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An Exclusive Platform for UPSC with Science Optional(Mathematics)

Differential Calculus

With PYQs Analysis

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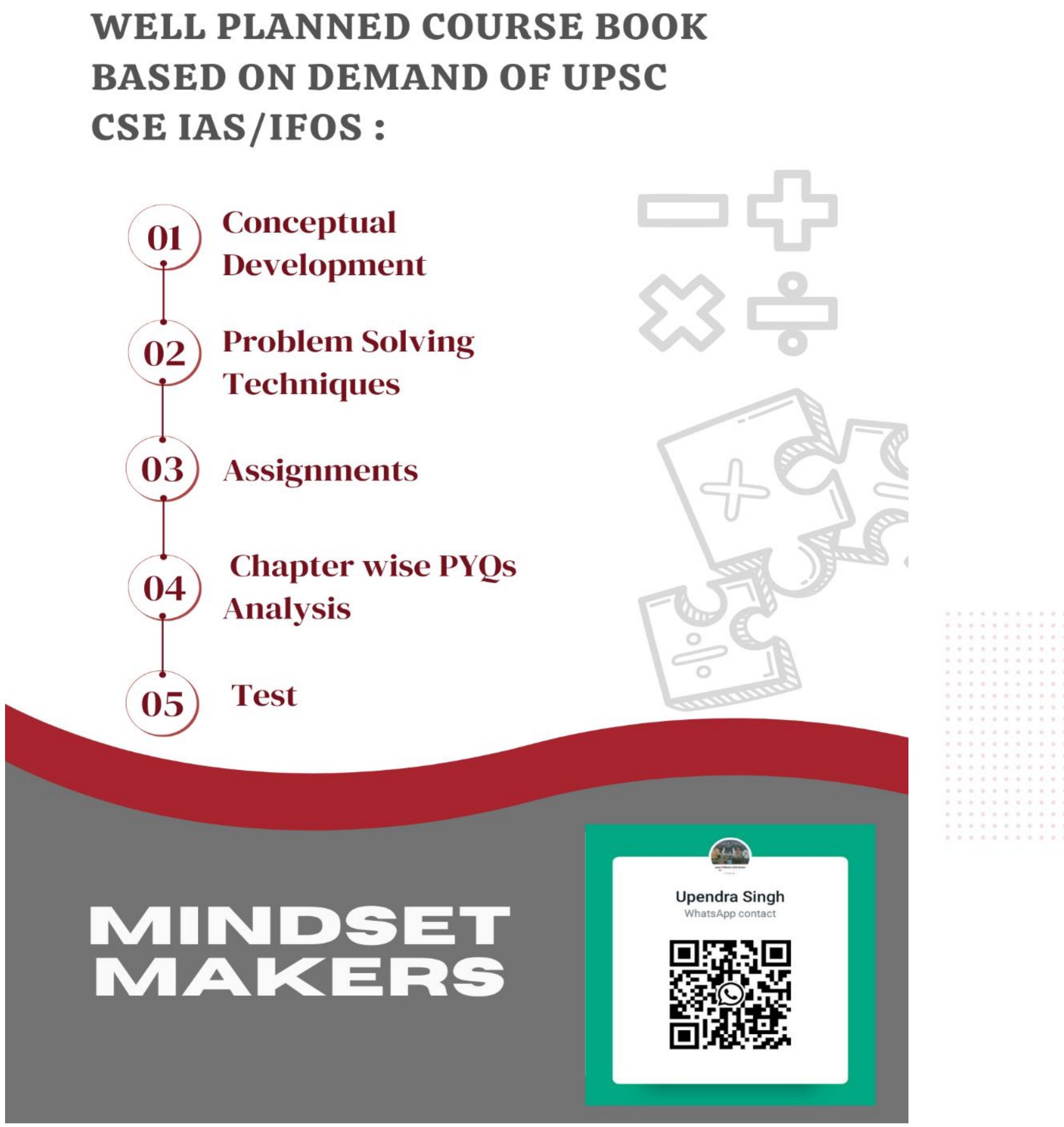
WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFOS :
01 Conceptual Development
02 Problem Solving Techniques
03 Assignments
04 Chapter wise PYQs Analysis
05 Test
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Planner: Lecture link: perfectly aligned lecture and book from Upendra Singh Sir
https://www.youtube.com/watch?v=S2DSeHIOIjI

Introduction: Just to recall basics. Your guide has done all parts from 10+2 level things here in this booklet itself with subsequent parts. So no need to waste your energy and time in revision from old books.

Decoding The Syllabus

Theme-1: Calculus of functions of one variable

Chapter-1: Limit, continuity and differentiability of a function at a point

- -Graphical approach
- -Left and right hand limit
- -Analytical definition approach
- -Indeterminate forms

Chapter-2: Mean value theorems(MVTs), Taylor's theorem, Graph tracing

- -Roll's, Lagrange's, Cauchy's MVTs
- -Taylor's, Maclaurin's, expansion of function about a point
- -Monotonic nature, Increasing / decreasing, maxima/minima, Graph Tracing

Theme-2: Calculus of functions of two or more variables

Chapter-3: Limit, continuity and differentiability of a function at a point

- -Graphical approach
- -epsilon delta definition
- -Analytical definition approach
- -Limit, continuity, partial derivatives, differentiability
- -Rule of differentiation, converting from one coordinate system to other
- -Mean value theorems, Taylor's theorem
- -Maxima/Minima, Lagrange's method of undetermined multipliers

Chapter-4: Jacobians

Chapter-5: Tangents and Normals, Curvature and asymptotes, curve tracing

Introduction to Calculus and getting meaning of limit of function at some point (out of curiosity only)

Newton (Mathematician)

Usain Bolt (Runner)

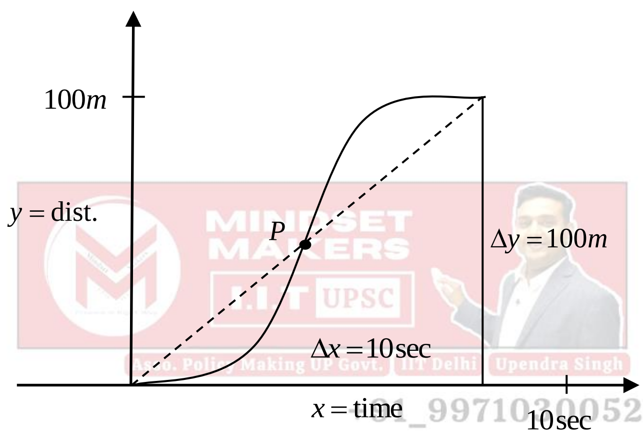
Leibnitz (Philospher Mathematician)

All of three concern about same fundamental question. The question is-

- What is the **instantaneous rate of change** of something with respect to other!
- Usain Bolt question how fast is he going right now? This is instantaneous.

This is what differential calculus is all about

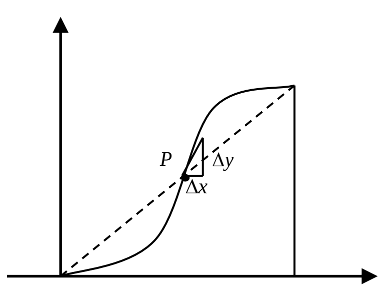
Differential calculus → instantaneous change



$$\therefore \quad \text{Average speed} = \frac{\Delta\text{Distance}}{\Delta\text{Time}} = \frac{100\,\text{m}}{10\,\text{sec}} = 10\,\text{m/sec}$$

Now where the differential calculus involved?

- Suppose we want to know the instantaneous speed of Bolt at some instant P, then



$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}}$$

Instantaneous slope

$\Downarrow$

Instantaneous speed. Infinitesimal small change in *y* with infinitesimal small change in *x*.

Functions: Let's have an idea about functions and bit feelings towards their geometry. It will help us in visualizing concepts of calculus like limits, continuity and differentiability.
(10+2 level revision)

- A mapping *f* from Set X to some set Y, which maps *each* element of X to some *unique* element of Y.
  - In our syllabus here, X & Y are subsets of set of Real numbers $(-\infty, \infty)$; **Real functions.**
- X is called Domain of *f* and collection of all those elements of Y which are mapping of some element of X; is called Range of *f*.

Examples: Let's take some natural domains and associated ranges of some simple functions

Function	**Domain** (x)	**Range** (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$(0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Note: notice, values must be real numbers in above table.

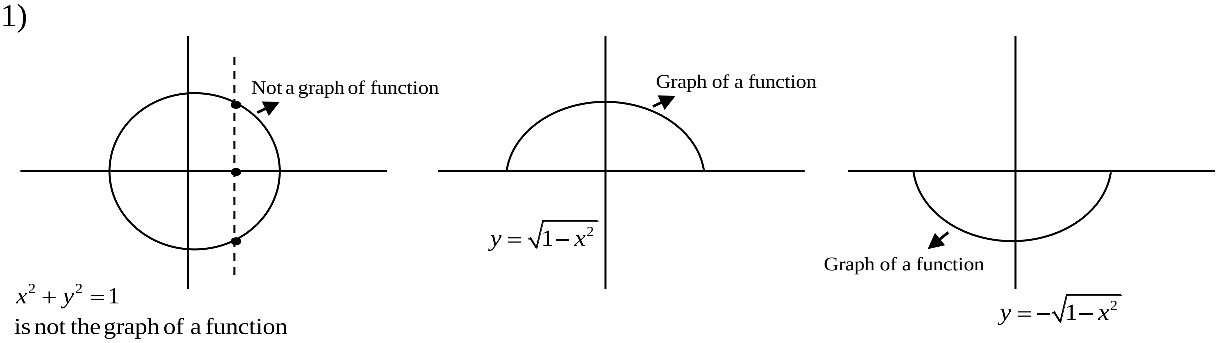
Graphs of Functions

The vertical line test for a function

Not every curve in the coordinate plane represents a function.

- A function *f* can have only one value $f(x)$ for *each* *x* in its domain So, no vertical line can intersect the graph of a function more than once.

(1)



Some Common Functions

A variety of important types of functions are frequently encountered.

(1) Linear function:

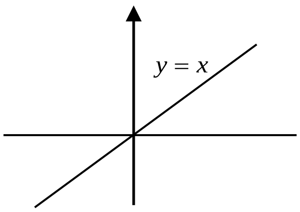
$f(x) = mx + b$

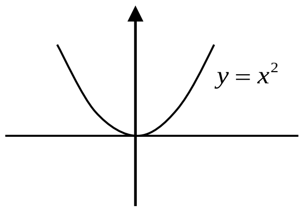
$f(x) = x$; identity function

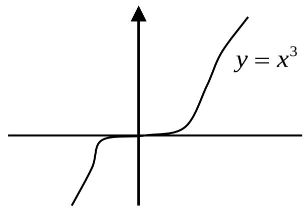
(2) Power functions:

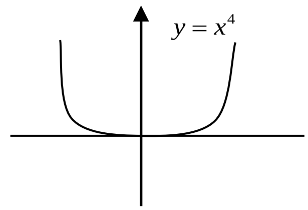
$f(x) = x^a$, where a is constant.

(i) $a = n$, a positive integer



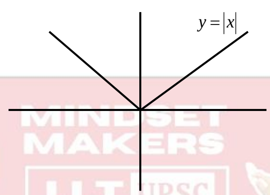






(3) The **absolute value function**(modulus function) has domain $(-\infty, \infty)$ and range $[0, \infty)$.

In General, it is defined as $|x-\alpha| = \begin{cases} x-\alpha & ; x \geq \alpha \\ -(x-\alpha) & ; x \leq \alpha \end{cases}$



4) The graph of the **greatest integer function** $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for *x*.

(5) The graph of the **least integer function** lies on or above the line $y = x$

Note:

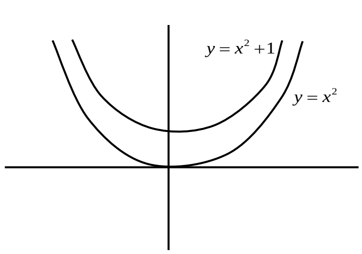
1. The graph of an even function is symmetric about the *y*-axis

$\because \quad f(-x) = f(x)$

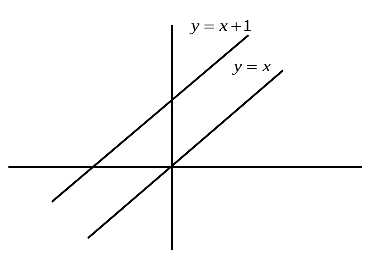
2. The graph of an odd function is symmetric about the origin.

$\therefore \quad f(-x) = -f(x)$

Example:



Adding constant term 1 to
$y = x^2$; still symmetric about y – axis



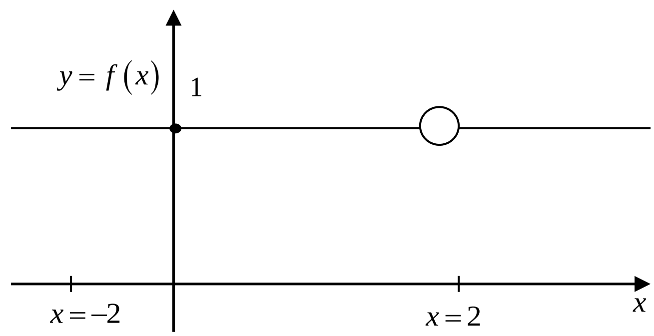
Not remains symmetric about origin

<u>Limit</u>:

Visualization of concept of *Limit of a function at some point*-

- Let us consider a function $f(x)=\dfrac{x-2}{x-2} \Rightarrow f(x)=1,\ x\neq 2,\qquad f(2)=\dfrac{0}{0} \leftarrow$ undefined

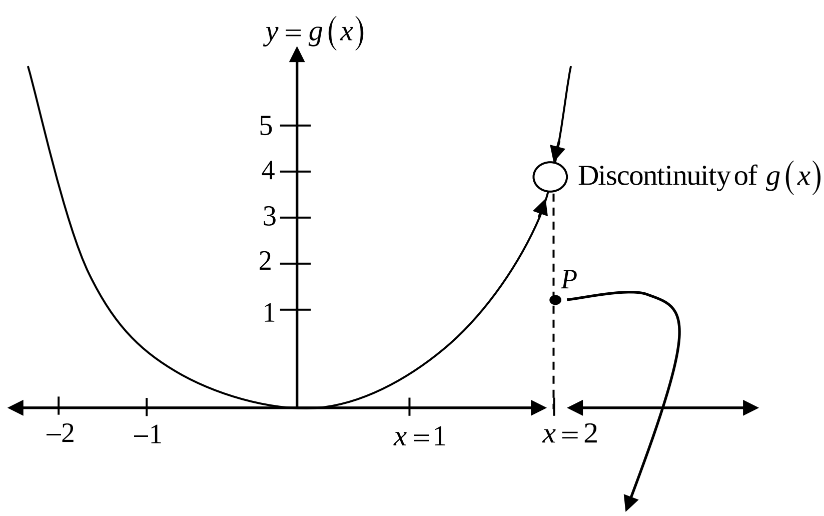
Now let us draw graph of $f(x)$



- *Question*: If I ask what about the value of function at $x=2$? I can ask what happens if x goes closer and closer to 2.
- This gives the idea of limit to what value, the function approaches if x gets closer and closer to $x=2$.
- Represented as : $\lim\limits_{x\to 2} f(x)$

- Let us take another example

$$g(x)=\begin{cases} x^2, & x\neq 2 \\ 1, & x=2 \end{cases}$$



It says that when $x=2$; the graph of $g(x)$ dropped to point $P;(g(x)=1)$; again it approached upward.

Let me ask you an interesting question what happens with $g(x)$ when x approaches to 2

i.e. in fancy notation

$$\lim_{x\to 2} g(x) = \begin{cases} (1.92)^2 = 3.61 \\ (1.999)^2 = 3.996001 \\ (1.999999)^2 = \text{Really closed to } 4 \end{cases}$$

$\Downarrow$

$$4 = \begin{cases} (2.1)^2 = 4.41 \\ (2.000001)^2 = \text{Really closed to } 4 \end{cases}$$

- This is the intention which motivated us to study the *concept of limit.*
- Also we can see here $\lim_{x\to 2} g(x) \neq g(2)$
- Intention: difference between the *limit of a function at a point* and *value of the function at that point. Leads us to study about CONTINUITY.*

Definition (Continuity at a point c).

https://www.youtube.com/watch?v=S2DSeHIOIjI&list=PL6ET_B1X78jVNCWLveD6bs7jtSXKvMhBc&index=145

- A function f is said to be continuous at a point c, if $\lim_{x\to c} f(x) = f(c)$

Note- A function f is said to be continuous from the left at c if $\lim_{x\to c-0} f(x) = f(c)$. Also written as $\lim_{h\to 0} f(c-h) = f(c)$

Also f is *continuous from the right at c* if $\lim_{x\to c+0} f(x) = f(c)$. Also written as $\lim_{h\to 0} f(c+h) = f(c)$

Clearly a function is continuous at c if and only if it is continuous from the left as well as from the right.

Continuity in an Interval

A function f is said to be continuous in an interval $[a,b]$ if it is continuous at every point of the interval (a, b) and continuous from right at point a and from left at b.

$\lim_{h\to 0} f(a+h) = f(a)$, $\lim_{h\to 0} f(b-h) = f(b)$

Discontinuous Functions

A function is said to be discontinuous at a point c of its domain if it is not continuous there at c. The point c is then called a point of discontinuity of the function.

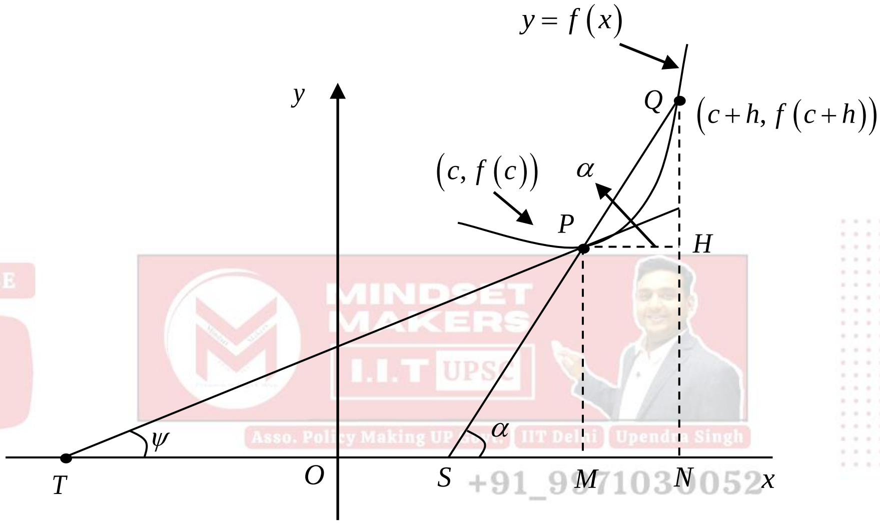
Types of discontinuities

(i). A function f is said to have a *removable discontinuity* at $x = c$ if $\lim_{x \to c} f(x)$ exists but is not equal to the value $f(c)$ (which may or may not exist) of the function. Such a discontinuity can be removed by assuming a suitable value to the function at $x = c$.

(ii). f is said to have a *discontinuity of the first kind* at $x = c$ if $\lim_{x \to c-0} f(x)$ and $\lim_{x \to c+0} f(x)$ both exist but are not equal.

Differentiability

Derivative of a function at a point:



PM and QN perpendicular to the *x*-axis and PH perpendicular to QN.

$$\tan\alpha = \frac{f(c+h) - f(c)}{h} \quad \text{....(1)}$$

Now as $h \to 0$, the point Q moving along the curve approaches the point P, the chord PQ approaches the tangent line TP as its limiting position and the angle α approaches the angle ψ.

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \tan\psi = f'(c)$$

Definition:

Geometrically, if there exist unique tangent to the curve at that point then it is said to be differentiable at that point.

Mathematically, Let f be a function defined on an open interval I, and let $c \in I$.

Progressive Derivative (RHD):

$$Rf'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}, \ h > 0$$

Regressive Derivative (LHD):

$$Lf'(c)=\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h},\ \ h>0$$

- f is differentiable at $x=c$ if and only if $Rf'(c)$ and $Lf'(c)$ both exists finitely and are equal.

It is also discussed as:

- Then f is said to be differentiable or derivable at c, if $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

Differentiability Vs Continuity:

Derivability at a point $\Rightarrow$ continuity at that point

Statement. *A function which is derivable at a point is necessarily continuous at that point.*

Let a function f be derivable at $x=c$.

Hence, $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

Now $f(x)-f(c)=\frac{f(x)-f(c)}{(x-c)}(x-c),\ \ (x\neq c)$

Taking limits as $x\to c$, we have

$$\lim_{x\to c}\{f(x)-f(c)\}=\lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}(x-c)\right\}=\lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}\right\}\cdot\lim_{x\to c}(x-c)$$

$$=f'(c)\cdot 0=0$$

so that $\lim_{x\to c}f(x)=f(c)$, and therefore, f is continuous at $x=c$.

Note- It is to be clearly understood that while continuity is a necessary condition for derivability at a point, it is not a sufficient condition. We come across functions which are continuous at a point without being derivable there at, and still many more functions may be constructed.

Consider the function f defined by

$f(x)=|x|,\ \ \forall x\in R$

$f(x)$ is continuous at $x=0$, for

$\lim_{x\to 0+}f(x)=\lim_{x\to 0-}f(x)=0=f(0)$

But $f'(0)$ does not exist. Thus, the function is continuous but not derivable at the origin. As there does not exist unique tangent at $x=0$.

Interesting Point- It was the genius of German mathematician Weierstrass, who gave a function which is continuous everywhere but not derivable anywhere, viz.,

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cos(3^n x),\ \ \forall x\in R$$

Some Counter Examples

1. $f(x)=|x|+|x-1|, \quad \forall x \in R$

   Continuous but not derivable at $x=0$ and $x=1$.

2. $f(x)=|x-\alpha|$

   Continuous but not derivable at $x=\alpha$.

3. $f(x)=x\sin 1/x$ if $x \neq 0$

   $=0$ if $x=0$

   Continuous but not derivable at the origin.

4. $f(x)=0$ if $x \leq 0$

   $=x$ if $x>0$

   Continuous but not derivable at $x=0$.

Some theoretical points- understanding and able to write mathematical language for these points will make you capable enough to deal with questions in exam which are bit theoretical in nature. So just have a look how these three points are being expressed.

1-The existence of the derivative of a function at a point depends on the existence of a limit, $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$. Therefore, keeping in view the corresponding theorems on limits, one can easily establish the following *fundamental theorem on derivatives*.

If the functions f, g are derivable at c, then the function $f+g, f-g, f\cdot g$ *and* $f/g\left(g(c)\neq 0\right)$ *are also derivable at c, and*

$$(f\pm g)'(c)=f'(c)\pm g'(c)$$

$$(f\cdot g)'(c)=f'(c)g(c)+f(c)g'(c)$$

$$(f/g)'(c)=\left\{f'(c)g(c)-f(c)g'(c)\right\}/\left(g(c)\right)^2 \text{ if } g(c)\neq 0$$

2-*If f is derivable at c and* $f(c)\neq 0$ *then the function* $1/f$ *is also derivable there at, and*

$$(1/f)'(c)=-f'(c)/\left\{f(c)\right\}^2$$

Since f is derivable at c, it is also continuous there at. Again since $f(c)\neq 0$, there exists a neighborhood N of c wherein f does not vanish.

Now

$$\frac{1/f(x)-1/f(c)}{x-c}=-\frac{f(x)-f(c)}{x-c}\cdot\frac{1}{f(x)f(c)}, x\in N$$

Proceeding to limits when $x\to c$, we get

$$\left(\frac{1}{f}\right)'(c)=\lim_{x\to c}\frac{1/f(x)-1/f(c)}{x-c}$$

$$=-f'(c)\cdot\frac{1}{f(c)\cdot f(c)}=-\frac{f'(c)}{\left\{f(c)\right\}^2}$$

Thus, the limit exists and equals, $-f'(c)/\{f(c)\}^2$.

3- Let $f: I \to \mathbf{R}$ and $g: J \to \mathbf{R}$, where $f(I) \subseteq J$ and $c \in I$. If *f* is differentiable at *c* and *g* is differentiable at *c*, and

$$(gof)'(c) = g'(f(c)) \cdot f'(c)$$

Proof: $\because f$ is differentiable at *c*, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f(c)$$

$$\Rightarrow f(x) - f(c) = (x - c)\{f'(c) + \lambda(x)\} \quad ....(1)$$

where $\lambda(x) \to 0$ as $x \to c$

Further $\because g$ is differentiable at $f(c)$, we have

$$\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c))$$

or $g(y) - g(f(c)) = (y - f(c))\{g'(f(c)) + h(y)\}$ ....(2)

where $h(y) \to 0$ as $y \to f(c)$

Now

$$(gof)(x) - (gof)(c) = g(f(x)) - g(f(c)) = \{f(x) - f(c)\} g'\{f(c) + h(f(x))\}$$
$$= (x - c)\{f'(c) + \lambda(x)\}\{g'(c) + h(f(x))\}$$

Thus, if $x \neq c$, we have

$$\frac{(gof)(x) - (gof)(c)}{(x - c)} = \{g'(f(c)) + h(f(x))\}\{f'(c) + \lambda(x)\} \quad ....(3)$$

Further, since *f* is differentiable at *c*, it is continuous there. So,

$$x \to c \Rightarrow f(x) \to f(c) \Rightarrow h(f(x)) \to 0$$

Taking limit as $x \to c$ in (3), we get

$$\boxed{(gof)'(c) = g'(f(c)) \cdot f'(c)}$$

<u>Analysis Point of View</u>

Limit

Let *f* be a function defined for all points in a neighborhood N of a point *c* except possibly at the point *c* itself.

Definition 1. The function *f* is said to tend to a limit l as *x* tends to (or approaches) *c* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

or $|f(x) - l| < \varepsilon$, when $0 < |x - c| < \delta$

or $f(x) \in]l-\varepsilon, l+\varepsilon[\; \forall x \in]c-\delta,\, c+\delta$ [except possibly c]

In symbols, we then write

$$\lim_{x \to c} f(x) = l$$

Definition 2. The function *f* is said to tend to $+\infty$ as *x* tends to *c* (or in symbols, $\lim_{x \to c} f(x) = +\infty$) if for each $G > 0$ (however large) there exists a $\delta > 0$ such that

$$f(x) > G, \text{ when } |x-c| < \delta$$

The function *f* is said to tend to $-\infty$ as *x* tends to *c* (or in symbols, $\lim_{x \to c} f(x) = -\infty$), if for each $G > 0$ (however large) there exists a $\delta > 0$ such that

$$f(x) < -G, \text{ when } |x-c| < \delta$$

Definition 3. The function *f* is said to tend to a limit *l* as *x* tends to ∞ (or in symbols, $\lim_{x \to \infty} f(x) = l$) if for each $\varepsilon > 0$, there exists a $k > 0$, such that

$$|f(x) - l| < \varepsilon, \text{ when } x > k$$

Definition 4. The function *f* is said to tend to $+\infty$ as *x* tends to ∞ (or in symbols, $\lim_{x \to \infty} f(x) = \infty$) if for each $G > 0$ (however large) there exists a $k > 0$, such that

$$f(x) > G, \text{ when } x > k$$

INDETERMINATE FORMS

Lecture link: https://www.youtube.com/live/5LWheRtBlXY?si=eHQdGDnqqJ_zFzSj

0/0, $\infty/\infty, 0 \times \infty, \infty - \infty, 0^0, 1^\infty$, and ∞^0.

In general, the limit of $\phi(x)/\psi(x)$ when $x \to a$, in case the limits of both the functions exist, is equal to the limit of the numerator divided by the limit of the denominator. But what happens when both these limits are zero? The division $(0/0)$ then becomes meaningless. A case like this is known as Indeterminate form. Other such forms are $\infty/\infty, 0 \times \infty, \infty - \infty, 0^0, 1^\infty$, and ∞^0.

Ordinary methods of evaluating the limits are of little help. Particular methods are required to evaluate these peculiar limits. We shall now discuss these particular methods, generally called *L' Hospital rule*.

Note: It should, however, be clearly understood, that in what follows, we do not find the value of 0/0 or of any of the other indeterminate forms. We only find the limits of combinations of functions which assume these forms when the limits of functions are taken separately.

Indeterminate Form 0/0

We shall now discuss some theorems concerning the indeterminate form 0/0.

Theorem 1. *If f, g be two functions such that*

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ *and*

(ii) $f'(a), g'(a)$ *exist and* $g'(a) \neq 0$ *then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}$$

Since the functions *f* and *g* are derivable at $x=a$, therefore, they are continuous there at, i.e.,

$$\lim_{x\to a} f(x)=f(a) \text{ and } \lim_{x\to a} g(x)=g(a)$$

Thus from condition (i), $f(a)=0=g(a)$.

Also

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{f(x)}{x-a}$$

and

$$g'(a)=\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=\lim_{x\to a}\frac{g(x)}{x-a}$$

$\therefore$

$$\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{f(x)}{g(x)}$$

Note: Condition (i) can be replaced by $f(a)=g(a)=0$.

L'Hospital's Rule for 0/0 form. *If f, g are two functions such that*

(i) $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$,

(ii) $f'(x), g'(x)$ *exist and* $g'(x)\neq 0, \forall x\in]a-\delta, a+\delta[, \delta>0$ *except possibly at a, and*

(iii) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *exists.*

then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Generalised L'Hospital's Rule for 0/0 form. If *f*, *g* be two functions such that

(i) $f^n(x), g^n(x)$ *exist, and* $g'(x)\neq 0$ $(r=0,1,2,.....,n)$ *for any x in* $]a-\delta, a+\delta[$ *except possibly at* $x=a$,

(ii) *when* $x\to a$, $\begin{cases}\lim f(x)=\lim f'(x)=...=\lim f^{(n-1)}(x)=0\\ \lim g(x)=\lim g'(x)=...=\lim g^{(n-1)}(x)=0\end{cases}$

and (iii) $\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$ *exists, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$$

L'Hospital's rule for infinite limits. If *f*, *g* be two functions such that

(i) $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=0$

(ii) $f'(x), g'(x)$ *exist, and* $g'(x) \neq 0$, $\forall x > 0$ *except possibly at* ∞, *and (iii)* $\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ *exists,*

then $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)}$

(a) Form $0 \times \infty$

When $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$, $f(x) \cdot g(x)$ takes $0 \times \infty$ form.

However $f(x) \cdot g(x)$ may be expressed as

$$\frac{f(x)}{1/g(x)} \text{ or } \frac{g(x)}{1/f(x)}$$

which has respectively 0/0 and ∞/∞ forms.

(b) Form $\infty - \infty$

This can be reduced to the form 0/0 or ∞/∞. Dividing numerator and denominator by $f(x)g(x)$,

$$f(x) - g(x) = \frac{\dfrac{1}{g(x)} - \dfrac{1}{f(x)}}{\dfrac{1}{f(x)g(x)}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

(c) Form $0^\circ, 1^\infty, \infty^0$

These forms will occur in $\lim\left(f(x)\right)^{g(x)}$ and can be made to depend upon one of the previous forms by putting $k = \left\{f(x)\right\}^{g(x)}$, so that

$$\log k = g(x) \cdot \log f(x)$$

$\therefore \lim \log k = \lim \left\{g(x) \log f(x)\right\}$

Also $\lim k = \lim e^{\log k} = e^{\lim \log k}$

Thus the limit may be evaluated by one of the previous methods.

- Let us now evaluate some limits which take up these forms, we shall not hesitate to make use of certain known limits, such as $\lim\limits_{x \to 0} \dfrac{\sin x}{x}, \lim\limits_{x \to 0} \dfrac{\tan x}{x}, \lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$ etc. or expansions of functions such as $\log(1+x), \sin x$, etc. either in the beginning or at some intermediate state because it simplifies and shortens the process of evaluation of a limit to a considerable extent.

Detailed Subjective Examples to substantiate above learnt concepts

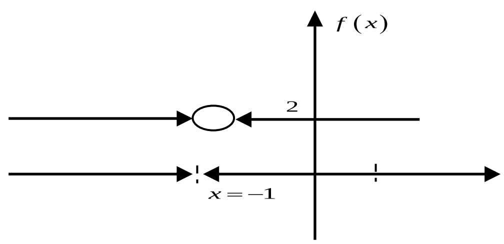
Limit

Example1: $\lim_{x\to -1}\left(\frac{2x+2}{x+1}\right)=\frac{0}{0}$ undefined

we can express it as

$f(x)=\frac{2(x+1)}{(x+1)}$ i.e. Rewriting the given function $f(x)=\begin{cases} 2, & x\neq -1 \\ \text{undefined}, & x=-1 \end{cases}$

Let us graph this function $f(x)$



Check in graph; $f(x)$ is approaching to 2 as x approaches to -1. So limit of given function at $x=-1$ is 2

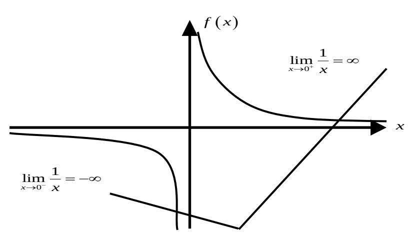
Now you can take a calculator and calculate $f(-1.001)$, $f(-1.000001)$, $f(-0.9999)$ check where $f(x)$ approaches.

Example2: $\lim_{x\to 0}\frac{1}{x}$; $f(x)=\frac{1}{x}$

Let us form a table to get graph:

x	$f(x)$
0	Undefined
- .01	- 100
- .001	- 1000
0.1	100
.001	1000

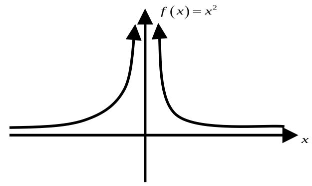
Now sketch $f(x)$



We say that $\lim_{x\to 0}\frac{1}{x}$ does not exist. (as ∞ and -∞ different)

Example3: $\lim_{x\to 0}\frac{1}{x^2}$

x	$x^2 = f(x)$
.1	100
-.1	100
.01	10,000
-.01	10,000



$$\lim_{x\to 0^-}\frac{1}{x^2} = \lim_{x\to 0^+}\frac{1}{x^2} = \infty$$

So, when you go from $-x$; $f(x)$ approaches to ∞ and from +x, $f(x)$ tends to ∞; so its limit is ∞; not a finite number. So limit exists over the *extended real line*.

What happens if you have really large value of x!

Question: $\lim_{x\to\infty}\frac{x^2+3}{x^3}$

∴ Denominator is fastesr than numerator as x is very large.

$$\lim_{x\to\infty}\frac{x^2+3}{x^3} = 0$$

Example4: $\lim_{x\to\infty}\frac{3x^2+5}{4x^2-3}$

It's always easy as x is going to be a larger and larger value. So 5 and 3 are useless as compare to $3x^2$ and $4x^2$.

∴ Just leave them simply

Now think about $3x^2/4x^2$ same pace

$$\lim_{x\to\infty}\frac{3x^2+5}{4x^2-3} = \frac{3}{4};$$

Fastest going term in numerator
Fastest going term in denominator

Example5: $\lim_{x\to\infty}\sqrt{(x^2+4x+1)} - x$

$$\lim_{x\to\infty}\frac{\cancel{x^2}+4x+1-\cancel{x^2}}{\sqrt{x^2+4x+1}+x}=\lim_{x\to\infty}\frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1}=\frac{4}{\sqrt{1}+1}=\frac{4}{2}=2$$

Note- Learnings from graphical approach of limit can also be reflected as Left Hand Limit and Right Hand Limit. Let's see now!

Example6. Find the right hand and the left hand limits of a function defined as follows:

$$f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$$

Now, when $x>4, |x-4|=x-4$

$$\therefore \lim_{x\to 4+0} f(x)=\lim_{x\to 4+0}\frac{|x-4|}{x-4}=\lim_{x\to 4+0}\frac{x-4}{x-4}=\lim_{x\to 4+0}1=1$$

Again when $x<4,\ |x-4|=-(x-4)$

$$\therefore \lim_{x\to 4-0} f(x)=\lim_{x\to 4-0}\frac{-(x-4)}{x-4}=\lim_{x\to 4-0}(-1)=-1$$

so that $\lim_{x\to 4+0} f(x)\neq \lim_{x\to 4-0} f(x)$; Hence $\lim_{x\to 4} f(x)$ does not exist.

Example7. Evaluate:

(i). $\lim_{x\to -1}\dfrac{(x+2)(3x-1)}{x^2+3x-2}$ (ii) $\lim_{x\to 0}\dfrac{\sqrt{4+x}-2}{x}$ (ii) $\lim_{x\to 0+}\dfrac{\sin x}{\sqrt{x}}$

Solution.

(i) $$\lim_{x\to -1}\frac{(x+2)(3x-1)}{x^2+3x-2}=\frac{\lim_{x\to -1}(x+2)\cdot\lim_{x\to -1}(3x-1)}{\lim_{x\to -1}\left(x^2+3x-2\right)}=\frac{1\cdot(-4)}{-4}=1$$

(ii) $$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}=\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}=\lim_{x\to 0}\frac{1}{\sqrt{4+x}+2}=\frac{1}{4}$$

(iii) $$\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}=\left(\lim_{x\to 0+}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0+}\sqrt{x}\right)=1\cdot 0=0$$

Example8. Evaluate $\lim_{x\to 0+}\dfrac{e^{1/x}}{e^{1/x}+1}=\lim_{x\to 0+}\dfrac{1}{1+e^{-1/x}}=1$ and $\lim_{x\to 0-}\dfrac{e^{1/x}}{e^{1/x}+1}=\dfrac{0}{1}=0$

so that the left hand limit is not equal to the right hand limit.

Hence $\lim_{x\to 0}\dfrac{e^{1/x}}{e^{1/x}+1}$ does not exist.

Example9. Find $\lim\limits_{x \to 0} e^x \operatorname{sgn}\left(x+[x]\right)$, where the signum function is defined as

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if} \quad x > 0 \\ 1, & \text{if} \quad x = 0 \\ -1, & \text{if} \quad x < 0 \end{cases}$$ and $[x]$ means the greatest integer $\le x$.

Solution. L.H.L. = $\lim\limits_{h \to 0-} e^{0-h} \operatorname{sgn}\left(0-h+[0-h]\right) = \lim\limits_{h \to 0-} e^{-h}\left(-h+(-1)\right) = 1(0-1) = -1$

R.H.L. = $\lim\limits_{h \to 0+} e^{0+h} \operatorname{sgn}\left[0+h+[0+h]\right] = \lim\limits_{h \to 0+} e^{h} \operatorname{sgn}(h+0) = 1.1 = 1$

$\therefore \lim\limits_{x \to 0} e^x \operatorname{sgn}\left(x+[x]\right)$ does not exist.

CONTINUITY

Example1. Examine the following function for continuity at the origin:

$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution.

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{xe^{1/x}}{1+e^{1/x}} = 0 \quad \left(\because e^{-\infty} = 0\right)$$

and

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x}{e^{-1/x}+1} = 0$$

Thus,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} f(x) = 0$$

Also

$$\lim_{x \to 0} f(x) = f(0)$$

Thus, the function is continuous at the origin.

Example2. Show that the function defined as:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when} \quad x \ne 0 \\ 0 & \text{when} \quad x = 0 \end{cases}$$

has removable discontinuity at the origin.

Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

so that

$$\lim_{x \to 0} f(x) \ne f(0)$$

Hence the limit exists, but is not equal to the value of the function at the origin. Thus the function has a removable discontinuity at the origin.

Note: The discontinuity can be removed by redefining the function at the origin such as $f(0)=2$

Example3. Show that the function defined by

$$f(x)=\begin{cases} x\sin 1/x, & \text{when} \quad x\neq 0 \\ 0, & \text{when} \quad x=0 \end{cases}$$

is continuous at $x=0$.

Now

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\left(x\sin\frac{1}{x}\right)=0$. (As $\sin\infty$ is finite real number belonging to $[-1,1]$)

so that

$\lim\limits_{x\to 0} f(x)=f(0)$. Hence, *f* is continuous at $x=0$.

Example4. A function *f* is defined on **R** by

$$f(x)=\begin{cases} -x^2 & \text{if} \quad x\le 0 \\ 5x-4 & \text{if} \quad 0<x\le 1 \\ 4x^2-3x & \text{if} \quad 1<x<2 \\ 3x+4 & \text{if} \quad x\ge 2 \end{cases}$$

Examine *f* for continuity at $x=0,1,2$. Also discuss the kind of discontinuity, if any.

Solution.

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\left(-x^2\right)=0=f(0)$$

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}(5x-4)=-4$$

so that $\lim\limits_{x\to 0-} f(x)=f(0)\neq\lim\limits_{x\to 0+} f(x)$

Thus the function has a discontinuity of the first kind from the right at the origin.

- $\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1-}(5x-4)=1$, $\lim\limits_{x\to 1+} f(x)=\lim\limits_{x\to 1+}\left(4x^2-3x\right)=1$

so that $\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1+} f(x)=1=f(1)\Rightarrow\lim\limits_{x\to 1} f(x)=f(1)$

Thus the function is continuous at $x=1$

- $\lim\limits_{x\to 2-} f(x)=\lim\limits_{x\to 2-}\left(4x^2-3x\right)=10$, $\lim\limits_{x\to 2+} f(x)=\lim\limits_{x\to 2+}(3x+4)=10$

Also $f(2)=10\Rightarrow\lim\limits_{x\to 2} f(x)=f(2)$. Thus, the function is continuous at $x=2$.

Example5. Is the function *f*, where $f(x)=\dfrac{x-|x|}{x}$ continuous?

For $x<0$, $f(x)=\dfrac{x+x}{2}=2$, continuous.

For $x>0$, $f(x)=\dfrac{x-x}{x}=0$, continuous.

The function is not defined at $x=0$. Thus $f(x)$ is continuous for all x expect at zero.

- Thus the function has discontinuity of the first kind from the right at $x=0$.

Differentiability

Example 1. Show that the function $f(x)=x^2$ is derivable on $[0, 1]$.

Solution. Let x_0 be any point of $]0, 1[$, then

$$f'(x_0)=\lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0}=\lim_{x\to x_0}(x+x_0)=2x_0$$

At the end points, we have

$$f'(0)=\lim_{x\to 0+0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0+}\frac{x^2}{x}=\lim_{x\to 0+}x=0$$

$$f'(1)=\lim_{x\to 1-0}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1-}\frac{x^2-1}{x-1}=\lim_{x\to 1-}(x+1)=2$$

Derivatives exist at all points of given interval. Thus, the function is derivable in the closed interval $[0, 1]$.

Example 2.a. A function f is defined on **R** by

$$f(x)=x \text{ if } 0\le x<1$$
$$=1 \text{ if } x\ge 1$$

Consider the derivability at $x=1$.

Solution.

$$Lf'(1)=\lim_{x\to 1-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1-}\frac{x-1}{x-1}=1$$

$$Rf'(1)=\lim_{x\to 1+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1+}\frac{1-1}{x-1}=0$$

$\therefore Lf'(1)\ne Rf'(1)$. Thus, $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$ does not exist, i.e., $f'(1)$ does not exist.

Example 2.b. Let the function f be defined by

$$f(t)=\begin{cases}0, & for\ t<0\\ t, & for\ 0\le t\le 1\\ 4, & for\ t>1\end{cases}$$

(i) Determine the function $F(x) = \int_0^x f(t)\,dt$

(ii) Where is F non-differentiable? Justify your answer.

Sol: The given function f is defined as

$$f(t)=\begin{cases}0, & for\ t<0\\ t, & for\ 0\le t\le 1\\ 4, & for\ t>1\end{cases}$$

Now we have to calculate

$$F(x)=\int_0^x f(t)\,dt$$

For: $0<x\le 1$

Then, $F(x)=\int_0^x f(t)\,dt=\int_0^x t\,dt=\dfrac{x^2}{2}$

For: $x>1$

Then, $F(x)=\int_0^x f(t)\,dt=\int_0^1 f(t)\,dt+\int_1^x f(t)\,dt=\int_0^1 t\,dt+\int_1^x 4\,dt=4x-7/2$

i.e., $F(x) = 4x-\dfrac{7}{2}\quad x>1$

Therefore, $$F(x)=\begin{cases}\dfrac{x^2}{2}, & for\ 0<x\le 1\\ 4x-\dfrac{7}{2}, & for\ x>1\end{cases}$$

Clearly the function $F(x)$ is not differential at $x = 1$.

Example 3. Consider the derivability of the function $f(x)=|x|$ at the origin.

- Left hand derivative = $\lim\limits_{x\to 0-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0-}\dfrac{|x|}{x}=\lim\limits_{x\to 0-}\dfrac{-x}{x}=-1$

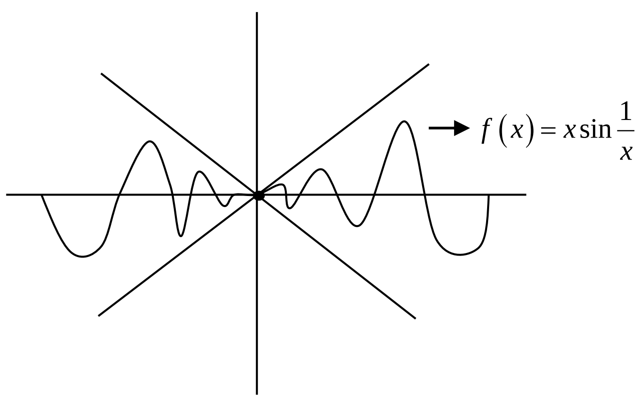
  Right hand derivative $=\lim\limits_{x\to 0+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0+}\dfrac{|x|}{x}=\lim\limits_{x\to 0+}\dfrac{x}{x}=1$

  Hence, the function is not derivable at $x=0$.

Example 4-a:

(1) Check continuity and differentiability of $f(x)=\begin{cases}x\sin\left(\dfrac{1}{x}\right), & x\ne 0\\ 0, & x=0\end{cases}$

so, $f(x)$ is continuous but not differential



Similarly $Lf'(0)$ does not exist

Note: Such graphs are easy to draw. If You want to learn see the curve tracing chapter in next chapter.

It is continuous at $x = 0$.
Now

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

which does not exist.
Similarly, $Lf'(0)$ does not exist.

• Now,

$$\because \quad f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}; & x \neq 0 \\ 0\,; & x = 0 \end{cases}$$

Extra- Now checking continuity of $f'(x)$ at $x = 0$.

$$f'(0+h) = \lim_{h \to 0} f'(0+h) = \lim_{h \to 0}\left(2h \sin\frac{1}{h} - \cos\frac{1}{h}\right)$$

$$= 0 - \lim_{h \to 0} \cos\left(\frac{1}{h}\right)$$

which does not exist.
Similarly we can show $f'(0-h)$ does not exist.

Example 4-b: A function f is defined as follows

$$f(x) = \begin{cases} x^p \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0\,, & x = 0 \end{cases}$$

What conditions should be imposed on *p* so that *f* may be
(i) Continuous at $x = 0$ (ii) Differentiable at $x = 0$
Solution:
(i) $f(0+0) = \lim_{h \to 0} h^p \cos(1/h)$

$f(0-0) = (-1)^p \lim_{h \to 0} h^p \cos(1/h)$

To make $f(0+0) = f(0-0) = f(0)$, condition required is $\boxed{p > 0}$

(ii) $Rf'(0) = \lim_{h \to 0} h^{p-1} \cos(1/h)$

$Lf'(0) = (-1)^{p-1} \lim_{h \to 0} h^{p-1} \cos(1/h)$

So for differentiability condition required is $\boxed{p > 1}$

Exam point: Continuity and Differentiability of derivative of a differentiable function:
Let's consider this question

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right); & x\neq 0 \\ 0\ ; & x=0 \end{cases}$$

Then show that $f(x)$ is differentiable everywhere, and that $f'(0)=0$.
Also show that the function f' is discontinuous at $x=0$.

It will be interesting to notice here If a function f is differential everywhere in $\mathbf{R}$ and $f'(0)=0$ then $\lim_{x\to 0} f'(x)=\ell$? For this to understand, let's have an example here-

function f defined as:

$$f(x)=\begin{cases} x^2\sin(1/x) & \text{if} \quad x\neq 0 \\ 0 & \text{if} \quad x=0 \end{cases}$$

is derivable at $x=0$ but $\lim_{x\to 0} f'(x)\neq f'(0)$.

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\sin 1/x}{x}$$

$$=\lim_{x\to 0}(x\sin 1/x)=0$$

From elementary calculus we know that for $x\neq 0$,

$f'(x)=2x\sin(1/x)-\cos(1/x)$

$\lim_{x\to 0} f'(x)$ does not exist $\left[\because \lim_{x\to 0}\cos\frac{1}{x} \text{ does not exist}\right]$ and therefore, there is no possibility of $\lim_{x\to 0} f'(x)$ being equal to $f'(0)$.

Thus, $f'(x)$ is not continuous at $x=0$ but $f'(0)$ exists.

Example 5: Discuss the continuity and differentiability of the function *f*, defined by

(1) $f(x)=\begin{cases} +1; & \text{if } x \text{ is rational} \\ -1; & \text{if } x \text{ is irrational} \end{cases}$

(2) $f(x)=\begin{cases} 2x+1\ ; & \text{if } x \text{ is rational} \\ x^2-2x+5\ ; & \text{if } x \text{ is irrational} \end{cases}$

Solution.

(1) We have to check it by epsilon delta definition because we can not be sure about value of function in nbd of a point at which, we're trying to check the continuity.

<u>Assumption:</u> Let it is continuous at any arbitrary rational number r_0.

So by definition, for **each** given positive but very small ε, there exists a positive and small δ s.t

$\left|f(x)-f(r_{0)}\right|<\varepsilon \ when \left|x-r_0\right|<\delta$ …..(1)

Now if this inequality fails to hold for at least one hoosen epsilon, then we say our assumption was wrong.

- It's important to notice here if x is an irrational number in above nbd of point r_0

then

$\left| f(i) - f(r_{0)} \right| = |-1-1| = 2$ which is greater than ½ if we take value of epsilon as ½.

i.e.

for some ε, there exists a δ s.t

$\left| f(x) - f(r_{0)} \right| > \varepsilon \text{ when } |x - r_0| < \delta$.

So given function is not continuous at r_0. Since r_0 was an arbitrary rational number, so given function is not continuous at any rational number.

Special note- Don't be confused by that above inequality holds for x rationals so it is continuous. Reason- If inequality does not hold for at least one epsilon, then it is said it does not hold. i.e. our assumption was wrong.

- Similarly, we can show that it is not continuous at any irrational number.

Therefore, *f(x)* is not continuous at any real number. So not differentiable at any real number.

(2) $\because x^2 - 2x + 5 = 2x + 1 \Leftrightarrow x^2 - 4x + 4 = 0 \Leftrightarrow x = 2$

We have $x^2 - 2x + 5 \neq 2x + 1$ except when $x = 2$ that means we have to check continuity at $x = 2$ only (except $x = 2$; they are unequal; (since, rational $\pm\delta$ = irrational).

Hence *f* is continuous at $x = 2$ only. (We can prove it by above method)

- Further, it follows that $x = 2$ is the only point at which *f* may be differentiable.

In this regard, we see that

$$\frac{f(x) - f(2)}{x - 2} = \begin{cases} \dfrac{(2x+1) - 5}{x - 2} = 2\ ; & \text{when } x \text{ is rational} \\ \dfrac{(x^2 - 2x + 5) - 5}{x - 2} = x\ ; & \text{when } x \text{ is irrational} \end{cases}$$

Therefore, in either case, we have

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 2$$

Hence *f* is differentiable at $x = 2$ and $f'(2) = 2$.

Example 6: Determine the set of all points where the function $f(x) = \dfrac{x}{1 + |x|}$ is differentiable.

Solution:

$$f(x)=\begin{cases}\frac{x}{1-x}; & x<0\\ 0; & x=0\\ \frac{x}{1+x}; & x>0\end{cases}$$

(i) $\because$ x and $1-x$ are differentiable (polynomial functions) and $1-x\neq 0$ when $x<0$

$\therefore$ $\frac{x}{1-x}$ is differentiable ; $x<0$

We know that $\boxed{\frac{f}{g}}$; $g\neq 0$; Also differential if f and g are differential.

(ii) Similarly, $\frac{x}{1+x}$ is also differential for $x>0$

i.e. $f(x)$ is differentiable for all values of $x\in\mathbf{R}$ except possibly at $x=0$.

For $x=0$; $Rf'(0)=1$; $Lf'(0)=1$

Hence f is differentiable for all $x\in\mathbf{R}$.

Example 7: Draw the graph of the function $y=|x-1|+|x-2|$ in the interval $[0,3]$ and discuss the continuity and differentiability of the function in this interval.

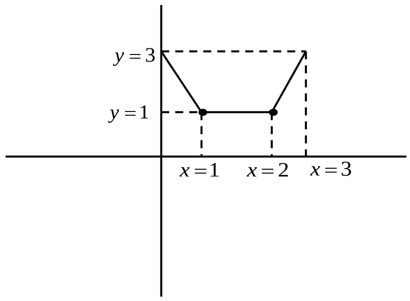
Solution:

$y=(1-x)+(2-x)=-2x+3$; $0=x\leq 1$

$y=(x-1)+(2-x)=1$, $1\leq x\leq 2$

$$y=\begin{cases}-2x+3; & 0\leq x=1\\ 1, & 1\leq x\leq 2\\ 2x-3; & 2\leq x\leq 3\end{cases}$$

$y=(x-1)+(x-2)=2x-3$, $2\leq x\leq 3$



Graph of y consists of the segments of 3 straight lines.

$y=-2x+3$ in $[0,1]$; $y=1$ in $[1,2]$; $y=2x-3$ in $[2,3]$

$\because$ there is no break in the graph so it is continuous in $[0,3]$

Further, we observe f is differential for all $x\in[0,3]$ except possibly at $x=1$ and $x=2$.

$Rf'(1)=0$, $Lf'(1)=-2$

$Rf'(2)=2$, $Lf'(2)=0$

$\therefore$ not differential at $x=1$ and $x=2$

we can observe that x=1; There does not exist unique tangent to the curve y=f(x) so not differentiable at x=1. Similarly, not diff. at x=2. Exam point- If there is a V shape then at bottom point, there cannot be unique tangent to the curve V.

Example 8: If $f(x)=\begin{cases} x\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right); & x\neq 0 \\ 0\ ; & x=0 \end{cases}$

Show that *f* is continuous but not differentiable at $x=0$.

Solution:

(i) $f(0+0)=0,\ f(0-0)=0,\ f(0)=0$

(ii) $Rf'(0)=1,\ Lf'(0)=-1$

Example-9: Let $f(x)=\begin{cases} e^{-1/x^2}\sin(1/x), & x\neq 0 \\ 0\ ; & x=0 \end{cases}$

Test differentiability of the function at $x=0$.

Solution:

$$Rf'(0)=\lim_{h\to 0}\frac{e^{-1/h^2}\sin(1/h)}{h}=\lim_{h\to 0}\frac{\sin(1/h)}{h\cdot e^{1/h^2}}$$

$$=\lim_{h\to 0}\frac{\sin(1/h)}{h+\frac{1}{h}+\frac{1}{2}h^3+\ldots}=\frac{\text{some finite quantity}}{\infty}=0$$

and $Lf'(0)=0$ (as above) $\because$ $Rf'(0)=Lf'(0)$ *f* is differentiable at $x=0$

Example 10: Show that the function *f* defined by

$$f(x)=\begin{cases} x\left\{1+\frac{1}{3}\sin\left(\log x^2\right)\right\}, & x\neq 0 \\ 0\ , & x=0 \end{cases}$$

is continuous everywhere but has no differential coefficient at $x=0$

Solution.

Hint: *f* is continuous & differential everywhere except possibly at $x=0$.

$$f(0+0)=\lim_{h\to 0}h\left\{1+\frac{1}{3}\sin\left(\log h^2\right)\right\}=\lim_{h\to 0}\left\{h+\frac{1}{3}h\sin\left(\log h^2\right)\right\}=0+(0\times\text{a finite quantity})=0$$

$\because$ $\sin\left(\log h^2\right)$ is a finite quantity oscillating between -1 and 1 as $h\to 0$.

$f(0-0)=0$

$\therefore \quad f(0+0) = f(0-0) = f(0)$

Further $Rf'(0) = 1 + \frac{1}{3}\lim_{h\to 0}\sin\left(\log h^2\right)$

The limit does not exist $\lim_{h\to 0}\sin\left(\log h^2\right)$ oscillating between -1 and +1

Hence *f* has no differential coefficient at the origin. Not differentiable.

Example11: Let $f(x) = \begin{cases} -1\,; & -2 \le x \le 0 \\ x-1\,; & 0 < x \le 2 \end{cases}$

and $g(x) = f(|x|) + |f(x)|$

Test the differentiability of $g(x)$ in $(-2,2)$

Solution:

$\because \; |x| = -x$ for $-2 \le x \le 0$ and $|x| = x$ for $0 < x \le 2$, we have

$$f(|x|) = \begin{cases} -x-1\,; & -2 \le x \le 0 \\ x-1\,; & 0 < x \le 2 \end{cases}$$

and $|f(x)| = \begin{cases} 1\,; & -2 \le x \le 0 \\ -x+1\,; & 0 < x \le 1 \\ x-1\,; & 1 < x \le 2 \end{cases}$

so $g(x) = f(|x|) + |f(x)| = \begin{cases} -x\,; & -2 \le x \le 0 \\ 0\,; & 1 < x \le 1 \\ 2x-2\,; & 1 < x \le 2 \end{cases}$

Answer: Differentiable for all *x* except at $x = 0$ and $x = 1$.

Example 12-a: Let $f(x+y) = f(x)\cdot f(y)$, $\forall x$ and *y*

If $f'(0) = 1$, Show that $f'(x) = f(x)$, for all *x*.

Solution:

Putting $y = x$ in $f(x+y) = f(x)\cdot f(y)$

$$f(2x) = \{f(x)\}^2$$

Differentiating w.r.t *x*, it gives

$$2f'(2x) = 2f(x)\cdot f'(x)$$

i.e. $f'(2x) = f(x)\cdot f'(x)$

Putting $x = 0$, we get

$$f'(0) = f(0)\cdot f'(0)$$

$\Rightarrow f(0) = 1 \;\because\; f'(0) = 1$ (Given)

Now

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x)\cdot f(h) - f(x)}{h} = f(x)\cdot\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$= f(x)\cdot\lim_{h\to 0}\frac{f(h)-f(0)}{h} = f(x)\cdot f'(0) = f(x)$$

$$\because\ f'(0)=1$$

Thus $\boxed{f'(x)=f(x)}$ for all x.

Example 12-b: Given that $f(x + y) = f(x)f(y)$, $f(0) \neq 0$, for all real x, y and $f'(0) = 2$. Show that for all real x, $f'(x) = 2f(x)$. Hence find $f(x)$.

Solution: Given, $f(x + y) = f(x)\,f(y)$.

Let $x = 0$, $y = 0$.

$\Rightarrow f(0) = f(0)\,f(0)$

$f(0) = 1$ …(i) $\qquad f(0) \neq 0$ given

Now, $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x).f(h)-f(x)}{h}$$

$$= f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$= f(x)\lim_{h\to 0}\frac{f(h)-f(0)}{h}\,[u\sin g\,(1)]$$

$f(x).f'(0) = 2f(x)$ (given $f'(0) = 2$)

$\therefore\ f'(x) = 2f(x)$

$$\Rightarrow \int\frac{f'(x)}{f(x)}dx = \int 2dx \Rightarrow \log f(x) = 2x + c$$

$f(0) = 1 \Rightarrow \log 1 = 2(0) + c \Rightarrow c = 0$

$\therefore\ \log f(x) = 2x \Rightarrow f(x) = e^{2x}$

Example 12-c: Let $f(x+y) = f(x)\cdot f(y)$ for all x and y. Suppose $f(5)=2$ and $f'(0)=3$. Find $f'(5)$.

Hint: Put $x=5,\ y=0$ to get $f(0)=1$

Then

$$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{f(5)\cdot f(h)-f(5)}{h}$$

Example(13) Let $f(x)$ be a real-valued function defined on the interval $(-5, 5)$ such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$ for all $x \in (-5, 5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$.

Sol: $$\frac{d}{dx}\left(f^{-1}(x)\right)(a) = \frac{1}{f'\left(f^{-1}(a)\right)}, \text{ where } f(t) = x$$

Here, $$e^{-x}f(x) = 2 + \int_0^x \sqrt{x^4+1}\,dt \qquad \ldots(i)$$

Differentiating both *w.r.t. x*,

$$-e^{-x} f(x) + e^{-x} f'(x) = 0 + \sqrt{x^4+1}$$

Put, $x = 0$

$$-f(0) + f'(0) = 1 \ldots \text{(ii)}$$

Also, putting $x = 0$ in (1), $f(0) = 2 + 0 \Rightarrow f(0) = 2 \Rightarrow f^{-1}(2) = 0$

Using $f(0)$ in (ii) $\quad \therefore f'(0) = 3$

$$\therefore \quad \left.\frac{d}{dx} f^{-1}(x)\right|_{x=2} = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(0)} = \frac{1}{3}$$

Example (14) If $\sqrt{x+y} + \sqrt{y-x} = c$; find $\frac{d^2y}{dx^2}$

Solution.

$$\left(\sqrt{y+x} + \sqrt{y-x}\right)^2 = c^2$$

$$(y+x) + (y-x) + 2\sqrt{y^2-x^2} = c^2$$

$$2y - c^2 = -2\sqrt{y^2-x^2}$$

$$4y^2 - 4c^2y + c^4 = 4(y^2 - x^2)$$

$$-4c^2y = -4x^2 - c^4$$

$$y = \frac{1}{c^2}x^2 + \frac{c^2}{4}$$

Differentiating *wrt* x,

$$\frac{dy}{dx} = \frac{2x}{c^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{c^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{2}{c^2}$$

Example 15: $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$

(a) $\frac{d}{dx}\Delta_1 = \Delta_2$

(b) $\frac{d}{dx}\Delta_1 = 3\Delta_2$

(c) $\frac{d}{dx}\Delta_1 = 0$

(d) $\frac{d}{dx} = \Delta_1 = \frac{d}{dx}\Delta_2 = 0$

Then

$$\frac{d}{dx}\Delta_1 = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(b) & \frac{d}{dx}(b) \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ \frac{d}{dx}(a) & \frac{d}{dx}(x) & \frac{d}{dx}(b) \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ \frac{d}{dx}a & \frac{d}{dx}a & \frac{d}{dx}x \end{vmatrix}$$

$$= 1\left(x^2 - ab\right) + \left(x^2 - ab\right) + x^2 - ab$$
$$= 3x^2 - ab = 3\Delta_2$$

Indeterminate Forms

Example 1. Evaluate $\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^3}$

- Let $\dfrac{x-\tan x}{x^3} = \dfrac{f(x)}{g(x)}$, where $f(x) = x - \tan x$ and $g(x) = x^3$.

Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(x - \tan x) = 0$$
$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^3 = 0$$

Hence, $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$ is of $\dfrac{0}{0}$ form, so that L'Hospital'r rule is applicable, *i.e.*,

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$\therefore \lim_{x\to 0}\frac{x-\tan x}{x^3} = \lim_{x\to 0}\frac{1-\sec^2 x}{3x^2} = -\frac{1}{3}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2 = -\frac{1}{3}$$

Example 2. Evaluate $\lim\limits_{x\to 0}\dfrac{xe^x - \log(1+x)}{x^2}$.

- It is a 0/0 form and therefore

$$\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2} = \lim_{x\to 0}\frac{xe^x + e^x - \dfrac{1}{1+x}}{2x} \qquad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\frac{xe^x + 2e^x + \dfrac{1}{(1+x)^2}}{2} = \frac{3}{2}$$

Example 3. Find $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x} - e}{x}$.

- Way 1: Since $\lim\limits_{x\to 0}(1+x)^{1/x} = e$, therefore it is a 0/0 form

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0}\frac{\dfrac{d}{dx}(1+x)^{1/x}}{1}$$
$$= \lim_{x\to 0}\frac{(1+x)^{1/x}\left\{x - (1+x)\log(1+x)\right\}}{x^2(1+x)};$$

Let $y=(1+x)^{1/x}$

$$\log y=\frac{1}{x}\log(1+x)\Rightarrow\frac{1}{y}.\frac{dy}{dx}=\frac{1}{x}.\frac{1}{1+x}+\left(\frac{-1}{x^2}\right)\log(1+x)$$

$$\Rightarrow\frac{dy}{dx}=(1+x)^{(1/x)}\left(\frac{1}{x}.\frac{1}{1+x}+\left(\frac{-1}{x^2}\right)\log(1+x)\right)$$

$$=e\cdot\lim_{x\to0}\frac{x-(1+x)\log(1+x)}{x^2(1+x)}\qquad\left(\frac{0}{0}\text{form}\right)$$

$$=e\cdot\lim_{x\to0}\frac{-\log(1+x)}{2x+3x^2}\qquad\left(\frac{0}{0}\text{form}\right)$$

$$=e\cdot\lim_{x\to0}\frac{-1}{(2+6x)(1+x)}=-\frac{e}{2}$$

Example 4. Find $\lim\limits_{x\to1-0}\frac{\log(1-x)}{\cot(\pi x)}$.

- It is a ∞/∞ form and therefore

$$\lim_{x\to1-0}\frac{\log(1-x)}{\cot(\pi x)}=\lim_{x\to1-0}\frac{\frac{-1}{1-x}}{-\pi\operatorname{cosec}^2(\pi x)}=\lim_{x\to1-0}\frac{\sin^2(\pi x)}{\pi(1-x)};\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to1-0}\frac{\pi\sin(2\pi x)}{-\pi}=0$$

Example 5. Evaluate $\lim\limits_{x\to0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$.

- Way 1: It is a $(\infty-\infty)$ form, we therefore write as

$$\lim_{x\to0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{\sin 2x-2x}{2x\sin^2 x+x^2\sin 2x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{2\cos 2x-2}{\left(2\sin^2 x+2x\sin 2x+2x^2\cos 2x+2x\sin 2x\right)}$$

$$=\lim_{x\to0}\frac{\cos 2x-1}{\left(\sin^2 x+2x\sin 2x+x^2\cos 2x\right)};\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{-2\sin 2x}{3\sin 2x+6x\cos 2x-2x^2\sin 2x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{-\cos 2x}{3\cos 2x-4x\sin 4x-x^2\cos 2x}=-\frac{1}{3}$$

Way 2: $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}\left(\frac{x}{\sin x}\right)^2 \ ; \left[\text{Using} \lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}\left(\frac{0}{0}\text{form}\right)=\lim_{x\to 0}\frac{\sin 2x-2x}{4x^3}\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\frac{\cos 2x-1}{6x^2}=\lim_{x\to 0}\left[-\frac{1}{3}\left(\frac{\sin x}{x}\right)^2\right]=-\frac{1}{3}$$

Example 6. Evaluate $\lim_{x\to 0+0}(\sin x\log x)$.

- It is a $(0\times\infty)$ form. Let us write

$$\lim_{x\to 0+0}(\sin x\log x)=\lim_{x\to 0+0}\frac{\log x}{\operatorname{cosec} x} \ ; \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=-\lim_{x\to 0+0}\frac{1/x}{\operatorname{cosec} x\cot x}=-\lim_{x\to 0+0}\left(\frac{\sin x}{x}\right)\tan x=0$$

Example 7. Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

- It is a (1^∞) form. Therefore, let $K=\left(\frac{\tan x}{x}\right)^{1/x^2}$, so that

$$\log K=\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$\therefore \lim_{x\to 0}\log K=\lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^2\tan x}=\lim_{x\to 0}\frac{\sec^2 x\tan x}{2\tan x+x\sec^2 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\tan x}{\sin 2x+x}=\lim_{x\to 0}\frac{\sec^2 x}{2\cos 2x+1}=\frac{1}{3}$$

i.e., $\lim_{x\to 0}\log K=\frac{1}{3}\Rightarrow \lim_{x\to 0}K=e^{1/3} \ \therefore \ \lim_{x\to 0}K=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}=e^{1/3}$

Example 8. Evaluate $\lim_{x\to 1-0}(1-x^2)^{1/(\log(1-x))}$.

- It is a (0°) form. Therefore, let $K=(1-x^2)^{1/(\log(1-x))}$, so that

$$\log K = \frac{\log(1-x^2)}{\log(1-x)} \therefore \lim_{x\to 1-0} \log K = \lim_{x\to 1-0} \frac{\log(1-x^2)}{\log(1-x)} \qquad \left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{x\to 1-0} \frac{2x(1-x)}{1-x^2} = \lim_{x\to 1-0} \frac{2x}{1+x} = 1 \Rightarrow \lim_{x\to 1-0} \log K = 1 \Rightarrow \lim_{x\to 1-0} K = e$$

Assignment-1

1. Find the limit, derivatives of the following functions at the indicated points:
   (i). $f(x) = K$, a constant, at $x = c$
   (ii). $f(x) = x$ at $x = 0$
   (iii). $f(x) = \sqrt{x}$ at $x = 4$
   (iv). $f(x) = e^x$ at $x = x_0$

2. Show that the function
$f(x) = |x| + |x-1|$
is continuous at all real points and derivable at all points except 0 and 1.

3. $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$
Prove that $f(x)$ has a derivative at $x = 0$ and that $f(x)$ and $f'(x)$ are continuous at $x = 0$.

4. $f(x) = (x-a)\sin\dfrac{1}{x-a}, \; x \neq a$
$\quad = 0, \qquad x = a$
Show that $f(x)$ is continuous but not derivable at $x = a$.

5. Discuss the derivability of the following functions:
(i). $f(x) = \begin{cases} 2, & x \le 1 \\ x, & x > 1 \end{cases}$ at $x = 1$
(ii). $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$ at $x = 1$
(iii). $f(x) = \begin{cases} 2x-3, & 0 \le x \le 2 \\ x^2-3, & 2 < x \le 4 \end{cases}$ at $x = 2, 4$

6. Show that the function $f(x) = x|x|$ is derivable at the origin.
7. Find the derivative of f at $x = 0$, where $f(x) = x^2|x|$.
8. Find $Lf'(0)$ and $Rf'(0)$ for the following functions:

(i) $f(x)=\begin{cases} x\tan^{-1}1/x, & x\neq 0 \\ 0, & x=0 \end{cases}$

(ii) $f(x)=\begin{cases} \dfrac{x\left(e^{1/x}-1\right)}{\left(e^{1/x}+1\right)}, & x\neq 0 \\ 0, & x=0 \end{cases}$

9. If $f(x)=\begin{cases} \dfrac{x\left(e^{1/x}-e^{-1/x}\right)}{\left(e^{1/x}+e^{-1/x}\right)}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Show that *f* is continuous but not derivable at $x=0$ and $Lf'(0)=-1, Rf'(0)=1$.

10. Examine the function *f*, where $f(x)=\begin{cases} x^m \sin 1/x, & x\neq \theta \\ 0, & x=0 \end{cases}$ for derivability at the origin. Also determine *m* where *f'* is continuous at the origin.

11. If functions *f* and *g* are defined on $[0,\infty]$ by

$$f(x)=\lim_{n\to\infty}\frac{x^n-1}{x^n+1} \text{ and } g(x)=\int_0^x f(t)\,dt\,;$$

then prove that g is continuous but not differentiable at $x=1$.

ANSWERS.

All answers can be found in above discussed examples

5. (i) Not derivable (ii) Not derivable, $Lf'(1)=0,\ Rf'(1)=1$

(iii) Not derivable; at $x=2,\ f'(4)=8$

7. 0 8. (i) $-\pi/2,\ \pi/2$ (ii) –1, 1

10. Derivable if $m>1$; $m>2$

11. Hint:

$$f(x)=\lim_{n\to\infty}\frac{x^n-1}{x^n+1}=\begin{cases} -1 & ; x\leq 1 \\ 1 & ; x>1 \end{cases}$$

$$for\ x\leq 1;\ g(x)=\int_0^x -1dt=-x$$

$$for\ x>1;\ g(x)=\int_0^x 1dt=x$$

So g(x) is continuous at x=1; can verify by finding LHL and RHL or by tracing the graph.
Not differentiable at *x=1*; There does not exist unique tangent at *x=1*. Or by finding RHD and LHD.

PREVIOUS YEARS QUESTIONS YEAR 2009 ONWARDS

LIMIT, CONTINUITY AND DIFFERENTIABILITY, INDETERMINATE FORMS

CSE 2022 Q1. Evaluate $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$.

IFoS 2021 Q2. Given that $f(x+y)=f(x)f(y)$, $f(0)\neq 0$, for all real *x*, *y* and $f'(0)=2$. Show that for all real *x*, $f'(x)=2f(x)$. Hence find $f(x)$.

IFoS 2020Q3. Evaluate $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$

CSE 2019 Q4. Let $f:\left[0,\frac{\pi}{2}\right]\to\mathbf{R}$ be a continuous function such that

$f(x)=\frac{\cos^2 x}{4x^2-\pi^2}$, $0\le x<\frac{\pi}{2}$. Find the value of $f\left(\frac{\pi}{2}\right)$.

CSE 2019 Q5. Is $f(x)=|\cos x|+|\sin x|$ differentiable at $x=\frac{\pi}{2}$? If yes, then find its derivative at $x=\frac{\pi}{2}$. If no, then give a proof of it.

CSE 2018Q6. Determine if $\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2}$ exists or not. If the limit exists, then find its value.

CSE 2015Q7. Evaluate the following limit:

$$\underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

IFoS 2015 Q8. Let $f(x)$ be a real-valued function defined on the interval $(-5,5)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}dt$ for all $x\in(-5,5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $\left(f^{-1}\right)'(2)$.

IFoS 2015 Q9. If $\sqrt{x+y}+\sqrt{y-x}=c$, find $\frac{d^2y}{dx^2}$

IFoS 2015 Q10. Evaluate $\lim_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x}-\frac{3}{x^4}\right)$

IFoS 2014 Q11. Show that the function given by

$$f(x)=\begin{cases}\dfrac{x\left(e^{1/x}-1\right)}{\left(e^{1/x}+1\right)}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous but not differentiable at $x=0$.

IFoS 2013 Q12. Evaluate:

$$\lim_{x\to 0}\left(\frac{e^{ax}-b^{bx}+\tan x}{x}\right)$$

IFoS 2012 Q13. Show that the function defined as

$$f(x)=\begin{cases}\dfrac{\sin 2x}{x} & \text{when } x\neq 0\\ 1 & \text{when } x=0\end{cases}$$

has removable discontinuity at the origin.

CSE 2011Q14. Evaluate:

(i) $\lim_{x\to 2}f(x)$, where $f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & x\neq 2\\ \pi, & x=2\end{cases}$

IFoS 2011Q15. Let the function *f* be defined by

$$f(t)=\begin{cases}0, \text{ for } t<0\\ t, \text{ for } 0\le t\le 1\\ 4, \text{ for } t>1\end{cases}$$

(i) Determine the function $F(x)=\int_0^x f(t)\,dt$

(ii) Where is F non-differentiable? Justify your answer.

CSE 2010 Q16. Let *f* be a function defined on $\mathbf{R}$ such that

$$f(x+y)=f(x)+f(y),\ x,y\in\mathbf{R}$$

If *f* is differentiable at one point of $\mathbf{R}$, then prove that *f* is differentiable on $\mathbf{R}$.

CSE 2009Q17. Let

$$f(x)=\begin{cases}\dfrac{|x|}{2}+1 \text{ if } x<1\\ \dfrac{x}{2}+1 \text{ if } 1\le x<2\\ -\dfrac{|x|}{2}+1 \text{ if } 2\le x\end{cases}$$

What are the points of discontinuity of *f*, if any? What are the points where *f* is not differentiable if any? Justify yours answers.

IFoS 2008 Q18. Obtain the values of the constants, *a*, *b* and *c* for which the function defined by

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}, & x<0\\ \dfrac{\left(x+bx^2\right)^{1/2}-x^{1/2}}{bx^{3/2}}, & x>0\end{cases}$$

is continuous at $x=0$.

CSE 2008 Q19. Prove that function

$$f(x)=\begin{cases}\dfrac{x\left(e^{1/x}-1\right)}{e^{rx+1}}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous but not differentiable at $x=0$

SOLUTIONS(HINTS) : See the video

https://www.youtube.com/live/mK8wBUrj6uo?si=Xp6h44LXqSFA9sQx

Ans2. $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$; 1^{∞} form.

Let $A=(\tan x)^{\tan 2x}$

$$\log_e A=\log_e(\tan x)^{\tan 2x}$$

$$\log_e A=\tan 2x\log_e(\tan x)$$

$$\lim_{x\to\frac{\pi}{4}}\log_e A=\lim_{x\to\frac{\pi}{4}}\tan 2x\cdot\log_e(\tan x)\ ;\infty\cdot 0 \text{ form}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\log_e(\tan x)}{1/\tan 2x} = \lim_{x\to\frac{\pi}{4}} \frac{\frac{d}{dx}(\log_e \tan x)}{\frac{d}{dx}(1/\tan 2x)}$$; L'Hospital Rule

$$= \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{-\frac{1\times 2}{\tan^2 2x}\cdot \sec^2 2x} = \lim_{x\to\frac{\pi}{4}} -\frac{1}{2}\frac{\sin^2 2x}{\sin x}\cdot \sec x$$

$$\lim_{x\to\frac{\pi}{4}} \log_e A = -\frac{1}{2}\times\frac{1}{1/\sqrt{2}}\cdot\sqrt{2} = -1 \Rightarrow \lim_{x\to\frac{\pi}{4}} A = e^{-1} \Rightarrow \lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = 1/e$$

Ans3. Given $f(x+y) = f(x)\cdot f(y)$ ....(1)

$f(0) \neq 0$ ....(2)

for all real x, y and

$f'(0) = 2$. ....(3)

From (1); taking $x = y$, we have

$f(x+x) = f(x)\cdot f(x)$

$f(2x) = (f(x))^2$

Differentiating w.r.t x, we get

$2f'(2x) = 2f(x)\cdot f'(x) \Rightarrow 2f'(0) = 2f(0)\cdot f'(0)$

$\Rightarrow 2\times 2 = 2\times f(0)\times 2 \Rightarrow f(0) = 1$ ....(4)

Now, by the definition of derivative of $f(x)$,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \text{ [using (1)]} = \lim_{h\to 0}\frac{f(x)\{f(h)-1\}}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\{f(h)-f(0)\}}{h} \text{ [using (4); } 1 = f(0)\text{]}$$

$$= f(x)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$f'(x) = f(x)\cdot f'(0)$. So $f'(x) = 2f(x)$ [using (3)]. Proved.

Ans4. Evaluate $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$

Solution: $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$; $0\cdot\infty$ form

$$=\lim_{x\to1}\frac{\tan\frac{\pi x}{2}}{1/(x-1)}=\lim_{x\to1}\frac{\frac{d}{dx}\left(\tan\frac{\pi x}{2}\right)}{\frac{d}{dx}\left(1/(x-1)\right)}\text{; applying L'Hospital Rule}=\lim_{x\to1}\frac{\frac{\pi}{2}\sec^2\frac{\pi x}{2}}{-\frac{1}{(x-1)^2}}\text{; complicated??}$$

<u>Let's try another method:</u>

$$\lim_{x\to1}(x-1)\tan\frac{\pi x}{2}=\lim_{x\to1}\frac{x-1}{1/\tan\frac{\pi x}{2}}=\lim_{x\to1}\frac{x-1}{\cot\frac{\pi x}{2}};\quad\frac{0}{0}$$

$$=\lim_{x\to1}\frac{1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}}=-\frac{2}{\pi}\times1=-\frac{2}{\pi}.$$

Ans5. Let $f:\left[0,\frac{\pi}{2}\right]\to\mathbf{R}$ be a continuous function such that $f(x)=\frac{\cos^2 x}{4x^2-\pi^2},\quad 0\le x<\frac{\pi}{2}$

Find the value of $f\left(\frac{\pi}{2}\right)$

$\because f\left(\frac{\pi}{2}\right)=\frac{0}{0}$ form (undefined). So let's try to find $\lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{4x^2-\pi^2}$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1+\cos2x}{2}}{4x^2-\pi^2}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{2}\times-\sin2x\times2}{8x}$$

$$\lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{-\sin\pi}{8\times\frac{\pi}{2}}=0$$

$\because$ Given that $f(x)$ is continuous in $[0,\pi/2]$

$\therefore\lim_{x\to\frac{\pi}{2}}f(x)=f(\pi/2)$

$$\boxed{0=f(\pi/2)}$$

Ans6. Is $f(x)=|\cos x|+|\sin x|$ differentiable at $x=\frac{\pi}{2}$? If yes then find its derivative at $x=\frac{\pi}{2}$.

If no, then give a proof of it.

Solution:

We know that

$$f(x)=|x-\alpha|=\begin{cases}x-\alpha; & x>\alpha\\ -(x-\alpha), & x<\alpha\end{cases}$$

$$\therefore \quad f(x) = |\cos x| + |\sin x| = \begin{cases} \cos x + \sin x, & x < \dfrac{\pi}{2} \\ -\cos x + \sin x, & x > \dfrac{\pi}{2} \end{cases} \qquad \ldots.(1)$$

Now, for differentiability of $f(x)$ at $x = \dfrac{\pi}{2}$;

$$Rf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-\cos\left(\frac{\pi}{2}+h\right) + \sin\left(\frac{\pi}{2}+h\right) - 1}{h} = \lim_{h\to 0} \frac{\sinh + \cosh - 1}{h}\ ;\ \frac{0}{0} \text{ form} = \lim_{h\to 0} \frac{\cosh - \sinh}{1} = 1 \qquad \ldots.(2)$$

$$Lf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\frac{\pi}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}-h\right) + \sin\left(\frac{\pi}{2}-h\right) - 1}{-h} = \lim_{h\to 0} \frac{\sinh + \cosh - 1}{-h}\ ;\ \frac{0}{0} \text{ form} = \lim_{h\to 0} \frac{\cosh - \sinh}{-1} = -1$$

$\because \quad Lf'(\pi/2) \neq Rf'(\pi/2)$

So given function $f(x)$ is not differentiable at $x = \pi/2$.

Ans7. Determine $\lim_{z\to 1} (1-z)\tan\dfrac{\pi z}{2}$

Solution:

$$\lim_{z\to 1} (1-z)\tan\frac{\pi z}{2} = \lim_{z\to 1} \frac{1-z}{\cot\frac{\pi z}{2}};\quad \frac{0}{0} \text{ form}$$

$$= \lim_{z\to 1} \frac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2 \frac{\pi}{2}\cdot z}; \text{ after applying L'Hospital Rule}$$

$$= \lim_{z\to 1} \frac{2}{\pi}\cdot\frac{1}{\operatorname{cosec}^2 \frac{\pi}{2}\cdot z}. \qquad \text{So } \lim_{z\to 1}(1-z)\tan\frac{\pi z}{2} = 2/\pi$$

Ans8. Evaluate the following limit

$$\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} \quad ;1^{\infty} \text{ form}$$

Solution:

Let $A=\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

$$\log_e A=\tan\left(\frac{\pi x}{2a}\right)\log_e\left(2-\frac{x}{a}\right)$$

$$\therefore \lim_{x\to a}\log_e A=\lim_{x\to a}\left[\tan\left(\frac{\pi x}{2a}\right)\log_e\left(2-\frac{x}{a}\right)\right];\ \infty\cdot 0\ \text{form}=\lim_{x\to a}\frac{\log_e\left(2-\frac{x}{a}\right)}{\cot\frac{\pi x}{2a}}$$

$$\lim_{x\to a}\log_e A=\lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\times\left(-\frac{1}{a}\right)}{-\frac{\pi}{2a}\text{cosec}^2\left(\frac{\pi x}{2a}\right)}\text{; after applying L'Hospital Rule}$$

$$=\lim_{x\to a}\frac{2\times\frac{1}{\left(2-\frac{x}{a}\right)}}{\pi\text{cosec}^2\left(\frac{\pi}{2a}x\right)}=\frac{2\times 1}{\pi}$$

$$\lim_{x\to a}\log_e A=2/\pi\Rightarrow\lim_{x\to a}A=e^{2/\pi}$$

Therefore, $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}=e^{2/\pi}$

Ans9. Let $f(x)$ be a real valued function defined on an interval $(-5,5)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}dt$ for all $x\in(-5,5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $\left(f^{-1}\right)'(2)$.

Solution: $\frac{d}{dx}\left(f^{-1}(x)\right)(a)=\frac{1}{f'\left(f^{-1}(a)\right)}$, where $f(t)=x$

Here, $e^{-x}f(x)=2+\int_0^x\sqrt{x^4+1}dt$ ...(*i*)

Differentiating both *w.r.t. x*,

$$-e^{-x}f(x)+e^{-x}f'(x)=0+\sqrt{x^4+1}$$

Put, $x=0$

$$-f(0)+f'(0)=1\ldots\text{(ii)}$$

Also, putting $x=0$ in (1), $f(0)=2+0\Rightarrow f(0)=2\Rightarrow f^{-1}(2)=0$

Using $f(0)$ in (ii) $\therefore f'(0)=3$

$$\therefore\quad \frac{d}{dx}f^{-1}(x)\Big|_{x=2}=\frac{1}{f'\left(f^{-1}(2)\right)}=\frac{1}{f'(0)}=\frac{1}{3}$$

Ans10. If $\sqrt{(x+y)}+\sqrt{(y-x)}=c$, find $\dfrac{d^2y}{dx^2}$.

Solution: $\left(\sqrt{y+x}+\sqrt{y-x}\right)^2=c^2$

$$(y+x)+(y-x)+2\sqrt{y^2-x^2}=c^2$$
$$2y-c^2=-2\sqrt{y^2-x^2}$$
$$4y^2-4c^2y+c^4=4(y^2-x^2)$$
$$-4c^2y=-4x^2-c^4$$
$$y=\frac{1}{c^2}x^2+\frac{c^2}{4}$$

Differentiating *wrt x*,

$$\frac{dy}{dx}=\frac{2x}{c^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2x}{c^2}\Rightarrow\frac{d^2y}{dx^2}=\frac{2}{c^2}$$

....(1)

Ans11. Evaluate $\lim\limits_{x\to0}\left(\dfrac{2+\cos x}{x^3\sin x}-\dfrac{3}{x^4}\right)$

Solution: $\lim\limits_{x\to0}\left(\dfrac{2+\cos x}{x^3\sin x}-\dfrac{3}{x^4}\right)=$

$$=\lim_{x\to0}\frac{2x+x\cos x-3\sin x}{x^4\sin x}$$
$$=\lim_{x\to0}\frac{x(2+\cos x)-3\sin x}{x^5}\cdot\frac{x}{\sin x};0/0$$
$$=\lim_{x\to0}\frac{2x+x\cos x-3\sin x}{x^5}\cdot1$$
$$=\lim_{x\to0}\frac{2+\cos x-x\sin x-3\cos x}{5x^4}\text{; applying L'hospital rule}$$
$$=\lim_{x\to0}\frac{2\sin x-\sin x-x\cos x}{20x^3}\text{; applying again}$$
$$=\lim_{x\to0}\frac{\cos x-\cos x+x\sin x}{60x^2}$$
$$=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1}{60}=\left(\frac{1}{60}\right)$$

Ans12. Show that the function given by

$$f(x)=\begin{cases}\dfrac{x\left(e^{1/x}-1\right)}{\left(e^{1/x}+1\right)}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous but not differentiable at $x=0$.

Solution:

(i) $$\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{h\left(e^{1/h}-1\right)}{e^{1/h}+1}=\lim_{h\to 0}\frac{h\left(1-e^{-1/h}\right)}{\left(1+e^{-1/h}\right)}=\frac{0\cdot(1-0)}{(1+0)}=0$$

$$\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{-h\left(e^{-1/h}-1\right)}{\left(e^{-1/h}+1\right)}=\frac{0(0-1)}{(0+1)}=0$$

Clear $\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(0-h)=f(0)$

So, $f(x)$ is continuous at $x=0$.

(ii) For differentiability at $x=0$

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h\left(e^{1/h}-1\right)}{h\left(e^{1/h}+1\right)}=\lim_{h\to 0}\frac{\left(1-e^{-1/h}\right)}{\left(1+e^{-1/h}\right)}=\frac{1-0}{1+0}=1$$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\dfrac{-h\left(e^{-1/h}-1\right)}{\left(e^{-1/h}+1\right)}-0}{-h}=\lim_{h\to 0}\frac{0-1}{0+1}=-1$$

$Rf'(0)\neq Lf'(0)\ \therefore\ f(x)$ is not differentiable at $x=0$.

Ans13. Evaluate $\lim_{x\to 0}\left(\dfrac{e^{ax}-e^{bx}+\tan x}{x}\right)$

Solution:

$$\lim_{x\to 0}\left(\frac{e^{ax}-e^{bx}+\tan x}{x}\right)=\lim_{x\to 0}\frac{ae^{ax}-be^{bx}+\sec^2 x}{1}=\frac{a\cdot e^0-b\cdot e^0+\sec^2 0}{1}=\frac{a-b+1}{1}=a-b+1$$

Ans14. Show that the function defined as

$$f(x)=\begin{cases}\dfrac{\sin 2x}{x} & \text{when } x\neq 0\\ 1 & \text{when } x=0\end{cases}$$

has removable discontinuity at $x=0$.

Solution:

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sin 2h}{h} = \lim_{h\to 0} 2\cdot\frac{\sin 2h}{2h} = 2\cdot 1 = 2$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{\sin(-2h)}{-h} = \lim_{h\to 0}\frac{-\sin 2h}{-h} = 2$$

$$\because \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0+h) \neq f(0)$$

$\therefore$ $f(x)$ has a removable discontinuity at $x=0$. As if we manage *f(0)*=2, this discontinuity can be removed.

Ans15. Evaluate

$$\lim_{x\to 2} f(x), \text{ where } f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x\neq 2 \\ \pi, & x=2 \end{cases}$$

Solution:

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\frac{(2-h)^2-4}{(2-h)-2} = \frac{0}{0} = \lim_{h\to 0}\frac{-2(2-h)}{-1} = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}\frac{(2+h)^2-4}{2+h-2}; \frac{0}{0}$$

$$= \lim_{h\to 0}\frac{2(2+h)}{1} = 4$$

$$\therefore \lim_{x\to 2} f(x) = 4$$

Ans16. Let the function *f* defined by

$$f(t) = \begin{cases} 0, & \text{for } t<0 \\ t, & \text{for } 0\le t\le 1 \\ 4, & \text{for } t>1 \end{cases}$$

(i) Determine the function $F(x) = \int_0^x f(t)\,dt$

(ii) Where is F non-differentiable? Justify your answer.

Solution:

Sol: The given function *f* is defined as

$$f(t) = \begin{cases} 0, & for\ t<0 \\ t, & for\ 0\le t\le 1 \\ 4, & for\ t>1 \end{cases}$$

Now we have to calculate;

$$F(x) = \int_0^x f(t)\,dt$$

For: $0 < x \le 1$

Then, $F(x) = \int_0^x f(t)dt = \int_0^x t dt = \frac{x^2}{2}$

For: $x > 1$

Then, $F(x) = \int_0^x f(t)dt = \int_0^1 f(t)dt + \int_1^x f(t)dt = \int_0^1 t dt + \int_1^x 4dt = 4x - 7/2$

i.e., $F(x) = 4x - \frac{7}{2} \; x > 1$

Therefore, $$F(x) = \begin{cases} \frac{x^2}{2}, & for\, 0 < x \le 1 \\ 4x - \frac{7}{2}, & for\, x > 1 \end{cases}$$

Clearly the function $F(x)$ is not differential at $x = 1$.

Ans17. Let f be a function defined on $\mathbf{R}$ such that $f(x+y) = f(x) + f(y)$, $x, y \in \mathbf{R}$. If f is differentiable at one point of $\mathbf{R}$, then prove that f is differentiable on $\mathbf{R}$.

Solution:

Let if $f(x)$ is differentiable at $x = 0$ and $f'(0) = 1$. (Note: Here may take any other point $x = \alpha$ and can take $f'(\alpha) = \beta$)

$\because$ given $f(x+y) = f(x) + f(-y)$ for $x = y$; $f(2x) = 2f(x)$

Not working like the question $f(x+y) = f(x) \cdot f(y)$ Question.

Method (II):

Let $f(x)$ is differentiable at $x = \alpha$. So is continuous at this point

i.e. $\lim_{h \to 0} f(\alpha + h) = f(\alpha) \Rightarrow \lim_{h \to 0} f(\alpha) + \lim_{h \to 0} f(h) = f(\alpha)$

$$\Rightarrow f(\alpha) + \lim_{h \to 0} f(h) = f(\alpha) \Rightarrow \lim_{h \to 0} f(h) = 0 \; \ldots (1)$$

$\because$ Given $f(x+y) = f(x) + f(y)$; $\forall x, y \in \mathbf{R}$

Now let's check differentiability of $f(x)$ at some arbitrary real number c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$Rf'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c) + f(h) - f(c)}{h} \text{ [using (2)] } = \lim_{h \to 0} \frac{f(h)}{h}$$

$$Lf'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c + (-h)) - f(c)}{-h}$$

$$=\lim_{h\to 0}\frac{f(c)+f(-h)-f(c)}{-h}=\lim_{h\to 0}\frac{f(-h)}{-h}$$

Now if we are able to show $f(-h)=-f(h)$; then $Rf'(c)$ and $Lf'(c)$ will be equal.

Note that from (1), f(x) may be a polynomial. *f(x)=x*

Asn17. Let $f(x)=\begin{cases}\dfrac{|x|}{2}+1 & \text{if } x<1\\ \dfrac{x}{2}+1 & \text{if } 1\le x<2\\ -\dfrac{|x|}{2}+1 & \text{if } 2\le x\end{cases}$

What are the points of discontinuity of *f*, if any? What are the points where *f* is not differentiable if any? Justify your answers.

Solution:

Probable points of discontinuity of $f(x)$ are $x=0, x=1$ and $x=2$.

• At $x=0$

$$\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{|0+h|}{2}+1=\lim_{h\to 0}\frac{h}{2}+1=1$$

$$\lim_{h\to 0} f(0-h)=\lim_{h\to 0}-\frac{h}{2}+1=\lim_{h\to 0}1=1$$

$$f(0)=\frac{0}{2}+1=1$$

$\because\ f(0+h)=f(0-h)=f(0)\ \therefore f(x)$ is continuous at $x=0$.

• At $x=1$

$$\lim_{h\to 0} f(1+h)=\lim_{h\to 0}\frac{1+h}{2}+1=\lim_{h\to 0}\frac{1}{2}+1=3/2$$

$$\lim_{h\to 0} f(1-h)=\lim_{h\to 0}\frac{|1+h|}{2}+1=\lim_{h\to 0}\frac{1+h}{2}+1=3/2$$

$$f(1)=\frac{1}{2}+1=3/2$$

$\because\ f(1+h)=f(1-h)=f(1)\ \therefore f(x)$ is continuous at $x=1$

• At $x=2$:

$$\lim_{h\to 0}(2+h)=\lim_{h\to 0}\frac{-|2+h|}{2}+1=-1+1=0$$

$$\lim_{h\to 0}(2-h)=\lim_{h\to 0}\frac{(2-h)}{2}+1=1+1=2$$

$\because\ \lim_{h\to 0} f(2+h)\ne\lim_{h\to 0} f(2-h)$

$\therefore f(x)$ is not continuous at $x=2$

For differentiability:

• $\because$ at $x=2$; function is not continuous so it is not differentiable at $x=2$.

• At $x=1$:

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\left(\frac{1+h}{2}+1\right)-3/2}{h}=1/2$$

$$Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{h}=\lim_{h\to 0}\frac{\left(\frac{|1-h|}{2}+1\right)-3/2}{-h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{1-h}{2}+1\right)-3/2}{-h}=1/2$$

Clearly $Rf'(1)=Lf'(1)$

$\therefore f(x)$ is differentiable at $x=1$.

Ans18. Obtain the values of the constants *a*, *b* and *c* for which the function defined by

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}; & x<0\\ c\,; & x=0\\ \dfrac{\left(x+bx^2\right)^{1/2}-x^{1/2}}{bx^{3/2}}; & x>0\end{cases}$$

is continuous at $x=0$.

Solution:

$$\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{-\sin(a+1)h+(-\sinh)}{-h}=\lim_{h\to 0}\frac{\sin(a+1)h+\sinh}{h}$$

$$=\lim_{h\to 0}\frac{\sin(a+1)h}{h}+\lim_{h\to 0}\frac{\sinh}{h}=(a+1)\cdot\lim_{h\to 0}\frac{\sin(a+1)h}{(a+1)h}+1$$

$$=(a+1)\cdot 1+1=a+2$$

$$\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{\left(h+bh^2\right)^{1/2}-h^{1/2}}{b\cdot h^{3/2}}=\lim_{h\to 0}\frac{(1+bh)^{1/2}-1}{b\cdot h}$$

$$=\lim_{h\to 0}\frac{1+\frac{1}{2}bh+\frac{1}{2}\left(\frac{1}{2}-1\right)b^2h^2+\ldots.-1}{bh}=\frac{1}{2}+0+0\ldots=1/2$$

$\because$ Given that $f(x)$ is continuous at $x=0$

$\therefore f(0+h)=f(0-h)=f(0)$ for $h\to 0\Rightarrow\frac{1}{2}=a+2=c\Rightarrow a=-3/2,\quad c=1/2$ and b is arbitrary.

REAL ANALYSIS AND CALCULUS

LIMIT, CONTINUITY & DIFFERENTIABILITY

Q1. Show that if a function f defined on an open interval (a,b) of $\mathbf{R}$ is convex, then f is continuous. Show, by example, if the condition of open interval is dropped, then the convex function need not be continuous. **[2c UPSC CSE 2018]**

Q2. Suppose $\mathbf{R}$ be the set of all real numbers and $f:\mathbf{R}\to\mathbf{R}$ is a function such that the following equations hold for all $x,y\in\mathbf{R}$:

(i) $f(x+y)=f(x)+f(y)$

(ii) $f(xy)=f(x)f(y)$

Show that $\forall x\in\mathbf{R}$ either $f(x)=0$ or $f(x)=x$. **[4a UPSC CSE 2018]**

Q3. A function $f:[0,1]\to[0,1]$ is continuous on $[0,1]$. Prove that there exists a point c in $[0,1]$ such that $f(c)=c$. **[1b 2018 IFoS]**

Q4. Find the supremum and the infimum of $\dfrac{x}{\sin x}$ on the interval $\left(0,\dfrac{\pi}{2}\right]$.

[1c UPSC CSE 2017]

Q5. A function $f:\mathbf{R}\to\mathbf{R}$ is defined as below:

$$f(x)=\begin{cases} x \text{ if } x \text{ is rational} \\ 1-x \text{ if } x \text{ is irrational}\end{cases}$$

Prove that f is continuous at $x=\dfrac{1}{2}$ but discontinuous at all other point in $\mathbf{R}$.

[1b 2017 IFoS]

Q6. For the function $f:(0,\infty)\to\mathbf{R}$ given by

$$f(x)=x^2\sin\frac{1}{x},0<x<\infty$$

show that there is a differentiable function $g:\mathbf{R}\to\mathbf{R}$ that extends f. **[1b UPSC CSE 2016]**

Q7. Examine the continuity of $f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y)\neq(0,0)\\ \dfrac{1}{2}, & (x,y)=(0,0)\end{cases}$ at the point $(0,0)$.

[3b 2016 IFoS]

Q8. Answer the following:

(a) Show that the function $f:\mathbf{R}\to\mathbf{R}$ defined by

$$f(x)=\begin{cases}1, & x \text{ is irrational}\\ -1, & x \text{ is rational}\end{cases}$$

is discontinuous at every point in $\mathbf{R}$. **[1a 2012 IFoS]**

Q9. Define the function

$f(x)=x^2\sin\frac{1}{x}$, if $x\neq 0$

$=0$, if $x=0$

Find $f'(x)$. Is $f'(x)$ continuous at $x=0$? Justify your answer. **[2c UPSC CSE 2010]**

Q10. If $f:\mathbf{R}\to\mathbf{R}$ is such that

$f(x+y)=f(x)f(y)$

for x,y in $\mathbf{R}$ and $f(x)\neq 0$ for any x in $\mathbf{R}$, show that $f'(x)=f(x)$ for all x in $\mathbf{R}$ given that $f'(0)=f(0)$ and the function is differentiable for all x in $\mathbf{R}$. **[1c 2010 IFoS]**

Q11. Show that if $f:[a,b]\to\mathbf{R}$ is a continuous function then $f([a,b])=[c,d]$ for some real numbers c and d, $c\leq d$. **[2d UPSC CSE 2009]**

UNIFORM CONTINUITY

Q1. Prove that the function $f(x)=\sin x^2$ is *not* uniformly continuous on the interval $[0,\infty[$.

[2b UPSC CSE 2020]

Q2. Show that the function $f(x)=\sin\left(\frac{1}{x}\right)$ is continuous and bounded in $(0,2\pi)$, but it is not uniformly continuous in $(0,2\pi)$. **[1b 2019 IFoS]**

Q3. Let $f:\mathbf{R}\to\mathbf{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on $\mathbf{R}$. **[4b UPSC CSE 2016]**

Q4. Let $X=(a,b]$. Construct a continuous function $f:X\to\mathbf{R}$ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on $[a+\varepsilon,b], a+\varepsilon<b$? Why? **[2b 2015 IFoS]**

Q5. Show that the function $f(x)=\sin\frac{1}{x}$ is continuous but not uniformly continuous on $(0,\pi)$.

[2b IFoS 2014]

Q6. Show that the function $f(x)=x^2$ is uniformly continuous in $\langle 0,1\rangle$ but not in $\mathbf{R}$.

[3a 2013 IFoS]

Q7. Let $S=(0,1]$ and f be defined by $f(x)=\frac{1}{x}$ where $0<x\leq 1$ (in $\mathbf{R}$). Is f uniformly continuous on S? Justify your answer. **[1b UPSC CSE 2011]**

Chapter-2 Rolle's and Mean Value Theorems
https://www.youtube.com/live/jQL-_9EeMwQ?si=mWXnI-JJRybL9c_W

Introduction

Rolle's theorem and MVTs are some of the most useful developments in calculus.
Lagrange's MVT provoked many mathematicians of eighteenth century to work on it.
Cauchy's MVT and Taylor's theorem are some of the important results of efforts made in this direction.
The present chapter is devoted to these theorems.

Rolle's Theorem

Statement: If $f(x)$ is a function defined in [a,b] such that

(i) $f(x)$ is continuous in the closed interval; $[a,b]$.

(ii) $f(x)$ is differentiable in the open interval; (a,b).

(iii) $f(a)=f(b)$.

Then there exists *at least one point* $c\in(a,b)$ such that $f'(c)=0$.

Note here c is between a and b.

Exam Point- Basically Rolle's theorem talks about a point where value of the derivative of the function is zero. In other words, it gives **root of derivative** of the function in between of some fixed points.

Note- The discussion below as proof or interpretation helps you in refining your understanding about definitions. No need to remember the proof but at least you must have an analytical look over it.

Proof: If $f(x)$ = constant, then

$f'(x)=0$ for all $x\in(a,b)$. Means Rolle's is verified.

• So let us assume that $f(x)$ is not a constant function.

$\because$ $f(a)=f(b)$. So, $f(x)$ should either increase or decrease when *x* takes values slightly greater than *a*.

For definiteness, suppose it increases.

$\because$ It again takes the value $f(b)=f(a)$, it must cease to increase and begin to decrease at some point $c\in[a,b]$. Evidently '*c*' cannot be equal to *a* or *b*.

$\therefore$ $c\in(a,b)$

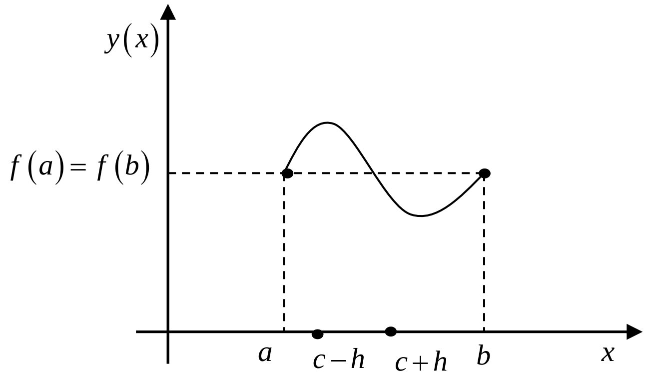
• Further, the value of $f(x)$ is maximum at $x=c$, and so

$$f(c+h)-f(c)\le 0 \Rightarrow \frac{f(c+h)-f(c)}{h}\le 0 \qquad ....(1)$$

and

$$f(c-h)-f(c)\le 0 \Rightarrow \frac{f(c-h)-f(c)}{-h}\ge 0 \qquad \text{....(2)}$$

Where h is any positive number such that $c+h,\ c-h\in[a,b]$.



From (1) and (2)

$Rf'(c)\le 0$ and $Lf'(c)\ge 0$

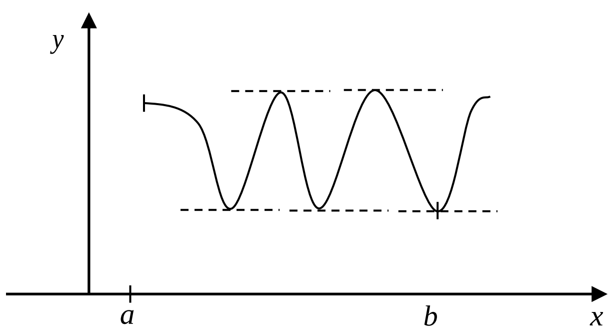
$\Rightarrow f'(c)\le 0$ and $f'(c)\ge 0$

$\because\ f'(c)=Rf'(c)=Lf'(c)$ as f is differentiable at c

Therefore, $f'(c)=0$

Geometrical interpretation of Rolle's theorem

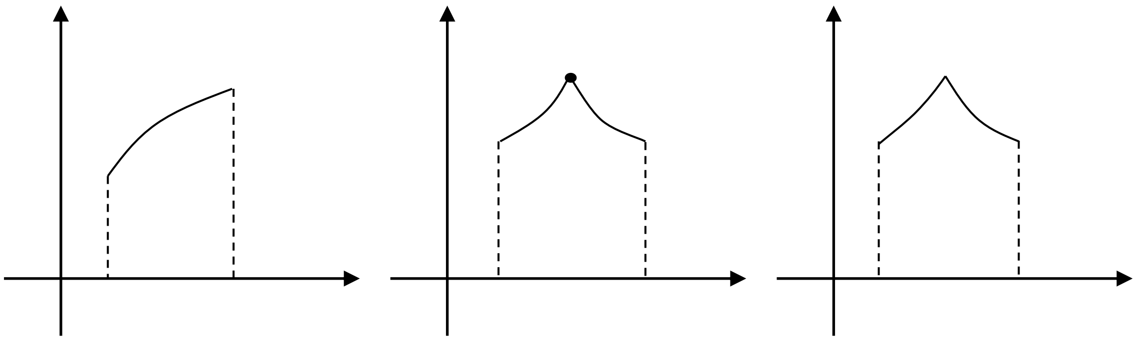
Geometrical interpretation of the result $f'(c)=0$ of Rolle's theorem is that under the given conditions there is at least one point $c\in(a,b)$ such that the tangent at the point $(c, f(c))$ is parallel to the x-axis.



It is clear from the above figure that $f'(x)$ may vanish at more than one points.

Note

Following All three figures demonstrate that the result $f'(c)=0$ of Rolle's theorem may not hold if any one of three conditions is not satisfied.



(i) $f(a) \neq f(b)$

(ii) $f(a)=f(b)$ but f is not continuous at $c \in (a,b)$

(iii) $f(a)=f(b)$ f is continuous but f is not differential at $c \in (a,b)$

Lagrange's MVT

https://www.youtube.com/live/8CvjYQ7mHAo?si=UjKIhq6OTz7w-tui

Lagrange's MVT is one of the most powerful tools in Differential calculus.

This theorem also known as the "Mean Value theorem of the first order" or simply MVT.

Statement:

If $f(x)$ is a function defined in [a, b] s.t

(i) f is continuous in $[a,b]$ (ii) f is differential in (a,b)

then there exists *at least one point* $c \in (a,b)$ s.t

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Proof: • Consider the function ϕ defined by $\phi(x)=f(x)+kx$, $\forall x \in [a,b]$ where k is any constant.

$\because f$ is differentiable and continuous in (a,b) and $[a,b]$ respectively. $\therefore \phi$ is too.

Also,

$$\phi(a)=f(a)+k(a)$$

$$\phi(b)=f(b)+k(b)$$

• If we choose k such that $\phi(a)=\phi(b)$, i.e. $k=\{f(b)-f(a)\}/(a-b)$

then ϕ satisfies all conditions of Rolle's theorem.

• Hence $\exists$ at least one $c \in (a,b)$ s.t $\phi'(c)=0$ i.e. $f'(c)+k=0 \therefore f'(c)=-k$

Therefore, $f'(c)=\frac{f(b)-f(a)}{b-a}$

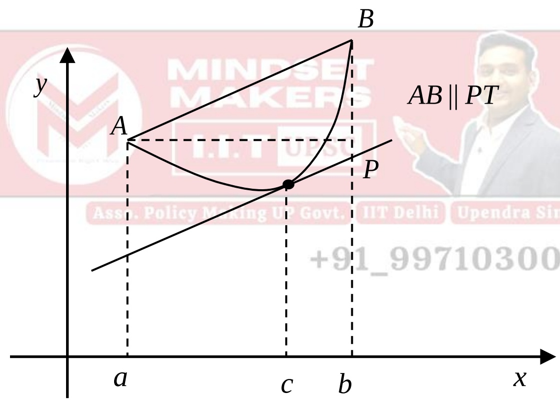
Mentor's advice: See the systematic approach- how by keeping the target in mind and instrument (Rolle's theorem conditions); we try to construct the function at very first step. This will help you in solving many PYQs.

Another form of MVT: If f is continuous in $[a,a+h]$ and differentiable in $(a,a+h)$, then ∃ at least one point $c=a+\theta h$, $0<\theta<1$ $\in(a,a+h)$ s.t

$$\frac{f(a+h)-f(a)}{h}=f'(a+\theta h)$$

Geometrical interpretation of MVT:

$\because \frac{f(b)-f(a)}{b-a}$ is the slope of the chord AB, Geometrical interpretation of Lagrange's MVT is that under the given conditions, there is at least one point $c\in(a,b)$ such that the tangent at the point $(c,f(c))$ is parallel to the chord AB.



In particular if $f(a)=f(b)$, then, $f'(c)=0$. Which is Rolle's theorem.

Cauchy's Mean Value Theorem

https://www.youtube.com/live/PswuL7cfLiY?si=EVMYo-aQMqaW2p3_

Cauchy generalized Lagrange's MVT with two functions defined on the same interval as below:

Statement

If two functions $f(x)$ and $g(x)$ are defined in [a,b] s.t

(i) Continuous in the closed interval $[a,b]$

(ii) Differentiable in (a,b)

(iii) $g'(x)\neq 0 \quad \forall x\in(a,b)$

Then there exists *at least one point* $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Proof: Obviously, $g(a)\neq g(b)$, for otherwise g would satisfy all condition of Rolle's theorem and so $g'(p)=0$ for some $p\in(a,b)$, thus contracting the (iii).

• Now consider the function $\phi(x)$ defined by

$\phi(x)=f(x)+kg(x)$, $\forall x\in[a,b]$

where k is any constant. Let us choose k such that $\phi(a)=\phi(b)$

i.e. $f(a)+kg(a)=f(b)+kg(b)$

$\therefore\ k=-\left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)$

• In view of (i) and (ii), $\phi(x)$ is continuous in $[a,b]$ and differentiable in (a,b).

Now $\phi(x)$ satisfies all the condition of Rolle's theorem.

• Hence there exists at least one point

$c\in(a,b)$ s.t $\phi'(c)=0$

i.e. $f'(c)+kg'(c)=0$

i.e. $\boxed{\frac{f'(c)}{g'(c)}=-k=\frac{f(b)-f(a)}{g(b)-g(a)}}$

Note: Lagrange's MVT follows by taking $g(x)=x$ in the above theorem.

Another Form of Cauchy's MVT

If f and g continuous in $[a,a+h]$ and differentiable in $(a,a+h)$ and $g'(x)\neq 0$ for all $x\in(a,a+h)$, then there exists at least one point $c=a+\theta h$, where $0<\theta<1$, s.t

$$\frac{f(a+b)-f(a)}{g(a+b)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)}$$

Generalized Mean Value Theorem

Statement

Let f,g,h be three functions defined on $[a,b]$ such that all of them are

(i) Continuous in $[a,b]$

(ii) Differentiable in (a,b)

Then there exists *at least one point* $c\in(a,b)$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Proof: Consider the function $\phi(x)$ defined by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}, \ \forall x \in [a,b]$$

i.e. $\phi(x)$ is of the form $k_1 f(x) + k_2 g(x) + k_3 h(x)$

so continuous in $[a,b]$ and differential in (a,b)

Further $\phi(a) = \phi(b) = 0$

Thus ϕ satisfies all condition of Rolle's theorem

$\therefore \ \phi'(c) = 0$

Note:

(1) Cauchy's M.V.T. is a particular case of above theorem $h(x) = 1$.

(2) Lagrange's MVT is a particular case of the above theorem $g(x) = x$ and $h(x) = 1$

Exam Points: Now we'll learn about different outcomes from Lagrange's MVT which are very important for exam.

(1): If a function f is continuous in $[a,b]$ and differentiable in (a,b) and if $f'(x) = 0$ for all $x \in (a,b)$, then f is a constant function on $[a,b]$.

Proof: Let p be any point in $[a,b]$. Then f is continuous in $[a, p]$ and differential in (a, p). Therefore by MVT there exists at least one point $c \in (a, p)$ s.t

$f(p) - f(a) = (p - a) f'(c)$

and so $f(p) - f(a) = 0$; As derivative of constant function is zero.

Thus $\boxed{f(p) = f(a)}$

(2): If two functions f and g are continuous in $[a,b]$ such that $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x)$ and $g(x)$ differentiated by a constant only.

Proof: Consider the function ϕ defined by

$\phi(x) = f(x) - g(x), \ \forall x \in [a,b]$

$\therefore f, g$ continuous and differential. So is ϕ

$\phi'(x) = f'(x) - g'(x) = 0, \ \forall x \in (a,b)$

Hence $\phi(x)$ is constant.

Monotonic property and Concavity
https://www.youtube.com/live/u57GoEffdmY?si=aAdIpA5Y9WjYzEzo
Topics: Increasing, decreasing, monotone, concave up, concave down.

Definition (1): A real valued function *f* defined on an interval *J* is increasing on *J* if $f(a) \le f(b)$ whenever $a, b \in J$ and $a \le b$. It is strictly increasing on *J* if $f(a) < f(b)$ whenever $a, b \in J$ and $a < b$. The function *f* is decreasing on *J* if $f(a) \ge f(b)$ whenever $a, b \in J$ and $a \le b$. It is strictly decreasing on *J* if $f(a) > f(b)$ whenever $a, b \in J$ and $a < b$.

(3): If *f* is continuous in $[a,b]$ and differential in (a,b), then *f* is increasing or decreasing according as $f'(x) \ge 0$ or $f'(x) \le 0$ for all $x \in (a,b)$

Proof: Let x_1 and x_2 be *any* two distinct points in $[a,b]$ such that $x_1 < x_2$ and so $[x_1, x_2] \le [a,b]$

• Then *f* is continuous in $[x_1, x_2]$ and differentiable in (x_1, x_2).

So by Lagrange's MVT there is a point $c \in (x_1, x_2)$ s.t

$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$

Now $\because$ $x_2 - x_1 > 0$, it follows that $f(x_2) \ge f(x_1)$ if $f'(c) \ge 0$.

As we have shown it for any arbitrary two points x_1, x_2.

Hence *f* is increasing or decreasing according as $f'(x) \ge 0$ or $f'(x) \le 0$, $\forall x \in (a,b)$

Definition (2): $f : A \to \mathbf{R}$ where $A \subseteq \mathbf{R}$. The function *f* has a local (or RELATIVE) MAXIMUM at a point $a \in A$ if there exists $r > 0$ such that $f(a) \ge f(x)$ whenever $|x - a| < r$ and $x \in \text{dom}f$.

It has a LOCAL (or RELATIVE) MINIMUM at a point $a \in A$ if there exists $r > 0$ such that $f(a) \le f(x)$ whenever $|x - a| < r$ and $x \in \text{dom}f$. The point a is relative extremum of f if it is either a relative maximum or relative minimum.

Note:

The function $f : A \to \mathbf{R}$ is said to attain a maximum at a if $f(a) \ge f(x)$ for all $x \in \text{dom}f$.

This is often called a GLOBAL (or ABASOLUTE) MAXIMUM.

It is clear that every global maximum is also a local maximum but not vice versa.

Exam Point: f' changes it's sign before and after the point of local maxima or minima.

Definition (3):

A real valued function *f* defined on an interval *J* is concave up on *J* if the chord line connecting any two points $\left(a, f(a)\right)$ and $\left(b, f(b)\right)$ on the curve (where $a, b \in J$) always lies on or above the curve. It is concave down if the chord line always lies on or below the curve.

Note:
A point on the curve where the concavity changes is a POINT OF INFLECTION when *f* is twice differentiable.
It is concave up on J if and only if $f''(c) \geq 0$ for all $c \in J$ and is concave down on *J* if and only if $f''(c) \leq 0$ for all $c \in J$.

Tracing of graphs
<u>Facts and definitions</u>:
(1) If the first derivative *f′* is positive, then the function is increasing (↑)
(2) If *f′* is negative, then function is decreasing (↓)
(3) If *f″* is +ve, then the function is concave up $(\cup)$
(4) If *f″* is -ve then the function is concave down $(\cap)$

Taylor's Theorem
https://www.youtube.com/live/t0JQmP2d1QY?si=AFHEuUq5vo-e9vxf

(1) Taylor's theorem with Lagrange's form of remainder.
Statement:
If *f* is a function defined in $[a, a+h]$ such that all the derivatives up to $(n-1)th$ are continuous in $[a, a+h]$ and $f^{(n-1)}$ is differentiable in $(a, a+h)$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h), \text{ where}$$

$0 < \theta < 1$.
Proof: Consider the function ϕ defined on $[a, a+h]$ by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots.$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{(a+h-x)^n}{n!} k \qquad \ldots.(1)$$

where *k* is a constant to be determined such that $\phi(a+h) = \phi(a)$

• Subjecting (1) to their condition

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots. + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{n^n}{\lfloor n} k \qquad \ldots.(2)$$

Now, by hypothesis, all $f', f'', ..., f^{n-1}$ are continuous in $[a, a+h]$ and differentiable in $(a, a+h)$.

Also $(a+h-x)$, $(a+h-x)^2$,...... $(a+h-x)^n$, all being polynomials, are continuous in $[a, a+h]$ and differentiable in $(a, a+h)$.

Therefore by algebra of functions $\phi(x)$ is continuous in $[a, a+h]$ and differentiable in $(a, a+h)$ by Rolle's theorem.

There exists at least one point $c \in (a, a+h)$ such that $\phi'(c) = 0$. This gives $c = a + \theta h$; $0 < \theta < 1$.

Now differentiating (1) w.r.t. *x*, we get

$$\phi'(x) = f'(x) + \left\{(a+h-x) f''(x) - f'(x)\right\}$$

$$+\left\{\frac{(a+h-x)^2}{2!} f'''(x) - (a+h-x) f''(x)\right\} + ...$$

$$+\left\{\frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(n) - \frac{(a+h-x)^{n-2}}{(n-2)!} f^{(n-1)} x\right\} - k \cdot \frac{(a+h-x)^{n-1}}{(n-1)!}$$

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}\left\{f^{(n)}(x) - k\right\}$$

$\because$ other terms cancel in pairs.

when $\phi'(a+\theta h) = 0$ gives $k = f^{(n)}(a+\theta h)$

Using this value of *k* in (2), we get the desired expansion of $f(a+h)$.

Note:

The last term, $\frac{h^n}{n!} f^{(n)}(a+\theta h)$ is called Lagrange's form of remainder after *n* terms in Taylor's expansion of $f(a+h)$ in ascending powers of *h*.

• By taking $h = x - a$, we obtain another useful form of the above theorem. Thus

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + ... + \frac{(x-a)^{n-1}}{(n-1)!} f(a)$$

$$+\frac{(x-a)^n}{n!} f^{(n)}(a + \theta(x-a)) \text{ where } 0 < \theta < 1$$

- Lagrange's MVT is a particular case of the above theorem with $n = 1$.
- Taking $n = 2$, we obtain the following result which is known as the Mean Value Theorem of the Second Order.

Corollary 1: If f is a function defined on $[a, a+h]$ such that f, f' are continuous in $[a, a+h]$ and f' is differentiable in $(a, a+h)$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h),\ 0 < \theta < 1$$

- Taking $a = 0$ and $h = x$ in the above theorem, we obtain the following result which is known as Maclaurin's Theorem with Lagrange's form of Remainder.

Corollary 2: If f is a function defined on $[0, x]$ such that all the derivatives up to $(n-1)^{th}$ are continuous in $[0, x]$ and $f^{(n-1)}$ is differentiable in $(0, x)$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots. \ + \dots. + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) \ + \frac{x^n}{n!} f^{(n)}(\theta x)$$

where $0 < \theta < 1$

- The following theorem is known as Taylor's Theorem with Cauchy's form of Remainder.

Theorem: If f is a function defined on $[a, a+h]$ such that all the derivatives upto $(n-1)th$ are continuous in $[a, a+h]$ and $f^{(n-1)}$ is differentiable in $(a, a+h)$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots. + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) \ + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$$

where $0 < \theta < 1$

Proof: Consider the function ϕ defined on $[a, a+h]$ by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \dots. \ + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x) \cdot k$$

....(1)

where k is a constant to be determined such that $\phi(a+h) = \phi(a)$

Subjecting (1) to this condition

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots.. + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + h \cdot k \qquad \text{....(2)}$$

All conditions of Rolle's theorem satisfied.

$\therefore \exists$ at least one point $c = a + \theta h$, where $0 < \theta < 1$, such that $\phi'(a+\theta h) = 0$

$$\because \ \phi'(x) = f'(x) + \{(a+h-x) f''(x) - f'(x)\} + \dots\dots\dots\dots - k = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - k$$

[$\because$ other terms cancel in pairs]

$\therefore \ \phi'(a+\theta h) = 0$ gives

$$\frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)-k=0$$

$$\therefore \quad k=\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$$

Using this value of k in (2), we obtain the desired expansion of $f(a+h)$.

Note: The last term $\frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$ is called Cauchy's form of remainder after n terms in Taylor's expansion of $f(a+h)$ is ascending powers of h.

- Mentor's advice: [Question they directly ask; Remainder after n terms]

- By taking $h=x-a$, we obtain

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+....+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

$$+\frac{(x-a)^n}{(n-1)!}(1-\theta)^{n-1}\cdot f^{(n)}(a+\theta(x-a)) \text{ where } 0<\theta<1$$

- Taking $a=0$ and $h=x$ in the above theorem we obtain Maclaurin's theorem with Cauchy's form of remainder.

Corollary: If f is a function defined on $[0,x]$ such that all the derivatives up to $(n-1)th$ are continuous in $[0,x]$ and $f^{(n-1)}$ is differentiable in $(0,x)$, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+....+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x)$$

where $0<\theta<1$

Note: The above two theorems provide the so called Taylor's development of a function in finite form.

The following theorem gives a necessary and sufficient condition under which a function can be expanded as an infinite series known as Taylor's series.

Theorem:
The series in

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+...+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+....$$

represents $f(x)$ for those values of *x*, and those only, for which the remainder after *n* terms in Taylor's theorem with Lagrange's form of remainder

$$R_n(x) = \frac{(x-a)^n}{(n)!} f^{(n)}\left(a+\theta(x-a)\right),\ 0<\theta<1$$

approaches zero as *n* tends to infinity.

Corollary:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots. \ + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \ldots..$$

represents $f(x)$ for those values of *x*, and those only, for which the remainder after *n* terms in Maclaurin's theorem with Lagrange's form of remainder

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x);\ 0<\theta<1$$

approaches zero as *n* tends to infinity.
The infinite series in this result called Maclaurin's series.

Exam Point: $$R_n(x) = \begin{cases} \dfrac{x^n}{n!} f^{(n)}(\theta x)\ ; & \text{Lagrange's form} \\ \dfrac{x^n}{(n-1)!}(1-\theta)^{n-1} \cdot f^{(n)}(\theta x) & \text{Cauchy's form} \end{cases}$$

<u>Power Series</u>:
The series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots..$$

where $a_0, a_1, a_2, \ldots..$ are constants, is called a power series.

<u>Some Standard Power Series</u>

1. Exponential Series

Let $f(x) = e^x;\ x \in \mathbf{R}$ so that $f(0) = e^0 = 1$. Then $f^{(n)}(x) = e^x,\ \forall x \in \mathbf{R}$

Also Lagrange's form of remainder is

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} e^{\theta x} \qquad \ldots.(1)$$

$\therefore\ \ f^{(n)}(0) = 1,\ \forall n \in \mathbf{N}$

Using Maclaurin's theorem, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

(2) The sine series

$f(x) = \sin x, \; x \in \mathbf{R}$

$f(0) = 0$

$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), \; \forall x \in \mathbf{R}$

$\because$ Lagrange's form of Remainder

$R_n(x) = \frac{x^n}{h!} f^{(n)}(\theta x) \; = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)$

= 0 × quantity oscillating between -1 and +1

= 0

Now $f^{(n)}(0) = \sin\frac{n\pi}{2}$

$f(0) = 1, \; f''(0) = 0, \; f'''(0) = -1, \; f^{(4)}(0) = 0, \; f^{(5)}(0) = 1, ...$

Using these values in Maclaurin's theorem,

We have

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^{n-1} \cdot \frac{x^{2n-1}}{(2n-1)!} + ...$

Similarly, we have for:

3. The cosine series
4. Natural logarithm series

$f(x) = \log(1+x)$

EXTREME VALUES (DEFINITIONS)

Let *c* be an interior point of the domain $[a,b]$ of a function *f*.

Definition 1. The point *c* is said to be the *stationary point* and $f(c)$, the *stationary value* of the function *f* if $f'(c) = 0$.

Definition 2. The function *f* is said to have a maximum value (a maxima or a maximum) at *c* if $f(c)$ is the greatest value of the function in a small neighborhood $(c-\delta, c+\delta), \delta > 0$ of *c*.

A Necessary Condition for Extreme Values

Theorem. If $f(c)$ is an extreme value of a function f then $f'(c)$, in case it exists, is zero.

Investigation of the Points of Maximum and Minimum Values

c is a point of maximum value if the function changes from an increasing to a decreasing function as x passes through c. Therefore, in case f is derivable, the derivative changes sign from positive to negative as x passes through c.

Theorem. *If c is an interior point of the domain of a function f and* $f'(c)=0$, *then the function has a maxima or a minima at c according as* $f''(c)$ *is negative or positive.*

Examples of Rolle's theorem

Example 1: Verify Rolle's theorem for the following functions:

(i) $f(x)=2x^3+x^2-4x-2$

(ii) $f(x)=x(x+3)e^{-x/2}$ in $[-3,0]$

(iii) $f(x)=(x-a)^m(x-b)^n$, where $x\in[a,b]$ and *m*, *n* are positive integers.

Solution:

(i) $f(x)$ is a polynomial $\therefore$ is continuous and is differentiable.

also $f(x)=0$ gives.(As it's not given here in which interval, we want to verify, So we find that interval first. Easy way is by taking two points where function takes the value zero)

$(x^2-2)(2x+1)=0$ i.e. $x=\pm\sqrt{2},\ -\dfrac{1}{2}$

Thus $f(\sqrt{2})=0\ =f(-\sqrt{2})\ =f(-1/2)$

Consider the interval $[-\sqrt{2},\sqrt{2}]$.

$\because$ All the conditions of Rolle's theorem are satisfied in this interval.

$\therefore\exists$ at least one $c\in(-\sqrt{2},\sqrt{2})$ for which $f'(c)=0$

Now $f'(x)=6x^2+2x-4$

$\therefore\ 6x^2+2c-4=0$

$3c^2+c-2=0$

$c=-1,\ 2/3$

$\because$ both these points $c=-1,\ 2/3$

belong to $(-\sqrt{2},\sqrt{2})$, Rolle's theorem is verified.

Note: The interval $[-\sqrt{2},\sqrt{2}]$ can be broken up into two parts $\left[-\sqrt{2},-\dfrac{1}{2}\right]$ and $\left[-\dfrac{1}{2},\sqrt{2}\right]$ in each of which Rolle's theorem can be verified seperately.

(ii) $f(x)=x(x+3)e^{-x/2}$

$f'(x)=-\dfrac{1}{2}(x^2-x-6)e^{-x/2}$

f' exists for all $x \in [-3, 0]$

$\therefore f$ is differential in $[-3, 0]$

$\Rightarrow f$ is continuous in $[-3, 0]$

Also, $f(-3) = 0 = f(0)$

$\therefore$ All condition of Rolle's theorem satisfied

$\therefore \exists$ at least one $x \in (-3, 0)$ for which $f'(c) = 0$

Now

$$f'(x) = 0 \Rightarrow -\frac{1}{2}(x^2 - x - 6)e^{-x/2} = 0$$

$$\Rightarrow x^2 - x - 6 = 0 \ \Rightarrow x = -2, 3$$

So there exists at least one $c = -2 \in [-3, 0]$. Hence Rolle's theorem is verified.

(iii) $\because$ m and n are positive integers, on being expanded by Binomial theorem, $(x-a)^m$ and $(x-b)^n$ are polynomials in x.

So $f(x)$ is differential and continuous in[a,b]. further $f(a) = f(b) = 0$

$\therefore$ All conditions of Rolle's theorem satisfied.

$\therefore \exists$ at least one $c \in (a, b)$ s.t $f'(c) = 0$

$$f'(x) = 0$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{n(x-a) + m(x-b)\} = 0$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{(m+n)x - (na+mb)\} = 0$$

$$\Rightarrow x = a, b, \frac{na+mb}{m+n}$$

Out of these values of x, $\frac{na+mb}{m+n}$ is the only point which lies in the (a, b).

- In fact, it divides the interval (a, b) internally in the ratio $m : n$. Hence Rolle's theorem is verified.

Example2. If $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = 0$.Then $C_0 + C_1 x + C_2 x^2 + ... + C_n x^n = 0$ has at least one real root between 0 and 1.

Solution.

Exam Point -Try; intuition comes from Rolle's theorem. But on which function?? To solve such questions, we need to consider a function by ourselves which can give desired result.

Considering such functions becomes easy game once you practice enough number of questions.

These examples in Rolle's, MVTs, Taylor's will help you in commanding over such important category of questions.

Let's consider a function

$$f(x)=C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+\ldots+\frac{C_nx^{n+1}}{n+1}$$

$\because$ $f(x)$ is polynomial $\therefore$ continuous in $[0,1]$ and differentiable in $(0,1)$

Also $f(0)=C_0 0^0+C_1 0^2+\ldots+C_n 0^{n+1}=0$

$$f(1)=C_0+C_1\times1^2+C_2\times1^3+\ldots+C_n\times1^{n+1}=C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots\frac{C_n}{n+1}=0 \text{ in (1)}$$

i.e. $f(0)=f(1)=0$

$\therefore f(x)$ satisfies all three conditions of Rolle's theorem

$\therefore\exists$ at least one point (real number) $c\in(0,1)$ s.t

$$f'(c)=0\Rightarrow C_0+C_1c+C_2c^2+\ldots+C_nc^n=0$$

$\because$ $f'(x)=C_0+C_1x+C_2x^2+\ldots+C_nx^n$

c is a real root of the equation $C_0+C_1x+C_2x^2+\ldots+C_nx^n$ =0 between 0 and 1.

Example 3. Determine whether the function $f(x)=\sec hx$ satisfies condition of Rolle's theorem for the interval $[-1,1]$. If so then find all numbers c in $(-1,1)$ which satisfy the conclusion of Rolle's theorem

Solution.

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i},\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\cosh\theta=\frac{e^{\theta}+e^{-\theta}}{2},\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}$$

$$\therefore f(x)=\sec hx=\frac{2}{e^x+e^{-x}}$$

Clearly $f(x)$ is continuous in $[-1,1]$

$\because e^x+e^{-x}\neq0$ in $[-1,1]$

$f(x)$ is differentiable in $(-1,1)$

$$f(-1)=\frac{2}{e^{-1}+e^{1}},\ f(1)=\frac{2}{e^{1}+e^{-1}} \Rightarrow f(-1)=f(1)$$

All conditions of Rolle’s theorem are satisfied (yes)

$\therefore \exists$ at least one $c\in(-1,1)$ s.t $f'(c)=0 \Rightarrow -\frac{2}{\left(e^{c}+e^{-c}\right)}\left(e^{c}-e^{-c}\right)=0$

$\because$ In $(-1,1)$; $e^{c}+e^{-c}\neq 0$ $\therefore$ $\frac{2}{\left(e^{c}+e^{-c}\right)}\neq 0$

$\therefore e^{c}-e^{-c}=0$ [must be] $e^{c}=e^{-c}$ Possible only for $c=0$

$\therefore$ Required $c=0$

Example 4. Show that under suitable condition there exits at least one real number where $a<\xi<b$

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}=(b-a)\begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}$$

Solution.

Applying Rolle’s theorem on a function $\phi(x)$ s.t $\phi'(x)$ can give required answer. So let’s try to construct $\phi(x)$

Let

$$\phi(x)=\begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix}-\frac{(x-a)}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

Now if (i) $f(x)$ and $g(x)$ are continuous in $[a,b]$ then $\phi(x)$ will also be continuous.

(ii) if $f(x)$ and $g(x)$ are differentiable in (a,b) then $\phi(x)$ will also be differentiable in (a,b)

$$\because\ \phi(a)=\begin{vmatrix} f(a) & f(x) \\ g(x) & g(x) \end{vmatrix}-\frac{(b-a)}{b-a}\begin{vmatrix} f(a) & f(a) \\ g(x) & g(x) \end{vmatrix}=0-0=0$$

Similarly $\phi(b)=0$

i.e. $\phi(a)=\phi(b)$

$\because$ $\phi(x)$ is continuous in $[a,b]$

$\phi(x)$ is differentiable in (a,b)

$\phi(a)=\phi(b)$

$\therefore$ All conditions of Rolle's theorem are satisfied in $[a,b]$ by $\phi(x)$

$\therefore\exists$ at least one real number $\xi\in(a,b)$ s.t $\phi'(\xi)=0$

$$\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b)\end{vmatrix} - \begin{vmatrix} f(a) & f(b) \\ g(x) & g(b)\end{vmatrix} = 0$$

Rough Work to construct ϕ

$$\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b)\end{vmatrix} = \begin{vmatrix} f(a) & f'(a) \\ g(a) & g'(b)\end{vmatrix}$$

and $\phi(a)=\phi(b)$

To get $\frac{1}{b-a}$, took x and to get $\phi(a)=\phi(b)$ took $\frac{x-a}{b-a}$

Example 5. Let $f:\mathbf{R}\to\mathbf{R}$ be twice differentiable function with $f(0)=(1)=f'(0)=0$

Then prove or disprove that $f''(x)=0$ for some $x\in(0,1)$

Solution.

$\because f(x)$ is twice differentiable

$\Rightarrow f(x)$ is differentiable and $f'(x)$ is also differentiable in $\mathbf{R}$

Let's think in $[0,1]$ for $f(x)$

$\because f(x)$ is differentiable in $(0,1)$ so also continuous in $[0,1]$

Also given $f(0)=f(1)$

All conditions of Rolle's theorem are satisfied

So there exists at least one point $c\in(0,1)$ s.t $f'(c)=0$

$\Rightarrow$ Now let's think in $[0,c]$ for the function $f'(x)$

$\because f'(x)$ is differentiable in $(0,c)$

$\therefore f'(x)$ is continuous in $[0,c]$

Also $f'(0)=0=f'(c)$

$\therefore$ All 3 conditions of Rolle's are satisfied by $f'(x)$ in $[0,c]$ $\therefore \exists$ at least one point are $c_1 \in (0,c)$

s.t $f''(c_1)=0 \Rightarrow f''(x)=0$ for some $x \in (0,1)$

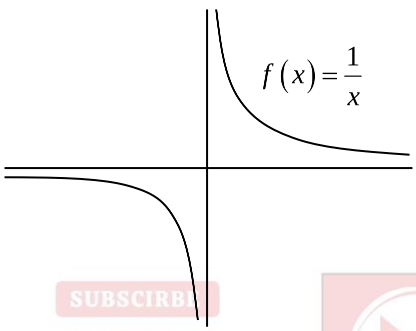
Examples of MVTs

Example 1: Discuss the applicability of Lagrange's MVT to the following functions:

(i) $f(x)=1/x$ in $[-1,1]$

(ii) $f(x)=x^{2/3}$ in $[-1,1]$

(i)



f is not defined at $x=0 \in [-1,1]$

$\therefore$ Lagrange's MVT not applicable.

(ii) The progressive derivative for $f(x)=x^{1/3}$ at $x=0$; does not exist finitely in $(-1,1)$

(2) Hence Lagrange's MVT not applicable.

Example 2:

Let

$$f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right); & x \neq 0 \\ 0; & x=0 \end{cases}$$

Now find a point *c* in $(-1,1)$ if possible by Lagrange's MVT for $g(x)$ where $g(x)=f'(x)$

Solution:

$g(x)$ is not continuous at $x=0$

$\therefore$ Lagrange's theorem not applicable.

Example 3: Find '*c*' of the MVT if $a<c<b$, $f(x)=x^2-3x-1$, where $a=-\frac{11}{7}$, $b=\frac{13}{7}$.

$\because f$ is a polynomial

$\therefore$ differential and continuous in $\left(-\frac{11}{7},\frac{13}{7}\right)$ & $\left[-\frac{11}{7},\frac{13}{7}\right]$ respectively.

$\therefore$ by Lagrange MVT; $\exists$ at least one $c\in(a,b)$ s.t $f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f\left(\frac{13}{7}\right)-f\left(-\frac{11}{7}\right)}{\frac{13}{7}-\left(-\frac{11}{7}\right)}$

$$=\frac{\left(\frac{169}{49}-\frac{39}{7}-1\right)-\left(\frac{121}{49}+\frac{33}{7}-1\right)}{\frac{28}{7}}$$

$c=33/196$ clearly $\frac{33}{196}\in\left(-\frac{11}{7},\frac{13}{7}\right)$

Example 4: Find 'c' of Lagrange's MVT if $f(x)=x(x-1)(x-2)$; $a=0,\ b=\frac{1}{2}$

Solution:

Hint: On solving as previous, we get: $c=1\pm\frac{1}{6}\sqrt{21}$

But only $c=1-\frac{1}{6}\sqrt{21}\in\left(0,\frac{1}{2}\right)$

Examples of Cauchy's MVT

Example 1: Find c in Cauchy MVT if $f(x)=x(x-1)(x-2)$, $g(x)=x(x-2)(x-3)$ defined in $\left[0,\frac{1}{2}\right]$.

Solution:

f, *g* are polynomials. So conditions of MVT are satisfied.

$$\therefore\ \frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f\left(\frac{1}{2}\right)-f(0)}{g\left(\frac{1}{2}\right)-g(0)}=\frac{\frac{3}{8}-0}{\frac{15}{8}-0}$$

i.e. $\frac{3c^2-6c+2}{3c^2-10c+6}=\frac{1}{5}$ i.e. $c=\frac{5\pm\sqrt{13}}{6}$

out of these two $c=\frac{5-\sqrt{13}}{6}$ is valid.

Example 2. Let us consider two functions. If in Cauchy's MVT we write,

(i) $f(x)=\frac{1}{\sqrt{x}}$ and $g(x)=\sqrt{x}$, then c is the geometric mean between a and b.

(ii) If $f(x)=\frac{1}{x}$ and $g(x)=\frac{1}{x^2}$, the c is the harmonic mean between a and b.

Solution.

Here if we considered that $f(x)$ and $g(x)$ are well defined in $[a,b]$ then we may proceed like:

At $x=0, f(x), g(x)$ will be tending to ∞

So a and b must be non-zero

(i) $\because f(x)$ and $g(x)$ continuous in $[a,b]$, differentiable in (a,b)

$\therefore$ Cauchy's MVT, $\exists$ at least one $c\in(a,b)$ s.t $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow \frac{-\frac{1}{2}c^{-3/2}}{\frac{1}{2}c^{3/2}}=\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}$

$$\Rightarrow -\frac{1}{c}=\frac{\sqrt{a}-\sqrt{b}}{\left(\sqrt{b}\cdot\sqrt{a}\right)}\left(\sqrt{b}-\sqrt{a}\right)$$

$\Rightarrow \frac{1}{c}=\frac{1}{\sqrt{ab}} \Rightarrow c=\sqrt{ab} \Rightarrow c$ is g.m. of a,b.

(ii): try by yourself. Show $\frac{1}{2c}=\frac{1}{a}+\frac{1}{b}$.

Example 3. The function $f(x)$ is differentiable on the interval $[a,b]$, where $ab>0$. Show that the following equality

$\frac{1}{a-b}\begin{vmatrix} a & b \\ f(a) & f(b) \end{vmatrix}=f(c)-cf'(c)$ holds for this function where $c\in(a,b)$

Solution.

Let's think:

Constructing some function $\phi(x)$ with the help of $f(x)$. But $f(x)$ itself is not given.

$\therefore$ Not by Rolle's or Lagrange's MVT

But we need to go other than those i.e. Cauchy's MVT

$\therefore$ We need to construct two functions.

Let's consider two functions; $F(x)=\frac{f(x)}{x}, G(x)=\frac{1}{x}$

Now let's apply Cauchy's MVT for $F(x)$ and $G(x)$ in $[a,b]$

$\because$ given $ab > 0 \Rightarrow a \neq 0, b \neq 0$

$\therefore F(x), G(x)$ are well defined $(\because \ x \neq 0)$

Also $F(x), G(x)$ are differentiable in (a,b) $\because f(x)$ is differentiable [given]

$\therefore \exists$ at least one point $c \in (a,b)$ s.t

$$\frac{F'(c)}{G'(c)} = \frac{F(b)-F(a)}{G(b)-G(a)} \Rightarrow \frac{\frac{1}{c}f'(c) - \frac{1}{c^2}f(c)}{-1/c^2} = \frac{f(b)/b - f(a)/a}{1/b - 1/a}$$

$$\Rightarrow \frac{1}{a-b}\begin{vmatrix} a & b \\ f(a) & f(b) \end{vmatrix} = f(c) - cf'(c)$$

Example **If $f : [a, b] \to R$ be continuous in $[a, b]$ and derivable in (a, b), where $0 < a < b$, show that for $c \in (a, b)$**

$$f(b) - f(a) = cf'(c)\,log(b/a)$$

Solution: Applying Cauchy's Mean Value Theorem Two functions f and g are: continuous on $[a, b]$, derivable in (a, b) (iii) $g'(x) \neq 0\ \forall x \in (a, b)$, then there exist at least one point

$c \in (a,b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

hint from the target $f(b) - f(a) = cf'(c)log(b/a)$;

take $g(x) = log\, x$ in $[a, b]$ $0 < a < b$

Applying Cauchy's MVT.

$$= \frac{f'(c)}{(1/c)} = \frac{f(b)-f(a)}{log\,b - log\,a}$$

$$\Rightarrow f(b) - f(a) = c \cdot f'(c) log$$

Example: **If ϕ and ψ be two functions derivable in $[a, b]$ and $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a, b]$, there lies exactly one root of $\psi(x) = 0$.**

Solution: Let α and β be two consecutive roots of $\phi(x) = 0$ in $[a, b]$ and $\alpha < \beta$.

required to prove:

only one root of $\psi(x) = 0$ lies between α and β.

Let if possible, $\psi(x) = 0$ has no root in (α, β).

Consider the function $F(x) = \dfrac{\phi(x)}{\Psi(x)}$

$$F(\alpha) = \frac{\phi(\alpha)}{\psi(\alpha)} = 0 \;\&\; F(\beta) = \frac{\phi(\beta)}{\psi(\beta)} = 0 \qquad (\because\ \phi(\alpha) = 0 = \phi(\beta))$$

∴ $F(x)$ is continuous in [α, β] and using given information,

$$F'(x)=\frac{\phi'(x)\psi(x)-\psi'(x)\phi(x)}{[\psi(x)]^2}$$ exisst in (α, β].

∴ $F(x)$ satisfies all conditions of Rolle's Theorem in [α, β]

∴ $F'(r) = 0$ where $\alpha < r < \beta$

but by given condition $\phi.\psi' - \psi.\phi' > 0$ ∴ $F'(x) \neq 0$ in (α, β) and we get contradiction.

Hence, $\psi(x)$ has at least one root in (α, β).

By similar argument, it can be shown that between two roots of $\psi(x) = 0$, there is a root of $\phi(x) = 0$

Now, we prove that there is exactly one root of $\psi(x) = 0$ between α, β.

If possible, let r and δ two roots of $\psi(x) = 0$ in (α, β), *i.e.*,

$\alpha < r < \delta < \beta$

Between r and δ, there would exist *a* root of $\phi(x) = 0$. This contradicts that roots of α and β are consecutive roots of $\phi(x) = 0$.

Hence, there is only one root of $\phi(x) = 0$ between α and β.

Examples of Monotonic functions, properties, applications

Example:

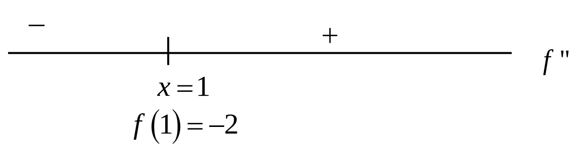
$f(x)=x^3-3x^2$

∵ $f'(x)=3x(x-2)$

- $f'(x)=0$ for $x=0$ and $x=2$

- Now determine a sign chart for f''

∵ $f''=6x-6=6(x-1)$



Now summarize the information from each of the sign chart from f'.

f is ↑ for $x<0$ and $x>2$

f is ↓ for $0<x<2$

f has a relative maximum at $x=0$

f has a relative minimum at $x=2$

From f''

f is ∪ for $x>1$

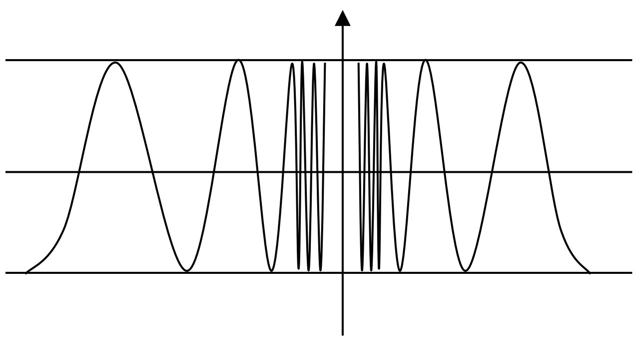
f is ∩ for $x<1$

Graphs of some functions

Category 1: Based on simple observation

(1) $f(x)=\sin\frac{1}{x}$

x	$\frac{1}{x}$	$\sin\frac{1}{x}$
$\frac{4}{\pi}$	$\frac{\pi}{4}$	$\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$
$\frac{1}{\pi}$	π	0
$\frac{2}{\pi}$	$\frac{\pi}{2}$	1



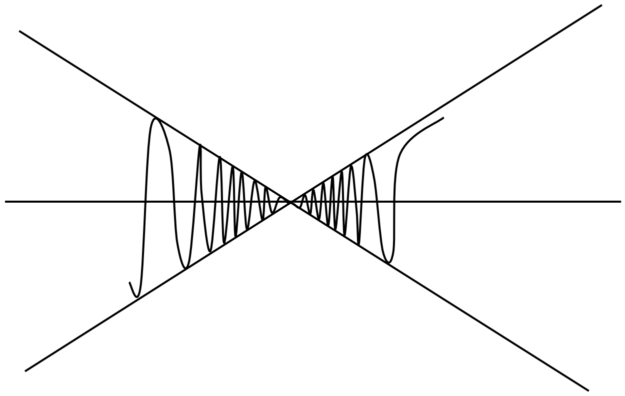
(2) $f(x)=x\sin\frac{1}{x}$

$x(-1)\le f(x)\le x(1)$; As $-1\le\sin x\le 1$

Note: 0 × Finite value = 0

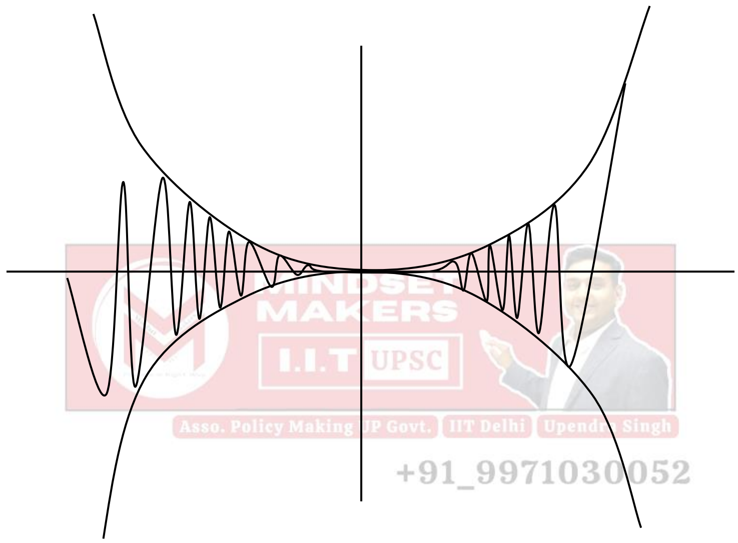
0 × infinite value need not to be zero.

$f'(0)=0\times\sin\infty=0$ × A finite value in between -1 & 1 = 0



(3) $f(x) = x^2 \sin\frac{1}{x}$

$-x^2 \le f(x) \le x^2$



Category 2: Procedure to trace any general function $f(x)$

Step 1: Monotonic nature of given function : (Increase/Decrease)

$\Rightarrow$ For what value of x ; $f(x)$ is decreasing just check the sign. If $f'(x) \le 0$

For what value of x. $f(x)$ increasing. If $f'(x) \ge 0$

Example:

$$f(x) = x^2$$

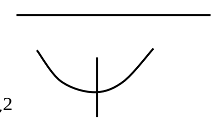
$$f'(x) = 2x$$

For x negative $f'(x)$ will negative so decreasing the $f(x)$

For x positive $f'(x)$ will positive so $f(x)$ increasing

2. Concavity: If $f''(x) \geq 0$ then for those values of x the function $f(x)$ will be concave up.

- If $f''(x) \leq 0$ then concave down



Example: $f(x) = x^2$

$f'(x) = 2x$

$\because$ $f''(x) = 2 > 0$ so always concave up.

Example:

$f(x) = \sin x$ $\quad \because f'(x) = \cos x$ $\qquad f''(x) = -\sin x$

For $0 \leq x \leq \dfrac{\pi}{2}$ for $\pi/2 \leq x \leq \pi$

$f'(x) \geq 0$ $\therefore f(x)$ is increasing $\qquad f'(x) \leq 0$

$f''(x) \leq 0$ $\qquad \therefore f(x)$ is decreases

$\therefore$ $f(x)$ is concave down

$\Rightarrow f''(x) \leq 0$

$f(x)$ concave down

At $x = 0$

$\because f(x)$ $\cos x$ & $\cos x$ is negative in second quadrant & 4th quadrant.

Now at $\pi \leq x \leq \dfrac{3\pi}{2}$, $f'(x) \leq 0$ decrease

$f''(x) \leq 0$, $f(x)$ concave down , $\because$ $\cos x$ is positive in 1st & 3rd quadrant

At $\dfrac{\pi}{2} \leq x \leq x$

$f'(x) \geq 0$ increase $f''(x) \leq 0$, $f(x)$ concave down.

Trace:

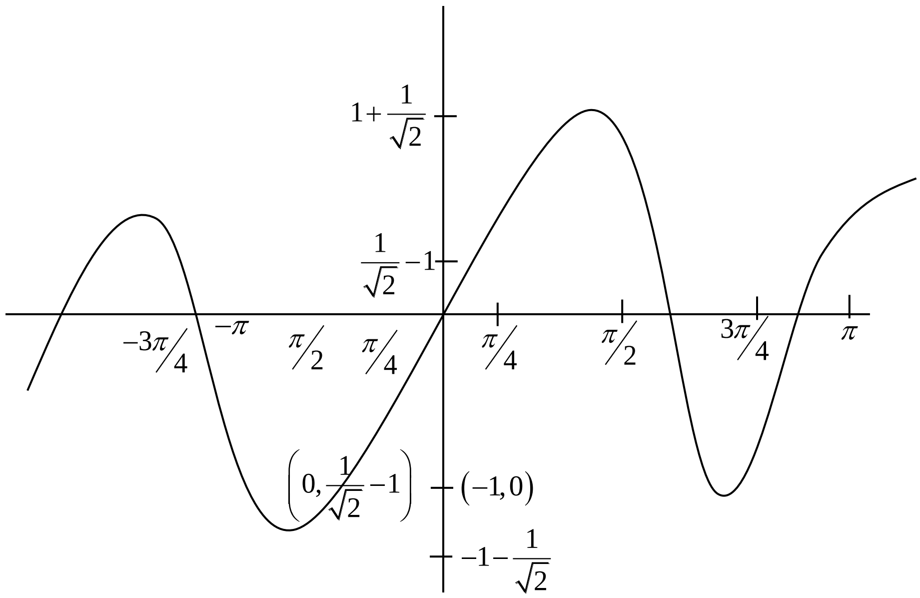
$f(x) = \sin x + \sin 2x$

$f'(x) = \cos x + 2\cos 2x$

$f''(x) = -\sin x + 4\sin x$

x	$2x$	$f(x)$
$0 < x < \pi/4$	$0 < 2x < \pi/2$	+ve
$\pi/4 < x < \pi/2$	$\pi/2 < x < \pi$	

$\pi/2 < x < \pi/4$	$\pi < 2x < \dfrac{3\pi}{2}$	



To trace the above graph we have to observe two things -

1st value of $f(x)$ at some particular value of *x*.

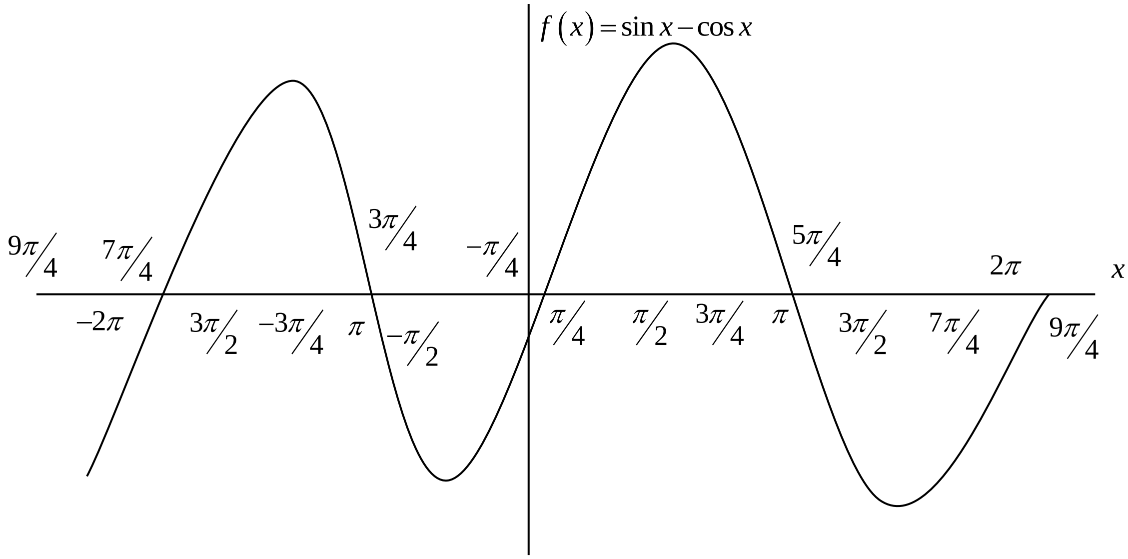
2nd sign of $f'(x)$($\because$ *f'* contains a term $2\times\cos 2x$ which will be dominating over the other term $\cos x$.

So we just observe if < x in first and fourth quadrant then *f'* will be positive mean $f(x)$ is increasing whereas if $2x$ is in 2nd & 3rd quadrant, *f'* is negative so *f* is decreasing.

$\because$ The given function is periodic with the period 2π so we need to observe above behaviour only in 0 to 2π and next it will be repeated.

Draw the graph of

(1) $f(x)=\sin x-\cos x$ (2) $f(x)=\sin x+\frac{1}{2}\sin 2x$


$f(x) = \sin x - \cos x$
x
$9\pi/4$
$7\pi/4$
$3\pi/4$
$-\pi/4$
$5\pi/4$
2π
-2π
$3\pi/2$
$-3\pi/4$
π
$-\pi/2$
$\pi/4$
$\pi/2$
$3\pi/4$
π
$3\pi/2$
$7\pi/4$
$9\pi/4$

Example 1. The equation $x^3+x-1=0$ has exactly one real root. Prove or disprove.

Solution.

Let $f(x)=x^3+x-1$

$\because \ f'(x)=3x^2+1>0$ for all real x

$\Rightarrow f(x)$ is strictly increasing

Also; $\because \ f(0)=0^3+0-1=-1$ Negative

$f(2)=2^3+2-1=9$ Positive

and $f(x)$ is a polynomial so it is continuous

$\therefore$ By Intermediate Value Theorem (IVT); $f(x)$ will take all values between negative to positive

$f(x)=0$ also will be taken for some real x.

$\because f(x)$ is strictly increasing so $f(x)=0$ will hold for exactly one real x [Proved]

Example 2. The equation $x^{19}+x^{13}+2x^7+3x^3+2x-5=0$ has exactly one real root prove or disprove?

$f(x)=x^{19}+x^{13}+2x^7+3x^3+2x-5$

$\because f'(x)=19x^{18}+13x^{12}+14x^6+9x^2+2$

$\because$ power of x in each term are even: $\therefore$ each term $\geq 0 \ \therefore f'(x)>0 \Rightarrow f(x)$ is strictly increasing

$\Rightarrow f(x)$ is one-one ….(1)

$\because f(x)$ is continuous also $f(0)$ is negative $f(x)$ is positive for some values of x

$\therefore$ By IVT; $f(x)$ will also take the value zero for at least one x ….(2)

On combining (1) and (2) we conclude

$f(x)=0$ is possible for exactly one value of x

Example 3.a. Let $P(x)=\left(\frac{5}{13}\right)^x+\left(\frac{12}{15}\right)^x-1$ for all $x\in\mathbf{R}$. Then prove or disprove that $P(x)$ is strictly decreasing for all $x\in\mathbf{R}$

Solution.

$$P(x)=\left(\frac{5}{13}\right)^x+\left(\frac{12}{13}\right)^x-1$$

$$P'(x)=\left(\frac{5}{13}\right)^x\log_e\left(\frac{5}{13}\right)+\left(\frac{12}{13}\right)^x\log_e\left(\frac{12}{13}\right) \qquad \text{....(1)}$$

$\because \frac{d}{dx}\left(a^x\right)=a^x\log_e a$ $\log_e\alpha$ is negative if $0<\alpha<1$

$$\left.\begin{array}{l}\because \left(\frac{5}{13}\right)^x>0; x\in\mathbf{R}\\ \left(\frac{12}{13}\right)^x>0' x\in\mathbf{R}\\ \because \log_e\frac{5}{13}<0,\log_e\frac{12}{13}<0\end{array}\right\} \qquad \text{....(2)}$$

Using (2) in (1), we get

$P'(x)<0$ for all $x\in\mathbf{R}$

$P(x)$ is strictly decreasing for all $x\in\mathbf{R}$[Proved]

Example 3.b. Show that the equation $3^x + 4^x = 5^x$ has exactly one root.

Sol: The given equation is

$$3^x + 4^x = 5^x$$

Dividing both the sides by 5^x, we get

$$\left(\frac{3}{5}\right)^x+\left(\frac{4}{5}\right)^x=1$$

Let $f(x)=\left(\frac{3}{5}\right)^x+\left(\frac{4}{5}\right)^x-1$

Clearly, f(x) is strict, so it will have exactly one real root. (like previous explanation)

Example 4 . Show that $\log(1+x)$ lies between

$$x-\frac{x^2}{2} \text{ and } x-\frac{x^2}{2(1+x)},\ \forall x>0$$

Solution- Consider $f(x)=\log(1+x)-\left(x-\frac{x^2}{2}\right)$

$$\therefore\ f'(x)=\frac{1}{1+x}-(1-x)=\frac{x^2}{1+x}>0,\ \forall x>0$$

Hence, $f(x)$ is an increasing function for all $x > 0$.

Also

$$f(0) = 0$$

Hence for $x > 0,\ f(x) > f(0) = 0$

Thus

$$\log(1+x) > x - \frac{x^2}{2}, \text{ for } x > 0$$

Similarly, by considering the function

$$F(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$$

it can be shown that

$$\log(1+x) < x - \frac{x^2}{2(1+x)}, \ \forall x > 0$$

Example 5. For $n \geq 2$ let $f_n : \mathbf{R} \to \mathbf{R}$ be a function given by $f_n(x) = x^2 \sin x$. Then at $x = 0$,

f_n has (i) local minima if n is odd. prove or disprove

(ii) local maxima if n is even. prove or disprove

Solution.

$f(x) = x^n \sin x$

For local maxima / minima ; We need to observe sign of $f'(x)$

$\because f'(x) = x^n \cos x + n \cdot x^{n-1} \cdot \sin x$ ....(ii)

Now $\because$ we want to check local maxima / minima at $x = 0$

$\because$ So let's observe the sign of $f'(x)$ in

Left nbd of $x = 0$; i.e. $x < 0$

Right nbd of $x = 0$; i.e. $x > 0$

If n is even sign of $f'(x)$	If n is odd
(+ve)(+ve) = _ve	Sign of $f'(x)$
+ve	(-ve) (-ve) =- ve
For $x < 0$	+ve
	$f'(x)$ is –ve to +ve $\therefore x = 0$ is a local minima

$f'(x)$ is positive for $x<0$ and positive for $x>0$ No sign change in $f'(x)$	

Example 6. By using the differentiation, show that $\tan x > x > \sin x, \forall x \in (0, \pi/2)$

Solution

Consider a function $f(x) = \tan x - x$ ….(1)

$\because f'(x) = \sec^2 x - 1 > 0$ for $x \in (0, \pi/2) \Rightarrow f(x)$ is increasing strictly

$\Rightarrow f(x) > f(0)$ for $x > 0 \Rightarrow \tan x - x > \tan 0 - 0$ for $x > 0 \Rightarrow \tan x - x > 0$

$\Rightarrow \tan x > x$ for $x > 0$ in $(0, \pi/2)$

Now let's consider another function $g(x) = x - \sin x$ for $x \in [0, \pi/2]$

$\because g(x) = 1 - \cos x > 0 \Rightarrow g(x)$ is strictly increasing

$\therefore$ By definition $g(x) > g(0)$ for $x > 0$

$x - \sin x > 0 - \sin 0$ for $x > 0$

$x > \sin x$ for $x > 0$ in $(0, \pi/2)$ ….(3)

$\therefore$ On combining (2) and (3) we get

$\tan x > x > \sin x$ for $x \in [0, \pi/2]$

Example 7. Prove or disprove in $(0, \pi/2)$

(i) $\cos x < \cos(\sin x)$

(ii) $\frac{1-x^2}{2} < \log(2+x)$

Solution.

We have used some fundamentals too

Let's consider a function

$f(x) = x - \sin x$

$\therefore f'(x) = 1 - \cos x > 0$ for $x \in (0, \pi/2)$

$\Rightarrow f(x)$ is strictly increasing in the interval $[0, \pi/2]$

$\therefore$ By definition $f(x) > f(0)$ for $x > 0$

$x - \sin x > 0 \cdot \sin 0$ in $(0, \pi/2)$

$x - \sin x > 0$ in $(0, \pi/2)$

$x > \sin x$ in $(0, \pi/2)$

$\therefore \cos(x) < \cos(\sin x)$ Hence Proved

Inequality reversed: On increasing x; $\cos x$ decreases in $(0, \pi/2)$

$$\begin{array}{ll} \cos\alpha\ , & \cos\beta \\ \downarrow & \downarrow \\ \alpha = x & \beta = \sin x \end{array}$$

If get some relation between α and β then it will be easy to get relation because $\cos\alpha$ and $\cos\beta$

$$\left.\begin{array}{l} \alpha < \beta ? \\ \alpha > \beta ? \\ \alpha = \beta ? \end{array}\right\} \Rightarrow \alpha \cdot \beta$$

Example 8. Prove or disprove in $(0, \pi/2)$

(i) $\cos x < \cos(\sin x)$

(ii) $\dfrac{1-x^2}{2} < \log(2+x)$

Solution (ii):

If we consider $f(x) = \log(1+x)$ then $f'(x) = \dfrac{1}{1+2x}$ which is not in inequality $\therefore$ so of no use

Let's try other way

Can we show : $\log(2+x) - \dfrac{(1-x^2)}{2} > 0$? So let's try

Consider $f(x) = \log(2+x) - \dfrac{(1-x^2)}{2}$ in $[0, x]$

$\because f'(x) = \dfrac{1}{2+x} + x > 0 \ \because x \in [0, \pi/2] \Rightarrow f(x)$ is strictly increasing in $(0, \pi/2)$

$\therefore$ By definition $f(x) > f(0)$ for $x > 0$

$$\log(2+x) - \frac{(1-x^2)}{2} > \log(2+0) - \frac{1}{2}$$

$$\Rightarrow \log(2+x) - \frac{(1-x^2)}{2} > 0.69 - 0.5 > 0$$

Example 9. Let $f:(-1,1) \to \mathbf{R}$ be the function defined by $f(x) = x^2 \cdot e^{\frac{1}{(1-x^2)}}$

Then show that

(i) f is decreasing in $(-1,0)$

(ii) f is increasing $(0,1)$

(iii) $f(x) = 1$ has two solutions in $(-1,1)$

Solution.

$$f(x) = x^2 \cdot e^{\frac{1}{(1-x^2)}}$$

$$\therefore f'(x) = x^2 \cdot e^{\frac{1}{(1-x^2)}} + \frac{2x}{(1-x^2)^2} + 2x \cdot e^{\frac{1}{(1-x^2)}}$$

$$f'(x) = x^2 \cdot e^{\frac{1}{(1-x^2)}} \times \frac{2x}{(1-x^2)^2} + 2x \cdot e^{\frac{1}{(1-x^2)}}$$

$$f'(x) = 2xe^{\frac{1}{(1-x^2)}} \left\{ \frac{x^2}{(1-x^2)^2} + 1 \right\}$$

$\because f'(x) < 0$ for $x < 0$

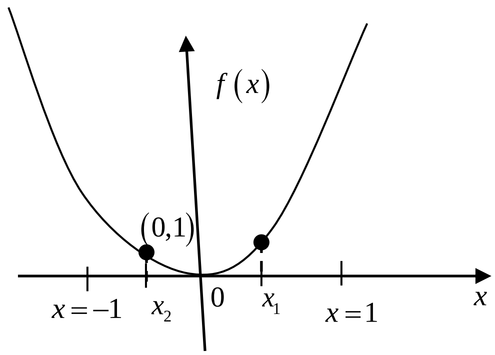
(i)

$\Rightarrow f(x)$ is decreasing in $(-1,0)$

$\because f'(x) > 0$ for $x > 0$

(ii) $\Rightarrow f(x)$ is increasing in $(0,1)$

(iii)



$\because f(0)=0$

$f(1)\to\infty$

$f(-1)\to\infty$

Clearly x_2, x_1 are two solution of $f(x)=1 \because f(x)$ is continuous.

Combined problems through considering a function

Example-1 Show that between any two roots of $e^x \cos x = 1$, there exists at least one root $e^x \sin x - 1 = 0$.

2-If ϕ and ψ be two functions derivable in $[a,b]$ and $\phi(x)\psi'(x)-\psi(x)\phi'(x)>0$ for any x in this interval, then show that between two consecutive roots of $\phi(x)=0$ in $[a,b]$, there lies exactly one root of $\psi(x)=0$.

3-If $f:[a,b]\to R$ be continuous in $[a,b]$ and derivable in (a,b), where $0<a<b$, show that for $c\in(a,b)$ $f(b)-f(a)=cf'(c)\log(b/a)$.

Hints

Let $f(x)=e^x\cos x+1=0$, $g(x)=e^x \sin x+1=0$

Method 1:- Let α and γ one two real roots of $f(x)=0$

$\Rightarrow$ $f(\alpha)=0$, $f(\beta)=0$

Also $f(x)$ is continuous in $[\alpha,\beta]$ and differentiable in (α,β)

$\therefore$ By Rolle's theorem, there exists at least one real $c\in(\alpha,\beta)$ such that $f'(c)=0$

$\Rightarrow \quad -e^{c}.\sin c + e^{c}\cos c = 0$. But it doesn't fulfil the demand of question. So let's try another way.

Method 2:- Let a and b are two roots of $e^{x}\cos x + 1 = 0$. So

$e^{a}\cos a + 1 = 0 \Rightarrow \cos a + e^{-a=0} = 0 and e^{b}\cos b + 1 = 0 \Rightarrow \cos b + e^{-b} = 0$ ...(i)

Let's consider a function f(x) which may give the desired function $e^{x}\sin x + 1$ after differentiation.

$f(x) = -e^{-x} - \cos x$

$f(a) = -e^{-a} - \cos a,\ f(b) = e^{-b} - \cos b.$ both are equal (using i)

Clearly this function satisfies all conditions of Rolle's theorem.

$\therefore\ f'(x) = e^{-x} + \sin x$

$\exists c \in (a,b) s.t. f'(c) = 0 \Rightarrow e^{-c} + \sin c = 0 \Rightarrow 1 + e^{c}\sin c = 0$

$e^{x}\sin x + 1 = 0$

Therefore we have proved that between two roots of $e^{x}\cos x + 1 = 0$ there is a root of $e^{x}\sin x + 1 = 0$.

- The following explanation, we may study later on.

Method 3):use of differential equation and linear algebra)- $f(x)$ and $g(x)$ are two linearly independent functions, so by sturm's separation theorem ($f(x), g(x)$ are differentiable linearly independent so their Wronskian is non zero by the definition of linear dependence)

$\therefore$ Between two consecutive roots of $f(x)$, there exists a exactly one roots of $f(x)$.

Proof of sturm's seperation theorem:

Let $W(f,g)(x) = W(x)$ without loss of generality, we can suppose that $W(x) < 0$, $W(x) \neq 0$.

Let x_0 and x_1 are two roots of $f(x)$ i.e. $f(x_0) = 0$, $f(x_1) = 0$

$\therefore$ From $W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$

$\therefore \quad W(x_0) = \begin{vmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{vmatrix}$

$= \begin{vmatrix} 0 & g(x_0) \\ f'(x_0) & g'(x_0) \end{vmatrix}$

$\therefore \quad W(x_0) = -f'(x_0).g(x_0)$ and $W(x_1) = -f'(x_1).g(x_1)$

$\because \quad x = x_0$ and $x = x_1$ are two consecutive zeros of $f(x)$ it caused $f'(x_1) < 0$. Thus to keep $W(x) < 0$ we must have $g(x_1) < 0$.

We see this by observing that $f'(x) > 0\ \forall x \in [x_0, x_1]$ then $f(x)$ would be increasing (away from the x-axis) which would never lead to zero at $x = x_1$, so for a zero to occur at $x = x_1$ at most $f'(x_1) \leq 0$ and it turns out from then Wronskian that $f'(x_1) \leq 0$ so somewhere in the interval (x_0, x_1) the sign of $g(x)$ changed. By the intermediate value theorem them exists $x^* \in (x_0, x_1)$ such that $g(x^*) = 0$.

On the other hand, there can be only one zero in (x_0, x_1) because otherwise $g(x)$ would have two zeros and there would be no zeros of $f(x)$ in between, and it was just proved that this is impossible.

2-Give ϕ and ψ are differentiable in $[a,b]$.

Also given $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$

$$\Rightarrow \quad \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix} > 0$$

$\Rightarrow$ Wronskian of ϕ and $\psi \neq 0$ so $\phi(x)$ and $\psi(x)$ are linearly independent function i.e. $\phi(x) \neq k.\psi(x)$ for constant k.

Let α and β are two roots of $\phi(x)$ in $[a,b]$ i.e. $\phi(\alpha) = 0$, $\phi(\beta)$ such that $\alpha < \beta$

.

Inequality type problems

Example- Show that $\dfrac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$

Solution- consider a function $f(x) = \log(1+x)$ over the interval [0,x]. So why we have choosen such function ! interesting to notice . See basically we want to use different outcomes learnt above in MVTs and at the same time the inequality asked in the question is also directing us to consider such function.

Evidently this function is continuous in [0,x] and differentiable in (0,x). So on applying Lagrange's MVT on this function we get

$f(x) = \log(1+x)$

$\exists c \in (0, x) s.t.$

$[f(x) - f(0)]/(x-0) = f'(c)$

$\Rightarrow \log(1+x) - \log 1/x = 1/(1+c)$

$\Rightarrow \log(1+x) = x/(1+c)$

$\because 0 < c < x \Rightarrow x/(1+x) < x/(1+c) < x$

$\Rightarrow 0 < x/(\log(1+x)) - 1 < x$

$\Rightarrow 1 < x/(\log(1+x)) < (1+x)$

$\Rightarrow \log(1+x) < x, x/(1+x) < \log(1+x)$

Question to Try

Q. 1. Show that

$$x^3-6x^2+15x+3>0,\quad \forall x>0$$

Q.2. Show that

(i) $\dfrac{x}{1+x}<\log(1+x)<x,\ \forall x>0$

(ii) $\dfrac{x}{1+x^2}<\tan^{-1}x<x,\ \forall x>0$

Q. 3. Show that

(i) $\tan x>x,\ 0<x<\pi/2$

(ii) $\dfrac{2}{\pi}\le\dfrac{\sin x}{x}<1,\ 0<|x|\le\pi/2$

Q. 4. Show that

$$2x<\log\frac{1+x}{1-x}<2x\left(1+\frac{x^2}{3(1-x^2)}\right),\ 0<x<1$$

Q.5. Show that

$$2/(2x+1)<\log(1+1/x)<1/\sqrt{x(x+1)},\ \forall x>0$$

Examples of Taylor's theorem

Example 1- Find Taylor's Series expansion for the function $f(x)=\log(1+x)$, $-1<x<\infty$ about $x=2$ with Lagrange's form of remainder after 3 terms.

Solution.

$\because$ Expansion about $x=2\ \Rightarrow$ power of $(x-2)$

$\therefore a=2$ and $h=x-2$

$$\therefore\quad f(x)=\underbrace{f(2)}_{\text{Ist Term}}+\underbrace{(x-2)\cdot f'(2)}_{\text{IInd Term}}+\underbrace{\frac{(x-2)^2}{2!}f''(2)}_{\text{IIIrd Term}}+\underbrace{\frac{(x-2)^3}{3!}f'''(2+\theta(x-2))}_{\text{Remainder term after 3 terms where } 0<\theta<1}$$

$\because f(x)=\log(1+x)\ \therefore f(2)=\log(1+2)=\log 3$

$f'(x)=\dfrac{1}{1+x}\quad\therefore f'(2)=\dfrac{1}{1+2}$

$f''(x)=-\dfrac{1}{(1+x)^2}\ \therefore f''(2)=-\dfrac{1}{9}$

$f'''(x)=+\dfrac{2}{(1+x)^3}\ \therefore f''(2)=\dfrac{2}{(1+2)^3}$

Use all values in (1).

Example. If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$. Find θ, when $h = 1$ and $f(x) = (1-x)^{5/2}$.

Solution. Given : $f(x) = (1-x)^{5/2} \Rightarrow f(h) = (1-h)^{5/2}$

Now,

$$f(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$$

Also,

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta h) = \frac{15}{4}(1-\theta h)^{1/2}$$

$$\therefore f(h) = f(0) + hf'(0) + \frac{h^2}{h!} f''(\theta h), 0 < \theta < 1$$

$$\Rightarrow \qquad (1-h)^{5/2} = 1 + h\left(\frac{-5}{2}\right) + \frac{h^2}{2!}\frac{15}{4}(1-\theta h)^{1/2}$$

When $h = 1$,

$$0 = 1 + \frac{-5}{2} + \frac{1}{2} \times \frac{15}{4}(1-\theta)^{1/2}$$

$$\Rightarrow \qquad 0 = 1 - \frac{5}{2} + \frac{15}{8} - (1-\theta)^{1/2}$$

$$\Rightarrow \qquad 0 = \frac{3}{8} - (1-\theta)^{1/2}$$

$$\Rightarrow \qquad (1-\theta)^{1/2} = 3/8$$

$$\Rightarrow \qquad (1-\theta) = \frac{9}{64}$$

$$\Rightarrow \qquad \theta = \frac{55}{64}$$

Examples of Maxima Minima

Critical Points and Inflection Points

Example:

$$f(x) = \begin{cases} x^2 + 2x, & x < -2 \\ -x^2 - 2x, & -2 \le x < 0 \\ x^2 - 2x, & 0 \le x < 2 \\ 2x - 4, & x \ge 2 \end{cases}$$

Before discussing critical points/inflection points; we need to check continuity of $f(x)$ at Junction points $x = -2$, $x = 0$, $x = 2$

At $x=-2$; $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(-2)$ it is ok $f(x)$ is continuous.
Similarly we can see it - at $x=0$, at $x=2$.
Now

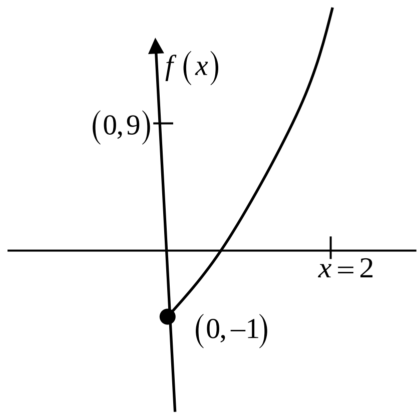
$$f'(x) = \begin{cases} 2x+2\ ; & x<-2 \\ -2x-2\ ; & -2<x<0 \\ 2x-2\ ; & 0<x<2 \\ 2\ ; & x>2 \end{cases}$$

<u>Note</u>:
Check for $x=-2$;
$\lim\limits_{x\to -2^-} f(x) = -2$
$\lim\limits_{x\to -2^+} f(x) = 2$, f' is discontinuous at $x=-2$
(ii) Similarly we can check for $x=+2$ get $f'(x)$ is continuous.
(iii) Also at $x=0$; $f'(x)$ is continuous

<u>Now Critical Points</u>:
[**Stationary Points & Non-differentiable points**]
$2x+2=0 \Rightarrow x=-1$ But outside the domain
$\therefore$ Reject $x=-1$
$-2x-2=0 \Rightarrow x=-1$
$2x-2=0 \Rightarrow x=1$
Critical points are $x=-1,1,-2$
Point of maxima / minima where $derivative f(x)$ changes its sign.

$$f''(x) = \begin{cases} 2\ ; & x<-2 \\ -2\ ; & -2<x<0 \\ 2\ ; & 0<x<2 \\ 0\ ; & x>2 \end{cases}$$

Inflection points $x=0$ is an inflection point ($\because$ f'' changes sign) but $x=-2$ is not an inflection point (at $x=-2$ tangent non-existence or not differentiable)



Example 1. Examine the function $\sin x+\cos x$ for extreme values.
Solution- Let

$$f(x)=\sin x+\cos x$$
$$f'(x)=\cos x-\sin x$$
$$f''(x)=-\sin x-\cos x$$

$f'(x)=0$ when $\tan x=1$, so that

$$x=n\pi+\frac{\pi}{4}$$

where n is zero or any integer.

$$f''\left(n\pi+\frac{1}{4}\pi\right)=-\left\{\sin\left(n\pi+\frac{1}{4}\pi\right)+\cos\left(n\pi+\frac{1}{4}\pi\right)\right\}$$
$$=(-1)^{n+1}\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)=(-1)^{n+1}\sqrt{2}$$
$$\left[\because\ \sin(n\pi+\alpha)=(-1)^n\sin\alpha \text{ and } \cos(n\pi+\alpha)=(-1)^n\cos\alpha\right]$$

Also $f\left(n\pi+\frac{1}{4}\pi\right)=\sin\left(n\pi+\frac{1}{4}\pi\right)+\cos\left(n\pi+\frac{1}{4}\pi\right)=(-1)^n\sqrt{2}$

When n is zero or an even integer, $f''(n\pi+\pi/4)$ is negative and therefore $x=n\pi+\frac{1}{4}\pi$ makes $f(x)$ a maxima with the maximum value $\sqrt{2}$.

When n is an odd integer, $f''\left(n\pi+\frac{1}{4}\pi\right)$ is positive and therefore $x=n\pi+\frac{1}{4}\pi$ makes $f(x)$ a minima with the minimum value $\sqrt{2}$.

Example. Find the difference between the maximum and the minimum of the function
$\left(a-\frac{1}{a}-x\right)\left(4-3x^2\right)$ where α is a constant and greater than zero.

Solution. Let $f(x) = \left(a - \frac{1}{a} - x\right)\left(4 - 3x^2\right)$ ...(1)

where a is a constant and greater than 0.

$\Rightarrow f'(x) = \left(a - \frac{1}{a} - x\right)(-6x) + (-1)\left(4 - 3x^2\right)$

$\Rightarrow f'(x) = -6ax + \frac{6x}{a} + 6x^2 - 4 + 3x^2$

$\Rightarrow f'(x) = 9ax^2 - 6\left(a - \frac{1}{a}\right)x - 4$ …(2)

For maxima or minima, $f' = 0$

$$\Rightarrow 9x^2 - 6\left(a - \frac{1}{a}\right)x - 4 = 0$$

$$\Rightarrow x = \frac{6\left(a - \frac{1}{a}\right) \pm \sqrt{36\left(a - \frac{1}{a}\right)^2 + 36 \times 4}}{2+9}$$

$$\Rightarrow x = \frac{\left(a - \frac{1}{a}\right) \pm \sqrt{\left(a - \frac{1}{a}\right)^2 + 4}}{3}$$

$$\Rightarrow x = \frac{\left(a - \frac{1}{a}\right) \pm \left(a + \frac{1}{a}\right)}{3}$$

$$\Rightarrow x = \frac{2a}{3}, \frac{-2}{3a}$$

From (2), $f''(x) = 18x - 6\left(a - \frac{1}{a}\right)$ …(3)

For $x = \frac{2a}{3}$,

$$f''\left(\frac{2a}{3}\right) = 18 \times \frac{2a}{3} - 6\left(a - \frac{1}{a}\right)$$

$$\Rightarrow \quad f''\left(\frac{2a}{3}\right) = 6a + \frac{6}{a} > 0 \because a > 0$$

$\Rightarrow$ f is minimum at $x = \frac{2a}{3}$.

$\therefore f_{min} = f\left(\frac{2a}{3}\right) = \left(a - \frac{1}{a} - \frac{2a}{3}\right)\left(4 - 3 \times \frac{4a^2}{9}\right)$

$$\Rightarrow \qquad f\,min = \left(\frac{a}{3} - \frac{1}{a}\right)\left(4 - \frac{4a^2}{3}\right)$$

$$\Rightarrow \qquad f\,min = \frac{4a}{3} - \frac{4}{a} - \frac{4a^3}{9} + \frac{4a}{3}$$

$$\Rightarrow \qquad f\,min = \frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9} \qquad \text{...(4)}$$

For $x = \frac{-2}{3a}$,

$$f''\left(-\frac{2}{3a}\right) = 18\left(\frac{-2}{3a}\right) - 6\left(a - \frac{1}{a}\right)$$

$$f''\left(-\frac{2}{3a}\right) = \frac{-12}{a} - 6a + \frac{6}{a} = \frac{-6}{a} - 6a < 0$$

$$\Rightarrow \qquad f \text{ is maximum at } x = \frac{-2}{3a}$$

$$\therefore \quad f\,max = f\left(-\frac{2}{3a}\right) = 4a - \frac{8}{3a} + \frac{4}{9a^3} \qquad \text{…(5)}$$

Subtracting (5) from (4) we get,

$$\left(4a - \frac{8}{3a} + \frac{4}{9a^3}\right) - \left(\frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9}\right) = \frac{4a}{3} + \frac{4}{3a} + \frac{4}{a}\left(\frac{1}{a^3} + a^3\right)$$

$$= \frac{4}{3}\left(a + \frac{1}{a}\right) + \frac{4}{9}\left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2} + 1\right)$$

$$= \left(a + \frac{1}{a}\right)\left[\frac{4}{3} + \frac{4}{9}\left(a^2 + \frac{1}{a^2} + 1\right)\right]$$

Examples on basic maxima minima but very important category of questions for UPSC

Example 1. Find the shortest distance from the point $(1,0)$ to the problem $y^2 = 4x$

Example 2. Show that the maximum rectangle inscribed n a circle is square.

Example 3. A conical tent is of given capacity. For the least amount of canvas required for it find out ratio of its height to the radius of its bare.

Solution 1. Let's take some arbitrary point on given parabola as $(at^2, 2at)$ and finding distance between this arbitrary point and given point (1,0)

$$p = \sqrt{\left(t^2 - 1\right)^2 + \left(2t - 0\right)^2} \Rightarrow p = \sqrt{t^4 - 2t^2 + 1 + 4t^2} \Rightarrow p = \sqrt{t^4 + 2t^2 + 1} = \left(t^2 + 1\right)^2$$

$$p = \left(t^2 + 1\right)^2 - f\left(t\right)$$

For minimum $p; dp/dt$ must be zero. $\Rightarrow (t^2+1)\times 2t = 0 \Rightarrow t^2+1=0 \Rightarrow t=\pm i$ [Not real]

$\therefore$ Minimum value of $p=(0+1)^2-1$ as $t=0$ [Possible]

Solution 2. $\because r^2 = x^2+y^2 \Rightarrow y^2 = r^2-x^2 \Rightarrow y=\sqrt{r^2-x^2}$ ….(1)

$\because$ Area of rectangle = $2x\times 2y \Rightarrow A = 2x\times 2\sqrt{r^2-x^2}$ $\because r$ constant

For maximum Area ; $\frac{dA}{dx}=0 \Rightarrow r=\sqrt{2}x$, $\therefore y = x$ from (1) $\Rightarrow$ ABCD is a square.

Solution 3. $\because$ $s^2 = \pi^2 r^2 (r^2+h^2)$

$s^2 = \pi^2 r^2 + \pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4}\right)$ using h in V

$\Rightarrow$ $s^2 = \pi^2 r^4 + \frac{9V^2}{r^2}$

Let $s^2 = z$

$\Rightarrow$ $z = \pi^2 r^4 + \frac{9V^2}{r^2}$

$\Rightarrow$ $\frac{dz}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3}$

$\Rightarrow$ $\frac{d^2z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4}$

Clearly z is maximum or minimum according as s is maximum or minimum.

$\therefore$ Stationary point of z are given by $\frac{dz}{dr}=0 \Rightarrow 4\pi^2 r^3 - \frac{18V^2}{r^3}=0$

$\Rightarrow$ $2\pi^2 r^6 = \pi^2 r^4 h^2$

$\Rightarrow$ $h=\sqrt{2r}$

$\because$ $\frac{d^2z}{dr^2} = 12\pi^2 r^2 + \frac{59V^2}{r^4} > 0$ for all V and r .

$\therefore$ Required ration $\frac{h}{r} = \sqrt{2/1}$

<u>Assignment</u>

1. Examine the validity of the hypothesis and the conclusion of Rolle's Theorem.
   (i). $f(x) = x^3-4x$ on $[-2,2]$.
   (ii). $f(x) = (x-a)^m (x-b)^n$, where m and n are positive integers on $[a,b]$,
   (iii). $f(x) = 1-(x-1)^{2/3}$ on $[0,2]$,
   (iv). $f(x)=|x|$ on $[-1,1]$,

(v). $f(x) = 1 - |x-1|$ on $[0, 2]$

2. Examine the validity of the hypothesis and the conclusion of Lagrange's Mean Value Theorem:

(i). $f(x) = |x|$ on $[-1, 1]$

(ii). $f(x) = \log x$ on $\left[\frac{1}{2}, 2\right]$

(iii). $f(x) = x(x-1)(x-2)$ on $\left[0, \frac{1}{2}\right]$,

(iv). $f(x) = x^{1/3}$ on $[-1, 1]$,

(v). $f(x) = 2x^2 - 7x + 10$ on $[2, 5]$.

<u>Assignment -</u>

1. Expand, if possible, $\sin x$ in ascending powers of *x*.

2. Assuming the validity of expansion, show that

(i). $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + ...$

(ii). $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + ...$

(iii). $\tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \frac{x - \pi/4}{1 + \pi^2/16} - \frac{\pi(x - \pi/4)^2}{4(1 + \pi^2/16)^2} + ...$

(iv). $\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + ...\right)$

(v). $f(x) = f(a) + 2\left[\frac{x-a}{2} f'\left(\frac{x+a}{2}\right) + \frac{(x-a)^3}{8\cdot(3)!} f'''\left(\frac{x+a}{2}\right) + \frac{(x-a)^5}{32(5)!} f^{v}\left(\frac{x+a}{2}\right) + ...\right]$

3. Use Taylor's theorem to show that

(i). $\cos x \geq 1 - \frac{x^2}{2}$, for all real *x*.

(ii). $x - \frac{x^3}{6} < \sin x < x$, for $x > 0$

(iii). $x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \forall x > 0$

(iv). $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2}e^x, \quad x > 0$

4. If $0 < x \le 2$, then prove that

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots.$$

Assignment

1) Show that the maximum value of the function $(x-1)(x-2)(x-3)$ is $\frac{2\sqrt{3}}{9}$ at $x = 2 - \frac{1}{\sqrt{3}}$.

2) Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maxima at $x = 1$ and a minima at $x = 3$ and neither at $x = 0$.

3) Find the maximum and the minimum as well as the greatest and the least value of $x^3 - 12x^2 + 45x$ in the interval $[0,7]$.

   [***Hint***: For greatest and least values, find $f(3), f(5), f(0)$ and $f(7)$]

4) Find the maximum or minimum of

   $$\frac{x^4}{(x-1)(x-3)^3}$$

5) Show that the maximum value of $(1/x)^x$ is $e^{1/e}$.

6) Show that the maximum value of $(\log x)/x$ in $0 < x < \infty$ is $1/e$.

7) Show that $\sin x(1+\cos x)$ is maximum at $x = \pi/3$.

8) If $(x-a)^{2n}(x-b)^{2m+1}$, where *m* and *n* are positive integers, is the derivatives of a function *f*, then show that $x = b$ gives a minimum but $x = a$ gives neither a maximum nor a minimum.

9) Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

10) Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $(4/27)\pi h^3 \tan^2 \alpha$.

11) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius *a* is $2a/\sqrt{3}$.

ANSWERS

3. Max 54, min 50, greatest 70, least 0

4. Max at $\frac{6}{5}$, min at $x = 0$

PREVIOUS YEARS QUESTIONS UPSC CSE/IFoS
(Years 2010 onwards)

ROLL'S, LAGRANGE'S, CAUCHY'S AND TAYLOR'S THEOREM

Q1 Edited(d) Show that between any two roots of $e^x \cos x = 1$, there exists at least one root $e^x \sin x - 1 = 0$.

[UPSC CSE 2021]

Q2 Edited(a) Does $f(x) = x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$ satisfy the conditions of the mean value theorem? If yes, then justify your answer and find $c \in (a,b)$ such that

$f'(c) = \frac{f(b) - f(a)}{b-a}\left(a = \frac{1}{2}, b = 3\right)$. **[IFoS 2021]**

IFoS 2020 Q1. Find the Taylor's series expansion for the function
$f(x) = \log(1+x), -1 < x < \infty$, about $x = 2$ with Lagrange's form of remainder after 3-terms.

IFoS 2019 Q2. Justify by using Rolle's theorem or mean value theorem that there is no number k for which the equation $x^3 - 3x + k = 0$ has two distinct solutions in the interval $[-1,1]$.

IFoS 2018Q3. If $f : [a,b] \to R$ be continuous in $[a,b]$ and derivable in (a,b), where $0 < a < b$, show that for $c \in (a,b)$ $f(b) - f(a) = cf'(c)\log(b/a)$.

IFoS 2018 Q4. If ϕ and ψ be two functions derivable in $[a,b]$ and $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a,b]$, there lies exactly one root of $\psi(x) = 0$.

IFoS 2017 Q5. Using the Mean Value Theorem, show that
(i) $f(x)$ is constant in $[a,b]$, if $f'(x) = 0$ in $[a,b]$
(ii) $f(x)$ is a decreasing function in (a,b), if $f'(x)$ exists and is < 0 everywhere in (a,b).

IFoS 2016 Q6. Show that $\frac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$

IFoS 2016 Q7. Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on $[2,3]$, where the tangent is parallel to the chord joining the end points of the curve.

CSE 2014 Q8. Prove that between two real roots of $e^x \cos x + 1 = 0$, a real root of $e^x \sin x + 1 = 0$ lies.

IFoS 2013Q9. Find C of the Mean value theorem, if $f(x) = x(x-1)(x-2), a = 0, b = \frac{1}{2}$ and C has usual meaning.

IFoS 2013 Q10. Prove that if $a_0, a_1, a_2, ..., a_a$ are the real numbers such that $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + ... + \frac{a_{n-1}}{2} + a_n = 0$ then there exists at least one real number x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n = 0$.

CSE 2011 Q11. Let f be a function on $\mathbf{R}$ such that **f**$f(0) = -3$ and $f'(x) \le 5$ for all values of x in $\mathbf{R}$. How large can $f(2)$ possibly be?

CSE 2010 Q12. A twice-differentiable function $f(x)$ is such that $f(a)=0=f(b)$ and $f(c)>0$ for $a<c<b$. Prove that there is at least one point $\xi, a<\xi<b$, for which $f''(\xi)<0$.

IFoS 2010Q13. Prove that between any two real roots of $e^x \sin x=1$, there is at least one real root of $e^x \cos x+1=0$.

CSE 2009Q14. Suppose that f'' is continuous on $[1,2]$ and that f has three zeros in the interval $(1,2)$. Show that f'' has at least one zero in the interval $(1,2)$.

IFoS 2009 Q15. If $f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h) \;\; 0<\theta<1$

Find θ, when $h=1$ and $f(x)=(1-x)^{5/2}$.

2. MAXIMA- MINIMA

CSE 2020 Q1. Consider the function $f(x)=\int_0^x (t^2-5t+4)(t^2-5t+6)\,dt$

(i) Find the critical points of the function $f(x)$

(ii) Find the points at which local minimum occurs

(iii) Find the points at which local maximum occurs

(iv) Find the number of zeros of the function $f(x)$ in $[0,5]$

CSE 2019 Q2. Find the maximum and the minimum value of the function $f(x)=2x^3-9x^2+12x+6$ on the interval $[2,3]$.

CSE 2018 Q3. Find the shortest distance from the point $(1,0)$ to the parabola $y^2=4x$.

IFoS 2018 Q4. Show that the maximum rectangle inscribed in a circle is a square.

CSE 2015Q5. A conical tent is of given capacity. For the least amount of Canvas required, for it, find the ratio of its height to the radius of its base.

CSE 2014 Q6. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

IFoS 2009Q7. Find the difference between the maximum and the minimum of the function $\left(a-\frac{1}{a}-x\right)(4-3x^2)$ where a is a constant and greater than zero.

IFoS 2008 Q8. A wire of length b is cut into two parts which are bent in the form of a square and a circle respectively. Find the minimum value of the sum of the areas so formed.

Solutions(hints)

1. Model to solve PYQ1 Edited, PYQ8 PYQ13, PYQ4

Let $f(x)=e^x\cos x+1=0$, $g(x)=e^x \sin x+1=0$

Method 1:- Let α and γ one two real roots of $f(x)=0$

$\Rightarrow \quad f(\alpha)=0,\ f(\beta)=0$

Also $f(x)$ is continuous in $[\alpha,\beta]$ and differentiable in (α,β)

$\therefore$ By Rolle's theorem, there exists at least one real $c \in (\alpha, \beta)$ such that $f'(c) = 0$

$\Rightarrow$ $-e^c.\sin c + e^c \cos c = 0$. But it doesn't fulfil the demand of question. So let's try another way.

Method 2:- Let a and b are two roots of $e^x \cos x + 1 = 0$. So

$e^a \cos a + 1 = 0 \Rightarrow \cos a + e^{-a=0} = 0 and e^b \cos b + 1 = 0 \Rightarrow \cos b + e^{-b} = 0$ …(i)

Let's consider a function f(x) which may give the desired function $e^x \sin x + 1$ after differentiation.

$f(x) = -e^{-x} - \cos x$

$f(a) = -e^{-a} - \cos a,\ f(b) = e^{-b} - \cos b.$ both are equal (using i)

Clearly this function satisfies all conditions of Rolle's theorem.

$\therefore\ f'(x) = e^{-x} + \sin x$

$\exists c \in (a,b) s.t. f'(c) = 0 \Rightarrow e^{-c} + \sin c = 0 \Rightarrow 1 + e^c \sin c = 0$

$e^x \sin x + 1 = 0$

Therefore we have proved that between two roots of $e^x \cos x + 1 = 0$ there is a root of $e^x \sin x + 1 = 0$.

Hint edited pyq2

Yes f(x) satisfies Lagrange's MVT. So on finding by solving the equation

$f'(c) = (f(3) - f(1/2))/(3 - 1/2)$

Ans1. Find Taylor's Series expansion for the function $f(x) = \log(1+x)$, $-1 < x < \infty$ about $x = 2$ with Lagrange's form of remainder after 3 terms.

Solution.

$\because$ Expansion around $x = 2$ $\Rightarrow$ power of $(x-2)$

$\therefore a = 2$ and $h = x - 2$

$$\therefore\ f(x) = \underbrace{f(2)}_{\text{Ist Term}} + \underbrace{(x-2)\cdot f'(2)}_{\text{IInd Term}} + \underbrace{\frac{(x-2)^2}{2!} f''(2)}_{\text{IIIrd Term}} + \underbrace{\frac{(x-2)^3}{3!} f'''(2 + \theta(x-2))}_{\text{Remainder term after 3 terms where } 0<\theta<1}$$

$\because f(x) = \log(1+x)\ \therefore f(2) = \log(1+2) = \log 3$

$f'(x) = \frac{1}{1+x}\ \therefore f'(2) = \frac{1}{1+2}$

$f''(x) = -\frac{1}{(1+x)^2}\ \therefore f''(2) = -\frac{1}{9}$

$f'''(x) = +\frac{2}{(1+x)^3}\ \therefore f''(2) = \frac{2}{(1+2)^3}$

Use all values in (1) to get required answer.

Ans2. Justify by using Rolle's theorem or mean value theorem that there is no number k for which the equation $x^3 - 3x + k = 0$ has two distinct solution in the interval $[-1,1]$

Sol:- $\phi(x) = x^3 - 3x + k$

$\because$ $\phi(x)$ is polynomial in x and so it is continuous and differential in $[-1,1]$ and $(-1,1)$ respectively.

Now let $\phi(x)$ has two roots (distinct) as α, β in $[-1,1]$ i.e. $\phi(\alpha) = 0$, $\phi(\beta) = 0$

$\therefore$ $\phi(x)$ is continuous in $[\alpha, \beta] \subseteq [-1,1]$

$\phi(x)$ is differentiable in $(\alpha, \beta) \subseteq (-1,1)$

Also $\phi(\alpha) = \phi(\beta)$

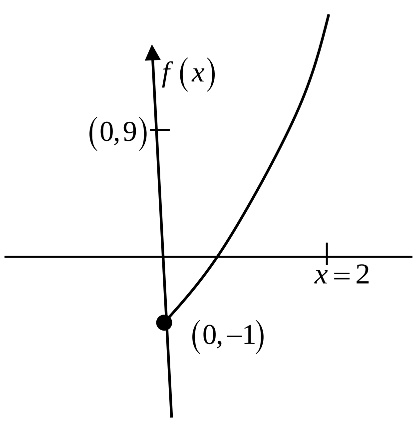
$\therefore$ Rolle's theorem is applicable for $\phi(x)$ in (α, β), so there exists at least one root $c \in (\alpha, \beta)$ such that $\phi'(c) = 0$

$\Rightarrow$ $3x^2 - 3 = 0 \Rightarrow c^2 = 1$

$\Rightarrow$ $c = \pm 1 \notin (-\alpha, \beta)$ $\quad\quad\quad$ $\because (\alpha, \beta) \subset [-1,1]$

This contradicts above explanation, so we can say that there exists no such k for which $\phi(x)$ has two distinct roots in $[-1,1]$.

* roots solutions



Ans4:- Give ϕ and ψ are differentiable in $[a,b]$.

Also given $\phi(x)\,\psi'(x) - \psi(x)\phi'(x) > 0$

$\Rightarrow \quad \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix} > 0$

$\Rightarrow$ Wronskian of ϕ and $\psi \neq 0$ so $\phi(x)$ and $\psi(x)$ are linearly independent function i.e. $\phi(x) \neq k.\psi(x)$ for constant k.

Let α and β are two roots of $\phi(x)$ in $[a,b]$ i.e. $\phi(\alpha)=0$, $\phi(\beta)$ such that $\alpha < \beta$.

Ans5 IFoS 2017Q5. Already we learnt in conceptual part Lagrange's MVT

Hint. Results(exam point 1 Lagrange's MVT): If a function *f* is continuous in $[a,b]$ and differentiable in (a,b) and if $f'(x)=0$ for all $x \in (a,b)$, then *f* is a constant function on $[a,b]$.

Proof: Let β be any point in $[a,b]$. Then *f* is continuous in $[a,p]$ and differential in (a,p). Therefore by MVT there exists at least one point $c \in (a,p)$ s.t.

$f(p) - f(a) = (p-a) f'(c)$

and so $f(p) - f(a) = 0$

Thus $\boxed{f(p) = f(a)}$

2: If two functions *f* and *g* are continuous in $[a,b]$ such that $f'(x) = g'(x)$ for all $x \in (a,b)$, then $f(x)$ and $g(x)$ differential by a constant only.

Proof: Consider the function ϕ defined by

$\phi(x) = f(x) - g(x),\ \forall x \in [a,b]$

$\therefore f, g$ continuous and differential. So is ϕ

$\phi'(x) = f'(x) - g'(x) = 0,\ \forall x \in (a,b)$

Hence from corollary (1), $\phi(x)$ is constant.

3: If *f* is continuous in $[a,b]$ and differential in (a,b), then *f* is increasing or decreasing according as $f'(x) \geq 0$ or $f'(x) \leq 0$ for all $x \in (a,b)$

Proof: Let x_1 and x_2 be any two distinct points in $[a,b]$ such that $x_1 < x_2$ and so $[x_1, x_2] \leq [a,b]$. Then *f* is continuous in $[x_1, x_2]$ and differential in (x_1, x_2).

So by Lagrange's MVT there is a point

$c \in (x_1, x_2)$ s.t.

$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$

Now $\because$ $x_2 - x_1 > 0$, it follows that $f(x_2) \geq f(x_1)$ if $f'(c) \geq 0$.

Hence *f* is increasing or decreasing according as $f'(x) \geq 0$ or $f'(x) \leq 0$, $\forall x \in (a,b)$

Ans6:- consider a function $f(x)=\log(1+x)$ over the interval [0,x].
So why we have chosen such function! interesting to notice.
See, basically we want to use different outcomes, learnt in MVTs and at the same time the inequality asked in the question is also directing us to consider such function.

Evidently this function is continuous in [0,x] and differentiable in (0,x). So on applying Lagrange's MVT on this function we get

$$f(x)=\log(1+x)$$

$$\exists c\in(0,x)s.t.$$

$$[f(x)-f(0)]/(x-0)=f'(c)$$

$$\Rightarrow \log(1+x)-\log 1/x=1/(1+c)$$

$$\Rightarrow \log(1+x)=x/(1+c)$$

$$\because 0<c<x \Rightarrow x/(1+x)<x/(1+c)<x$$

$$\Rightarrow 0<x/(\log(1+x))-1<x$$

$$\Rightarrow 1<x/(\log(1+x))<(1+x)$$

$$\Rightarrow \log(1+x)<x, x/(1+x)<\log(1+x)$$

Ans7:- $y=\sqrt{x-2}$ defined on $[2,3]$

$\because$ y is continuous in $[2,3]$

y is differentiable in $(2,3)$

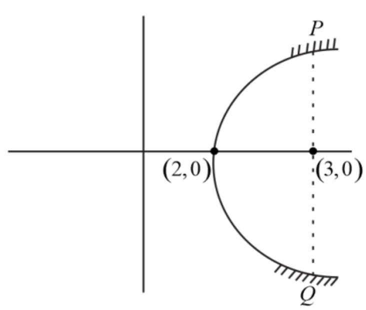
$\therefore$ by Lagrange's MVT, there exist at least one $c\in(2,3)$ such that $y'(c)=\dfrac{y(3)-y(2)}{3-2}$

$$y'(c)=\frac{1-0}{1}$$

$y'(c)=1\Rightarrow 1=1/\sqrt{c-2}\Rightarrow c=2+1/4$. So we get c=2+1/4=2.25 in [2,3]

$\because$ $y=\sqrt{x-2} \Rightarrow y^2=(x-2)$

$\therefore$ Curve is parabola



Ans9:- $f(x)=x(x-1)(x-2)$, $a=0,\ b=\dfrac{1}{2}$

$$=x(x^2-3x+2)=x^3-3x^2+2x$$

Here $f(x)$ is polynomial, so it is continuous in $\left[0, \frac{1}{2}\right]$

Differentiable in $\left(0, \frac{1}{2}\right)$

$\therefore$ By Lagrange's MVT. There exists at least one point $c \in (0, 1/2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\therefore \quad f'(c) = \frac{\frac{1}{2} \times \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) - 0}{\frac{1}{2}} = \frac{3}{4}$$

<u>**Ans10 IFoS 2013** Already we solved this example, just replace C_n by a_n in our example</u>

Example. If $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = 0$ .Then $C_0 + C_1 x + C_2 x^2 + ... + C_n x^n = 0$ has at least one real

Root between 0 and 1.

Solution.

Try; intuition comes from Rolle's theorem. But on which function??

Let's consider a function

$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}$$

$\because$ $f(x)$ is polynomial $\therefore$ continuous in $[0,1]$ and differentiable in $(0,1)$

Also $f(0) = C_0 x^0 + C_1 x_0^2 + ... + C_n x_0^{n+1} = 0$

$f(1) = C_0 + C_1 \times 1^2 + C_2 \times 1^3 + ... + C_n \times 1^{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = 0$ in (1)

i.e. $f(0) = f(1) = 0$

$\therefore f(x)$ satisfies all three conditions of Rolle's theorem

$\therefore \exists$ at least one point (real number) $c \in (0,1)$ s.t

$f'(c) = 0$

$\Rightarrow C_0 + C_1 c + C_2 c^2 + ... + C_n c^n = 0$

$\because$ $f'(x) = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$

C is a real root of the equation $C_0 + C_1 x + C_2 x^2 + ... + C_n x^n = 0$ between 0 and 1.

Ans12:- Given $f(x)$ is twice differentiable and $f(a)=0$, $f(b)=0$ and $f(c)>0$ for $a<c<b$. Prove that there is at least one point ξ, $a<\xi<b$ for which $f''(\xi)<0$.

$\because$ $f(x)$ is differentiable and continuous in $[a,b]$ also $f(a)=0=f(b)$

$\therefore$ Rolle's theorem is application for $f(x)$ in $[a,b]$

$\therefore$ There exists at least one point $\alpha \in (a,b)$ such that $f'(\alpha)=0$ (1)

Now let's consider $f'(x)$ in $[\alpha, c]$

$\because$ $f;(x)$ is continuous and differentiable in $[\alpha, c]$

$\therefore$ By Lagrange's MVT, there exists a real number $\xi \in (\alpha, c)$ such that $f''(\xi) = \dfrac{f'(c)-f'(\alpha)}{c-\alpha}$

$$f''(\xi) = \frac{f'(c)}{c-\alpha} \quad (2)$$

Notice that $f(a)=0$, $f(1)>0$and $f(b)=0$. It show s that $f(x)$has at least for some (β, c) is increasing function i.e.

Ans14:- f'' is continuous on $[1,2]$

$\Rightarrow$ $f(x)$ is continuous in $[1,2]$

$\Rightarrow$ & $f(x)$ is differentiable in $[1,2]$

Also given $f(x)$has three zeros in $(1,2)$

Let $x_0, x_1, x_2 \in (1,2)$ such that $f(x_0)=0$, $f(x_1)=0$, $f(x_2)=0$such that $x_0<x_1<x_2$

$\rightarrow$ Applying Rolle's theorem on $f(x)$ in (x_0, x_1) $\exists$ at least one point $c_1 \in (x_0, x_1)$ such that $f'(c_1)=0$ (1)

$\rightarrow$ Applying Rolle's theorem on $f(x)$in (x_1, x_2) $\exists$ at least one point $c_2 \in (x_1, x_2)$ such that $f'(c_2)=0$ (2)

$\because$ f' is also continuous in $[1,2]$

f' is also differentiable in $(1,2)$and $f'(c_1)=f'(c_2)$ from (1) & (2)

$\rightarrow$ $\therefore$ Applying Rolle's theorem on $f'(x)$in (c_1, c_2) $\exists$ at least one point $c \in (c_1, c_2) \subseteq (1,2)$ such that $(f')'(c)=0$

$\Rightarrow$ $f''(c)=0$

$\Rightarrow$ $f''(x)$ but at least one zero in $(1,2)$

Ans15IFoS 2009 Q. If $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f(\theta x)$

Find the value of θ as $x \to 1$, given $f(x) = (1-x)^{5/2}$

Solution.

$$f(x) = (1-x)^{5/2} \therefore f(0) = 1$$

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \therefore f'(0) = -5/2$$

$$f''(x) = \frac{5}{2} \times \frac{3}{2}(1-x)^{1/2} \therefore f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$$

$\therefore$ By Taylor's theorem: [Maclaurin's, Lagrange's form of remainder]

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$$

$$(1-x)^{5/2} = 1 + x\left(-\frac{5}{2}\right) + \frac{x^2}{2} \times \frac{15}{4}(1-\theta x)^{1/x}$$

$$\therefore \lim_{x\to 1}(1-x)^{1/2} = \lim_{x\to 1}(1) + \lim_{x\to 1}(1)\left(-\frac{5}{2}x\right) + \lim_{x\to 1}\frac{15x^2}{\theta}(1-\theta x)^{1/2}$$

$$0 = 1 - \frac{5}{2} + \frac{15\times 1}{\theta}(1-\theta)^{1/2}$$

$$\frac{3}{2} = \frac{15}{\theta}(1-\theta)^{1/2}$$

$$\frac{4}{5} = (1-\theta)^{1/2} \Rightarrow 1-\theta = \frac{16}{25} \Rightarrow \theta = 9/25$$

MAXIMA MINIMA

Critical point : points (value of x) for which derivative $f'(x)$ becomes zero then that value point is known as critical points for $f(x)$; Stationary Point

Q. Consider the function $f(x) = \int_0^x (t^2 - 5t + 4)(t^2 - 5t + 6)\,dt$

(i) Find critical point of $f(x)$

(ii) Find the points at which local minimum occurs

(iii) Find the points at which local maxima occurs

(iv) Find the number of zeros of $f(x)$ in $[0,5]$

Solution.

$\because$ We need to check sign of $f'(x)$

$\because f(x)=\int_0^x (t^2-5t+4)(t^2-5t+6)\,dt$

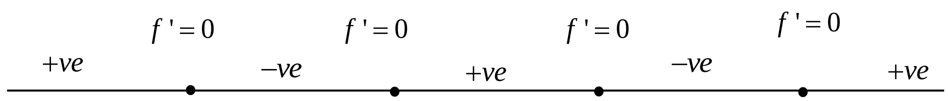
Differentiating w.r.t x by the Leibnitz of differentiation under the sign of integration.

$$f'(x)=\int_0^x \frac{\partial}{\partial x}(t^2-5t+4)(t^2-5t+8)\,dt+(x^2-5x+4)(x^2-5x+6)\cdot\frac{\partial}{\partial}(x)-0$$

$$f'(x)=0+(x^2-5x+4)(x^2-5x+6)-0$$

$$\because f'(x)=(x-1)(x-4)(x-3)(x-2)$$

$$f'(x)=(x-1)(x-2)(x-3)(x-4)$$



Sign chart of $f'(x)$

(i) $x=1,2,3,4$ are critical point

Based on change in sign of derivative:

(ii) $x=1$ is a point of local maxima

(iii) $x=2$ is a point of local minima

$x=3$ is a point of local maxima

$x=4$ is a point of local minima

(iv) Roots of $f(x)$ i.e. values of x in $[0,5]$ at when $f(x)=0$

Possible only at $x=0$. As in given integration integrand is polynomial and it's value will be non zero if x takes any non zero value.

HINT: PYQ 3, 4, 6, 8, 10, 11, 12

Example 1. Find the shortest distance from the point $(1,0)$ to the problem $y^2=4x$

Example 2. Show that the maximum rectangle inscribed in a circle is square.

Example 3. A conical tent is of given capacity. For the least amount of canvas required for it find out ratio of its height to the radius of its bare.

Solution 1. Let's take some arbitrary point on given parabola as $(at^2, 2at)$ and finding distance between this arbitrary point and given point (1,0)

$$p = \sqrt{\left(t^2-1\right)^2 + \left(2t-0\right)^2} \Rightarrow p = \sqrt{t^4-2t^2+1+4t^2} \Rightarrow p = \sqrt{t^4+2t^2+1} = \left(t^2+1\right)^2$$

$$p = \left(t^2+1\right)^2 - f\left(t\right)$$

For minimum $p; dp/dt$ must be zero. $\Rightarrow \left(t^2+1\right)\times 2t = 0 \Rightarrow t^2+1 = 0 \Rightarrow t = \pm i$ [Not real]

$\therefore$ Minimum value of $p = \left(0+1\right)^2 - 1$ as $t = 0$ [Possible]

Solution 2. $\because r^2 = x^2 + y^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$ ….(1)

$\because$ Area of rectangle = $2x \times 2y \Rightarrow A = 2x \times 2\sqrt{r^2 - x^2} \because r$ constant

For maximum Area ; $\frac{dA}{dx} = 0 \Rightarrow r = \sqrt{2}x$, $\therefore y = x$ from (1) $\Rightarrow$ABCD is a square.

Solution 3:

$\because \quad s^2 = \pi^2 r^2 \left(r^2 + h^2\right)$

$s^2 = \pi^2 r^2 + \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4}\right)$ using h in V

$\Rightarrow \quad s^2 = \pi^2 r^4 + \frac{9V^2}{r^2}$

Let $s^2 = z$

$\Rightarrow \quad z = \pi^2 r^4 + \frac{9V^2}{r^2}$

$\Rightarrow \quad \frac{dz}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3}$

$\Rightarrow \quad \frac{d^2z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4}$

Clearly z is maximum or minimum according as s is maximum or minimum.

$\therefore$ Stationary point of z are given by $\frac{dz}{dr} = 0 \Rightarrow 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0$

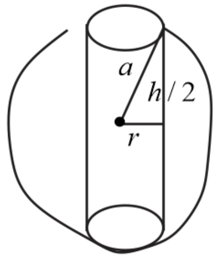
$\Rightarrow \quad 2\pi^2 r^6 = \pi^2 r^4 h^2$

$\Rightarrow \quad h = \sqrt{2r}$

$\because \quad \frac{d^2z}{dr^2} = 12\pi^2 r^2 + \frac{59V^2}{r^4} > 0$ for all V and r .

$\therefore$ Required ratio $\frac{h}{r} = \sqrt{2/1}$

PYQ. 6:- Let r and h be the radius and height of the cylinder respectively.



$$h = 2\sqrt{a^2 - r^2} \quad \text{volume of cylinder } V = \pi r^2 h$$

$$= \pi r^2 . 2\sqrt{a^2 - r^2}$$

$$= 2\pi r^2 . \sqrt{a^2 - r^2}$$

$$\therefore \quad \frac{dV}{dr} = 4\pi r\sqrt{a^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{a^2 - r^2}}$$

$$= \frac{4\pi r^2(a^2 - r^2) - 2\pi r^3}{\sqrt{a^2 - r^2}}$$

Now $\frac{dV}{dr} = 0 \Rightarrow 4\pi ra^2 - 6\pi r^2 = 0$

$$\Rightarrow \quad r^2 = \frac{2a^2}{3}$$

$$\frac{d^2V}{dr^2} = \frac{\sqrt{(a^2 - r^2)}(4\pi ra^2 - 18\pi r^2) - (4\pi ra^2 - 6\pi r^3).\frac{(-2r)}{2\sqrt{a^2 - r^2}}}{(a^2 - r^2)}$$

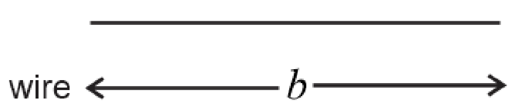
$$= \frac{4\pi a^4 - 22\pi r^2 a^2 + 12\pi r^4 + 4\pi r^2 a^2}{(a^2 - r^2)^{3/2}}$$

Now it can be observed that $r^2 = \frac{2a^2}{3}, \frac{d^2V}{dr^2} < 0$

$\therefore$ Height of cylinder $= 2\sqrt{a^2 - \frac{2a^2}{3}}$

$$= \frac{2a}{\sqrt{3}}$$

PYQ. 8:-



Let x length goes to from square.

$\therefore$ $(b - x)$ ———————— circle

$\therefore$ Area $= A_1 + A_2$

$$A = \left(\frac{\pi}{4} \times \frac{x}{4}\right) + \pi\left(\frac{b-x}{2\pi}\right)^2$$

$\because$ $2\pi r^4 = b - x$

$\therefore$ $r = \dfrac{b-x}{2\pi}$

Put $\dfrac{dA}{dx} = 0$ then $\dfrac{d^2A}{dx^2}$ sign.

Differential Calculus Booklet Part-2

Chapter-1: Limit, Continuity, Differentiability, Partial Derivatives, Euler's equation for homogeneous functions, Total Differentials in Multiple Coordinate Systems
Chapter-2: Taylor's, Maxima/Minima, Lagrange's Method of Undetermined Multipliers
Chapter-3: Jacobians
Chapter-4: Curvature, Tangents and normal, Asymptotes and Curve Tracing

Chapter-1

Functions of Several Variables

1. https://www.youtube.com/live/DTSimWow1Xw?si=H8P0-iYqND6CTT_F
2. https://www.youtube.com/live/v2q3Wf3FSCg?si=4JMp4rzoonFDEDmJ

Neighbourhood of a Point

The set of values x_i, y_i other than *a*, *b* that satisfy the conditions

$$|x_i - a| < \delta, |y_i - b| < \delta$$

where δ is an arbitrarily small positive number, is said to form a *neighbourhood* of the point (a,b).

Thus a neighbourhood is the square: $(a-\delta, a+\delta; b-\delta, b+\delta)$

where *x* takes any value from $a-\delta$ to $a+\delta$ except *a*, and *y* from $b-\delta$ to $b+\delta$ except *b*.

Note: This is not the only way of specifying a neighbourhood of a point. There can be many other; for example the points inside the circle $x^2 + y^2 = \delta^2$ may be taken as a neighbourhood of the point $(0,0)$.

Limit of a Function

• A function *f* is said to tend to a limit *l* as a point (x, y) tends to the point (a,b) if for every arbitrarily small positive number *ε*, there exists a positive number δ (depending on *ε*), such that

$$|f(x,y) - l| < \varepsilon,$$

for every point (x, y), [different from (a,b)] which satisfies $|x-a| < \delta, |y-b| < \delta$

• *Represented by* $\lim_{(x,y)\to(a,b)} f(x,y) = l$

l is the *limit* (the *double limit* or the *simultaneous limit*) of *f* when x, y tend to *a* & *b* simultaneously.

• Note: The above definition implies that there must be no assumption of any relation between the independent variables as they tend to their respective limits.

For instance take $f(x, y)$ where

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

and find the limit when $(x, y) \to (0,0)$.

If we put $y = m_1 x$ and let $x \to 0$, we get the limit to be equal to $\frac{m_1}{1+m_1^2}$, while putting $y = m_2 x$ leads to a limit $\frac{m_2}{1+m_2^2}$. Similarly letting $x \to 0$, while *y* remains constant or vice-versa leads to zero limit.

- Thus, we are led to erroneous results. Geometrically speaking when we approach the point $(0,0)$ along different paths, first along lines with slopes m_1 and m_2 and then along lines parallel to the coordinate axes, the function reaches different limits.

- The simultaneous limit postulates that by whatever path the point is approached, the function f attains the same limit. **In general, the determination whether a simultaneous limit exists or not is a difficult matter but very often a simple consideration enables us to show that the *limit does not exist*. [Exam Point]**

Note: It may however be noted that

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \;\Rightarrow \lim_{x\to a} f(x,b) = l = \lim_{y\to b} f(a,y)$$

Non-existence of limit. The above note makes it clear that if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ and if $y=\phi(x)$ is any function such that $\phi(x)\to b$, when $x\to a$, then $\lim\limits_{x\to a} f(x,\phi(x))$ must exist and should be equal to l.

Thus, if we can find two functions $\phi_1(x)$ and $\phi_2(x)$ such that the limits of $f(x,\phi_1(x))$ and $f(x,\phi_2(x))$ are different, then the simultaneous limit in question does not exist.

Algebra of Limits

If f and g are two functions with a domain N, we define four functions, $f\pm g$, fg, f/g on N by setting

$$(f+g)(x,y) = f(x,y)+g(x,y)$$
$$(f-g)(x,y) = f(x,y)-g(x,y)$$
$$f\cdot g(x,y) = f(x,y)\cdot g(x,y)$$
$$(f/g)(x,y) = f(x,y)/g(x,y),\ \text{if } g(x,y)\neq 0,\ \text{for } (x,y)\in N$$

Theorem. *If f, g be two functions defined on some neighbourhood of a point (a,b) such that $\lim f(x,y)=l$, $\lim g(x,y)=m$, when $(x,y)\to(a,b)$, then*

(i) $\lim(f+g) = \lim f + \lim g = l+m$

(ii) $\lim(f-g) = \lim f - \lim g = l-m$

(iii) $\lim(f\cdot g) = \lim f\cdot\lim g = l\cdot m$

(iv) $\lim\dfrac{f}{g} = \dfrac{\lim f}{\lim g} = \dfrac{l}{m}$, *provided* $m\neq 0$, *when* $(x,y)\to(a,b)$

The proofs are exactly similar to those of the corresponding theorems for a single variable.

Repeated Limits

• If a function *f* is defined in some neighbourhood of (a,b), then the limit, $\lim_{y\to b} f(x,y)$, if it exists, is a function of *x*, say $\phi(x)$. If then the limit $\lim_{x\to a}\phi(x)$ exists and is equal to λ, we write

$$\lim_{x\to a}\lim_{x\to b} f(x,y)=\lambda$$

and say that λ is a *repeated limit* of *f* as $y\to b,\ x\to a$.

• If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y\to b}\lim_{x\to a} f(x,y)=\lambda' \text{ (say)}$$

when first $x\to a$, and then $y\to b$.

• These two limits may or may not be equal.

Exam Point: In case the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. However *if the repeated limits are not equal, the simultaneous limit cannot exist.*

<u>CONTINUITY</u>

https://www.youtube.com/live/ykYo5SbpjFA?si=Akl_MUzTtIKbbtFy

A function *f* is said to be continuous at a point (a,b) of its domain of definition, if for every arbitrarily small positive number *ε*, there exists a positive number δ (depending on *ε*), such that

$$\left|f(x,y)-f(a,b)\right|<\varepsilon,$$

for every point (x,y), which satisfies $|x-a|<\delta,\ |y-b|<\delta$

i.e. $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$

Exam Point: A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of a single variable. To be more specific if a function *f* of two variables *x*, *y* is continuous at (a,b) then $f(x,b)$ is a continuous function of *x* at $x=a$ and $f(a,y)$ that of y at $y=b$.

The converse however is not true, i.e., a function may be a continuous function of one variable when the others remain constant and yet not be a continuous function of all the variables.

For instance, consider a function *f*, where $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y}, & (x,y)\neq(0,0)\\ 0, & \text{at}(0,0)\end{cases}$

The function is not continuous at $(0,0)$ for $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. But

$$\lim_{x\to 0} f(x,0) = 0 = f(0,0), \text{ and } \lim_{y\to 0} f(0,y) = 0 = f(0,0)$$

so that f is continuous at $(0,0)$, when considered as a function of a single variable *x* or that of *y*.

A function is said to be continuous in a region if it is continuous at every point of the same.

PARTIAL DERIVATIVES

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function with respect to the variable.

• Partial derivative of $f(x,y)$ with respect to *x* is generally denoted by $\partial f/\partial x$ or f_x or $f_x(x,y)$, while those with respect to *y* are denoted by $\partial f/\partial y$ or f_y or $f_y(x,y)$.

$\therefore$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

and

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

when these limits exist.

The partial derivatives at a particular point (a,b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a,b)}, \frac{\partial f(a,b)}{\partial x} \text{ or } f_x(a,b)$$

and

$$\left[\frac{\partial f}{\partial y}\right]_{(a,b)}, \frac{\partial f(a,b)}{\partial y} \text{ or } f_y(a,b)$$

$\therefore$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

in case the limit exists.

Mean Value Theorem

If f_x *exists throughout a neighbourhood of a point* (a,b) *and* $f_y(a,b)$ *exists, then for any point* $(a+h,b+k)$ *of this neighbourhood,*

$$f(a+h,b+k) - f(a,b) = hf_x(a+\theta h,b+k) + k\left[f_y(a,b) + \eta\right]$$

where $0<\theta<1$, *and* η *is a function of k, tending to zero with k.*

Proof:

$\because f(a+h,b+k)-f(a,b)=f(a+h,b+k)-f(a,b+k)+f(a,b+k)-f(a,b)$ ....(1)

- As, f_x exists in a neighbourhood of (a,b), therefore by Lagrange's mean value theorem,

$$f(a+h,b+k)-f(a,b+k)=hf_x(a+\theta h,b+k),\ 0<\theta<1 \ldots\ldots\ldots\ldots(2)$$

- Also $f_y(a,b)$ exists, so that; $\lim\limits_{k\to 0}\dfrac{f(a,b+k)-f(a,b)}{k}=f_y(a,b)$

$$\Rightarrow f(a,b+k)-f(a,b)=k\left[f_y(a,b)+\eta\right]\ldots\ldots.(3)$$

by using $\lim\limits_{x\to a} f(x)=l \Rightarrow f(x)=l+\lambda$; where λ is very small.

where η is a function of k and tends to zero as $k\to 0$.

From equation (1), (2) and (3) we get the required result.

Sufficient Condition for Continuity

A sufficient condition that a function f be continuous at (a,b) *is that one of the partial derivatives exists and is bounded in a neighbourhood of* (a,b) *and that the other exist at* (a,b).

- Let f_x exist and be bounded in a neighbourhood of (a,b) and let $f_y(a,b)$ exist, then for any point $(a+h,b+k)$ of this neighbourhood we have

$$f(a+h,\ b+k)-f(a,b)=hf_x(a+\theta h,\ b+k)+k\left[f_y(a,b)+\eta\right]$$

where $0<\theta<1$, and $\eta\to 1$ as $k\to 0$.

- Proceeding to limits as $(h,k)\to(0,0)$, since $f_x(a+\theta h,b+k)$ is bounded, we have

$\lim\limits_{(h,k)\to(0,0)} f(a+h,\ b+k)=f(a,b)\Rightarrow f$ is continuous at (a,b).

DIFFERENTIABILITY

https://www.youtube.com/live/Q4yH_fhT52o?si=tPoKmN9EKmPyjkwG

- Let $(x,y),(x+\delta x,y+\delta y)$ be two neighbouring points in the domain of definition of a function *f*. The change δf in the function as the point changes from (x,y) to $(x+\delta x,y+\delta y)$ is given by

$$\delta f=f(x+\delta x,y+\delta y)-f(x,y). \quad ....(1)$$

• The function *f* is said to be *differentiable* at (x, y) if the change δf can be expressed in the form

$$\delta f = A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y) \qquad ....(2)$$

where A and B are constants independent of $\delta x, \delta y$ and ϕ, ψ are function of $\delta x, \delta y$ tending to zero as $\delta x, \delta y$ tend to zero simultaneously.

Exam point: $A\delta x + B\delta y$ is then called the *differential* of *f* at (x, y) and is denoted by df.
Thus

$$df = A\delta x + B\delta y$$

From (1) and (2), when $(\delta x, \delta y) \to (0,0)$, we get

$$f(x+\delta x, y+\delta y) - f(x, y) \to 0$$

or

$$f(x+\delta x, y+\delta y) \to f(x, y)$$

$\Rightarrow$ The function *f* is continuous at (x, y)

• Thus every *differentiable function is continuous*.

• Again from (2), when $\delta y = 0$ (i.e., y remains constant)

$$\delta f = A\delta x + \delta x\phi(\delta x, 0)$$

Dividing by δx and proceeding to limits as $\delta x \to 0$, we get

$\frac{\partial f}{\partial x} = A$. Similarly, $\frac{\partial f}{\partial y} = B$

• Thus, the constants *A* and *B* are respectively the partial derivatives of *f* with respect to *x* and *y*.

Hence, *a function which is differentiable at a point possesses the first order partial derivatives threat*.

Converse, of course is not true, so that functions exist which are continuous and may even possess partial derivatives at a point but are not differentiable there at.

Exam Point: Again the differential of *f* is given by

$$df = A\delta x + B\delta y = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

• Taking $f = x$, we get $df = dx, \frac{\partial f}{\partial x} = 1, \frac{\partial f}{\partial y} = 0$ and hence $dx = \delta x$

Similarly taking $f = y$, we obtain $dy = \delta y$.

Thus, the differentials dx, dy of x, y are respectively δx and δy, and

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy \qquad ....(3)$$

is the differential of *f* at (x, y).

Notes:

- If we replace $\delta x, \delta y$, by h, k in equations (1) and (2), we say that the function is differentiable at a point (a,b) of the domain of definition if *df* can be expressed as

$$df = f(a+h, b+k) - f(a,b)$$
$$= Ah + Bk + h\phi(h,k) + k\psi(h,k) \qquad ....(4)$$

where $A = f_x,\ B = f_y$ and ϕ, ψ are function of h, k tending to zero as $h,\ k$ tend to zero simultaneously.

- We have that a function differentiable at a point is necessarily continuous and possesses partial derivatives there at. Not only that, we talk of differentiability at a point of a function only when it is continuous and has partial derivatives there at, for it is only then that it can be expressed in the form of equation (1).

PARTIAL DERIVATIVES OF HIGHER ORDER

- If a function *f* has partial derivatives of the first order at each point (x, y) of a certain region, then f_x, f_y are themselves functions of x, y and may also possess partial derivatives. These are called *second order partial derivatives of f* and are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{x^2}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_{y^2}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

- In a similar manner higher order partial derivatives are defined. For example $\frac{\partial^3 f}{\partial x \partial x \partial y} = f_{xxy}$ and so on.

The second order partial derivatives at a particular point (a,b) are often denoted by

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(a,b)}, \frac{\partial^2 f(a,b)}{\partial x^2}, f_{xx}(a,b) \text{ or } f_{x^2}(a,b)$$
$$\left[\frac{\partial^2 f}{\delta x \partial y}\right]_{(a,b)}, \frac{\partial^2 f(a,b)}{\partial x \partial y} \text{ or } f_{xy}(a,b)$$

and so on.

Thus

$$f_{xx}(a,b)=\lim_{h\to 0}\frac{f_x(a+h,b)-f_x(a,b)}{h}$$
$$f_{xy}(a,b)=\lim_{h\to 0}\frac{f_y(a+h,b)-f_y(a,b)}{h}$$
$$f_{yx}(a,b)=\lim_{h\to 0}\frac{f_x(a,b+h)-f_x(a,b)}{h}$$
$$f_{yy}(a,b)=\lim_{h\to 0}\frac{f_y(a,b+h)-f_y(a,b)}{h}$$

in case the limits exist.

- **Theorem**. *If (a,b) be a point of the domain of definition of a function f such that*

(i) f_x is continuous at (a,b),

(ii) f_y exists at (a,b),

then f is differentiable at (a,b).

<u>Change in the Order of Partial Derivation</u>

- $f_{xy}(a,b)=\lim_{h\to 0}\frac{f_y(a+h,b)-f_y(a,b)}{h}$

$$=\lim_{h\to 0}\frac{1}{h}\left[\lim_{k\to 0}\frac{f(a+h,b+k)-f(a+h,b)}{k}-\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}\right]$$
$$=\lim_{h\to 0}\lim_{k\to 0}\frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk}=\lim_{h\to 0}\lim_{k\to 0}\frac{\phi(h,k)}{hk}$$

where $\phi(h,k)=f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)$.

- Similarly,

$$f_{yx}(a,b)=\lim_{k\to 0}\lim_{h\to 0}\frac{\phi(h,k)}{hk}$$

Thus we see that $f_{xy}(a,b)$ and $f_{yx}(a,b)$ are the repeated limits of the same expression taken in different orders. There is therefore no a *priori* reason why they should always be equal.

Young's theorem. *If f_x and f_y are both differentiable at a point (a,b) of the domain of definition of a function f, then*

$f_{xy}(a,b)=f_{yx}(a,b)$

Schwarz's theorem. *If f_y exists in a certain neighbourhood of a point (a,b) of the domain of definition of a function f, and f_{yx} is continuous at (a,b), then $f_{xy}(a,b)$ exists and is equal to $f_{yx}(a,b)$.*

Euler's theorem on homogenous functions :-

Statement:- If it is a homogenous function of degree n in two independent variable x and y, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Proof:- $\because \quad u=x^n \, . \, \phi(y/x)$

$$\therefore \quad \frac{\partial u}{\partial x x^2}=n\,x^{n-1}.\,\phi(y/x)+x^n.\left(\frac{-y}{x^2}\right)\phi'(y/x)$$

$$\Rightarrow \quad x.\frac{\partial u}{\partial x}=n\,x^n\,\phi(y/x)-x^{n-1}y\,\phi'(y/x) \qquad (1)$$

Similarly $y\frac{\partial u}{\partial y}=x^{n-1}.\,y\,\phi'(y/x)$ (2)

On adding (1) & (2), we get $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,x^n.\phi(y/x)$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u$$

Note:- (1) The converse of Euler's theorem is also correct i.e. if u is a function of two independent variable x and y such that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u$, for all values of x and y then u must be a homogenous function of degree n.

Note:- (2) If u is a homogenous function of m variables $x_1.\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+....+x_m\frac{\partial u}{\partial x_m}=n\,u$

Examples on Limits and Continuity

Example 1(a). Let

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & \text{if} \quad x^4+y^2\neq 0\\ 0, & \text{if} \quad x+y=0\end{cases}$$

- If we approach the origin along any axis, $f(x,y)=0$.
- If we approach $(0,0)$ along any line $y=mx$, then

$$f(x,y)=f(x,mx)=\frac{mx^3}{x^4+m^2x^2}=\frac{mx}{x^2+m^2}\to 0,\text{ as } x\to 0$$

So any straight line approach gives,

$$\lim_{(x,y)\to(0,0)} f(x,y)=0$$

- By putting $y=mx^2$,

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0} f(x,mx^2)=\frac{m}{1+m^2}$$

which is different for the different m selected.

Hence, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 1(b). Show that

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} \text{ does not exist.}$$

- If we put $x=my^2$ and let $y\to 0$, we get

$$\lim_{y\to 0}\frac{2my^4}{(m^2+1)y^4}=\frac{2m}{1+m^2}$$

which is different for different values of m.
Hence, the limit does not exist.

Example 2(a). Show that

$$\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}=0$$

- Put $x=r\cos\theta, y=r\sin\theta$

$$\left|xy\frac{x^2-y^2}{x^2+y^2}\right|=\left|r^2\sin\theta\cos\theta\cos 2\theta\right|$$

$$=\left|\frac{r^2}{4}\sin 4\theta\right|\leq\frac{r^2}{4}=\frac{x^2+y^2}{4}<\varepsilon,$$

if

$$\frac{x^2}{4} < \frac{\varepsilon}{2}, \frac{y^2}{4} < \frac{\varepsilon}{2}$$

or if

$$|x| < \sqrt{2\varepsilon} = \delta, \ |y| < \sqrt{2\varepsilon} = \delta$$

Thus for $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \varepsilon, \text{ when } |x| < \delta, |y| < \delta$$

$$\Rightarrow \qquad \lim_{(x,y)\to(0,0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

Example 2(b). Show that

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2 + 1} - 1}{x^2 + y^2} = 0$$

- Since x, y are small

$$\frac{\sqrt{x^2y^2 + 1} - 1}{x^2 + y^2} = \frac{\left(1 + x^2y^2\right)^{1/2} - 1}{x^2 + y^2} = \frac{\frac{1}{2}x^2y^2}{x^2 + y^2}$$

Now changing to polars, we can show, as in the above example, that

$$\lim_{(x,y)\to(0,0)} \frac{\frac{1}{2}x^2y^2}{x^2 + y^2} = 0$$

Hence the required result.

Ex.1. Show that

(i) $\lim_{(x,y)\to(0,0)} \left(\frac{1}{|x|} + \frac{1}{|y|} \right) = \infty$, (ii) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$,

(iii) $\lim_{(x,y)\to(0,0)} (x + y) = 0$, (iv) $\lim_{(x,y)\to(0,0)} (1/xy)\sin\left(x^2y + xy^2\right) = 0$

Ex.2. Show that the limit, when $(x, y) \to (0,0)$ does not exist in each case

(i) $\lim \frac{2xy}{x^2 + y^2}$, (ii) $\lim \frac{xy^3}{x^2 + y^6}$,

(iii) $\lim \frac{x^2y^2}{x^2y^2 + \left(x^2 - y^2\right)^2}$, (iv) $\lim \frac{x^3 + y^3}{x - y}$

[***Hint***: (iv) Put $y = x - mx^3$]

Ex. 3. Show that the limit, when $(x, y) \to (0,0)$ exist in each case.

(i) $\lim \dfrac{xy}{\sqrt{x^2+y^2}}$, (ii) $\lim \dfrac{x^3y^3}{x^2+y^2}$,

(iii) $\lim \dfrac{x^3-y^3}{x^2+y^2}$, (iv) $\lim \dfrac{x^4+y^4}{x^2+y^2}$

Example 3(a). Show that

(i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\sin\left(x^2+y^2\right)}{x^2+y^2}=0$, (ii) $\lim\limits_{(x,y)\to(2,1)} \dfrac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}=\dfrac{1}{3}$

- (i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\sin\left(x^2+y^2\right)}{x^2+y^2}=\lim\limits_{(x,y)\to(0,0)} x\cdot \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}=0\cdot 1=0$

  (ii) $\lim\limits_{(x,y)\to(2,1)} \dfrac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}=\lim\limits_{t\to 0}\dfrac{\sin^{-1} t}{\tan^{-1} 3t}$, where $t=(xy-2)=\lim\limits_{t\to 0}\dfrac{1/\sqrt{1-t^2}}{3/\left(1+9t^2\right)}=\dfrac{1}{3}$

  (L'Hospital Rule)

Ex.3(b). Show that $\lim\limits_{(x,y)\to(0,1)} \tan^{-1}(y/x)$, does not exist.

[***Hint***: Limit from the left is $-\dfrac{\pi}{2}$ and that from the right $\dfrac{\pi}{2}$].

Example 4. Let

$$f(x,y)=\frac{xy}{x^2+y^2}\text{, then}$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y)=\lim_{y\to 0}(0)=0,$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y)=0$$

Thus, the repeated limits exist and are equal. But the simultaneous limit does not exist which may be seen by putting $y=mx$.

(ii) Let

$$f(x,y)=\frac{y-x}{y+x}\cdot\frac{1+x}{1+y}\text{, then}$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{x\to 0}\left(-\frac{1+x}{1}\right)=-1,$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y)=\lim_{y\to 0}\left(\frac{1}{1+y}\right)=1$$

Thus, the two repeated limits exist but are unequal, consequently the simultaneous limit cannot exist, which may be verified by putting $y=mx$.

Example 5. Show that the limit exists at the origin but the repeated limits do not, where

$$f(x,y)=\begin{cases} x\sin\left(\dfrac{1}{y}\right)+y\sin\left(\dfrac{1}{x}\right), & xy\neq 0 \\ 0, & xy=0 \end{cases}$$

- Here $\lim\limits_{y\to 0} f(x,y), \lim\limits_{x\to 0} f(x,y)$ do not exist and therefore $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$; $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ do not exist.

Again

$$\left|x\sin\frac{1}{y}+y\sin\frac{1}{x}\right|\leq |x|\left|\sin\frac{1}{y}\right|+|y|\left|\sin\frac{1}{x}\right|\leq |x|+|y|<\varepsilon$$

if

$$|x|<\frac{\varepsilon}{2}=\delta,\ |y|<\frac{\varepsilon}{2}=\delta$$

Thus for $\varepsilon>0$, $\exists\delta>0$ such that

$$\left|x\sin\frac{1}{y}+y\sin\frac{1}{x}\right|<\varepsilon, \text{ when } |x|<\delta, |y|<\delta$$

$\Rightarrow$
$$\lim_{(x,y)\to(0,0)}\left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)=0$$

Example 6: Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where

$$f(x,y)=\begin{cases} 1, \text{ if} & xy\neq 0 \\ 0, \text{ if} & xy=0 \end{cases}$$

- Here

$$\lim_{y\to 0} f(x,y)=\begin{cases} 1, \text{ if} & x\neq 0 \\ 0, \text{ if} & x=0 \end{cases}$$

$\therefore$
$$\lim_{x\to 0}\lim_{y\to 0} f(x,y)=1$$

Similarly,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y)=1$$

Hence, the repeated limits exist and are equal.

Again, since there are points arbitrarily near $(0,0)$ at which f is equal to 0 and points arbitrarily near $(0,0)$ at which f is equal to 1, therefore, there is an $\varepsilon>0$, such that

$$|f(x,y)-f(0,0)|=|f(x,y)| \text{Ž } \varepsilon,$$

for all points in any neighbourhood of $(0,0)$.

Hence, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 7. Investigate the continuity at $(0,0)$ of

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

- Since $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (put $y=mx$), therefore the function is not continuous at $(0,0)$.

Example 8. Show that the function

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is continuous at the origin.

- Let $x=r\cos\theta,\quad y=r\sin\theta$

$\therefore$ $$\left|\frac{xy}{\sqrt{x^2+y^2}}\right|=r\left|\cos\theta\sin\theta\right|\le r=\sqrt{x^2+y^2}<\varepsilon,$$

if

$$x^2<\frac{\varepsilon^2}{2},\quad y^2<\frac{\varepsilon^2}{2}$$

or, if

$$|x|<\frac{\varepsilon}{\sqrt{2}},\quad |y|<\frac{\varepsilon}{\sqrt{2}}$$

Thus

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|<\varepsilon,\text{ when } |x|<\frac{\varepsilon}{\sqrt{2}},\quad |y|<\frac{\varepsilon}{\sqrt{2}}$$

$\Rightarrow$ $$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0$$

$\therefore$ $$\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)$$

Hence, *f* is continuous at $(0,0)$.

Examples on partial derivatives

Example 1. If $f(x,y)=2x^2-xy+2y^2$, then find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(1,2)$.

- Now

  $\dfrac{\partial f}{\partial x}=4x-y=2$, at $(1,2)$

  $\dfrac{\partial f}{\partial y}=-x+4y=7$, at $(1,2)$

2. If

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

show that both the partial derivatives exist at $(0,0)$ but the function is not continuous there at.

- Putting $y=mx$, we see that

$$\lim_{x\to 0} f(x,y)=\frac{m}{1+m^2}$$

so that the limit depends on the value of m, i.e., on the path of approach and is different for the different paths followed and therefore does not exist. Hence the function $f(x,y)$ is not continuous at $(0,0)$.

Again

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0\cdot 0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0}{k}=0$$

Examples on Differentiability

Example 1. Let

$$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Solution- Put $x=r\cos\theta, y=r\sin\theta$.

$$\therefore \left|\frac{x^3-y^3}{x^2+y^2}\right|=\left|r\left(\cos^3\theta-\sin^3\theta\right)\right|\le|r||\cos\theta|^3+|r||\sin\theta|^3\le 2|r|=2\sqrt{x^2+y^2}<\varepsilon,$$

if

$$x^2<\frac{\varepsilon^2}{8},\ y^2<\frac{\varepsilon^2}{8}$$

or, if

$$|x|<\frac{\varepsilon}{2\sqrt{2}},\ |y|<\frac{\varepsilon}{2\sqrt{2}}$$

$\therefore \left|\dfrac{x^3-y^3}{x^2+y^2}-0\right|<\varepsilon$, when $|x|<\dfrac{\varepsilon}{2\sqrt{2}}, |y|<\dfrac{\varepsilon}{2\sqrt{2}}$

$$\Rightarrow \qquad \lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x^2+y^2}=0$$

$$\Rightarrow \qquad \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence the function is continuous at $(0,0)$.

Again

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1$$

Thus, the function possesses partial derivatives at $(0,0)$.

If the function is differentiable at $(0,0)$, then by definition

$$df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi \qquad \text{....(1)}$$

when A and B are constants $\left(A = f_x(0,0) = 1, B = f_y(0,0) = -1\right)$ and ϕ, ψ tend to zero as $(h,k) \to (0,0)$.

Putting $h = \rho\cos\theta, k = \rho\sin\theta$, and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta \qquad \text{....(2)}$$

For arbitrary $\theta = \tan^{-1}(h/k), \rho \to 0$ implies that $(h,k) \to (0,0)$. Thus we get the limit,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

or $$(\cos\theta - \sin\theta)\left(\cos^2\theta + \sin^2\theta + \cos\theta + \sin\theta\right) = \cos\theta - \sin\theta$$

or $$\cos\theta\sin\theta(\cos\theta - \sin\theta) = 0$$

which is plainly impossible for arbitrary θ.

Thus, the function is not differentiable at the origin.

Note: The method used to show that the function is not differentiable, can also be used to show that the function is not continuous at $(0,0)$; for example,

The function *f*, where

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, \text{ if} & x = y = 0 \end{cases}$$

is not differentiable at the origin because it is discontinuous threat.

Example 2. Show that the function *f*, where

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, \text{if} & x^2+y^2 \neq 0 \\ 0, \quad \text{if} & x = y = 0 \end{cases}$$

is continuous, possess partial derivatives but is not differentiable at the origin.

- *f* is continuous at the origin, also it may be easily shown that

$$f_x(0,0) = 0 = f_y(0,0)$$

If the function is differentiable at the origin, then by definition

$$df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi \qquad ....(1)$$

where $A = f_x(0,0) = 0,\ B = f_y(0,0) = 0$, and ϕ, ψ tend to zero as $(h,k) \to (0,0)$.

$$\therefore \ \frac{hk}{\sqrt{h^2+k^2}} = h\phi + k\psi \qquad ....(2)$$

Putting $k = mh$ and letting $h \to 0$, we get

$$\frac{m}{\sqrt{1+m^2}} = \lim_{h\to 0}(\phi + m\psi) = 0$$

which is impossible for arbitrary *m*.

Hence, the function is not differentiable at $(0,0)$.

Note: If we put $h = r\cos\theta,\ k = r\sin\theta$ in (2) we get

$$\cos\theta\sin\theta = \phi\cos\theta + \psi\sin\theta$$

For arbitrary $\theta, r \to 0$ implies $(h,k) \to (0,0)$.

Thus when $r \to 0$, we get

$\cos\theta \cdot \sin\theta = 0$

which is impossible for arbitrary θ. So *f* is not differentiable at the origin.

Try by yourself

Q1. Show that the function *f*, where

$$f(x,y) = \begin{cases} x\sin 1/x + y\sin 1/y, & xy \neq 0 \\ x\sin 1/x, & y = 0, x \neq 0 \\ y\sin 1/x, & x = 0, y \neq 0 \\ 0, & x = 0 = y \end{cases}$$

is continuous but not differentiable at the origin.

Q2. Show that the function $|x| + |y|$ is continuous, but not differentiable at the origin.

Q3. Discuss the following functions for continuity and differentiability at the origin.

(i) $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 0$

(ii) $f(x,y) = y\sin 1/x$, if $x \neq 0,\ f(0,y) = y$

Combined Examples

Example1. Consider the function

$$f(x,y) = \begin{cases} x^2\sin 1/x + y^2\sin 1/y, & \text{if } xy \neq 0 \\ x^2\sin 1/x, & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2\sin 1/y, & \text{if } x = 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

- The partial derivatives,

$$f_x(x,y) = \begin{cases} 2x \sin 1/x - \cos 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f_y(x,y) = \begin{cases} 2y \sin 1/y - \cos 1/y, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

are discontinuous at the origin, so that both the partial derivatives exist at the origin, but none is continuous threat.

Let us show that the function is differentiable at the origin. Here,

$$f(h,k) - f(0,0) = h^2 \sin 1/h + k^2 \sin 1/k$$
$$= 0h + 0k + h(h \sin 1/h) + k(k \sin 1/k)$$

Now $(h \sin 1/h)$ and $(k \sin 1/k)$ both tend to zero when $(h,k) \to (0,0)$ so that f is differentiable at the origin.

Example 2. Prove that the function

$$f(x,y) = \sqrt{|xy|}$$

is not differentiable at the point $(0,0)$, but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

- Now at $(0,0)$,

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

If the function is differentiable at $(0,0)$ then by definition

$$f(h,k) - f(0,0) = 0h + 0k + h\phi + k\psi$$

where ϕ and ψ are functions of h and k, and tend to zero as $(h,k) \to (0,0)$.

Putting $h = \rho \cos\theta, k = \rho \sin\theta$ and dividing by ρ, we get

$$|\cos\theta \sin\theta|^{1/2} = \phi \cos\theta + \psi \sin\theta$$

Now for arbitrary $\theta, \rho \to 0$ implies that $(h,k) \to (0,0)$.

Taking the limit as $\rho \to 0$, we get

$$|\cos\theta \sin\theta|^{1/2} = 0,$$

which is impossible for all arbitrary θ.

Hence, the function is not differentiable at $(0,0)$ and consequently the partial derivatives f_x, f_y cannot be continuous at $(0,0)$, for otherwise the function would be differentiable threat,

Let us now see that it is actually so

For $(x,y) \neq (0,0)$.

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{|x+h||y|}-\sqrt{|x||y|}}{h}$$

Taking $|y|$ common from the numerator and rationalizing,

$$=\lim_{h\to 0}\sqrt{|y|}\,\frac{|x+h|-|x|}{h\left[\sqrt{|x+h|}+\sqrt{|x|}\right]}$$

Now as $h\to 0$, we can take $x+h>0$, i.e., $|x+h|=x+h$, when $x>0$ and $x+h<0$ or $|x+h|=-(x+h)$, when $x<0$.

$$\therefore \qquad f_x(x,y)=\begin{cases}\dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x>0\\[2ex] -\dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x<0\end{cases}$$

Similarly,

$$f_y(x,y)=\begin{cases}\dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y>0\\[2ex] -\dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y<0\end{cases}$$

which are, obviously, not continuous at the origin.

Example3. Show that the function *f*, where

$$f(x,y)=\begin{cases}xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } x^2+y^2\neq 0\\[1ex] 0, & \text{if } x=y=0\end{cases}$$

is differentiable at the origin.

- It may be easily shown that

$$f_x(0,0)=0=f_y(0,0)$$

Also when $x^2+y^2\neq 0$,

$$|f_x|=\frac{|x^4y+4x^2y^3-y^5|}{(x^2+y^2)^2}=|y|\frac{|x^4+4x^2y^2-y^4|}{(x^2+y^2)^2}\le\frac{\sqrt{x^2+y^2}\,2(x^2+y^2)^2}{(x^2+y^2)^2}=2(x^2+y^2)^{1/2}$$

Evidently

$$\lim_{(x,y)\to(0,0)} f_x(x,y)=0=f_x(0,0)$$

Thus f_x is continuous at $(0,0)$ and $f_y(0,0)$ exists.

$\Rightarrow$ *f* is differentiable at $(0,0)$.

Example4. Let

$$f(x,y)=\frac{xy\left(x^2-y^2\right)}{x^2+y^2},\ (x,y)\neq(0,0),\ f(0,0)=0\text{, then}$$

show that at the origin $f_{xy}\neq f_{yx}$.

- Now

$$f_{xy}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}$$

But $$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0}{k}=0$$

and $$f_y(h,0)=\lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k}=\lim_{k\to 0}\frac{hk\left(h^2-k^2\right)}{k\cdot\left(h^2+k^2\right)}=h$$

$\therefore$ $$f_{xy}(0,0)=\lim_{h\to 0}\frac{h-0}{h}=1$$

Again

$$f_{yx}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$$

But

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$$

$$f_x(0,k)=\lim_{h\to 0}\frac{f(h,k)-f(0,k)}{h}=\lim_{h\to 0}\frac{hk\left(h^2-k^2\right)}{h\left(h^2+k^2\right)}=-k$$

$\therefore$ $$f_{yx}(0,0)=\lim_{k\to 0}\frac{-k-0}{k}=-1$$

$\therefore$ $$f_{xy}(0,0)\neq f_{yx}(0,0).$$

Example5. Examine the equality of f_{xy} and f_{yx}, where

$$f(x,y)=x^3y+e^{xy^2}$$

- Now

$$f_y=x^3+2xye^{xy^2}$$

$\therefore$ $$f_{xy}=3x^2+2ye^{xy^2}+2xy^3e^{xy^2}$$

Again

$$f_x=3x^2y+y^2e^{xy^2}$$

$$f_{yx}=3x^2+2ye^{xy^2}+2xy^3e^{xy^2}$$

$\Rightarrow$ $$f_{xy}=f_{yx}$$

Example 6. Show that for the function

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$f_{xy}(0,0)=f_{yx}(0,0)$, even though the conditions of Schwarz's theorem and also of Young's theorem are not satisfied.

- Now

  $$f_x(0,0)=\lim_{x\to0}\frac{f(h,0)-f(0,0)}{h}=0$$

  Similarly, $f_y(0,0)=0$.

  Also, for $(x,y)\neq(0,0)$,

  $$f_x(x,y)=\frac{(x^2+y^2)\cdot 2xy^2-x^2y^2\cdot 2x}{(x^2+y^2)^2}=\frac{2xy^4}{(x^2+y^2)^2}$$

  $$f_y(x,y)=\frac{2x^4y}{(x^2+y^2)^2}$$

  Again

  $$f_{yx}(0,0)=\lim_{y\to0}\frac{f_x(0,y)-f_x(0,0)}{y}=0$$

  and

  $f_{xy}(0,0)=0$, so that $f_{xy}(0,0)=f_{yx}(0,0)$

  For $(x,y)\neq(0,0)$, we have

  $$f_{yx}(x,y)=\frac{8xy^3(x^2+y^2)^2-2xy^4\cdot 4y(x^2+y^2)}{(x^2+y^2)^4}=\frac{8x^3y^3}{(x^2+y^2)^3}$$

  and it may be easily shown (by putting $y=mx$) that

  $$\lim_{(x,y)\to(0,0)} f_{yx}(x,y)\neq 0=f_{yx}(0,0)$$

  so that f_{yx} is not continuous at $(0,0)$, i.e., the conditions of Schwarz's theorem are not satisfied.

  Let us now show that the conditions of Young's theorem are also not satisfied.

  Now

  $$f_{xx}(0,0)=\lim_{x\to0}\frac{f_x(x,0)-f_x(0,0)}{x}=0$$

  Also f_x is differentiable at $(0,0)$ if

  $$f_x(h,k)-f_x(0,0)=f_{xx}(0,0)\cdot h+f_{yx}(0,0)\cdot k+h\phi+k\psi$$

  or

$$\frac{2hk^4}{\left(h^2+k^2\right)^2} = h\phi + k\psi$$

where ϕ, ψ tend to zero as $(h,k) \to (0,0)$.

Putting $h = \rho\cos\theta$ and $k = \rho\sin\theta$, and dividing by ρ, we get

$$2\cos\theta\sin^4\theta = \cos\theta\cdot\phi + \sin\theta\cdot\psi$$

and $(h,k)\to(0,0)$ is same thing as $\rho\to 0$ and θ is arbitrary. Thus proceeding to limits, we get

$$2\cos\theta\sin^4\theta = 0$$

which is impossible for arbitrary θ.

$\Rightarrow$ f_x is not differentiable at $(0,0)$

Similarly, it may be shown that f_y is not differentiable at $(0,0)$.

Thus the conditions of Young's theorem are also not satisfied but, as shown above,

$$f_{xy}(0,0) = f_{yx}(0,0).$$

Example 7. Show that the function

$$f(x,y) = \frac{xy\left(x^2-y^2\right)}{x^2+y^2}, \quad (x,y)\neq(0,0)$$

$$f(0,0) = 0$$

does not satisfy the conditions of Schwarz's theorem and

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

- It may be shown, as in example 15, that

$$f_{xy}(0,0) = 1, \ f_{yx}(0,0) = -1$$

so that

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

Now, for $(x,y)\neq(0,0)$ we have

$$f_x(x,y) = \frac{\left(x^2+y^2\right)y\left(3x^2-y^2\right) - 2x^2y\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} = \frac{y\left\{x^4+4x^2y^2-4^4\right\}}{\left(x^2+y^2\right)^2}$$

$$\therefore \ f_{yx}(x,y) = \frac{\left(x^2+y^2\right)^2\left\{x^4+12x^2y^2-5y^4\right\} - 4y^2\left(x^2+y^2\right)\left\{x^4+4x^2y^2-y^4\right\}}{\left(x^2+y^2\right)^4}$$

$$= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{\left(x^2+y^2\right)^3}$$

By putting $y = mx$ or $x = r\cos\theta, \ y = r\sin\theta$, it may be shown that

$$\lim_{(x,y)\to(0,0)} f_{yx}(x,y) \neq -1 = f_{yx}(0,0)$$

Thus f_{yx} is not continuous at $(0,0)$.

It may similarly be shown that f_{xy} is also not continuous at $(0,0)$.

Thus, the conditions of Schwarz's theorem are not satisfied.

Examples on Euler's theorem on homogeneous functions

Example:- 1 Verify Euler's theorem for the following functions

(i) $u = x\left(x^3 - y^3\right)/\left(x^3 + y^3\right)$

(ii) $u = axy + byz + czx$

$$u = \frac{x\left(x^3 - y^3\right)}{x^3 + y^3} = \frac{x.x^3\left(1 - \left(\frac{y}{x}\right)^3\right)}{x^3\left(1 + \left(\frac{y}{x}\right)^3\right)} = x.\phi\left(y/x\right) \qquad (1)$$

Clearly u is a homogenous function of degree 1 .

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u \qquad (2)$$

For L.H.S of (2). Let's find partial derivatives

$\because$ From (1) $\log u = \log x + \log\left(x^3 - y^3\right) - \log\left(x^3 + y^3\right)$ differentiating partially w.r.t x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{3x^2}{x^3 - y^3} - \frac{3x^2}{x^3 + y^3} \qquad (3)$$

For y; $\frac{1}{u}\frac{\partial u}{\partial y} = 0 - \frac{3y^2}{x^3 - y^3} - \frac{3y^2}{x^3 + y^3}$ (4)

Example:- (2) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 24$

Solution:- $\tan u = \frac{x^3 + y^3}{x + y} = x^2\,\phi\left(y/x\right)$

$\therefore$ $\tan u$ is a homogenous function of degree 2.

$\therefore$ By Euler's theorem $x\frac{\partial u}{\partial x}\left(\tan u\right) + y\frac{\partial u}{\partial y}\left(\tan u\right) = 2\tan u$

$$x\sec^2 u\frac{\partial u}{\partial x} + y.\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Example:- (3) If $u = \tan^{-1}\frac{x^3 + y^3}{x - y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ and find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Solution:- $\because$ $\tan u$ is a homogenous function of degree 2.

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x}(\tan u)+y\frac{\partial u}{\partial y}(\tan u)=2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \qquad (1)$$

Differentiating (1) w.r.t. x, y separately, we get $x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}+0=2\cos 2u\frac{\partial u}{\partial x}$

Multiplying (iii) by x and (iv) by y ,then adding columnuise, we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3\left(x^3+y^3\right)}{x^3-y^3}-\frac{3\left(x^3+y^3\right)}{\left(x^3+y^3\right)}=1+3-3$$

$\Rightarrow$ $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.u$ Euler's theorem is verified

(i) $\because$ u is a homogenous function in x, y, z of degree 2.

$$\therefore \quad \frac{\partial u}{\partial x}=ay+cz,\ \frac{\partial u}{\partial y}=ax+bz,\ \frac{\partial u}{\partial z}=by+cx$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\left(axy+byz+czx\right)=2u$$

Hence Euler's theorem is verified

Example:- (4) If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

Solution:- Here $\sin u=\frac{x^2+y^2}{x+y}=x\left\{\frac{1+\left(y/x\right)^2}{1+\left(y/x\right)}\right\}=x.\phi\left(y/x\right)$

$\therefore$ $\sin u$ is a homogenous function of degree 1.

$\therefore$ By Euler's theorem

$$x.\frac{\partial}{\partial x}(\sin u)+y.\frac{\partial}{\partial y}(\sin u)=1.\sin u\,.$$

$$x\cos u\frac{\partial u}{\partial x}+y.\cos u\frac{\partial u}{\partial y}=\sin u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u\,.$$

<u>Assignment</u>

<u>(Students are suggested to try to develop their skills to use what they've learnt above. All the questions are Show That kind of, So without giving detailed answers and leaving it on your practice.)</u>

Type-1

1. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exist, but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ does not, where

$$f(x,y)=\begin{cases} y+x\sin\left(\dfrac{1}{y}\right), \text{if} & y\neq 0 \\ 0, \qquad\qquad \text{if} & y=0 \end{cases}$$

2. Show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ exists, but the other repeated limit and the double limit do not exist at the origin, when

$$f(x,y)=\begin{cases} y\sin(1/x)+xy/\left(x^2+y^2\right), & x\neq 0 \\ 0, & x=0 \end{cases}$$

3. Show that the repeated limits exist but the double limit does not when $(x,y)\to(0,0)$:

(i) $f(x,y)=\dfrac{x-y}{x+y}$, (ii) $f(x,y)=\dfrac{x^2y^2}{x^4+y^4-x^2y^2}$

(iii) $f(x,y)=\begin{cases} \dfrac{x^3+y^3}{x-y}, & x\neq y \\ 0, & x=y \end{cases}$ (iv) $f(x,y)=\begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & x\neq y \\ 0, & x=y \end{cases}$

4 Show that the limit and the repeated limits exist when $(x,y)\to(0,0)$:

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Type-2

1. Show that the following functions are discontinuous at the origin:

(i) $f(x,y)=\begin{cases} \dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

(ii) $f(x,y)=\dfrac{x^4-y^4}{x^4+y^4},\ (x,y)\neq(0,0),\ f(0,0)=0$

(iii) $f(x,y)=\dfrac{\left(x^2y^2\right)}{\left(x^4+y^4\right)},\ (x,y)\neq(0,0),\ f(0,0)=0$

2. Show that the following functions are continuous at the origin:

(i) $f(x,y)=\dfrac{x^2y^2}{(x^2+y^2)},\ (x,y)\neq(0,0),\ f(0,0)=0$

(ii) $f(x,y)=\begin{cases}\dfrac{x^3y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

3. Show that the following functions are discontinuous at $(0,0)$.

(i) $f(x,y)=\begin{cases}\dfrac{x^2y}{x^3+y^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(ii) $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$

(iii) $f(x,y)=\dfrac{xy^3}{x^2+y^6},\ (x,y)\neq(0,0),\ f(0,0)=0$

4. Discuss the following functions for continuity at $(0,0)$.

(i) $f(x,y)=\begin{cases}\dfrac{x^2y}{x^3+y^3}, & x^2+y^2\neq 0\\ 0, & x+y=0\end{cases}$

(ii) $f(x,y)=\begin{cases}2xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

(iii) $f(x,y)=\begin{cases}0, & (x,y)=(2y,y)\\ \exp\left\{|x-2y|/\left(x^2-4xy+4y^2\right)\right\}, & (x,y)\neq(2y,y)\end{cases}$

5. Show that f has a removable discontinuity at $(2,3)$:

$$f(x,y)=\begin{cases}3xy, & (x,y)\neq(2,3)\\ 6, & (x,y)=(2,3)\end{cases}$$

Suitably redefine the function to make it continuous.

6. Show that the function f is continuous at the origin, where

$$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

7. Can the given functions be appropriately defined at $(0,0)$ in order to be continuous there?

(i) $f(x,y)=|x|^y$, (ii) $f(x,y)=\sin\dfrac{x}{y}$

(iii) $f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}$, (iv) $f(x,y)=x^2\log\left(x^2+y^2\right)$

Type-3

1. If $f(x,y)=x^3y+e^{xy^2}$, find f_x and f_y.

2. If $f(x,y)=xy\dfrac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)}$, when $x^2+y^2\neq 0$, and $f(0,0)=0$, show that

$f_x(x,0)=0=f_y(0,y)$

$f_x(0,y)=-y,\ f_y(x,0)=x$

3. If $f(x,y)=\begin{cases}\dfrac{x^2-xy}{x+y}, \text{ if } & (x,y)\neq(0,0)\\ 0, \text{ if } & (x,y)=(0,0)\end{cases}$, find $f_x(0,0)$ and $f_y(0,0)$

4. If $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$, show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including the origin.

5. If $f(x,y)=\begin{cases}xt\tan(y/x), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$, show that $xf_x+yf_y=2f$.

6. Calculate $f_x, f_y, f_x(0,0), f_y(0,0)$ for the following:

(i) $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & x\neq 0,\ y\neq 0\\ 0, & x=0=y\end{cases}$

(ii) $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, \text{ if } & x^2+y^2\neq 0\\ 0, \text{ if } & x=y=0\end{cases}$

7. Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & x^2+y^2\neq 0\\ 0, & x=0=y\end{cases}$$

possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

8. If $f(x,y)=\sqrt{|xy|}$, find $f_x(0,0), f_y(0,0)$

Type-4

1. Verify that $f_{xy} = f_{yx}$ for the functions:

   (a) $\dfrac{2x-y}{x+y}$, (b) $x\tan xy$, (c) $\cosh(y+\cos x)$, (d) x^y

   indicating possible exceptional points and investigate these points.

2. Show that $z=\log\left\{(x-a)^2+(y-b)^2\right\}$ satisfies $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$, except at (a,b).

3. Show that $z=x\cos(y/x)+\tan(y/x)$ satisfies $x^2z_{xx}+2xyz_{xy}+y^2z_{yy}=0$, except at points for which $x=0$.

4. Prove that $f_{xy} \neq f_{yx}$ at the origin for the function:

   $f(x,y)=x^2\tan^{-1}(y/x)-y^2\tan^{-1}(x/y),\ x\neq 0,\ y\neq 0$

   $f(x,y)=0$, elsewhere.

5. If $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$, show that

   $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=0$.

6. Examine for the change in the order of derivation at the origin for the functions:

   (i) $f(x,y)=e^x(\cos y+x\sin y)$

   (ii) $f(x,y)=\sqrt{x^2+y^2}\sin 2\phi$,

   where $f(0,0)=0$ and $\phi=\tan^{-1}(y/x)$,

   (iii) $f(x,y)=\left|x^2-y^2\right|$

7. Examine the equality of $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function:

   $f(x,y)=\left(x^2+y^2\right)\tan^{-1}(y/x),\ x\neq 0,\ f(0,y)=\pi y^2/2$

8. Given $u = e^x \cos y + e^y \sin z$, find all first partial derivatives and verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x};\ \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x};\ \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}$$

DIFFERENTIALS OF HIGHER ORDER

Note: This chapter deals with all basics which we use frequently in mathematics optional topics. Also here proper justifications are given. Students are suggested to read and just remember results only according to demand of exam.

TOTAL DIFFERENTIAL

If $u = f(x, y)$, then to show that $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$.

We have $u = f(x, y)$. Let

$u + \delta u = f(x + \delta x, y + \delta y)$.

Then $\delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\delta x + \frac{f(x, y+\delta y) - f(x, y)}{\delta y}\delta y.$$

Now in the limiting case when δx becomes indefinitely small,

$\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}$ becomes $\frac{\partial}{\partial x} f(x, y+\delta y)$,

and further when δy diminishes indefinitely,

$\frac{\partial}{\partial x} f(x+y+\delta y)$ becomes $\frac{\partial f(x, y)}{\partial x}$, i.e., $\frac{\partial u}{\partial x}$.

Also, when δy becomes indefinitely small,

$\frac{f(x, y+\delta y) - f(x, y)}{\delta y}$ becomes $\frac{\partial f(x, y)}{\partial y}$, i.e., $\frac{\partial u}{\partial y}$.

Therefore, when δx and δy diminish indefinitely, it follows from (1) that

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

This value of du is called the *total differential* of u with respect to x and y.

In general, if $u = f(x_1, x_2, ..., x_n)$, where $x_1, x_2, ..., x_n$ are independent variables, we can show that

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + ... + \frac{\partial u}{\partial x_n} dx_n.$$

TOTAL AND PARTIAL DIFFERENTIAL COEFFICIENTS

If $u = f(x, y)$, where *x* and *y* are function of a single variable *t*, we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

But $du = \frac{du}{dt} dt, dx = \frac{dx}{dt} dt$ and $dy = \frac{dy}{dt} dt$. Therefore

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

This value of du/dt is called the *total differential coefficient*.

In general, if $u = f(x_1, x_2, ..., x_n)$, where $x_1, x_2, ..., x_n$ are functions of *t*, we can show that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + ... + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}.$$

Similarly, if $u = f(x, y)$, where *x* and *y* are functions of two other variables t_1 and t_2, then we have

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t_1} \quad ....(1)$$

and $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t_2}. \quad ....(2)$$

These results can be extended to any number of variables.

CHANGE OF TWO INDEPENDENT VARIABLES

In this section, we consider the problem of changing two independent variables *x* and *y* to t_1 and t_2 with the help of relations

$$x = f_1(t_1, t_2) \text{ and } y = f_2(t_1, t_2). \quad ....(1)$$

Let $u = f(x, y)$. Then solving the equation (1) and (2) from previous segment ; $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ we get

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial u}{\partial t_1}\frac{\partial y}{\partial t_2} - \frac{\partial u}{\partial t_2}\frac{\partial y}{\partial t_1}}{\frac{\partial x}{\partial t_1}\frac{\partial y}{\partial t_2} - \frac{\partial x}{\partial t_2}\frac{\partial y}{\partial t_1}} \quad ....(2)$$

and $$\frac{\partial u}{\partial y} = \frac{\frac{\partial u}{\partial t_2}\frac{\partial y}{\partial t_1} - \frac{\partial u}{\partial t_1}\frac{\partial y}{\partial t_2}}{\frac{\partial x}{\partial t_1}\frac{\partial y}{\partial t_2} - \frac{\partial x}{\partial t_2}\frac{\partial y}{\partial t_1}}. \quad ....(3)$$

On the other hand, if the equations given by (1) are easily solvable for t_1 and t_2 in terms of x and y, say

$$t_1 = F_1(x, y) \text{ and } t_2 = F_2(x, y),$$

then we can find the values of

$$\frac{\partial t_1}{\partial x}, \frac{\partial t_1}{\partial y}, \frac{\partial t_2}{\partial x}, \frac{\partial t_2}{\partial y}$$

and substitute in the relations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} \quad ....(4)$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial y}. \quad ....(5)$$

The higher order partial derivatives of *u* with respect to *x* and *y* can be obtained by repeated application of the formulae (2), (3), (4) and (5).

The above formulae can be easily extended for more than two independent variables.

Caution. While changing two independent variables, one should be careful to detect which variables are to be regarded as constants in performing partial differentiation. For instance, to find the value of $\frac{\partial u}{\partial x}$, the variable *y* is to be regarded as a constant and not t_1 or t_2, since there is independence between *x* and *y*, and not between *x* and t_1 or *x* and t_2.

TRANSFORMATION FROM CARTESIAN TO POLAR CO-ORDINATES

If V is a function of *x* and *y*, $x = r\cos\theta$ and $y = r\sin\theta$, then to show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, ...(1)

we have $r = \left(x^2 + y^2\right)^{1/2}$ and $\theta = \tan^{-1}(y/x)$. ....(2)

From relations (1) and (2), we have

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r},\ \frac{\partial r}{\partial x} = \frac{x}{\left(x^2+y^2\right)^{1/2}} = \frac{x}{r} = \cos\theta,$$

$$\frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r},\ \frac{\partial r}{\partial y} = \frac{y}{\left(x^2+y^2\right)^{1/2}} = \frac{y}{r} = \sin\theta,$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y,\ \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r},$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta = x,\ \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.$$

Using these equations, we have

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta},$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta},$$

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r} = \frac{x}{r}\frac{\partial V}{\partial x} + \frac{y}{r}\frac{\partial V}{\partial y},$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta} = -y\frac{\partial V}{\partial x} + x\frac{\partial V}{\partial y}.$$

Therefore, we have the following equivalence of Cartesian and polar operators

$$\left.\begin{aligned} \frac{\partial}{\partial x} &\equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{d\theta}, \\ \frac{\partial}{\partial y} &\equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{aligned}\right\} \quad ....(3)$$

and
$$\left.\begin{aligned} r\frac{\partial}{\partial r} &\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \theta} &\equiv x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x} \end{aligned}\right\} \qquad \text{....(4)}$$

We now transform $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial^2 V}{\partial y^2}$ to polar co-ordinates. Using (3), we have

$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$=\cos\theta\left(\cos\theta\frac{\partial^2 V}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial V}{\partial \theta}-\frac{\sin\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta}\right)-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 V}{\partial \theta\partial r}-\sin\theta\frac{\partial V}{\partial r}-\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}-\frac{\sin\theta}{r}\frac{\partial^2 V}{\partial \theta^2}\right)$$

$$=\cos^2\theta\frac{\partial^2 V}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta}. \qquad \text{....(5)}$$

Also,

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$=\sin\theta\left(\sin\theta\frac{\partial^2 V}{\partial r^2}-\frac{\cos\theta}{r^2}\frac{\partial V}{\partial \theta}+\frac{\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta}\right)+\frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 V}{\partial \theta\partial r}+\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}+\frac{\cos\theta}{r}\frac{\partial^2 V}{\partial \theta^2}\right)$$

$$=\sin^2\theta\frac{\partial^2 V}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta}. \qquad \text{....(6)}$$

Adding (5) and (6), we obtain

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Exercise. Transform the following equation to polar co-ordinates

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0.$$

Note. It should be noted that $\frac{\partial r}{\partial x}$ denotes the partial derivative of r when it is expressed as a function of x and y. While $\frac{\partial x}{\partial r}$ denotes the partial derivative of x when it is expressed as a function of r and θ. Here it is worth mentioning that

$$\frac{\partial r}{\partial x}\frac{\partial x}{\partial r}\neq 1, \text{ i.e., } \frac{\partial r}{\partial x}\neq 1\Big/\frac{\partial x}{\partial r}.$$

Corollary. If V be a function of r alone, where $r^2=x^2+y^2$, then

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}.$$

TRANSFORMATION OF $\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}$ TO SPHERICAL POLAR CO-ORDINATES

From *three dimensional co-ordinate geometry*, we know that the transformation formulae from Cartesian to spherical polar co-ordinates are

$x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$.

Putting $r\sin\theta=u$, we have

$x=u\cos\phi$ and $y=u\sin\phi$.

Now using the result of past discussion, we obtain

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}.$$

Adding $\frac{\partial^2 V}{\partial z^2}$ to both the sides, we get

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}. \quad ....(1)$$

Now for z and u, we have

$z=r\cos\theta$ and $u=r\sin\theta$.

Therefore, again using the result of past discussion, we have

$$\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial u^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}. \quad ....(2)$$

Also, $\frac{\partial V}{\partial u}=\frac{\partial V}{\partial r}\frac{\partial r}{\partial u}+\frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial u}$, where $r^2=z^2+u^2, \theta=\tan^{-1}\left(\frac{u}{z}\right)$

$$=\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta},$$

so that $\frac{1}{u}\frac{\partial V}{\partial u}=\frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$

i.e., $\frac{1}{u}\frac{\partial V}{\partial u}=\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta}$. ....(3)

Substituting from (2) and (3) in (1), we get

$$\nabla^2 V=\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

ORTHOGONAL TRANSFORMATION OF $\nabla^2 V$

To show that the expression $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}$ transforms to $\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+\frac{\partial^2 V}{\partial \zeta^2}$ by changing to any other set of axes $O\xi, O\eta, O\zeta$, mutually at right angles, the origin being the same.

Let $(l_1,m_1,n_1),(l_2,m_2,n_2),(l_3,m_3,n_3)$ be the direction cosines of $O\xi, O\eta, O\zeta$, referred to Ox, Oy, Oz as axes. Then we know (from *three dimensional co-ordinate geometry*) that

$l_1^2+m_1^2+n_1^2=1,\ l_2^2+m_2^2+n_2^2=1, l_3^2+m_3^2+n_3^2=1,$ ....(1)

and $l_1l_2+m_1m_2+n_1n_2=0$ etc.

Again, since $(l_1,l_2,l_3),(m_1,m_2,m_3),(n_1,n_2,n_3)$ are the direction cosines of Ox,Oy,Oz, respectively, referred to $O\xi,O\eta,O\zeta$, as axes, we have

$l_1^2+l_2^2+l_3^2=1$ etc.

and $l_1m_1+l_2m_2+l_3m_3=0$ etc.

Now

$\xi=$ projection of the line joining $(0,0,0)$ and (x,y,z) on $O\xi$, i.e.,

$\xi=l_1x+m_1y+n_1z$.

Similarly, $\eta=l_2x+m_2y+n_2z$ and $\zeta=l_3x+m_3y+n_3z$.

Now $\dfrac{\partial V}{\partial x}=\dfrac{\partial V}{\partial \xi}\dfrac{\partial \xi}{\partial x}+\dfrac{\partial V}{\partial \eta}\dfrac{\partial \eta}{\partial x}+\dfrac{\partial V}{\partial \zeta}\dfrac{\partial \zeta}{\partial x}$

$$=l_1\frac{\partial V}{\partial \xi}+l_2\frac{\partial V}{\partial \eta}+l_3\frac{\partial V}{\partial \zeta}.$$

Therefore, $\dfrac{\partial^2 V}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial x}\right)$,

i.e., $\dfrac{\partial^2 V}{\partial x^2}=\left(l_1\dfrac{\partial}{\partial \xi}+l_2\dfrac{\partial}{\partial \eta}+l_3\dfrac{\partial}{\partial \zeta}\right)\left(l_1\dfrac{\partial V}{\partial \xi}+l_2\dfrac{\partial V}{\partial \eta}+l_3\dfrac{\partial V}{\partial \zeta}\right)$

$$=l_1^2\frac{\partial^2 V}{\partial \xi^2}+l_2^2\frac{\partial^2 V}{\partial \eta^2}+l_3^2\frac{\partial^2 V}{\partial \zeta^2}+2l_1l_2\frac{\partial^2 V}{\partial \xi\partial \eta}+2l_2l_3\frac{\partial^2 V}{\partial \eta\partial \zeta}+2l_3l_1\frac{\partial^2 V}{\partial \xi\partial \zeta}. \quad ....(3)$$

Similarly, $\dfrac{\partial^2 V}{\partial y^2}=m_1^2\dfrac{\partial^2 V}{\partial \xi^2}+\dots$ etc. ....(4)

and $\dfrac{\partial^2 V}{\partial z^2}=n_1^2\dfrac{\partial^2 V}{\partial \xi^2}+\dots$ etc. ....(5)

Adding (3), (4) and (5), and using (1) and (2), we get

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+\frac{\partial^2 V}{\partial \zeta^2}.$$

Example 1. If $x=r\cos\theta$ and $y=r\sin\theta$, show that

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0.$$

Solution.

From $x=r\cos\theta$ and $y=r\sin\theta$, we have $\theta=\tan^{-1}(y/x)$. Then

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$

so that $\dfrac{\partial^2\theta}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial\theta}{\partial x}\right) = -\dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{2xy}{\left(x^2+y^2\right)^2}$. ....(1)

Also, $\dfrac{\partial\theta}{\partial y} = \dfrac{1}{1+\left(y/x\right)^2}\cdot\dfrac{1}{x} = \dfrac{x}{x^2+y^2}$,

so that $\dfrac{\partial^2\theta}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial\theta}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right) = -\dfrac{2xy}{\left(x^2+y^2\right)^2}$. ....(2)

Adding (1) and (2), we get $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0$.

Example 2. If $x = r\cos\theta, y = r\sin\theta$, prove that the equation

$$xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) - \left(x^2-y^2\right)\frac{\partial^2 u}{\partial x\partial y} = 0$$

becomes $r\dfrac{\partial^2 u}{\partial r\partial\theta} - \dfrac{\partial u}{\partial\theta} = 0$.

Solution.
We know that

$r\dfrac{\partial}{\partial r} = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial\theta} = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$.

Therefore,

$$r\frac{\partial^2 u}{\partial r\partial\theta} = r\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial\theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right)$$

$$= x^2\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial y} - xy\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial y^2} - y^2\frac{\partial^2 u}{\partial y\partial x} - y\frac{\partial u}{\partial x}$$

$$= \left(x^2-y^2\right)\frac{\partial^2 u}{\partial x\partial y} - xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial\theta}, \text{ since } x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\theta},$$

or $r\dfrac{\partial^2 u}{\partial r\partial\theta} - \dfrac{\partial u}{\partial\theta} = -\left[xy\left(\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2}\right) - \left(x^2-y^2\right)\dfrac{\partial^2 u}{\partial x\partial y}\right] = 0$.

Example3. If $x = \xi\cos\alpha - \eta\sin\alpha, y = \xi\sin\alpha + \eta\cos\alpha$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2}$.

OR

Prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is invariant for change of rectangular axes.

Solution.
We have

$$\frac{\partial u}{\partial\xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\xi} = \cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y},$$

so that $\dfrac{\partial^2 u}{\partial \xi^2} = \dfrac{\partial}{\partial \xi}\left(\dfrac{\partial u}{\partial \xi}\right) = \left(\cos\alpha \dfrac{\partial}{\partial x} + \sin\alpha \dfrac{\partial}{\partial y}\right)\left(\cos\alpha \dfrac{\partial u}{\partial x} + \sin\alpha \dfrac{\partial u}{\partial y}\right)$,

i.e., $\dfrac{\partial^2 u}{\partial \xi^2} = \cos^2\alpha \dfrac{\partial^2 u}{\partial x^2} + 2\sin\alpha\cos\alpha \dfrac{\partial^2 u}{\partial x \partial y} + \sin^2\alpha \dfrac{\partial^2 y}{\partial y^2}$. ....(1)

Also, $\dfrac{\partial u}{\partial \eta} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \eta} = -\sin\alpha \dfrac{\partial u}{\partial x} + \cos\alpha \dfrac{\partial u}{\partial y}$,

so that $\dfrac{\partial^2 u}{\partial \eta^2} = \dfrac{\partial}{\partial \eta}\left(\dfrac{\partial u}{\partial \eta}\right) = \left(-\sin\alpha \dfrac{\partial}{\partial x} + \cos\alpha \dfrac{\partial}{\partial y}\right)\left(-\sin\alpha \dfrac{\partial u}{\partial x} + \cos\alpha \dfrac{\partial u}{\partial y}\right)$,

i.e., $\dfrac{\partial^2 u}{\partial \eta^2} = \sin^2\alpha \dfrac{\partial^2 u}{\partial x^2} - 2\sin\alpha\cos\alpha \dfrac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \dfrac{\partial^2 u}{\partial y^2}$. ....(2)

Adding (1) and (2), we get $\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

Example 4. If $u = c\cosh x \cos y, v = c \sinh x \sin y$, prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{c^2}{2}(\cosh 2x - \cos 2y)\left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right).$$

Solution.

We have

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x} = c\sinh x \cos y \frac{\partial V}{\partial u} + c\cosh x \sin y \frac{\partial V}{\partial v}$$

and $\dfrac{\partial V}{\partial y} = \dfrac{\partial V}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial V}{\partial v}\dfrac{\partial v}{\partial y} = -c\cosh x \sin y \dfrac{\partial V}{\partial u} + c\sinh x \cos y \dfrac{\partial V}{\partial v}$.

Therefore

$$\frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y} = c(\sinh x \cos y - i\cosh x \sin y)\frac{\partial V}{\partial u} + c(\cosh x \sin y + i\sinh x \cos y)\frac{\partial V}{\partial v}$$

$$= c\sinh(x - iy)\frac{\partial V}{\partial u} - \frac{c}{i}\sin(x - iy)\frac{\partial V}{\partial y}$$

$$= c\sinh(x - iy)\left(\frac{\partial V}{\partial u} - i\frac{\partial V}{\partial v}\right),$$

so that $\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = c\sinh(x - iy)\left(\dfrac{\partial}{\partial u} - i\dfrac{\partial}{\partial v}\right)$. ....(1)

Replacing *i* by -*i*, we get

$\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} = c\sinh(x + iy)\left(\dfrac{\partial}{\partial u} + i\dfrac{\partial}{\partial u}\right)$. ....(2)

Multiplying (1) and (2) we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = c^2 \sinh(x - iy)\sinh(x + iy)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$$

$$= \frac{c^2}{2}(\cosh 2x - \cosh 2iy)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$$

$$=\frac{c^2}{2}(\cosh 2x-\cos 2y)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right).$$

Now operating both sides on the function V, we get

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{c^2}{2}(\cosh 2x-\cos 2y)\left(\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}\right).$$

Example 5. Transform the equation

$$\frac{\partial^2 z}{\partial x^2}+2xy^2\frac{\partial z}{\partial x}+2\left(y-y^3\right)\frac{\partial z}{\partial y}+x^2y^2z=0$$

by the substitutions $x=uv, y=1/v$, and hence show that *z* is the same function of *u*, *v* as of *x*, *y*.

Solution.

Since $x=uv$ and $y=1/v$, we have $u=xy$ and $v=1/y$.

Therefore

$$\frac{\partial u}{\partial x}=y,\frac{\partial u}{\partial y}=x,\frac{\partial v}{\partial x}=0,\frac{\partial v}{\partial y}=-\frac{1}{y^2}. \qquad ....(1)$$

Now $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}$, using (1)

i.e., $\frac{\partial z}{\partial x}=\frac{1}{v}\frac{\partial z}{\partial u}$, since $y=1/v$, ...(2)

so that $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{1}{v}\frac{\partial}{\partial u}\left(\frac{1}{v}\frac{\partial z}{\partial u}\right)=\frac{1}{v^2}\frac{\partial^2 z}{\partial u^2}$. ....(3)

Also, $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}-\frac{1}{y^2}\frac{\partial z}{\partial v}$, using (1),

so that $y\frac{\partial z}{\partial y}=xy\frac{\partial z}{\partial u}-\frac{1}{y}\frac{\partial z}{\partial v}=u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}$. ....(4)

Using (2), (3) and (4) the given equation transforms into

$$\frac{1}{v^2}\frac{\partial^2 z}{\partial u^2}+2u\frac{1}{v}\left(\frac{1}{v}\frac{\partial z}{\partial u}\right)+2\left(1-\frac{1}{v^2}\right)\left(u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}\right)+u^2z=0$$

i.e., $\frac{\partial^2 z}{\partial u^2}+\left\{2u+2u\left(v^2-1\right)\right\}\frac{\partial z}{\partial u}-2v\left(v^2-1\right)\frac{\partial z}{\partial v}+u^2v^2z=0$,

i.e., $\frac{\partial^2 z}{\partial u^2}+2uv^2\frac{\partial z}{\partial u}+2\left(v-v^3\right)\frac{\partial z}{\partial v}+u^2v^2z=0$.

This equation is exactly similar to the given equation, which shows that *z* is the same function of *u*, *v* as of *x*, *y*.

Example 6. If the equation $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$ is satisfied when V is a function of *x* and *y*, show that it will also be satisfied when V is the same function of u and v, where

$$u=\frac{1}{2}\log\left(x^2+y^2\right) \text{ and } v=\tan^{-1}(y/x).$$

Solution.

Multiplying v by i and then adding to u, we have

$u+iv=\frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\left(y/x\right)=\log\left(x+iy\right)$,

so that $x+iy=e^{u+iv}=e^u e^{iv}=e^u\left(\cos v+i\sin v\right)$.

Equating real and imaginary parts, we get

$x=e^u\cos v$ and $y=e^u\sin v$.

Whence $\frac{\partial x}{\partial u}=e^u\cos v=x$, $\frac{dx}{dv}=-e^u\sin v=-y$

$\frac{\partial y}{\partial u}=e^u\sin v=y$, $\frac{\partial y}{\partial v}=e^x\cos v=x$.

Using these values of partial derivatives, we have

$$\frac{\partial V}{\partial u}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial u}=x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}.$$

and $\frac{\partial V}{\partial v}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial v}=-y\frac{\partial V}{\partial x}+x\frac{\partial V}{\partial y}$.

Therefore,

$$\frac{\partial^2 V}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial V}{\partial u}\right)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\left(x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}\right),$$

i.e., $\frac{\partial^2 V}{\partial u^2}=x^2\frac{\partial^2 V}{\partial x^2}+x\frac{\partial V}{\partial x}+2xy\frac{\partial^2 V}{\partial x\partial y}+y^2\frac{\partial^2 V}{\partial y^2}+y\frac{\partial V}{\partial y}$; ....(1)

and $\frac{\partial^2 V}{\partial v^2}=\frac{\partial}{\partial v}\left(\frac{\partial V}{\partial v}\right)=\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial V}{\partial x}+x\frac{\partial V}{\partial y}\right)$,

i.e., $\frac{\partial^2 V}{\partial x^2}=y^2\frac{\partial^2 V}{\partial x^2}-2xy\frac{\partial^2 V}{\partial x\partial y}-y\frac{\partial V}{\partial y}-x\frac{\partial V}{\partial x}+x^2\frac{\partial^2 V}{\partial y^2}$. ....(2)

Adding (1) and (2), we have

$$\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}=\left(x^2+y^2\right)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)=0, \text{ as } \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0.$$

Hence the given equation is also satisfied when V is the same function of u and v.

Example 7. If $f\left(x,y\right)$ has continuous partial derivatives of the first two orders, and $\left(x+y\right)=\left(u+v\right)^3, x-y=\left(u-v\right)^3$, show that

$$9\left(x^2-y^2\right)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)=\left(u^2-v^2\right)\left(\frac{\partial^2 f}{\partial u^2}-\frac{\partial^2 f}{\partial v^2}\right).$$

Solution.

We are given that

$x+y=u^3+3u^2v+3uv^2+v^3$ and $x-y=u^3-3u^2v+3uv^2-v^3$.

On addition and subtraction, these equations give

$x=u^3+3uv^2$ and $y=3u^2v+v^3$.

Whence $\frac{\partial x}{\partial u}=3\left(u^2+v^2\right)=\frac{\partial y}{\partial v}, \frac{\partial x}{\partial v}=6uv=\frac{\partial y}{\partial u}$.

Using these values of partial derivatives, we have

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}=3\left(u^2+v^2\right)\frac{\partial f}{\partial x}+6uv\frac{\partial f}{\partial y} \qquad ....(1)$$

and $$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}=6uv\frac{\partial f}{\partial x}+3\left(u^2+v^2\right)\frac{\partial f}{\partial y}. \qquad ....(2)$$

Relations (1) and (2) on addition and subtraction give

$$\frac{\partial}{\partial u}+\frac{\partial}{\partial v}=3\left(u^2+v^2+2uv\right)\frac{\partial}{\partial x}+3\left(u^2+v^2+2uv\right)\frac{\partial}{\partial y},$$

i.e., $$\frac{\partial}{\partial u}-\frac{\partial}{\partial v}=3(u+v)^2\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$$

and $$\frac{\partial}{\partial u}-\frac{\partial}{\partial y}=3(u-v)^2\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right).$$

Therefore,

$$(u+v)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)=3(u+v)^3\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)=3(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$$

and $$(u-v)\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)=3(u-v)^3\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)=3(x-y)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right).$$

Hence $$(u+v)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left\{(u-v)\left(\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\right)\right\}$$

$$=3(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left\{3(x-y)\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)\right\},$$

i.e., $$\left(u^2-v^2\right)\left(\frac{\partial^2 f}{\partial u^2}-\frac{\partial^2 f}{\partial v^2}\right)=9\left(x^2-y^2\right)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right).$$

Example 8. If $x=e^v\sec u, y=e^v\tan u$ and ϕ is a function of x and y, show that

$$\cos u\left(\frac{\partial^2\phi}{\partial u\partial v}-\frac{\partial\phi}{\partial u}\right)=xy\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\left(x^2+y^2\right)\frac{\partial^2\phi}{\partial x\partial y}.$$

Solution.

From the given relations, we have

$$\frac{\partial x}{\partial u}=e^v\sec u\tan u,\ \frac{\partial x}{\partial v}=e^v\sec u,$$

$$\frac{\partial y}{\partial u}=e^v\sec^2 u,\ \frac{\partial y}{\partial v}=e^v\tan u.$$

Using these values of partial derivatives, we get

$$\frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u}=e^v\sec u\tan u\frac{\partial\phi}{\partial x}+e^v\sec^2 u\frac{\partial\phi}{\partial y},$$

so that $$\cos u\frac{\partial\phi}{\partial u}=e^v\tan u\frac{\partial\phi}{\partial x}+e^v\sec u\frac{\partial\phi}{\partial y},$$

$$= y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y}. \qquad ....(1)$$

Also, $\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v} = e^v \sec u\frac{\partial\phi}{\partial x} + e^v \tan u\frac{\partial\phi}{\partial y}$

$$= x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}. \qquad ....(2)$$

Now using (1) and (2), we obtain

$$\cos u\frac{\partial^2\phi}{\partial u\partial v} = \cos u\frac{\partial}{\partial u}\left(\frac{\partial\phi}{\partial v}\right) = \left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right)$$

$$= xy\frac{\partial^2\phi}{\partial x^2} + y\frac{\partial\phi}{\partial x} + y^2\frac{\partial^2\phi}{\partial x\partial y} + x^2\frac{\partial^2\phi}{\partial x\partial y} + xy\frac{\partial^2\phi}{\partial y^2} + x\frac{\partial\phi}{\partial y}$$

$$= xy\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left(x^2 + y^2\right)\frac{\partial^2\phi}{\partial x\partial y} + y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y}$$

$$= xy\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left(x^2 + y^2\right)\frac{\partial^2\phi}{\partial x\partial y} + \cos u\frac{\partial\phi}{\partial u},$$

i.e., $\cos u\left(\frac{\partial^2\phi}{\partial u\partial v} - \frac{\partial\phi}{\partial u}\right) = xy\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left(x^2 + y^2\right)\frac{\partial^2\phi}{\partial x\partial y}$.

Example 9. If $x + y = 2e^\theta \cos\phi$ and $x - y = 2ie^\theta \sin\phi$, show that

$$\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial\phi^2} = 4xy\frac{\partial^2 V}{\partial x\partial y}$$

Solution.

On adding and subtracting, the given relations provide

$$x = e^\theta\left(\cos\phi + i\sin\phi\right) = e^\theta e^{i\phi} = e^{\theta+i\phi}$$

and $y = e^\theta\left(\cos\phi - i\sin\phi\right) = e^\theta e^{-i\phi} = e^{\theta-i\phi}$.

Whence $\frac{\partial x}{\partial\theta} = e^{\theta+i\phi} = x, \frac{\partial x}{\partial\phi} = ie^{\theta+i\phi} = ix$,

$$\frac{\partial y}{\partial\theta} = e^{\theta-i\phi} = y, \frac{\partial y}{\partial\phi} = -ie^{\theta-i\phi} = -iy.$$

Using these partial derivatives, we have

$$\frac{\partial V}{d\theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial\theta} = x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} \qquad ....(A)$$

and $\frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial\phi} = i\left(x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right)$. ....(B)

Therefore,

$$\frac{\partial^2 V}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial\theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right),$$

i.e., $\dfrac{\partial^2 V}{\partial \theta^2} = x^2 \dfrac{\partial^2 V}{\partial x^2} + x\dfrac{\partial V}{\partial x} + 2xy\dfrac{\partial^2 V}{\partial x \partial y} + y^2 \dfrac{\partial^2 V}{\partial y^2} + y\dfrac{\partial V}{\partial y}$, ...(1)

and $\dfrac{\partial^2 V}{\partial \phi^2} = \dfrac{\partial}{\partial \phi}\left(\dfrac{\partial V}{\partial \phi}\right) = i^2\left(x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}\right)\left(x\dfrac{\partial V}{\partial x} - y\dfrac{\partial V}{\partial y}\right)$,

i.e., $\dfrac{\partial^2 V}{\partial \phi^2} = -\left(x^2 \dfrac{\partial^2 V}{\partial x^2} + x\dfrac{\partial V}{\partial x} - 2xy\dfrac{\partial^2 V}{\partial x \partial y} + y^2 \dfrac{\partial^2 V}{\partial y^2} + y\dfrac{\partial V}{\partial y}\right)$ ....(2)

Adding (1) and (2), we get

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial \phi^2} = 4xy\frac{\partial^2 V}{\partial x \partial y}.$$

[**other way]**. From (A) and (B), we have

$$\frac{\partial V}{\partial \theta} + i\frac{\partial V}{\partial \phi} = 2y\frac{\partial V}{\partial y} \text{ and } \frac{\partial V}{\partial \theta} - i\frac{\partial V}{\partial \phi} = 2x\frac{\partial V}{\partial x}.$$

Therefore, $\dfrac{\partial^2 V}{\partial \theta^2} + \dfrac{\partial^2 V}{\partial \phi^2} = \left(\dfrac{\partial}{\partial \theta} + i\dfrac{\partial}{\partial \phi}\right)\left(\dfrac{\partial}{\partial \theta} - i\dfrac{\partial}{\partial \phi}\right)V$.

$= \left(2y\dfrac{\partial}{\partial y}\right)\left(2x\dfrac{\partial}{\partial x}\right)V = 4xy\dfrac{\partial^2 V}{\partial x \partial y}$.]

Example 10. If $u = f(x, y), x^2 = \xi\eta$ and $y^2 = \xi/\eta$, change independent variables to ξ, η in the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + 2y\frac{\partial u}{\partial y} = 0.$$

Solution.

From the given relations, we have

$\xi = xy$ and $\eta = x/y$.

Whence $\dfrac{\partial \xi}{\partial x} = y$, $\dfrac{\partial \xi}{\partial y} = x$, $\dfrac{\partial \eta}{\partial x} = \dfrac{1}{y}, \dfrac{\partial \eta}{\partial y} = -\dfrac{x}{y^2}$.

Using these partial derivatives, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = y\frac{\partial u}{\partial \xi} + \frac{1}{y}\frac{\partial u}{\partial \eta}$$

so that $x\dfrac{\partial}{\partial x} = xy\dfrac{\partial}{\partial \xi} + \dfrac{x}{y}\dfrac{\partial}{\partial \eta} = \xi\dfrac{\partial}{\partial \xi} + \eta\dfrac{\partial}{\partial \eta}$; ....(1)

also $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial y} = x\dfrac{\partial u}{\partial \xi} - \dfrac{x}{y^2}\dfrac{\partial u}{\partial \eta}$

so that $y\dfrac{\partial}{\partial y} = xy\dfrac{\partial}{\partial \xi} - \dfrac{x}{y}\dfrac{\partial}{\partial \eta} = \xi\dfrac{\partial}{\partial \xi} - \eta\dfrac{\partial}{\partial \eta}$. ....(2)

Subtracting (2) from (1), we get

$x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y} = 2\eta\dfrac{\partial}{\partial \eta}$. ....(3)

Now $\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)$

$$=x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)-y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)$$

$$=x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}-xy\frac{\partial^2 u}{\partial x\partial y}-xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}$$

$$=x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$=x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2y\frac{\partial u}{\partial y}+\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)$$

on adding and subtracting $y(\partial u/\partial y)$

Therefore,

$$x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2y\frac{\partial u}{\partial y}$$

$$=\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)-\left(x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)$$

$$=\left(2\eta\frac{\partial}{\partial \eta}\right)\left(2\eta\frac{\partial u}{\partial \eta}\right)-2\eta\frac{\partial u}{\partial \eta}, \text{ using (3)}$$

$$=2\eta\left(2\eta\frac{\partial^2 u}{\partial \eta^2}+2\frac{\partial u}{\partial \eta}\right)-2\eta\frac{\partial u}{\partial \eta}$$

$$=4\eta^2\frac{\partial^2 u}{\partial \eta^2}+4\eta\frac{\partial u}{\partial \eta}-2\eta\frac{\partial u}{\partial \eta}=4\eta^2\frac{\partial^2 u}{\partial \eta^2}+2\eta\frac{\partial u}{\partial \eta}.$$

Hence the transformed equation is

$$4\eta^2\frac{\partial^2 u}{\partial \eta^2}+2\eta\frac{\partial u}{\partial \eta}=0, \text{ i.e., } 2\eta\frac{\partial^2 u}{\partial \eta^2}+\frac{\partial u}{\partial \eta}=0.$$

PYQ Example: If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left(u^2+v^2\right)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$$

Solution: $u = e^x \cdot \cos y$

$$\frac{\partial u}{\partial x}=e^x.\cos y=u$$

$$\frac{\partial u}{\partial y}=-e^x.\sin y=-u$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial u}{\partial y}.=-e^x.\cos y=-u$$

$v = e^x \cdot \sin y$

$$\frac{\partial u}{\partial x}=e^x.\sin y=u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} = u$$
$$\frac{\partial u}{\partial y} = e^x.\cos y = u$$
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial y} = -e^x.\sin y = -u$$

$$\frac{\partial^2 f}{\partial x^2} = \left[\frac{\partial f}{\partial u,}.\frac{\partial^2 u}{\partial x2} + \frac{\partial^2 f}{\partial u^2}.\left(\frac{\partial u}{\partial x}\right)^2\right] + \left[\frac{\partial f}{\partial u}.\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $\frac{\partial^2 f}{\partial x^2} = \left[\frac{\partial f}{\partial u,}.\frac{\partial^2 u}{\partial x2} + \frac{\partial^2 f}{\partial u^2}.\left(\frac{\partial u}{\partial x}\right)^2\right] + \left[\frac{\partial f}{\partial u}.\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial v}{\partial y}\right)^2\right]$

Using above values, we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 u}{\partial u^2}\right) + \frac{\partial^2 f}{\partial u^2}.\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial f}{\partial v}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$
$$= \frac{\partial f}{\partial u}(u-u) + \frac{\partial^2 f}{\partial u^2}(u^2+v^2) + \frac{\partial f}{\partial u}(v-v) + \frac{\partial f}{\partial v^2}(u^2+v^2) = (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

PYQ Example If $u = Ae^{-gx}\,sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\left(\frac{n}{2\mu}\right)}$

Sol: $u = Ae^{-gx}\,sin\,(nt - gx)$, where A, g, n positive constants.

finding $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from give expression of u, we get

$$\frac{\partial u}{\partial t} = n\ Ae^{-gx}\cos(nt-gx),$$
$$\frac{\partial u}{\partial x} = A\left(-ge^{-gx}\cos(nt-gx) - ge^{-tx}\sin(nt-gx)\right) = -Age^{-gx}\left[\cos(nt-gx) + \sin(nt-gx)\right]$$
$$\therefore \frac{\partial^2 u}{\partial x^2} = -Ag\begin{pmatrix} e^{-gx}\left[\left(g\sin(nt-gx)\right) - g\cos(nt-gx)\right] \\ -ge^{-gx}\left[\cos(nt-gx) + \sin(nt-gx)\right] \end{pmatrix}$$
$$\frac{\partial^2 u}{\partial x^2} = -Ag^2e^{-gx}\left[\sin(nt-gx) - \cos(nt-gx) - \sin(nt-gx) - \cos(nt-gx)\right]$$
$$\frac{\partial^2 u}{\partial x^2} = 2Ag^2e^{-gx}\cos(nt-gx)$$

Substituting values of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from (ii) and (iii) in (i), we get

$$n\ Ae^{-gx}\ cos(nt - gx) = 2Ag^2e^{-gx}\ \mu[cos(nt - gx)$$
$$n = 2\mu g^2$$
$$\therefore g = \sqrt{\left(\frac{n}{2\mu}\right)}$$

PYQs - FUNCTIONS OF TWO VARIABLES since year 2009

LIMIT, CONTINUITY AND DIFFERENTIABILITY

Q1. Given that $f(x,y)=\left|x^2-y^2\right|$. Find $f_{xy}(0,0)$ and $f_{yz}(0,0)$. Hence show that $f_{xy}(0,0)=f_{yz}(0,0)$. **[2b UPSC CSE 2021]**

Q2. Let $f: D\left(\subseteq \mathbf{R}^2\right)\to \mathbf{R}$ be a function and $(a,b)\in D$. If $f(x,y)$ is continuous at (a,b) then show that the functions $f(x,b)$ and $f(a,y)$ are continuous at $x=a$ and at $y=b$ respectively.

[1b UPSC CSE 2019]

Q3. Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x-y}, & (x,y)\neq(1,-1),(1,1)\\ 0, & (x,y)=(1,1),(1,-1)\end{cases}$$

is continuous and differentiable at $(1,-1)$. **[1b P-2 UPSC CSE 2019]**

Q4. Let

$f(x,y)=xy^2$, if $y>0$

$=-xy^2$, if $y\le 0$

Determine which of $\dfrac{\partial f}{\partial x}(0,1)$ and $\dfrac{\partial f}{\partial y}(0,1)$ exists and which does not exist.

[3b UPSC CSE 2018]

Q5. Consider the function *f* defined by

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{where } x^2+y^2\neq 0\\ 0, & \text{where } x^2+y^2=0\end{cases}$$

Show that $f_{xy}\neq f_{yx}$ at $(0,0)$. **[2b 2018 P-2 IFoS]**

Q6. If

$$f(x,y)=\begin{cases}\dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases},$$

calculate $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at $(0,0)$. **[3c UPSC CSE 2017]**

Q7. Let

$$f(x,y)=\begin{cases}\dfrac{2x^4y-5x^2y^2+y^5}{\left(x^2+y^2\right)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Find a $\delta>0$ such that $\left|f(x,y)-f(0,0)\right|<.01$, whenever $\sqrt{x^2+y^2}<\delta$.

[3b UPSC CSE 2016]

Q8. For the function

$$f(x,y)=\begin{cases}\dfrac{x^2-x\sqrt{y}}{x^2+y}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Examine the continuity and differentiability. **[4d UPSC CSE 2015]**

Q9. Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Also, discuss the continuity of f_{xy} and f_{yx} at $(0,0)$. **[3b UPSC CSE 2013]**

Q10. Define a function *f* of two real variables in the *xy*-plane by

$$f(x,y)=\begin{cases}\dfrac{x^3\cos\frac{1}{y}+y^3\cos\frac{1}{x}}{x^2+y^2} & \text{for } x,y\neq 0\\ 0, & \text{otherwise}\end{cases}$$

Check the continuity and differentiability of *f* at $(0,0)$. **[1a UPSC CSE 2012]**

Q11. Find $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y}{x^3+y^3}$ if it exists. **[1c UPSC CSE 2011]**

Q12. Let $f:\mathbf{R}^2\to\mathbf{R}$ be defined as

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Is *f* continuous at $(0,0)$? Compute partial derivatives of *f* at any point (x,y) if exist.

[3b UPSC CSE 2009]

Q13. A function $f(x,y)$ is defined as follows:

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Show that $f_{xy}(0,0)=f_{yx}(0,0)$. **[2017 4b IFoS]**

Q14. Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$ given that

$$f(x,y)=\begin{cases}x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y} & \text{if } xy\neq 0\\ 0, & \text{otherwise}\end{cases}$$. **[3a 2017 P-2 IFoS]**

Q15. Examine if the function $f(x,y)=\dfrac{xy}{x^2+y^2},(x,y)\neq(0,0)$ and $f(0,0)=0$ is continuous at $(0,0)$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points other than origin. **[2016 1c IFoS]**

Q16. Examine the continuity of $f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y)\neq(0,0)\\ \dfrac{1}{2}, & (x,y)=(0,0)\end{cases}$ at the point $(0,0)$.

[3b 2016 P-2 IFoS]

Q17. Obtain $\dfrac{\partial^2 f(0,0)}{\partial x\partial y}$ and $\dfrac{\partial^2 f(0,0)}{\partial y\,\partial x}$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy\left(3x^2-2y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Also, discuss the continuity of $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at $(0,0)$.

[3b P-2 UPSC CSE 2014]

Q18. Let $f(x,y)=\begin{cases}\dfrac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 1, & \text{if } (x,y)=(0,0)\end{cases}$

Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ through $f(x,y)$ is not continuous at $(0,0)$.

[2b P-2 UPSC CSE 2012]

Q19. Show that the function defined by

$$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat. **[2011 1c IFoS]**

Q20. Let

$$f(x,y)=\begin{cases}\dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}, & \text{if } (x,y)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Show that:

(i) $f_{xy}(0,0)\neq f_{yx}(0,0)$

(ii) f is differentiable at $(0,0)$ **[2010 3c IFoS]**

<u>PARTIAL DERIVATIVES</u>

Q1. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, $x\neq y$ then show that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=\left(1-4\sin^2 u\right)\sin 2u$.

[3c P-2 UPSC CSE 2020]

Q2. If $u=u\left(y-z,z-x,x-y\right)$, then find the value of $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$.

If $u\left(x,y,z\right)=\dfrac{x}{y+z}+\dfrac{y}{z+x}+\dfrac{z}{x+y}$, then find the value of $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$.

[1b 2020 P-2 IFoS]

Q3. If

$$u=\sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$$

then show that $\sin^2 u$ is a homogeneous function of x and y of degree $-\dfrac{1}{6}$. Hence show that

$x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\tan u}{12}\left(\dfrac{13}{12}+\dfrac{\tan^2 u}{12}\right)$. **[4c(i) UPSC CSE 2019]**

Q4. If $f=f\left(u,v\right)$, where $u=e^x\cos y$ and $v=e^x\sin y$, show that

$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=\left(u^2+v^2\right)\left(\dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 f}{\partial v^2}\right)$. **[2018 3b IFoS]**

Q5. If $u\left(x,y\right)=\cos^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}, 0<x<1,\ 0<y<1$ then find the value of $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$.

[3c 2016 P-2 IFoS]

Q6. If the three thermodynamic variables P,V,T are connected by a relation, $f\left(P,V,T\right)=0$. Show that, $\left(\dfrac{\partial P}{\partial T}\right)_V\left(\dfrac{\partial T}{\partial V}\right)_P\left(\dfrac{\partial V}{\partial P}\right)_T=-1$. **[2012 1c IFoS]**

Q7. If $u=Ae^{-gx}\sin\left(nt-gx\right)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\dfrac{\partial u}{\partial t}=\mu\dfrac{\partial^2 u}{\partial x^2}$ then show that $g=\sqrt{\left(\dfrac{n}{2\mu}\right)}$. **[2012 1d IFoS]**

Q8. If

$u=x^2\tan^{-1}\left(\dfrac{y}{x}\right)-y^2\tan^{-1}\left(\dfrac{x}{y}\right)$ show that $x^2\dfrac{\partial u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=2u$.

[2012 P-2 IFoS]

Q9. If $f\left(x,y\right)$ is a homogeneous function of degree n in x and y, and has continuous first and second-order partial derivatives, then show that

(i) $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf$

(ii) $x^2\dfrac{\partial^2 f}{\partial x^2}+2xy\dfrac{\partial^2 f}{\partial x\partial y}+y^2\dfrac{\partial^2 f}{\partial y^2}=n\left(n-1\right)f$ **[4b UPSC CSE 2010]**

Solutions(hints) **https://www.youtube.com/live/Uj8jz3M6sLA?si=JzI8iY0cOYRWHGLz**
Note- Maximum PYQs are solved in examples itself. Here are those which are to be guided.

Ans1: Hint- we know that

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} = 0$$

$$\because\ f_x(0,k) = \lim_{h\to 0} f[(0+h,k) - f(0,0)]/h = 0$$

$$fx(0,0) = 0$$

Ans2- Hint
Repeated Limits

If a function *f* is defined in some neighbourhood of (a,b), then the limit

$$\lim_{y\to b} f(x,y),$$

if it exists, is a function of *x*, say $\phi(x)$. If then the limit $\lim_{x\to a}\phi(x)$ exists and is equal to λ, we write

$$\lim_{x\to a}\lim_{x\to b} f(x,y) = \lambda$$

and say that λ is a *repeated limit* of *f* as $y\to b,\ x\to a$.
If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y\to b}\lim_{x\to a} f(x,y) = \lambda' \text{ (say)}$$

when first $x\to a$, and then $y\to b$.
These two limits may or may not be equal.

Now let if function is continuous at some point (a,b). that means limit of function at (a,b) equals to f(a,b). So limit f(x,b) will also be equal to f(a,b), so this is continuous too.

<u>Theatrically Thinking</u>:

If some function $f(x,y)$ is continuous at (a,b)

$\therefore$ by definition

$$|f(x,y) - f(a,b)| < \varepsilon \text{ whenever } |x-a| < \delta,\ |y-b| < \delta$$

or $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ ....(1)

$\because$ (1) holds for arbitrary values y in nbd of (a,b)

$\therefore$ (1) will also hold for particular $y = b$

i.e. $\lim\limits_{(x,b)\to(a,b)} f(x,b) = f(a,b)$

$\Rightarrow \lim\limits_{x\to a} f(x,b) = f(a,b)$

$\Rightarrow f(x,b)$ is also continuous at $x = a$

Ans3 CSE 2019. Prove or disprove $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ where $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x-y} & ; x-y \neq 0 \\ 0 & ; x-y = 0 \end{cases}$

Steps (i)

$$\left|\frac{x^2-y^2}{x-y} - 0\right| = \left|\frac{r^2\left(\cos^2\theta - \sin^2\theta\right)}{r(\cos\theta - \sin\theta)}\right| = \left|r\cdot(\cos\theta + \sin\theta)\right| = \left|r\cos\theta + r\sin\theta\right|$$

$\boxed{|a+b| \le |a| + |b|}$

$$= \le |r\cos\theta| + |r\sin\theta| \le |r|\cdot|\cos\theta| + |r|\cdot|\sin\theta|$$

$$\le |r|\cdot 1 + |r|\cdot 1 \le 2|r| = 2\sqrt{x^2+y^2} < \varepsilon$$

Step (ii)

i.e. $\sqrt{x^2+y^2} < \dfrac{\varepsilon}{2}$; $|x-0| < \dfrac{\varepsilon}{2\sqrt{2}}$, $|y-0| < \dfrac{\varepsilon}{2\sqrt{2}}$; $|x-0| < \delta$, $|y-0| < \delta$

$$\delta = \frac{\varepsilon}{2\sqrt{2}}; x = \frac{\varepsilon}{2\sqrt{2}}, y = \frac{\varepsilon}{2\sqrt{2}}; \; x^2 = \frac{\varepsilon^2}{4\times 2}, y^2 = \frac{\varepsilon^2}{4\times 2}$$

$$\therefore x^2 + y^2 = \frac{\varepsilon^2}{4} \;\therefore \sqrt{x^2+y^2} = \frac{\varepsilon}{2} \quad \boxed{\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = 0}$$

Ans. Show that the limit exists at origin i.e. $(0,0)$

$$f(x,y) = \begin{cases} x\sin\left(\dfrac{1}{y}\right) + y\sin\left(\dfrac{1}{x}\right) & ; xy \neq 0 \\ 0 & ; \quad xy = 0 \end{cases}$$

Solution.

We want limit at $(0,0)$

$\therefore f(x,y)$ may approach to the value $0\times\sin\frac{1}{2}+0+\sin\frac{1}{2}=0=\ell$

$\therefore$ Now we want to check limit is 0 or not??

Step (i)

$$|f(x,y)-0|=\left|x\sin\frac{1}{y}+y\sin\frac{1}{x}\right|\le\left|x\sin\frac{1}{y}\right|+\left|y\sin\frac{1}{x}\right|$$

$$\le|x|\cdot\left|\sin\frac{1}{y}\right|+|y|\cdot\left|\sin\frac{1}{x}\right|\le|x|\cdot1+|y|\cdot1\le|x|+|y|<\varepsilon$$

Step (ii)

$$|x|<\frac{\varepsilon}{2},\ |y|<\frac{\varepsilon}{2}\Rightarrow|x-0|<\delta,\ |y-0|<\delta \text{ where } \delta=\frac{\varepsilon}{2}$$

Therefore $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$

Ans4 IFoS 2018. Consider the function $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\ne0 \\ 0, & x^2+y^2=0\end{cases}$

Find a $\delta>0$ such that $|f(x,y)-f(0,0)|<0.01$ whenever $x^2+y^2<\delta$

Solution.

Step (i)

$$|f(x,y)-f(0,0)|=\left|xy\frac{x^2-y^2}{x^2+y^2}-0\right|=\left|r^2\cos\theta\sin\theta\frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\right|$$

$$=\left|\frac{r^2}{4}\sin4\theta\right|<\frac{r^2}{4}\le\frac{x^2+y^2}{4}<0.01=\varepsilon$$

$$\frac{x^2}{4}<\frac{\varepsilon}{2},\frac{y^2}{2}\frac{\varepsilon}{2}\ x^2<2\varepsilon;\ |x|<\sqrt{2\varepsilon},\ |y|<\sqrt{2\varepsilon};\ \delta=\sqrt{2\varepsilon};\ \delta=\sqrt{2\times0.01}$$

Ans6 CSE 2016. Let $f(x,y)=\begin{cases}\dfrac{2x^4y-5x^2y^3+y^5}{(x^2+y^2)^2} & ;\ (x,y)-(0,0) \\ 0 & ;\ (x,y)=(0,0)\end{cases}$

Find a $\delta > 0$ s.t. $\left| f(x,y) - f(0,0) \right| < 0.01$ whenever $\sqrt{x^2+y^2} < \delta$

Solution.

$$\left| f(x,y) - f(0,0) \right| = \left| \frac{2r^4\cos^4\theta\sin\theta - 5r^5\cos^2\theta\sin^3\theta + r^5\sin^5\theta}{\left(r^2\cos^2\theta + r^2\sin^2\theta\right)^2} - 0 \right|$$

$x = r\cos\theta$, $y = r\sin\theta$

$$= \left| \frac{2r^5\left(\cos^4\theta\sin\theta\right) - 5\cos^2\theta\sin^3\theta + \sin^5\theta}{r^4 \cdot 1} \right|$$

$$\leq 2\left| r\cos^4\theta\sin\theta \right| + \left| -5r\cos^2\theta\sin^2\theta \right| + \left| r\sin^5\theta \right| \leq 2|r|\cdot 1 + 5|r|\cdot 1 + |r|\cdot 1 \leq 8|r|$$

Using $|a+b+c| \leq |a|+|b|+|c|$; $|a-b+c| \leq |a|+|b|+|c|$

$$\leq 8\sqrt{x^2+y^2} < \varepsilon = 0.01 \therefore \sqrt{x^2+y^2} < \frac{\varepsilon}{8} = \frac{0.01}{8} = \delta$$

CSE 2015. Discuss the limit of the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Solution.

Way 1:

$$\left| f(x,y) - 0 \right| = \left| \frac{x^2 - x\sqrt{y}}{x^2+y} \right| = \left| \frac{r^2\cos\theta - r^{3/2}\cos\theta\sqrt{\sin\theta}}{r^2\cos^2\theta + r\sin\theta} \right| = \left| \frac{r^2\left(\cos^2\theta - \frac{1}{r^{1/2}}\cos\theta\sqrt{\sin\theta}\right)}{r^2\left(\cos^2\theta + \frac{1}{r}\sin\theta\right)} \right|$$

$\because r$ is present in denominator

When $\therefore (x,y) \to (0,0) \Rightarrow r^2 = x^2+y^2 \;\; \therefore r \to 0 \;\; \therefore \left| f(x,y) - 0 \right|$

May tend to infinity $\Rightarrow \left| f(x,y) - 0 \right| > \varepsilon$ when $|x-0| < \delta$, $|y-0| < \delta$

$\therefore \lim\limits_{(x,y)\to y(0,0)} f(x,y)$ does not exist.

Exam Point

If after implications $\left|f(x,y)-\ell\right|$ has either r in the denominator of $\sin\theta, \cos\theta$ in the denominator

then $\left|f(x,y)-\ell\right| > \varepsilon$

e.g. $\dfrac{1}{|\sin\theta|}$; $\dfrac{1}{1/2} = 2$, $\dfrac{1}{1/2000} = 2000, \ldots\ldots$ $\therefore \dfrac{1}{\sin\theta}\, may \to \infty$

Way 2:

$$\lim_{(x,y)\to(0.0)} \frac{x^2 - x\sqrt{y}}{x^2+y} = \lim_{(x,m,x^2)} \frac{x^2 - x\sqrt{m}\cdot x}{x^2+mx^2} = \frac{1-\sqrt{m}}{1+m}$$

$\therefore$ limit does not exist

CSE 2009. Let $f:\mathbf{R}^2 \to \mathbf{R}^2$ be defined as

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, \ if \ (x,y) \neq (0,0) \\ 0 \quad \text{if} \quad (x,y) = (0,0) \end{cases}$$

Discuss the continuity if $f(x,y)$ at $(0,0)$

Solution.

If $f(x,y)$ is continuous at $(0,0)$

Then $\left|f(x,y) - f(0,0)\right| < \varepsilon$ whenever $|x-0| < \delta$, $|y-0| < \delta$ ….(1)

$$\therefore \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{r\cos\theta\cdot r\sin\theta}{r\sqrt{\cos^2\theta+\sin^2\theta}}\right| = |r\cos\theta\sin\theta| = \left|\frac{r}{2}\sin 2\theta\right| \le \left|\frac{r}{2}\right| = \frac{1}{2}\sqrt{x^2+y^2} < \varepsilon$$

$$\sqrt{x^2+y^2} < 2\varepsilon \,;\; x^2+y^2 < 4\varepsilon^2 \,;\; x^2 < \frac{4\varepsilon^2}{2} \,,\; y^2 < \frac{4\varepsilon^2}{2} \Rightarrow |y| < \sqrt{2}\varepsilon$$

Therefore (1) holds well $\therefore$ given function is continuous at $(0,0)$

UPSC CSE 2015. Check the continuity of $f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y^2} \ ; (x,y) \neq (0,0) \\ 0 \quad , \quad (x,y) = (0,0) \end{cases}$

Solution.

$\because f(x,y)$ may be undefined or infinitely at $x=0, y=0$ so we need to check continuity of $f(x,y)$ at $(0,0)$ only. At Rest of the points in $\mathbf{R}\times\mathbf{R}$ plane, $f(x,y)$ will be continuous.

$\Rightarrow$ Let's check continuity at $(0,0)$

$\because$ for continuity $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

But we have already shown in previous example that

$$\lim_{(x,y)\to(0.0)} \frac{x^2 - x\sqrt{y}}{x^2+y} = \lim_{(x,m,x^2)} \frac{x^2 - x\sqrt{m}\cdot x}{x^2+mx^2} = \frac{1-\sqrt{m}}{1+m}$$

$\therefore$ limit does not exist

Function cannot be continuous.

CSE 2019. Let $f(x,y) = \begin{cases} xy^2 : \text{if } y>0 \\ -xy^2 : \text{if } y \le 0 \end{cases}$

Determine which of $\left(\frac{\partial f}{\partial x}\right)_{(0,1)}$ and $\left(\frac{\partial f}{\partial y}\right)_{(0,1)}$ exists and which does not exist.

Solution.

We know that

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\therefore \left(\frac{\partial f}{\partial x}\right)_{(0,1)} = \lim_{h\to 0} \frac{f(0+h,1) - f(0,1)}{h} = \lim_{h\to 0} \frac{f(h,1) - f(0,1)}{h} = \lim_{h\to 0} \frac{h\times 1^2 - 0\times 1^2}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1$$

$\because y=1, \therefore y>0, \therefore f(x,y) = xy^2$

$\therefore \left(\frac{\partial f}{\partial x}\right)_{(0,1)}$ exists and equal to 1.

$$\therefore \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

$$\therefore \left(\frac{\partial f}{\partial y}\right)_{(0,1)} = \lim_{h\to 0} \frac{f(0,1+h) - f(0,1)}{h} = \lim_{h\to 0} \frac{0\times(1+h)^2 - 0\times 1^2}{h} = 0$$

$\therefore \left(\dfrac{\partial f}{\partial y}\right)_{(0,1)}$ also exists and is equal to 0.

CSE 2009. Let $f:\mathbf{R}^2 \to \mathbf{R}$ be defined as

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & ;\ \text{if } (x,y)\neq(0,0)\\ 0 & ;\ \text{if } (x,y)=(0,0)\end{cases}$$

Is f continuous at $(0,0)$? Compute partial derivatives of f at any point (x,y) if exist.

Solution.

$\because \lim\limits_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$ [From previous question]

$\therefore f(x,y)$ is continuous at $(0,0)$

$$\left(\frac{\partial f}{\partial x}\right)_{(x,y)}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{\dfrac{(x+h)\cdot y}{\sqrt{(x+h)^2+y^2}}-\dfrac{xy}{\sqrt{x^2+y^2}}}{h}:\frac{0}{0}$$

$\therefore$ Let's apply L'Hospital rule [w.r.t. h]

$$=\lim_{h\to 0}\frac{\dfrac{\left(\sqrt{(x+h)^2+y^2}\right)\times y-(x+h)y\times\dfrac{1}{2\sqrt{(x+h)^2+y^2}}\times 2(x+h)}{(x+h)^2+y^2}-0}{1}$$

$$\frac{\sqrt{\left(x^2+y\right)}\times y-\dfrac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{\left(x^2+y^2\right)y-x^2}{\left(x^2+y^2\right)^{3/2}}$$

CSE 2014. Obtain $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy\left(3x^2-2y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0\ , & (x,y)=(0,0)\end{cases}$$

Also discuss the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)_{(0,0)}$$

$$= \lim_{h\to\infty} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} - \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{h\cdot k\left(\frac{3h^2 - 2k^2}{h^2 + k^2}\right) - 0 - 0}{k}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{h\cdot\left(3h^2 - 2\times 0^2\right)}{h^2}\right] = 3$$

Similarly, we can find

HINT: $\left(\frac{\partial^2 f}{\partial y \partial x}\right)_{(0,0)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]_{(0,0)}$

Let's check continuity of $\frac{\partial^2 f}{\partial y \partial x}$

$$g(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \begin{cases} \phi & :(x,y) \neq (0,0) \\ 3: & (x,y) = (0,0) \end{cases}$$

$$\because \frac{\partial f}{\partial x} = \frac{\left(x^2+y^2\right)\left(9yx^2 - 2y^3 - xy\left(3x^2 - 2y^2\right)\cdot 2x\right)}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \ldots\ldots\ldots..$$

Now check continuity of $g(x,y)$ at $(0,0)$

IFoS 2016. Examine the continuity of $f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, (x,y) \neq (0,0) \\ 1/2\ ; (x,y) = (0,0) \end{cases}$

At the point $(0,0)$

Solution.

Putting $x+2y=t$ for $(x,y)\to(0,0)\Rightarrow t\to 0$ [Single variable function]

$$f(t)=\begin{cases}\dfrac{\sin^{-1}t}{\tan^{-1}2t}\ ;\ t\neq 0\\ 1/2\ ;\ t=0\end{cases}$$

$$\because\ \lim_{t\to 0} f(t)=\lim_{t\to 0}\frac{\sin^{-1}t}{\tan^{-1}t}=\lim_{t\to 0}\frac{\frac{1}{\sqrt{1-t^2}}}{1/(1+t^2)}=1$$

$\because\ \lim_{t\to 0} f(t)=f(0)\ \therefore\ f(t)$ i.e. $f(xy)$ is not continuous at $(0,0)$

IFoS 2010. Let $f(x,y)=\begin{cases}\dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}\ ;\ \text{if } (x,y)\to(0,0)\\ 0\quad ,\quad \text{if}\quad (x,y)=(0,0)\end{cases}$

Show that (i) $f_{xy}(0,0)\neq f_{yx}(0,0)$

(ii) f is differentiable at $(0,0)$

Solution.

$$f_{xy}(0,0)=\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(0,0)}=\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]_{(0,0)}$$

$$=\lim_{h\to 0}\frac{f_y(0+h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{1}{h}\left[\lim_{k\to 0}\frac{f(h,0+k)-f(h,0)}{k}-\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\lim_{k\to 0}\frac{\frac{h\cdot k\left(h^2-k^2\right)}{h^2+k^2}-0}{k}-\lim_{k\to 0}\frac{0-0}{k}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{h\cdot\left(h^2-0\right)}{h^2+0}-0\right]=\lim_{h\to 0}\frac{h^3}{h^3}=1\qquad\ldots.(1)$$

$$f_{yx}(0,0)=\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]_{(0,0)}$$

$$=\lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h}=\lim_{h\to 0}\frac{1}{h}\left[\lim_{k\to 0}\frac{f(k,h)-f(0,h)}{k}-\lim_{k\to 0}\frac{f(k,0)-f(0,0)}{k}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\frac{k \cdot h\left(k^2 - h^2\right)}{k^2 + h^2} - 0}{k} \right] - 0 = \lim_{h \to 0} \frac{-h^3}{h} = -1 \qquad \ldots.(2)$$

From (1) and (2)

$f_{xy}(0,0) \neq f_{yx}(0,0)$

IFoS 2011. Show that the function $f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$ is discontinuous at the origin but posseses partial derivatives f_x and f_y there at

Solution.

Mental Exercise:

Let $y = x - mx^3$ be the path through which this (x, y) approaches to origin, then. Clearly $y \to 0$ when $x \to 0$. Then,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x \to 0} \frac{x^3 + \left(x - mx^3\right)^3}{x - x + mx^3}$$

$$\lim_{x \to 0} \frac{x^3 + x^3 - 3x^2 \cdot n^2 x^3 + 3x \cdot m^2 x^6 - m^3 x^9}{mx^3}$$

$$= \lim_{x \to 0} \frac{2x^3 - 3mx^5 + 3m^2 x^7 - m^3 x^9}{mx^3}$$

$$= \frac{2}{m}$$

$\Rightarrow$ It approaches to different values depending on the value of m.

$\Rightarrow$ The function is discontinuous at the origin.

Thinking by another approach:

Let's check $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$?

$\varepsilon - \delta$ Definition:

$$\left| f(x,y) - f(0,0) \right| = \left| \frac{x^3 + y^3}{x - y} - 0 \right| = \left| \frac{r^3\left(\cos^3\theta + \sin^3\theta\right)}{r\left(\cos\theta - \sin\theta\right)} \right| > \varepsilon$$

$\because$ $\cos\theta - \sin\theta$ is available in denominator [for some $\varepsilon > 0$]

$\therefore$ Not continuous at $(0,0)$ for $|x - 0| < \delta, |y - 0| < \delta$

For partial derivatives:

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^3+0^3}{h-0}-0}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{0^3+h^3}{0-h}}{h}=0$$

UPSC CSE 2012. Let $f(x,y)=\begin{cases}\dfrac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases}$

Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$

$$\left(\frac{\partial f}{\partial x}\right)_{(0,0)}=\lim_{h\to\infty}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^2}{h^2}-1}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

$$\left(\frac{\partial f}{\partial y}\right)_{(0,0)}=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^2}{h^2}-1}{h}=0$$

$$\left|f(x,y)-f(0,0)\right|=\left|\frac{(r\cos\theta+r\sin\theta)^2}{r^2\left(\cos^2\theta+\sin^2\theta\right)}\right|=\left|\frac{(\cos\theta+\sin\theta)^2}{1}\right|\geq 2 \text{ [may be]}$$

If we choose $\varepsilon=\frac{1}{2}$

We get $\left|f(x,y)-f(0,0)\right|>\varepsilon$ for $|x-0|<\delta$, $|y-0|<\delta$

Where δ towards on ε

$\therefore f(x,y)$ is not continuous at $(0,0)$

UPSC CSE 2014. Obtain $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy\left(3x^2-2y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Also discuss the continuity of $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at $(0,0)$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)_{(0,0)} = \lim_{h\to\infty} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} - \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{h\cdot k\left(\frac{3h^2 - 2k^2}{h^2 + k^2}\right) - 0 - 0}{k}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{h\cdot\left(3h^2 - 2\times 0^2\right)}{h^2}\right] = 3$$

Similarly we can find

$$\left(\frac{\partial^2 f}{\partial y \partial x}\right)_{(0,0)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]_{(0,0)}$$

Let's check continuity of $\frac{\partial^2 f}{\partial y \partial x}$

$$g(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \begin{cases} \quad :(x,y) \neq (0,0) \\ 3:\ (x,y) = (0,0) \end{cases}$$

$$\because \frac{\partial f}{\partial x} = \frac{\left(x^2+y^2\right)\left(9yx^2 - 2y^3 - xy\left(3x^2 - 2y^2\right)\cdot 2x\right)}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} =$$

Now check continuity of $g(x,y)$ at $(0,0)$

UPSC IFoS 2016. Examine if the function $f(x,y) = \frac{xy}{x^2+y^2}, (x,y) \neq (0,0)$ and $f(0,0) = 0$ is continuous at $(0,0)$ Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at points other than main.

Solution.

Method I

$$\left|f(x,y) - f(0,0)\right| = \left|\frac{xy}{x^2+y^2} - 0\right| = \left|\frac{r\cos\theta \cdot r\sin\theta}{r^2\left(\cos^2\theta + \sin^2\theta\right)}\right|$$

$$= |\cos\theta \sin\theta| = \left|\frac{1}{2}\sin 2\theta\right| \le \left|\frac{1}{2}\right| \cdot |\sin 2\theta|$$

$$\le \frac{1}{2} \cdot 1 \le \frac{1}{2}$$

But if we choose $\varepsilon = 1/\delta$

Then

$$|f(x,y) - f(0,0)| > \varepsilon$$

For $|x-0| < \delta$, $|y-0| < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) \ne f(0,0)$$

$\therefore f(x,y)$ is not continuous at $(0,0)$

Method II

$\because$ along $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,\sin x)\to(0,0)} \frac{x \cdot mn}{x^2 + m^2 x^2} = \frac{m}{1+m^2}$$ depending on m

$$\therefore \lim_{(x,y)\to(0,0)} f(xy)$$ does not exist

$\therefore f(x,y)$ is not continuous at $(0,0)$

UPSC SCE 2012. Define a function f of two real variables in the $x-y$ -plane by

$$f(x,y) = \begin{cases} \dfrac{x^3 \cos\frac{1}{y} + y^3 \cos\frac{1}{x}}{x^2+y^2} & \text{for } (x,y) \ne (0,0) \\ 0 \text{ , otherwise} \end{cases}$$

Check the continuity and differentiability of f at $(0,0)$

Solution.

$$|f(x,y) - f(0,0)| = \left| \frac{x^3 \cos\frac{1}{y} + y^3 \cos\frac{1}{x}}{x^2+y^2} - 0 \right|$$

$$=\left|\frac{r^3\cos^3\theta\cdot\cos\frac{1}{(r\sin\theta)}+r^3\sin^3\theta\cdot\cos\frac{1}{r\cos\theta}}{r^2\left(\cos^2\theta+\sin^2\theta\right)}\right|$$

$$=\left|\frac{r^3\left(\cos^3\theta\cdot\cos\frac{1}{(r\sin\theta)}+\sin^3\theta\cdot\cos\frac{1}{(r\cos\theta)}\right)}{r^2}\right|$$

$$=\left|r\left(\cos^3\theta\cdot\cos\frac{1}{r\sin\theta}+\sin^3\theta\cos\theta\frac{1}{2\times\cos\theta}\right)\right|$$

$$\le|r|\cdot\left|\cos^3\theta\cos\frac{1}{\sin\theta}+\sin^3\theta\cdot\cos\frac{1}{x\cos\theta}\right|$$

$$\le|r|\cdot(1+1)\le 2|2|\le 2\sqrt{x^2+y^2}<\varepsilon$$

Define a function f of two real variables in the $x-y$ plane by

$$f(x,y)=\begin{cases}\dfrac{x^3\cos\frac{1}{y}+y^3\cos\frac{1}{x}}{x^2+y^2} & \text{for } (x,y)\ne(0,0)\\ 0 \text{ , otherwise}\end{cases}$$

To check differentiability of $f(x,y)$ at $(0,0)$

Step (i)

Let $f(x,y)$ is differentiable $(0,0)$ $\therefore$ by definition

$$f(h,k)-f(0,0)=df=Ah+Bk+h\phi+k\cdot\psi \text{ where } A=\left(\frac{\partial f}{\partial x}\right)_{(0,0)},\ B=\left(\frac{\partial f}{\partial y}\right)_{(0,0)}$$

and ϕ,ψ both tend to zero as $(h,k)\to(0,0)$

$$A=\left(\frac{\partial f}{\partial x}\right)_{(0,0)}=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{h^3\cos\frac{1}{0}+0-0}{h}=\lim_{h\to0}h^2\cos\infty=0\times\text{ some finite value}$$

= 0

$$B=\left(\frac{\partial f}{\partial y}\right)_{(0,0)}=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{h^3\cos\frac{1}{0}}{h}=0$$

$\therefore$ From (1)

$$\frac{h^3\cos\frac{1}{k}+k^3\cos\frac{1}{h}}{h^2+k^2}-0=0\times h+0\times k+h\phi+k\cdot\psi$$

Putting $g=\delta\cos\theta, k=\delta\sin\theta$ $\therefore$ for arbitrary $\theta:\delta\to 0$ when $(h,k)\to(0,0)$ $\because \delta^2=h^2+k^2$

$$\not\delta\left[\cos\theta\cdot\cos\frac{1}{\delta\sin\theta}+\sin\theta\cdot\cos\frac{1}{\delta\cos\theta}\right]=\not\delta\left(\cos\theta\cdot\phi+\sin\theta\cdot\psi\right)$$

$\because$ R.H.S. $\to 0$ But L.H.S. may not be zero for arbitrary values at θ

$\therefore$ Our Assumption was wrong.

UPSC IFoS 2011. Show that the function $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$

Is discontinuous at the origin but possesses partial derivatives f_x and f_y there at

Solution.

Let's check $\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)=0$?

$\varepsilon-\delta$ Definition:

$$\left|f(x,y)-f(0,0)\right|=\left|\frac{x^3+y^3}{x-y}-0\right|=\left|\frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r\left(\cos\theta-\sin\theta\right)}\right|>\varepsilon$$

$\because$ $\cos\theta-\sin\theta$ is available in denominator [for some $\varepsilon>0$]

$\therefore$ Not continuous at $(0,0)$ for $|x-0|<\delta, |y-0|<\delta$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^3+0^3}{h-0}-0}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{0^3+h^3}{0-h}}{h}=0$$

Q6. If the three thermodynamic variables P,V,T are connected by a relation, $f\left(P,V,T\right)=0$. Show that, $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$. **[2012 1c IFoS]**

Ans. Hint;

As *f* is the function of three variables P,V,T. So the total differential of *f* is given by;

$$df = \frac{\partial f}{\partial P} d_P + \frac{\partial f}{\partial V} d_V + \frac{\partial f}{\partial T} d_T \ldots(1)$$

Now, it's given $f(P,V,T)=0 \Rightarrow df = 0 \ldots(2)$

So from (1) and (2), we have

$$\frac{\partial f}{\partial P} d_P + \frac{\partial f}{\partial V} d_V + \frac{\partial f}{\partial T} d_T = 0 \ldots(3)$$

Differentiating (3) w.r.t P, treating T as constant, we get

$$\frac{\partial f}{\partial P}.1 + \frac{\partial f}{\partial V}\left(\frac{\partial V}{\partial P}\right)_T + \frac{\partial f}{\partial T}.0 = 0$$

$$\Rightarrow -\frac{\partial f}{\partial P} = \frac{\partial f}{\partial V}\left(\frac{\partial V}{\partial P}\right)_T \Rightarrow \left(\frac{\partial V}{\partial P}\right)_T = -\frac{f_P}{f_V} \ldots(4)$$

Similarly, we get $\left(\frac{\partial P}{\partial T}\right)_V$ & $\left(\frac{\partial T}{\partial V}\right)_P$ and then multiplying three equations, we get the required proof as done.

CHAPTER- TAYLOR'S THEOREM, MAXIMA/MINIMA

Taylor's theorem
Statement:
If $f(x,y)$ *is a function which possesses continuous partial derivatives of order n in any domain of a point* (a,b), *and the domain is large enough to contain a point* $(a+h,\, b+k)$ *within it, then there exists a positive number* $0<\theta<1$, *such that*

$$f(a+h,\, b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

$$+\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n,$$

where $R_n=\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0<\theta<1$.

Just have a look at proof, it's an opportunity to revise your concepts.

Proof:
Let $x=a+th,\ y=b+tk\ \ldots(1)$, where $0\le t\le 1$ is a parameter, and
$f(x,y)=f(a+th,\, b+tk)=\phi(t)$
Since the partial derivatives of $f(x,y)$ of order n are continuous in the domain under consideration, $\phi^{(n)}(t)$ is continuous in $[0,1]$, and also

$$\phi'(t)=\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f;$$

using (1) for $h=dx/dt$ & $k=dy/dt$

$$\phi''(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

$\vdots$

$$\phi^{(n)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f$$

Therefore, by Maclaurin's theorem

$$\phi(t)=\phi(0)+t\phi'(0)+\frac{t^2}{2!}\phi''(0)+\ldots+\frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0)+\frac{t^n}{n!}\phi^{(n)}(\theta t),$$

where $0<\theta<1$.
Now on putting $t=1$, we get

$$\phi(1)=\phi(0)+\phi'(0)+\frac{1}{2!}\phi''(0)+\ldots+\frac{1}{(n-1)!}\phi^{(n-1)}(0)+\frac{1}{n!}\phi^{(n)}(0)$$

But $\phi(1)=f(a+h,b+k)$, and $\phi(0)=f(a,b)$

$$\phi'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)$$

$$\phi''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

$$\vdots$$

$$\phi^{(n)}(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k)$$

Exam Point: $\therefore\ f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n$$

where $R_n=\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k),\ 0<\theta<1$.

R_n is called the *remainder after n terms*, and the theorem, *Taylor's theorem with remainder* or *Taylor's expansion* about the point (a,b).

Note: If we put $a=b=0$; $h=x,k=y$, we get

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\ldots+\frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n-1} f(0,0)+R_n$$

where $R_n=\frac{1}{n!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f(\theta x,\theta y), 0<\theta<1$, is called the *Maclaurin's theorem* or *Maclaurin's expansion*.

Note: It is easy to see that Taylor's theorem can also be put in the form:

$$f(a+h,\ b+k)=f(a,b)+df(a,b)+\frac{1}{2!}d^2 f(a,b)+\ldots$$

$$+\frac{1}{(n-1)!}d^{n-1}f(a,b)+\frac{1}{n!}d^n f(a+\theta h,b+\theta k)$$

The reasoning in the general case of several variables is precisely the same and so the theorem can be easily extended to any number of variables.

Maxima and Minima of Several Variables

Let $f(x, y, z, ...)$ be a function of several independent variables $x, y, z, ...$.

Further, let f be continuous and finite for all values of $x, y, z, ...$ in the neighbourhood of their values $a, b, c, ...$ respectively.

Then the value $f(a, b, c, ...)$ is said to be a *maximum* or a *minimum* value of $f(x, y, z, ...)$ if $f(a+h, b+k, c+l, ...) - f(a, b, c, ...)$, maintains an invariant sign, negative or positive, for all small values, positive or negative, of $h, k, l, ...$.

Necessary condition for existence of maxima minima

- Let $f(x, y, z, ...)$ be a continuous and differentiable function of variables $x, y, z, ...$.

Expanding $f(x+h, y+k, z+l, ...)$ by extended Taylor's theorem, we have

$$f(x+h, y+k, z+l, ...) = \left[e^{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + ...} \right] f$$

$$= \left[1 + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + ... \right) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + ... \right)^2 + ... \right] f,$$

or

$f(x+h, y+k, z+l, ...) - f(x, y, z, ...) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + ... \right) +$ terms of second and higher orders. ....(1)

- Now for a maximum or a minimum, the difference

$f(x+h, y+k, z+l, ...) - f(x, y, z, ...)$ ....(2)

must preserve an invariant sign. Since $h, k, l, ...$ are sufficiently small, the first degree terms, namely, $h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + ...$, can be made to govern the sign of difference (2).

- Evidently, by changing the sign of each of $h, k, l, ...$, the sign of the first degree terms can be changed. Hence unless the first degree term on the right hand side of (1) is zero, the sign of the left hand side can be changed by changing the signs of $h, k, l, ...$.

- Therefore, a necessary condition for the existence of a maximum or a minimum is

$h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + ... = 0$. ....(3)

Since $h, k, l, ...$ are arbitrary and independent, (3) holds if

Exam Point $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, ...$ ....(4)

If there are n independent variables, we have n simultaneous equations in (4). Solving them for $x, y, z...$, we get the values of variables for which $f(x, y, z, ...)$ may have a maximum or a minimum value.

Note. The equations in (4) are necessary but not sufficient for the existence of maxima or minima. In the coming articles, we shall investigate sufficient conditions in the case of two and three independent variables.

• *An important theory of numbers to discuss for sufficient condition*

Case 1. Two Variables. The quadratic expression $I_2 \equiv ax^2 + 2hxy + by^2$ in two variables x and y can be written as

$$I_2 \equiv \frac{1}{a}\left(a^2x^2 + 2ahxy + aby^2\right)$$

$$= \frac{1}{a}\left\{(ax+hy)^2 + \left(ab-h^2\right)y^2\right\}.$$

Therefore, if $ab-h^2$ is positive, the sign of I_2 will be the same as that of a.

But if $ab-h^2$ is negative, the expression within the curly brackets may be positive or negative and therefore we cannot say anything about the sign of the expression I_2.

When $ab-h^2=0$, the case is doubtful.

Case 2. Three Variables. The quadratic expression in three variables x, y and z can be written as

$$I_3 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$= \frac{1}{a}\left(a^2x^2 + aby^2 + acz^2 + 2afyz + 2agzx + 2ahxy\right)$$

$$= \frac{1}{a}\left\{a^2x^2 + 2ax(gz+hy) + aby^2 + acz^2 + 2afyz\right\}$$

$$= \frac{1}{a}\{ax+hy+gz\}^2 + aby^2 + acz^2 + 2afyz - (gz+hy)^2$$

$$= \frac{1}{a}\left\{(ax+hy+gz)^2 + \left(ab-h^2\right)y^2 + 2(af-gh)yz + \left(ac-g^2\right)z^2\right\}.$$

Therefore, I_3 will have the same sign as a if

$\left(ab-h^2\right)y^2 + 2(af-gh)yz + \left(ac-g^2\right)z^2$ be positive,

i.e., $\left(ab-h^2\right)$ and $a\left(abc+2fgh-af^2-bg^2-ch^2\right)$ be positive.

But $ab-h^2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix}$

and $abc+2fgh-af^2-bg^2-ch^2=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Thus the expression I_3 will be positive if

$$a, \begin{vmatrix} a & h \\ h & b \end{vmatrix}; \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

are all positive, and will be negative if these are alternately negative and positive.

Lagrange's condition for maxima minima of functions of two variables
https://www.youtube.com/live/ySxl6vhPs9Y?si=OEv8x61zQJqvmpBw

Mentor's advice: Students can read below article just to have feel for how this maxima minima method works so. After all they have to remember working rule to solve questions in their exam.

- Let $f(x,y)$ be a function of two independent variables x and y.

Further, let

$$r=\frac{\partial^2 f}{\partial x^2}, s=\frac{\partial^2 f}{\partial x \partial y}, t=\frac{\partial^2 f}{\partial y^2} \text{ at } x=a, y=b.$$

- As a set of necessary conditions for a maximum or a minimum to exist at (a,b), we have
  $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ at (a,b).

- Then by extended Taylor's theorem, we have

$$f(a+h,b+k)-f(a,b)=\frac{1}{2!}\left(rh^2+2shk+tk^2\right)+R_3, \quad \text{....(1)}$$

where R_3 consists of terms of the third and higher orders of small quantities.

Now by taking h and k sufficiently small, the second degree terms in (1) can be made to govern to sign of the right hand side and therefore that of the left hand side also. If for all such values of h and k, these terms be of a fixed sign, then we shall have a maximum or a minimum for $f(x,y)$ according as that sign is negative or positive.

- Therefore, by Case 1, the expression $rh^2+2shk+tk^2$ will be of the same sign as r, provided $rt-s^2$ is positive.

Exam Point: Hence if $rt-s^2$ is positive, we have a maximum or minimum according as r is negative or positive.

This is Lagrange's condition for maxima and minima of a function of two variables.

• If $rt - s^2 < 0$, $f(x, y)$ is not a maximum or minimum at $x = a, y = b$. In this case, $f(a+h, b+k) - f(a, b)$ has one sign for some values of h and k (e.g., $k = 0, h \neq 0$), while it has another sign for other values h and k (e.g., $h = -ks/r, k \neq 0$). Such a point is called a ***saddle point.***

• If $rt - s^2 = 0$, further investigation is needed to determine whether $f(x, y)$ is a maximum or minimum at $x = a, y = b$.

• (CSE & IFoS) Remember This Procedure

Working Rule Exam Point. In order to determine a maximum or minimum of a function $u = f(x, y)$, we proceed as follows:

(i) Find $\partial f/\partial x$ and $\partial f/\partial y$, and equate them to zero.

Solve these simultaneous equations for x and y.

Let the roots be $x = a_i, y = b_i$, where $i = 1, 2, \ldots$.

Find $\partial^2 f/\partial x^2, \partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y^2$, and substitute in them by turns $x = a_i, y = b_i$. Calculate the value of $rt - s^2$ for each pair of values.

(ii) If $rt - s^2 > 0$ and $r < 0$ for a pair of roots, $f(x, y)$ is a maximum for this pair.

(iii) If $rt - s^2 > 0$ and $r > 0$, it is a minimum.

(iv) If $rt - s^2 < 0$, the function has a saddle point there.

(v) If $rt - s^2 = 0$, the case is undecided, and further investigation is necessary to decide it. We shall leave this case.

Lagrange's condition for maxima minima of functions of three variables

Let a, b, c be the values of x, y, z respectively obtained by solving the equations

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$

Also, let the corresponding values of

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y}$$

be denoted by A, B, C, F, G, H respectively.

Exam Point: If the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum value of $f(x, y, z)$ and if they be alternately negative and positive, we shall have a maximum value of $f(x, y, z)$.

Note- These are Lagrange's conditions for maxima and minima of a function of three variables. However, if none of these two conditions is satisfied, we shall, in general, have neither a maximum nor a minimum.

Lagrange's method of undetermined multipliers: finding maxima minima of a function subject to given conditrions
https://www.youtube.com/live/bb7Hk56fDdA?si=uxV7KAKzexEx2mnH

• The sufficient conditions that the function $f(x_1, x_2, ..., x_n)$ *has an extreme value subject to the side conditions* $\phi_r(x_1, x_2, ..., x_n) = 0, r = 1, 2, ..., m$ *are the same as for the function F to have an extreme value without any condition on the variables*.

• *Remember*

$$dF = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + ... + \frac{\partial F}{\partial x_n} dx_n,$$

and $d^2F = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + ... \frac{\partial}{\partial x_n} \right)^2 F$; where $F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + ...$

Then function F and hence f has a maximum value if $d^2F < 0$ and a minimum value if $d^2F > 0$.

Working Rule. Let us consider the function $f(x, y, z)$ subject to the conditions $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$. We follow the under mentioned steps to find maximum and minimum values of f :

Step 1. Put $F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2$.

Step 2. Find the equations

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda_1 \frac{\partial \phi_1}{\partial x} + \lambda_2 \frac{\partial \phi_2}{\partial x} = 0,$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda_1 \frac{\partial \phi_1}{\partial y} + \lambda_2 \frac{\partial \phi_2}{\partial y} = 0,$$

and $\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda_1 \frac{\partial \phi_1}{\partial z} + \lambda_2 \frac{\partial \phi_2}{\partial z} = 0$.

Solve these equations for x, y, z by using the given conditions,

Step 3. If $x = a, y = b, z = c$ be a solution in Step 2, then find the value of d^2F at this point and conclude that at this point,

f is maximum if $d^2F < 0$,

and f is minimum if $d^2F > 0$.

• It may be noted that

$$d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \right)^2 F$$

$$= \frac{\partial^2 F}{\partial x^2}dx^2 + \frac{\partial^2 F}{\partial y^2}dy^2 + \frac{\partial^2 F}{\partial z^2}dz^2 + 2\left(\frac{\partial^2 F}{\partial x \partial y}dxdy + \frac{\partial^2 F}{\partial y \partial z}dydz + \frac{\partial^2 F}{\partial z \partial x}dzdx \right)$$

$$= \sum \frac{\partial^2 F}{\partial x^2}dx^2 + 2\sum \frac{\partial^2 F}{\partial x \partial y}dxdy.$$

Categories of problems

Category 1. Problems in which the points of maxima/minima are obtained and the sign of d^2F is also determined at each of these points.

Category 2. Problems in which points of maxima/minima are not obtained and the sign of d^2F is not determined.

(i) Equation giving maximum and minimum values of the function is obtained.

(ii) A relation among x, y, z is obtained.

Category 3. Problems in which points of maxima/minima are not obtained but the sign of d^2F is determined.

(i) Equation giving maximum and minimum values of the function is obtained.

(ii) A relation among x, y, z is obtained.

Note. If u is positive, then the following rules may be helpful while solving the problems:

(i) u is maximum or minimum according as log u is maximum or minimum.

(ii) u is maximum or minimum according as $1/u$ is minimum or maximum.

Examples

Example 1. Discuss the maxima or minima of the function

$u = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$.

Solution.

• For maxima and minima, we must have

$\dfrac{\partial u}{\partial x} = y - \dfrac{a^3}{x^2} = 0$ and $\dfrac{\partial u}{\partial y} = x - \dfrac{a^3}{y^2} = 0$.

Solving these equations, we get

$x^2 y = a^3 = xy^2$, so that $x = y = a$.

• Now at the point given by $x = y = a$, we have

$r = \dfrac{\partial^2 u}{\partial x^2} = \dfrac{2a^3}{x^3} = 2, s = \dfrac{\partial^2 u}{\partial x \partial y} = 1, t = \dfrac{\partial^2 u}{\partial y^2} = \dfrac{2a^3}{y^3} = 2$.

• Therefore, $rt - s^2 = 2 \times 2 - 1^2 = 3 > 0$.

Now, since $r > 0$, by Lagrange's condition, the function u has a minima at $x = y = a$.

The minimum value of u is given by

$$u_{\min} = aa + \frac{a^3}{a} + \frac{a^3}{a} = 3a^2.$$

Example 2. Find the maximum or minimum values of

$y = x^2 y^2 (1 - x - y)$.

Solution.

For maxima and minima of u, we must have

$\dfrac{\partial u}{\partial x} = 2xy^2(1 - x - y) - x^2 y^2 = 0$, i.e., $xy^2(2 - 3x - 2y) = 0$

and $\dfrac{\partial u}{\partial y} = 2x^2 y(1 - x - y) - x^2 y^2 = 0$ i.e., $x^2 y(2 - 2x - 3y) = 0$.

• The factors x and y present in both these equations provide the solutions $x = 0, -\infty < y < \infty$ and $-\infty < x < \infty, y = 0$.

Further, the other factors give the equations

$2 - 3x - 2y = 0$ and $2 - 2x - 3y = 0$.

Solving these, we get $x = \dfrac{2}{5}, y = \dfrac{2}{5}$.

Now, $r = \dfrac{\partial^2 u}{\partial x^2} = 2y^2 - 6xy^2 - 2y^3$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = 4xy - 6x^2 y - 6xy^2$ and $t = \dfrac{d^2 u}{\partial y^2} = 2x^2 - 2x^3 - 6x^2 y$.

When $x = 0,\ -\infty < x < \infty$, we have $r = 2y^2(1 - y), s = 0$ and $t = 0$. Therefore, $rt - s^2 = 0$. So this case is doubtful and needs further investigation.

When $-\infty < x < \infty, y = 0$, we have $r = 0, s = 0$ and $t = 2x^2(1-x)$. Therefore, $rt - s^2 = 0$. So this case is also doubtful and needs further investigation.

When $x = \frac{2}{5}, y = \frac{2}{5}$, we have

$$r = 2\left(\frac{2}{5}\right)^2 - 6 \cdot \frac{2}{5}\left(\frac{2}{5}\right)^2 - 2\left(\frac{2}{5}\right)^3 = -\frac{24}{125},$$

$$s = 4 \cdot \frac{2}{5} \cdot \frac{2}{5} - 6\left(\frac{2}{5}\right)^2 \cdot \frac{2}{5} - 6 \cdot \frac{2}{5}\left(\frac{2}{5}\right)^2 = -\frac{16}{125}$$

and $t = 2\left(\frac{2}{5}\right)^2 - 2\left(\frac{2}{5}\right)^3 - 6\left(\frac{2}{5}\right)^2 \cdot \frac{2}{5} = -\frac{25}{125}$.

So, $rt - s^2 = \left(-\frac{24}{125}\right)\left(-\frac{24}{125}\right) - \left(-\frac{16}{125}\right)^2 = \frac{320}{15625}$, which is positive.

Now, since r is negative, by Lagrange's condition, the function u is maximum at $x = \frac{2}{5}, y = \frac{2}{5}$.

The maximum value of u is

$$u_{\max} = \left(\frac{2}{5}\right)^2 \left(\frac{2}{5}\right)^2 \left(1 - \frac{2}{5} - \frac{2}{5}\right) = \frac{16}{3125}.$$

Example 3. Discuss the maximum or minimum values of $u = x^3 + y^3 - 3axy$.

Solution.

For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay = 0, \text{ i.e., } x^2 - ay = 0 \quad \text{....(1)}$$

and $\frac{\partial u}{\partial y} = 3y^2 - 3ax = 0$, i.e., $y^2 - ax = 0$. ....(2)

Putting the value of x from (2) in (1), we obtain

$(y^2/a)^2 - ay = 0$, i.e., $y^4 - a^3y = 0$, i.e., $y(y^3 - a^3) = 0$, which gives $y = 0, a$.

• From (1), when $y = 0$, we get $x = 0$, and when $y = a$, we get $x = \pm a$.

But $x = -a, y = a$ do not satisfy (2). So we reject these values. Therefore, the only solutions of (1) and (2) are $x = 0, y = 0$ and $x = a, y = a$.

• Now $r = \frac{\partial^2 u}{\partial x^2} = 6x$, $s = \frac{\partial^2 u}{\partial x \partial y} = -3a$ and $t = \frac{\partial^2 u}{\partial y^2} = 6y$.

At $x = 0, y = 0$, we have $r = 0, s = -3a$ and $t = 0$. So, $rt - s^2 = -9a^2 < 0$.

Hence u is neither maximum nor minimum at $x = 0, y = 0$.

At $x = a, y = a$, we have $r = 6a, s = -3a$ and $t = 6a$. So,

$$rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 = 27a^2.$$

Since $rt-s^2$ is positive and r is positive or negative according as a is positive or negative, we have maximum or minimum according as a is negative or positive. The maximum or minimum value of u is $=a^3+a^3-3aaa=-a^3$.

Example 4. Find the maximum value of u, where $u=\sin x\sin y\sin(x+y)$.

Solution.

• Since the function u is periodic with period π for x and y both, it suffices to consider the values of x and y satisfying $0\le x<\pi, 0\le y\le\pi$

• For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x}=\sin y\left\{\sin x\cos(x+y)+\cos x\sin(x+y)\right\}=0,$$

i.e., $\sin y\sin(2x+y)=0$, ...(1)

and $\dfrac{\partial u}{\partial y}=\sin x\left\{\sin y\cos(x+y)+\cos y\sin(x+y)\right\}=0$,

i.e., $\sin x\sin(x+2y)=0$. ....(2)

• To find the values of x and y lying between 0 and π, and satisfying (1) and (2), we need to consider the following pairs of equations:

$x=0, y=0;\ 2x+y=\pi;\ 2x+y=2\pi,\ 2y+x=2\pi$.

Solving these, we get the following pairs of values of x and y between 0 and π:

$$x=0,\ y=0;\ x=\frac{1}{3}\pi,\ y=\frac{1}{3}\pi;\ x=\frac{2}{3}\pi,\ y=\frac{2}{3}\pi.$$

Now, $r=\dfrac{\partial^2 u}{\partial x^2}=2\sin y\cos(2x+y)$,

$$s=\frac{\partial^2 u}{\partial x\partial y}=\sin x\cos(x+2y)+\cos x\sin(x+2y)=\sin(2x+2y)$$

and $t=\dfrac{\partial^2 u}{\partial y^2}=2\sin x\cos(x+2y)$.

At $x=0, y=0$, we have $r=0, s=0, t=0$ so that $rt-s^2=0$.

Therefore, at the point $(0,0)$, the case is doubtful, and further investigation is needed.

At $x=\frac{1}{3}\pi, y=\frac{1}{3}\pi$, we have

$$r=2\sin\frac{1}{3}\pi\cos\pi=2\left(\frac{1}{2}\sqrt{3}\right)(-1)=-\sqrt{3},$$

$$s=\sin\frac{4}{3}\pi=\sin\left(\pi+\frac{1}{3}\pi\right)=-\sin\frac{1}{3}\pi=-\frac{1}{2}\sqrt{3}$$

and $t=2\sin\frac{1}{3}\pi\cos\pi=2\left(\frac{1}{2}\sqrt{3}\right)(-1)=-\sqrt{3}$.

So, $rt-s^2=\left(-\sqrt{3}\right)\left(-\sqrt{3}\right)-\left(-\frac{1}{2}\sqrt{3}\right)^2=3-\frac{3}{4}=\frac{9}{4}>0$. Also, since $r=-\sqrt{3}<0, u$ is maximum at $x=\frac{1}{3}\pi, y=\frac{1}{3}\pi$.

At $x=\frac{2}{3}\pi, y=\frac{2}{3}\pi$, we have

$r=2\sin\frac{2}{3}\pi\cos 2\pi=2\left(\frac{1}{2}\sqrt{3}\right)(1)=\sqrt{3}$,

$s=\sin\frac{8}{3}\pi=\sin\left(2\pi+\frac{2}{3}\pi\right)=\sin\frac{2}{3}\pi=\frac{1}{2}\sqrt{3}$,

and $t=2\sin\frac{2}{3}\pi\cos 2\pi=2\left(\frac{1}{2}\sqrt{3}\right)(1)$.

So, $rt-s^2=\left(\sqrt{3}\right)\left(\sqrt{3}\right)-\left(\frac{1}{2}\sqrt{3}\right)=3-\frac{3}{4}=\frac{9}{4}>0$. Also, since $r=\sqrt{3}>0, u$ is minimum at $x=\frac{2}{3}\pi, y=\frac{2}{3}\pi$.

Example 5. Discuss the maxima or minima of the function

$u=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+\cos(x+y)$.

Solution.

- The given function can be written in the form $u=\sin x+\sin y+\cos(x+y)$, which is more convenient for the purpose.

- For maxima or minima of u, we must have

$\frac{\partial u}{\partial x}=\cos x-\sin(x+y)=0$, ...(1)

and $\frac{\partial u}{\partial y}=\cos y-\sin(x+y)=0$, ...(2)

These equations give $\cos x=\cos y$, i.e., $x=y$. Putting $y=x$ in (1) (or in (2)), we get, $\cos x-\sin 2x=0$, i.e., $\cos x-2\sin x\cos x=0$,

i.e., $\cos(1-2\sin x)$, so that $\cos x=0, \sin x=\frac{1}{2}$.

Now $\cos x=0$ gives $x=2n\pi\pm\frac{1}{2}\pi$,

and $\sin x=\frac{1}{2}$ gives $x=n\pi+(-1)^n\frac{1}{6}\pi$, where n is any integer.

- Further, $r=\frac{\partial^2 u}{\partial x^2}=-\sin x-\cos(x+y)$,

$s=\dfrac{\partial^2 u}{\partial x \partial y}=-\cos(x+y)$, $t=\dfrac{\partial^2 u}{\partial y^2}=-\sin y-\cos(x+y)$.

When $x=y=2n\pi+\dfrac{1}{2}\pi$, we have

$r=0, s=1, t=0$ so that $rt-s^2=-1<0$.

When $x=y=2n\pi-\dfrac{1}{2}\pi$, we have

$r=2,\ s=1,\ t=2$ so that $rt-s^2=2\times 2-1=3>0$.

Since $r=2>0, u$ is minimum at $x=y=2n\pi-\dfrac{1}{3}\pi$.

Lastly, when $x=y=n\pi+(-1)^n\dfrac{1}{6}\pi$, we have

$r=-1,\ s=-\dfrac{1}{2},\ t=-1$ so that $rt-s^2=\dfrac{3}{4}>0$.

Since $rt-s^2$ is positive and r is negative, u is maximum at $x=y=n\pi+(-1)^n\dfrac{1}{6}\pi$.

Example 6. Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Solution.

• Let $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ be the co-ordinates of the vertices of the triangle and let the co-ordinates of the required point be (x,y).

If u denote the sum of the squares of the distances of (x,y) from these vertices, then

$u=\left\{(x-x_1)^2+(y-y_1)^2\right\}+\left\{(x-x_2)^2+(y-y_2)^2\right\}+\left\{(x-x_3)^2+(y-y_3)^2\right\}$.

For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x}=2(x-x_1)+2(x-x_2)+2(x-x_3)=0$$

and $\dfrac{\partial u}{\partial y}=2(y-y_1)+2(y-y_2)+(y-y_3)=0$.

Solving these equations, we have

$x=\dfrac{1}{3}(x_1+x_2+x_3)$ and $y=\dfrac{1}{3}(y_1+y_2+y_3)$

Now $r=\dfrac{\partial^2 u}{\partial x^2}=6$, $s=\dfrac{\partial^2 u}{\partial x \partial y}=0$ and $t=\dfrac{\partial^2 u}{\partial y^2}=6$, so that

$rt-s^2=36$, which is positive.

Since r is positive, by Lagrange's condition the function u is minimum at the point

$\left(\dfrac{x_1+x_2+x_3}{3},\dfrac{y_1+y_2+y_3}{3}\right)$; which is nothing but the centroid of the triangle.

For three variables

Example 1. Show that the function $u=(x+y+z)^3-3(x+y+z)-24xyz+a^3$ has a minimum at $(1,1,1)$ and maximum at $(-1,-1,-1)$.

Solution.

• For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x}=3(x+y+z)^2-3-24yz=0.$$

i.e., $8yz=(x+y+z)^2-1$, ...(1)

$$\frac{\partial u}{\partial y}=3(x+y+z)^2-3-24xz=0,$$

i.e., $8xz=(x+y+z)^2-1$, ...(2)

and $\frac{\partial u}{\partial z}=3(x+y+z)^2-3-24xy=0$,

i.e., $8xy=(x+y+z)^2=1$. ....(3)

From (1), (2) and (3), we have

$yz=xz=xy$, i.e., $x=y=z$.

Then (1) gives $8x^2=9x^2-1$, i.e., $x^2=1$ or $x=\pm1$.

Therefore, $(1,1,1)$ and $(-1,-1,-1)$ are the points of maxima or minima.

• Now to discuss maxima or minima, we have

$$A=\frac{\partial^2 u}{\partial x^2}=6(x+y+z)=\begin{cases}18 \text{ at} & (1,1,1)\\ -18 \text{ at} & (\text{-}1,\text{-}1,\text{-}1)\end{cases}$$

and $F=\frac{\partial^2 u}{\partial y\partial z}=6(x+y+z)-24x=\begin{cases}-6 \text{ at} & (1,1,1)\\ 6 \text{ at} & (\text{-}1,\text{-}1,1)\end{cases}$

By symmetry, we have

$A=B=C=18$ and $F=G=H=-6$ at $(1,1,1)$

and $A=B=C=-18$ and $F=G=H=6$ at $(-1,-1,-1)$.

Now we can see that the values of the expressions

$$A,\begin{vmatrix}A & H\\ H & B\end{vmatrix},\begin{vmatrix}A & H & G\\ H & B & F\\ G & F & C\end{vmatrix}$$

are $18, 288, 3456$ respectively at $(1,1,1)$

and $-18, 288, -3456$ respectively at $(-1,-1,-1)$.

Hence by Lagrange's conditions, there is a minimum at $(1,1,1)$ and maximum at $(-1,-1,-1)$.

Example 2. Find the maximum value of

$$u=\frac{xyz}{(a+x)(x+y)(y+z)(z+b)},$$

where all the letters denote positive real numbers.

Solution.

• Taking logarithms, we have

$$\log u=\log x+\log y+\log z-\log(a+x)-\log(x+y)-\log(y+z)-\log(z+b).$$

Differentiating partially with respect to x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}-\frac{1}{a+x}-\frac{1}{x+y}=\frac{ay-x^2}{x(a+x)(x+y)},$$

so that $\dfrac{\partial u}{\partial x}=\dfrac{(ay-x^2)u}{x(a+x)(x+y)}$. ....(1)

Similarly, $\dfrac{\partial u}{\partial y}=\dfrac{(zx-y^2)u}{y(x+y)(y+z)}$ and $\dfrac{\partial u}{\partial z}=\dfrac{(by-z^2)u}{z(y+z)(z+b)}$.

• Now for maxima and minima, we have

$$\frac{\partial u}{\partial x}=0,\frac{\partial u}{\partial y}=0,\frac{\partial u}{\partial z}=0,$$

so that $ay-x^2=0$, $zx-y^2=0$, $by-z^2=0$.

These equations give

$$\frac{x}{a}=\frac{y}{x}=\frac{z}{y}=\frac{b}{z}.$$

Therefore, each of these fraction is

$$=\left(\frac{x}{a}\cdot\frac{y}{x}\cdot\frac{z}{y}\cdot\frac{b}{z}\right)^{1/4}=\left(\frac{b}{a}\right)^{1/4}=\gamma \text{ say.}$$ *

Hence $x=a\gamma, x\gamma=a\gamma^2=a\gamma^3$ and $b=z\gamma=a\gamma^4$.

Substituting these values of x, y and z in the given equation, we have

$$u=\frac{a\gamma\cdot a\gamma^2\cdot a\gamma^3}{a(1+\gamma)\cdot a\gamma(1+\gamma)\cdot a\gamma^2(1+\gamma)\cdot a\gamma^3(1+\gamma)}$$

$$=\frac{1}{a(1+\gamma)^4}=\frac{1}{a\left\{1+(b/a)^{1/4}\right\}^4}=\frac{1}{\left\{a^{1/4}+b^{1/4}\right\}^4}.$$ ....(2)

• Now we decide whether this value of u is maximum or minimum. For this, we proceed to find the second order derivatives of u.

From relation (1), we have

$$x(a+x)(x+y)\frac{\partial u}{\partial x}=(ay-x^2)u.$$

Differentiating partially with respect to x, this gives

$$x(a+x)(x+y)\frac{\partial^2 u}{\partial x^2}+\psi'\frac{\partial u}{\partial x}=(ay-x^2)\frac{\partial u}{\partial x}-2xu,$$

where ψ' is some function of x and y. Substituting the values of x, y and z obtained above and using $\partial u/\partial x = 0$, we get

$$A = \frac{\partial^2 u}{\partial x^2} = -\frac{2}{a^3\gamma(1+\gamma)^6}.$$

Similarly, we can show that

$$B = -\frac{2}{a^3\gamma^3(1+\gamma)^6}, C = -\frac{2}{a^3\gamma^5(1+\gamma)^6},$$

$$F = \frac{1}{a^3\gamma^4(1+\gamma)^6}, G = 0, H = \frac{1}{a^3\gamma^2(1+\gamma)^5}.$$

From these values, we find that

A = a negative quantity,

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = AB - H^2 = \frac{3}{a^6\gamma^4(1+\gamma)^{12}} = a \text{ a positive quantity}$$

and $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = -\frac{4}{a^9\gamma^9(1+\gamma)^{18}}$ = a negative quantity.

Hence by Lagrange's conditions, the value of u found in (2) is maximum.

Example 3. Find the maximum and minimum values of

$(ax+by+cz)e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$.

Solution.

Let $u = (ax+by+cz)e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$. Taking logarithm on both sides, we have

$$\log u = \log(ax+by+cz) - (\alpha^2x^2+\beta^2y^2+\gamma^2z^2). \quad \text{....(1)}$$

For maxima and minima, we must have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0.$$

Differentiating (1) partially with respect to x, y and z respectively, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{a}{ax+by+cz} - 2\alpha^2 x = 0, \quad \text{....(2)}$$

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{b}{ax+by+cz} - 2\beta^2 y = 0, \quad \text{....(3)}$$

and $\frac{1}{u}\frac{\partial u}{\partial z} = \frac{c}{ax+by+cz} - 2\gamma^2 z = 0$ ....(4)

These equations give

$$x(ax+by+cz) = \frac{a}{2\alpha^2}, \quad \text{....(2')}$$

$$y(ax+by+cz) = \frac{b}{2\beta^2}, \quad \text{....(3')}$$

and $z(ax+by+cz)=\dfrac{c}{2\gamma^2}$. ....(4')

Multiplying (2'), (3'), (4') by a,b,c respectively and adding the resulting equations, we obtain

$$(ax+by+cz)^2=\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right),$$

i.e., $ax+by+cz=\pm\sqrt{\left\{\dfrac{1}{2}\left(\dfrac{a^2}{\alpha^2}+\dfrac{b^2}{\beta^2}+\dfrac{c^2}{\gamma^2}\right)\right\}}=\pm\lambda$, say. ....(5)

It follows from (2'), (3') and (4') that

$$x=\frac{a}{2\alpha^2\lambda},\quad y=\frac{b}{2\beta^2\lambda},\quad z=\frac{c}{2\gamma^2\lambda} \qquad ....(6)$$

and $x=-\dfrac{a}{2\alpha^2\lambda},\quad y=-\dfrac{b}{2\beta^2\lambda},\quad z=-\dfrac{c}{2\gamma^2\lambda}$. ....(7)

Now from (2), differentiating partially with respect to x, we have

$$\frac{1}{u}\frac{\partial^2 u}{\partial x^2}-\frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2=-\frac{a^2}{(cx+by+cz)^2}-2\alpha^2,$$

so that $A=\dfrac{\partial^2 u}{\partial x^2}=-u\left[\dfrac{a^2}{(ax+by+cz)^2}+2\alpha^2\right]$, as $\dfrac{\partial u}{\partial x}=0$.

Case 1. Since u is positive for the values of x,y,z obtained in (6), we find that A is negative. Similarly, we can show that

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix}>0 \text{ and } \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}<0$$

for the values of x,y,z obtained in (6). Hence u is maximum for these values.

Case 2. Since u is negative for the values of x,y,z obtained in (7), we find that A is positive. Similarly, we can show that

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix}>0 \text{ and } \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}>0$$

for the values of x,y,z obtained in (7). Hence u is minimum for these values.

Now putting the values of x,y,z from (6) and using (5), the maximum value of u is

$$u_{\max}=\sqrt{\left\{\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}\exp\left\{-\frac{1}{4\gamma^2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}$$

$$=\sqrt{\left\{\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}e^{-1/2}=\sqrt{\left\{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}$$

Similarly, putting the values of x,y,z from (7) and using (5), the minimum value of u is

$$u_{\min} = -\sqrt{\left\{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}.$$

Assignment

1. Find the conditions of optimality of a function of two variables.
2. Find the maxima and minima values of the following functions:

(i) $u = xy(1-x-y)$

(ii) $u = x^3y^2(1-x-y)$

(iii) $u = y^2 + x^2y + x^4$

3. Discuss the maxima and minima of u in the following cases:

(i) $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

(ii) $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

(iii) $u = x^4 + 2x^2y - x^2 + 3y^2$

(iv) $u = xy^2(3x+6y-2)$.

ANSWERS

2. (i) Maximum at $\left(\frac{1}{3},\frac{1}{3}\right)$; neither maximum nor minimum at $(0,0),(0,1)$ and $(1,0)$.

(ii) Maximum at $x=\frac{1}{2}, y=\frac{1}{3}$; maximum value $=\frac{1}{432}$.

(iii) Doubtful case at $(0,0)$.

3. (i) Maximum at $x=y=0$.

(ii) Minimum at $x=\pm\sqrt{2}, y=\mp\sqrt{2}$.

Doubtful case when $x=0, y=0$.

(iii) Minimum at $x=\pm\frac{1}{2}\sqrt{3}, y=-\frac{1}{4}$.

(iv) Minimum at $x=\frac{1}{6}, y=\frac{1}{6}$.

Maximum at $x=-\frac{1}{2}, y=\frac{3}{2}$ and $x=\frac{3}{2}, y=-\frac{1}{2}$.

Examples on Lagrange's method of undetermined multipliers

Mentor's advice: In the following all questions, answers include Geometrical interpretation. Students are advised to learn these interpretations as well because, in UPSC CSE & IFoS, language of questions are direct and based on geometry as well.

Category 1.

Example 1. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $ax+by+cz=p$.

Solution.

• Define a function F, where $F=\left(x^2+y^2+z^2\right)+\lambda\left(ax+by+cz-p\right)$.

For maxima and minima of F, we have

$$\frac{\partial F}{\partial x}=2x+a\lambda=0, \qquad \text{....(1)}$$

$$\frac{\partial F}{\partial y}=2y+b\lambda=0, \qquad \text{....(2)}$$

and $$\frac{\partial F}{dz}=2z+c\lambda=0 \qquad \text{....(3)}$$

• Multiplying (1) by a, (2) by b, (3) by c and adding the resulting equations column wise, we have

$$2\left(ax+by+cz\right)+\lambda\left(a^2+b^2+c^2\right)=0,$$

i.e., $2p+\lambda\left(a^2+b^2+c^2\right)=0$, since $ax+by+cz=p$,

so that $\lambda=-2p/\left(a^2+b^2+c^2\right)$ ....(4)

• Therefore, for maximum or minimum values of F, using (4) in (1), (2) and (3), we have

$$x=\frac{ap}{a^2+b^2+c^2},\ y=\frac{bp}{a^2+b^2+c^2},\ z=\frac{cp}{a^2+b^2+c^2} \qquad \text{....(5)}$$

• Now $d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2 F$

$$=\sum\frac{\partial^2 F}{\partial x^2}dx^2+2\sum\frac{\partial^2 F}{\partial x\partial y}dxdy$$

$=2\left(dx^2+dy^2+dz^2\right)$, since $\partial^2F/\partial x\partial y=0$ etc.

• Since $d^2F>0$, the function F and therefore $x^2+y^2+z^2$ is minimum for the values of x,y,z given by (5).

Also, the minimum value of $x^2+y^2+z^2$ is

$$= \frac{a^2 p^2}{\left(a^2+b^2+c^2\right)^2} + \frac{b^2 p^2}{\left(a^2+b^2+c^2\right)^2} + \frac{c^2 p^2}{\left(a^2+b^2+c^2\right)^2}$$

$$= \frac{p^2\left(a^2+b^2+c^2\right)}{\left(a^2+b^2+c^2\right)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Example 2. In a plane triangle, find the maximum value of $u = \cos A \cos B \cos C$.
Solution.

- Since A, B, C are angles of a triangle, we have

$A+B+C=\pi$. ....(1)

- Now define a function F, where

$F = \cos A \cos B \cos C + \lambda\left(A+B+C-\pi\right)$.

For maxima or minima of F, we have

$\partial F/\partial A = -\sin A \cos B \cos C + \lambda = 0$, ...(2)

$\partial F/\partial B = -\cos A \sin B \cos C + \lambda = 0$, ...(3)

and $\partial F/\partial C = -\cos A \cos B \sin C + \lambda = 0$ ....(4)

Equating the three values of λ obtained from (2), (3) and (4) we have
$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$.
Dividing by $\cos A \cos B \cos C$, it follows that
$\tan A = \tan B = \tan C$,
so that $A = B = C = \frac{1}{3}\pi$, using (1).

- To identify whether F is maximum or minimum for these values of A, B, C, we see that

$$d^2F = \left(dA\frac{d}{\partial A} + dB\frac{\partial}{\partial B} + dC\frac{\partial}{\partial C}\right)^2 F$$

$$= \sum \frac{\partial^2 F}{\partial A^2} dA^2 + 2\sum \frac{\partial^2 F}{\partial A \partial B} dAdB$$

$$= -\Sigma \cos A \cos B \cos C dA^2 + 2\Sigma \sin A \sin B \cos C dAdB$$

$$= -\frac{1}{8}\Sigma dA^2 + \frac{3}{4}\Sigma dAdB, \text{ since } A = B = C = \frac{1}{3}\pi$$

$$= -\frac{1}{8}\left(dA^2 + dB^2 + dC^2\right) + \frac{3}{4} dAdB + dBdC + dCdA. \quad ....(5)$$

- But $dA + dB + dC = 0$, for $A + B + C = \pi$, so that

$\left(dA + dB + dC\right)^2 = 0$

$\Rightarrow dA^2 + dB^2 + dC^2 + 2\left(dAdB + dBdC + dCdA\right) = 0$

$\Rightarrow dA^2 + dB^2 + dC^2 = -2\left(dAdB + dBdC + dCdA\right)$

Therefore, from (5), we have

$$d^2F = -\frac{1}{8}\left(dA^2 + dB^2 + dC^2\right) - \frac{3}{8}\left(dA^2 + dB^2 + dC^2\right)$$
$$= -\frac{1}{2}\left(dA^2 + dB^2 + dC^2\right).$$

Since $d^2F < 0$, the function F and hence u is maximum when $A = B = C = \frac{1}{3}\pi$. The maximum value of u is given by

$$u_{\max} = \cos\frac{1}{3}\pi\cos\frac{1}{3}\pi\cos\frac{1}{3}\pi = \frac{1}{8}.$$

Example 3. Find the dimensions of the rectangular parallelopiped of maximum volume that can be cut from the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

OR

Prove that of all rectangular parallelopipeds inscribed in the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the parallelopiped having volume $8abc/3\sqrt{3}$ is the greatest.

Solution.

• Let (x, y, z) be the co-ordinates of a vertex, lying in the positive octant, of the rectangular parallelopiped. Then the length of the sides of inscribed parallelopiped are $2x, 2y, 2z$.

Therefore, the volume V is $(2x)(2y)(2z)$, i.e., $V = 8xyz$,….(1)

subject to the condition that the point (x, y, z) lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad ....(2)$$

Therefore, we need to find the maximum value of V subject to the condition (2).

• Let

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

For maxima or minima of F, we must have

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda x}{a^2} = 0,\ \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda y}{b^2} = 0,\ \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda z}{c^2} = 0.$$

Therefore, $\frac{x}{a^2yz} = \frac{y}{b^2xz} = \frac{z}{c^2xy} = -\frac{4}{\lambda}$. ....(3)

Multiplying by *xyz* these equations give

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \text{ i.e., } \frac{x^2/y^2}{1} = \frac{y^2/b^2}{1} = \frac{z^2/c^2}{1} = \frac{x^2/a^2 + y^2/b^2 + z^2/c^2}{1+1+1}$$

so that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\dfrac{1}{\sqrt{3}}$. using (2)

Whence $x=a/\sqrt{3}, y=b/\sqrt{3}, z=c/\sqrt{3}$. ....(4)

Using these values of x, y, z, we have from (3),

$$\lambda=-\frac{4a^2yz}{x}=-\frac{4abc}{\sqrt{3}} \qquad ....(5)$$

• Now to identify the maxima or minima, we have

$$d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2F=\sum\frac{\partial^2F}{\partial x^2}dx^2+2\sum\frac{\partial^2F}{\partial x\partial y}dxdy$$

$$=2\lambda\left(\frac{dx^2}{a^2}+\frac{dy^2}{b^2}+\frac{dz^2}{c^2}\right)+16\left(z\,dxdy+x\,dydz+y\,dzdx\right)$$

$$=-\frac{8abc}{\sqrt{3}}\left(\frac{dx^2}{a^2}+\frac{dy^2}{b^2}+\frac{dz^2}{c^2}\right)+\frac{16}{\sqrt{3}}\left(c\,dxdy+a\,dydz+b\,dzdx\right),$$

using (4) and (5)

$$=-\frac{8abc}{\sqrt{3}}\left\{\frac{dx^2}{a^2}+\frac{dy^2}{b^2}+\frac{dz^2}{c^2}-2\left(\frac{dxdy}{ab}+\frac{dydz}{bc}+\frac{dzdx}{ca}\right)\right\} \qquad ....(6)$$

But the differentiation of (2) gives

$\dfrac{2x\,dx}{a^2}+\dfrac{2y\,dy}{b^2}+\dfrac{2z\,dz}{c^2}=0$, i.e., $\dfrac{dx}{a}+\dfrac{dy}{b}+\dfrac{dz}{c}=0$, using (4).

Squaring this equation, we get

$$\frac{(dx)^2}{a^2}+\frac{(dy)^2}{b^2}+\frac{(dz)^2}{c^2}+2\left(\frac{dx}{a}\frac{dy}{b}+\frac{dy}{b}\frac{dz}{c}+\frac{dz}{c}\frac{dx}{a}\right)=0$$

which gives $\dfrac{dx^2}{a^2}+\dfrac{dy^2}{b^2}+\dfrac{dz^2}{c^2}=-2\left(\dfrac{dxdy}{ab}+\dfrac{dydz}{bc}+\dfrac{dzdx}{ca}\right)$, since $(dx)^2=dx^2$ etc.

Using this equation in (6), we get

$$d^2F=-\frac{16abc}{\sqrt{3}}\left(\frac{dx^2}{a^2}+\frac{dy^2}{b^2}+\frac{dz^2}{c^2}\right).$$

Obviously, $d^2F<0$. Therefore, F and hence V is maximum at the point given by (4). The maximum value of V is:

$$V_{\max}=8\cdot\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}=\frac{8abc}{3\sqrt{3}}.$$

The dimensions (i.e., the lengths of sides) of the rectangular parallelopiped of maximum volume are $2a/\sqrt{3}, 2b/\sqrt{3}, 2c/\sqrt{3}$.

Example 4. Find the maximum value of $u=(x+1)(y+1)(z+1)$, where $a^xb^yc^z=A$. Interpret the result geometrically.

Solution.

• Taking logarithms, the given condition becomes

$$x\log a + y\log b + z\log c = \log A \qquad ....(1)$$

• Now define the function F, where

$$F = (x+1)(y+1)(z+1) + \lambda(x\log a + y\log b + z\log c) = \log A.$$

• For maxima or minima of F, we must have

$$\partial F/\partial x = (y+1)(z+1) + \lambda\log a = 0,$$

$$\partial F/\partial y = (x+1)(z+1) + \lambda\log b = 0,$$

and $\partial F/\partial z = (x+1)(y+1) + \lambda\log c = 0$.

These equations can be written as

$$(y+1)(z+1) = -\lambda\log a, \qquad ....(2)$$

$$(x+1)(z+1) = -\lambda\log b, \qquad ....(3)$$

and $(x+1)(y+1) = -\lambda\log c$. ....(4)

• Multiplying (2) by $x+1$, (3) by $y+1$, (4) by $z+1$ and equating the three values of u thus obtained, we get

$$(x+1)\log a = (y+1)\log b = (z+1)\log c$$

$$= \frac{(x+1)\log a + (y+1)\log b + (z+1)\log c}{1+1+1}$$

$$= \frac{\log A + (\log a + \log b + \log c)}{3}, \text{ using (1)}$$

$$= \frac{1}{3}\log(Aabc).$$

• Therefore

$$x+1 = \frac{\log(Aabc)}{3\log a}, y+1 = \frac{\log(Aabc)}{3\log b}, z+1 = \frac{\log(Aabc)}{3\log c}. \qquad ....(5)$$

Now to identify whether the values of x, y, z given by (5) provide a maximum or minimum value of F, we have

$$d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 F = \sum\frac{\partial^2 F}{\partial x^2}dx^2 + 2\sum\frac{\partial^2 F}{\partial x\partial y}dxdy$$

$= 2\{(z+1)dxdy + (x+1)dydz + (y+1)dzdx\}$, since $\partial^2 F/\partial x^2 = 0$ etc.

$$= \frac{2\log(Aabc)}{3}\left(\frac{dxdy}{\log c} + \frac{dydz}{\log a} + \frac{dzdx}{\log b}\right), \text{ using (5)}$$

$$= 2\mu(\log a\log b\,dxdy + \log b\log c\,dydz + \log c\log a\,dzdx), \qquad ....(6)$$

where $\mu = \dfrac{\log(Aabc)}{3\log a \log b \log c}$.

• But differentiation of (1) yields

$\log a\, dx + \log b\, dy + \log c\, dz = 0$

Squaring this equation, we get

$\Sigma(\log a)^2 (dx)^2 + 2\Sigma \log a\, dx \cdot \log b\, dy = 0$

i.e., $2\Sigma \log a \log b\, dxdy = -\Sigma(\log a)^2 dx^2$, since $(dx)^2 = dx^2$ etc.

Using this relation in (6), we find that

$d^2F = -\mu\left\{(\log a)^2 dx^2 + (\log b)^2 dy^2 + (\log c)^2 dz^2\right\}$

Evidently, $d^2F < 0$ provided the positive real numbers a, b, c and A are such that $\mu > 0$. Therefore, the function F and hence u is maximum at the point given by (5), provided $\mu > 0$.

Using (5) the maximum value of u is given by

$u_{\max} = \dfrac{\{\log(Aabc)\}^3}{27\log a \log b \log c}$.

Geometrical Interpretation. Consider a rectangular parallelopiped whose sides are parallel to the co-ordinate axes and are of lengths $x+1, y+1, z+1$. Clearly, the end points of one of the diagonals of the parallelopiped will be $(-1,-1,-1)$ and (x, y, z). Then

$u = (x+1)(y+1)(z+1)$ = volume of the parallelopiped.

Also, $a^x b^y c^z = A$ denotes some surface.

Hence geometrically the problem aims at finding out the maximum volume of the rectangular parallelopiped whose sides are parallel to the co-ordinate axes, one end of one of whose diagonals is $(-1,-1,-1)$ and the other end (x, y, z) moves on the surface $a^x b^y c^z = A$.

Example 5. Find the maximum value of $x^m y^n z^p$ with the condition $x + y + z = a$.

Solution.

• Let $u = x^m y^n z^p$.

The function u will be maximum if its logarithm is also maximum.

Therefore, taking the logarithm of u, we have

$\log u = m\log x + n\log y + p\log z$.

• Now define a function F, where

$F = (m\log x + n\log y + p\log z) + \lambda(x + y + z - a)$.

For maxima or minima of F, we must have

$\dfrac{\partial F}{\partial x} = \dfrac{m}{x} + \lambda = 0$, $\dfrac{\partial F}{\partial y} = \dfrac{n}{y} + \lambda = 0$, $\dfrac{\partial F}{\partial z} = \dfrac{p}{z} + \lambda = 0$.

These equations give

$m/x = n/y = p/z = -\lambda$

so that $\frac{x}{m} = \frac{y}{n} = \frac{z}{p} = \frac{x+y+z}{m+n+p} = \frac{a}{m+n+p}$.

Thus $x = \frac{ma}{m+n+p}$, $y = \frac{na}{m+n+p}$, $z = \frac{pa}{m+n+p}$. ....(1)

• To discuss whether F is maximum or minimum for these values, we have

$$\partial^2 F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \right)^2 F = \sum \frac{\partial^2 F}{\partial x^2} dx^2 + 2\sum \frac{\partial^2 F}{\partial x \partial y} dxdy$$

$$= -\left(\frac{m}{x^2}dx^2 + \frac{n}{y^2}dy^2 + \frac{p}{z^2}dz^2 \right), \text{ since } \frac{\partial^2 F}{\partial x \partial y} = 0 \text{ etc.}$$

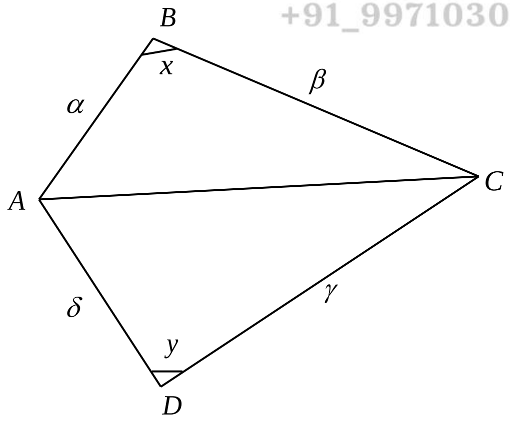
$$= -(m+n+p)^2 \left(\frac{dx^2}{ma^2} + \frac{dy^2}{na^2} + \frac{dz^2}{pa^2} \right), \text{ using (1).}$$

Since $d^2F < 0$, the function F and hence the function u has a maximum value at the point given by (1). The maximum value of u is:

$$u_{\max} = \left(\frac{ma}{m+n+p} \right)^m \left(\frac{na}{m+n+p} \right)^n \left(\frac{pa}{m+n+p} \right)^p = \frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}.$$

Example 6. Determine the greatest quadrilateral which can be formed with four given sides $\alpha, \beta, \gamma, \delta$ taken in this order.

Solution.



Here we first construct the function to be maximized and the side condition.

• Let x be the angle between the sides of lengths α and β, and y that between the sides of lengths γ and δ. Evidently, the area of quadrilateral ABCD will change by changing x and y. So, the area will be function of x and y.

• Now, if u denotes the area of the quadrilateral ABCD, we have (from geometry),

$u = \frac{1}{2}(\alpha\beta \sin x + \gamma\delta \sin y)$. ....(1)

Also, from ΔABC, we have

$(AC)^2 = \alpha^2 + \beta^2 - 2\alpha\beta \cos x$.

and from ΔADC, we have

$(AC)^2 = \gamma^2 + \delta^2 - 2\gamma\delta \cos y$.

Equating the two values of $(AC)^2$, we get

$\alpha^2 + \beta^2 - 2\alpha\beta \cos x = \gamma^2 + \delta^2 - 2\gamma\delta \cos y$,

i.e., $\alpha\beta \cos x - \gamma\delta \cos y + c = 0$, ...(2)

where $c = \frac{1}{2}(\gamma^2 + \delta^2 - \alpha^2 - \beta^2)$, which is a constant.

Equation (2) gives us a relation between x and y. Our problem is to find out the maximum value of the function u of variables x and y, as given in (1), subject to the condition (2).

• Let us define a function F, where

$F = \frac{1}{2}(\alpha\beta \sin x + \gamma\delta \sin y) + \lambda(\alpha\beta \cos x - \gamma\delta \cos y + c)$

For maxima and minima of F, we must have

$\partial F/\partial x = \frac{1}{2}\alpha\beta \cos x - \lambda\alpha\beta \sin x = 0$, ...(3)

and $\partial F/\partial y = \frac{1}{2}\gamma\delta \cos y + \lambda\gamma\delta \sin y = 0$. ....(4)

From (3), $\lambda = \frac{1}{2}\cot x$, and from (4), $\lambda = -\frac{1}{2}\cot y$.
Equating these two values of λ, we have $\cot x = -\cot y$ so that either $\cot x = \cot(-y)$ or $\cot x = \cot(\pi - y)$.
Thus either $x = -y$ or $x = \pi - y$. But $x = -y$ is not possible as both x and y must be positive.

• Therefore, $x = \pi - y$, i.e., $x + y = \pi$. Now

$$d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^2 F = \frac{\partial^2 F}{\partial x^2}dx^2 + \frac{\partial^2 F}{\partial y^2}dy^2 + 2\frac{\partial^2 F}{\partial x \partial y}dxdy$$

$$= -\frac{1}{2}\alpha\beta(\sin x + 2\lambda\cos x)dx^2 - \frac{1}{2}\gamma\delta(\sin y + 2\lambda\cos y)dy^2 + 0$$

$$= -\frac{1}{2}\left\{\alpha\beta(\sin x + \cot x\cos x)dx^2 + \gamma\delta(\sin y + \cot y\cos y)dy^2\right\}$$

$$= -\frac{1}{2}\left(\alpha\beta\,\mathrm{cosec}\,x\,dx^2 + \gamma\delta\,\mathrm{cosec}\,y\,dy^2\right)$$

$= a$ negative quantity,
since $\mathrm{cosec}\,x$ and $\mathrm{cosec}\,y$ are positive for $0 < x, y < \pi$.

Therefore, the function F and hence u is maximum when $x + y = \pi$. Thus the area of the quadrilateral is greatest when $x + y = \pi$, i.e., when it can be inscribed in a circle.

Examples of Category 2.

Example 8. Find the maxima and minima of $u = x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically.

Solution.
• Let us define a functions F as

$$F = x^2 + y^2 + z^2 + \lambda_1\left(ax^2 + by^2 + cz^2 - 1\right) + \lambda_2\left(lx + my + nz\right).$$

For maxima and minima of F, we must have

$\partial F/\partial x = 2x + 2ax\lambda_1 + l\lambda_2 = 0$, ...(1)

$\partial F/\partial y = 2y + 2by\lambda_1 + m\lambda_2 = 0$, ...(2)

and $\partial F/\partial z = 2z + 2cz\lambda_1 + n\lambda_2 = 0$. ....(3)

Multiplying (1) by x, (2) by y, (3) by z and adding the resulting equations column wise, we have

$$2\left(x^2 + y^2 + z^2\right) + 2\lambda_1\left(ax^2 + by^2 + cz^2\right) + \lambda_2\left(lx + my + nz\right) = 0,$$

i.e., $2u + 2\lambda_1 + \lambda_2 \cdot 0 = 0$, which gives $\lambda_1 = -u$.

Substituting this value of λ_1 in (1), we get

$2x - 2axu + l\lambda_2 = 0$, which gives $x = \dfrac{l\lambda_2}{2(au-1)}$.

Similarly, $y = \dfrac{m\lambda_2}{2(bu-1)}$ and $z = \dfrac{n\lambda_2}{2(cu-1)}$.

Putting these values in the relation $lx + my + nz = 0$, we have

$$\frac{l^2\lambda_2}{2(au-1)} + \frac{m^2\lambda_2}{2(bu-1)} + \frac{n^2\lambda_2}{2(cu-1)} = 0.$$

i.e., $\dfrac{l^2}{au-1} + \dfrac{m^2}{bu-1} + \dfrac{n^2}{cu-1} = 0$, since $\lambda_2 \neq 0$.

This equation gives the maximum and minimum values of u.

• **Geometrical Interpretation**. Here $x^2 + y^2 + z^2$ is the square of the distance of any point (x, y, z) from the origin. Thus we have discussed maxima and minima of the square of the distance of the origin from the point of intersection of the central conicoid $ax^2 + by^2 + cz^2 = 1$ by the central plane $lx + my + nz = 1$.

Example 9. Find the maximum and minimum values of $u = x^2 + y^2 + z^2$ when $px + qy + rz = 0$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Interpret the result geometrically.

Or

Using the method of Lagrange's multipliers find the length of the axis of the section of ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ by the plane $px + qy + rz = 0$

Solution.

• Let us define a function F, where

$$F = (x^2 + y^2 + z^2) + \lambda_1 (px + qy + rz) + \lambda_2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right).$$

For maximum and minimum values of F, we must have

$\partial F/\partial x = 2x + p\lambda_1 + 2x\lambda_2/a^2 = 0$, ...(1)

$\partial F/\partial y = 2y + q\lambda_1 + 2y\lambda_2/b^2 = 0$, ...(2)

and $\partial F/\partial z = 2z + r\lambda_1 + 2z\lambda_2/c^2 = 0$. ....(3)

Multiplying (1) by x, (2) by y, (3) by z and adding the resulting equations column wise, we get

$$2(x^2 + y^2 + z^2) + \lambda_1 (px + qy + rz) + 2\lambda_2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 0,$$

i.e., $2u+\lambda_1 \cdot 0+2\lambda_2=0$, which gives $\lambda_2=-u$.
Substituting this value of λ_2 in (1), we get

$2x+p\lambda_1-2xu/a^2=0$, which gives $x=\dfrac{\lambda_1 pa^2}{2\left(u-a^2\right)}$.

Similarly, $y=\dfrac{\lambda_1 qb^2}{2\left(u-b^2\right)}$ and $z=\dfrac{\lambda_1 rc^2}{2\left(u-c^2\right)}$.

• Putting these values in the relation $px+qy+rz=0$, we have

$$\frac{\lambda_1 p^2a^2}{2\left(u-a^2\right)}+\frac{\lambda_1 q^2b^2}{2\left(u-b^2\right)}+\frac{\lambda_1 r^2c^2}{2\left(u-c^2\right)}=0,$$

i.e., $\dfrac{p^2a^2}{u-a^2}+\dfrac{q^2b^2}{u-b^2}+\dfrac{r^2c^2}{u-c^2}=0$.

This equation gives the maximum and minimum values of u.

Geometrical Interpretation. A point (x,y,z) satisfying the given condition lies on the section of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ by the central plane $20\,px+qy+rz=0$, and $u=x^2+y^2+z^2$ is the square of the distance of (x,y,z) from the centre $(0,0,0)$ of the ellipsoid. In this problem, we have discussed determination of the axes of the section of ellipsoid by the central plane $px-qy+rz=0$.

Example 10. Find the maxima and minima of $u=x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$ and $lx+my+nz=0$.

Solution.

• Let us define a function F, where

$$F=\left(x^2+y^2+z^2\right)+\lambda_1\left(ax^2+by^2+cz^2+3fyz+2gx+2hxy-1\right)+\lambda_2\left(lx+my+nz\right).$$

For maxima and minima of F, we must have

$\partial F/\partial x=2x+2\lambda_1\left(ax+gz+hy\right)+\lambda_2 l=0$, ...(1)

$\partial F/\partial y=2y+2\lambda_1\left(by+fz+hx\right)+\lambda_2 m=0$, ...(2)

and $\partial F/\partial z=2z+2\lambda_1\left(cz+fy+gx\right)+\lambda_2 n=0$. ....(3)

Multiplying (1) by x, (2) by y, (3) by z and adding the resulting equations column wise, we obtain

$$2\left(x^2+y^2+z^2\right)+2\lambda_1\left(ax^2+by^2+cz^2+2fyz+2gzx+2hxy\right)+\lambda_2\left(lx+my+nz\right)=0,$$

i.e., $2u+2\lambda_1+\lambda_2\cdot 0=0$, which gives $\lambda_1=-u$.

• Putting this value of λ_1 in (1), we get

$2x - 2u(ax + gz + hy) + \lambda_2 l = 0$,

i.e., $2(au-1)x + 2uhy + 2ugz - l\lambda_2 = 0$, ...(4)

on changing signs.

Similarly, equations (2) and (3) yield

$2uhx + 2(bu-1)y + 2ufz - m\lambda_2 = 0$, ...(5)

and $2ugx + 2ufy + 2(cu-1)z - n\lambda_2 = 0$. ....(6)

Also the given relation $lx + my + nz = 0$ can be written as

$lx + my + nz - 0 \cdot \lambda_2 = 0$. ....(7)

Eliminating $2x, 2y, 2z$ and $-\lambda_2$ from equations (4), (5), (6) and (7), we have

$$\begin{vmatrix} au-1 & uh & ug & l \\ uh & bu-1 & uf & m \\ ug & uf & cu-1 & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Dividing R_1, R_2 and R_3 by u and then multiplying C_4 of the resulting determinant by u, we obtain

$$\begin{vmatrix} a-1/u & h & g & l \\ h & b-1/u & f & m \\ g & f & c-1/u & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

This equation gives the maximum and minimum values of u.

Example 11. Find the maxima and minima of $u = x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$ and $lx + my + nz = 0$.

Solution.

Let us define a function F, where

$F = (x^2 + y^2 + z^2) + \lambda_1 (ax^2 + by^2 + cz^2 + 3fyz + 2gx + 2hxy - 1) + \lambda_2 (lx + my + nz)$.

For maxima and minima of F, we must have

$\partial F/\partial x = 2x + 2\lambda_1 (ax + gz + hy) + \lambda_2 l = 0$, ...(1)

$\partial F/\partial y = 2y + 2\lambda_1 (by + fz + hx) + \lambda_2 m = 0$, ...(2)

and $\partial F/\partial z = 2z + 2\lambda_1 (cz + fy + gx) + \lambda_2 n = 0$. ....(3)

Multiplying (1) by x, (2) by y, (3) by z and adding the resulting equations column wise, we obtain

$2(x^2 + y^2 + z^2) + 2\lambda_1 (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) + \lambda_2 (lx + my + nz) = 0$,

i.e., $2u + 2\lambda_1 + \lambda_2 \cdot 0 = 0$, which gives $\lambda_1 = -u$.

Putting this value of λ_1 in (1), we get

$2x - 2u(ax + gz + hy) + \lambda_2 l = 0$,

i.e., $2(au-1)x + 2uhy + 2ugz - l\lambda_2 = 0$, ...(4)

on changing signs.

Similarly, equations (2) and (3) yield

$2uhx + 2(bu-1)y + 2ufz - m\lambda_2 = 0$, ...(5)

and $2ugx + 2ufy + 2(cu-1)z - n\lambda_2 = 0$. ....(6)

Also the given relation $lx + my + nz = 0$ can be written as

$lx + my + nz - 0\cdot\lambda_2 = 0$. ....(7)

Eliminating $2x, 2y, 2z$ and $-\lambda_2$ from equations (4), (5), (6) and (7), we have

$$\begin{vmatrix} au-1 & uh & ug & l \\ uh & bu-1 & uf & m \\ ug & uf & cu-1 & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Dividing R_1, R_2 and R_3 by u and then multiplying C_4 of the resulting determinant by u, we obtain

$$\begin{vmatrix} a-1/u & h & g & l \\ h & b-1/u & f & m \\ g & f & c-1/u & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

This equation gives the maximum and minimum values of u.

Example 12. Show that the maximum and minimum radii vectors of the section of the surface $\left(x^2+y^2+z^2\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ by the plane $lx + my + nz = 0$ are given by the equation $\frac{a^2l^2}{1-a^2r^2} + \frac{b^2m^2}{1-b^2r^2} + \frac{c^2n^2}{1-c^2r^2} = 0$

Solution.

We need to find the maximum and minimum values of r, where $r^2 = x^2 + y^2 + z^2$.

Clearly, r will be maximum or minimum according as r^2 is maximum or minimum.

Let us define a function F, where

$$F = \left(x^2+y^2+z^2\right) + \lambda_1\left[\left(x^2+y^2+z^2\right)^2 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\right] + \lambda_2\left(lx + my + nz\right).$$

For maxima and minima of F, we have

$$\frac{\partial F}{\partial x} = 2x + \lambda_1\left[2\left(x^2+y^2+z^2\right)2x - \frac{2x}{a^2}\right] + \lambda_2 l = 0, \qquad \text{....(1)}$$

$$\frac{\partial F}{\partial y} = 2y + \lambda_1\left[2\left(x^2+y^2+z^2\right)2y - \frac{2y}{b^2}\right] + \lambda_2 m = 0, \qquad \text{....(2)}$$

and $$\frac{\partial F}{\partial z} = 2z + \lambda_1\left[2\left(x^2+y^2+z^2\right)2z - \frac{2z}{c^2}\right] + \lambda_2 n = 0. \qquad \text{....(3)}$$

Multiplying (1) by x, (2) by y, (3) by z and adding the resulting equations column wise, we get

$$2\left(x^2+y^2+z^2\right) + \lambda_1\left[4\left(x^2+y^2+z^2\right)^2 - 2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right] + \lambda_2\left(lx+my+nz\right) = 0,$$

i.e., $2r^2 + \lambda_1\left[2r^4 + 2\left\{\left(x^2+y^2+z^2\right)^2 - \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right\}\right] + \lambda_2 \cdot 0 = 0,$

since $4\left(x^2+y^2+z^2\right)^2 = 2r^4 + 2\left(x^2+y^2+z^2\right)^2$,

i.e., $2r^2 + \lambda_1\left(2r^4 + 2\times 0\right) = 0$, which gives $\lambda_1 = -1/r^2$.

Substituting this value of λ_1 is (1) we have

$$2x - \frac{1}{r^2}\left[2r^2\cdot 2x - \frac{2x}{a^2}\right] + \lambda_2 l = 0,$$

i.e., $2x\left[1-\left(2-\frac{2}{a^2r^2}\right)\right] + \lambda_2 l = 0$, which gives $x = \frac{\lambda_2 l a^2 r^2}{2\left(a^2r^2-1\right)}$.

Similarly, $y = \frac{\lambda_2 m b^2 r^2}{2\left(b^2r^2-1\right)}$ and $z = \frac{\lambda_2 n c^2 r^2}{2\left(c^2r^2-1\right)}$.

Putting these values of x, y and z in the relation $lx+my+nz=0$, we get

$$\frac{\lambda_2 l^2 a^2 r^2}{2\left(a^2r^2-1\right)} + \frac{\lambda_2 m^2 b^2 r^2}{2\left(b^2r^2-1\right)} + \frac{\lambda_2 n^2 c^2 r^2}{2\left(c^2r^2-1\right)} = 0,$$

which on multiplying by $-2/\lambda_2 r^2$ gives

$$\frac{a^2l^2}{1-a^2r^2} + \frac{b^2m^2}{1-b^2r^2} + \frac{c^2n^2}{1-c^2r^2} = 0.$$

Example of Category 3.

Example 13. Find a plane triangle ABC such that $u = \sin^m A \sin^n B \sin^p C$ has a maximum value.

Solution.

• If a function is maximum, then its logarithm will also be maximum. therefore, taking the logarithm of the given function, we have
$\log u = m\log\sin A + n\log\sin B + p\log\sin C$.

Also, since A, B, C are the angles of a plane triangle, we have
$A + B + C = \pi$. ....(1)

• Now define a function F, where
$F = m\log\sin A + n\log\sin B + p\log\sin C + \lambda(A + B + C - \pi)$.
For maxima or minima of F, we have
$\partial F/\partial A = m\cot A + \lambda = 0$, ...(2)
$\partial F/\partial B = n\cot B + \lambda = 0$, ...(3)
and $\partial F/\partial C = p\cot C + \lambda = 0$. ....(4)

Equating the three values of λ obtained from (2), (3) and (4), we get
$m\cot A = n\cot B = p\cot C$,
i.e., $\dfrac{\tan A}{m} = \dfrac{\tan B}{n} = \dfrac{\tan C}{p}$. ....(5)

• To identify whether F is maximum or minimum for the values of A, B, C given by (5), we see that
$$d^2F = \left(dA\frac{\partial}{\partial A} + dB\frac{\partial}{\partial B} + dC\frac{\partial}{\partial C}\right)^2 F = \sum \frac{\partial^2 F}{\partial A^2}dA^2 + 2\sum \frac{\partial^2 F}{\partial A\partial B}dAdB$$
$= -\left\{\left(m\operatorname{cosec}^2 A\right)dA^2 + n\operatorname{cosec}^2 B\right)dB^2 + \left(p\operatorname{cosec}^2 C\right)dC^2\right\}$ since $d^2F/\partial A\partial B = 0$ etc.
Obviously, $d^2F < 0$.
Thus the function F and hence u is maximum for the values of A, B, C given by (5).

Example 14. Divide a number n into three parts x, y, z such that $u = ayz + bzx + cxy$ shall have a maximum or minimum value and determine which it is. (a, b, c may be taken as the sides of a triangle).
Solution.
Since n is divided into three parts x, y, z, we have
$x + y + z = n$. ....(1)

Now define a function F, where
$F = (ayz + bzx + cxy) + \lambda(x + y + z - n)$.

For maxima or minima of F, we have
$\partial F/\partial x = bz + cy + \lambda = 0$, ...(2)
$\partial F/\partial y = az + cx + \lambda = 0$, ...(3)
and $\partial F/\partial z = ay + bx + \lambda = 0$. ....(4)

Multiplying (2) by x, (3) by y, (4) by z and adding the resulting equations column wise, we obtain

$2(ayz+bzx+cxy)+\lambda(x+y+z)=0$,

i.e., $2u+n\lambda=0$, which gives $\lambda=-2u/n$.

Substituting this value of λ, equations (2), (3) and (4) can be written as

$0x+cy+bz-2u/n=0$,

$cx+0y+az-2u/n=0$,

$bx+ay+0z-2u/n=0$.

Also, (1) is: $x+y+z-n=0$.

Eliminating x, y, z and -1 from these four equations, the maximum or minimum value of u is given by

$$\begin{vmatrix} 0 & c & b & 2u/n \\ c & 0 & a & 2u/n \\ b & a & 0 & 2u/n \\ 1 & 1 & 1 & n \end{vmatrix}=0 \qquad \text{....(5)}$$

• Now we proceed to show whether it is a maximum or minimum value of u. We have

$$d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2F=\sum\frac{\partial^2 F}{\partial x^2}dx^2+2\sum\frac{\partial^2 F}{\partial x\partial y}dxdy$$

$=2(c\,dxdy+a\,dydz+b\,dzdx)$, since $\partial^2F/\partial x^2=0$ etc.

$=2\{c\,dxdy+(a\,dy+b\,dx)dz\}$

$=\{c\,dxdy-(a\,dy+b\,dx)(dx+dy)\}$, since $dx+dy+dz=0$, from (1)

$=-2\{b\,dx^2+(a+b-c)dxdy+a\,dy^2\}$.

• Therefore d^2F will have the same sign as -2 if

$ba-\frac{1}{2}(a+b-c)^2>0$,

i.e., $4ab-(a^2+b^2+c^2+2ab-2bc-2ca)>0$,

i.e., $2ab+2bc+2ca-a^2-b^2-c^2>0$,

i.e., $a(b+c-a)+b(c+a-b)+c(a+b-c)>0$.

The above inequality will be true if we assume that a,b,c are such that a triangle could be constructed with these sides so that $b+c-a, c+a-b$ and $a+b-c$ are all positive. With this assumption, we find that $d^2F<0$.

Hence the value of u given by (5) is maximum under the assumption that a, b, c form the sides a triangle.

Example 15. Find the minimum value of $u = x^2 + y^2 + z^2$ subject to the conditions $ax + by + cz = 1 = a'x + b'y + c'z$.

Solution.

• Let us define a function F, where

$F = \left(x^2 + y^2 + z^2\right) + \lambda_1\left(ax + by + cz - 1\right) + \lambda_2\left(a'x + b'y + c'z - 1\right)$.

For maxima and minima of F, we have

$\partial F/\partial x = 2x + \lambda_1 a + \lambda_2 a' = 0$, ...(1)

$\partial F/\partial y = 2y + \lambda_1 b + \lambda_2 b' = 0$, ...(2)

and $\partial F/\partial z = 2z + \lambda_1 c + \lambda_2 c' = 0$. ....(3)

Multiplying (1), (2), (3) by x, y, z respectively and adding the resulting equations column wise, we get $2\left(x^2 + y^2 + z^2\right) + \lambda_1\left(ax + by + cz\right) + \lambda_2\left(a'x + b'y + c'z\right) = 0$,

i.e., $2u + \lambda_1 + \lambda_2 = 0$.

Further, multiplying (1), (2), (3) by a, b, c respectively and adding the resulting equations column wise, we get $2\left(ax + by + cz\right) + \lambda_1\left(a^2 + b^2 + c^2\right) + \lambda_2\left(aa' + bb' + cc'\right) = 0$,

i.e., $2 + \lambda_1 \Sigma a^2 + \lambda_2 \Sigma aa' = 0$ ....(5)

Similarly, multiplying (1), (2), (3) by a', b', c' respectively and adding the resulting equations column wise, we get $2\left(a'x + b'y + c'z\right) + \lambda_1\left(aa' + bb' + cc'\right) + \lambda_2\left(a'^2 + b'^2 + c'^2\right) = 0$,

i.e. $2 + \lambda_1 \Sigma aa' + \lambda_2 \Sigma a'^2 = 0$. ....(6)

Eliminating λ_1, λ_2 from (4), (5) and (6), the maximum or minimum value of u is given by

$\begin{vmatrix} 2u & 1 & 1 \\ 2 & \Sigma a^2 & \Sigma aa' \\ 2 & \Sigma aa' & \Sigma a'^2 \end{vmatrix} = 0$, i.e., $\begin{vmatrix} u & 1 & 1 \\ 1 & \Sigma a^2 & \Sigma aa' \\ 1 & \Sigma aa' & \Sigma a'^2 \end{vmatrix} = 0$, dividing C_1 by 2.

To identify whether it is a maximum or minimum value of u, we see that

$$d^2F = \left(dx\frac{\partial}{\partial x} + \partial y\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 F$$

$$= \sum \frac{\partial^2 F}{\partial x^2} dx^2 + 2\sum \frac{\partial^2 F}{\partial x \partial y} dxdy$$

$= 2dx^2 + 2dy^2 + 2z^2$, since $\partial^2 F/\partial xdy = 0$ etc.

$= 2\left(dx^2 + dy^2 + dz^2\right)$.

Obviously, $d^2F > 0$. Hence the value of u determined above is its minimum value.

<u>Miscellaneous Examples.</u>

Example 16. Obtain the extreme values of $u = x^3 + y^3 + z^3 + 3kxyz$, $k \neq 2$, where x, y, z are subject to the condition $x + y + z = 1$, and show that the symmetrical extreme value is a maximum or minimum according as $k >$ or < 2.

Solution.

• Let us define a function F, where

$$F = \left(x^3 + y^3 + z^3 + 3kxyz\right) + \lambda\left(x + y + z - 1\right).$$

For extreme values of u, we must have

$$\partial F/\partial x = 3\left(x^2 + kyz\right) + \lambda = 0, \quad ....(1)$$

$$\partial F/\partial y = 3\left(y^2 + kzx\right) + \lambda = 0, \quad ....(2)$$

and $\partial F/\partial z = 3\left(z^2 + kxy\right) + \lambda = 0$. ....(3)

Subtracting (2) from (1), we get

$$(x - y)(x + y - kz) = 0 \quad ....(4)$$

and subtracting (3) from (1), we get

$$(x - z)(x + z - ky) = 0. \quad ....(5)$$

Equations (4) and (5) give the following:

$$x - y = 0, \quad x + y - kz = 0, \quad ....(6)$$

and $x - z = 0, \quad x + z - ky = 0$. ....(7)

Solving these equations, taking one form the pair (6) and one from the pair (7), and using the given side condition, we obtain the following four solutions:

(i) $x = y = z = \frac{1}{3}$,

$$\left.\begin{array}{l} \text{or (ii) } x = y = \dfrac{z}{k-1} = \dfrac{1}{k+1}, \\ \text{or (iii) } x = z = \dfrac{y}{k-1} = \dfrac{1}{k+1}, \\ \text{or } (ii)\ y = z = \dfrac{x}{k-1} = \dfrac{1}{k+1} \end{array}\right\} k \neq 2$$

For instance, (ii) is obtained by solving

$x - y = 0$ (the first equation of (6))

and $x - ky + z = 0$ (the second equation of (7))

to get $\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-k+1}$, i.e., $x = y = \dfrac{z}{k-1} = \dfrac{x+y+z}{1+1+(k-1)} = \dfrac{1}{k+1}$, using the given side condition.

The solution (i) is symmetrical in x, y, z and gives the extreme value $u = (k+1)/9$. While the solutions (ii), (iii) and (iii) are non-symmetrical, but give the extreme value $u = (k^3+1)/(k+1)^3$ in each case. It is worth noting that if $k = 2$, there is only one solution, viz. (i), and the two extreme values are equal.

• Further, we have

$$d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \right)^2 F$$

$$= \sum \frac{\partial^2 F}{\partial x^2} dx^2 + 2\sum \frac{\partial^2 F}{\partial x \partial y} dxdy$$

$$= 6\left(x\,dx^2 + y\,dy^2 + z\,dz^2 \right) + 6k\left(z\,dxdy + x\,dydz + y\,dzdx \right)$$

$$= 2\left(dx^2 + dy^2 + dz^2 \right) + 2k\left(dxdy + dydz + dzdx \right) \text{, in Case (i).}$$

But the differentiation of the given relation $x + y + z = 1$ gives
$dx + dy + dz = 0$.
Squaring this equation, we have
$2(dxdy + dydz + dzdx) = -\left(dx^2 + dy^2 + dz^2 \right)$.
Using this relation in (4), we have

$$d^2F = 2\left(dx^2 + dy^2 + dz^2 \right) - k\left(dx^2 + dy^2 + dz^2 \right)$$

$$= (2-k)\left(dx^2 + dy^2 + dz^2 \right).$$

It follows that $d^2F < 0$ where $k > 2$ and $d^2F > 0$ where $k < 2$.
Hence the value of u in the symmetrical case is a maximum or minimum according as $k > \text{or} < 2$
.

Example 17. If $u = a^3x^2 + b^3y^2 + c^3z^2$, where $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, show that an extreme value of u is given by $ax = by = cz$ and this gives a true maximum or minimum if $abc(a+b+c)$ is positive.

Solution.

• Let us define a function F, where

$$F = \left(a^3x^2 + b^3y^2 + c^3z^2 \right) + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \right)$$

For an extreme value of u, we must have

$$\frac{\partial F}{\partial x} = 2a^3x - \frac{\lambda}{x^2} = 0; \ \frac{\partial F}{\partial y} = 2b^3y - \frac{\lambda}{y^2} = 0; \ \frac{\partial F}{\partial z} = 2c^3z - \frac{\lambda}{z^2} = 0.$$

Equating the values of λ obtained from these equations, we get

$$2a^3x^3 = 2b^3y^3 = 2c^3z^3 = \lambda, \qquad \text{....(1)}$$

so that $ax = by = cz = \mu$ (say).

These equations give

$$x = \frac{\mu}{a},\ y = \frac{\mu}{b},\ z = \frac{\mu}{c}. \qquad \text{....(2)}$$

• Using these values in the given relation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, we get

$\frac{a}{\mu}+\frac{b}{\mu}+\frac{c}{\mu}=1$, which gives $\mu = a+b+c$.

Substituting this value of μ in (2), we have

$$x = \frac{a+b+c}{a}, y = \frac{a+b+c}{b}, z = \frac{a+b+c}{c}. \qquad \text{....(3)}$$

• Now to identify whether u is maximum or minimum for these values of x, y, z, we have

$$d^2F = \left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2 F = \sum\frac{\partial^2 F}{\partial x^2}dx^2 + 2\sum\frac{\partial^2 F}{\partial x\partial y}dxdy$$

$$= 2\left(a^3+\frac{\lambda}{x^3}\right)dx^2 + 2\left(b^3+\frac{\lambda}{y^3}\right)dy^2 + 2\left(c^3+\frac{\lambda}{z^3}\right)dz^2, \text{ since } \partial^2 F/\partial x\partial y = 0 \text{ etc.}$$

$$= 6\left(a^3dx^2+b^3dy^2+c^3dz^2\right), \qquad \text{....(4)}$$

since by (1), $\lambda/x^3 = 2a^3, \lambda/y^3 = 2b^3, \lambda/z^3 = 2c^3$.

• But differentiating the relation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, we have

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}=0.$$

Using (3), this equation gives

$a^2dx+b^2dy+c^2dz=0$, so that $dz = -\left(\frac{a^2dx+b^2dy}{c^2}\right)$.

Substituting this value of dz in (4), we have

$$d^2F = 6\left\{a^3dx^2+b^3dy^2+c^3\left(\frac{a^2dx+b^2dy}{c^2}\right)^2\right\}$$

$$= 6\left\{\left(a^3+\frac{a^4}{c}\right)dx^2+\frac{2a^2b^2}{c}dxdy+\left(b^3+\frac{b^4}{c}\right)y^2\right\}.$$

• For a true maximum, we must have

$$\left(a^3+\frac{a^4}{c}\right)\left(b^3+\frac{b^4}{c}\right)-\left(\frac{a^2b^2}{c}\right)^2>0,$$

i.e., $a^3b^3+\dfrac{a^3b^3}{c}(a+b)+\dfrac{a^4b^4}{c^2}-\dfrac{a^4b^4}{c^2}>0$,

i.e., $\dfrac{a^3b^3}{c}(a+b+c)>0$ i.e., $\dfrac{a^2b^2}{c^2}\{(abc(a+b+c))\}>0$,

i.e., $abc(a+b+c)>0$, since a^2b^2/c^2 is positive.

Hence the result.

Example 18. Prove that if $x+y+z=1$, the function $u=ayz+bzx+cxy$ has an extreme value equal to $abc/\left(2bc+2ca+2ab-a^2-b^2-c^2\right)$.

Prove also that if a,b,c are all positive and c lies between $a+b\pm2\sqrt{(ab)}$, this value is a true maximum and that if a,b,c are all negative and c lies between $a+b\pm2\sqrt{(ab)}$, it is a true minimum.

Solution.

Let us define a function F, where

$F=(ayz+bzx+cxy)+\lambda(x+y+z-1)$.

For an extreme value of u, we must have

$\partial F/\partial x=bz+cy+\lambda=0$, ...(1)

$\partial F/\partial y=az+cx+\lambda=0$, ...(2)

and $\partial F/\partial z=ay+bx+\lambda=0$. ....(3)

Subtracting (3) from (1), we get

$(bz+cy)-(ay+bx)=0$, which gives $z=x+\dfrac{a-c}{b}y$.

Further, subtracting (3) from (2), we get

$(az+cx)-(ay+bx)=0$, which gives $z=y+\dfrac{b-c}{a}x$.

Equating the two values of z, we have

$x+\dfrac{a-c}{b}y=y+\dfrac{b-c}{a}x$, i.e., $\dfrac{x}{a(b+c-a)}=\dfrac{y}{b(c+a-b)}$.

By symmetry, we have

$$\frac{x}{a(b+c-a)}=\frac{y}{b(c+a-b)}=\frac{z}{c(a+b-c)}$$

$$=\frac{x+y+z}{a(b+c-a)+b(c+a-b)+c(a+b-c)}$$

$$=\frac{1}{2bc+2ca+2ab-a^2-b^2-c^2}, \text{ since } x+y+z=1,$$

$= \mu$, (say).

Then $x = \mu a(b+c-a), y = \mu b(c+a-b), z = \mu c(a+b-c)$.

Using these values, an extreme value of u is

$= a\mu b(c+a-b)\mu c(a+b-c) + b\mu c(a+b-c)\mu a(b+c-a) + c\mu a(b+c-a)\mu b(c+a-b)$

$= \mu^2 abc\{(c+a-b)(a+b-c) + (a+b-c)(b+c-a) + (b+c-a)(c+a-b)\}$

$= \mu^2 abc(2ab + 2bc + 2ca - a^2 - b^2 - c^2)$

$= abc/(2bc + 2ca + 2ab - a^2 - b^2 - c^2)$.

Now $d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 F = \sum \frac{\partial^2 F}{\partial x^2}dx^2 + 2\sum \frac{\partial^2 F}{\partial x \partial y}dxdy$

$= 2(c\,dxdy + a\,dydz + b\,dzdx)$, since $\partial^2 F/\partial x^2 = 0$ etc.

$= 2\{c\,dxdy + (a\,dy + b\,dx)dz\}$

$= 2\{c\,dxdy - (a\,dy + b\,dx)(dx + dy)\}$, since $dx + dy + dz = 0$, from $x + y + z = 1$

$= -\{2a\,dy^2 - 2(a+b-c)dxdy + 2b\,dx^2\}$.

Therefore, d^2F will have the same sign as $-2a$, if

$(2a)(2b) - (a+b-c)^2 > 0$,

i.e., $\left\{2\sqrt{(ab)}\right\}^2 - (a+b-c)^2 > 0$, provided $ab > 0$ ....(4)

i.e., $\left\{2\sqrt{(ab)} + (a+b-c)\right\}\left\{2\sqrt{(ab)} - (a+b-c)\right\} > 0$,

i.e., $\left[\left\{a+b+2\sqrt{(ab)}\right\} - c\right]\left[c - \left\{a+b-2\sqrt{(ab)}\right\}\right] > 0$,

which is true if c satisfies

$a+b-2\sqrt{(ab)} < c < a+b+2\sqrt{(ab)}$. ....(5)

Thus if c lies between $a+b-2\sqrt{(ab)}$ and $a+b+2\sqrt{(ab)}$, then $d^2F < 0$ when $a > 0$ and $d^2F > 0$ when $a < 0$.

Hence u has a true maximum of minimum according as a is positive or negative, and c lies between $a+b-2\sqrt{(ab)}$ and $a+b+2\sqrt{(ab)}$.

It should be noted that the condition $ab > 0$ in (4) is satisfied when both a and b are either positive or negative. In case, both a and b are positive, it follows from (5) that c is also positive. While if both a and b are negative, c is also negative. Thus for the feasibility of (5), a, b, c should be either all positive or all negative.

Example 19. Show that $u = yz + zx + xy$ has no extreme value when considered as a function of three independent variables x, y, z but has a maximum when the three variables are connected by the relation $ax + by + cz = 1$ and a, b, c are positive constants such that $2(ab + bc + ca) > (a^2 + b^2 + c^2)$.

Solution.

With usual notations, we know that the function u will have a minimum value if the expressions

$$A, \quad \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \quad \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, and a maximum value if they be alternatively negative and positive.

Now $A = \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}(z + y) = 0$.

Since A is neither negative nor positive, the function u has neither a maximum nor a minimum value. Thus u has no extreme value. **(First Result)**

When x, y, z are connected by the relation

$ax + by + cz = 1$, ...(1)

we define a function F, where

$F = (yz + zx + xy) + \lambda(ax + by + cz - 1)$.

For extreme values of u, we must have

$\partial F/\partial x = y + z + a\lambda = 0$, ...(2)

$\partial F/\partial y = z + x + b\lambda = 0$, ...(3)

and $\partial F/\partial z = x + y + c\lambda = 0$. ....(4)

Multiplying (2) by x, (3) by y, (4) by z and adding the resulting equations column wise, we get

$2(yz + zx + xy) + \lambda(ax + by + cz) = 0$,

i.e., $2u + \lambda = 0$, which gives $u = -\dfrac{1}{2}\lambda$. ....(5)

Further, adding (2), (3) and (4), we obtain

$2(x + y + z) + (a + b + c)\lambda = 0$,

so that $x + y + z + \dfrac{1}{2}(a + b + c)\lambda = 0$.

Putting one by one the values of $y + z, z + x, x + y$ obtained from (2), (3) and (4), this equation yields

$x - a\lambda + \dfrac{1}{2}(a + b + c)\lambda = 0$, ...(6)

$y - b\lambda + \dfrac{1}{2}(a + b + c)\lambda = 0$, ...(7)

and $z - c\lambda + \dfrac{1}{2}(a + b + c)\lambda = 0$. ....(8)

Multiplying (6) by a, (7) by b, (8) by c and adding the resulting equations column wise, we obtain

$$(ax+by+cz)-(a^2+b^2+c^2)\lambda+\frac{1}{2}(a+b+c)\lambda(a+b+c)=0,$$

i.e., $1+\lambda\left\{\frac{1}{2}(a^2+b^2+c^2+2ab+2bc+2ca)-(a^2+b^2+c^2)\right\}=0$,

which gives $\lambda=2/(a^2+b^2+c^2-2ab-2bc-2ca)$.

Using this value of λ in (5), the extreme value of u is

$1/(2ab+2bc+2ca-a^2-b^2-c^2)$.

Now $d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2F=\sum\frac{\partial^2F}{\partial x^2}dx^2+2\sum\frac{\partial^2F}{\partial x\partial y}dxdy$

$=2(dxdy+dydz+dzdx)$, since $\partial^2F/\partial x^2=0$ etc.

$=2(dxdy+(dx+dy)dz)$

$$=2\left\{dxdy-(dx+dy)\left(\frac{a\,dx+b\,dy}{c}\right)\right\}$$

since, from (1), $a\,dx+b\,dy+c\,dz=0$, so that $dz=-(a\,dx+b\,dy)/c$.

$$=-\frac{1}{c}\left\{2a\,dx^2+2(a+b-c)dxdy+2b\,dy^2\right\}.$$

Since a,b,c are positive, d^2F will be negative if $(2a)(2b)-(a+b-c)^2>0$,

i.e., $2(ab+bc+ca)>(a^2+b^2+c^2)$.

Hence u is maximum under the given conditions.

Example 20. If two variables x and y are connected by the relation $ax^2+by^2=ab$, show that the maximum and minimum values of the function x^2+y^2+xy will be the values of u given by the equation $4(u-a)(u-b)=ab$.

Solution.

We have $u=x^2+y^2+xy$ subject to the condition $ax^2+by^2=ab$.

Let us define a function F, where

$F=(x^2+y^2+xy)+\lambda(ax^2+by^2-ab)$.

For maxima and minima of u, we must have

$\partial F/\partial x=2x+y+2\lambda ax=0$, ...(1)

and $\partial F/\partial y=2y+x+2\lambda by=0$. ....(2)

Multiplying (1) by x, (2) by y and adding the resulting equations column wise, we get

$2(x^2+y^2+xy)+2\lambda(ax^2+by^2)=0$,

i.e., $u+\lambda ab=0$, or $\lambda=-u/ab$. ....(3)

Further, let us write equations (1) and (2) as
$2(1+\lambda a)x=-y$ and $x=-2(1+\lambda b)y$.
Since both x and y cannot be zero, it follows that
$4(1+\lambda a)(1+\lambda b)=1$ ....(4)

Putting the value of λ from (3), this gives
$4(1-u/b)(1-u/a)=1$,
or $4(u-a)(u-b)=ab$, ...(5)
which gives maximum and minimum values of u

Now $d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 F$

$$=\frac{\partial^2 F}{\partial x^2}dx^2+\frac{d^2F}{\partial y^2}dy^2+2\frac{\partial^2 F}{\partial x\partial y}dxdy$$

$$=2(1+\lambda a)dx^2+2(1+\lambda b)dy^2+2dxdy$$

$$=2(1+\lambda a)\{dx+2(1+\lambda b)dy\}^2\text{, by (4).}$$

Thus d^2F is positive or negative according as $1+\lambda a$ (or $1+\lambda b$) is positive or negative.
From (5), we obtain
$4a^2-4(a+b)u+3ab=0$,
i.e., $2u=(a+b)\pm\left\{(a-b)^2+ab\right\}^{1/2}$.

Therefore, supposing a and b to be positive, we see that with the upper sign,
$2u>(a+b)+|a-b|$,
i.e., $u<a$ or b whichever is less.
Further, since $u+\lambda ab=0$, we find that
if $u>a(a>b)$, then $1+\lambda b<0$,
and if $u<b\ (b<a)$, then $1+\lambda a>0$.
It follows that the value of u with the upper sign is the maximum, while the value with the lower sign is the minimum.

Again, since $2(1+\lambda a)x=-y$, we observe that when $1+\lambda a<0$, x and y have the same sign, and there are two points in the first and the third quadrants at which u is maximum. Similarly, u is minimum at two points lying in the second and the fourth quadrants.

PYQs: Taylor's, APPLICATIONS- MAXIMA-MINIMA, TOTAL DIFFERENTIATION

Q1. Using differentials, find an approximate value of $f(4\cdot1, 4\cdot9)$ where $f(x,y)=(x^3+x^2y)^{\frac{1}{2}}$.

[2c P-2 UPSC CSE 2019]

Q2. If $x=3\pm0.01$ and $y=4\pm0.01$, with approximately what accuracy can you calculate the polar coordinates r and θ of the point $P(x,y)$? Express your estimates as percentage changes of the values that r and θ have at the point $(3,4)$. **[3b 2009 IFoS]**

Q3. Find the maxima and minima for the function

$f(x,y)=x^3+y^3-3x-12y+20$

Also find the saddle points (if any) for the function. **[2017 2c IFoS]**

Q4. Find the relative maximum and minimum values of the function

$f(x,y)=x^4+y^4-2x^2+4xy-2y^2$. **[3b P-2 UPSC CSE 2016]**

Q5. Find the maxima and minima of the function $f(x,y)=x^3+y^3-3x-12y+20$.

[2016 1c P-2 IFoS]

Q6. Locate the stationary points of the function $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature. **[2013 2b IFoS]**

Q7. Find the points of local extrema and saddle points of the function f of two variables defined by

$f(x,y)=x^3+y^3-63(x+y)+12xy$. **[3a UPSC CSE 2012]**

Q8. Let $f(x,y)=y^2+4xy+3x^2+x^3+1$. At what points will $f(x,y)$ have a maximum or minimum? **[3c P-2 UPSC CSE 2013]**

Q9. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. **[2012 3b IFoS]**

Q10. Show that the function defined by

$f(x,y,z)=3\log(x^2+y^2+z^2)-2x^2-2y^3-2z^3$, $(x,y,z)\neq(0,0,0)$ has only one extreme value, $\log\left(\frac{3}{e^2}\right)$. **[2011 3d IFoS]**

Q11. Find the maxima, minima and saddle points of the surface $Z=(x^2-y^2)e^{(-x^2-y^2)/2}$.

[3d P-2 UPSC CSE 2010]

LAGRANGE'S MULTIPLYER'S METHOD

Q1. Using Lagrange's undetermined multipliers method, find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. [2b**IFoS 2022]**

Ans. Refer Example 3: Category-1

Q2. A wire length l is cut into two parts which are bent in the form of a square and a circle respectively. Using Lagrange's method of undetermined multipliers, find the least value of the sum of the areas so formed.

Solution. Let side of square is x and radius of circle is r. So the given condition is:
$\phi : 4x + 2\pi r = l \ldots(1)$
And, we have to find least value of $A = x^2 + \pi r^2 \ldots.(2)$
i.e. we have to find minima of A subject to condition (1).

Let's find $F = A + \lambda\phi = x^2 + \pi r^2 + \lambda(4x + 2\pi r - l)$

On putting equals zero $\partial F / \partial x$ & $\partial F / \partial y$; we get $\frac{x}{2} = r$

So from (1), we get $8r + 2\pi r = l \Rightarrow r = \frac{l}{2(4+\pi)}$

Therefore, minimum value of A is $\left\{\frac{l}{4+\pi}\right\}^2 + \pi\left\{\frac{l}{2(4+\pi)}\right\}^2 = \frac{l^2}{(4+\pi)^2}\left\{1 + \frac{\pi}{4}\right\} = \frac{l^2}{4(4+\pi)}$

Note. If not asked by Lagrange's method, then it can be interpreted as : finding minima of $A = \left\{\frac{(l - 2\pi r)}{4}\right\}^2 + \pi r^2$; A is one variable function. So put *da/dr* as zero, get $r = \frac{l}{2(4+\pi)}$. Now check sign of d^2A/dr^2, it's positive. So minimum is $\frac{l^2}{4(4+\pi)}$

[2b**UPSC CSE 2022]**

Q3. Find the shortest distance between the line $y = 10 - 2x$ and the ellipse
$\frac{x^2}{4} + \frac{y^2}{9} = 1$
using Lagrange's method of multipliers. **[2bIFoS 2021]**

Hint. Let (x, y) be a point on ellipse and u be the perpendicular distance from (x, y) on the given straight line: where $u = \frac{2x + y + 10}{\sqrt{2^2 + 1^2}} = \frac{2x + y + 10}{\sqrt{5}}$, $\phi = \frac{x^2}{4} + \frac{y^2}{9} = 1$

Q4. Find an extreme value of the function $u = x^2 + y^2 + z^2$, subject to the condition $2x + 3y + 5z = 30$, by using Lagrange's method of undetermined multiplier. **[4c UPSC CSE 2020]**
Ans. Refer Example-1: Category-1; take *a=2, b=3, c=5 and p=30*

Q5. Using Lagrange's multiplier, show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube. **[2b 2020 IFoS]**
Ans. Refer Example-3: Category-1; take $a = 1, b = 1, c = 1$.

Q6. Find the extreme values of $f(x,y,z)=2x+3y+z$ such that $x^2+y^2=5$ and $x+z=1$.

[3a 2020 P-2 IFoS]

Q7. Find the maximum value of $f(x,y,z)=x^2y^2z^2$ subject to the subsidiary condition $x^2+y^2+z^2=c^2,\ (x,y,z>0)$. **[4a P-2 UPSC CSE 2019]**

Ans. Take help from Example-5: Category-1;

Q8. Determine the extreme values of the function $f(x,y)=3x^2-6x+2y^2-4y$ in the region $\{(x,y)\in \mathbf{R}^2 : 3x^2+2y^2 \le 20\}$. **[2019 2a IFoS]**

Ans. Refer Example-20: Miscellaneous

Q9. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $ax+by+cz=p$.

Ans. Refer Example-1: Category-1

[2018 3a P-2 IFoS]

Q10. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the condition $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$. **[2017 3b P-2 IFoS]**

Ans. Refer Example-3: Category-1; take $a^2=4, b^2=5, c^2=25$

Q11. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $x+y-z=0$. **[3a UPSC CSE 2016]**

Ans. Refer Example-3: Category-1; take $a^2=4, b^2=5, c^2=25$

Q12. Using Lagrange's method of multipliers, find the point on the plane $2x+3y+4z=5$ which is closest to the point $(1,0,0)$. **[2016 3b IFoS]**

Ans. Refer Example-1: Category-1; take $u=x^2+y^2+z^2$; $a=2, b=3, c=4; p=5$

Q13. Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point $(2,1,3)$?

[3b UPSC CSE 2015]

Hint: Take $u=(x-2)^2+(y-1)^2+(z-3)^2$; $\phi; x^2+y^2+z^2=1$

On putting first order partial derivatives equal to zero, and managing terms as $(\lambda+1)x=2, (\lambda+1)y=1, (\lambda+1)z=3$; On solving , we get $x=2, y=1, z=3$; get max. at this point.

Q14. Find the absolute maximum and minimum values of the function $f(x,y)=x^2+3y^2-y$ over the region $x^2+2y^2 \le 1$. **[4b P-2 UPSC CSE 2015]**

Ans. Refer Example-20: Missellenеous

Q15. A rectangular box, open at the top, is said to have a volume of 432 cubic metres. Find the dimensions of the box so that the total surface is a minimum. **[2015 3c IFoS]**

Sol: Let the dimensions of the rectangular box be x, y and z.

Then volume, $V = xyz$

surface area of the rectangular box (open at the top) = $xy + 2z(x + y) = 432$

$F = xyz + \lambda(xy + 2z(x + y) - 432$

For points of maxima/minima;

$$yz + \lambda(y + 2z) = 0$$

$$xz + \lambda(x + 2z) = 0$$

$$xy + 2\lambda(x + y) = 0$$

On subtracting second from one;

$\Rightarrow (y - x)z + \lambda(y - x) = 0 \Rightarrow \quad (y - x)(z + \lambda) = 0 \Rightarrow y - x = 0, \ \therefore y = x$

Now multiplying second equation (by 2 and the subtracting the resulting equation from third equation, we get

$x(y - 2z) + 2\lambda(x + y - x - 2z) = 0 \Rightarrow (x + 2\lambda)(y - 2z) = 0 \Rightarrow y = 2z$

$\therefore$ The dimensions of the box are of the form; $x = y = 2z$

From $xy + 2z(x + y) = 432 \Rightarrow 12z^2 = 432 \Rightarrow z = 6$

Hence, the dimensions of the box are (12, 12, 6) cm respectively.

Q16. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

[3a UPSC CSE 2014]

Q17. Find the maximum or minimum values of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically.

Ans. Refer Example-8: Category-2

[3b UPSC CSE 2014]

Q18. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ the method of Lagrange multipliers.

Hint:

$F = (x^2 + y^2 + z^2) + \lambda(\log x + \log y + \log z - 3\log a)$

On putting first order partial derivatives equal to zero, and managing terms as *x=y=z; So taking x=y=z=a; As it satisfies given condition as well.*

$$d^2F = 8-\lambda\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) = 8-(-2a^2)(\frac{3}{a^2}) > 0.$$

Min. value is $3a^2$

[4b P-2 UPSC CSE 2014]

Q19. If $xyz = a^3$ then show that the minimum value of $x^2+y^2+z^2$ is $3a^2$.

Ans. Refer Example-3: Category-1; take $a^2=1, b^2=1, c^2=1$; take $u = xyz$ and proceed. Finally put required values as $u-a^3$.

[2014 2b IFoS]

Q20. Using Lagrange's multiplier method find the shortest distance between the line $y = 10-2x$ and the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. **[3a UPSC CSE 2013]**

Same as [2bIFoS 2021]

Hint. Let (x, y) be a point on ellipse and u be the perpendicular distance from (x, y) on the given straight line: where $u = \frac{2x+y+10}{\sqrt{2^2+1^2}} = \frac{2x+y+10}{\sqrt{5}}$, $\phi = \frac{x^2}{4}+\frac{y^2}{9}=1$

Q21. Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ from the origin, by the method of Lagrange's multipliers.

Ans. Refer Example-15: Category-3

[2d P-2 UPSC CSE 2012]

Q22. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$. **[3b UPSC CSE 2011]**

Hint: Same as [3b UPSC CSE 2015]

Q23. Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2+8xy+7y^2=225$.

Ans. Refer Example-12: Category-2 **[2d P-2 UPSC CSE 2011]**

Q24. Show that a box (rectangular parallelepiped) surface area is a cube.

Ans. Refer Example 3: Category-1

[2b UPSC CSE 2010]

Q25. A rectangular box open at the top is to have a surface area of 12 square units. Find the dimensions of the box so that the volume is maximum. **[2010 2a P-2 IFoS]**

Q26. Find the extreme value of xyz if $x+y+z=a$. **[2010 3b IFoS]**

Ans. Refer Example 3: Category-1

Q27. A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z)

on the probe surface is given by $T(x,y,z)=8x^2+4yz-16z+600$. Find the hottest point on the probe surface.
Hint: finding maxima of T subject to the condition $4x^2+y^2+4z^2=16$.

[3c UPSC CSE 2009]

Solutions(hints). NOTE- Some PYQs are left unsolved here just to avoid repetition. Those are already solved in examples of respective chapter with supporting examples. [Lagrange's method, maxima minima]

Hints for few answers

1. **Hint for Taylor's question**

We know that
If $f(x,y)$ is a function which possesses continuous partial derivatives of order n in any domain of a point (a,b), and the domain is large enough to contain a point $(a+h,\, b+k)$ within it, then there exists a positive number $0<\theta<1$, such that

$$f(a+h,\, b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$
$$+\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n,$$

where $R_n=\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0<\theta<1$.

Now for given question, take $a=4, h=0.1, b=5, k=-0.1$

Now find $f(2,5), \left(\frac{\partial f}{\partial x}\right)_{(4,5)}, \left(\frac{\partial f}{\partial y}\right)_{(4,5)}, \left(\frac{\partial^2 f}{\partial x^2}\right)_{(4,5)}, \left(\frac{\partial^2 f}{\partial y^2}\right)_{(4,5)}, \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(4,5)}$

Since h and k are small so neglecting higher powers after third term.
Use above values in

$$f(a+h,\, b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

Ans. 2 Use polar coordinates in above answer.

Ans.Show that the function defined by

$$f(x, y, z) = 3\log(x^2 + y^2 + z^2) - 2x^2 - 2y^3 - 2z^3,\ (x, y, z) \neq (0, 0, 0)$$

has only one extreme value, log $\left(\frac{3}{e^2}\right)$

Sol: The given function is

$$f(x, y, z) = 3\log(x^2 + y^2 + z^2) - 2x^2 - 2y^3 - 2z^3,\ (x, y, z) \neq (0, 0, 0)$$

for maxima/minima;

$$f_x = 3\cdot\frac{2x}{x^2+y^2+z^2} - 6x^2 = \frac{6x}{x^2+y^2+z^2} - 6x^2 = 0$$

$$\Rightarrow \quad \frac{6x\left[1 - x\left(x^2+y^2+z^2\right)\right]}{\left(x^2+y^2+z^2\right)} = 0$$

$$\Rightarrow \quad 1 - x\left(x^2+y^2+z^2\right) = 0$$

$$x\left(x^2+y^2+z^2\right) = 1$$

Similarly, $f_y = 0 \Rightarrow y\left(x^2+y^2+z^2\right) = 1$

And $f_z = 0 \Rightarrow z\left(x^2+y^2+z^2\right) = 1$

From above equations, we have; $x = y = z$, *i.e.* $x\left(x^2+x^2+x^2\right) = 1 \Rightarrow x = \frac{1}{3^{1/3}}$

So, $x = y = z = \frac{1}{3^{1/3}}$

Hence, the value of $f(x, y, z)$ at the point $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$ is given by

$$f\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right) = 3\log\left(\frac{1}{3^{2/3}} + \frac{1}{3^{2/3}} + \frac{1}{3^{2/3}}\right) - 2\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)$$

$$= 3\log\left(\frac{3}{3^{2/3}}\right) - 2 = 3\log 3^{1/3} - 2 = \frac{3}{3}\log 3 - 2 = \log 3 - 2 = \log\left(\frac{3}{e^2}\right)$$

Therefore, the only extreme value of $f(x, y, z)$ is $\log\left(\frac{3}{e^2}\right)$

Ans. A rectangular box, open at the top, is said to have a volume of 32 cubic meters. Find the dimensions of the box so that the total surface is minimum.

Sol: Let x, y, z are dimensions of this box

So volume is $\quad V = xyz = 32 \ldots(1)$

Surface area is $\quad S = xy + 2yz + 2zx \ldots.(2)$

Way1: By Lagrange's method-

Finding minima of (2) under the condition (1): **Refer Category 3 example 14**

Way 2: By treating this problem as finding maxima/minima of functions of two variables.

$$S = xy + 2y\cdot\frac{32}{xy} + 2x - \frac{32}{xy} = xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2};\quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$\frac{\partial S}{\partial x} = 0,\ \frac{\partial S}{\partial y} = 0$ gives $y - \frac{64}{x^2} = 0$, $x - \frac{64}{y^2} = 0$. We get $x=4$, $y=4$

So, $r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $s = \frac{\partial^2 S}{\partial x \partial y} = 1$, $t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

For stationary points, (4,4); $rt - s^2 = 4 - 1 = 3 > 0$ & $r > 0$

$\therefore$ (4, 4) is a point of minima.

$\therefore$ $x = 4,\ y = 4,\ z = \frac{32}{4 \times 4} = 2$

Ans. Determine the extreme values of the function $f(x, y) = 3x^2 - 6x + 2y^2 - 4y$ in the region $\{(x, y) \in \mathbb{R}^2 : 3x^2 + 2y^2 \le 20\}$

$f(x, y) = 3x^2 - 6x + 2y^2 - 4y$. So for points of maxima/minima

$f_x = 0 \Rightarrow 6x - 6 = 0 \Rightarrow x = 1$; $f_y = 0 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1$. $\therefore$ $P(1, 1)$ is the only critical point

notice; $3(1)^2 + 2(1)^2 = 5 < 20$ $\Rightarrow$ The point $P(1, 1)$ lies in the given region.

$$f(1, 1) = 3 - 6 + 2 - 4 = -5 \qquad \ldots\text{(i)}$$

$$rt - s^2 = (6)(4) - 0^2 = 24 > 0 \qquad \text{and}$$

$r = f_{xx} = 6 > 0$ *at* $P(1, 1)$. So, point (1,1) is a point of local minima.

Let us check at boundaries of the ellipse *i.e.* $3x^2 + 2y^2 = 20$

$$\therefore\ f(x, y) = 3x^2 - 6x + 2y^2 - 4y = 20 - 6x - 4y = 20 - 6x \pm 2\sqrt{2}\sqrt{20 - 3x^2}$$

Let $\phi(x) = 20 - 6x + 2\sqrt{2}\sqrt{20 - 3x^2}$

$\phi'(x) = -6 + 2\sqrt{2}\frac{(-6x)}{2\sqrt{20 - 3x^2}} = 0$; gives $x = \pm 2 \Rightarrow y = \mp 2$

At (2, −2), (-2,2), (2,2), (-2,-2); get the maximum value of *f(x,y)*.

Ans. At (-2,-2); value is 40.

Ans. Find the maxima and minima for the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. Also find the saddle points [if any] for the function

Step (i): To get points of maxima / minima

We put $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 12 = 0 \Rightarrow y = \pm 2$

Stationary point are: $A(1, 2), B(-1, 2), C(1, -2), D(-1, -2)$

Step (ii)

$r = \frac{\partial^2 f}{\partial x^2} = 6x$, $t = \frac{\partial^2 f}{\partial y^2} = 6y$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$(r)_A = 6 \times 1 = 6, (t)_A = 6 \times 2 = 12$

$\therefore$ At the point $A(1, 2)$

$rt - s^2 = 6\times12 - 0^2 = 72 > 0$ also $r > 0$

$\therefore$ A is a point of minima

$\therefore f_{\min A} = 1^3 + 2^3 - 3\times1 - 12\times2 + 20$

$(r)_B = 6\times(-1) = -6$, $(t)_B = 6\times2 \therefore rt - s^2 = -72 < 0$

$\therefore B(-1,2)$ is saddle point

$(r)_C = 6\times1 = 6, (t)_C = 6\times(-2) \;\therefore rt - s^2 < 0$

$\therefore C(1,-2)$ is saddle point

$(r)_D = 6\times(-1) = -6$, $(t)_0 = -12$, $\therefore rt - s^2 > 0$ and $r < 0 \;\therefore$ Maxima

Ans.: Find relative maximum and minimum values of the function

$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Step (i): To find point of maximum / minimum

We put $\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0$ ....(1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0$ ....(2)

On adding (1) and (2)

$\Rightarrow 4x^3 + 4y^3 = 0$

$\Rightarrow x^3 = -y^3$

$\Rightarrow x = -y$

Putting $x = -y$

$-4y^3 + 4y + 4y = 0$

$-4y^3 + 5y = 0$

$4y(-y^2 + 2) = 0$

$y = 0$, $y^2 = 2,\; y = \pm\sqrt{2}$

Points are $A(0,0), B(-\sqrt{2},\sqrt{2}), C(\sqrt{2},\sqrt{2})$

Step (ii)

$r=\frac{\partial^2 f}{\partial x^2}=12x^2-4$, $t=\frac{\partial^2 f}{\partial y^2}=12y^2-4$, $s=\frac{\partial^2 f}{\partial x\partial y}=4$

$rt-s^2=\left(12x^2-4\right)\left(12y^2-4\right)-4^2$

$=16\left[\left(3x^2-1\right)\left(3y^2-1\right)-1\right]$

$\Rightarrow\left(rt-s^2\right)A=0$; Doubtful care

$\Rightarrow\left(rt-s^2\right)_B>0$; $(r)_B>0\;\therefore B$ is a minima

$\Rightarrow\left(rt-s^2\right)_C>0$; $(r)_C>0\;\therefore C$ is a minima

$f_B=\left(-\sqrt{2}\right)^4+\left(\sqrt{2}\right)^4-2\times\left(-\sqrt{2}\right)^2+4\left(-\sqrt{2}\right)\sqrt{2}-2\left(\sqrt{2}\right)^2$

$f_C=$

Ans. Find relative maximum and minimum values of the function $f(xy)=x^4+y^4-2x^2+4xy-2y^2$

How to investigate doubtful care??

[Although its not easy]

$\because$ Point $(0,0)$ is a doubtful case

We would like to get the sign of $f(x,y)-f(0,0)$

Where (x,y) denote arbitrary point in nbd of $(0,0)$ i.e. x and y are small closer to zero

Let's do it:

$f(x,y)-f(0,0)=x^4+y^4-2x^2+4xy-2y^2-0$

$=x^4+y^4-\left(2x^2-4xy+2y^2\right)$

$=\left(x^4+y^4\right)-\left[\left(\sqrt{2}x\right)^2-2\times\sqrt{2}x\cdot\sqrt{2}y+\left(\sqrt{2}y\right)^2\right]$

$=\left(x^4+y^4\right)-\left(\sqrt{2}x-\sqrt{2}y\right)^2$

$=\left(x^4+y^4\right)-2(x-y)^2$

$f(x,y)-f(0,0)<0$

$\Rightarrow$ the point $(0,0)$ is a point of relative maxima

Ans.: **Show that the function** $u=(x+y+z)^3-3(x+y+z)-24xy^2+a^3$ **has a minimum at** $(1,1,1)$ **and maximum at** $(-1,-1,-1)$

Step (i) Already given in question

(i) At the point $(1,1,1)$

$$(A)_{(1,1,1)}=6(1+1+1)=18>0$$

Finding B, C, F, G, H as $\partial^2 f/\partial y^2, \partial^2 f/\partial z^2, \partial^2 f/\partial y\partial z, \partial^2 f/\partial z\partial x, \partial^2 f/\partial x\partial y$ at the point (1,1,1). Then observing

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix}=\begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix}>0$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}>0$$

Therefore, it's minimum at given point.

Chapter- Implicit Functions, JACOBIANS

JACOBIANS

https://www.youtube.com/live/VzMxocY1M9k?si=3sRaeI-m8ocxdktX

If $u_1, u_2, ..., u_n$ be n differentiable functions of n variables $x_1, x_2, ..., x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the *Jacobian* or the *Functional Determinant* of the functions $u_1, u_2, ..., u_n$ with respect to $x_1, x_2, ..., x_n$ and is denoted by

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} \text{ or } J\left(\frac{u_1, u_2, ..., u_n}{x_1, x_2, ..., x_n}\right)$$

Some Properties

Jacobians have the remarkable property of behaving like the derivatives of functions of one variable. A few of the important relations are given here and the proofs depend upon the algebra of determinants.

For $n = 1$, the determinant is simply $\frac{\partial y_1}{\partial x_1}$ or $\frac{dy_1}{dx_1}$, the derivative of y_1 with respect to x_1; the first of the notations for a Jacobian is suggested by a certain analogy between the properties of the Jacobian and the derivative.

Theorem 1. *If $u_1, u_2, ..., u_n$ are functions of $y_1, y_2, ..., y_n$ and $y_1, y_2, ..., y_n$ are themselves functions of $x_1, x_2, ..., x_n$, then*

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(u_1, u_2, ...u_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} \quad ....(1)$$

For $n = 1$, the theorem reduces to the usual notation

$$\frac{du_1}{dx_1} = \frac{du_1}{dy_1}\frac{dy_1}{dx_1}$$

The proof of the theorem depends on the "row by column" rule of multiplication of determinants combined with the rule for the derivative of a function of a function.

Thus for determinants on the right hand side of (1), r^{th} row of the first is $\frac{\partial u_r}{\partial y_1}, \frac{\partial u_r}{\partial y_2}, ..., \frac{\partial u_r}{\partial y_n}$, s^{th} column of the second is $\frac{\partial y_1}{\partial x_s}, \frac{\partial y_2}{\partial x_s}, ..., \frac{\partial y_n}{\partial x_s}$, so that the element in the r^{th} row and the s^{th} column of the product is

$$\frac{\partial u_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial u_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial u_r}{\partial y_n}\frac{\partial y_n}{\partial x_s}$$

and this is equal to $\frac{\partial u_r}{\partial x_s}$, which is the element in the r^{th} row and the s^{th} column of the Jacobian on the left hand side. Hence the theorem.

Corollary. If $x_r = u_r$, $r = 1, 2, ..., n$ and assuming the existence of inverse functions $x_1, x_2, ..., x_n$ (that is, assuming that the equations which define $y_1, y_2, ..., y_n$ as functions of $x_1, x_2, ..., x_n$ determine $x_1, x_2, ..., x_n$ as functions of $y_1, y_2, ..., y_n$) we find

$$\frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)} \cdot \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(x_1, x_2, ..., x_n)} = 1 \qquad ....(2)$$

since $\frac{\partial x_i}{\partial x_j} = 0$, for $i \neq j$ and $= 1$, for $i = j$

Theorem 2. *If $y_1, y_2, ..., y_n$ are determined as functions of $x_1, x_2, ..., x_n$ by the equations*

$$\phi_r(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) = 0, \quad r = 1, 2, ..., n$$

then

$$\frac{\partial(\phi_1, \phi_2, ..., \phi_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\partial(\phi_1, \phi_2, ...\phi_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ...y_n)}{\partial(x_1, x_2, ..., x_n)} \qquad ....(3)$$

[Theorem 1 is a particular form of this theorem.]

Differentiating the equations $\phi_r = 0$ with respect to x_s, we get

$$\frac{\partial \phi_r}{\partial x_s} + \frac{\partial \phi_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = 0$$

or

$$\frac{\partial \phi_r}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = -\frac{\partial \phi_r}{\partial x_s}$$

so that the element in the r^{th} row and the s^{th} column of the determinant which is the product of the two determinants on the right of (3) is $-\dfrac{\partial \phi_r}{\partial x_s}$, from which the result follows.

Theorem 3. *(i) If* $y_{m+1}, y_{m+2}, ..., y_n$ *are constant with respect to* $x_1, x_2, ..., x_m$, *or (ii) if* $y_1, y_2, ..., y_m$ *are constant with respect to* $x_{m+1}, x_{m+2}, ..., x_n$, *then*

$$\frac{\partial(y_1, y_2, ..., y_m, ...y_n)}{\partial(x_1, x_2, ..., x_m, ..., x_n)} = \frac{\partial(y_1, y_2, ..., y_m)}{\partial(x_1, x_2, ..., x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, ..., y_n)}{\partial(x_{m+1}, x_{m+2}, ..., x_n)} \qquad ....(4)$$

(i) $\dfrac{\partial y_r}{\partial x_s} = 0$, when $r = m+1, m+2, ..., n;\ s = 1, 2, ..., m$

$$\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \cdots & \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial y_{m+2}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+2}}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial(y_1, y_2, ..., y_m)}{\partial(x_1, x_2, ..., x_n)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, ..., y_n)}{\partial(x_{m+1}, x_{m+2}, ..., x_n)}$$

(ii) may also be proved similarly.

Corollary. *In particular,*

$$\frac{\partial(y_1, ..., y_m, x_{m+1}, ..., x_n)}{\partial(x_1, ..., x_m, x_{m+1}, ..., x_n)} = \frac{\partial(y_1, ..., y_m)}{\partial(x_1, ..., x_m)} \quad ....(5)$$

Theorem 4. If u, v are functions of ξ, η, ζ and the variables ξ, η, ζ are themselves functions of the independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\xi,\eta)} \cdot \frac{\partial(\xi,\eta)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\eta,\zeta)} \cdot \frac{\partial(\eta,\zeta)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\zeta,\xi)} \cdot \frac{\partial(\zeta,\xi)}{\partial(x,y)} \quad ....(6)$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} \quad ....(7)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} \quad ....(8)$$

and if we substitute these values in the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$, we get

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial \xi}\frac{\partial(\xi,v)}{\partial(x,y)} + \frac{\partial u}{\partial \eta}\frac{\partial(\eta,v)}{\partial(x,y)} + \frac{\partial u}{\partial \zeta}\frac{\partial(\zeta,v)}{\partial(x,y)} \quad ....(9)$$

which is a linear expression of the Jacobians of $(\xi,v),(\eta,v)$ and (ζ,v) with respect to *x* and *y*.

Now in each Jacobian on the right of equation (9), substitute the expressions for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ which are similar to (7) and (8).

Each of these Jacobians will be given as a linear expression of the Jacobians of (ξ,η), (η,ζ) and (ζ,ξ) since those of (ξ,ξ), (η,η) and (ζ,ζ) have two identical parallel lines and so vanish. Thus we see that the terms which involve the Jacobian of (ξ,η) are

$$\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta}\frac{\partial(\xi,\eta)}{\partial(x,y)} + \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi}\frac{\partial(\eta,\xi)}{\partial(x,y)}$$

which is equal to $\frac{\partial(u,v)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(x,y)}$, the first terms on the right of equation (6).

Similarly, we obtain the remaining two terms and the formula is established.

Exam Point: (Applications of Jocobian) – PYQs 60 – 70%. **CSE & IFoS**

If $\frac{\partial(u_1,u_2,......u_n)}{\partial(x_1,x_2,....x_n)}=0$ Then we say u_1, u_2 … u_n are functionally related (or dependent on x_1, x_2, …. x_n).

u_1, u_2, ….u_n will be some functions of each other

If J is non zero; then, u_1, u_2, ….u_n are functionally independent.

Examples

Example1. If $x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta$, then show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$$

- $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

  Taking r common from C_2 and $r\sin\theta$ common from C_3

$$=r^2\sin\theta\begin{vmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{vmatrix}$$

  Adding $(\cos\phi)R_1$ to $(\sin\phi)R_2$,

$$=\frac{r^2\sin\theta}{\sin\phi}\begin{vmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta & \cos\theta & 0\\ \cos\theta & -\sin\theta & 0\end{vmatrix}=r^2\sin\theta$$

Example2. **4**. If $y_1+y_2+...+y_n=x_1,\ y_2+y_3+...+y_n=x_1x_2,...,y_r+y_{r+1}+...+y_n=x_1x_2...x_r,...,$ $y_n=x_1x_2...x_n$, then show that

$$\frac{\partial(y_1,y_2,...,y_n)}{\partial(x_1,x_2,...x_n)}=x_1^{n-1}x_2^{n-2}...x_{n-2}^2x_{n-1}$$

- Solving for $y_1,y_2,....,y_n$, we get

$$y_1 = x_1 - x_1x_2 = x_1(1-x_2)$$
$$y_2 = x_1x_2 - x_1x_2x_3 = x_1x_2(1-x_3)$$
$$\vdots$$
$$y_{n-1} = x_1x_2 \ldots x_{n-1}(1-x_n)$$
$$y_n = x_1x_2 \ldots x_n$$

$$\therefore \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} =$$

$$\begin{vmatrix} 1-x_2 & -x_1 & 0 & \cdots & 0 \\ x_2(1-x_3) & x_1(1-x_3) & -x_1x_2 & \cdots & 0 \\ x_2x_3(1-x_4) & x_1x_3(1-x_4) & x_1x_2(1-x_4) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2x_3 \ldots x_{n-1}(1-x_n) & x_1x_3 \ldots x_{n-1}(1-x_2) & x_1x_2x_4 \ldots x_{n-1}(1-x_n) & \cdots & x_1x_2 \ldots x_{n-1} \\ x_3x_3 \ldots x_n & x_1x_2x_4 \ldots x_n & x_1x_3 \ldots x_n & \cdots & x_1x_2 \ldots x_{n-1} \end{vmatrix}$$

Adding R_n to R_{n-1}, then R_{n-1} to R_{n-2},..., then R_2 to R_1 and expanding by last column

$$= (x_1x_2 \ldots x_{n-1})(x_1x_2 \ldots x_{n-2}) \ldots (x_1x_2)(x_1)$$
$$= x_1^{n-1}x_2^{n-2} \ldots x_{n-2}^2 x_{n-1}$$

Example3. The roots of the equation in λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are *u*, *v*, *w*. Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

- Here *u*, *v*, *w* are roots of the equation

$$\lambda^3 - (x+y+z)\lambda^2 + (x^2+y^2+z^2)\lambda - \frac{1}{3}(x^3+y^3+z^3) = 0$$

Let $x+y+z = \xi,\ x^2+y^2+z^2 = \eta,\ \frac{1}{3}(x^3+y^3+z^3) = \zeta$ ....(1)

$\therefore\ u+v+w = x+y+z = \xi$, $vw+wu+uv = x^2+y^2+z^2 = \eta$, $uvw = \frac{1}{3}(x^3+y^3+z^3) = \zeta$ ....(2)

Hence from (1),

$$\therefore \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 2(y-z)(z-x)(x-y) \qquad ....(3)$$

and from (2),

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= -(v-w)(w-u)(u-v) \qquad ....(4)$$

Hence from (3) and (4) and using theorem 1, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)} \cdot \frac{\partial(\xi,\eta,\zeta)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Example4. If $y_r = \frac{u_r}{u}$, $r = 1, 2, ..., n$, and if u and u_r are functions of the n independent variables $x_1, x_2, ..., x_n$ prove that

$$\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{1}{u^{n+1}} \begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

- Now

$$\frac{\partial y_r}{\partial x_s} = \frac{1}{u}\frac{\partial u_r}{\partial x_s} - \frac{u_r}{u^2}\frac{\partial u}{\partial x_s} \qquad \left(\because y_r = \frac{u_r}{u} \text{ (given)}\right)$$

$$\therefore \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{1}{u}\frac{\partial u_1}{\partial x_1} - \frac{u_1}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_1}{\partial x_n} - \frac{u_1}{u^2}\frac{\partial u}{\partial x_n} \\ \frac{1}{u}\frac{\partial u_2}{\partial x_1} - \frac{u_2}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_2}{\partial x_n} - \frac{u_2}{u^2}\frac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{1}{u}\frac{\partial u_n}{\partial x_1} - \frac{u_n}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_n}{\partial x_n} - \frac{u_n}{u^2}\frac{\partial u}{\partial x_n} \end{vmatrix}$$

Taking out $\frac{1}{u}$ from each column and bordering the determinant, we get

$$\frac{\partial(y_1,y_2,....,y_n)}{\partial(x_1,x_2,....,x_n)}=\frac{1}{u^n}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ u_1 & \frac{\partial u_1}{\partial x_1}-\frac{u_1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n}-\frac{u_1}{u}\frac{\partial u}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1}-\frac{u_2}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n}-\frac{u_2}{u}\frac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1}-\frac{u_n}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n}-\frac{u_n}{u}\frac{\partial u}{\partial x_n} \end{vmatrix}$$

Operating $C_2+\frac{1}{u}\frac{\partial u}{\partial x_1}C_1,\ C_3+\frac{1}{u}\frac{\partial u}{\partial x_2}C_1,....,C_{n+1}+\frac{1}{u}\frac{\partial u}{\partial x_n}C_1$

$$=\frac{1}{u^n}\begin{vmatrix} 1 & \frac{1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Taking $\frac{1}{u}$ common from R_1,

$$=\frac{1}{u^{n-1}}\begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Example 5. If $u=\frac{x^2+y^2+z^2}{x}$, $v=\frac{x^2+y^2+z^2}{y}$, and $w=\frac{x^2+y^2+z^2}{z}$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} 1-\dfrac{y^2+z^2}{x^2} & \dfrac{2y}{x} & \dfrac{2z}{x} \\ \dfrac{2x}{y} & 1-\dfrac{x^2+z^2}{y^2} & \dfrac{2z}{y} \\ \dfrac{2x}{z} & \dfrac{2y}{z} & 1-\dfrac{x^2+y^2}{z^2} \end{vmatrix}$$

- Applying $C_1 \to C_1 + \frac{y}{x}C_2 + \frac{z}{x}C_3$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} \dfrac{x^2+y^2+z^2}{x^2} & \dfrac{2y}{x} & \dfrac{2z}{x} \\ \dfrac{x^2+y^2+z^2}{xy} & 1-\dfrac{x^2+z^2}{y^2} & \dfrac{2z}{y} \\ \dfrac{x^2+y^2+z^2}{xz} & \dfrac{2y}{z} & 1-\dfrac{x^2+y^2}{z^2} \end{vmatrix}$$

Taking $x^2+y^2+z^2$ common from C_1; multiplying R_1 by x^2, R_2 by xy and R_3 by xz,

$$=\frac{\left(x^2+y^2+z^2\right)}{x^2\cdot xy\cdot xz}\begin{vmatrix} 1 & 2xy & 2xz \\ 1 & xy-\dfrac{x}{y}\left(x^2+z^2\right) & 2xz \\ 1 & 2yz & xz-\dfrac{x}{z}\left(x^2+y^2\right) \end{vmatrix}$$

Operating R_2-R_1 and R_3-R_1,

$$=\frac{\left(x^2+y^2+z^2\right)}{x^4yz}\begin{vmatrix} 1 & 2xy & 2xz \\ 0 & -\dfrac{x\left(x^2+y^2+z^2\right)}{y} & 0 \\ 0 & 0 & -\dfrac{x}{z}\left(x^2+y^2+z^2\right) \end{vmatrix}$$

$$=\frac{\left(x^2+y^2+z^2\right)^3}{x^2y^2z^2}$$

$$\therefore\ \frac{\partial(x,y,z)}{\partial(u,v,w)}=\frac{x^2y^2z^2}{\left(x^2+y^2+z^2\right)^3}$$

Q.1. If $u = \cos x, v = \sin x \cos y, w = \sin x \sin y \cos z$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^2 \sin^3 x \sin^2 y \sin z$$

Q.2. If $u = a\cosh x \cos y, v = a \sinh x \sin y$, then show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}a^2(\cosh 2x - \cos 2y)$$

Q.3. If $x+y+z=u, y+z=uv, z=uvw$, then show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v$$

Q4. If α,β,γ are the roots of the equation in t, such that

$$\frac{u}{a+t}+\frac{v}{b+t}+\frac{w}{c+t}=1,$$

then prove that

$$\frac{\partial(u,v,w)}{\partial(\alpha,\beta,\gamma)}=-\frac{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(b-c)(c-a)(a-b)}.$$

Q5. If $u=x/(1-r^2)^{1/2}$, $v=y/(1-r^2)^{1/2}$, $w=z/(1-r^2)^{1/2}$ where $r^2=x^2+y^2+z^2$, then show that

$$J\begin{pmatrix} u, & v, & w \\ x, & y, & z \end{pmatrix}=\frac{1}{(1-r^2)^{5/2}}.$$

PYQs Jacobian

Q1. If $x+y+z=u, y+z=uv, z=uvw$, then determine $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. [1d**IFoS 2022]**

Q2.If $u=x^2+y^2, v=x^2-y^2$, where $x=r\cos\theta, y=r\sin\theta$, then find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

[3aUPSC CSE 2021]

Q3. Using the Jacobian method, show that if $f'(x)=\frac{1}{1+x^2}$ and $f(0)=0$, then

$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$. **[4c (ii) UPSC CSE 2019]**

Q4. Show that the functions $u=x+y+z, v=xy+yz+zx$ and $w=x^3+y^3+z^3-3xyz$ are dependent and find the relation between them. **[2018 4b IFoS]**

Q5. Let $u(x,y)=ax^2+2hxy+by^2$ and $v(x,y)=Ax^2+2Hxy+By^2$. Find the Jacobian $J=\frac{\partial(u,v)}{\partial(x,y)}$, and hence show that u,v are independent unless $\frac{a}{A}=\frac{b}{B}=\frac{h}{H}$.

[2017 1d IFoS]

Q6. Show that the functions:

$u=x^2+y^2+z^2$

$v=x+y+z$

$w = yz + zx + xy$

are not independent of one another. **[2012 1d P-2 IFoS]**

Q7. The roots of the equation in λ

$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v, w. Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = -2\dfrac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$.

Q8. Show that the function u, v, w given by $u = \dfrac{x}{y-z}, v = \dfrac{y}{z-x}$ and $\omega = \dfrac{z}{x-y}$ are not independent of one another. Also find the relation between them.

Let's start analyzing now

$$J\left(\frac{(u_1, u_2, \ldots\ldots u_n)}{(x_1, x_2, \ldots\ldots x_n)}\right) = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \ldots\ldots\ldots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \ldots\ldots & \frac{\partial u_2}{\partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Jacobians
Asymptotes
} With PYQs discussion.

Questions:- The roots of the eq^n in λ; $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v, w. Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-2(x-y)(y-z)(z-x)}{(v-w)(w-u)(u-v)}$$

Ans:- $\because$ u, v, w are roots of $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

$$\lambda^3 - (x + y + z)\lambda^2 - (x^2 y^2 z^2)\lambda - \frac{1}{3}(x^3 + y^3 + z^3) = 0$$

We know that;

$u + v + w = (x + y + z)$

$uv + vw + wu = (x^2 + y^2 + z^2)$

$uvw = \frac{1}{3}(x^3 + y^3 + z^3)$

Note:-

Here neither u, v, w nor x, y, z are explicit functions.

$u = \phi_1(x, y, z)$

$v = \phi_2(x, y, z)$

$w = \phi_3(x, y, z)$

or

$x = f_1(u, v, w)$

$y = f_2(u, v, w)$

$z = f_3(u, v, w)$

Let's come out……. solution for it.

Let $\boxed{\eta = x + y + z}$, $\boxed{\xi = x^2 + y^2 + z^2}$, $\boxed{\phi = \frac{1}{3}\left(x^3 + y^3 + z^3\right)}$

$\Rightarrow$ η, ξ, ϕ, are explicit functions of x, y, z. So calculating Jacobians is easy pass.

Also we have now

$\boxed{\eta = u + v + w}$, $\boxed{\xi = uv + vw + wu}$, $\boxed{\phi = uvw}$

$\Rightarrow \eta, \xi, \phi$, are exclusive functions of x, y, z.

$$\frac{\partial(\eta,\xi,\phi)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

$$\frac{\partial(\eta,\xi,\phi)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ (v+w) & (u+v) & (v+u) \\ vw & uw & uv \end{vmatrix} = (u-w)\,(w-u)\,(u-v)$$

Required $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(\eta,\xi,\phi)} . \frac{\partial(\eta,\xi,\phi)}{\partial(x,y,z)} = \frac{-2(x-y)(y-2)(z-x)}{(u-w)(w-u)(u-v)}$$

Ex.4.: If $u = \frac{x^2 + y^2 + z^2}{x}, v = \frac{x^2 + y^2 + z^2}{y}$

$w = \frac{x^2 + y^2 + z^2}{z}$. Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Ans:- $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} \dfrac{1}{x}\times 2x-\dfrac{1}{x^2}\left(x^2+y^2+z^2\right) & \dfrac{1}{x}\times 2y & \dfrac{1}{x}\times 2z \\ \dfrac{1}{y}\times 2x & \dfrac{1}{y}\times 2y+\dfrac{1}{y^2} & \dfrac{1}{y}\times 2z \\ \dfrac{1}{z}\times 2x & \dfrac{1}{z}\times 2y & \dfrac{1}{z}\times 2z-\dfrac{1}{z^2}\left(x^2+y^2+z^2\right) \end{vmatrix}$

$$1-\frac{y^2+2^2}{x^2}\begin{vmatrix} 1-\dfrac{y^2+z^2}{x^2} & \dfrac{2y}{x} & \dfrac{2z}{x} \\ \dfrac{2x}{y} & 1-\dfrac{z^2+x^2}{y^2} & \dfrac{2z}{y} \\ \dfrac{2x}{z} & \dfrac{2y}{z} & 1-\dfrac{\left(x^2+y^2\right)}{z^2} \end{vmatrix}$$

$$c_1\to c_1+\frac{y}{x}c_2+\frac{z}{x}c_3;\begin{vmatrix} \dfrac{x^2+y^2+z^2}{x^2} & \dfrac{2y}{x} & \dfrac{2z}{x} \\ \dfrac{x^2+y^2+z^2}{x.y} & 1-\dfrac{x^2+z^2}{y^2} & \dfrac{2z}{y} \\ \dfrac{x^2+y^2+z^2}{x.z} & \dfrac{2y}{z} & 1-\dfrac{x^2+y^2}{z^2} \end{vmatrix}$$

R_2-R_1, R_3-R_1

$$=\frac{x^2+y^2+z^2}{x^4yz}\begin{vmatrix} 1 & 2xy & 2xz \\ 0 & \dfrac{-x\left(x^2+y^2+z^2\right)}{y} & 0 \\ 6 & 0 & \dfrac{-x}{2}\left(x^2+y^2+z^2\right) \end{vmatrix}$$

$$=\frac{\left(x^2+y^2+z^2\right)}{x^2.y^2.z^2}$$

Another type of problems :-

(Applications of Jocobian) – PYQs 60 – 70%.

If $\dfrac{\partial(u_1,u_2,......u_n)}{\partial(x_1,x_2,....x_n)}=0$ Then we say u_1, u_2 ... u_n are functionally related (or dependent on x_1, x_2, ... x_n).

u_1, u_2, ...u_n will be some functions of each other

If J is non zero; then, $u_1, u_2, \ldots u_n$ are functionally independent.

Examples: Show that the functions $u = x^2 + y^2 + z^2$,

$v = x + y + z,\ w = yz + zx + xy$ are not independent of one another.

Ans:- we know that if Jacobian of u, v, w w.r.t, x, y, z is zero then u, v, w will be functional dependent otherwise independent.

$\therefore \quad u = x^2 + y^2 + z^2, \qquad v = x + y + z,\ \ w = yz + zx + xy$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ z+y & z+x & x+y \end{vmatrix} = 0$$

So, u, v, w are not independent. hence proved.

Q: Show that the function $u = x + y + z$, $v = xy + yz + zx$, $w = x^3 + y^3 + z^3 - 3xyz$ are dependent and find the relation between them;

$$\begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ 3x^2-3yz & 3y^2-3zx & 3z^2-3xy \end{vmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \left|\frac{\partial u}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial u}{\partial z}\right|$$

$= (z + x)\ \{3(z^2 - xy)\} - 3(y^2 - zx),\ (x + y) - 3\ (z^2 - xy)\ (y + z) + 3\ (x^2 - yz)\ (x + y) + 3\ (y^2 - zx)\ (y + z)$

$- 3\ (x^2 - yz)\ (z + x)$

$= 3\{z^2x - x^2y - z^2y + x^2y + y^2z - z^2x - xy^2 + zx^2 + x^2y - y^2z - x^2z + yz^2\}$

$= 0$

$$\underbrace{x^3 + y^3 + z^3 - 3xyz}_{w} = (x + y + z)\ \{x^2 + y^2 + z^2 - xy - yz - zx\}$$

$$= \underbrace{\underbrace{(x+y+z)}_{u}\left\{\underbrace{(x+y+z)^2}_{u^2} - 3\underbrace{(xy+yz+zx)}_{v}\right\}}_{\boxed{w=u(u^2-3v)} \Rightarrow w=\phi(u,v)}$$

Hint:

$$\left.\begin{matrix} x+y+z \\ xy+yz+zx \\ x^3+y^3+z^3-3xyz \end{matrix}\right\}$$ Just observe and try to manage terms to get relation between u, v, w.

No standard rule here; Hit and trial works.

As this is not a linear relation between u, v and w. So the function u, v, and w are independent.

Q. Using the Jacobians method, show that if $f'(x) = \frac{1}{1+x^2}$ and $f(0) = 0$, then $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Approach:- Our target is to show $f(x) + f(y)$ is some function of $\frac{x+y}{1-xy}$.

Exam point

$\Rightarrow$ Let if $u = f(x) + f(y)$, $v = \frac{x+y}{1-xy}$ then u and v are functionally dependent. (i.e. Jacobians of u, v must be zero)

Now check $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} f'(x) & f'(y) \\ \left(\frac{1}{(1-xy)}.1 + \frac{(x+y)y}{(1-xy)^2}\right) & \left(\frac{1}{z-xy}\cdot 1 + \frac{(x+y)x}{(1-xy)^2}\right) \end{vmatrix} = \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix} = 0$$

$\because$ Jacobians of u, v is zero.

$\therefore u = \phi(v)$

$f(x) + f(y) = \phi\left(\frac{xy}{1-xy}\right)$

$\because$ using $f(0) = 0$; we can get $f(x) = \phi(x)$

So; $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Q. Let $u(x, y) = ax^2 + 2hxy + by^2$ and $v(x, y) = Ax^2 + 2Hxy + By^2$

Find the Jacobian J = $J = \frac{\partial(u,v)}{\partial(x,y)}$ and hence show that u, v are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$.

Ans. $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} 2ax + 2hy & 2hx + 2by \\ 2Ax + 2Hy & 2Hx + 2By \end{vmatrix}$$

$= 4(ax + 2hy)\,(Hx + By) - 4(Ax + Hy)\,(hx + by)$

$= 4\{aHx^2 + xy(hH + qB) + hBy^2) - (Abx^2 + xy\,(Hh + Ab) + bHy^2\}$

$= -4\{(aH - Ah)x^2 + (aB - Ab)xy + (Bb - bH)\,y^2\}$

Now if $J = 0$; then u and v will be dependent. Which Possible only when

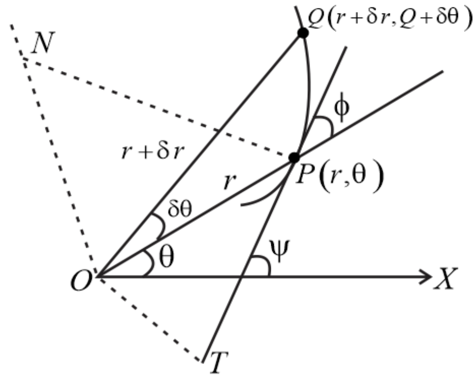
$$\left.\begin{aligned} aH - Ah &= 0 \\ aB - Ab &= 0 \\ Bh - bH &= 0 \end{aligned}\right\} \Rightarrow \frac{a}{A} = \frac{b}{B} = \frac{h}{H}$$

Therefore u and v are independent unless; $\boxed{\frac{a}{A} = \frac{b}{B} = \frac{h}{H}}$

Some Basics required at different topics to study in mathematics optional.

1. **Tangents and Normals:-**

Polar Form:-



ϕ ; angle between the tangent and radius vector for the curve $r = f(\theta)$

. $\tan\phi = \dfrac{rd\theta}{dr}$

. $\psi = \theta + \phi$

. $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$

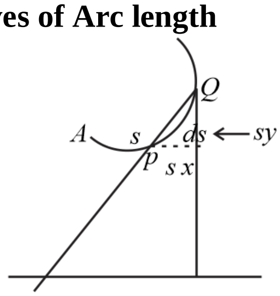
. $OT =$ Polar sub tangent $= r^2 \dfrac{d\theta}{dr}$,

. $ON =$ Polar subnormal $= \dfrac{dr}{d\theta}$

. PT =Polar tangent $= r\sqrt{1+\left(r\dfrac{d\theta}{dr}\right)^2}$

. Perpendicular from the pole on tangent $p = r\sin\phi$; known as Pedal equation of a curve.

2. **Derivatives of Arc length**



Cartesian Form:-

Simply $ds^2 = dy^2 + dx^2$

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Or $$\frac{ds}{dy}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

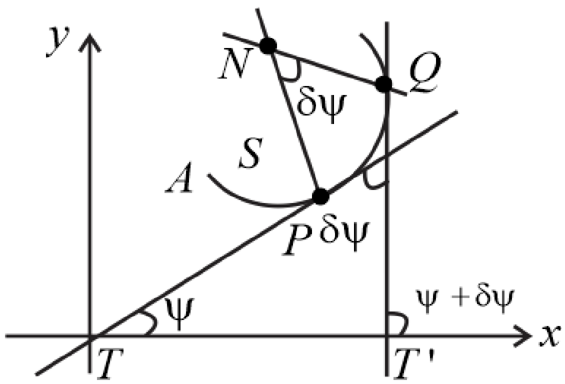
Parametric Form:- $x=f(t),\ y=g(t)$

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

Polar Form:- $$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{ds}{dr}=\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}$$

3. CURVATURE: [Bending of a curve]



arc $AP=s$, arc $PQ=\delta s$

In ΔPNQ

$$\frac{PN}{\sin NQP}=\frac{Chord\ PQ}{\sin PNQ}\Rightarrow\frac{PN}{\sin NQP}=\frac{Chord\ PQ}{\sin\delta\psi}$$

$$\therefore\ PN=\frac{Chord\ PQ}{\sin\delta\psi}\sin NQP$$

The radius of curvature of $\rho=\lim\limits_{\delta\psi\to 0}PN$

$$\therefore\qquad \rho=\lim_{\delta\psi\to 0}\frac{Chord\ PQ}{\sin\delta\psi}\sin NQP$$

$$=\lim_{\delta\psi\to 0}\frac{\delta s}{\delta\psi}.\frac{\delta\psi}{\sin\delta\psi}.\sin NPQ$$

But as $\delta\psi\to 0$ $Chord\ \dfrac{PQ}{\sin\delta\psi}\to 1$ and $\dfrac{\delta\psi}{\sin\delta\psi}\to 1$.

Also $\angle NQP = \pi/2$ $\qquad \because \angle NQP = \dfrac{\pi}{2} - \angle PQT'$,where $\angle PQT' \to 0$

So we found $\boxed{\rho = \dfrac{ds}{d\psi}}$

$\because$ any equation containing s and ψ ; for any curve is called intrinsic equation for that curve

$\therefore$ this formula is called intrinsic formula for radius of curvature.

Cartesian From:-

$$\because \quad \frac{dy}{dx} = \tan\psi$$

$$\therefore \quad \frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx} = \left(1+\tan^2\psi\right)\frac{d\psi}{ds}.\frac{ds}{dx}$$

$$\frac{d^2y}{dx^2} = \left\{\{1+\left(\frac{dy}{dx}\right)^2\}.\frac{1}{\rho}\left\{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right\}\right\}$$

$$\therefore \quad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2y/dx^2}$$

$$\because \quad ds^2 = dy^2 + dy^2$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Also $\rho = \dfrac{\left\{1+\left(dx/dy\right)^2\right\}^{3/2}}{d^2x/dy^2}$

For Pedal Equations:- Let Pedal equation of curve is $r = f\left(p\right)$, where p is the length of normal from origin to the curve,

$$\because \quad p = r\sin\phi \qquad \therefore \frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}.\frac{d\phi}{dr} \qquad \because \sin\phi = r\frac{d\theta}{ds}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) \qquad \because \cos\phi = \frac{dr}{ds}$$

$$= r\frac{d}{ds}\left(\theta+\phi\right) = r\frac{d\psi}{ds} \qquad \because \psi = \theta + \phi$$

$$\therefore \quad \frac{dp}{dr} = r/\rho \qquad \therefore \quad \rho = ds/d\psi$$

$$\therefore \quad \rho = r\frac{dr}{dp}$$

Central orbit (Dynamics)

→ To obtain p in terms of r and θ

$\because \quad p = r\sin\phi \qquad \therefore \frac{1}{p^2} = \frac{1}{r^2}\cos ec^2\phi = \frac{1}{r^2}\left(1+\cot^2\phi\right)$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad \because \sin\phi = r\frac{d\theta}{ds} \ \& \ \because \cos\phi = \frac{dr}{ds}$$

Sometimes it is convenient to use u to denote $1/r$. Then we have $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Polar form Equation of Radius of Curvature:

- Let equation of curve be $r = f(\theta)$ we know that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

Differentiating w.r.t r ,we get $-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2$

$+\frac{1}{r^4}.2\frac{dr}{d\theta}\frac{d}{dr}\left(\frac{dr}{d\theta}\right)$

$$\therefore \quad \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{d^2r}{d\theta^2}$$

$$\therefore \quad \rho = r\frac{dr}{dp} = \frac{\left(r^2 + (dr/d\theta)^2\right)^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)}$$

Asymptotes and Curve Tracing

https://www.youtube.com/live/SczBjJrwdXc?si=4hX0gjVx98x9D16h

https://www.youtube.com/live/4N4TYQZhMGI?si=0hvXie9CLgYv_kt2

Curve Tracing:

To trace the curve $f(x,y)=0$, we need to know:

(i) symmetry
(ii) tangents at origin (if curve passes from origin)
(iii) Asymptotes parallel to axes, arbitrary asymptotes as well sometimes
(iv) points on axes; where curve cuts. Observing from what points the curve passes.

1. Symmetry

(i) Replace x by $-x$ to get symmetry about y-axis

(ii) Replace y by $-y$ to get symmetry about x-axis

So $y^2(2a-(-x))=(-x)^3$, curve changed. So no symmetry about y-axis and

$(-y)^2(2a-x)=x^3$, curve remains unchanged.

$\therefore$ So it is symmetric about x-axis.

Note:

If the curve remains unchanged on exchanging x and y then the curve is symmetrical about the line $y=x$.

E.g. $x^3+y^3=3axy$ is symmetrical about the line $y=x$.

But $y^3+x^3=3ayx \Rightarrow x^3+y^3=3axy$

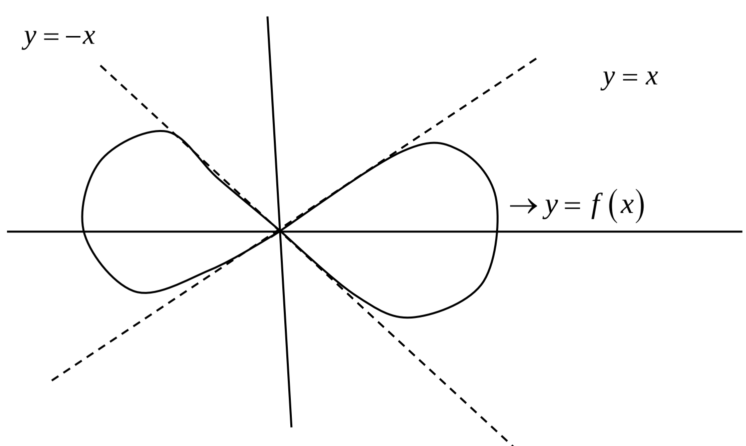
2.Tangents of Origin:

Multiple Points:

The points through which more than one branches of the curve pass, is called a multiple point.

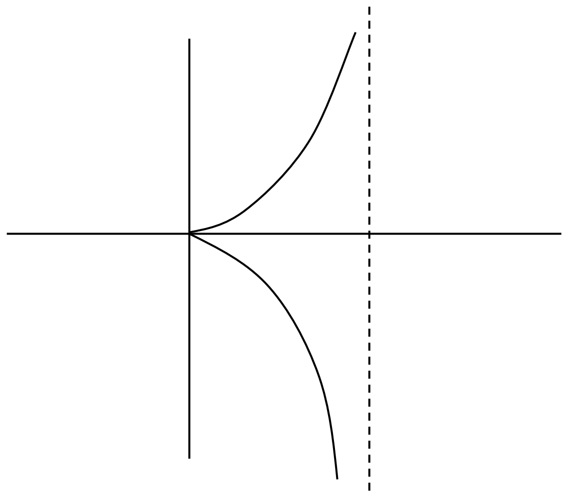
Double Points: Two branches of curve pass through this point.

(a) Node:



When two tangents at a double point are real and distinct, the point is called a node.

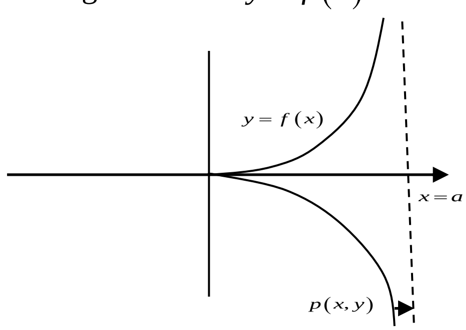
E.g. In the above figure we have two tangents $y = x$ and $y = -x$ at origin. So origin is a node here.
(b) Cusp: When the two tangents at a double point are real and coincident then that point is known as cusp.



Here x-axis is the only tangent for the curve at origin. So origin is a cusp here.

Note:
To get tangents of origin we equate the lowest degree terms to zero.

3.Asymptotes:
A straight line at a finite distance from the origin to which a given curve approaches infinitely near as we move along the curve to an infinite distance.
E.g.
Here $x = a$ is an asymptote to some given curve $y = f(x)$



- In other words, a straight line is said to be an asymptote, if for distance of any point $p(x, y)$ on the curve from the s.t line tends to zero as the point p moves to infinite along the curve.

Finding Asymptotes (if exists) of some Algebraic curve $f(x, y) = 0$
E.g. Find asymptote for the curve

Important Note to move further:

(1) The equation $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ ....(1)

will have one root as $x = 0$ if $a_n = 0$.

Putting $x = \frac{1}{y}$ in ...(1)

$$a_ny^n + a_1y^{n-1} + ... + a_{n-1}y + a_0 = 0 \quad ....(2)$$

equation (2) has one root as $y = 0$ if $a_0 = 0$.

Now it is clear that equation (1) may have one root tending to infinity is that $a_0 = 0$.

(2) The asymptote (if exists), it must meet the curve in at least two points at infinity i.e. to have two roots as infinity for the equation (1), the condition is $a_0 = 0$ and $a_1 = 0$.

Method to find asymptotes

Let's consider a curve $f(x, y) = 0$ ...(1); which is algebraic expression of nth degree.

Let $y = mx + c$ ...(2) is an asymptote of equation (1)

Putting this value of y in the given curve we get an equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0 \quad ....(3)$$

Here $a_0, a_1, ...a_n$ are constants (containing m and c)

Now, since $y = mx + c$ is an asymptote, it must meet at least two points of the given curve.

Thus, two roots of equation (3) must be infinity

i.e. we must have

$$a_0 = 0 \quad ....(4)$$

$$a_1 = 0 \quad ....(5)$$

Now find values of m from equation (4) and get values of c for different values of m by using in (5).

Example. Find all asymptotes of the curve

$$x^3 + 2x^2y - xy^2 + 2y^3 + 4y^2 + 2xy + y - 1 = 0 \quad ....(1)$$

Ans. Putting $y = mx + c$ in equation (1)

$$x^3 + 2x^2(mx+c) - x(mx+c)^2 - 2(mx+c)^3 + 4(mx+c)^2 + 2x(mx+c) + (mx+c) - 1 = 0$$

$$\Rightarrow x^3 + 2mx^3 + 2cx^2 - x(m^2x^2 + 2mxc + c^2)$$

$$-2(m^3x^3 + 3m^2x^2c + 3mxc^2 + c^3) + 4(m^2x^2 + 2mxc + c^2) \ +2(mx^2 + cx) + mx + c - 1 = 0$$

$$\Rightarrow \{x^3 + 2mx^3 - m^2x^3 - 2m^3x^3\} + \{2cx^2 - 2mcx^2 - 6m^2cx^2 + 4m^2x^2 + 2mx^2\}$$

$$\{-c^2x + 6mc^2x + 8mxc + 2cx + mx\} + \{-2c^3 + 4c^2 + c - 1\} = 0$$

$$\Rightarrow (1 + 2m - m^2 - 2m^3)x^3 + \{2c - 2mc - 6m^2c + 4m^2 + 2m\}x^2 + ... = 0$$

Now coefficient of x^3; $a_0 = 0 \Rightarrow 1 + 2m - m^2 - 2m^3 = 0$ ....(1)

$\Rightarrow m = 1, -1, -½$.

Now

coefficient of x^2; $a_1 = 2c - 2mc - 6m^2c + 4m^2 + 2m = 0$ ....(2)

Putting $m = 1$; $2c - 2c - 6c + 4 + 2 = 0 \Rightarrow c = 1$.

Putting $m = -1$ in equation (2); $2c + 2c - 6c + 4 - 2 = 0 \Rightarrow c = 1$

Putting $m = -½$ in equation (2); $2c + c - \frac{6}{4}c + 4 \cdot \frac{1}{4} + 2(-½) = 0 \Rightarrow 3c - \frac{3c}{2} = 0 \Rightarrow c = 0$

Therefore, there are 3 asymptotes of the given curve given by
$y = x + 1$; $y = -x + 1$ and $y = -½x$.

<u>Another Method</u>.

Let the equation of the curve be

$$\left(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_ny^n\right) + \left(b_1x^{n-1} + b_2x^{n-2}y + ... + b_ny^{n-1}\right) +$$
$$\left(c_2x^{n-2} + c_3x^{n-3}y + ... + c_ny^{n-2}\right) + ... = 0$$

The above equation can be put in the form

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + ... = 0 \quad ....(1)$$

where $\phi_n\left(\frac{y}{x}\right)$ represents a term of nth degree in y/x.

Let $y = mx + c$ be an asymptote of (1) putting $\frac{y}{x} = m + \frac{c}{x}$ in (1), we get

$$x^n\phi_n\left(m + \frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m + \frac{c}{x}\right) + ... = 0 \quad ....(2)$$

Applying Taylor's expansion in (2)

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n(m)\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi_{n-1}(m)\right\} + ... = 0$$

Arranging the above expression in descending powers of x

$$x^n\phi_n(m) + x^{n-1}\left\{c\phi_n'(m) + \phi_{n-1}(m)\right\} + x^{n-2}\left\{\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m)\right\} + ... = 0 \quad ...(iii)$$

Now as $y = mx + c$ is an asymptote so 2 roots of equation (3) must be infinity.

i.e., $\phi_n(m) = 0$ ....(4)

$c \cdot \phi_n'(m) + \phi_{n-1}(m) = 0$ ....(5)

Note:

The equation (4) is of nth degree so it has n values of m.
If any k values of n are repeated, we say the given curve has k parallel asymptotes.

<u>Working Rule</u>

(1) Find $\phi_n(m)$ by putting $x = 1$ and $y = m$ in the nth degree term.

Similarly find $\phi_{n-1}(m)$ by putting $x=1$ and $y=m$ in $(n-1)$th degree term of the given curve and so on.
(2) Solve $\phi_n(m)=0$. If all roots of m are real and distinct get value for c from the equation (5).

Note: If we get two parallel asymptote, i.e. two values of m are same we find the value of c from the equation

$$\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)=0$$

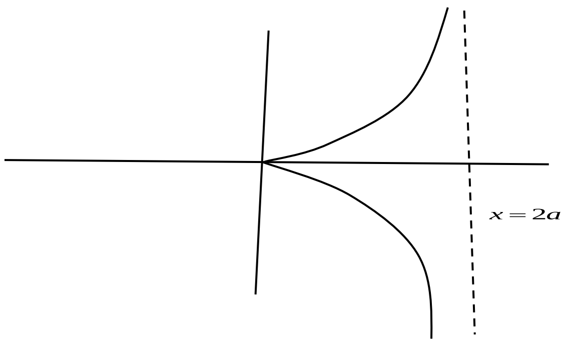
Example. Find asymptotes of the curve $y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0$.
Solution.
Find $\phi_3(m)$;
Putting $x=1, y=m$ in 3rd degree term of (1)
$\phi_3(m)=m^3-m-2m^2+2$ and $\phi_2(m)=-7m+3m^2+2$
Now $\phi_3(m)=0 \Rightarrow m^3-2m^2-m+2=0$
$\Rightarrow m^2(m-2)-1(m-2)=0$
$\Rightarrow (m+1)(m-1)(m-2)=0$
$\Rightarrow m=-1,1,2$
Now find c from this equation
$c\phi_n'(m)+\phi_{n-1}(m)=0$
$\Rightarrow c\{3m^2-4m-1\}+2-7m+3m^2=0$
Now for $m=1$, $c\{3-4-1\}=0 \Rightarrow -2c=2 \Rightarrow c=-1$
For $m=-1$; $c=-2$
For $m=2$; $c=0$
$\therefore$ Asymptotes are $y=x-1$; $y=-x-2$; $y=2x$

<u>Asymptotes parallel to axes</u>

To get an asymptote parallel to the *x*-axis put the coefficient of the highest power of ***x*** in the given expression as equal to zero.

Similarly, asymptotes parallel to the *y*-axis can be obtained by putting put the coefficient of the highest power of ***y*** in the given expression as equal to zero.

Example. Find asymptotes of the cissoid $y^2(2a-x)=x^3$.
Solution.
There is no asymptote parallel to *x*-axis
As, coefficient of $x^3=1\neq 0$
Asymptote parallel to y-axis is ; $2a-x=0 \Rightarrow x=2a$.



Now we are ready to have examples for Curve Tracing

Example 1. Trace the curve $y^2(2a-x)=x^3$.

Solution.

<u>Symmetry</u>

(i) Replace x by $-x$ to get symmetry about y-axis

(ii) Replace y by $-y$ to get symmetry about x-axis

So $y^2(2a-(-x))=(-x)^3$, curve changed. So no symmetry about y-axis and

$(-y)^2(2a-x)=x^3$, curve remains unchanged.

$\therefore$ So it is symmetric about x-axis.

<u>Note</u>:

If the curve remains unchanged on exchanging x and y then the curve is symmetrical about the line $y=x$.

E.g. $x^3+y^3=3axy$ is symmetrical about the line $y=x$.

But $y^3+x^3=3ayx$

$\Rightarrow x^3+y^3=3axy$

(ii) <u>Tangents of origin</u>

Equating the lowest degree term to zero.

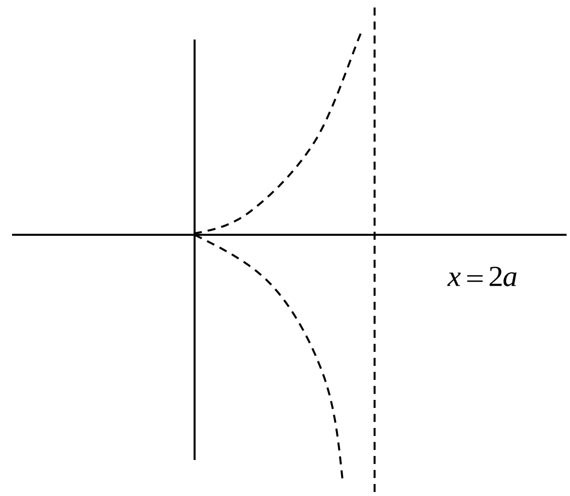
$\therefore y^2\times 2a=0 \Rightarrow y^2=0 \Rightarrow y=0$

$\therefore$ Real and coincident tangents (so, origin is cusp). Tangent is x-axis.

(iii) <u>Asymptotes parallel to axes</u>

Parallel to y-axis: $2a-x=0 \Rightarrow x=2a$

Parallel to x-axis: As, $1=0$ (not possible). So no asymptote parallel to x-axis



(iv) <u>Points on the axes (cut by curve)</u>

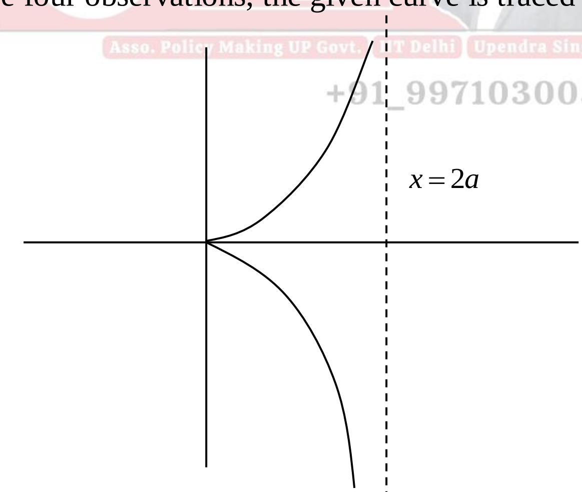
on x-axis; put $y=0$ in the given curve, we get $x=0$ i.e. one point on curve is $(0,0)$

on y-axis put $x=0 \Rightarrow y^2(2a-0)=0 \Rightarrow y=0$ i.e. $(0,0)$

There is no point on the curve in the 2nd and 3rd quadrant.

$\because$ *for* $x<0$; $y=\sqrt{\dfrac{x^3}{2a-x}}$ (complex numbers)

Therefore, based on above four observations, the given curve is traced as:



Example 2. Trace the curve $x^3+y^3=3axy$.

(i) <u>Symmetry</u>

It is symmetrical about the line $y=x$ because by putting $y=x$ in given equation, the curve remains unchanged.

(ii) <u>Tangent/s at origin</u>

Equating the lowest degree term to term

We get $3axy = 0 \Rightarrow xy = 0$

$\Rightarrow x = 0$ or $y = 0$

Two Real and distinct tangents

$\therefore$ Origin is NODE

(iii) <u>Asymptotes Parallel to Axes</u>:

Parallel to y-axis; $1 = 0$ (not possible)

Parallel to x-axis; $1 = 0$ (not possible)

$\therefore$ There is no asymptotes parallel to coordinate axes.

By Method of finding asymptotes, we get that

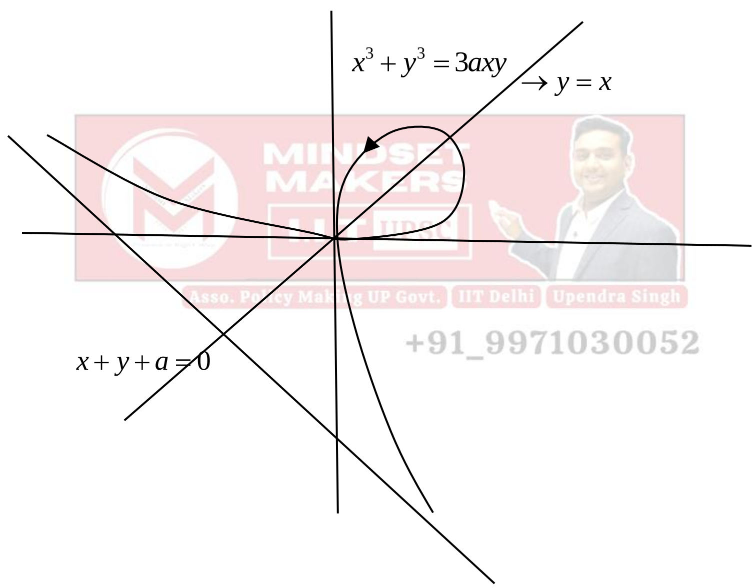
$x + y + a = 0$ is an asymptote. (try by the method of finding asymptotes)

(iv) <u>Points on the axes</u>:

On x-axis ; put $y = 0$, we get $x = 0$ i.e. $(0,0)$

on y-axis; put $x = 0$, we get $y = 0$ i.e. $(0,0)$

In 3rd quadrant, $\therefore$ $x < 0, y < 0$, R.H.S. > 0 but L.H.S. < 0, 3rd quadrant. So no part of the curve lies in third quadrant.



Q. Trace the curve (1) $a^2y^2 = x^3(2a - x)$

(2) $x^2(a^2 + y^2) = y^2(a^2 - y^2)$

(3) $y^2 = \dfrac{x^2(1 + x^2)}{(1 - x^2)}$

(4) $x^2y^2 = (a + y)^2(b^2 - y^2),\ b > a$

Ans. (1) $a^2y^2 = x^3(2a - x)$

(i) It is symmetrical about x-axis

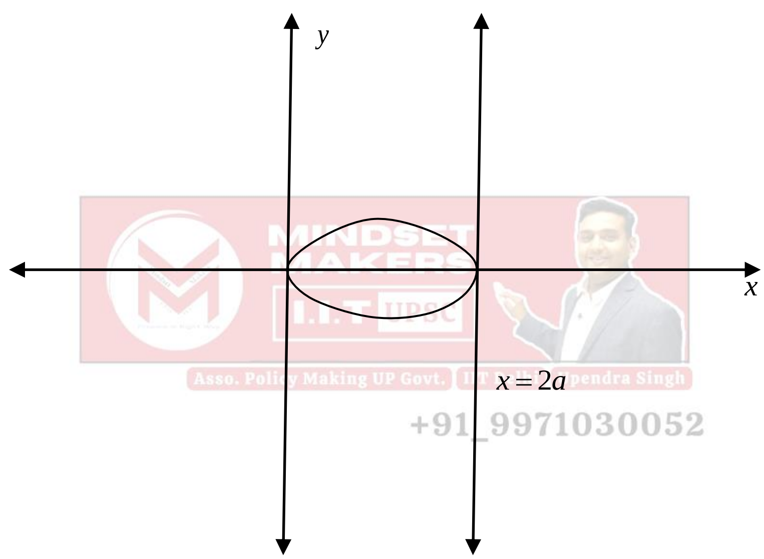
(ii) Tangent at origin; $a^2y^2 = 0 \Rightarrow y = 0, y = 0$ i.e., x-axis

Two Real and coincident tangents at origin. $\therefore$ Origin is a cusp.

(iii) Asymptotes parallel to x-axis
No asymptotes parallel to x-axis

Asymptotes parallel to y-axis
No asymptotes parallel to y-axis.
And also the curve passes through origin.

Points:
$$a^2y^2 = x^3(2a-x)$$
$$y = \pm\frac{\sqrt{x^3(2a-x)}}{a}$$
Hence No part of the curve will lie on 2nd and 4th quadrant.
Putting $x = 2a,\ y = 0$



(3) Trace the curve $y^2 = \dfrac{x^2(1+x^2)}{1-x^2}$

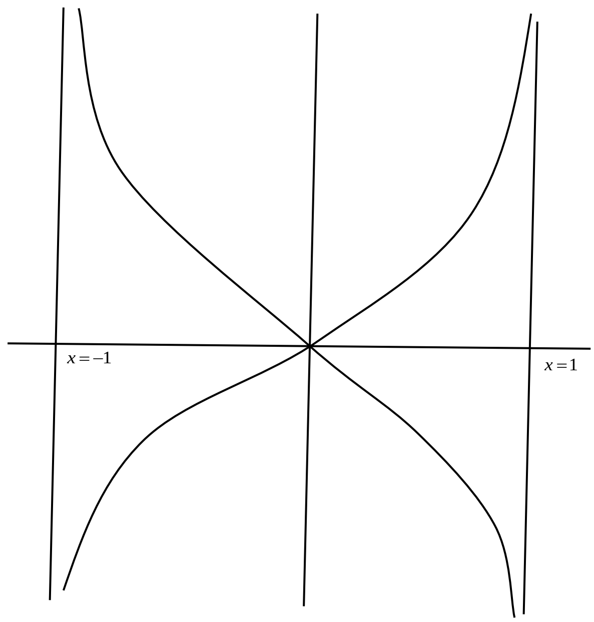
(i) Symmetry:
Replacing x by $-x$, y by $-y$; curve remains unchanged.
$\therefore$ Symmetrical about both axes.
(ii) The curve passes through origin? Yes.

<u>Tangent at origin</u>

Putting the lowest degree term equal to zero;
$y^2 - x^2 = 0 \Rightarrow y = \pm x$; Two real and distinct tangents at origin. $\therefore$ Origin is a node.

(iii) Asymptotes parallel to y-axis; $1 - x^2 = 0 \Rightarrow x = \pm 1$

(iv) Curve cuts no points on the axes, except $(0,0)$.



No point of the curve lies out side $|x|\le 1$. As, if $|x|>1, y$ is imaginary.

Q. Trace the curve $x^2y^2=(a+y)^2(b^2-y^2), b>a$

(i) It is symmetrical about y-axis as putting x by -x the curve remains unchanged.

(ii) Tangents at origin

Putting $x=0, y=0$ in the equation $0=a^2b^2$ origin does not satisfy the equation.

i.e., The curve does not pass through origin

$\therefore$ No tangent at origin.

$$\because \quad x=\pm\left(1+\frac{a}{y}\right)\sqrt{b^2-y^2}$$

$$\frac{dx}{dy}=\pm\left\{\left(1+\frac{a}{y}\right)\left(\frac{-2y}{2\sqrt{b^2-y^2}}\right)+\sqrt{b^2-y^2}\left(-\frac{a}{y^2}\right)\right\}$$

$$\therefore \quad \left.\frac{dx}{dy}\right|_{(0,0)}=\pm\frac{\sqrt{b^2-a^2}}{a} \quad \therefore \frac{dy}{dx}=\pm\frac{a}{\sqrt{a^2-b^2}}$$

i.e., There are two tangents at $(0,-a)$. So the point $(0,-a)$ is a node.

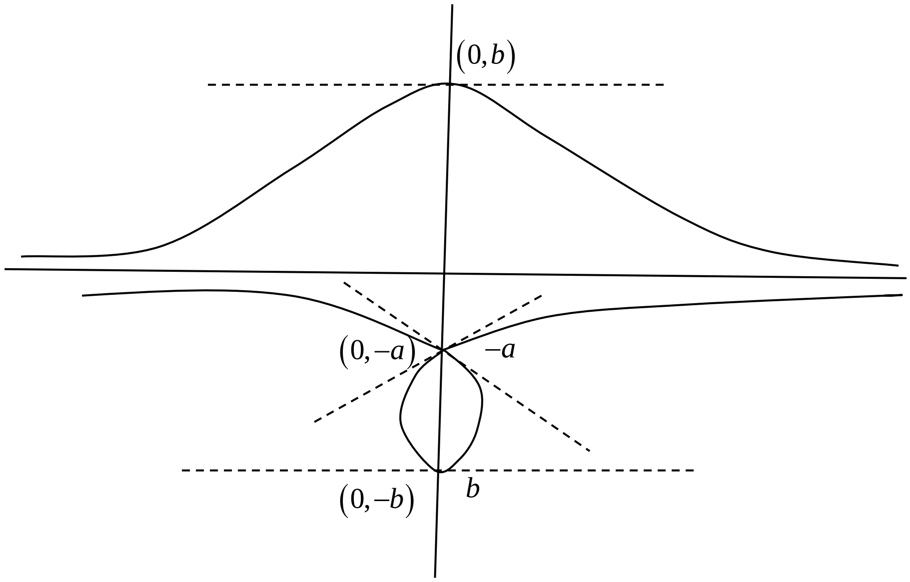
(iii) **Points on axes**:

The curve cuts

- y-axis at $(0,-a)$ and $(0,\pm b)$
- Does not cut x-axis at any point

(iv) Asymptotes parallel to x-axis

Parallel to x-axis. $y=0$ i.e., y-axis only



Note. For above, Tangents parallel to x-axis at $(0,b)$ and $(0,-b)$

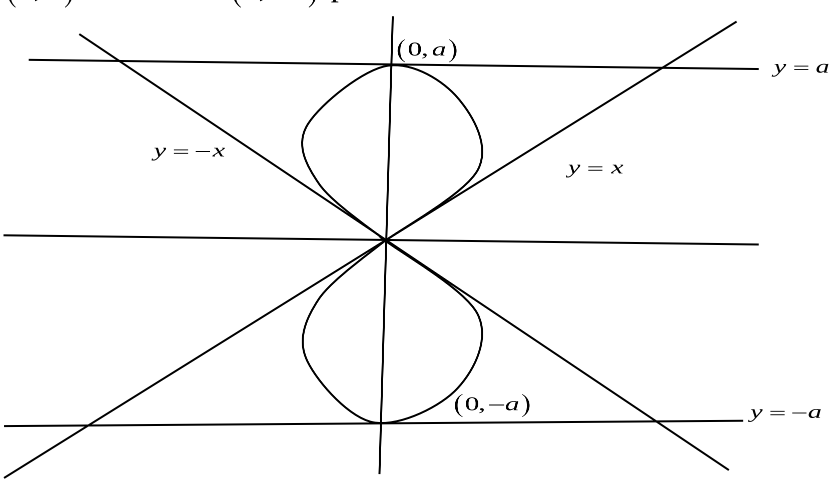
$\frac{dx}{dy} \to \infty$ and $\frac{dy}{dx} \to 0$.

Q. (2) Trace $x^2\left(a^2+y^2\right)=y^2\left(a^2-y^2\right)$

Symmetrical about both the axes. curve passes through origin.

$y=\pm x$ are tangents at origin.

The curve passes through the points $(0,a)$ and $(0,-a)$ and no parts of the curve will lie above $(0,a)$ and below $(0,-a)$ points.



PYQs: ASYMPTOTES, CURVE TRACING

Q1. Trace the curve $y^2(x^2 - 1) = 2x - 1$. **[UPSC CSE 2023]**

Q2. Trace the curve $y^2x^2 = x^2 - a^2$, where a is a real constant.

[4bUPSC CSE 2022]

Q3. Find all the asymptotes of the curve $(2x+3)y = (x-1)^2$.

[1d UPSC CSE 2020]

Q4. Find the asymptotes of the curve $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$. **[3a 2020 IFoS]**

Q5. Find all the asymptotes of the curve $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$.

[4d 2013 IFoS]

Q6. Determine the asymptotes of the curve
$x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$.

PYQs Analysis:

Answer 1. Trace the curve $y^2(x^2 - 1) = 2x - 1$. **[UPSC CSE 2023]**

Curve is symmetrical about x-axis.

• Curve does not pass through origin.

• $x^2 - 1 = 0 \Rightarrow x = 1, x = -1$ are two real asymptotes parallel to y-axis.

and $y^2 = 0 \Rightarrow y = 0$ is a real asymptote parallel to x-axis

• **Interpretation;**

• $\because y = 0$ at $x = \frac{1}{2}$ $(\because 0 = 2x - 1 \Rightarrow x = \frac{1}{2})$

• $\because y = \pm\sqrt{\frac{2x-1}{x^2-1}}$

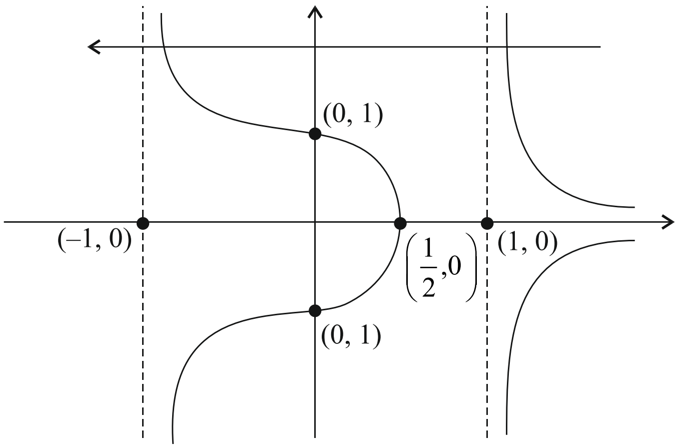
So y is imaginary if $\frac{1}{2} < x < 1 \rightarrow x > 1$; y is not imaginary

• $x = 0$; $y = \pm\sqrt{\frac{-1}{-1}} = \pm 1$

$\Rightarrow$ curve passes through (0, 1) & (0, –1).

• For $x \rightarrow \infty$; $y \rightarrow 0$

(observe/solve)



Answer 2. Trace the curve $y^2x^2 = x^2 - a^2 \Rightarrow (y^2 - 1)x^2 = -a^2$, where a is a real constant.

Following observations can be done from $y^2x^2 = x^2 - a^2$:

(i) $\because$ on replacing $\boxed{x}$ by $\boxed{-x}$ or $\boxed{y}$ by $\boxed{-y}$; curve remains unchanged. So it is symmetrical about both axes.

(ii) $\because$ (0, 0) does not satisfy given equation. So curve does not pass through the origin.
$\because$ L.H.S = 0, R.H.S = $-a^2$.

(iii) Parallel asymptotes; To the Y-axis

Coefficients of y^2 =0 ; $\boxed{x^2 = 0} \Rightarrow \boxed{x = 0}$ is an asymptote parallel to y-axis

Coefficients of x^2

$$1 - y^2 = 0 \Rightarrow y = 1, y = -1 \text{ are two asymptotes parallel to x-axis}$$

(iv) points:

Equation clearly indicating that we should think for $x = \pm a$.

$$\therefore\ y^2 \cdot a^2 = a^2 - a^2 \Rightarrow y^2 = \frac{0}{a^2} = 0$$

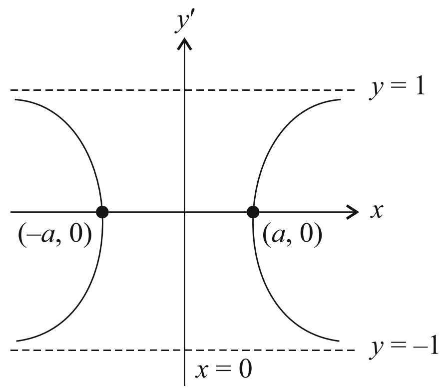
$\therefore$ curve (1) passes through the points $(a, 0)$, $(-a, 0)$

- Equation also indicates that if we put $y = 0$; we get

$x = \pm a$

- If $x < a$; y imaginary or $x > -a$; y imaginary

So, No part of curve lies between $-a < x < a$



Answer 3. Find all asymptotes of the curve $(2x + 3)y = (x – 1)$

$\because$ given curve is $(2x + 3)y = x^2 + 1 – 2x$

$\Rightarrow x^2 – 2xy – 2x – 3y + 1 = 0$

Now; $\phi_2(m) = 1 – 2m$, $\phi_1(m) = –2 – 3m$

To get values of m; putting $\phi_2(m) = 0$

$1 – 2m = 0 \Rightarrow m = \frac{1}{2}$

To get values of c;

$c \cdot \phi'_2(m) + \phi_1(m) = 0$

$c \cdot (–2) – 2 – 3m = 0$

$\therefore –2c – 2 – 3 \times \frac{1}{2} = 0 \Rightarrow c = \frac{7}{4}$

Therefore given curve has exactly one asymptote; $\boxed{y = \frac{x}{2} – \frac{7}{4}}$

Answer 4. Find all asymptotes of the curve $(x^3 + 3x^2y – 4y^3) + (–x + y) + 3 = 0$

To get asymptotes;

$(x^3 + 3x^2y – 4y^3) + (–x + y) + 3 = 0$

$\phi_3(m) = 1 + 3m – 4m^3$, $\phi_2(m) = 0$, $\phi_1(m) = –1 + m$

To get slopes (values of m):

Put $\phi_3(m) = 0 \Rightarrow 1 + 3m – 4m^3 = 0$; $(m – 1)(2m + 1)^2 = 0$

$m = 1, –\frac{1}{2}, –\frac{1}{2}.$

• For $m = 1$; $\therefore$ c is given by $c \cdot \phi_3'(m) + \phi(m) = 0 \Rightarrow c \cdot (3 – 12m^2) = 0 \therefore c = 0$;

$\therefore y = x + 0 \Rightarrow y = x$ is one asymptote;

For $m = –\frac{1}{2}, –\frac{1}{2}$;

c is given by: $\frac{c^2}{2!}\phi_3''(m) + c\cdot\phi_2'(m) + \phi_1(m) = 0$

$\frac{c^2}{2}(-24\text{m}) + c \times 0 + (-1 + m) = 0$

$\therefore \frac{c^2}{2}\left(-24\times\left(-\frac{1}{2}\right)\right) + 0 + \left(-1-\frac{1}{2}\right) = 0$; $c^2 \times 6 - \frac{2-1}{2} = 0$; $\boxed{c = -\frac{1}{2}, \frac{1}{2}}$

$\therefore$ $y = (-1/2)x - 1/2$, $\therefore$ $y = (-1/2)x + ½$ are two asymptotes.

Answer 5. Find all asymptotes of the curve;

$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$

Given curve is;

$(x^4 - y^4) + (3x^2y + 3xy^2) + xy = 0$

$\therefore \phi_4(m) = 1 - m^4$, $\phi_3(m) = 3m + 3m^2$, $\phi_2(m) = m$, $\phi_1(m) = 0$

To get values of m;

$\phi_4(m) = 0$; $1 - m^4 = 0$; $m = 1, -1, i, i$

so real asymptotes are with the slopes $m = 1, -1$

To get values of c;

$c\phi'_4(m) + \phi_3(m) = 0$

$c(-4m^3) + 3m(1 + m) = 0$

for $m = 1$; $-4c + 6 = 0 \Rightarrow 4c = 6$; $c = \frac{3}{2}$

for $m = -1$; $4c = 0 \Rightarrow c = 0$

Therefore all real asymptotes of given curve are:

$\boxed{y = x + \frac{3}{2}}, \boxed{y = -x}$.

Answer 6. Determine the asymptotes of the curve;

$x^3 + x^3y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$

Given curve is;

$(x^3 + x^3y - xy^2 - y^3) + (2xy + 2y^2) + (-3x + y) = 0$

$\therefore$ $\phi_3(m) = 1 + m - m^2 - m^3$, $\phi_2(m) = 2m + 2m^2$, $\phi_1(m) = -3 + m$

For values of m;

$1 - m - m^2 - m^3 = 0$ gives $m = 1, -1, -1$

for $m = 1$; $c\cdot\phi'_3(m) + \phi_2(m) = 0$

$c\cdot(1 - 2m - 3m^2) + 2m + 2m^2 = 0$

$c\cdot(1 - 2 \times 1 - 3 \times 1^2) + 2 \times 1 + 2 \times 1^2 = 0 \Rightarrow c(-4) + 4 = 0$

for $m = -1$; $\boxed{c = 1}$

$$\frac{c^2}{2!}\phi_3'(m) + c\cdot\phi_2'(m) + \phi_1(m) = 0 \Rightarrow \frac{c^2}{2}(-2 - 6m) + c\cdot(2 + 4m) - 3 + m = 0$$

$$\frac{c^2}{2}(-2 - 6 \times -1) + c\cdot\ (2 + 4 \times -1) - 3 + (-1) = 0;\ \ \frac{c^2}{2}(4) - 26 - 4 = 0$$

$2c^2 - 2c - 4 = 0;\ c^2 - c - 2 = 0;\ c^2 - (2 - 1)c - 2 = 0;\ c^2 - 2c + c - 2 = 0$

$c(c - 2) + 1\ (c - 2) = 0;\ (c - 2)\ (c + 1) = 0;\ \boxed{c = 2, -1}$

Ans. $\boxed{y = x + 1, y = -x + 2, y = -x + 1}$

TEST- Differential Calculus

Time allowed: Three Hours **Maximum Marks : 250**

Question Paper Specific Instructions:

1- **There are eight questions divided in two sections.**
2- **Candidate has to attempt five questions in all.**
3- **Question nos. 1 and 5 are compulsory and out of remaining, three are to be attempted choosing at least one question from each section.**
4- **The number of marks carried by a question / part is indicated against it.**
5- **Answers must be written in the medium authorized.**
6- **Assume suitable data, if considered necessary, and indicate the same clearly.**
7- **Unless and otherwise indicated, symbols and notations carry their usual standard meanings.**
8- **Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partially. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.**

SECTION-A

Q1. Each question is of 10 marks.

a. Discuss the kind of discontinuity, if any of the function is defined as follows:

$$f(x)=\begin{cases}\dfrac{x-|x|}{x} & \text{when } x\neq 0\\ 2 & \text{when } x=0\end{cases}$$

b. A function f is defined as follows

$$f(x)=\begin{cases}x^p\cos\left(\dfrac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

What conditions should be imposed on *p* so that *f* may be

(i) Continuous at $x=0$ (ii) Differentiable at $x=0$

c. Discuss the continuity and differentiability of the function *f* defined by

$$f(x)=\begin{cases}2x+1; & \text{if } x \text{ is rational}\\ x^2-2x+5; & \text{if } x \text{ is irrational}\end{cases}$$

d. Verify Rolle's theorem for the following function $f(x)=(x-a)^m(x-b)^n$, where $x\in[a,b]$ and *m*, *n* are positive integers.

e. Show that the maximum rectangle inscribed in a circle is square.

Each question is of 25marks

Q2. a. If $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = 0$ .Then $C_0 + C_1 x + C_2 x^2 + ... + C_n x^n = 0$ has at least one real root between 0 and 1.

b. Find the maximum value of u, where $u = \sin x \sin y \sin(x+y)$.

Q3. a. If $u = \frac{x^2+y^2+z^2}{x}$, $v = \frac{x^2+y^2+z^2}{y}$, and $w = \frac{x^2+y^2+z^2}{z}$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

b. Tracing the curve $x^3 + y^3 = 3axy$.

Q4. a. Show that under suitable condition there exits at least one real number where $a < \xi < b$

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(x) \end{vmatrix}$$

b. Find Taylor's Series expansion for the function $f(x) = \log(1+x)$, $-1 < x < \infty$ about $x = 2$ with Lagrange's form of remainder after 3 terms.

SECTION-B

Each question is of 10 marks

Q5. a. . If

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

show that both the partial derivatives exist at $(0,0)$ but the function is not continuous there at.

b. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ and find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

c. Check continuity and differentiability of the function

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

d. Show that $\log(1+x)$ lies between

$$x-\frac{x^2}{2} \text{ and } x-\frac{x^2}{2(1+x)}, \forall x>0$$

e. Let $P(x)=\left(\frac{5}{13}\right)^x+\left(\frac{12}{15}\right)^x-1$ for all $x\in\mathbf{R}$. Then prove or disprove that $P(x)$ is strictly decreasing for all $x\in\mathbf{R}$.

Each question is of 25 marks

Q6.a. Examine the function $\sin x+\cos x$ for extreme values.

b. Define a function f of two real variables in the $x-y$ -plane by

$$f(x,y)=\begin{cases}\dfrac{x^3\cos\frac{1}{y}+y^3\cos\frac{1}{x}}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0\text{, otherwise}\end{cases}$$

Check the continuity and differentiability of f at $(0,0)$

Q7.a Discuss the maximum or minimum values of $u=x^3+y^3-3axy$.

b. Prove that of all rectangular parallelopipeds inscribed in the ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the parallelopiped having volume $8abc/3\sqrt{3}$ is the greatest.

Q8.a If two variables x and y are connected by the relation $ax^2+by^2=ab$, show that the maximum and minimum values of the function x^2+y^2+xy will be the values of u given by the equation $4(u-a)(u-b)=ab$. [25 marks]

b. By using the differentiation, show that $\tan x>x>\sin x, \forall x\in(0,\pi/2)$.

c. Prove or disprove in $(0,\pi/2)$[12.5+12.5 marks]

(i) $\cos x<\cos(\sin x)$

(ii) $\frac{1-x^2}{2}<\log(2+x)$

SOLUTIONS

Q1. a

The function is continuous at all points expect possibly the origin.

Let us test at $x=0$.

Now

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{x+x}{x} = 2\,;\ \lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x-x}{x} = 0$$

Thus the function has discontinuity of the first kind from the right at $x=0$.

b. **Solution**:

(i) $f(0+0) = \lim_{h\to 0} h^p \cos(1/h)$

$f(0-0) = (-1)^p \lim_{h\to 0} h^p \cos(1/h)$

To make $f(0+0) = f(0-0) = f(0)$, condition required is $\boxed{p>0}$

(ii) $Rf'(0) = \lim_{h\to 0} h^{p-1} \cos(1/h)$

$Lf'(0) = (-1)^{p-1} \lim_{h\to 0} h^{p-1} \cos(1/h)$

So for differentiability condition required is $\boxed{p>1}$

c. **Solution**: $\because\ x^2-2x+5 = 2x+1 \Leftrightarrow x^2-4x+4=0 \Leftrightarrow x=2$

We have $x^2-2x+5 \neq 2x+1$ except when $x=2$ that means we have to check continuity at $x=2$ only (except $x=2$; they are unequal; (since rational $\pm\delta$ = irrational).

Hence *f* is continuous at $x=2$ only.

Further, it follows that $x=2$ is the only point at which *f* may be differentiable.

In this regard, we see that

$$\frac{f(x)-f(2)}{x-2} = \begin{cases} \dfrac{(2x+1)-5}{x-2} = 2\,; & \text{when } x \text{ is rational} \\ \dfrac{(x^2-2x+5)-5}{x-2} = x\,; & \text{when } x \text{ is irrational} \end{cases}$$

Therefore, in either case, we have

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = 2$$

Hence *f* is differentiable at $x=2$ and $f'(2)=2$.

d. $\because$ m and n are positive integers, on being expanded by Binomial theorem, $(x-a)^m$ and $(x-b)^n$ are polynomials in *x*.

So $f(x)$ is differential and continuous in (a,b) further $f(a)=f(b)=0$

$\therefore$ All conditions of Rolle's theorem satisfied.

$\therefore \exists$ at least one $c \in (a,b)$ s.t $f'(c)=0$

$$f'(x)=0$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{n(x-a)+m(x-b)\}=0$$

$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{(m+n)x-(na+mb)\}=0$$

$$\Rightarrow x=a,b,\frac{na+mb}{m+n}$$

Out of these values of x, $\frac{na+mb}{m+n}$ is the only point which lies in the (a,b).

- In fact, it divides the interval (a,b) internally in the ratio $m:n$. Hence Rolle's theorem is verified.

e. $\because r^2=x^2+y^2 \Rightarrow y^2=r^2-x^2 \Rightarrow y=\sqrt{r^2-x^2}$ ....(1)

$\because$ Area of rectangle = $2x\times 2y \Rightarrow A=2x\times 2\sqrt{r^2-x^2}$ $\because r$ constant

For maximum Area ; $\frac{dA}{dx}=0 \Rightarrow r=\sqrt{2}x$, $\therefore y=x$ from (1) $\Rightarrow$ ABCD is a square.

Ans. 2 a. Solution.

Try; intuition comes from Rolle's theorem. But on which function??

Lets consider a function

$$f(x)=C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+...+\frac{C_nx^{n+1}}{n+1}$$

$\because$ $f(x)$ is polynomial $\therefore$ continuous in $[0,1]$ and differentiable in $(0,1)$

Also $f(0)=C_0x^0+C_1x_0^2+...+C_nx_0^{n+1}=0$

$$f(1)=C_0+C_1\times 1^2+C_2\times 1^3+...+C_n\times 1^{n+1}=C_0+\frac{C_1}{2}+\frac{C_2}{3}+...\frac{C_n}{n+1}=0 \text{ in (1)}$$

i.e. $f(0)=f(1)=0$

$\therefore f(x)$ satisfies all three conditions of Rolle's theorem

$\therefore \exists$ at least one point (real number) $C\in(0,1)$ s.t

$$f'(C)=0$$

$$\Rightarrow C_0+C_1C+C_2C^2+...+C_nC^n=0$$

$$\because f'(x)=C_0+C_1x+C_2x^2+...+C_nx^n$$

C is a real root of the equation $C_0+C_1x+C_2x^2+...+C_nx^n$ =0 between 0 and 1.

b. Solution.

Since the function u is periodic with period π for x and y both, it suffices to consider the values of x and y satisfying $0 \le x < \pi, 0 \le y \le \pi$

For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x} = \sin y \{\sin x \cos(x+y) + \cos x \sin(x+y)\} = 0,$$

i.e., $\sin y \sin(2x+y) = 0$, ...(1)

and $\frac{\partial u}{\partial y} = \sin x \{\sin y \cos(x+y) + \cos y \sin(x+y)\} = 0$,

i.e., $\sin x \sin(x+2y) = 0$. ....(2)

To find the values of x and y lying between 0 and π, and satisfying (1) and (2), we need to consider the following pairs of equations:

$x = 0, y = 0;\ 2x + y = \pi;\ 2x + y = 2\pi,\ 2y + x = 2\pi$.

Solving these, we get the following pairs of values of x and y between 0 and π:

$x = 0,\ y = 0;\ x = \frac{1}{3}\pi,\ y = \frac{1}{3}\pi;\ x = \frac{2}{3}\pi,\ y = \frac{2}{3}\pi$.

Now, $r = \frac{\partial^2 u}{\partial x^2} = 2\sin y \cos(2x+y)$,

$s = \frac{\partial^2 u}{\partial x \partial y} = \sin x \cos(x+2y) + \cos x \sin(x+2y) = \sin(2x+2y)$

and $t = \frac{\partial^2 u}{\partial y^2} = 2\sin x \cos(x+2y)$.

At $x = 0, y = 0$, we have $r = 0, s = 0, t = 0$ so that $rt - s^2 = 0$.

Therefore, at the point $(0,0)$, the case is doubtful, and further investigation is needed.

At $x = \frac{1}{3}\pi, y = \frac{1}{3}\pi$, we have

$r = 2\sin\frac{1}{3}\pi \cos\pi = 2\left(\frac{1}{2}\sqrt{3}\right)(-1) = -\sqrt{3}$,

$s = \sin\frac{4}{3}\pi = \sin\left(\pi + \frac{1}{3}\pi\right) = -\sin\frac{1}{3}\pi = -\frac{1}{2}\sqrt{3}$

and $t = 2\sin\frac{1}{3}\pi \cos\pi = 2\left(\frac{1}{2}\sqrt{3}\right)(-1) = -\sqrt{3}$.

So, $rt - s^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{1}{2}\sqrt{3}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$. Also, since $r = -\sqrt{3} < 0, u$ is maximum at $x = \frac{1}{3}\pi, y = \frac{1}{3}\pi$.

At $x = \frac{2}{3}\pi, y = \frac{2}{3}\pi$, we have

$r = 2\sin\frac{2}{3}\pi\cos 2\pi = 2\left(\frac{1}{2}\sqrt{3}\right)(1) = \sqrt{3}$,

$s = \sin\frac{8}{3}\pi = \sin\left(2\pi + \frac{2}{3}\pi\right) = \sin\frac{2}{3}\pi = \frac{1}{2}\sqrt{3}$,

and $t = 2\sin\frac{2}{3}\pi\cos 2\pi = 2\left(\frac{1}{2}\sqrt{3}\right)(1)$.

So, $rt - s^2 = \left(\sqrt{3}\right)\left(\sqrt{3}\right) - \left(\frac{1}{2}\sqrt{3}\right) = 3 - \frac{3}{4} = \frac{9}{4} > 0$. Also, since $r = \sqrt{3} > 0, u$ is minimum at $x = \frac{2}{3}\pi, y = \frac{2}{3}\pi$.

Ans.3 a

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1-\frac{y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{2x}{y} & 1-\frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{2x}{z} & \frac{2y}{z} & 1-\frac{x^2+y^2}{z^2} \end{vmatrix}$$

- Applying $C_1 \to C_1 + \frac{y}{x}C_2 + \frac{z}{x}C_3$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{x^2+y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{x^2+y^2+z^2}{xy} & 1-\frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{x^2+y^2+z^2}{xz} & \frac{2y}{z} & 1-\frac{x^2+y^2}{z^2} \end{vmatrix}$$

Taking $x^2+y^2+z^2$ common from C_1; multiplying R_1 by x^2, R_2 by xy and R_3 by xz,

$$= \frac{\left(x^2+y^2+z^2\right)}{x^2 \cdot xy \cdot xz} \begin{vmatrix} 1 & 2xy & 2xz \\ 1 & xy - \frac{x}{y}\left(x^2+z^2\right) & 2xz \\ 1 & 2yz & xz - \frac{x}{z}\left(x^2+y^2\right) \end{vmatrix}$$

Operating $R_2 - R_1$ and $R_3 - R_1$,

$$= \frac{\left(x^2+y^2+z^2\right)}{x^4 yz} \begin{vmatrix} 1 & 2xy & 2xz \\ 0 & -\dfrac{x\left(x^2+y^2+z^2\right)}{y} & 0 \\ 0 & 0 & -\dfrac{x}{z}\left(x^2+y^2+z^2\right) \end{vmatrix}$$

$$= \frac{\left(x^2+y^2+z^2\right)^3}{x^2y^2z^2}$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{x^2y^2z^2}{\left(x^2+y^2+z^2\right)^3}$$

3. b solution

$x^3+y^3=3axy$ is symmetrical about the line $y=x$ .But $y^3+x^3=3ayx \Rightarrow x^3+y^3=3axy$ Tracing the curve $x^3+y^3=3axy$.

(i) <u>Symmetry</u>
If is symmetric about the line $y=x$ because by putting $y=x$ the curve remains unchanged.
(ii) <u>Tangent/s at origin</u>
Equating the lowest degree term to term
We get $3axy=0 \Rightarrow xy=0$

$\Rightarrow x=0$ or $y=0$

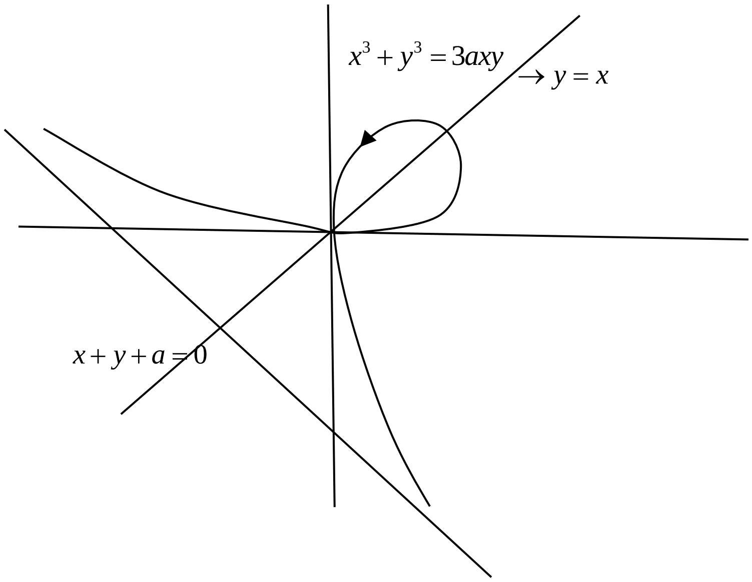
Real and distinct tangents
$\therefore$ Origin is NODE
(iii) <u>Asymptotes Parallel to Axes</u>:
Parallel to y-axis
$1=0$ (not possible)
Parallel to x-axis
$1=0$ (not possible)
$\therefore$ There is no asymptotes parallel to coordinate axes.
By Method of finding asymptotes we get that
$x+y+a=0$ is an asymptote.
(iv) <u>Points on the axes</u>:
x-axis ; put $y=0$, we get $x=0$

i.e. $(0,0)$

y-axis; put $x=0$, we get $y=0$

i.e. $(0,0)$

In 3rd ODE, $\therefore$ $x<0, y<0$, R.H.S. > 0 but L.H.S. < 0, 3rd ODE



Ans4.
Solution.

Applying Rolle's theorem on a function $\phi(x)$ s.t $\phi'(x)$ can give required answer. So let's try to construct $\phi(x)$

Let

$$\phi(x) = \begin{vmatrix} f'(a) & f(x) \\ g(a) & g(x) \end{vmatrix} - \frac{(x-a)}{(b-a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

Now if (i) $f(x)$ and $g(x)$ are continuous in $[a,b]$ then $\phi(x)$ will also be continuous.

(ii) if $f(x)$ and $g(x)$ are differentiable in (a,b) then $\phi(x)$ will also be differentiable in (a,b)

$$\because \ \phi(a) = \begin{vmatrix} f(a) & f(x) \\ g(x) & g(x) \end{vmatrix} - \frac{(b-a)}{b-a} \begin{vmatrix} f(a) & f(a) \\ g(x) & g(x) \end{vmatrix} = 0 - 0 = 0$$

Similarly $\phi(b) = 0$

i.e. $\phi(a) = \phi(b)$

$\because \ \phi(x)$ is continuous in $[a,b]$

$\phi(x)$ is differentiable in (a,b)

$\phi(a) = \phi(b)$

$\therefore$ All condition of Rolle's theorem are satisfied in $[a,b]$ by $\phi(x)$

$\therefore \exists$ at least one real number $\xi \in (a,b)$ s.t $\phi'(\xi) = 0$

$$\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} - \begin{vmatrix} f(a) & f(b) \\ g(x) & g(b) \end{vmatrix} = 0$$

Rough Work to construct ϕ

$$\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = \begin{vmatrix} f(a) & f'(a) \\ g(a) & g'(b) \end{vmatrix}$$

and $\phi(a) = \phi(b)$

To get $\frac{1}{b-a}$, took x and to get $\phi(a) = \phi(b)$ took $\frac{x-a}{b-a}$

b. Solution.

$\because$ Expansion around $x = 2 \Rightarrow$ power of $(x-2)$

$\therefore a = 2$ and $h = x - 2$

$$\therefore \quad f(x) = \underbrace{f(2)}_{\text{Ist Term}} + \underbrace{(x-2)\cdot f'(2)}_{\text{IInd Term}} + \underbrace{\frac{(x-2)^2}{2!} f''(2)}_{\text{IIIrd Term}} + \underbrace{\frac{(x-2)^3}{3!} f'''(2+\theta(x-2))}_{\text{Remainder term after 3 terms where } 0<\theta<1}$$

$\because f(x) = \log(1+x) \;\therefore f(2) = \log(1+2) = \log 3$

$$f'(x) = \frac{1}{1+x} \quad \therefore f'(2) = \frac{1}{1+2}$$

$$f''(x) = -\frac{1}{(1+x)^2} \;\therefore f''(2) = -\frac{1}{9}$$

$$f'''(x) = +\frac{2}{(1+x)^3} \;\therefore f''(2) = \frac{2}{(1+2)^3}$$

Use all values in (1).

Ans5 a. Solution- Putting $y = mx$, we see that

$$\lim_{x \to 0} f(x,y) = \frac{m}{1+m^2}$$

so that the limit depends on the value of m, i.e., on the path of approach and is different for the different paths followed and therefore does not exist. Hence the function $f(x,y)$ is not continuous at $(0,0)$.

Again

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

b. **Solution:-** $\because$ $\tan u$ is a homogenous function of degree 2.

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x}(\tan u)+y\frac{\partial u}{\partial y}(\tan u)=2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$\therefore$ $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \qquad (1)$$

Differentiating (1) w.r.t. x, y separately, we get $x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}+0=2\cos 2u\frac{\partial u}{\partial x}$

Multiplying (iii) by x and (iv) by y ,then adding columnuise, we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3+y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{(x^3+y^3)}=1+3-3$$

$\Rightarrow$ $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.u$ Euler's theorem is verified

(i) $\because$ u is a homogenous function in x, y, z of degree 2.

$\therefore$ $$\frac{\partial u}{\partial x}=ay+cz, \frac{\partial u}{\partial y}=ax+bz, \frac{\partial u}{\partial z}=by+cx$$

$\therefore$ $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2(axy+byz+czx)=2u$$

Hence Euler's theorem is verified

c. Solution- Put $x=r\cos\theta, y=r\sin\theta$.

$\therefore$ $$\left|\frac{x^3-y^3}{x^2+y^2}\right|=\left|r\left(\cos^3\theta-\sin^3\theta\right)\right|\le|r||\cos\theta|^3+|r||\sin\theta|^3\le 2|r|=2\sqrt{x^2+y^2}<\varepsilon,$$

if

$$x^2<\frac{\varepsilon^2}{8},\ y^2<\frac{\varepsilon^2}{8}$$

or, if

$$|x|<\frac{\varepsilon}{2\sqrt{2}},\ |y|<\frac{\varepsilon}{2\sqrt{2}}$$

$\therefore$ $\left|\frac{x^3-y^3}{x^2+y^2}-0\right|<\varepsilon$, when $|x|<\frac{\varepsilon}{2\sqrt{2}}, |y|<\frac{\varepsilon}{2\sqrt{2}}$

$\Rightarrow$ $$\lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x^2+y^2}=0$$

$$\Rightarrow \qquad \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence the function is continuous at $(0,0)$.

Again

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{-k}{k} = -1$$

Thus, the function possesses partial derivatives at $(0,0)$.

If the function is differentiable at $(0,0)$, then by definition

$$df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi \qquad \text{....(1)}$$

when A and B are constants $(A = f_x(0,0) = 1, B = f_y(0,0) = -1)$ and ϕ, ψ tend to zero as $(h,k)\to(0,0)$.

Putting $h = \rho\cos\theta, k = \rho\sin\theta$, and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta \qquad \text{....(2)}$$

For arbitrary $\theta = \tan^{-1}(h/k), \rho \to 0$ implies that $(h,k)\to(0,0)$. Thus we get the limit,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

or $$(\cos\theta - \sin\theta)(\cos^2\theta + \sin^2\theta + \cos\theta + \sin\theta) = \cos\theta - \sin\theta$$

or $$\cos\theta\sin\theta(\cos\theta - \sin\theta) = 0$$

which is plainly impossible for arbitrary θ.

Thus, the function is not differentiable at the origin.

d. Solution- Consider $f(x) = \log(1+x) - \left(x - \dfrac{x^2}{2}\right)$

$$\therefore \quad f'(x) = \frac{1}{1+x} - (1-x) = \frac{x^2}{1+x} > 0, \ \forall x > 0$$

Hence, $f(x)$ is an increasing function for all $x > 0$.

Also

$$f(0) = 0$$

Hence for $x > 0, \ f(x) > f(0) = 0$

Thus

$$\log(1+x) > x - \frac{x^2}{2}, \text{ for } x > 0$$

Similarly by considering the function

$$F(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$$

it can be shown that

$$\log(1+x) < x - \frac{x^2}{2(1+x)},\ \forall x > 0$$

e. Solution.

$$P(x) = \left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x - 1$$

$$P'(x) = \left(\frac{5}{13}\right)^x \log_e\left(\frac{5}{13}\right) + \left(\frac{12}{13}\right)^x \log_e\left(\frac{12}{13}\right) \quad \ldots.(1)$$

$\because \frac{d}{dx}(a^x) = a^x \log_e a$ $\log_e \alpha$ is negative if $0 < \alpha < 1$

$$\left.\begin{array}{l} \because \left(\frac{5}{13}\right)^x > 0; x \in \mathbf{R} \\ \left(\frac{12}{13}\right)^x > 0' x \in \mathbf{R} \\ \because \log_e \frac{5}{13} < 0, \log_e \frac{12}{13} < 0 \end{array}\right\} \quad \ldots.(2)$$

Using (2) in (1), we get

$P'(x) < 0$ for all $x \in \mathbf{R}$

$P(x)$ is strictly decreasing for all $x \in \mathbf{R}$ [Proved]

6.a Solution- Let

$f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f'(x) = 0$ when $\tan x = 1$, so that

$$x = n\pi + \frac{\pi}{4}$$

where n is zero or any integer.

$$f''\left(n\pi + \frac{1}{4}\pi\right) = -\left\{\sin\left(n\pi + \frac{1}{4}\pi\right) + \cos\left(n\pi + \frac{1}{4}\pi\right)\right\}$$

$$= (-1)^{n+1}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = (-1)^{n+1}\sqrt{2}$$

$$\left[\because\ \sin(n\pi+\alpha)=(-1)^n\sin\alpha \text{ and } \cos(n\pi+\alpha)=(-1)^n\cos\alpha\right]$$

Also $f\left(n\pi+\frac{1}{4}\pi\right)=\sin\left(n\pi+\frac{1}{4}\pi\right)+\cos\left(n\pi+\frac{1}{4}\pi\right)=(-1)^n\sqrt{2}$

When n is zero or an even integer, $f''(n\pi+\pi/4)$ is negative and therefore $x=n\pi+\frac{1}{4}\pi$ makes $f(x)$ a maxima with the maximum value $\sqrt{2}$.

When n is an odd integer, $f''\left(n\pi+\frac{1}{4}\pi\right)$ is positive and therefore $x=n\pi+\frac{1}{4}\pi$ makes $f(x)$ a minima with the minimum value $\sqrt{2}$.

b. Solution.

$$|f(x,y)-f(0,0)|=\left|\frac{x^3\cos\frac{1}{y}+y^3\cos\frac{1}{x}}{x^2+y^2}-0\right|$$

$$=\left|\frac{r^3\cos^3\theta\cdot\cos\frac{1}{(r\sin\theta)}+r^3\sin^3\theta\cdot\cos\frac{1}{r\cos\theta}}{r^2\left(\cos^2\theta+\sin^2\theta\right)}\right|$$

$$=\left|\frac{r^3\left(\cos^3\theta\cdot\cos\frac{1}{(r\sin\theta)}+\sin^3\theta\cdot\cos\frac{1}{(r\cos\theta)}\right)}{r^2}\right|$$

$$=\left|r\left(\cos^3\theta\cdot\cos\frac{1}{r\sin\theta}+\sin^3\theta\cos\theta\frac{1}{2\times\cos\theta}\right)\right|$$

$$\leq|r|\cdot\left|\cos^3\theta\cos\frac{1}{\sin\theta}+\sin^3\theta\cdot\cos\frac{1}{x\cos\theta}\right|$$

$$\leq|r|\cdot(1+1)\leq 2|2|\leq 2\sqrt{x^2+y^2}<\varepsilon$$

Define a function f of two real variables in the $x-y$ plane by

$$f(x,y)=\begin{cases}\dfrac{x^3\cos\frac{1}{y}+y^3\cos\frac{1}{x}}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\\ 0\text{ , otherwise}\end{cases}$$

To check differentiability of $f(x,y)$ at $(0,0)$

Step (i)

Let $f(x,y)$ is differentiable $(0,0)$ $\therefore$ by definition

$$f(h,k)-f(0,0)=df=Ah+Bk+h\phi+k\cdot\psi \text{ where } A=\left(\frac{\partial f}{\partial x}\right)_{(0,0)},\ B=\left(\frac{\partial f}{\partial y}\right)_{(0,0)}$$

and ϕ,ψ both tend to zero as $(h,k)\to(0,0)$

$$A=\left(\frac{\partial f}{\partial x}\right)_{(0,0)}=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h^3\cos\frac{1}{0}+0-0}{h}=\lim_{h\to 0}h^2\cos\infty=0\times\text{ some finite value}$$

$= 0$

$$B=\left(\frac{\partial f}{\partial y}\right)_{(0,0)}=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{h^3\cos\frac{1}{0}}{h}=0$$

$\therefore$ From (1)

$$\frac{h^3\cos\frac{1}{k}+k^3\cos\frac{1}{h}}{h^2+k^2}-0=0\times h+0\times k+h\phi+k\cdot\psi$$

Putting $g=\delta\cos\theta, k=\delta\sin\theta$ $\therefore$ for arbitrary $\theta:\delta\to 0$ when $(h,k)\to(0,0)$ $\because \delta^2=h^2+k^2$

$$\not{\delta}\left[\cos\theta\cdot\cos\frac{1}{\delta\sin\theta}+\sin\theta\cdot\cos\frac{1}{\delta\cos\theta}\right]=\not{\delta}\left(\cos\theta\cdot\phi+\sin\theta\cdot\psi\right)$$

$\because$ R.H.S. $\to 0$ But L.H.S. may not be zero for arbitrary values at θ

$\therefore$ Our Assumption was wrong.

7.a Solution.

For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x}=3x^2-3ay=0\text{, i.e., } x^2-ay=0 \qquad ....(1)$$

and $\dfrac{\partial u}{\partial y} = 3y^2 - 3ax = 0$, i.e., $y^2 - ax = 0$. ....(2)

Putting the value of x from (2) in (1), we obtain

$\left(y^2/a\right)^2 - ay = 0$, i.e., $y^4 - a^3 y = 0$, i.e., $y\left(y^3 - a^3\right) = 0$, which gives $y = 0, a$.

From (1), when $y = 0$, we get $x = 0$, and when $y = a$, we get $x = \pm a$.

But $x = -a, y = a$ do not satisfy (2). So we reject these values. Therefore, the only solutions of (1) and (2) are $x = 0, y = 0$ and $x = a, y = a$.

Now $r = \dfrac{\partial^2 u}{\partial x^2} = 6x$, $s = \dfrac{\partial^2 u}{\partial x \partial y} = -3a$ and $t = \dfrac{\partial^2 u}{\partial y^2} = 6y$.

At $x = 0, y = 0$, we have $r = 0, s = -3a$ and $t = 0$. So, $rt - s^2 = -9a^2 < 0$.

Hence u is neither maximum nor minimum at $x = 0, y = 0$.

At $x = a, y = a$, we have $r = 6a, s = -3a$ and $t = 6a$. So,

$rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 = 27a^2$.

Since $rt - s^2$ is positive and r is positive or negative according as a is positive or negative, we have maximum or minimum according as a is negative or positive. The maximum or minimum value of u is $= a^3 + a^3 - 3aaa = -a^3$.

b. Solution.

Let (x, y, z) be the co-ordinates of a vertex, lying in the positive octant, of the rectangular parallelopiped. Then the length of the sides of inscribed parallelopiped are $2x, 2y, 2z$. Therefore, the volume V is $(2x)(2y)(2z)$, i.e.,

$V = 8xyz$, ...(1)

subject to the condition that the point (x, y, z) lies on the ellipsoid

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. ....(2)

Therefore, we need to find the maximum value of V subject to the condition (2). Let

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

For maxima or minima of F, we must have

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda x}{a^2} = 0, \quad \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda y}{b^2} = 0, \quad \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda z}{c^2} = 0.$$

Therefore, $\dfrac{x}{a^2 yz} = \dfrac{y}{b^2 xz} = \dfrac{z}{c^2 xy} = -\dfrac{4}{\lambda}$. ....(3)

Multiplying by x, y, z, these equations give

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \text{ i.e., } \frac{x^2/y^2}{1} = \frac{y^2/b^2}{1} = \frac{z^2/c^2}{1} = \frac{x^2/a^2 + y^2/b^2 + z^2/c^2}{1+1+1}$$

so that $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{1}{\sqrt{3}}$. using (2)

Whence $x = a/\sqrt{3}, y = b/\sqrt{3}, z = c/\sqrt{3}$. ....(4)

Using these values of x, y, z, we have from (3),

$$\lambda = -\frac{4a^2yz}{x} = -\frac{4abc}{\sqrt{3}} \qquad ....(5)$$

Now to identify the maxima or minima, we have

$$d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)^2 F = \sum\frac{\partial^2 F}{\partial x^2}dx^2 + 2\sum\frac{\partial^2 F}{\partial x \partial y}dxdy$$

$$= 2\lambda\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right) + 16\left(z\,dxdy + x\,dydz + y\,dzdx\right)$$

$$= -\frac{8abc}{\sqrt{3}}\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right) + \frac{16}{\sqrt{3}}\left(c\,dxdy + a\,dydz + b\,dzdx\right),$$

using (4) and (5)

$$= -\frac{8abc}{\sqrt{3}}\left\{\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2} - 2\left(\frac{dxdy}{ab} + \frac{dydz}{bc} + \frac{dzdx}{ca}\right)\right\} \qquad ....(6)$$

But the differentiation of (2) gives

$\frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} + \frac{2z\,dz}{c^2} = 0$, i.e., $\frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0$, using (4).

Squaring this equation, we get

$$\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2} + 2\left(\frac{dx}{a}\frac{dy}{b} + \frac{dy}{b}\frac{dz}{c} + \frac{dz}{c}\frac{dx}{a}\right) = 0$$

which gives $\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2} = -2\left(\frac{dxdy}{ab} + \frac{dydz}{bc} + \frac{dzdx}{ca}\right)$, since $(dx)^2 = dx^2$ etc.

Using this equation in (6), we get

$$d^2F = -\frac{16abc}{\sqrt{3}}\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right).$$

Obviously, $d^2F < 0$. Therefore, F and hence V is maximum at the point given by (4). The maximum value of V is:

$$V_{\max} = 8\cdot\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}.$$

The dimensions (i.e., the lengths of sides) of the rectangular parallelopiped of maximum volume are $2a/\sqrt{3}, 2b/\sqrt{3}, 2c/\sqrt{3}$.

Ans8 a. Solution.

We have $u = x^2 + y^2 + xy$ subject to the condition $ax^2 + by^2 = ab$.

Let us define a function F, where

$$F = \left(x^2 + y^2 + xy\right) + \lambda\left(ax^2 + by^2 - ab\right).$$

For maxima and minima of u, we must have

$\partial F/\partial x = 2x + y + 2\lambda ax = 0$, ...(1)

and $\partial F/\partial y = 2y + x + 2\lambda by = 0$. ....(2)

Multiplying (1) by x, (2) by y and adding the resulting equations column wise, we get

$$2\left(x^2+y^2+xy\right)+2\lambda\left(ax^2+by^2\right)=0,$$

i.e., $u+\lambda ab=0$, or $\lambda=-u/ab$. ....(3)

Further, let us write equations (1) and (2) as

$2(1+\lambda a)x=-y$ and $x=-2(1+\lambda b)y$.

Since both x and y cannot be zero, it follows that

$$4(1+\lambda a)(1+\lambda b)=1 \quad ....(4)$$

Putting the value of λ from (3), this gives

$$4(1-u/b)(1-u/a)=1,$$

or $4(u-a)(u-b)=ab$, ...(5)

which gives maximum and minimum values of u

Now $d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 F$

$$=\frac{\partial^2 F}{\partial x^2}dx^2+\frac{d^2F}{\partial y^2}dy^2+2\frac{\partial^2 F}{\partial x\partial y}dxdy$$

$$=2(1+\lambda a)dx^2+2(1+\lambda b)dy^2+2dxdy$$

$=2(1+\lambda a)\left\{dx+2(1+\lambda b)dy\right\}^2$, by (4).

Thus d^2F is positive or negative according as $1+\lambda a$ (or $1+\lambda b$) is positive or negative.

From (5), we obtain

$$4a^2-4(a+b)u+3ab=0,$$

i.e., $2u=(a+b)\pm\left\{(a-b)^2+ab\right\}^{1/2}$.

Therefore, supposing a and b to be positive, we see that with the upper sign,

$$2u>(a+b)+|a-b|,$$

i.e., $u<a$ or b whichever is less.

Further, since $u+\lambda ab=0$, we find that

if $u>a\,(a>b)$, then $1+\lambda b<0$,

and if $u<b\,(b<a)$, then $1+\lambda a>0$.

It follows that the value of u with the upper sign is the maximum, while the value with the lower sign is the minimum.

Again, since $2(1+\lambda a)x=-y$, we observe that when $1+\lambda a<0$, x and y have the same sign, and there are two points in the first and the third quadrants at which u is maximum. Similarly, u is minimum at two points lying in the second and the fourth quadrants.

b. Solution

Consider a function $f(x)=\tan x-x$ ....(1)

$\because f'(x)=\sec^2 x-1>0$ for $x\in(0,\pi/2)\Rightarrow f(x)$ is increasing strictly

$\Rightarrow f(x) > f(0)$ for $x > 0 \Rightarrow \tan x - x > \tan 0 - 0$ for $x > 0 \Rightarrow \tan x - x > 0$

$\Rightarrow \tan x > x$ for $x > 0$ in $(0, \pi/2)$

Now let's consider another function $g(x) = x - \sin x$ for $x \in [0, \pi/2]$

$\because g(x) = 1 - \cos x > 0 \Rightarrow g(x)$ is strictly increasing

$\therefore$ By definition $g(x) > g(0)$ for $x > 0$

$x - \sin x > 0 - \sin 0$ for $x > 0$

$x > \sin x$ for $x > 0$ in $(0, \pi/2)$ ….(3)

$\therefore$ On combining (2) and (3) we get

$\tan x > x > \sin x$ for $x \in [0, \pi/2]$

c. Solution.

We have used some fundamentals too

Let's consider a function

$f(x) = x - \sin x$

$\therefore f'(x) = 1 - \cos x > 0$ for $x \in (0, \pi/2)$

$\Rightarrow f(x)$ is strictly increasing in the interval $[0, \pi/2]$

$\therefore$ By definition $f(x) > f(0)$ for $x > 0$

$x - \sin x > 0 \cdot \sin 0$ in $(0, \pi/2)$

$x - \sin x > 0$ in $(0, \pi/2)$

$x > \sin x$ in $(0, \pi/2)$

$\therefore \cos(x) < \cos(\sin x)$ Hence Proved

Inequality reversed: On increasing x; $\cos x$ decreases in $(0, \pi/2)$

$\cos\alpha$, $\cos\beta$
$\downarrow$ $\downarrow$
$\alpha = x$ $\beta = \sin x$

If get some relation between α and β then it will be easy to get relation because $\cos\alpha$ and $\cos\beta$

$$\left.\begin{matrix}\alpha < \beta ? \\ \alpha > \beta ? \\ \alpha = \beta ?\end{matrix}\right\} \Rightarrow \alpha \cdot \beta$$