx$ (13)

Eq. (11) along with Eq.(13) represents the solution of given $IBVP$

<u>Consider a heat conduction equation in infinite bar</u>

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2};\quad -\infty < x < \infty;\quad t > 0 \tag{1}$$

$$IC: \quad u(x,0) = f(x); \quad -\infty < x < \infty; \quad t = 0 \tag{2}$$

Let $u = XT$ be a variable separable solution of Eq.(1) Then, $\dfrac{X''}{X} = \dfrac{1}{k}\dfrac{T'}{T} =$ Separation constant

If separation constant is zero then Eq.(3) gives $T = C_1,\ X = C_2 x + C_3$

Hence $\qquad u = Ax + B, \quad t \ge 0$ (3)

Form realistic physical considerations its is reasonable to assume that $f(x) \to 0$ as $|x| \to \infty$

So we discard this solution if separation constant is $+ve$ then Eq.(3) gives

$$u = (C_1 \cosh\lambda x + C_2 \sinh\lambda x)C_3 e^{-\lambda^2 kt}$$

(4)

Which shows that $u(x,t)$ grows exponentially with time $i.e.,\ u \to \infty$ as $t \to \infty$ But $u(x,t)$ is Bounded for bounded $u(x,t)$ separation constant should be $-ve$

Now from Eq. (3) we have, $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-k\lambda^2 t}$ (5)

• Function $f(x)$, $i.e.,\ u(x,0) = f(x)$ is either continuous or piecewise continuous on $(-\infty, \infty)$ and non-periodic in general $*$, therefore use of Fourier integral is advisable instead of Fourier series for the principle of superposition we shall use following relation instead of

Summation $u(x,t) = \int_0^{\infty} u(x,t,\lambda)\,d\lambda = \int_0^{\infty}\left(A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right)e^{-k\lambda^2 t}d\lambda$ (7)

$$\left(\because \sum_{n=1}^{\infty} \to \int_0^{\infty},\ A_n, B_n \to A(\lambda), B(\lambda)\right)$$

Which is general solution of Eq.(1) Use initial condition in Eq.(7) to obtain

$$u(x,0) = f(x) = \int_0^{\infty} \big(A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\big)\,d\lambda \qquad (8)$$

On comparing with Fourier integral theorem $f(t) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(x)\cos\omega(t-x)\,dx\right]d\omega$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos\lambda(x-y)\,dy\right]d\lambda$$

We get, $A(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(y)\cos\lambda y\ dy,\ B(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(y)\sin\lambda y\ dy$..(9)

Using Eq. (9) in Eq.(7) we get $u(x,t) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos\lambda(x-y)e^{-\lambda^2 kt}dy\right]d\lambda$

Changing order of integration, we get $u(x,t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(y)\left[\int_0^{\infty} e^{-k\lambda^2 t}\cos\lambda(x-y)\,d\lambda\right]dy$ (10)

Using the standard result $\displaystyle\int_0^{\infty} e^{-z^2}\cos(2bz)\,dz = \frac{\sqrt{\pi}}{2}e^{-b^2}$ (11)

In Eq. (10), we obtain $u(x,t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(y)\left[\frac{\sqrt{\pi}}{2}e^{-\frac{(x-y)^2}{4kt}}\frac{1}{\sqrt{kt}}\right]dy$ $\quad[\because k\lambda^2 t = z^2]$

$u(x,t) = \dfrac{1}{\sqrt{4kt\pi}}\displaystyle\int_{-\infty}^{\infty} f(y)e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2}dy$ .. (12); Eq. (12) is solution of Eq. (1) if $f(y)$ is bounded.

***Example8*:-** in a one-dimensional infinite solid $-\infty < x < \infty$, the surface $a < x < b$ is initially maintained at temperature T_0 and at zero temperature everywhere outside the surface find temperatures distribution.

***Solution*:-** the problem is defined as

$$u_t = ku_{xx} \quad ; \quad -\infty < x < \infty$$

$$IC: \quad u(x,0) = \begin{cases} T_0 & : \ a<x<b \\ 0 & : \ otherwise \end{cases}$$

The general solution of problem is $u(x,t) = \dfrac{1}{\sqrt{4\pi kt}}\displaystyle\int_{-\infty}^{\infty} f(y)e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2}dy$

Using initial condition $f(y) = T_0; a < y < b$, we get, $u(x,t) = \dfrac{T_0}{\sqrt{4\pi kt}}\displaystyle\int_a^b e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2 dy}$ (A)

Let $-\dfrac{x-y}{\sqrt{4\pi kt}} = z$. then dy $= \sqrt{4kt}$ dz using in (A), we get

$$u(x,t) = \frac{T_0}{\sqrt{4\pi kt}}\int_{(a-x)/\sqrt{4k-t}}^{(b-x)/\sqrt{4kt}} e^{-z^2}\sqrt{4kt}\ dz \ = \frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{\frac{b-x}{\sqrt{4kt}}} e^{-z^2}dz - \frac{2}{\sqrt{\pi}}\int_0^{\frac{a-x}{\sqrt{4kt}}} e^{-z^2}\ dz\right]$$

$$u(x,t) = \frac{T_0}{2}\left[erf\left(\frac{b-x}{\sqrt{4kt}}\right) - erf\left(\frac{a-x}{\sqrt{4kt}}\right)\right]$$

Consider a heat conduction in a semi-infinite bar

$$\frac{\partial u}{\partial t} = k\frac{\partial^{2u}}{\partial x_2};\ x>0\ ;\ t>0 \qquad (1)$$

$$BC\ :\ u(x,0)=0;\ t>0 \qquad (2)$$

$$IC\ :\ u(x,0)=f(x) \qquad (3)$$

Proceeding on the same line as in previous section, we get

$$u(x,t)=(A\cos\lambda x+B\sin\lambda x)e^{-k\lambda^2 t} \qquad (6)$$

Applying BC, we get, $0=A$

Hence $\quad u(x,t)=B\sin\ \lambda x e^{-k\lambda^2 t} \qquad (7)$

Function $f(x)$ i.e., $u(x,0)=f(x)$ is either continuous or piecewise on $(-\infty,\infty)$ and non-periodic in general therefore use of Fourier sine integral is advisable instead of Fourier series. For the principle of superposition we shall use following relation instead of summation.

$$u(x,t)=\int_0^\infty B(\lambda)\sin\lambda x e^{-\lambda^{2kt}d\lambda} \qquad (8)$$

Using IC, we get, $f(x)=\int_0^\infty B(\lambda)\sin\lambda x\,d\lambda \qquad (9)$

On comparing with Fourier sine integral formula, we get, $B(\lambda)=\frac{2}{\pi}\int_0^\infty f(y)\sin\lambda y\,dy \qquad (10)$

Using Eq. (10) in Eq. (8), we get $u(x,t)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(y)\sin\lambda\sin\lambda x e^{-\lambda kt}dy\ d\lambda$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\infty f(y)\left[\cos\lambda(x-y)-\cos\lambda(x+y)\right]e^{-\lambda kt}dy\,d\lambda$$

Using $\int_0^\infty e^{-z^2}\cos(2bz)dz=\frac{\sqrt{\pi}}{2}e^{-b^2}$, we get

$$u(x,t)=\frac{1}{\pi}\int_0^\infty f(y)\frac{\sqrt{\pi}}{2}\left[e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2}-e^{-\left(\frac{x+y}{\sqrt{4kt}}\right)^2}\right]\frac{dy}{\sqrt{kt}}$$

$$u(x,t)=\frac{1}{\sqrt{4kt}}\int_0^\infty f(y)\left[e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2}-e^{-\left(\frac{x+y}{\sqrt{4kt}}\right)^2}\right]dy \qquad (11) \qquad \left[\because \lambda^2 kt=z^2\right]$$

Substituting $-\frac{x-y}{\sqrt{4kt}}=z$ and $\frac{x+y}{\sqrt{4kt}}=z$ in first and second integrals respectively, we get

$$u(x,t)=\frac{1}{\sqrt{\pi}}\left[\int_{\frac{-x}{\sqrt{4kt}}}^{\infty} f\left(x+z\sqrt{4kt}\right)e^{-z^2}-\int_{\frac{x}{\sqrt{4kt}}}^{\infty} f\left(-x+z\sqrt{4kt}\right)e^{-z^2}dz\right] \quad (12)$$

Example8: solve $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$; $x>0,\ t>0$

$$BC: u(0,t)=0,\ \ IC: u(x,0)=f(x)=1$$

Solution: proceeding on the same line as in previous derivation, we get

$$u(x,t)=\frac{1}{\sqrt{\pi}}\left[\int_{\frac{-x}{\sqrt{4kt}}}^{\infty} e^{-z^2}dz-\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-z^2}dz\right] \qquad \left[\because f(x)=1\right]$$

$$=\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-z^2}dz=\frac{2}{\sqrt{\pi}}\int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-z^2}dz=erf\left(\frac{x}{\sqrt{4kt}}\right)$$

Example9: (Heat loss due to radioactive decay). Solve the following heat equation

$$\frac{\partial u}{\partial t}=k\,u_{xx}+Ne^{-ax} \qquad (1) \qquad \text{With}$$

$$BC: \quad u(0,t)=0=u(l,t); \quad t>0$$
$$IC: \quad u(x,0)=f(x) \quad ; \quad 0<x<l \qquad (2),(3)$$

Where Ne^{-ax} represent the heat loss due to radioactive decay in the bar.

Solution: Consider a new unknown function $v(x,t)$ such that $u(x,t)=v(x,t)+w(x)$ (4)

which reduces the given non-homogeneous pde to homogeneous pde.

Substituting Eq. (4) in (1), we get $v_t=k\left(v_{xx}+w_{xx}\right)+Ne^{-ax}$ (5)

For homogeneous pde, we get $kw_{xx}=-Ne^{-ax}$ (6)

The two point boundary condition on $u(x,t)$ becomes $v(x,t)+w(0)=0, v(l,t)+w(l)=0$ (7)

They are homogeneous if $w(0)=0,\quad w(l)=0$ (8)

Eq. (6) along with Eq. (8) gives $w=-\frac{N}{ka^2}e^{-ax}+c_1x+c_2$ where $c_2=\frac{N}{lka^2}\left(e^{-al}-1\right)$

Hence $$w=\frac{N}{ka^2}\left(-e^{-ax}+\frac{x}{1}\left(a^{-al}-1\right)+1\right)=\frac{N}{ka^2}\left(\left(1-e^{-ax}\right)-\frac{x}{l}\left(1-e^{-al}\right)\right) \qquad (9)$$

The boundary value problem in $v(x,t)$ becomes

$$\left.\begin{aligned} &v_t=kv_{xx}\\ BC\ &:\ v(0,t)=0=v(l,t)\\ IC\ &:\ v(x,0)=u(x,0)-w(x)=f(x)-w(x)\end{aligned}\right] \qquad (10)$$

The general solution of Eq.(10) by method separation of variable is

$$v(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{kn^2\pi^2 t}{t^2}} \qquad (11)$$

Where $\quad B_n=\frac{2}{l}\int_0^l \left(f(x)-w(x)\right)\sin\left(\frac{n\pi x}{l}\right)dx$

Hence complete solution of original problem is $u(x,t)=v(x,t)+w(x)$

i.e., $\quad u(x,t)=\frac{N}{ka^2}\left[\left(1-e^{-ax}\right)-\frac{x}{l}\left(1-e^{-al}\right)\right]+\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{kn^2\pi^2 t}{l^2}} \qquad (12)$

***Example10*:** the end A of a rod of length l is kept at zero temperature and heat is supplied at the end B with constant heat flux q_0. rod is initially at zero temperatures. Find temperature distribution.

***Solution*:** the given IBVP can be defined as $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$; $\quad 0\le x\le l$, $\quad t>0 \qquad (1)$

BC : $\quad u(0,t)=0, \quad \frac{\partial u}{\partial x}(l,t)=q_0\,, \quad t>0 \qquad (2)$

IC : $\quad u(x,0)=0; \quad 0\le x\le l \qquad (3)$

Here BC are non-homogenous. Therefore we consider a new function $v(x,t)$ s.t.

$$u(x,t)=v(x,t)+w(x) \qquad (4)$$

Which reduces the non-homogeneous p.d.e to homogenous BC.

Substituting Eq. (4) in Eqs. (1)-(3), we get $v_t=ku_{xx}+kw_{xx}$

$$v(0,t)+w(0)=0;\quad \frac{\partial v}{\partial x}(l,t)+\frac{dw}{dx}(l)=q_0$$

$$v(x,0)+w(x)=0$$

For homogenous pde, we set $kw_{xx}=0 \qquad (5)$

The two point BC on $v(x,t)$ become homogeneous if $w(0)=0$, $\quad \frac{dw(l)}{dx}=q_0 \qquad (6)$

Eq. (5) along with Eq. (6) gives $w=q_0 x \qquad (7)$

The IBVP in $v(x,t)$ is become

$$\left.\begin{array}{l} v_t=kv_{xx} \\ BC\ : v(0,t)=0=\frac{\partial v}{\partial x}(l,t) \end{array}\right] \qquad (8)$$

$$IC\ :\ v(x,0)=u(x,0)-w(x)=-q_0 x$$

The general solution of Eq. (8) is $v(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{2n-1}{2}\right)\pi x e^{-\left(\frac{2n-1}{2l}\pi\right)^2 kt} \qquad (9)$

Where $\quad B_n=-\frac{2q_0}{l}\int_0^l x\sin\left(\frac{2n-1}{2l}\right)\pi x\, dx$

$$= -\frac{2q_0}{l}\left[-\frac{2lx}{(2n-1)\pi}\cos\left(\frac{2n-1}{2l}\right)\pi x + \left(\frac{2l}{(2n-1)\pi}\right)^2 \sin\left(\frac{2\pi}{2l}\right)\pi x\right]_0^l$$

$$= -\frac{2q_0}{l}\left(\left(\frac{2l}{(2n-1)\pi}\right)^2 .\sin\left(\frac{2n-1}{2}\pi\right)\right) = -\frac{8q_0 l}{(2n-1)^2 \pi^2}(-1)^{n-1}$$

Hence $$v(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n 8q_0 l}{((2n-1)\pi)^2} \sin\frac{(2n-1)\pi x}{2l} e^{-\left(\frac{2n-1}{2l}\pi\right)^2 kt} \quad (10)$$

The solution of original problem is

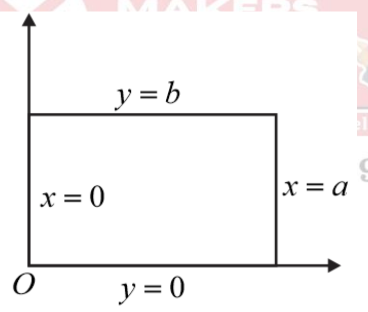
$$u(x,t) = q_0 x + \sum_{n=1}^{\infty} \frac{(-1)^n 8q_0 l}{((2n-1)\pi)^2} \sin\left(\frac{2n-1}{2l}\pi x\right) e^{-\left(\frac{2n-1}{2e}\pi\right)^2 kt} \quad (11)$$

<u>Rectangular plate surface: heat flow: Examples</u>

Example11: A thin rectangular plate whose surface is impervious to heat flow, has an arbitrary function $f(x,y)$ at $t=0$.its four edges $x=0, x=a, y=0, y=b$ are kept at zero temperature. Determine temperature distribution.

Solution: the given IBVP is defined as

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2}\right);\ 0 \le x \le a,\ 0 \le y \le b \quad (1)$$



$$BC: \quad u(0,y,t) = 0 = u(a,y,t) \quad (2)$$

$$u(x,0,t) = 0 = u(x,b,t) \quad (3)$$

$$IC: \quad u(x,y,0) = f(x,y) \quad (4)$$

Let u=XYT be variables separable solution. Then Eq. (1) gives

$$\frac{x''}{x} + \frac{y''}{y} = \frac{1}{k}\frac{T'}{T} \quad (5)$$

Eq.(5) hold good if each side is equal to separation constant. Since BC(2) and (3) are periodic in x and y therefore we set $\frac{x''}{x} = -\lambda^2, \frac{y''}{y} = -\mu^2$ (6)

Using in (5) we get $\frac{T'}{kT} = -P^2;\ P^2 = \lambda^2 + \mu^2$

On solving, we get $u(x,y,t) = (A\cos\lambda x + B\sin\lambda x)(c\cos\mu y + D\sin\mu y)e^{-p^2 ky}$ (7)

Applying BC (2), we get

$$A=0, B\sin\lambda a=0$$
$$A=0,\quad B\neq 0,\quad \lambda=\frac{n\pi}{a};\quad n\in l \tag{8}$$

Applying BC (3), we get

$$C=0, D\sin\mu b=0$$
$$C=0,\quad D\neq 0\quad \mu=\frac{m\pi}{b};\ m\in l \tag{9}$$

Using Eqs. (8) And (9) in Eq.(7), we get

$$u(x,y,t)=BD\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-p^2nm^{kt}} \tag{10}$$

Where $p^2nm=\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)$

Using principle of superposition, we get

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}E_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-p^2nm^{kt}} \tag{11}$$

Applying IC, we get

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}E_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \tag{12}$$

Which is Fourier sine series in two dimensions? Therefore,

$$E_{n,m}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy \tag{13}$$

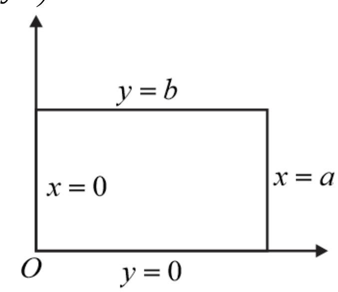
Eq.(11) along with Eq.(13) represents the solution of Eq.(1).

Example12: (a) The four edges of a thin square of a thin square plate of area π^2 are kept at temperature zero and the faces are perfectly insulated. The initial temperature is assumed to be $u(x,y,0)=xy(\pi-y)$ by applying the method of separation of variables to the two dimensional heat equation $u_{t=}k\nabla^2u$, determine the temperature $u(x,y,t)$ in the plate.

(b) A rectangular plate bounded by the lines $x=0, y=0, x=a, y=b$ has an initial distribution given by $u=a\sin(\pi y/a).\sin(\pi y/b)$.the edges are kept at zero temperature and the plane faces are impervious to heat .find the temperature at any point.

Solution: (a) The given IBVP is defined as

$$u_t=k\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right) \tag{1}$$



$$BC: \quad u(0,y,t) = 0 = u(a,y,t) \qquad (2)$$

$$u(x,0,t) = 0 = u(x,b,t) \qquad (3)$$

$$IC: \quad u(x,y,0) = f(x,y) = xy(\pi - x)(\pi - y) \qquad (4)$$

Proceeding on the same line as in example 1, we get

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-p^2 nm^{kt}} \qquad (11)$$

Applying IC ,we get

$$xy(\pi - x)(\pi - y) = \sum\sum E_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad (12)$$

Which is half-Fourier sine series? Therefore.

$$E_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy(\pi - x)(\pi - y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

$$= \left(\frac{4}{\pi^2}\right)\int_0^{\pi} x(\pi - x)\sin\left(\frac{n\pi x}{a}\right)dx\int_0^{\pi} y(\pi - y)\sin\left(\frac{m\pi y}{b}\right)dy$$

$\because$ integral is separable function

$$= \frac{4}{\pi^2}\frac{2}{n^3}\left(1-(1-)^n\right)\frac{2}{m^3}\left(1-(-1)^m\right) = \frac{16\left(1-(-1)^n\right)\left(1-(-1)^m\right)}{n^3m^3\pi^2}$$

$$E_{st} = \frac{64}{\pi^2(2_{s-1})^3(2t-1)^3}$$

Using Eq.(11) we get

$$u(x,y,t) = \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{64\sin\frac{(2s-1)\pi x}{a}\sin\frac{(2t-1)\pi y}{b}}{\pi^2(2s-1)^3(2t-1)^3}e^{-p^2{}_{st}kt}$$

Where $p^2{}_{st} = \pi^2\left(\frac{(2s-1)^2}{a^2} + \frac{(2t-1)^2}{b^2}\right)$

Proceeding on the same line as above,we get

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-p^2{}_{nm}kt}$$

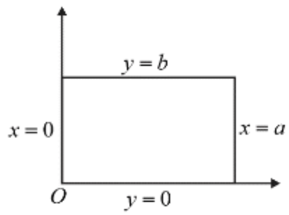
Applying IC we get

$$A\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad (12)$$

Equating coefficients of similar terms, we get

$$E_{11} = A \text{ and all other } E_{nm} = 0$$

Using in Eq.(11), we get



$$u(x,y,t) = A\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)e^{-p^2_{11}kt}$$

Where $p^2_{11} = \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

Consider a three-dimensional diffusion/heat conduction equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u \qquad (1)$$

In cylindrical coordinates (r,θ,z), Eq.(1) becomes

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \qquad (2)$$

Where $u = u(r,\theta,z,t)$

Let $u(r,\theta,z,t) = R(r)H(\theta)Z(z)T(t)$ be variable separable solution of Eq.(2) substituting into Eq.(2),it becomes

$$R''HZT + \frac{1}{r}R'HZT + \frac{1}{r^2}H''RZT + Z''RHT = \frac{T'}{K}RHZ$$

Or $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \frac{Z''}{Z} = \frac{1}{k}\frac{T'}{T} = -\lambda^2 \qquad (3)$$

Which $-\lambda^2$ is a separation constant. Then Eq.(3) $\Rightarrow$

$$T' + k\lambda^2 T = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2\,(say) \qquad (4)$$

Thus, the equations determining Z, R and H become

$$Z'' - \mu^2 Z = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 + \mu^2 = 0 \qquad (5)$$

i.e., $$r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda^2+\mu^2)r^2 = -\frac{H''}{H} = v^2\,(say)$$

Therefore $$H'' + v^2 H = 0 \qquad (6)$$

$$R'' + \frac{1}{r}R' + \left[(\lambda^2+\mu^2) - \frac{v^2}{r^2}\right]R = 0 \qquad (7)$$

Equation (4) – (6) have particular solution of the from

$$\left.\begin{aligned} T &= e^{-k\lambda^2 t} \\ H &= c\cos v\theta + D\sin v\theta \\ Z &= Ae^{\mu z} + Be^{-\mu z} \end{aligned}\right] \quad (8)$$

The differential equation (7) is called Bessel's equation of order v and its general solution is

$$R(r) = c_1 j_v\left(\sqrt{\lambda^2 + \mu^2 r}\right) + c_2 Y_v\left(\sqrt{\lambda^2 + \mu^2 r}\right) \quad (9)$$

Where $j_v(r)$ and $Y_v(r)$ are Bessel function of order v of the first and second kind respectively since Eq.
(7) is singular at $r = 0$ therefore physically meaningful solution must be twice continuously Differentiable in $0 \le r \le a$. Hence Eq.(7) has only one bounded solution given by

$$R(r) = j_v\left(\sqrt{\lambda^2 + \mu^2}\right) \qquad \because Y_v \to \infty as\, r \to 0 \quad (9)$$

Finally the general solution of Eq.(2) is given by

$$u(r,\theta,z,t) = e^{k\lambda^2 t}\left[Ae^{\mu z} + Be^{-\mu z}\right]\left[C\cos v\theta + D\sin v\theta\right] j_v\left(\sqrt{\lambda^2 + \mu^2 r}\right) \quad (10)$$

Heat flow in the infinite cylinder

Example13:- Determine the temperature $u(r,t)$ in the infinite cylinder $0 \le r \le a$ when the initial Temperature is $u(r,0) = f(r)$, and the surface $r = a$ is maintained at 0° temperature

Solution:- The given $IBVP$ is defined as $\dfrac{\partial u}{\partial t} = k\nabla^2 u$

(1)

Where u is a function of r and t only therefore

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{k}\frac{\partial u}{\partial t} \quad (2)$$

Subject to $\qquad BC: u(a,t) = 0 \quad (3)$

$\qquad IC: u(r,0) = f(r) \quad (4)$

The general solution of Eq.(2) is $u(r,t) = A\exp\left(-k\lambda^2 t\right) j_0(\lambda r)$ (5)

Using the BC(3) we obtain $j_0(\lambda a) = 0$ (6)

Which has an infinite number of roots, $a\xi_n\left(n = 1, 2, ...\infty\right)$. Thus use of principle of superposition gives $u(r,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\alpha\xi_n^2 t\right) j_0(\xi_n r)$ (7)

Now using the $IC: u(r,0) = f(r)$, we get $f(r) = \sum_{n=1}^{\infty} A_n j_0(\xi_n r)$

To compute A_n, we multiply both sides of Eq.(8) by $rJ_0(\xi_m r)$ and integrate with respect to r to a; Get (i.e., orthogonality of Bessel's function)

$$\int_0^a rf(r)J_0(\xi_m r)dr=\sum_{n=1}^{\infty}A_n\int_0^a rJ_0(\xi_m r)J_0(\xi_n r)dr=\begin{cases}0 & for\, n\neq m\\ A_m\left(\frac{a^2}{2}\right)J_1^2(\xi_m a) & for\, n=m\end{cases}$$

$$\Rightarrow \qquad A_m=\frac{2}{a^2J_1^2(\xi_m a)}\int_0^a uf(u)J_0(\xi_m u)du$$

Hence the complete solution of the given problem is

$$u(r,t)=\frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_0(\xi_m r)}{J_1^2(\xi_m a)}\exp\left(-\alpha\xi_m^2 t\right)\left[\int_0^a uf(u)J_0(\xi_m u)du\right] \qquad (10)$$

Heat flow in the Sphere

Example14:- Find the temperature in a sphere of radius a, when its surface is kept at zero temperature And its initial temperature is $f(r,\theta)$

Solution:- Here the temperature is governed by the three-dimensional heat equation in spherical polar

Coordinates independent of ϕ therefore we have to find the solution of $pde.$

$$\frac{1}{k}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) \qquad (1)$$

$$BC: \quad u(a,\theta,t)=0 \qquad (2)$$

$$IC: \quad u(r,\theta,0)=f(r,\theta) \qquad (3)$$

The general solution of Eq.(1) with the help of previous derivation can be written as

$$u(r,\theta,t)=\sum_{\lambda,n}A_{\lambda n}(\lambda r)^{-1/2}J_{n+1/2}(\lambda r)P_n(\cos\theta)e^{-\alpha\lambda^2 t} \qquad (4)$$

Applying the BC(2), we get $J_{n+1/2}(\lambda a)=0$

This equation has infinitely many positive roots; denoting them by $\xi_i a$, we have

$$u(r,\theta,t)=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}(\xi_i r)^{-1/2}J_{n+1/2}(\xi_i r)P_n(\cos\theta)\exp\left(-\alpha\xi_i^2 i\right) \qquad (5)$$

Applying the IC,we get

$$f(r,\theta)=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}(\xi_i r)^{-1/2}J_{n+1/2}(\xi_i r)P_n(\cos\theta) \qquad (6)$$

Multiplying both sides by $P_n^2(\cos\theta)d(\cos\theta)$ and integrating between-1to1,we get

$$\int_{-1}^{1}P_n^2(\cos\theta)f(r,\theta)d(\cos\theta)=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}(\xi_i r)^{-1/2}J_{n+1/2}(\xi_i r)\int_{-1}^{1}P_n^2(\cos\theta)d(\cos\theta)$$

$$=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}(\xi_i r)^{-1/2}J_{n+1/2}(\xi_i r)\left(\frac{2}{2n+1}\right)$$

Using the orthogonality property of Legendre polynomials we have

$$\frac{2n+1}{2}\int_{-1}^{1}P_n(\mu)f(r,\theta)d\mu=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_{ni}(\xi_i r)^{-1/2}J_{n+1/2}(\xi_i r)\left(\frac{2}{2n+1}\right) \qquad (7)$$

Where $\mu=\cos\theta$. in order to evaluate A_{ni} we multiply both sides of the Eq.(7) by

$r^{-3/2}J_{n+1/2}(\xi_i r)$ and integrate $w.r.t.\ r$ from 0 to a to obtain

$$\frac{2n+1}{2}\int_0^a r^{3/2}J_{n+1/2}(\xi_i r)\,dr\int_{-1}^{1}P_n(\mu)\,f(r,\theta)\,d\mu = A_{ni}\int_0^a rJ_{n+1/2}^2(\xi_i r).\frac{dr}{\sqrt{\xi_i}} \qquad (8)$$

Eq.(5) along with Eq.(8) constitute the solution of given $IBVP$.

Example15:- A homogeneous solid sphere of radius R has the initial temperature distribution $f(r), 0 \le r \le R,$ where r is the distance measured from centre the surface temperature Is maintained at 0° show that temperature $u(r,t)$ in the sphere is the solution of $u_t = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$ where c^2 is a constant show also that the temperature in the sphere For $t > 0$ is given by $u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)\exp\left(-\lambda_n^2 t\right), \lambda_n = \frac{n\pi c}{R}$

Solution:- The temperature distribution in a solid sphere is governed by the parabolic heat equation $u_t = c^2\nabla^2 u$ (1)

From the given data u is a function of r and t alone Due to spherical symmetry the Eq.(1) Can be written as $u_t = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$ (2)

Setting $v = ru$ the given BC gives $v(R,t) = ru(R,t) = 0$ (3)

While the IC gives $v(r,0) = ru(r,0) = rf(r)$ (4)

Since u must be bounded at $r = 0,$ we need $v(0,t) = 0$ (5)

Now $v_t = ru_t \quad u_r = \left(\frac{v}{r}\right)_r = \frac{rv_r - v}{r^2}$

Similarly finding u_{rr} and substituting into Eq.(2) we obtain $v_t = c^2 v_{rr}$ (6)

Using the variables separable method we may write $v(r,t) = R(r)T(t)$ and get

$$\left.\begin{aligned} R(r) &= A\cos kr + B\sin kr \\ T(t) &= \exp\left(-c^2k^2t\right) \end{aligned}\right] \qquad (7)$$

Thus using the principle of superposition we get

$$v(r,t) = \sum_{n=1}^{\infty}\left(A_n\cos kr + B_n\sin kr\right)\exp\left(-c^2k^2t\right)$$

Also using $v(0,t) = 0,$ we have

$$\left(A_n\cos kr + B_n\sin kr\right)_{r=0} = 0 \quad \Rightarrow A_n = 0$$

For non-trivial solution we get $B_n \ne 0, \sin kr = 0$

i.e., $kR = n\pi, \quad k = \frac{n\pi}{R}, \quad n = 1, 2, \ldots.$

Thus, the possible solution is $v(r,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)\exp\left(-\frac{c^2n^2\pi^2t}{R^2}\right)$

Finally, applying the $v(r,0)=rf(r)$ we obtain $rf(r)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)$

Which is a half-range Fourier sine series. Therefore $B_n = \frac{2}{R}\int_0^R rf(r)\sin\left(\frac{n\pi}{R}r\right)dr$

But $v(r,t)=ru(r,t)$. Hence the temperature in the sphere is given by

$$u(r,t)=\frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)\exp\left(-\frac{n^2\pi^2c^2t}{R^2}\right) \qquad (8)$$

Example16:- A circular cylinder of radius a has its surface kept at a constant temperature u_0 if the initial Temperature is zero throughout the cylinder prove that for $t>0$,

$u(r,t)=u_0\left\{1-\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\xi_n a)}{\xi_n J_1(\xi_n a)}\exp\left(-\xi_n^2 kt\right)\right\}$ Where ξ_n's are the positive roots of $J_0(\xi a)=0$

and k is the thermal conductivity which is a Constant

Solution:- It is evident that u is a function of r and t alone and therefore the $p.d.e.$ to be solved is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{k}\frac{\partial u}{\partial t} \qquad (1)$$

Subject to $IC:\quad u(r,0)=0 \quad 0\le r\le a$ (2)

$BC:\quad u(a,t)=T_0, t\ge 0$ (3)

Let $u(r,t)=u_0+u_1(r,t)$. Then

So that $u_1(r,0)=-u_0$ (4)

$u_1(a,t)=0$ (5)

Where u_1 is the solution of Eq.(1) by the variables separable method we have

$$u_1(r,t)=AJ_0(\lambda r)\exp\left(-\lambda^2 kt\right)$$

Using the $BC:\ u_1(a,t)=0$, we get $AJ_0(\lambda a)\exp\left(-\lambda^2 kt\right)=0$

Which gives $J_0(\lambda a)=0$ as $A\ne 0$. Let $\xi_1,\xi_2,....,\xi_n$, be the roots of $J_0(\lambda a)=0$. Then the

Possible solution bythe principle of is superposition $T_1(r,t)=\sum_{n=1}^{\infty}A_n J_0(\xi_n r)\exp\left(-\xi_n^2 kt\right)$ (6)

Using the $IC: u_1(r,0)=-u_0$ into Eq. (6) we get $\sum_{n=1}^{\infty}A_n J_0(\xi_n r)=-u_0$

Multiplying both sides by $rJ_0(\xi_m r)$ and integrating we get

$$-u_0\int_0^a rJ_0(\xi_m r)dr=\sum_{n=1}^{\infty}A_n\int_0^a rJ_0(\xi_m r)rJ_0(\xi_n r)dr = A_m\int_0^a rJ_0^2(\xi_m r)\, if\ m=n; otherwise\, 0$$

$$=A_m\frac{a^2}{2}(\xi_m a)$$

But $-u_0\int_0^a rJ_0(\xi_m r)dr=\int_0^{\xi_m a}\frac{x}{\xi_m}J_0(x)\frac{dx}{\xi_m} \qquad (x=\xi_m r)$

$$= -\frac{u_0}{\xi_m^2}\int_0^{\xi_m a}\frac{d}{dx}\left[xJ_1(x)\right]dx = -\frac{u_0}{\xi_m^2}\left[xJ_1(x)\right]_0^{\xi_m a} = -\frac{a u_0}{\xi_m^2}J_1(\xi_m a)$$

Therefore, $A_m \frac{a^2}{2}J_1^2(\xi_m a) = -\frac{a u_0}{\xi_m}J_1(\xi_m a);\quad A_n = -\frac{2u_0}{a\xi_n}\frac{1}{J_1(\xi_n a)}$

Finally the complete solution is $u(r,t) = u_0\left[1-\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{J_1(\xi_n a)}\frac{\exp(-\xi_n^2 kt)}{\xi_n}\right]$

Heat conduction equation in 3D

Example:- *If* $\psi_1(x,t),\psi_2(y,y),\psi_3(z,t)$ be solution of three linear heat conduction equation in x,y,z respectively prove that $\psi=\psi_1\psi_2\psi_3$ is necessarily a solution of 3D heat conduction Equation

Solution:- Since *If* $\psi_1(x,t),\psi_2(y,y),\psi_3(z,t)$ are solution of linear conduction equation in $x,y,z,$ Therefore

$$\frac{\partial\psi_1}{\partial t}=k\frac{\partial^2\psi_1}{\partial x^2};\quad \frac{\partial\psi_2}{\partial t}=k\frac{\partial^2\psi_2}{\partial y^2};\quad \frac{\partial\psi_3}{\partial t}=k\frac{\partial^2\psi_3}{\partial z^2}\qquad (1)\text{-}(3)$$

Multiplying by $\psi_2\psi_3\psi_1\psi_3\psi_1\psi_2$ in Eq.(1)-(3) respectively and adding we get

$$\psi_2\psi_3\frac{\partial\psi_1}{\partial t}+\psi_1\psi_3\frac{\partial\psi_2}{\partial t}+\psi_1\psi_2\frac{\partial\psi_3}{\partial t}=k\left[\psi_2\psi_3\frac{\partial^2\psi_1}{\partial x^2}+\psi_1\psi_3\frac{\partial^2\psi_2}{\partial y^2}+\psi_1\psi_2\frac{\partial^2\psi_3}{\partial z^2}\right]$$

Using product rule of differentiation in

$$\frac{\partial}{\partial t}(\psi_1\psi_2\psi_3)=k\left[\psi_2\psi_3\frac{\partial^2\psi_1}{\partial x^2}+\psi_1\psi_3\frac{\partial^2\psi_2}{\partial y^2}+\psi_1\psi_2\frac{\partial^2\psi_3}{\partial z^2}\right]=k\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right](\psi_1\psi_2\psi_3)$$

$$\frac{\partial\psi}{\partial t}=k\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)\qquad [\because \psi_1,\psi_2\neq fun.of\ x]\qquad (4)$$

Which shows that ψ is solution of 3D heat conduction equation

Elliptic Partial Differential Equation

Example1:- Find the steady temperature distribution in a thin rectangular plate bounded by lines $x=0, x=a\quad y=0, y=b$ the edges $x=0, x=a,\ \ y=0$ are kept at zero temperature while the Edge $y=b$ kept at $100^\circ C$.

Solution:- The given problem is Dirichlet type and con be defined as

$$\nabla^2 u = u_{xx}+u_{yy}=0\qquad (1)$$

$$BC:\quad u(0,y)=0=u(a,y)=u(x,0),\quad u(x,b)=100^\circ C\qquad (2)$$

The general solution of given problem is

$$u(x,y)=(A\cos\lambda x+B\sin\lambda x)\left(Ce^{\lambda y}+De^{-\lambda y}\right)\qquad (3)$$

Applying homogeneous BC, we get

$$0 = A\cos 0 + B\sin 0, \quad 0 = A\cos\lambda a + B\sin\lambda a$$

And $$0 = C + D$$

$$\Rightarrow \quad A = 0, \quad B\sin\lambda a = 0, \quad D = -C$$

For non-zero solution we set $B \neq 0, \ \sin\lambda a = 0$ *i.e.*, $\lambda = \dfrac{n\pi}{a}$

Using in Eq. (3) we get $u(x,y) = 2BC\sin\left(\dfrac{n\pi x}{a}\right)\sinh\left(\dfrac{n\pi y}{a}\right)$

Using principle of superposition, we get

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi x}{a}\right) \qquad (4)$$

Using $BC \ u(x,b) = 100$, we get

$$100 = \sum E_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)$$

Which is half-range Fourier sine series. Therefore

$$E_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a 100\sin\left(\frac{n\pi x}{a}\right)dx = \frac{200}{n\pi\sin\left(\frac{n\pi b}{a}\right)}.\left(1-(-1)^n\right) = \begin{cases} \dfrac{400}{n\pi\sinh\left(\frac{n\pi b}{a}\right)}; & n = odd \\ 0 & ; n = even \end{cases}$$

Using in Eq. (4) we get required solution $u(x,y) = \displaystyle\sum_{n=1}^{\infty} \frac{400\left(\frac{(2n-1)\pi x}{a}\right)\sinh\left(\frac{(2n-1)\pi y}{a}\right)}{(2n-1)\pi\sinh\left(\frac{(2n-1)\pi b}{a}\right)}$

Example2:- A thin rectangular homogeneous thermally conducting plate lies. In $x-y$ plane $0 \le x = a, 0 \le y \le b$. The edge $y = 0$ is held at the temperature $Tx(x-a)$ where T is Constant while the remaining edges are held at 0°. the other faces are insulated and no Internal sources and sinks are present find temperature distribution

Solution:- The given problem is of Dirichlet type and can be defined as

$$u_{xx} + u_{yy} = 0 \qquad (1)$$

$$BC: \quad u(0,y) = 0 = u(a,y) = u(x,b), \quad u(x,0) = Tx(x-a) \qquad (2)$$

The variables separable solution of Eq.(1) is

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right) \qquad (3)$$

Applying homogeneous BC, we get

$$0 = A. \quad 0 = B\sin\lambda a, \quad Ce^{\lambda b} + De^{-\lambda b} = 0$$

For non-trivial solution we set $B \neq 0, \lambda = \frac{n\pi}{a}$. Using these values in Eq.(3) ,we get

$$u(x,y) = BC\sin\left(\frac{n\pi x}{a}\right)\left(\frac{e^{\lambda(y-b)} - e^{\lambda(b-y)}}{e^{-\lambda b}}\right)$$

$$u(x,y) = E\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n}{a}(b-y)\pi\right)$$

Where $\qquad E = \frac{-2BC}{e^{n\pi b/a}}$

Using principle of superposition, we get

$$u(x,y) = \sum E_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y)\right)$$

Using BC $u(x,0) = Tx(x-a)$, we get

$$Tx(x-a) = \sum E_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y)\right)$$

Which is half range Fourier sine series Therefore

$$E_n = \frac{2T}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a x(x-a)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{4a^2T}{n^3\pi^2}\frac{\left((-1)^n - 1\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

$$= \begin{cases} -\dfrac{8a^2T}{\pi^3 n^3 \sinh\left(\frac{n\pi b}{a}\right)} & ; n = odd \\ 0 & ; n = even \end{cases} \tag{5}$$

Using in Eq. (4) we get required solution

$$u(x,y) = -\sum_{n=1}^{\infty}\frac{8a^2T}{\pi^3(2n-1)^3}\frac{\sin\frac{(2n-1)\pi x}{a}}{\sinh\left(\frac{(2n-1)\pi b}{a}\right)}\sinh\left(\frac{(2n-1)(b-y)\pi}{a}\right) \tag{6}$$

Example3:- Solve :

(a) $u_{xx} + u_{yy} = 0$

$$u(0,y) = u(a,y) = u(x,0) = 0; \qquad u(x,b) = \sin\left(\frac{3\pi x}{\alpha}\right)$$

(b) $u_{xx} + u_{yy} = 0;$

$$u(0,y) = 0 = u(a,y), \quad u(x,y) \to 0\, as\, y \to \infty\ u(x,b) = u_0$$

(c) $u_{xx} + u_{yy} = 0;$

$$u(0,y) = 0 = u(\pi,y), \ u(x,y) \to 0\, as\, y \to \infty,\ u(x,0) = \sin x$$

Solution Ans. $u(x,y)=\dfrac{\sin\left(\frac{3\pi x}{a}\right)\sinh\left(\frac{3\pi y}{a}\right)}{\sinh\left(\frac{3\pi y}{a}\right)}$

(b) $u_0=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n\pi y}{a}}$ (4)

$$E_n=\frac{2}{-\frac{n\pi b}{a}}\int_0^a u_0\sin\left(\frac{n\pi x}{a}\right)dx=\frac{2u_0}{n\pi}e^{\frac{n\pi b}{a}}\left(1-(-1)^n\right)=\begin{cases}\frac{4u_0}{n\pi}e^{\frac{n\pi b}{a}} & ;\quad n=odd\\ 0 & ;\quad n=even\end{cases}$$

Using in Eq. (3) we get required solution

$$u(x,y)=\sum_{n=1}^{\infty}\frac{4u_0e^{(2n-1)\frac{\pi b}{a\pi}}}{(2n-1)\pi}.\sin\left(\frac{(2n-1)\pi x}{a}\right)e^{-\frac{(2n-1)\pi y}{a}}=\frac{4u_0}{\pi}\sum_{n=1}^{\infty}\sin\left(\frac{(2n-1)\pi x}{a}\right)e^{-\frac{(2n-1)\pi y}{a}}$$

(c) $u(x,y)=\sum_{n=1}^{\infty}E_n\sin(nx)e^{-ny}$ (3)

$\sin x=\sum_{n=1}^{\infty}E_n\sin(nx)\quad\Rightarrow\quad E_1=1$ and all other $E_n=0$

Using in Eq. (3) we get required solution $u(x,y)=e^{-y}\sin x$ (4)

Example4:- (a) Evaluate the steady temperature in rectangular plate of length a and with b, the sides of

Which are kept at temperature zero the lower and is kept at temperature $f(x)$ and the upper

Edge is kept insulated

(b) Solve $\nabla^2u=0;\quad 0\le x\le a,\quad 0\le y\le b$

$u(0,y)=0,\ u(x,0)=,\ u(x,b)=0$

$\dfrac{\partial u(a,y)}{\partial x}=T\sin^3\left(\dfrac{\pi y}{b}\right)$

Solution:- $u=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi}{a}(b-y)\right)$ (6)

Using $BC\ u(x,0)=f(x)$ we get , $f(x)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi}{a}\right)$

$$E_n=\frac{2}{a\cosh\left(\frac{n\pi b}{a}\right)}\int_0^a\sin\left(\frac{n\pi x}{a}\right)f(x)dx \quad (7)$$

Eq.(6) along with Eq.(7) represents the solution of Eq.(1)

(b) Since BC in y are periodic therefore solution of given problem is

$$u(x,y)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)\left(C\cos\lambda y+D\sin\lambda y\right) \quad (1)$$

Applying $BC\ u(x,0)=0,\ u(x,b)=0,$ we get

$$C=0,\quad D\sin\lambda b=0 \tag{2}$$

For non-zero solution we get $D\neq 0,\ \lambda=\frac{n\pi}{b}$

Applying $BC\ \ u(0,y)=0,$ we get $0=A+B,\ i.e.,\ B=-A$ (3)

Using Eqs.(2) and (3) in Eq.(1),we get $u=2AD\sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi x}{a}\right)$

Using principle of superposition we get $u=\sum E_n\sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi x}{a}\right)$ (4)

Using condition $u_x(a,y)=T\sin^3\left(\frac{\pi y}{b}\right)$, we get

$$T\sin^3\left(\frac{\pi y}{b}\right)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi y}{b}\right)\cosh\left(\frac{n\pi x}{a}\right).\frac{n\pi}{b}$$

$$\frac{T}{4}\left(3\sin\frac{\pi y}{b}-\sin\frac{3\pi y}{b\pi}\right)=\sum_{n=1}^{\infty}E_n\sin\left(\frac{n\pi y}{b}\right)\cosh\left(\frac{n\pi x}{a}\right).\frac{n\pi}{b}$$

$$\Rightarrow\qquad E_1=\frac{3Tb}{4\pi\cosh\left(\frac{\pi a}{b}\right)};E_3=-\frac{Tb}{12\pi\cosh\left(\frac{3\pi a}{b}\right)}\text{ and all other } E_n=0.$$

Hence Eq.(4) gives required solution

$$u(x,y)=\frac{Tb}{4\pi}\left[\frac{3\sin\left(\frac{\pi y}{b}\right)\sinh\left(\frac{\pi x}{b}\right)}{\cosh\frac{\pi a}{b}}-\frac{\sin\left(\frac{3\pi y}{b}\right)\sinh\left(\frac{3\pi x}{b}\right)}{3\cosh\left(\frac{3\pi a}{b}\right)}\right]$$

Example5:- Solve $\nabla^2u=0;\ 0\le x\le\pi,\ \ 0\le y\le 1$ subject to $u(x,0)=u_0\cos x,\ u(x,1)=u_0\sin^2 x,$ $u_x(0,y)=0=u_x(\pi,y).$

Solution:- Hint. $u(x,y)=A_0y+u_0\cosh y\cos+\sum_{n=1}^{\infty}D\sinh ny\cos nx$ (4)

Using $BC\ u(x,1)=u_0\sin^2 x,$ we get $u_0\sin^2 x=A_0+u_0\cosh 1\cos x+\sum_{n=1}^{\infty}D_n\sinh(n)\cos nx$

$$\Rightarrow\qquad A_0=\frac{u_0}{2},\qquad D_1=-u_0\coth 1,\qquad D_2=-\frac{u_0}{2}\operatorname{cosech} 2 \tag{5}$$

Using Eq. (5) in Eq. (4) we get required solution

$$u(x,y)=\frac{u_0 y}{2}+u_0\cosh y\cos x-u_0\cosh 1\sinh y\cos x-\frac{u_0}{2}\cos ech\,2\sinh(2y)\cos 2x \qquad (6)$$

Example6:- Find the steady state temperature distribution in a rectangular bar in which heat is generated At a constant rate q per unit volume and no temperature gradient in the z-direction

Solution:- The given BVP is defined as $K\left(u_{xx}+u_{yy}\right)-q=0$ (1)

Where K is constant thermal conductivity

$$BC:\quad u_x(0,y)=0=u(a,y)=0 \qquad (2)$$

$$u_y(x,0)=0=u(x,b)=0 \qquad (3)$$

Eq.(1) is non-homogeneous, therefore separation of variables method is not applicable. If we

Assume the solution as $u(x,y)=\nu(x,y)+w(x)$ (4)

Then problem Eq.(1) and (3) reduced to the following two problems

$$\frac{d^2w}{dx^2}+\frac{q}{k}=0.\quad \left(\frac{dw}{dx}\right)_0=0\quad w(a)=0 \qquad (5)$$

And $\quad \upsilon_{xx}+\upsilon_{yy}=0,\quad \upsilon_x(0,y)=0=\upsilon(a,y)=\upsilon_y(x,0);\quad \upsilon(x,b)=-w(x)$ (6)

$$\because\quad K\left(\upsilon_{xx}+\upsilon_{yy}+w_{xx}\right)=q\ \Rightarrow\ K\left(\upsilon_{xx}+\upsilon_{yy}\right)=0;\ w_{xx}=-\frac{q}{K}$$

Solution Eq.(5) is $\quad w(x)=c_1x+c_2-\frac{qx^2}{2k}$ (7)

Applying conditions $w(0)=0=w(a)$, we get $c_1=0; 0=c_1a+c_2-\frac{qa^2}{2k}\Rightarrow c_1=0, c_2=\frac{qa^2}{2k}$

Hence Eq.(7) $\quad w(x)=\frac{qa^2}{2k}\left(1-\frac{x^2}{a^2}\right)$ (8)

Solution of Eq.(6) is $\upsilon(x,y)=\left(A\cos x+B\sin\lambda x\right)\left(Ce^{\lambda y}+De^{-\lambda y}\right)$ (9)

Applying $BC\ \upsilon_x(0,a)=0=\upsilon(a,y)$, we get $0=B,\quad 0=A\cos\lambda a+B\sin\lambda a$

(10)

$\Rightarrow\quad B=0,\quad A\cos\lambda a=0;\quad$ For non- trivial solution we get $\quad A=0,\ \lambda=\frac{(2n-1)}{2}\frac{\pi}{a}$

Applying $BC\ \upsilon_y(x,0)=0$, we get $0=C\lambda-D\lambda$, *i.e.*, $D=C$

Hence Eq. (9) becomes $\upsilon(x,y)=2AC\cos\left((2n-1)\frac{\pi x}{a}\right)\cosh\left(\frac{(2n-1)}{2}\frac{\pi y}{a}\right)$

principle of superposition; $\upsilon(x,y)=\sum_{n=1}^{\infty}E_n\cos\left((2n-1)\frac{\pi x}{a}\right)\cosh\left(\frac{(2n-1)}{2}\frac{\pi y}{a}\right)$ (11)

Using $BC\ \upsilon(x,b)=-w(x)=-\frac{qa^2}{2k}\left(1\frac{x^2}{a^2}\right)$, we get

$$-\frac{qa^2}{2k}\left(1\frac{x^2}{a^2}\right)=\sum_{n=1}^{\infty}E_n\cos\frac{(2n-1)\pi x}{2a}.\cosh\frac{(2n-1)\pi b}{2a}$$

Which is half-range Fourier cosine series. Therefore

$$E_n = \frac{2}{a\cosh\left(\frac{(2n-1)\pi b}{2a}\right)}\int_0^a \frac{-qa^2}{2k}\left(1\frac{x^2}{a^2}\right)\cos\left(\frac{2n-1}{2a}\pi x\right)dx = \frac{2q}{ka}\frac{(-1)^{n+1}}{ka\left(\frac{(2n-1)\pi}{2a}\right)^3\cosh\left(\frac{(2n-1)\pi b}{2a}\right)}$$

Hence $\upsilon(x,y) = E_n = \frac{2q}{ka}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\left(\frac{2n-1}{2a}.\pi x\right)\cosh\left(\frac{2n-1}{2a}\pi y\right)}{\left(\frac{(2n-1)\pi}{2a}\right)^3\cosh\left(\frac{(2n-1)\pi b}{2a}\right)}$ (12)

The solution of problem is obtained by adding Eqs. (8) and (12) we get

$$\upsilon(x,y) = \frac{qa^2}{2k}\left(1-\frac{x^2}{a^2}\right) + \frac{16qa^2}{k\pi^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\left(\frac{2n-1}{2a}.\pi x\right).\cos\left(\frac{2n-1}{2a}\pi y\right)}{(2n-1)^3\cosh\left(\frac{(2n-1)\pi b}{2a}\right)} \quad (13)$$

NOTE: (1) Above problem can also be solved by using $u(x,y) = \upsilon(x,y) + w(y)$.

(2) If q is variable rate $i.e., q = q(x)$ then we shall use $u(x,y) = \upsilon(x,y) + w(x)$. only

(3) If q is variable rate $i.e., q = q(y)$ then we shall use $u(x,y) = \upsilon(x,y) + w(y)$. only

Example7:- Solve the following Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$

Subject to $BC\ u(0,y) = u(5,y) = x(x,0) = u(x,4) = 0.$

Solution:- Given equation is not homogeneous therefore separation of variables method is not Applicable if we assume the solution as $u(x,y) = \upsilon(x,y) + w(x)$ (1)

Then given problem reduced to the following two problem $w_{xx} = 2,\ \ w(0) = (0) = w(5)$ (2)

$$\upsilon_{xx} + \upsilon_{yy} = 0;\ \upsilon(0,y) = 0 = \upsilon(5,y)\ \ \upsilon(x,0) = -w(x)$$

$$\upsilon(x,4) = -w(x) \quad (3)$$

$\left(\because\ \upsilon_{xx} + \upsilon_{yy} + w_{xx} = 2 \Rightarrow \upsilon_{xx} + \upsilon_{yy} = 0,\ \ w_{xx} = 2\right)$

Solution of Eq.(2) is $w = x^2 + c_1 x + c_2.$

Applying $BC\ \ w(0) = 0 = w(5)$, we get $0 = c_2,\ \ 0 = 25 + 5c_1, i.e.,\ c_2 = 0, c_1 = -5$

Hence $w(x) = x^2 - 5x$ (4)

Solution of Eq.(3) by separation of variables method

$$\upsilon(x,y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y) \quad (5)$$

Applying $BC\ \upsilon(0,y) = 0 = \upsilon(5,y)$, we get ; $0 = A,\ \ B\sin 5\lambda = 0$

For non-trivial solution we set $B \neq 0,\ \ \lambda = \frac{n\pi}{5}.$

Hence $$\upsilon(x,y) = B\sin\left(\frac{n\pi x}{5}\right)\left(C\cosh\left(\frac{n\pi y}{5}\right) + D\sinh\left(\frac{n\pi y}{5}\right)\right)$$

principle of superposition; $$\upsilon(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{5}\left(C\cosh\left(\frac{n\pi y}{5}\right) + D\sinh\left(\frac{n\pi y}{5}\right)\right) \quad (6)$$

Applying BC, we get $$\upsilon(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{5} \quad (7)$$

And $$\upsilon(x,4) = \sum_{n=1}^{\infty}\left(C_n\cosh\left(\frac{4n\pi}{5}\right) + D_n\sinh\left(\frac{4n\pi}{5}\right)\right)\sin\left(\frac{n\pi}{5}\right) \quad (8)$$

Which are half-range Fourier series therefore from Eq. (7),

$$C_n = \frac{2}{5}\int_0^5 \upsilon(x,0)\sin\left(\frac{n\pi x}{5}\right)dx = \frac{2}{5}\int_0^5 (5x - x^2)\sin\left(\frac{n\pi x}{5}\right)dx \qquad [\because w(x) = \upsilon(x,0)]$$

$$C_n = \begin{cases} \dfrac{200}{\pi^3 n^3}\ ; & n = odd \\ 0\ ; & n = even \end{cases}$$

From Eq. (8), $$\left(C_n\cosh\frac{4n\pi}{5} + D_n\sinh\frac{4n\pi}{5}\right) = \frac{2}{5}\int_0^5 \upsilon(x,4)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{2}{5}\int_0^5 (5x - x^2)\sin\left(\frac{n\pi x}{5}\right)dx$$

$$C_n\cosh\frac{4n\pi}{5} + D_n\sinh\left(\frac{4n\pi}{5}\right) = C_n$$

$$\Rightarrow D_n = \left(\frac{1-\cosh\frac{4n\pi}{5}}{\sinh\left(\frac{4n\pi}{5}\right)}\right) = C_n \Rightarrow = \begin{cases} \dfrac{200}{n^3\pi^3}\left(\dfrac{1-\cosh\frac{4n\pi}{5}}{\sinh\left(\frac{4n\pi}{5}\right)}\right) ; & n = odd \\ 0 ; & n = even \end{cases} \quad (10)$$

Using in Eq. (6), we get

$$\upsilon(x,y) = \sum_{n=odd}^{\infty}\frac{200}{n^3\pi^3}\sin\left(\frac{n\pi x}{5}\right)\left[\cosh\frac{n\pi y}{5} + \frac{\left(1-\cosh\frac{4n\pi}{5}\right)}{\sinh\frac{4n\pi}{5}}.\sinh\frac{n\pi x}{5}\right]$$

$$= \sum_{n=odd}^{\infty}\frac{200}{n^3\pi^3}\frac{\sin\left(\frac{n\pi x}{5}\right)}{\sin\left(\frac{4n\pi}{5}\right)}\left[\sinh\frac{n\pi}{5}(4-y) + \sinh\frac{n\pi y}{5}\right] \quad (11)$$

Complete solution of given problem is

$$u(x,y)=\upsilon(x,y)+w(x)=\left(x^2-5x\right)+\frac{200}{\pi^3}\sum_{n=1,3,5,..}^{\infty}\frac{\sin\frac{n\pi x}{5}}{\sinh\frac{4n\pi}{5}}\left[\sinh\frac{n\pi}{5}(4-y)+\sinh\frac{n\pi y}{5}\right] \quad (12)$$

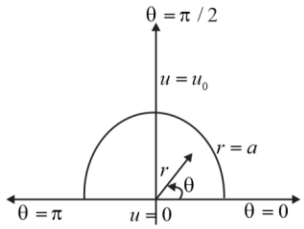
Example8:- Find the steady state temperature in a semi-circular plate of radius a, insulated on both the Faces with its curved boundary kept at constant temperature u_0 and its bounding diameter is Kept at zero temperature

Solution:- The given problem (BVP) is defined as

$$\nabla^2 u=0, \quad i.e., \quad u_{rr}+\frac{u_r}{r}+\frac{u_\theta\theta}{r^2}=0 \quad (1)$$

$$BC: \quad u(a,\theta)=u_0, \;\; u(r,0)=0=u(r,\pi) \quad (2)$$

Let $u=R(r)H(\theta)$ be variables separable solution of Eq. (1).then Eq.(1) becomes



$$\frac{r^2R''+rR'}{R}+\frac{H''}{H}=0 \quad (3)$$

The general solution of Eq. (3)(As discussed in Interior Dirichlet Problem) is

$$u(r,\theta)=(A\cos\lambda\theta+B\sin\lambda\theta)\left(Cr^{\lambda}+Dr^{-\lambda}\right) \quad (4)$$

Applying $BC: u(r,\theta)=0,$ we get $A=0,$ while $BC\; u(r,\pi)=0$ gives $B\sin\lambda\pi=0.$

For a non-trivial solution we get $B\neq 0, \;\; \sin\lambda\pi=0, \;\; i.e., \quad \lambda\pi=n\pi, \quad n=1,2,.....$

Hence the possible solution is $u(r,\theta)=B\sin n\theta\left(Cr^{\lambda}+Dr^{-\lambda}\right)$ (5)

Since $u(r,\theta)\to$ finite as $r\to 0,$ therefore constant D must be zero Hence using

Principle of super position in Eq.(5) we get $u(r,\theta)=\sum_{n=1}^{\infty}B_n r^n\sin n\theta$ (6)

Using $BC: \quad u(a,\theta)=u_0$, in Eq.(6), we get $u(a,\theta)=u_0=\sum_{n=1}^{\infty}B_n a^n\sin n\theta$

Which is a half-range Fourier sine series therefore

$$B_n a^n=\frac{2}{\pi}\int_0^{\pi}u_0\sin n\theta\; a\theta=\begin{cases}\frac{4u_0}{n\pi} & for\; n=odd\\ 0 & for\; n=even\end{cases}$$

Using in Eq. (6) we get required solution

$$u(r,\theta)=\sum_{n=1}^{\infty}\frac{4u_0 r^{(2n-1)}}{(2n-1)\pi}\sin(2n-1)\theta \quad (7)$$

Example9:- A thin annulus occupies the region $0 \le a \le r \le b, 0 \le \theta \le 2\pi$ whose faces are insulated The temperature along the inner edge is 0 while along outer edge temperature is held at $u = k\cos(\theta/2)$, where K is a constant determine the temperature distribution the annulus.

Solution:- The given problem (BVP) is defined as

$$\nabla^2 U = 0, \quad a \le r \le b, \quad 0 \le \theta \le 2\pi \qquad (1)$$

Subject to BCs : $\quad u(a,\theta) = 0 \qquad (2)$

$$u(b,\theta) = k\cos\frac{\theta}{2} \qquad (3)$$

The general solution of the problem is $u(r,\theta) = (c_1 r^n + c_2 r^{-n})(c_3 \cos n\theta + c_4 \sin n\theta) \qquad (4)$

Using BC, (2) we get $0 = (c_1 a^n + c_2 a^{-n})(c_3 \cos n\theta + c_4 \sin n\theta) \Rightarrow c_1 a^n + c_2 a^{-4} = 0, or\ c_2 = -c_1 a^{2n}$

Using in Eq. (4), we get $u(r,\theta) = \left(r^n - \dfrac{a^{2n}}{r^n}\right)(A\cos n\theta + B\sin n\theta)$

principle of superposition gives $u(r,\theta) = \sum_{n=1}^{\infty} \left(r^n - \dfrac{a^{2n}}{r^n}\right)(A_n \cos n\theta + B_n \sin n\theta) \qquad (5)$

Using BC (2), we obtain $u(b,\theta) = k\cos\dfrac{\theta}{2} = \sum_{n=1}^{\infty}(b^n - b^{-n}a^{2n})(A_n \cos n\theta + B_n \sin n\theta)$

Which is a full-range Fourier series. Therefore,

$$A_n(b^n - b^{-n}a^{2n}) = \frac{1}{\pi}\int_0^{2\pi} K\cos\frac{\theta}{2}\cos n\theta\, d\theta = \frac{k}{2\pi}\int_0^{2\pi}\left[\cos\left(n+\frac{1}{2}\right)\theta + \cos\left(n-\frac{1}{2}\right)\theta\right]d\theta = 0$$

$$\Rightarrow \qquad A_n = 0 \qquad (6)$$

And $B_n(b^n - b^{-n}a^{2n}) = \dfrac{k}{\pi}\displaystyle\int_0^{2\pi}\cos\frac{\theta}{2}\sin n\theta\, d\theta = \frac{k}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)\theta + \sin\left(n-\frac{1}{2}\right)\theta\right]d\theta$

$$= -\frac{k}{2\pi}\left[\frac{\cos\left(n+\frac{1}{2}\right)\theta}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_{\theta}^{2\pi}$$

$$= -\frac{k}{2\pi}\left(-\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right) = +\frac{k}{\pi}\left(-\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right) = \frac{k}{\pi}\frac{2n}{n^2 - \frac{1}{4}}$$

Or $\qquad B_n(b^n - b^{-n}a^{2n}) = \dfrac{8kn}{\pi(4n^2 - 1)} \qquad (7)$

Using Eq. (6) and (7) in Eq.(5) we get required solution of problem

$$u(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2 - 1}\left[\frac{(r/a)^n - (a/r)^n}{(b/a)^n - (a/b)^{-n}}\right]\sin n\theta$$

Example10:- A function u of r and θ satisfying the equation $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0$

Within the region of the plane bounded by $r=a, r=b, \theta=0, \theta=\dfrac{\pi}{2}$. its value along the

Boundary $r=a$ is $\theta\left(\dfrac{\pi}{2}-\theta\right)$, along the other boundaries is zero prove that

$$u=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(r/b)^{4n-2}-(b/r)^{4n-2}}{(a/b)^{4n-2}-(b/a)^{4n-2}}\left[\frac{\sin(4n-2)\theta}{(2n-1)^3}\right]$$

Solution:- The given problem can be defined as $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0$ (1)

Subject to the following boundary conditions

(i) $u(b,\theta)=0, \quad 0<\theta<\dfrac{\pi}{2}$

(ii) $u(r,\pi/2)=0 \quad a<r\le b$ (2)

(iii) $u(r,0)=0, \quad a<r\le b$

(iv) $u(a,\theta)=\theta(\pi/2-\theta), \quad 0<\theta<\pi/2$

The general solution by separable of variables method is

$u=\left(c_1 r^{\lambda}+c_2 r^{-\lambda}\right)\left(c_3\cos\lambda\theta+c_4\sin\lambda\theta\right)$ (As discussed in derivation)

Use of BC (iii) gives $0=c_3\left(c_1 r^{\lambda}+c_2 r^{-\lambda}\right)$ (3) $\Rightarrow c_3=0$

Use of BC (ii) gives $0=c_4\sin\lambda\dfrac{\pi}{2}\left(c_1 r^{\lambda}+c_2 r^{-\lambda}\right)$

For non-trivial solution we set $c_4\ne 0$, $\sin\dfrac{\pi\lambda}{2}=0$, *i.e.,* $\lambda=2n$, $n=1,2,\ldots$. thus the

Possible solution of the given equation takes the form $U(r,\theta)=c_4\sin(2n\theta)\left(c_1 r^{2n}+c_2 r^{-2n}\right)$ ..(4)

Applying the boundary condition (i), we get $0=c_4\sin(2n\theta)\left(c_1 b^{2n}+c_2 b^{-2n}\right)$

Which gives $c_2=-c_1 b^{4n}$. Therefore, Eq. (4) becomes $U(r,\theta)=c_1c_4\sin(2n\theta)\left[r^{2n}-r^{-2n}b^{4n}\right]$

Using principle of superposition we get $U(r,\theta)=\displaystyle\sum_{n=1}^{\infty}c_n\sin(2n\theta)\left(r^{2n}-r^{-2n}b^{4n}\right)$ (5)

Using boundary condition (iv), we get $\theta\left(\dfrac{\pi}{2}-\theta\right)=\sum\ \sin(2n\theta)\left(\dfrac{a^{4n}-b^{4n}}{a^{2n}}\right)$

Which is a half-range Fourier sine series. Therefore,

$$\frac{2}{\pi/2}\int_0^{\pi/2}\theta\left(\frac{\pi}{2}-\theta\right)\sin(2n\theta)=c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)-\frac{1}{4n^3}\left\{(-1)^n-1\right\}=\frac{\pi}{4}c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)$$

i.e.,

$$\frac{\pi}{4}c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)=\begin{cases}\dfrac{1}{2n^3}\ ; & \textit{for n odd}\\ 0\ ; & \textit{for n even}\end{cases}$$

Using in Eq. (5), we get required solution

$$u(r,\theta)=\sum_{1}^{\infty}\frac{2}{\pi}\frac{1}{(2n-1)^3}\left(\frac{a}{r}\right)^{4n-2}\sin(4n-2)\theta\left(\frac{r^{8n-4}-b^{8n-4}}{a^{8n-4}-b^{8n-4}}\right)$$

Example11:- A homogeneous thermally conducting cylinder occupies the region $0\le r\le a$, $0\le\theta\le\pi$, $0\le z\le h$, where r,θ,z are cylindrical coordinates the top $z=h$ and the lateral surfaces $r=a$ are held at 0°, while the base $z=0$ is held at 100°. Assuming that there are no sources of heat generation within the cylinder find the steady-temperature distribution within the Cylinder

Solution:- The gives BVP is defined as $\nabla^2 u=0$ (1)

Subject to $BCs: u=0^\circ\ on\ z=h,;\quad u=0^\circ\ on\ r=a,\quad u=100^\circ\ on\ z=0,$ (2)

The general solution of the Laplace equation in cylindrical coordinate is

$$r(r,\theta,z)=j_n(\lambda r)(c_1\cos\mu\theta+c_2\sin\mu\theta)\left(c_3e^{\lambda z}+c_4e^{\lambda z}\right) \quad (3)$$

BC shows that temperature $u(r,\theta,z)$ is independent of θ this is possible only when $\mu=0$ in Eq.(3) hence $u(r,z)=J_0(\lambda r)\left(Ae^{\lambda z}+Be^{-\lambda z}\right)$ (4)

Using the $BC:\quad u=0\ on\ z=h$, we get $0=J_0(\lambda r)\left(Ae^{\lambda h}+Be^{-\lambda h}\right)\Rightarrow B=-\frac{Ae^{\lambda h}}{e^{-\lambda h}}$

Therefore the solution is $u(r,z)=\frac{J_0(\lambda r)A}{e^{-\lambda h}}\left[e^{\lambda(z-h)}-e^{-\lambda(z-h)}\right]=J_0(\lambda a)\sinh\lambda(z-h)$ ..(5)

Now using the $BC: u=0\,on\,r=a$, we have $0=A_1J_0(\lambda a)\sinh\lambda(z-h)\Rightarrow J_0(\lambda a)=0$

$J_0(\lambda a)=0$, which has infinitely many positive roots. Let $\xi_n=a\lambda$, then solution given by

Eq. (5) becomes $u(r,z)=A_1J_0\left(\frac{\xi_n r}{a}\right)\sinh\left[\frac{\xi_n}{a}(z-h)\right],\quad n=1,2,3,...$

Using the principle of superposition we obtain $u(r,z)=\sum_{n=1}^{\infty}A_nJ_0\left(\frac{\xi_n r}{a}\right)\sinh\left[\frac{\xi_n}{a}(z-h)\right]$ (6)

The $BC:\quad u=100^\circ\ on\ z=0$ gives $100=\sum_{n=1}^{\infty}A_n\sinh\left(-\frac{\xi_n h}{a}\right)J_0\left(\frac{\xi_n r}{a}\right)$

Which is a Fourier-Bessel series multiplying both sides with $rJ_0(\xi_m r/a)$ and integrating $w.r.t.\ r$ From 0 to a, we get $100\int_0^a rJ_0\left(\frac{\xi_m r}{a}\right)dr=\sum_{n=1}^{\infty}A_n\sinh\left(-\frac{\xi_n h}{a}\right)\int_0^a rJ_0\left(\frac{\xi_m r}{a}\right)J_0\left(\frac{\xi_m r}{a}\right)dr$

Using the orthogonality property of Bessel's function

$$\int_0^a xJ_n(\alpha_i x)J_n(\alpha_j x)dr=\begin{cases}0 & if\ i=j\\ \frac{a^2}{2}J_{n+1}^2(\alpha_i) & if\ i\ne j\end{cases}$$

Where α_i,α_j are the zeros of $J_n(x)=0$, we have

$$100\int_0^a rJ_0\left(\frac{\xi_m r}{a}\right)dr=\sum_{n=1}^{\infty}A_n\sinh\left(-\frac{\xi_n h}{a}\right)\frac{a^2}{2}J_1^2(\xi_n)$$

So,

$$A_n = \frac{200}{a^2 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a r J_0\left(\frac{\xi_n r}{a}\right) dr = \frac{200}{\xi_n^2 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} x J_0(x)\, dx \quad \because x = \frac{\xi_n}{a} \quad (7)$$

Using the recurrence relation $xJ_1(x) = \int x J_0(x)\, dx$ in Eq. (7) we get

$$A_n = \left[\frac{200x\, J_1(x)}{\xi_n^2 \sinh(-\xi_n h/a) J_1^2(\xi_n)}\right]_0^{\xi_n} = \frac{200}{\xi_n \sinh(-\xi_n h/a) J_1(\xi_n)}$$

Hence the required temperature distribution inside the cylinder is

$$u(r,z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\xi_n r/a) \sinh\left[\xi_n \frac{(z-h)}{a}\right]}{\xi_n \sinh(-\xi_n h/a) J_1(\xi_n)}$$ ..(8)Where ξ_n are the positive zeros of $J_0(\xi)$.

Example12:- In a solid sphere of radius 'a', the surface is maintained at the temperature given by $f(\theta) = \begin{cases} k\cos\theta, & 0 \le \theta < \pi/2 \\ 0 \quad, & \pi/2 < \theta < \pi \end{cases}$. Prove that the steady state temperature within the solid is

$$u(r,\theta) = k\left[\frac{1}{4}P_0(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) + \frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{3}{32}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + ...\right]$$

Solution:- The given BVP is defined as

$$\nabla^2 u = 0 \text{ where } \nabla^2 = \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad (1)$$

Subject to $u(a,\theta) = f(\theta)$,we get

Since u is independent of ϕ and finite at $r = 0$ therefore general solution of Eq. (1) is

$$u(a,\theta) = \sum_{n=1}^{\infty} A_n r^n P_n(\cos\theta) \quad (2)$$

Using the given $BC: u(a,\theta) = f(\theta)$ we get $u(a,\theta) = f(\theta) \sum_{n=1}^{\infty} A_n a^n P_n(\cos\theta) = \sum_{n=1}^{\infty} b_n P_n(\cos\theta)$

Where $A_n a^n = b_n$. This is a Fourier-Legendre series where

$$b_n = \frac{2n+1}{2} \int_{-1}^{1} f(\theta) P_n(\cos\theta)\, dr \qquad \text{(use of orthogonality) (3)}$$

In this problem limits are 0 to 1 because f is zero when $-1 \le x \le 0$, therefore

$$b_n = \frac{2n+1}{2} \int_0^1 f(\theta) P_n(\cos\theta)\, dr = \frac{2n+1}{2} \int_0^1 kx P_n(x)\, dx; \quad x = \cos\theta$$

If $n = 0$ *then* $b_0 = \frac{1}{2}\int_0^1 kx.P_0(x)\, dr = \frac{1}{2}\int_0^1 kx.1.dx = \frac{k}{4} = A_0 \qquad \because P_0(x) = 1$

If $n = 1$ *then* $b_1 = \frac{3}{2}\int_0^1 kx.x.dx = \frac{k}{4} = A_1 a \qquad \because P_1(x) = x$

If $n = 2$ *then* $b_2 = \frac{5}{2}\int_0^1 kx P_2(x)\, dx = \frac{5}{2}\int_0^1 kx \frac{3x^2-1}{2} dx = \frac{5}{16}k = A_2 a^2 \qquad \because P_2(x) = \frac{3x^2-1}{2}$

If $n=3$ then $b_3=\frac{7}{2}\int_0^1 kxP_3(x)\,dx=\frac{7}{2}\int_0^1 kx\frac{5x^2-3x}{2}dx=0$ $\qquad \because P_3(x)=\frac{5x^2-3x}{2}$

$$b_4=-\frac{3}{32}k=A_4a^4 \qquad \because P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$$

Substituting these values into Eq. (2) we obtain the required temperature distribution

$$u(r,\theta)=k\left[\frac{1}{4}P_0(\cos\theta)+\frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta)+\frac{5}{16}\left(\frac{r}{a}\right)^2P_2(\cos\theta)\right.$$
$$\left.+\left(-\frac{3}{32}\right)\left(\frac{r}{a}\right)^4P_4(\cos\theta)+\ldots\ldots\right] \quad (4)$$

Example13:- A thermally conducting solid bounded by two concentric spheres of radii a and $b(a<b)$ is Such that internal boundary is kept at $f_1(\theta)$ and the outer boundary at $f_2(\theta)$ find steady State temperature in the solid

Solution:-The given problem is defined as $\nabla^2u=0$ (1)

Subject to the boundary conditions $u=f_1(\theta)$ at $r=a$, $\quad u=f_2(\theta)$ at $r=a$ (2)

For axially symmetric case the solution of the Laplace equation (1) is

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(A_nr^n+\frac{B_n}{r^n+1}\right)P_n(\cos\theta) \quad (3)$$

Using the BCs: $\quad f_1(\theta)=\sum_{n=0}^{\infty}\left(A_na^n+\frac{B_n}{r^n+1}\right)P_n(\cos\theta)$ (4)

$$f_2(\theta)=\sum_{n=0}^{\infty}\left(A_nb^n+\frac{B_n}{r^n+1}\right)P_n(\cos\theta) \quad (5)$$

Which are Fourier-Legendre's series Applying orthogonality property

$$\int_0^{\pi}P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta=\begin{cases}0 & ;\ m\neq n\\ \frac{2}{2n+1} & ;\ m=n\end{cases}$$

We get $\int_0^{\pi}f_1(\theta)P_m(\cos\theta)\sin\theta\ d\theta=\sum_{n=0}^{\infty}\left(A_na^n+\frac{B_n}{a^{n+1}}\right)\int_0^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta$

$$=\left(A_ma^m+\frac{B_b}{a^{m+1}}\right)\frac{2}{2m+1} \quad (6)$$

And $\int_0^{\pi}f_2(\theta)P_m(\cos\theta)\sin\theta\ d\theta=\sum_{n=0}^{\infty}\left(A_nb^n+\frac{B_n}{b^{n+1}}\right)\int_0^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta$

$$=\left(A_mb^m+\frac{B_m}{b^{m+1}}\right)\frac{2}{2m+1} \quad (7)$$

Let
$$\frac{2}{2m+1}\int_0^{\pi}f_1(\theta)P_m(\cos\theta)\sin\theta\,d\theta=C_m$$
$$\frac{2}{2m+1}\int_0^{\pi}f_2(\theta)P_m(\cos\theta)\sin\theta\,d\theta=D_m$$

Then Eq. (6) and (7) transformed to $A_m a^m + \frac{B}{a^{m+1}} = C_m \quad \Rightarrow \quad A_m b^m + \frac{B}{b^{m+1}} = D_m$

On solving we obtain

$$A_m = \frac{C_m a^{m+1} - D_m b^{m+1}}{a^{2m+1} - b^{2m+1}} \qquad (8) \qquad B_m = \frac{a^{m+1} - b^{m+1}\left(C_m b^m - D_m a^m\right)}{b^{2m+1} - a^{2m+1}} \qquad (9)$$

Hence the required steady temperature distribution is

$U(r,\theta) = \sum_{n=0}^{\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)P_m(\cos\theta)$ Where A_m and B_m are given by Eq. (8) and (9).

Hyperbolic Differential Equations

Example:-1 A. A solve the following wave equations

(a) $\frac{\partial^2 y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}; \quad 0 \le x \le l, \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(l,t)$

$IC: \quad u(x,0) = f(x) = \mu x(l-x), \quad u_1(x,0) = 0$

(b) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le \pi, \ t \ge 0$

$BC: \quad u(0,t) = 0 = u(\pi,t)$

$IC: \quad u(x,0) = x, \quad u(x,0) = 0$

(c) $\frac{\partial^2 u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le 1; \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(1,t)$

$$IC: \quad u(x,0) = f(x) = \begin{cases} x & ; \ 0 < x < \frac{1}{4} \\ \frac{1}{2} - x & ; \ \frac{1}{4} < x < \frac{1}{2} \\ 0 & ; \ \frac{1}{4} < x < 1 \end{cases}, \quad u_t(x,0) = 0$$

(d) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le l, \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(l,t)$

$$IC: \quad u(x,0) = \begin{cases} \frac{kx}{b} & ; \ 0 < x < b \\ \frac{k(l-x)}{(l-b)} & ; \ b < x < l \end{cases}$$

Solution:- (a) We have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le l, \ \ t \ge 0 \tag{1}$$

$$BC: \quad u(0,t) = 0 = u(l,t)$$

$$IC: \quad u(X,0) = f(x) = \mu x(l-x); \quad u_t(x,0) = 0 \tag{2}$$

Let $u = X(x)T(t)$ be variables separable solution of Eq. (1) then (3)

$$\frac{T''}{c^2 T} = \frac{X''}{X} \tag{4}$$

Eq. (4) holds goods if each side is equal to same separation constant since BC in x are periodic

Therefore X must be periodic for this we set

$$\frac{X''}{X} = -\lambda^2, \frac{T}{c^2 T} = -\lambda^2$$

On solving we get

$$u = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct) \tag{5}$$

Applying BC (2) we get

$$A = 0, \qquad B\sin\lambda l = 0$$

For non-trivial solution we get $\quad B \ne 0, \quad \sin\lambda l = 0$

i.e., $$B \ne 0, \quad \lambda = \frac{n\pi}{l} \tag{6}$$

Using in Eq. (5) we get

$$u = \sin\left(\frac{n\pi x}{l}\right)\left(C\cos\left(\frac{n\pi ct}{l}\right) + D\sin\left(\frac{n\pi ct}{l}\right)\right) \tag{7}$$

Using principle of superposition we get

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left(C_n\cos\left(\frac{n\pi ct}{l}\right) + D_n\sin\left(\frac{n\pi ct}{l}\right)\right)$$

Applying BC $u_t(x,0) = 0$, we get $D_n = 0$

Hence Eq. (7) becomes

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \tag{8}$$

Applying $I.C.$ $u(x,0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

Which is half-range Fourier sine series therefore

$$C_n = \frac{2}{l}\int_0^1 f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$C_n = \frac{2}{l}\int_0^1 \mu x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2\mu}{l}\left[-x(l-x)\frac{1}{n\pi}\cos\frac{n\pi x}{l}+\int(l-2x)\frac{l}{n\pi}\cos\frac{n\pi x}{l}\,dx\right]_0^l$$

$$= \frac{2\mu}{0}\left[\left(\frac{1}{n\pi}\right)(l-2x)\sin\frac{n\pi x}{l}-2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2\mu}{l}\left[2\left(\frac{1}{n\pi}\right)^3\left(1-(-1)^n\right)\right] = \begin{cases} \dfrac{8\mu l^2}{(n\pi)^3}; & n = odd \\ 0 \quad ; & n = even \end{cases}$$

Hence $$u(x,t) = \sum_{n=1}^{\infty}\frac{8ul^2}{((2n-1)\pi)^3}\sin\left(\frac{(2n-1)\pi x}{l}\right)\cos\left(\frac{(2n-1)\pi ct}{l}\right)$$

(b) We have $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}; \quad 0 \le x \le \pi; \quad t \ge 0$ (1)

$BC: \quad u(0,t) = 0 = u(\pi,t)$ (2)

$IC: \quad u(x,0) = x\ ; \ u_1(x,0) = 0$ (3)

Proceeding on the same line as in (a) we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\cos(n,ct) \tag{8}$$

Applying $IC \ \ u(x,0) = x$, we get

$$x = \sum_{n=1}^{\infty} C_n \sin(nx)$$

Which is half-range Fourier sine series.

Therefore $$C_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)+\left(\frac{1}{n}\right)^2\sin^2(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n\right] = \frac{2}{n}(-1)^{n+1}$$

Hence $$u(x,t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)\cos(nct)$$

(c) We have $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}; \qquad 0 \le x \le 1$ (1)

$BC: \quad u(x,0) = 0 = u(l,t)$ (2)

$IC: \quad u(x,0) = f(x); \quad u_t(x,0) = 0$ (3)

Proceeding on the same line as in (a) we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\cos(n\pi ct) \tag{8}$$

Applying $I.C.\ u(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Which is half-range Fourier sine series therefore

$$C_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^{1/4} x\sin(n\pi x)\,dx + 2\int_{1/4}^{1/2}\left(\frac{1}{2} - x\right)\sin(n\pi x)\,dx$$

$$= 2\left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{\sin n\pi x}{(n\pi)^2}\right]_0^{\frac{1}{4}} + 2\left[-\frac{\left(\frac{1}{2} - x\right)\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{(n\pi)^2}\right]_{1/4}^{1/2}$$

$$= 2\left[-\frac{1}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{\sin\left(\frac{n\pi}{4}\right)}{(n\pi)^2} - \frac{\sin\frac{n\pi}{2}}{(n\pi)^3} + \frac{1}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \frac{1}{(n\pi)^2}\sin\frac{n\pi}{4}\right]$$

$$C_n = 2\left[\frac{2\sin\left(\frac{n\pi}{4}\right)}{(n\pi)^2} - \frac{\sin\frac{n\pi}{2}}{(n\pi)^2}\right] = \frac{2}{(n\pi)^2}\left[2\sin\left(\frac{n\pi}{4}\right) - 2\sin\frac{n\pi}{4}\cos\frac{n\pi}{4}\right]$$

$$= \frac{4}{(n\pi)^2}\left[1 - \cos\frac{n\pi}{4}\right]\sin\left(\frac{n\pi}{4}\right)$$

Hence $$u(x,t) = \sum_{n=1}^{\infty} \frac{4\sin\left(\frac{n\pi}{4}\right)}{(n\pi)^2}\left[1 - \cos\frac{n\pi}{4}\right]\sin(n\pi x)\cos(n\pi ct)$$

(d) We have $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$$BC: \quad u(0,t) = 0 = u(l,t) \qquad (2)$$

$$IC: \quad u(x,0) = \begin{cases} \dfrac{kx}{b} & ;\ 0 < x < b \\ \dfrac{k(l-x)}{(l-b)} & ;\ b < x < l \end{cases}, \qquad u_t(x,0) = 0 \qquad (3)$$

Proceeding on the same line as in (a) we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \qquad (8)$$

Where $C_n = \frac{2}{l}\int_0^{1/2}\frac{kx}{b}\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\int_{1/2}^{l}\frac{k(l-x)}{(l-b)}\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2kl^2 \sin\left(\frac{n\pi b}{l}\right)}{n^2\pi^2 b(l-b)}$$

Hence $$u(x,t) = \frac{2kl^2}{n^2 b(l-b)} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi b}{l}\right)}{n^2} \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \qquad (9)$$

Example:-2. Solve the following wave equations

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $\quad 0 < x < l; \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(l,t)$

$IC: \quad u(x,0) = k\sin\left(\frac{\pi x}{l}\right); \quad u_t(x,0) = 0$

(b) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $\quad 0 < x < 1; \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(l,t)$

$IC: \quad u(x,0) = k\sin 2\pi x; \quad u_t(x,0) = 0$

(c) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $\quad 0 < x < 1; \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(l,t)$

$IC: \quad u(x,0) = k\sin^3 \frac{\pi x}{l}; \quad u_t(x,0) = 0$

(d) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $\quad 0 < x < \pi; \quad t \ge 0$

$BC: \quad u(0,t) = 0 = u(\pi,t)$

$$IC: \quad u(x,0) = \begin{cases} \sin x \ ; & 0 < x < \pi/2 \\ 0 \ ; & \pi/2 < x < \pi \end{cases} \ ; \quad u_t(x,0) = 0$$

Solution:- (a) Proceeding on the same line as in example 1 A (a) we have

$$u(x,t)\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \qquad (8)$$

Using $IC: \quad u(x,0) = f(x) = k\sin\left(\frac{\pi x}{l}\right)$ we get

$$k\sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

In this case do not use Euler's integral formula but compare directly hence

$C_1 = k$ and all other $C_n = 0; \quad n \ge 2.$

Hence required solution of the problem

$$u(x,t) = k\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right) \qquad (7)$$

(b) Proceeding on the same line as in $1B$ (a) we get

$$u(x,t)\sum C_n \sin(n\pi x)\cos(n\pi ct) \quad (8)$$

Using $IC\ u(x,0) = f(x) = k\sin(2\pi x)$ we get

$$k\sin(2\pi x) = \sum C_n \sin(n\pi x)$$

$\Rightarrow$ $C_2 = k$, and all other $C_n = 0$.

Hence required solution of the problem

$$u(x,t) = k\sin(2\pi x)\cos(2\pi ct) \quad (9)$$

(c) Proceeding on the same line as in $1B$ (a) we get

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \quad (8)$$

Using IC : $u(x,0) = k\sin^3\left(\frac{\pi x}{l}\right)$, we get

$$k\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{k}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$\Rightarrow$ $C_1 = \frac{3k}{4}, C_3 = -\frac{k}{4}$ and all other $C_n = 0$.

Hence required solution of the problem is

$$u(x,t) = \frac{k}{4}\left(3\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right) \quad (9)$$

(d)Proceeding on the same line as in $2B$ (a) we get

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(\pi x)\cos(nct)$$

Using $IC\ u(x,0) = \begin{bmatrix} \sin x & ; & 0 < x < \pi/2 \\ 0 & ; & \pi/2 < x < \pi \end{bmatrix}$, *we get*

$$\left.\begin{matrix} \sin x \\ 0 \end{matrix}\right] = \sum_{n=1}^{\infty} C_n \sin(nx)$$

$\Rightarrow$ $C_1 = 1$ and all other $C_n = 0;\quad 0 < x < \pi/2$

And all $C_n = 0$ when $\pi/2 < x < \pi.$

Hence required solution of problem is

$$u(x,t)=\begin{cases}\sin x\cos(ct)\,; & 0<x\le \pi/2\\ 0\,; & \pi/2<x<\pi\end{cases} \quad (9)$$

Example:-3 Solve the following wave equations

(a) $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2};\quad 0<x<l;\quad t\ge 0$

$BC:\quad u(0,t)=0=u(l,t)$

$IC:\quad u(x,0)=0,\quad u_t(x,0)=k\sin^3\left(\dfrac{\pi x}{l}\right)$

(b) $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2};\quad 0<x<\pi;\quad t\ge 0$

$BC:\quad u(0,t)=0=u(\pi,t)$

$IC:\quad u(x,0)=0,\quad u_t(x,0)=k\sin x$

Solution:- (a) We have

$$\frac{\partial^2 u}{\partial t^2}c^2=\frac{\partial^2 u}{\partial x^2};\quad 0<x<l;\quad t\ge 0 \quad (1)$$

$$BC:\quad u(0,t)=0=u(l,t) \quad (2)$$

$$IC:\quad u(x,0)=0,\quad u_t(x,0)=k\sin^3\left(\frac{\pi x}{l}\right) \quad (3)$$

Let $u=X(x)T(t)$ be variables separable solution of Eq. (1) then Eq. (1) reduced to

$$\frac{X''}{X'}=\frac{T''}{c^2T} \quad (4)$$

Eq. (4) holds good if each side is equal to same separation constant since BC are periodic in x

Therefore X must be periodic. For this we consider $\dfrac{X''}{X}=\lambda^2,\dfrac{T''}{c^2T}=-\lambda^2.$

Hence $\qquad u=(A\cos\lambda x+B\sin\lambda x)(C\cos\lambda ct+D\sin\lambda ct) \quad (5)$

Applying BC (2) we get

$$A = 0, \qquad B \sin \lambda l = 0$$

For non-trivial solution we set $\qquad B \neq 0, \;\; \sin \lambda l = 0$

$$i.e., \qquad B = 0, \;\; \lambda = \frac{n\pi}{l} \tag{6}$$

Using in Eq. (5) we get

$$u(x,t) = \left[C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

Using principle of superposition we get

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right) \right) \sin\left(\frac{n\pi x}{l}\right) \tag{7}$$

Using $IC \quad u(x,0) = 0,$ we get $C_n = 0.$ hence

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \tag{8}$$

Applying $IC \;\; u_t(x,0) = k \sin^2(\pi x / l)$ we get

$$k \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{k}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \qquad \frac{3k}{4} = D_1 \frac{\pi c}{l}, -\frac{k}{4} = D_3\left(\frac{3\pi c}{l}\right) \text{ and all other } D_n = 0.$$

Hence required solution of the problem is

$$u(x,t) = \frac{kl}{4\pi c}\left[3 \sin \frac{\pi ct}{l} \sin \frac{\pi x}{l} - \frac{1}{3} \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l} \right] \tag{9}$$

(b)Proceeding on the same line as in (a) we get

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin(nct) \sin(nx) \tag{8}$$

Applying $IC \;\; u_t(x,0) = k\sin x$, we get

$$k\sin x = \sum_{n=1}^{\infty} D_n(nc)\sin nx$$

$\Rightarrow \qquad K = D_1.c$ and all other $D_n = 0$

Hence required solution of the problem is

$$u(x,t) = \frac{k}{c}\sin(ct)\sin(x) \qquad (9)$$

Example:-4.Solve the wave equation (in elastic bar)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}; \quad 0 < x < l; \quad t \geq 0$$

$BC: \quad u_x(0,t) = 0 = u_x(l,t)$

$IC: \quad u(x,0) = kx, \quad u_t(x,0) = 0$

Solution:- Proceeding on the same line as in derivation of elastic bar (II),we get

$$u = C_0 + \sum_{n=1}^{\infty} C_n\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \qquad (9)$$

Applying $IC \; u_t(x,0) = bx$, we get

$$bx = C_0 + \sum_{n=1}^{\infty} C_n\cos\left(\frac{n\pi x}{l}\right) \qquad (10)$$

Which is half-range Fourier cosine series. Therefore

$$C_0 = \frac{1}{l}\int_0^1 bx\;dx = \frac{bl}{2}$$

$$C_n = \frac{2}{l}\int_0^1 bx\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2b}{l}\left[x\sin\frac{n\pi x}{l}.\frac{l}{n\pi} + \left(\frac{l}{n\pi}\right)^2\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2b}{l}\left[\left(\frac{l}{n\pi}\right)^2(-1)^n - \left(\frac{l}{n\pi}\right)^2\right] = \frac{2b}{l}\left(\frac{l}{n\pi}\right)^2\left[(-1)^n - 1\right]$$

$$C_n = \begin{cases} -\dfrac{4bl}{(n\pi)^2} & ;\ n = odd \\ 0 & ;\ n = even \end{cases}$$

Hence Eq.(9) becomes

$$u(x,t) = \frac{bl}{2} + \sum_{n=1}^{\infty} \frac{-4bl}{((2n-1)\pi)^2} \cos\frac{(2n-1)\pi x}{l} . \cos\frac{(2n-1)\pi ct}{l} \qquad (11)$$

Example:-5. The points of trisection of a string are pulled aside through h on opposite sides of the Position of equilibrium and the string is released from rest. Derive an expression for the string At any subsequent time and show that the middle point of the middle point of the string always Remains at rest

Solution:-Give $IBVP$ can be defined as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} ; \qquad 0 < x < 3l \qquad (1)$$

$$BC : \quad u(0,t) = 0 = u(3l,t)$$

$$IC : \quad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(x,0) = f(x) \qquad (3)$$

Proceeding on the same line as in example (1) we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi ct}{3l}\right) \sin\left(\frac{n\pi x}{3l}\right) \qquad (8)$$

Applying $IC\ u(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{3l}\right)$$

Which is half-range Fourier series therefore

$$C_n = \frac{2}{3l}\int_0^{3l} f(x) \sin\left(\frac{n\pi x}{3l}\right) dx \qquad (9)$$

Where
$$f(x) = \begin{cases} y(OA) ;\ 0 \le x \le l \\ y(AC) ;\ l \le x \le 2l \\ y(CD) ;\ 2l \le x \le 3l \end{cases}$$

$y(OA)$ Represents Eq. of straight line OA.

Eq. of OA : $\quad y - 0 = \dfrac{h-0}{l-0}(x-0) \quad i.e., \quad y = \dfrac{hx}{l}$

Eq. of AC : $\quad y - h = \dfrac{h-h}{2l-l}(x-2l)$

i.e., $$y = h - \frac{2h}{l}(x-l) = \frac{h}{l}(3l-2x)$$

Eq. of CD : $$y + h = \frac{0+h}{3l-2l}(x-2l)$$

i.e., $$y = -h + \frac{h}{l}(x-2l)$$

$$y = \frac{h}{l}(x-3l)$$

Hence $$f(x) = \begin{cases} \dfrac{hx}{l} & ;\ 0 \le x \le l \\ \dfrac{h}{l}(3l-2l); & l \le x \le 2l \\ \dfrac{h}{l}(x-3l)\ ; & 2l \le x \le 3l \end{cases}$$

Using in Eq. (9) we get

$$C_n = \frac{2}{3l}\left[\int_0^l \frac{hx}{l}\sin\left(\frac{n\pi x}{3l}\right)dx + \int_l^{2l}\frac{h}{l}(3l-2x)\sin\left(\frac{n\pi x}{3l}\right)dx + \int_{2l}^{3l}\frac{h}{l}(x-3l)\sin\left(\frac{n\pi x}{3l}\right)dx\right]$$

$$= \frac{2h}{3l^2}\left[\int_0^1 x\sin\left(\frac{n\pi x}{3l}\right)dx + \int_l^{2l}(3l-2x)\sin\left(\frac{n\pi x}{3l}\right)dx + \int_{2l}^{3l}(x-3l)\sin\left(\frac{n\pi x}{3l}\right)dx\right]$$

$$= \frac{2h}{3l^2}\left[-\frac{3lx}{n\pi}\cos\left(\frac{n\pi x}{3l}\right) + \frac{9l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{3l}\right)\right]_0^l$$

$$+\frac{2h}{3l^2}\left[-(3l-2x)\frac{3l}{n\pi}\cos\left(\frac{n\pi x}{3l}\right) - \frac{18l^2}{n^2\pi^2}\sin\frac{n\pi x}{3l}\right]_l^{2l}$$

$$+\frac{2h}{3l^2}\left[-(3l-2x)\frac{3l}{n\pi}\cos\frac{n\pi x}{3l} + \frac{9l^2}{n^2\pi^2}\sin\frac{n\pi x}{3l}\right]_{2l}^{3l}$$

$$= \frac{2h}{3l^2}\left[-\frac{3l^2}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{9l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) + \frac{3l^2}{n\pi}\cos\left(\frac{2n\pi}{3}\right) - \frac{18l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right) + \frac{3l^2}{n\pi}\cos\left(\frac{n\pi}{3}\right) + \frac{18l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{3}\right) - \frac{3l^2}{n\pi}\cos\left(\frac{2n\pi}{3}\right) - \frac{9l^2}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right)\right]$$

$$=\frac{2h}{3l^2}\left[\frac{27l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-\frac{27l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$=\frac{18h}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{3}\right)-\sin\left(\frac{2n\pi}{3}\right)\right]=\frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3}+(-1)^2\sin\frac{n\pi}{3}\right]$$

$$C_n=\begin{cases}\dfrac{18h}{n^2\pi^2}\sin\dfrac{n\pi}{3} & ;\ n=even\\ 0 & ;\ n=odd\end{cases}$$

Using in Eq. (8) we get

$$u(x,t)=\sum_{n=1}^{\infty}\frac{9h}{n^2\pi^2}\cos\left(\frac{2n\pi ct}{3l}\right)\sin\left(\frac{2n\pi x}{3l}\right)\sin\left(\frac{2n\pi}{3}\right) \qquad (10)$$

If $x=\frac{3l}{2}$, *i.e.*, mid point then Eq. (10) gives

$$u\left(\frac{3l}{2},t\right)=\sum_{n=1}^{\infty}\frac{9h}{n^2\pi^2}\cos\left(\frac{2n\pi ct}{3l}\right)\sin(n\pi)\sin\left(\frac{2n\pi}{3}\right)$$

$$u\left(\frac{3l}{2},t\right)=0$$

Which shows that mid-point remains at rest always.

Example:-6. Solve the following non-homogeneous wave equation

(a) $u_{tt}=u_{xx}+Ax;\quad 0<x1;\quad t>0$

$BC:\quad u(0,t)=u(l,t)=0;\quad t>0$

$IC:\quad u(x,0)=0=u_t(x,0);\ 0<x<l$

(b) $u_{tt}=u_{xx}+A;\quad 0<x<l;\ t>0$

$BC:\quad u(0,t)=u(l,t)=0;\quad t>0$

$IC:\quad u(x,0)=0=u_t(x,0);\ 0<x<l$

(c) $u_{tt}=u_{xx}-\sin x;\quad 0<x<\pi/2, t>0$

$BC:\quad u(0,t)=0=u(\pi/2,t)$

$IC:\quad u(x,0)=0=u_t(x,0)=0;\ 0<x<\pi/2$

Solution:- (a) We have

$$u_{tt}=u_{xx}+w_{xx}+Ax;\qquad 0<x<1,\ t>0 \qquad (1)$$

$$BC:\quad u(0,t)=0=u(1,t);\qquad t>0 \qquad (2)$$

$$IC:\quad u(x,0)=0=u_t(x,0);\qquad 0<x<1 \qquad (3)$$

Suppose $\quad u(x,t)=\upsilon(x,t)+w(x)$

Substituting in Esq. (1) –(3) we get

$$\upsilon_{tt} = \upsilon_{xx} + w_{xx} + Ax$$

$$\upsilon(0,t) + w(0) = 0, \qquad \upsilon(1,t) + w(1) = 0$$

$$\upsilon(x,0) + w(x) = 0, \qquad \upsilon_t(x,0) = 0$$

$$\Rightarrow \quad \left.\begin{aligned} \upsilon_{tt} &= \upsilon_{xx} \\ \upsilon(0,t) &= 0 = \upsilon(1,t) \\ \upsilon(x,0) &= -w(x) \\ \upsilon_t(x,0) &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} w_{xx} &= -Ax \\ w(0) &= 0 = w(1) \end{aligned}\right\} \qquad \text{(4)-(5)}$$

Which are two *BVP*.

Solution of Eq. (5) is $w(x) = -\frac{Ax^3}{6} + C_1 x + C_2; \qquad w(0) = 0 = w(t)$

i.e., $$w(x) = \frac{Ax(1-x^2)}{6} \qquad (6)$$

Using Eq. (6) Eq. (4) we get

$$\left.\begin{aligned} &\upsilon_{tt} = \upsilon \\ &BC : \upsilon(0,t) = 0 = \upsilon(1,t) \\ &IC : \upsilon_t(x,0) = 0, \quad \upsilon(x,0) = -\frac{Ax}{6}(1-x^2) \end{aligned}\right] \qquad (7)$$

The variables separable solution of Eq. (7) is

$$\upsilon(x,t) = \sum_{n=1}^{\infty} C_n \cos(n\pi t)\sin(n\pi x)$$

Applying IC $\upsilon(x,0) = -\frac{Ax}{6}(1-x^2)$, we get

$$-\frac{Ax}{6}(1-x^2) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

Which is half-range Fourier sine series therefore

$$C_n = \frac{2}{1}\int_0^1 -\frac{Ax}{6}(1-x^2)\sin(n\pi x)\,dx$$

$$= \frac{A}{3}\int_0^1 x(1-x^2)\sin n\pi x\,dx$$

$$= \frac{A}{3}\left[\frac{x(1-x^2)}{n\pi}\cos n\pi x + \int \frac{(1-3x^2)\cos n\pi x}{n\pi}dx\right]_9^1$$

$$= -\frac{A}{3n\pi}\left[\frac{(1-3x^2)\cos n\pi x}{n\pi} + \int \frac{6x\sin n\pi x}{n\pi}dx\right]_0^1$$

$$= -\frac{2A}{n^2\pi^2}\left[-\frac{x\cos n\pi x}{n\pi}+\frac{\sin n\pi x}{(n\pi)}\right]_0^1$$

$$C_n = \frac{2A\cos(n\pi)}{n^2\pi^2} = \frac{2A(-1)^n}{n^2\pi^2}$$

Hence $$\upsilon(x,t) = \sum_{n=1}^{\infty}\frac{2A(-1)^n}{n^2\pi^2}\cos(n\pi t)\sin(n\pi x) \qquad (9)$$

Thus solution of original problem is

$$\upsilon(x,t) = \frac{Ax}{6}(1-x^2) + \frac{2A}{\pi^3}\sum_{n=1}^{\infty}(-1)^n\frac{\cos(n\pi t)\sin(n\pi x)}{n^3} \qquad (10)$$

(b)Proceeding on the same line as in (a) we get

$$w_{xx}(x) = -A; \quad w(0) = 0 = w(1) \qquad (4)$$

And $$\upsilon_{tt} = \upsilon_{xxx} \qquad (5)$$

$$\upsilon(0,t) = 0 = \upsilon(1,t)$$

$$\upsilon(x,0) = -w(x), \qquad \upsilon_t(x,0) = 0$$

Solution of Eq. (4) is $$w(x) = \frac{A}{2}x(1-x) \qquad (6)$$

Using in Eq. (5) we get

$$\upsilon_{tt} = \upsilon_{xx} \qquad (7)$$

$$BC: \quad \upsilon(0,t) = 0 = \upsilon(1,t)$$

$$IC: \quad \upsilon(x,0) = 0, \quad \upsilon(x,0) = \frac{x}{2}(1-x)$$

The variables separable solution of Eq. (7) is

$$\upsilon(x,t) = \sum_{n=1}^{\infty}C_n\cos(n\pi t)\sin(n\pi x) \qquad (8)$$

Applying IC $\upsilon(x,0) = \frac{Ax(1-x)}{2}$, we get

$$-\frac{Ax}{2}(1-x) = \sum_{n=1}^{\infty}C_n\sin(n\pi x)$$

Which is half-range Fourier sine series therefore

$$C_n = \frac{2}{1}\int_0^1 -\frac{Ax}{2}(1-x)\sin(n\pi x)\,dx$$

$$= -A\left[\frac{x(1-x)}{n\pi}\cos n\pi x + \int\frac{(1-2x)}{n\pi}\cos n\pi x\,dx\right]_0^1$$

$$= -A\left[\frac{1-2x}{(n\pi)^2}\sin nx - \frac{2}{(n\pi)^3}\cos(n\pi x)\right]_0^1$$

$$= \frac{2A}{(n\pi)^3}\left((-1)^n - 1\right) = \begin{bmatrix} -\dfrac{4A}{(n\pi)^3}; & n = odd \\ 0 \quad ; & n = even \end{bmatrix}$$

Hence
$$\upsilon(x,t) = -\frac{4A}{\pi^3}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t}{(2n-1)^3}\sin\left((2n-1)\pi x\right) \qquad (9)$$

Thus solution of original problem is

$$u(x,t) = \frac{Ax}{2}(1-x) - \frac{4A}{\pi^3}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t}{(2n-1)^3}\sin(2n-1)\pi x \qquad (10)$$

(c)Proceeding on the same line as in (a) we get

$$w_{xx} = \sin x; \qquad w(0) = 0 = w(\pi/2) \qquad (4)$$

And
$$\left.\begin{array}{l} \upsilon_{tt} = \upsilon_{xx} \\ BC : \upsilon(0,t) = 0 = \upsilon(\pi/2,t) \\ IC : \upsilon(x,0) = 0 - w(x), \quad \upsilon_t(x,0) = 0 \end{array}\right] \qquad (5)$$

Solution of Eq. (4) is

$$w(x) = \frac{2}{\pi}x - \sin x \qquad (6)$$

Solution of Eq. (5) by variables separable method is

$$\upsilon(x,t) = \sum_{n=1}^{\infty} C_n \cos(2nt)\sin(2nx) \qquad (7)$$

Applying $IC\ \upsilon(x,0) = -w(x) = -\frac{2}{\pi}x + \sin x$, we get

$$-\frac{2x}{\pi} + \sin x = \sum_{n=1}^{\infty} C_n \sin(2nx)$$

Which is half-range Fourier sine series therefore

$$C_n = \frac{2}{\pi/2}\int_0^{\pi/2}\left(-\frac{2}{\pi}x + \sin x\right)\sin(2nx)\,dx$$

$$= -\frac{8}{\pi^2}\int_0^{\pi/2} x\sin(2nx)\,dx + \frac{4}{\pi}\int_0^{\pi/2}\sin\ x\ \sin(2nx)\,dx$$

$$= -\frac{8}{\pi^2}\left[-\frac{x}{2n}\cos 2nx + \frac{\sin 2nx}{(2n)^2}\right]_0^{\pi/2} + \frac{2}{\pi}\int_0^{\pi/2}\left(\cos(2n-1)x - \cos(2n+1)x\right)dx$$

$$= -\frac{8}{\pi^2}\left[-\frac{\pi}{4n}\cos\ n\pi\right] + \frac{2}{\pi}\left[\frac{\sin(2n-1)\frac{\pi}{2}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1}\right]$$

$$= \frac{2}{n\pi}(-1)^n + \frac{2}{\pi}\left(\frac{(-1)^{n+1}}{2n-1} - \frac{(-1)^n}{2n+1}\right) = \frac{2(-1)^n}{n\pi} - \frac{2}{\pi}(-1)^n\left(\frac{4n}{4n^2-1}\right)$$

$$= \frac{(-1)^n}{\pi}\left[\frac{1}{n} - \frac{4n}{4n^2-1}\right]$$

$$C_n = \frac{2(-1)^{n+1}}{n\pi(4n^2-1)}$$

Hence $$\upsilon(x,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cos(2nt)\sin(2nx)}{n(4n^2-1)} \quad (8)$$

Thus solution of original problem is

$$\upsilon(x,t) = \frac{2}{\pi}x - \sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cos(2nt)\sin 2nx}{n(4n^2-1)} \quad (9)$$

PREVIOUS YEARS QUESTIONS

METHOD OF SEPARATION OF VARIABLES

Q1. Find the solution of the initial-boundary value problem

$u_t - u_{xx} + u = 0,\ 0 < x < l,\ t > 0$

$u(0,t) = u(l,t) = 0,\ t \geq 0$

$u(x,0) = x(l-x),\ 0 < x < l$. **[7c UPSC CSE 2023]**

Q2. Solve the differential equation $u_x^2 = u_y^2$ by variable separation method. **[(6b) 2015 IFoS]**

HEAT EQUATION: PARABOLIC PDE

Q3. Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < l, t > 0$ subject to the conditions

$u(0,t) = u(l,t) = 0$

$u(x,0) = x(l-x), 0 \leq x \leq l$. **[6a UPSC CSE 2022]**

Q4. A thin annulus occupies the region $0 < a \leq r \leq b, 0 \leq \theta \leq 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0°, while along the outer edge the temperature is held at $T = K\cos\frac{\theta}{2}$, where K is a constant. Determine the temperature distribution in the annulus.

[8c UPSC CSE 2018]

Q5. Find the temperature $u(x,t)$ in a bar of silver of length 10 cm and constant cross-section of area 1 cm^2. Let density $\rho = 10.6$ g/cm^3, thermal conductivity K = 1.04 cal / (cm sec°C) and specific heat $\sigma = 0.56$ cal/g°C. The bar is perfectly isolated laterally, with ends kept at 0°C and initial temperature $f(x) = \sin(0.1\pi x)$ °C. Note that $u(x,t)$ follows the heat equation $u_t = c^2 u_{xx}$, where $c^2 = K/(\rho\,\sigma)$. **[8a UPSC CSE 2016]**

Q6. A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x,t)$.

[UPSC CSE (6c) 2016]

Q7. Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0$$

subject to the conditions $u(0,t) = u(1,t) = 0$ for $t > 0$ and $u(x,0) = \sin \pi x,\ 0 < x < 1$.

[UPSC CSE (7a) 2015]

Q8. Solve the following heat equation, using the method of separation of variables:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0$$

subject to the conditions

$u = 0$ at $x = 0$ and $x = 1$, for $t > 0$

$u = 4x(1-x)$, at $t = 0$ for $0 \le x \le 1$. **[(8a) 2013 IFoS]**

Q9. The edge $r = a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.

[7b UPSC CSE 2012]

Q10. Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Also it is given that $y(x,0) = 1 - x,\ 0 < x < 1$.

[6c UPSC CSE 2011]

Q11. Solve the following heat equation $u_t - u_{xx} = 0,\ 0 < x < 2,\ t > 0$

$u(0,t) = u(2,t) = 0,\ t > 0$

$u(x,0) = x(2-x),\ 0 \le x \le 2$.

[6c UPSC CSE 2010]

Q12. Solve $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$

given the conditions

(i) $u(0,t) = u(\pi,t) = 0,\ t > 0$

(ii) $u(x,0) = \sin 2x,\ 0 < x < \pi$. **[(6c) UPSC CSE 2010]**

LAPLACE EQUATION: ELLIPTIC PDE

Q13.1 Show that the solution of the two-dimensional Laplace's equation

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0, \qquad x \in (-\infty, \infty), y \ge 0$$

Subject to the boundary condition

$$\phi(x,0)=f(x), x\in(-\infty,\infty)$$

Along with $\phi(x,y)\to 0$ for $|x|\to\infty$ and $y|\to\infty$ can be written in the form

$$\phi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{y^2+(x-\xi)^2}.$$ **[6a UPSC CSE 2024]**

Q13.2. Let Γ be a closed curve in *xy*-plane and let S denote the region bounded by the curve Γ. Let

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y)\forall(x,y)\in S.$$

If f is prescribed at each point (x,y) of S and w is prescribed on the boundary Γ of S, then prove that any solution $w=w(x,y)$, satisfying these conditions, is unique. **[5d UPSC CSE 2017]**

Q14. Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the conditions $u(0,y)=u(l,y)=u(x,0)=0$ and $u(x,a)=\sin\left(\frac{n\pi x}{l}\right)$. **[(8d) 2017 IFoS]**

Q15. Using Method of Separation of variables, Solve Laplace Equation in three dimensions. **[(6a) 2012 IFoS]**

Q16. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0\le x\le a,\ 0\le y\le b$

satisfying the boundary conditions

$u(0,y)=0,\ u(x,0)=0,\ u(x,b)=0$

$\frac{\partial u}{\partial x}(a,y)=T\sin^3\frac{\pi y}{a}$. **[6b UPSC CSE 2011]**

WAVE EQUATION: HYPERBOLIC PDE

Q17. Solve the partial differential equation $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\ 0<x<L,\ t>0$

subject to the conditions $u(0,t)=0,\ u(L,t)=0,\ t>0$

$u(x,0)=x,\ \left(\frac{\partial u}{\partial t}\right)_{t=0}=1,\ 0<x<L$ **(20 marks)**

Q18. Solve the wave equation $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\quad 0<x<L,\ t>0$

subject to the conditions

$u(0,t)=0,\ u(L,t)=0$

$u(x,0)=\frac{1}{4}x(L-x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=0$. **[6a UPSC CSE 2021]**

Q19. One end of a tightly stretched flexible thin string of length *l* is fixed at the origin and the other at $x = l$. It is plucked at $x = \frac{l}{3}$ so that it assumes initially the shape of a triangle of height *h* in the $x - y$ plane. Find the displacement *y* at any distance *x* and at any time *t* after the string is released from rest. Take, $\frac{\text{horizontal tension}}{\text{mass per unit length}} = c^2$. **[8a UPSC CSE 2020]**

Q20. Given the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2};\ t > 0$,

where $c^2 = \frac{T}{m}$, T is the constant tension in the string and *m* is the mass per unit length of the string.

(i) Find the appropriate solution of the above wave equation.

(ii) Find also the solution under the conditions $y(0,t) = 0,\ y(l,t) = 0$ for all *t* and

$\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0, y(x,0) = a\sin\frac{\pi x}{l},\ 0 \le x \le l, a > 0$. **[8a UPSC CSE 2017]**

Q21. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position, find the displacement $y(x,t)$.

[(6d) 2017 IFoS]

Q22. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for a string of length *l* fixed at both ends. The string is given initially a triangular deflection

$$u(x,0) = \begin{cases} \frac{2}{l}x, & \text{if } 0 < x < \frac{l}{2} \\ \frac{2}{l}(l - x), & \text{if } \frac{l}{2} \le x < l \end{cases}$$

with initial velocity $u_t(x,0) = 0$. **[(8c) 2015 IFoS]**

Q23. Find the deflection of a vibrating string (length $= \pi$, ends fixed, $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$) corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$. **[7a UPSC CSE 2014]**

Q24. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0$, given that

(i) $u(x,0) = 0,\ 0 \le x \le 1$

(ii) $\frac{\partial u}{\partial t}(x,0) = x^2,\ 0 \le x \le 1$

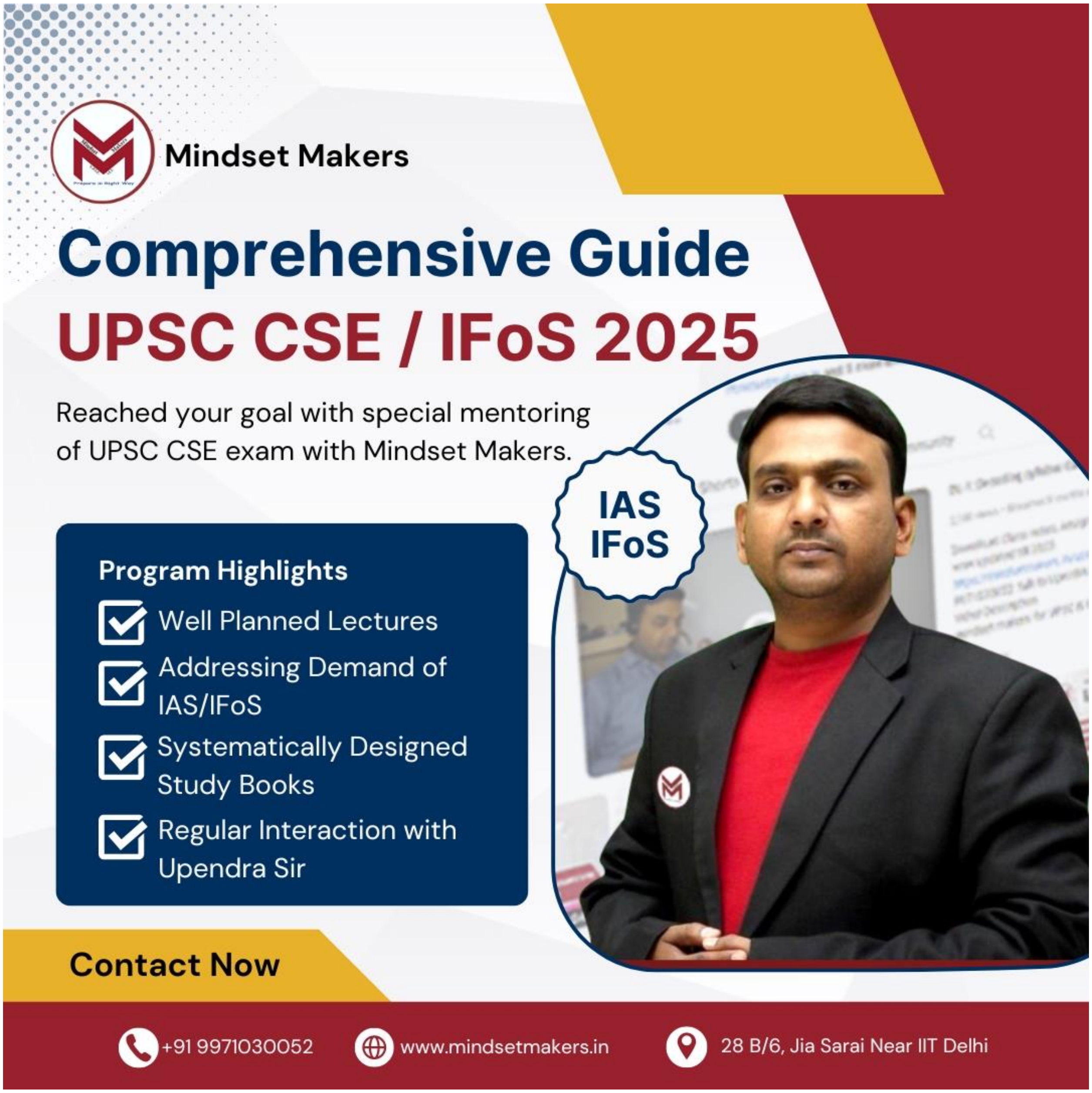
(iii) $u(0,t) = u(1,t) = 0$, for all *t*. **[8a UPSC CSE 2014]**

Q25. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating by giving end point a velocity $\lambda.x(l-x)$, find the displacement of the string at any distance *x* from one end at any time t. **[6c UPSC CSE 2013]**

Q26. A string of length *l* is fixed at its ends. The string from the mid-point is pulled up to a height *k* and then released from rest. Find the deflection $y(x,t)$ of the vibrating string. **[6b UPSC CSE 2012]**

Q27. A uniform string of length *l* is held fixed between the points $x=0$ and $x=l$. The two points of trisection are pulled aside through a distance ε on opposite sides of the equilibrium position and is released from rest at time $t=0$. Find the displacement of the string at any latter time $t>0$. What is the displacement of the string at the midpoint? **[(6a) UPSC CSE 2011]**

Note: Analysis and answers are detailed through examples of each category. Refer examples for answers of PYQs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