MINDSET MAKERS

UPSC
IAS / IFoS
Mathematics
Optional

Vector Calculus Book

With PYQs till year 2023

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WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFOS :

01 Conceptual Development

02 Problem Solving Techniques

03 Assignments

04 Chapter wise PYQs Analysis

05 Test

MINDSET MAKERS

I.I.T UPSC

This book consists of:

Part-1: Conceptual clarity for Vector Analysis, Differentiation, Integration and related theorems

Part-2: Examples and PYQs Analysis: Gradient, Divergence and Curl

Part-3: Examples and PYQs Analysis: line Integral, Green's theorem

Part-4: Examples: Surface Integral

Part-5: Examples: Gauss's Divergence theorem

Part-6: Examples: Stoke's theorem

Part-7: PYQs Analysis: Gauss div & Stoke's theorems

Part-8: Examples and PYQs Analysis: Curvature & Torsion

Part-9: Examples and PYQs Analysis: Curvilinear coordinates

VECTOR ANALYSIS & CALCULUS

Vector Analysis: 5% Qs

Dot Product, Cross Product, Properties of vectors and their modulus

Example PYQs-

Let $\vec{a}, \vec{b}, \vec{c}$ are some given vectors. Show that they possibly make a triangle. Also Find medians of this triangle.

Vector Calculus = 95% Qs

(I) Differential

- Gradient
- Directional derivative
- Greatest rate of increase
- Angle between two surfaces
- Divergence, Solenoid Field, Change in per unit volume per unit time (Rate)
- Curl, Rotation, Work done etc. Exactness.

(II) Integral

- Line Integral, Surface Integral, Volume Integral
- Three Important Theorems
  Green's, Stokes, Gauss Divergence Theorem.

Chapter 1: Vector Analysis

Vectors: (Vector Space) : V(IF)

- A non-empty set of vector
- A field of scalar IF
- Vector Addition
- Scalar Multiplication

e.g. A special kind of vector spaces : $\mathbf{R}^n(\mathbf{R})$ [Euclidean Space]

$$\mathbf{R}^n = \{(a_1, a_2 \ldots a_n);\ a_1, a_2 \ldots \text{ are real numbers}$$

Field is Real Numbers.

Vector Addition: $(a_1, a_2, \ldots a_n) + (b_1, \ldots b_n)$

$= (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n)$

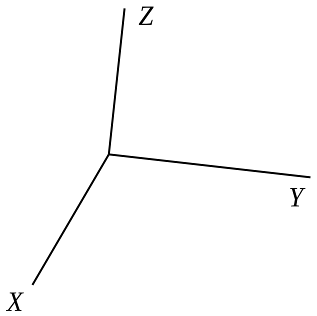
Scalar Multiplication

$\alpha(a_1, a_2 \ldots a_n) = (\alpha a_1, \alpha a_2 \ldots \alpha a_n)$

More Specifically, Here we will deal with $\mathbf{R}^3(\mathbf{R})$

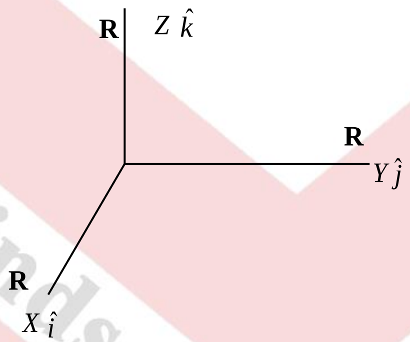
$\mathbf{R}^3 = \{(a_1, a_2, a_3); a_1, a_2, a_3 \text{ are real number}\}$

(3- Dimensional Euclidean Space)



Representation

$\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$



$\vec{a} = a_1\hat{i} + a_2\hat{j} + a\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Operation on Vectors:

(1) Scalar product:

$\vec{a} \cdot \vec{b}$ for two vectors

For three vectors-

(Scalar triple product) $\vec{a}, \vec{b}, \vec{c}$

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ = Volume of parallelepiped having edges $\vec{a}, \vec{b}, \vec{c}$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

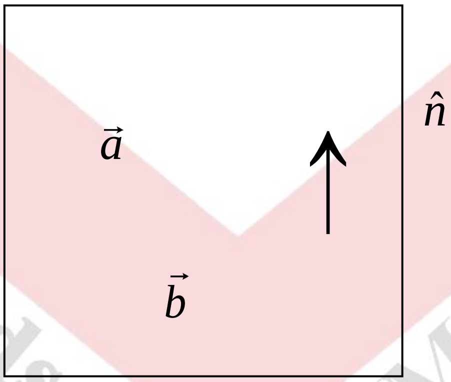
$$\vec{b}\cdot(\vec{c}\times\vec{a})=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left[\vec{b}\ \vec{c}\ \vec{a}\right]=\left[\vec{c}\ \vec{a}\ \vec{b}\right]$$

(2) <u>Vector Product</u>

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}, 0<\theta<\pi$$

$\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.



<u>Vector Triple Product</u>

Formula

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$\left(\vec{a}\times\vec{b}\right)\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}$$

<u>Reciprocal Seat of Vectors</u>

$\vec{a},\vec{b},\vec{c}$ are set to form reciprocal set of vector if

$$\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$$

$$\vec{a}\cdot\vec{b}=\vec{a}'\cdot\vec{c}=\vec{b}'\vec{a}=\vec{b}'\vec{c}'=\vec{c}'\cdot\vec{a}=\vec{c}'\cdot\vec{b}=0$$

Note: $\vec{a}',\vec{b}',\vec{c}'$, $\vec{a},\vec{b},\vec{c}$ are said to be reciprocal

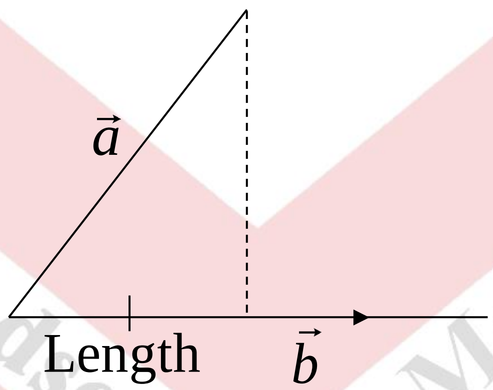
if $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)}=\dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\vec{b}\vec{c}\right]}$

$\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{\vec{b}\cdot(\vec{c}\times\vec{a})}=\dfrac{\vec{c}\times\vec{a}}{\left[\vec{b}\vec{c}\vec{a}\right]}=\dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\vec{b}\vec{c}\right]}$

Projection of $\vec{a}$ on $\vec{b}$

is given by $\vec{a}\cdot\hat{b}$

where $\hat{b}=\frac{\vec{b}}{|\vec{b}|}$



Example: Find projection of

$\vec{A}=\hat{i}-2\hat{j}+3\hat{k}$ on the vector $\hat{i}+2\hat{j}+2\hat{k}=\vec{B}$

$$\vec{A}\cdot\hat{B}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\frac{\left(\hat{i}+2\hat{j}+2\hat{k}\right)}{\left|\hat{i}+2\hat{j}+2\hat{k}\right|}=\frac{1-4+6}{\sqrt{1^2+4+4}}=\frac{3}{3}=1$$

Q1. Without making use of cross product find a vector perpendicular to the plane of

$\vec{A}=2\hat{i}-6\hat{j}-3\hat{k}$

$\vec{B}=4\hat{i}+3\hat{j}-\hat{k}$

Solution.

Let $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ is required unit vector

$\therefore\ \vec{c}\cdot\vec{A}=0$

$\Rightarrow\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)\cdot\left(2\hat{i}-6\hat{j}-3\hat{k}\right)=0$

$2c_1-6c_2-3c_3=0$ ....(i)

$\vec{c}\cdot\vec{B}=0\Rightarrow 4c_1+3c_2-c_3=0$ ....(ii)

On solving (i) and (ii) we get

$c_1=\frac{1}{2}c_3, c_2=-\frac{1}{3}c_3$

$$\therefore\ \vec{c} = \frac{1}{2}c_3\hat{i} - \frac{1}{3}c_3\hat{j} + 3c_3\hat{k}$$

$$\therefore\ \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\frac{1}{2}c_3\hat{i} - \frac{1}{3}c_3\hat{j} + c_3\hat{k}}{\left|\sqrt{\frac{1}{4}c_3^2 + \frac{1}{9}c_3^2 + c_3^2}\right|} = \frac{\frac{1}{2}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}}{\sqrt{\frac{1}{4} + \frac{1}{9} + 1}}$$

Formula

(1) Area of parallelogram with touching side as $\vec{A}, \vec{B}$

$= |\vec{A} \times \vec{B}|$

(2) Area of triangle with two adjacent sides $\vec{A}$, $\vec{B}$

$= \frac{1}{2}|\vec{A} \times \vec{B}|$

Q. Prove that the necessary and sufficient condition for $\vec{A}, \vec{B}, \vec{C}$ to be coplaner is $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

Solution.

The necessary part -

Let if $\vec{A}, \vec{B}, \vec{C}$ are coplaner then $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ must hold.

As we know that $\vec{B} \times \vec{C}$ represents a vector perpendicular to plane containing $\vec{B}$ and $\vec{C}$

$\therefore\ \vec{A}$ must be $\perp$ to $\vec{B} \times \vec{C}$

$\therefore\ \vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

<u>Sufficient Part -</u>

Let if $\vec{A}.(\vec{B} \times \vec{C}) = 0$ then volume of parallelepiped with edges $\vec{A}, \vec{B}, \vec{C}$ must be zero.

$\Rightarrow \vec{A}, \vec{B}, \vec{C}$ must lie in same plane.

Q. Find the equation of the plane containing three vectors $P_1(2,-1,1), P_2(3,2,-1), P_3(-1,3,2)$

Solution. We know that

Equation of a plane is given as

$Ax + by + cz + d = 0$

*Two planes together represent straight line in 3D (if they intersect) represented by

$a_1x + a_2y + a_3z + a_4 = 0$

$b_1x + b_2y + b_3z + b_4 = 0$

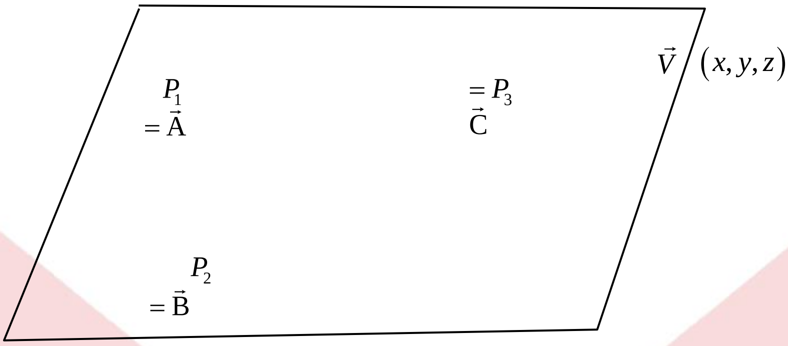
or in symmetrical form, Line passing through $(x_1, y_1, z_1), (x_2, y_2, z_2)$ is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$P_1(2,-1,1) = 2\hat{i} - \hat{j} + \hat{k} = \vec{A}$

$P_2(3,2,-1) = 3\hat{i} + 2\hat{j} - \hat{k} = \vec{B}$

$P_3(-1,3,2) = -\hat{i} + 3\hat{j} + 2\hat{k} = \vec{c}$



$(\vec{r} - \vec{A})(\vec{r} - \vec{B})(\vec{r} - \vec{C})$ are co-planer

$(\vec{r} - \vec{A}) \cdot ((\vec{r} - \vec{B}) \times (\vec{r} \times \vec{C})) = 0$

$\Rightarrow ((x-2)\hat{i} + (y+1)\hat{j} + (z-1)\hat{k}).$

$.[(x-3)\hat{i} + (y-2)\hat{j} + (z+1)\hat{k} \times (x+1)\hat{i} + (y-3)\hat{j} + (z-z)\hat{k}] = 0$

$\Rightarrow 11x + 5y + 13z = 30$

Q. Find the constant a so that the following vectors are co-planer

$2\hat{i} - \hat{j} + \hat{k}\ ,\hat{i} + 2\hat{j} - 2\hat{k},\ 3\hat{i} + a\hat{j} + 5\hat{k}$

A B C

Solution.

$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \quad (\vec{a}\ \vec{b}\ \vec{c})$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

$\Rightarrow 2(10+30) + 1(5+9) + 1(a-6) = 0$

$\Rightarrow 20 + 6a + 14 + a - 6 = 0$

$\Rightarrow 34 - 6 + 7a = 0$

$\Rightarrow 28 + 7a = 0$

$\boxed{a = -4}$

CONCEPTUAL CLARITY FOR VECTOR CALCULUS

Vector Differentiation

Type (1) Problems

Simple Differentiation

e.g. Velocity, Acceleration, Momentum, Work done, K.E.

if position vector

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Parametric Form

e.g. if $\vec{r} = \sin t\hat{i} + e^{t}\hat{j} + e^{2t}\cos t + \hat{k}$ then find velocity at $t = 0$ and acceleration at $t = 0$

Note- Some added information from Vector Analysis will be needed here too.

Type II Problem

(1) Gradient

- Finding gradient at some point
- Finding normal vector to cover surface
- Angle of intersection between two level surfaces
- Gradient and greatest rate of increase/decrease

(2) Divergence

(3) Curl

Type - I

(i) $d\left(\vec{A}+\vec{B}\right) = d\vec{A} + d\vec{B}$

(ii) $d\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\cdot d\vec{B} + d\vec{A}\cdot\vec{B}$

(iii) $d\left(\vec{A}\times\vec{B}\right) = \vec{A}\times d\vec{B} + d\vec{A}\times\vec{B}$

(iv) $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\cdot\vec{B}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\vec{A}\cdot\vec{B}\right)\right)$

$$= \frac{\partial}{\partial y}\left(\vec{A}\cdot\frac{\partial}{\partial x}\vec{B} + \left(\frac{\partial}{\partial x}\vec{A}\right)\vec{B}\right)$$

I II I II

$$= \vec{A}\cdot\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\vec{B}\right) + \left(\frac{\partial}{\partial y}\vec{A}\right)\left(\frac{\partial}{\partial z}\vec{B}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\vec{A}\right)\cdot\vec{B} + \left(\frac{\partial}{\partial y}\vec{B}\right)\left(\frac{\partial}{\partial y}\vec{A}\right)$$

Similarly we can do for $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right)$

PYQ [2012]

If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

$\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, then find the value of $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right)$ at $(1, 0, -2)$

Solution. Hint:

$$\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\vec{A}\times\vec{B}\right)$$

Method-(1) First find $\vec{A}\times\vec{B}$ then derivative

Method-(2) Applying formula

$$\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right)\right)$$

$$=\frac{\partial}{\partial x}\left(\vec{A}\times\frac{\partial}{\partial y}\times\vec{B}+\vec{B}\times\frac{\partial}{\partial y}\vec{A}\right)$$

$$=\vec{A}\times\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\vec{B}\right)+\left(\frac{\partial}{\partial y}B\right)\frac{\partial}{\partial y}A+B\times\left(\frac{\partial}{\partial y}\vec{A}\right)\frac{\partial}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\vec{A}\right)\cdot\vec{B}$$

Q. For two vectors $\vec{a}$, $\vec{b}$ given by

[2017]

$$\vec{a}=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$$
$$\vec{b}=\sin t\,\hat{i}-\cos t\,\hat{j}$$

Determine $\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$, $\frac{d}{dt}\left(\vec{A}\times\vec{B}\right)$

Solution.

$$\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)=\vec{a}\,\frac{d\vec{b}}{dt}+\vec{b}\cdot\frac{d\vec{\theta}}{dt}$$

$$=\left(5t^2\hat{i}+t\hat{j}-t^3\hat{k}\right)\cdot\left(\cos t\,\hat{i}+\sin t\,\hat{j}\right)+\left(10t\,\hat{i}+\hat{j}-3t^2\hat{k}\right)\cdot\left(\sin t\,\hat{i}-\cos t\,\hat{i}\right)$$

$$=5t^2\cos t+t\sin t-t^3\cdot 0+10t\sin t-\cos t-3t^2\cdot 0$$

$$=5t^2\cos t+11t\sin t-\cos t$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\frac{d\vec{b}}{dt}+\vec{b}\times\frac{d\vec{a}}{dt}$$

$$=\begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix}+\begin{vmatrix} i & j & k \\ 10t & 1 & -3t \\ \sin t & -\cos t & 0 \end{vmatrix}$$

Q. The position vector of a moving particle at time t is

[2017]

$$\vec{v}=\sin t\,\hat{i}+\cos 2t\,\hat{j}+\left(t^2+2t\right)\hat{k}$$

Find the component of acceleration $\vec{a}$ in the direction parallel to the velocity vector $\vec{v}$ and perpendicular to the plane of $\vec{v}$ and $\vec{v}$ at time $t=0$.

Solution. Hint:

$$\because\quad \vec{v}=\sin t\,\hat{i}+\cos 2t\,\hat{j}+\left(t^3+2t\right)\hat{k}$$

$$\vec{v}=\frac{d\vec{r}}{dt}=\cos t\,\hat{i}-2\sin 2t\,\hat{j}+\left(2t+2\right)\hat{k}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$$

Dynamics

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{v}}{dt^2}$$
$$= -\sin t\,\hat{i} - y\cos 2t\,\hat{j} + 2\hat{k}$$

Vector Analysis required here

If $c = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Here c_1, c_2, c_3 are component of $\vec{c}$

Let's say if it is given $\vec{c}$ is parallel to $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\Rightarrow \frac{c_1}{b_1} = \frac{c_2}{b_2} = \frac{c_3}{b_3} \quad ....(1)$$

Let's say if $\vec{c}$ is perpendicular to $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

$$c_1d_1 + c_2d_2 + c_3d_3 = 0 \quad ....(2)$$

Using those condition (1) and (2)

Can we try to figure out

$c_1 =$

$c_2 =$

$c_3 =$

We need

$$\frac{a_1}{v_1} = \frac{a_2}{v_2} = \frac{a_3}{v_3} \quad ....(1)$$

$$\because \quad \vec{\gamma} \times \vec{v} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$$

$$\therefore \quad \alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3 = 0 \quad ....(2)$$

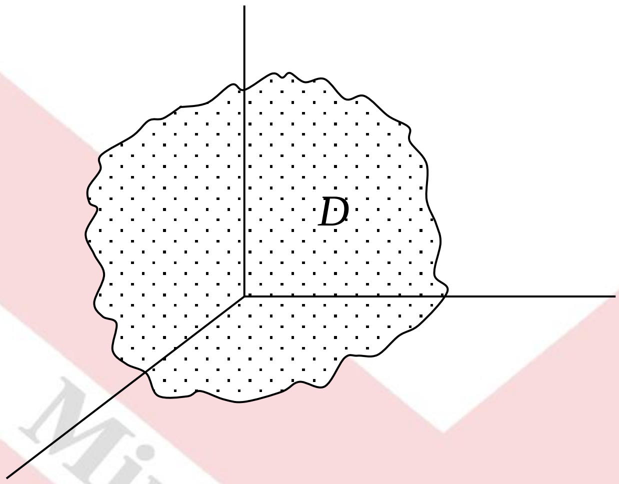
<u>Vector Calculus</u>

Scalar Field: If we can assign some particular scalar value to each point of a region D in some space then this scalar valued function is called scalar function of the position and we say $f(x, y, z)$ is a scalar field defined on region D.

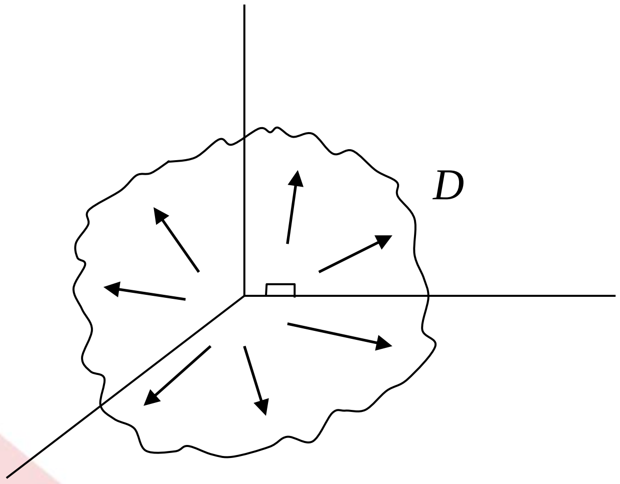
e.g.

Temperature $T(x, y, z) = x^2 y + yz^3$ on earth's surface is a scalar field because we can assign a particular scalar value to each point on surface.

$T(1,2,3) = 1^2 - 2 + 2 \cdot 3^2 = 56$



Vector Field: Suppose to each point (x, y, z) in the region D in space there corresponds a vector $\vec{f}(x, y, z)$ then $\vec{f}$ is known as vector function of the position (x, y, z) and we say that a vector field $\vec{f}$ has been defined on D.



Level Surfaces: Let's consider a function of 3 variables $f(x,y,z)$ whose inputs are points in $\mathbf{R}^3$ and whose outputs are numbers.

e.g.

$f(x,y,z)=x^2+y^2+z^2$

or $f(x,y,z)=x^2+y^3$

or $f(x,y,z)=z-(x^2+y^2)$

A function $f(x,y,z)$ is said to be of level $\mathbf{K}$ to be the set of all points in $\mathbf{R}^3$ which are solution of $f(x,y,z)=\mathbf{K}$.

e.g.

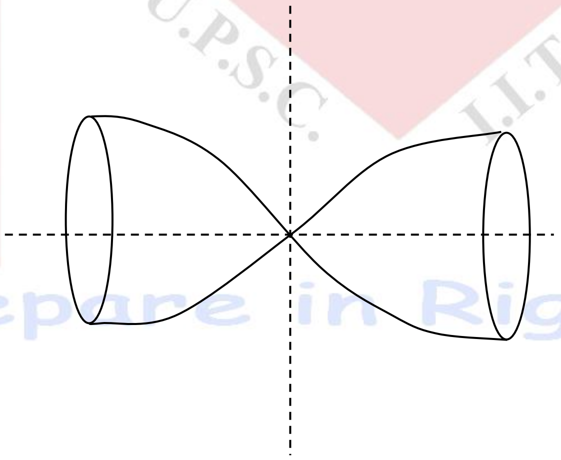
$\therefore \quad z=\phi(x^2,y^2)$

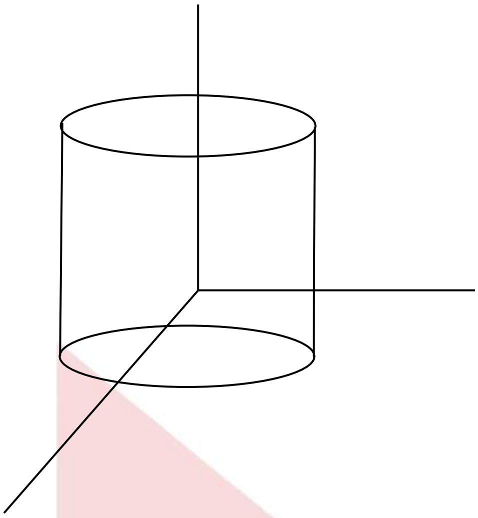
$a \le t \le b$ cylinder.

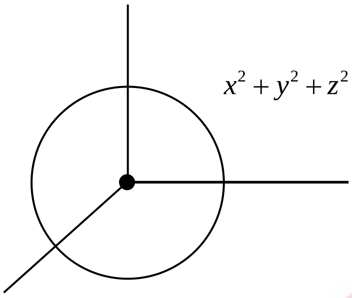
$\therefore \quad z=\phi(x^2,y^2)$

$a \le z \le b$

cylinder







(i) Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(ii) Elliptic paraboloids: $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

(iii) Hyperbolic paraboloids: $\frac{z}{c}=\frac{x^2}{a^2}-y^2/b^2$

(iv) Hyperboloid in one-shed: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(v) Hyperboloid in two-shed: $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Directional Derivative

- $\dfrac{\partial f}{\partial x}$ is the directional derivatives of f along the direction of unit normal vector $\hat{i}$.
- $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are directional derivatives of f along $\hat{j}$ and $\hat{k}$.

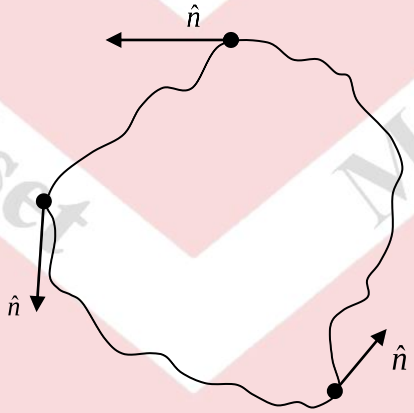
  i.e., $\dfrac{\partial f}{\partial n}$ is the directional derivative of the function f in an arbitrary direction n (along unit normal vector $\hat{n}$)

Gradient and Level Surfaces

For a scalar function f the gradient vector is defined as $\dfrac{\partial f}{\partial n}\cdot\hat{n}$ where $\hat{n}$ is the unit normal vector to the level surface f at some point in the direction of increasing f and $\dfrac{\partial f}{\partial n}$ is called the normal derivative at that point.

Grad f :

$$\vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f = \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$



Note:

$$|\text{grad } f| = \left|\frac{\partial f}{\partial n}\right| |\hat{n}|$$

$$\therefore \quad |\vec{\nabla} f| = \frac{\partial f}{\partial n}\cdot 1 = \frac{\partial f}{\partial n}$$

For a function f, the gradient vector $\vec{\nabla} f$ has the properties:

- It points in the direction in which f increases most rapidly (fastest).
- It is perpendicular to level curves or surface of f.

Divergence

"Loss" per unit volume, per unit time

Vector valued function

$$\nabla\cdot\vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}} \qquad \text{....(1)}$$

Let's consider a vector valued function

$\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$ which is defined and differentiable at each point (x,y,z) in a region of space. Then div$\left(\vec{f}\right)$ is defined by equation (1).

Although $\vec{f}$ is a vector valued function but div $\left(\vec{f}\right)$ is a scalar.

e.g.

Find the Divergence of $\vec{F} = \left(e^{x\log z} + \cos y\right)\hat{i} + \left(z^2 + \log x\right)\hat{j} + e^{2z}\hat{k}$ at $\left(1, e^{2\log 7}, \log 5\right)$ over a region in $\mathbf{R}^3$ in which $\vec{F}$ is defined and differentiable.

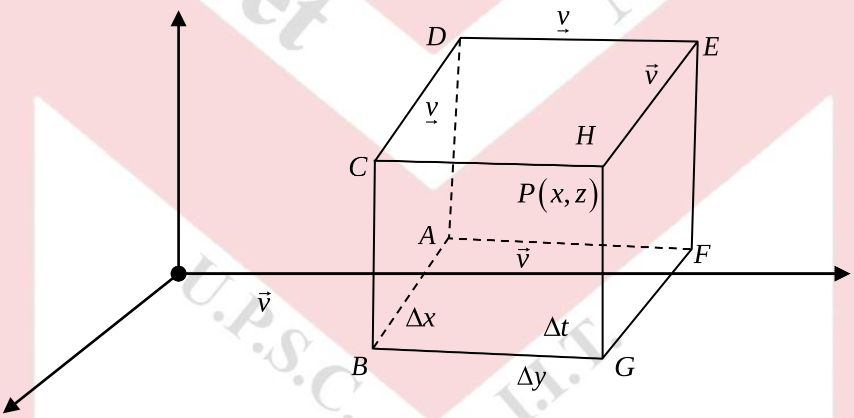
$$\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}\left(e^{x\log z} + \cos y\right) + \frac{\partial}{\partial y}\left(z^2 + \log x\right) + \frac{\partial}{\partial z}\left(e^{2z}\right)$$

$$= e^{x\log z}\log z + 0 + 2e^{2z}$$

$$= \log z\, e^{x\log z} + 2e^{2z} \text{ at } \left(1, e^{\sqrt{2}\log 7}, \log 5\right)$$

$$\vec{\nabla}\cdot\vec{f} = \log(\log 5)e^{\log(\log 5)} + 2e^{2\log 5} = \log 5\log(\log 5) + 50$$

Q. A fluid moves so that it's velocity at any point $P(x,y,z)$ is $\vec{v}(x,y,z)$. Show that the loss of fluid per unit volume per unit time in a small parallelepiped having centre at $P(x,y,z)$ and edge parallel to the coordinate axes and having magnitude $\Delta x, \Delta y$ and Δz respectively, is given approximately by div $\vec{v}$.



Let x component of velocity $\vec{v}$ at $P = v_1$

x component of $\vec{v}$ at centre of the face

AFED $= v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x$ approx.

- x component of $\vec{v}$ at centre of the face

  GHCB $= v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x$ approx.

Now it is clear volume of the fluid entering through the face GHCB per unit time

$$= \left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x \right)\Delta y \Delta z$$

$\because \quad v = \frac{d}{t}$

$t = 1, v = d$

Volume $= d\Delta y \Delta z$

Volume of the fluid existing through the face

AFED $= \left(v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x \right)\Delta y \Delta z$

So loss in volume per unit time in x-direction

$$= \left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x \right)\Delta y \Delta z - \left(v_1 - \frac{\partial v_1}{\partial x}\Delta x \right)\Delta y \Delta z = \frac{\partial v_1}{\partial x}\Delta x \Delta y \Delta z \qquad ....(1)$$

Similarly loss in volume of the fluid per unit time in the y-direction

$$= \frac{\partial v_2}{\partial y}\Delta x \Delta y \Delta z$$

and in z-direction

$$= \frac{\partial v_3}{\partial z}\Delta x \Delta y \Delta z$$

$\therefore$ Total loss in volume of the fluid per unit volume per unit time equal to

$$= \frac{\frac{\partial v_1}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial v_2}{\partial y}\Delta x \Delta y \Delta z + \frac{\partial v_3}{\partial z}\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \vec{\nabla} \cdot \vec{\nabla} \;\; \text{div}\, \vec{v}$$, where

$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Note:

The above article is true exactly only in the limit as the parallelepiped shrinks to P i.e., $\Delta x, \Delta y, \Delta z$ approaches to '0'. If there is no loss of fluid anywhere then $\boxed{\text{div}\, \vec{v} = \vec{\nabla} \cdot \vec{v} = 0}$. This is known as equation of continuity for an incompressible fluid.

i.e., neither source nor sinks such vector $\vec{v}$ is known as **Solenoidal.**

<u>**Curl**</u>

Let's consider a vector valued function $\vec{f} = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$

If $\vec{F}$ is differentiable, then the curl or rotation of $\vec{F}$ is defined as

$$curl\, \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial t} \right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial t} \right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

Note:

At the time of numerical solution we should take care of e.g. f_3, f_1 are functions free from y and f_2, f_3 are free from x and f_1, f_2 are from z, then the calculation is very easy.

e.g. curl $\vec{F}$, $\vec{F} = \left(e^{x^2} \cos x\right)\hat{i} + e^{2y}\hat{j} + e^{z^2} \log z\,\hat{k}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial t}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) = 0 \text{ (zero vector)}$$

Note:

Suppose ϕ and $\vec{A}$ are differentiable scalar and vector functions respectively and both have continuous 2nd partial derivatives, then following laws hold.

(i) $\vec{\nabla} \cdot \left(\vec{\nabla}\phi\right) = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$ where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called Laplacian operator.

(ii) $\vec{\nabla} \times \left(\vec{\nabla}\phi\right) = 0$ i.e., curl grad $\phi = 0$

(iii) $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$ i.e., div curl $\vec{A} = 0$

(iv) ϕ satisfies Laplacian equation if $\nabla^2\phi = 0$

Note:

This problem indicates that the curl of a vector field has something to do with the rotational properties of the field (Because $\vec{\omega}$ is present).

- If the field $\vec{F}$ is that due to a moving fluid e.g. a paddle wheel placed at various points in the field would tend to rotate in regions where curl $\vec{F} \neq 0$, while curl $\vec{F} = 0$ in the region, there would be no rotation and in this case, the vector field $\vec{F}$ is called Irrational.
- If a field is not irrotational then sometimes it is also called as a "Vortex Field".

__Vector Integration__

Let's consider a vector valued function $\vec{F} = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$ in a region D of some space.

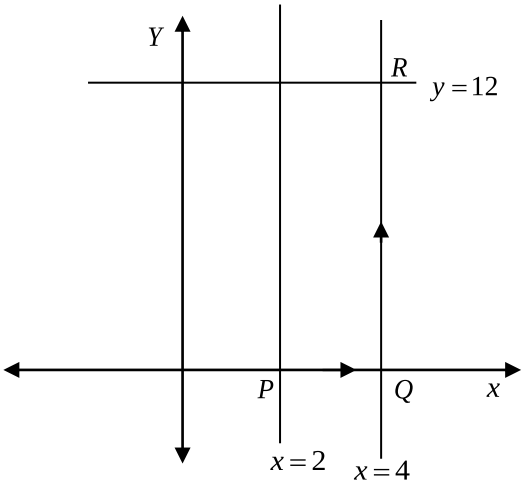
If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of some point in this region D.

Let c be a curve in this region and we want to find the value of the integral $\int_c \vec{F} \cdot d\vec{r}$ i.e., integration along the curve c.

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \left\{ f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k} \right\} \cdot \left\{ dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right\}$$

$$\Rightarrow \boxed{\int_c \vec{F} \cdot d\vec{r} = \int_c f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz}$$

e.g. Evaluate the line integral $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = xy\,\hat{i} + \left(x^2 + y^2\right)\hat{j}$ and the curve c is the x-axis from $x = 2$ to $x = 4$ and the line $x = 4$ from $y = 0$ to $y = 12$.



$$\int_c \vec{F}\cdot d\vec{r} = \int_c xy\,dx + \left(x^2+y^2\right)dy$$

Along the line PQ; $x=2$ to $x=4$, $y=0$ and $dy=0$

$$\int_{PQ} \vec{F}\cdot d\vec{r} = \int_{x=2}^{x=4} \left(x\times 0\times dx\right)+\left(x^2+0^2\right)\cdot 0 = 0$$

Along the line QR

$x=4 \Rightarrow dx=0, y=0$ to $y=12$

$$\int_{QR} \vec{F}\cdot d\vec{r} = \int_{y=0}^{y=12} 4\times y\times 0+\left(4^2+y^2\right)dy = \left[16y+\frac{y^3}{3}\right]_0^{12} = \left[192+\frac{(12)^3}{3}\right] = 192+576=768$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_{PQ} \vec{F}\cdot d\vec{r} + \int_{QR} \vec{F}\cdot d\vec{r} = 0+768=768$$

Conservative Fields

Suppose $\vec{F}=\vec{\nabla}\phi$ everywhere in a region $\mathbf{R}$ of the space where R is defined by $a_1\le x\le a_2, b_1\le y\le b_2, c_1\le z\le c_2$ and $\phi(x,y,z)$ is a single valued function and has continuous partial derivatives in the region $\mathbf{R}$. Then

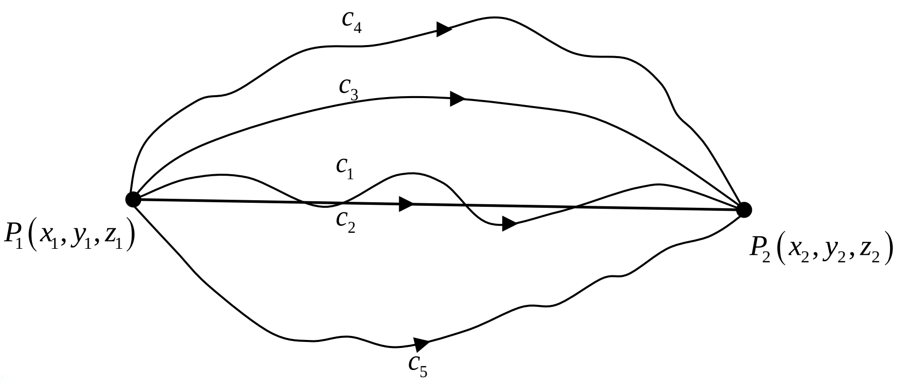
(i) $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ is independent of the path c in $\mathbf{R}$ joining the points P_1 and P_2 in $\mathbf{R}$.

(ii) $\oint_c \vec{F}\cdot d\vec{r}=0$ around any closed curve c in $\mathbf{R}$.

In such a case $\vec{F}$ is called conservative vector field and ϕ is its scalar potential.

Q. Suppose $\vec{F}=\nabla\phi$, where ϕ is single valued and has continuous partial derivatives. Show that the work done in moving a particle from a point P_1 to $P_2(x,y,z)$ in this vector field is independent of the path joining P_1 and P_2. Conversely suppose $\int_c \vec{F}\cdot d\vec{r}$ is independent of the path c joining two points. Show that $\exists$ a function ϕ s.t. $\vec{F}=\vec{\nabla}\phi$.

Proof:



Let $\vec{F} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

Work done $= \int_{c_1} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right)$

$= \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \right) = \int_{P_1}^{P_2} d(\phi) = \phi(P_2) - \phi(P_1)$

Since it is given that ϕ is single valued so whatever the path $c_1 / c_2 / c_3 ...$ joining the points P_1 and P_2 is chosen, we get

Work done $= \int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r} = \int_{c_3} \vec{F} \cdot d\vec{r} = \phi(P_2) - \phi(P_1)$

Therefore if $\vec{F} = \vec{\nabla}\phi$ the work done or the line integral $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the path.

<u>Conversely</u>

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

We have to show

If $\int_c \vec{F} \cdot d\vec{r}$ is independent of path c.

Then $\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

i.e., we have to show $F_1 = \frac{\partial \phi}{\partial x}, F_2 = \frac{\partial \phi}{\partial y}, F_3 = \frac{\partial \phi}{\partial z}$

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{F} \cdot d\vec{r} = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \left(F_1 dx + F_2 dy + F_3 dz \right) \qquad ....(1)$$

$$\therefore \quad \phi(x + \Delta x, y, t) = \int_{(x_1, y_1, z_1)}^{(x + \Delta x, y, t)} F_1 dx + F_2 dy + F_3 dz \qquad ....(2)$$

Target

$$\frac{\partial \phi}{\partial x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y, z) - \phi(x, y, z)}{\Delta x}$$

On subtracting (1) from (2), we get

$\phi(x + \Delta x, y, t) - \phi(x, y, z)$

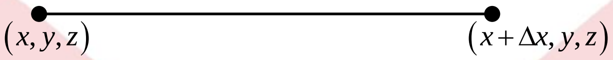
$$= \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} F_1dx + F_2dy + F_3dz - \int_{(x_1,y_1,z_1)}^{(x,y,z)} F_1dx + F_2dy + F_3dz$$

$$= \int_{(x,y,z)}^{(x_1,y_1,z_1)} F_1dx + F_2dy + F_3dz + \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} F_1dx + F_2dy + F_3dz$$

$$= \int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1dx + F_2dy + F_3dz$$

$$\Rightarrow \frac{\phi(x+\Delta x, y, t) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)} F_1dx + F_2dy + F_3dz \qquad ....(3)$$

Since we have taken the integral in the R.H.S. of equation (3) is independent of the path joining points (x, y, z) and $(x+\Delta x, y, z)$. So, let's choose the path as straight line



$$\therefore \; dy = 0,\; dz = 0$$

So equation (3) becomes

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\phi(x+\Delta x, y, t) - \phi(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1dx + 0 + 0}{\Delta x} = d\int F_1dx = F_1$$

$$\therefore \; \frac{\partial \phi}{\partial x} = F_1 \qquad ....(4)$$

Similarly we can find $\frac{\partial \phi}{\partial y} = F_2, \frac{\partial \phi}{\partial z} = F_3$

Therefore, we have $\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$\therefore \; \boxed{\vec{F} = \vec{\nabla}\phi}$

Theorem

Suppose $\vec{F}$ is a conservative field then curl $\vec{F} = 0$ (i.e. $\vec{F}$ is irrotational) and conversely if curl $\vec{F} = 0$ then $\vec{F}$ is conservative.

Proof:

Let $\vec{F}$ is conservative field, then by definition $\vec{F} = \vec{\nabla}\phi$

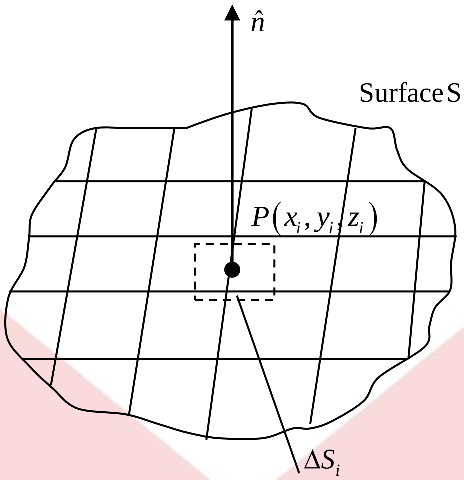
$\therefore$ curl $\vec{F} = \vec{\nabla} \times (\vec{\nabla}\phi) = 0$

i.e., $\vec{F}$ is conservative $\Rightarrow$ curl $\vec{F} = 0$.

Surface Integral

The surface integral of a function ϕ over a surface S (which need not be closed surface or phase surface). May be defined as:

Divide the surface S into m small elements $\Delta S_1, \Delta S_2, ..., \Delta S_n$ and form the expression $\phi_1 \Delta S_1 + \phi_2 \Delta S_2 + ... + \phi_m \Delta S_m$, where ϕ_i is the value of the function ϕ at point P_i. Now if $m \to \infty$ we land up with the surface integral $\int_S \phi \cdot ds$ or for vector valued functions $\int_S \vec{F} \cdot d\vec{s}$.



Let if $\hat{n}$ is the unit normal vector drawn outward to the surface S then we can define the vector elementary surface area by

$$d\vec{S} = |d\vec{S}| \cdot \hat{n}$$

$$\Rightarrow \boxed{d\vec{S} = \hat{n} \cdot ds}$$

$\downarrow$

Elementary surface area

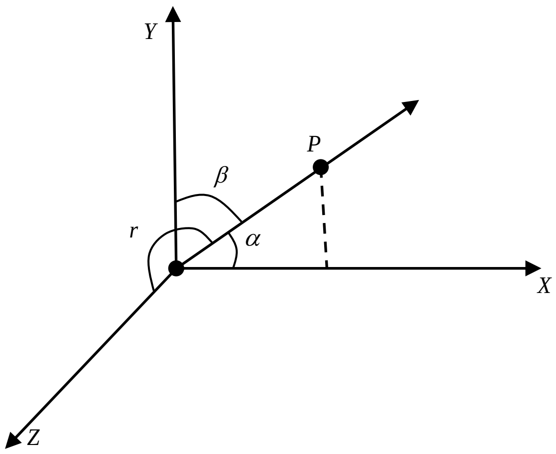
Note:

$$\therefore \int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \hat{n}\, dS$$

Here $\vec{F} \cdot \hat{n}$ is the component of $\vec{F}$ along $\hat{n}$ i.e., normal to the surface S and $\int_S \vec{F} \cdot \hat{n}\, dS$ is called the total Flux across the surface S.

How to calculate $\int_S \vec{F} \cdot d\vec{s}$?

We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = r\cos\alpha\hat{i} + r\cos\beta\hat{j} + r\cos r\hat{k}$



The same concept we apply for the elementary surface area.
If $ds\cos\alpha, ds\cos\beta, ds\cos r$ are the orthogonal projections of the elementary area ds on the YZ-plane, ZX-plane and XY-plane respectively.

Therefore, now we can write

$$d\vec{S} = ds\cos\alpha\hat{i} + ds\cos\beta\hat{j} + ds\cos r\hat{k}$$

Here α, β, r are direction angles of ds with x-axis, y-axis and z-axis respectively.

$$\hat{n}\cdot ds = ds\cos\alpha\hat{i} + ds\cos\beta\hat{j} + ds\cos r\hat{k}$$

$$\hat{i}\cdot\hat{n}ds = ds\cos\alpha$$

$$\hat{i}\cdot\hat{n}\,ds = dy\cdot dz \Rightarrow ds = \frac{dy\cdot dz}{\hat{i}\cdot\hat{n}} \qquad ....(1)$$

$$\hat{j}\cdot\hat{n}\,ds = ds\cos\beta \Rightarrow ds = \frac{dz\cdot dx}{\hat{j}\cdot\hat{n}} \qquad ....(2)$$

$$\hat{k}\cdot\hat{n}\,ds = ds\cos r \Rightarrow ds = \frac{dx\cdot dy}{\hat{k}\cdot\hat{n}} \qquad ....(3)$$

$$\int_S \vec{F}\cdot d\vec{s} = \int_S \vec{F}\cdot\hat{n}\,ds = \int_x\int_y \vec{F}\cdot\hat{n}\frac{dx\,dy}{\hat{k}\cdot\hat{n}} = \int_y\int_z \frac{\vec{F}\cdot\hat{n}\,dy\,dz}{\hat{i}\cdot\hat{n}} = \int_z\int_x \frac{\vec{F}\cdot\hat{n}\,dt\,dx}{\hat{j}\cdot\hat{n}} \qquad ....(4)$$

Whichever form in (4) suits you to easily integrate (According to given condition); Apply that

<u>Green's Theorem</u>

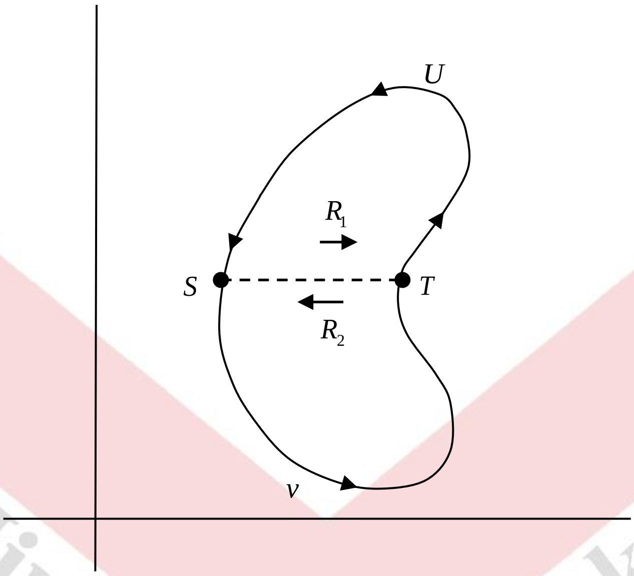
Let's consider a closed region **R** in the xy-plane bounded by a simple closed curve c and suppose $P(x,y), Q(x,y)$ are continuous function with continuous derivatives in the region R.

Then

$$\boxed{\int_c P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy}$$

Note:
Unless otherwise stated, we will always consider that the line integral is described in the positive sense (i.e. the curve c is transverse in the counterclockwise direction.

Note:
We can extend the proof of Green's Theorem in the plane to the curve c for which lines parallel to the coordinate axis may cut the curve c in more than 2 points.



$$\therefore \int_{STUS} Mdx + Ndy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad ....(1)$$

$$\int_{SVTS} Mdx + Ndy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad ....(2)$$

On adding (1) and (2), we get

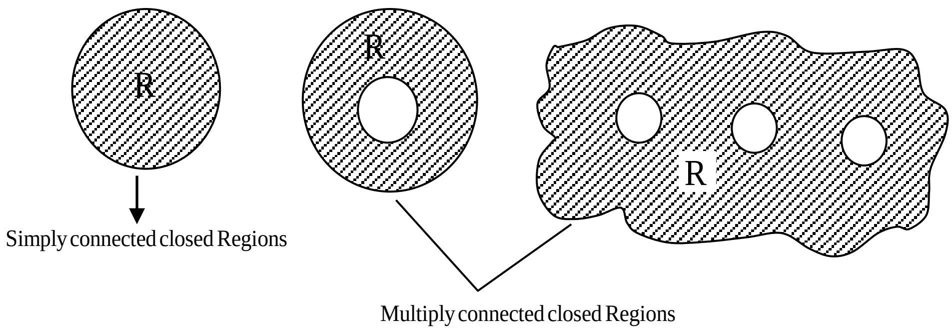
L.H.S.

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

R.H.S.

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

Note:
From the above description we just try to show that the Green's Theorem in the plane is applicable for simply connected closed Regions.



Note:
The Green's Theorem in the plane is also applicable for the multiply connected Regions. It can also be shown by the similar process as above.

Q. Express the Green's Theorem in the plane in vector notation.
Solution.
Let's consider a vector field function
$\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ and the position vector in the plane as $\vec{r} = x\hat{i} + y\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

Now, curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \hat{i}\left(-\frac{\partial N}{\partial z}\right) + \hat{j}\left(\frac{\partial M}{\partial y}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$\therefore$ curl $\vec{F}\cdot\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Now the Green's Theorem in the plane can be written as $\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\cdot\hat{k}\cdot dr}$

- A Generalization of this phenomena to the surface S in the space having a curve c as a boundary leads quite naturally to Stoke's Theorem.

Q. Show that a necessary and sufficient condition for $F_1dx + F_2dy + F_3dz$ to be an exact differential is that $\boxed{\vec{\nabla}\times\vec{F} = 0}$, where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

Q. Show that $\left(y^2z^3\cos x - 4x^3z\right)dx + 2z^3y\sin x\,dy + \left(3y^2z^2\sin x - x^4\right)dt$ is an exact differential of a function ϕ and find such ϕ.
Solution.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3\cos x - 4x^3z & 2z^3y\sin x & 3y^2z^2\sin x - x^4 \end{vmatrix}$$

Proof:
Let $F_1dx + F_2dy + F_3dz$ is an exact differential

i.e., $F_1dx + F_2dy + F_3dz = d(\phi) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

i.e., $F_1 = \frac{\partial\phi}{\partial x}, F_2 = \frac{\partial\phi}{\partial y}, F_3 = \frac{\partial\phi}{\partial z}$

i.e., $\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \vec{\nabla}\phi$

$\therefore \quad \vec{\nabla}\times\vec{F} = \vec{\nabla}\times\left(\vec{\nabla}\phi\right) = 0$

Let $\vec{\nabla}\times\vec{F} = 0$

Then $\vec{F}$ must be of the form $\vec{\nabla}\phi$

i.e., $\vec{F}\cdot d\vec{r} = \vec{\nabla}\phi\cdot d\vec{r}$

$\Rightarrow F_1 dx + F_2 dy + F_3 dz = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$\Rightarrow F_1 dx + F_2 dy + F_3 dz = d\phi$

$\therefore \quad F_1 dx + F_2 dy + F_3 dz = d\phi$ is an exact differential equation.

Solution.

$\left(y^2 z^3 \cos x - 4x^3 z\right)dx + 2z^3 y \sin x\, dy + \left(3y^2 z^2 \sin x - x^4\right)dz$

$= F_1 dx + F_2 dy + F_3 dz$

i.e., $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\because \quad \vec{\nabla}\times\vec{F} = 0$ (on calculating)

$\Rightarrow \vec{F}$ is an exact differential i.e., $\exists$ a function ϕ s.t.

$F_1 dx + F_2 dy + F_3 dz = d\phi$

i.e., $\frac{\partial \phi}{\partial x} = F_1, \frac{\partial \phi}{\partial y} = F_2, \frac{\partial \phi}{\partial z} = F_3$

$\Rightarrow \phi_1 = y^2 z^3 \sin x - x^3 z$

$\phi_2 = z^3 y^2 \sin x$

$\phi_3 = y^2 z^3 \sin x - x^4 z$

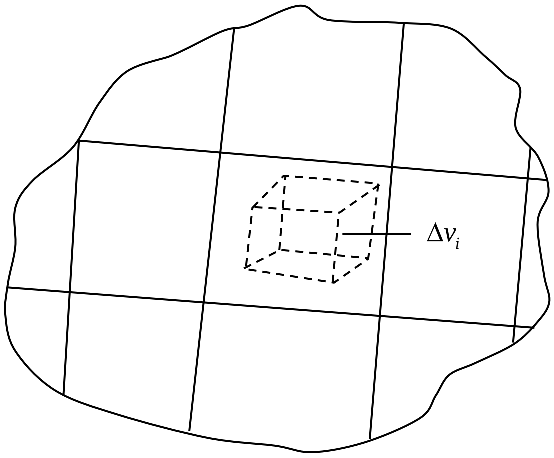
$\therefore \quad \boxed{\phi(x, y, z) = y^2 z^3 \sin x - x^4 z}$

Result

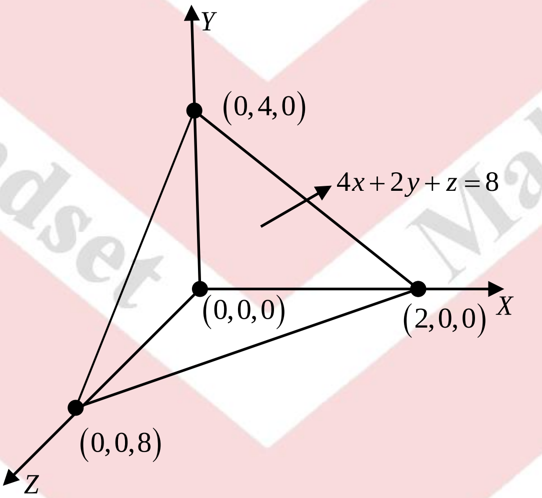
Consider a closed curve c in a simply connected region then $\oint_c M dx + N dy = 0$

iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ everywhere in the region.

Volume Integral



Q. Let's consider some scalar valued function $\phi(x, y, z) = 45x^2 y$ and let v be the closed region bounded by the planes $4x + 2y + z = 8$ and $x = 0, y = 0, z = 0$. Then evaluate the volume integral $\iiint_v \phi dv$



$$v = \int_{x=0}^{2} \int_{y=0}^{\frac{8-4x}{2}} \int_{z=0}^{8-(4x+2y)} 45x^2 y \, dxdydt$$

$$= \int_{x=0}^{2} \int_{y=0}^{4} 45x^2 y [z]_0^{8-(4x+2y)} dxdy$$

$$= \int_{x=0}^{4} \int_{y=0}^{4-2x} 45x^2 y (8 - 4x - 2y) dxdy$$

$$= \int_{x=0}^{4} \int_{y=0}^{4-2x} \left(360x^2 y - 180x^3 y - 90x^2 y^2\right) dxdy$$

$$= \int_{x=0}^{4} \left[\frac{360x^2 y^2}{2} - \frac{180x^3 y^2}{2} - \frac{90x^2 y^3}{3} \right]_0^{4-2x} dx$$

$$= \int_{x=0}^{4} \left[180x^2 (4-2x)^2 - 90x^3 (4-2x)^2 - 30x^2 (4-2x)^3 \right] dx$$

$$= \int_{x=0}^{4} \left[180x^2\left(16+4x^2-16x\right)-90x^3\left(16+4x^2-16x\right)-30x^2\left(64-96x+48x^2-8x^3\right)\right]dx$$

Gauss Divergence Theorem

Let v is the volume bounded by the closed surface S and $\vec{F}$ is a vector valued function of position with continuous derivative then

$$\boxed{\iiint_v \text{div}\,\vec{F} = \iint_S \vec{F}\cdot\hat{n}\,dS}$$

- Applicable only for closed surface.

e.g. If the surface S is $x^2+y^2=4$, $z=5$ we cannot apply Gauss's Divergence Theorem here.

But if the surface S is $x^2+y^2=4$, $z=5$ to $z=8$; yes we can apply Gauss Divergence Theorem.

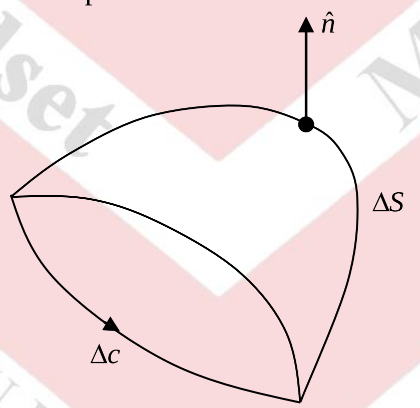
Stoke's Theorem

Alternative definition of curl:

Suppose ΔS is a surface element at a point P, the boundary of the element being the closed curve Δc and $\hat{n}$ is the unit normal vector at the point P drawn outward to the surface. Then we define a limit

$$\left(\text{curl}\,\vec{F}\right)_n = \lim_{\Delta S\to 0}\frac{\int_{\Delta c}\vec{F}\cdot d\vec{r}}{\Delta S}$$

If this limit exists independent of the shape of the curve.



Here $\left(\text{curl}\,\vec{F}\right)_n$ is the component of a certain vector curl $\vec{F}$ along the normal $\hat{n}$ to the surface.

Statement

The line integral of a vector field $\vec{F}$ around any closed curve is equal to $\iint_S \text{curl}\,\vec{F}$ (i.e., the surface integral of curl $\vec{F}$ taken over any surface of which the curve is a boundary edge.

Mathematically if $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve c, then

$$\boxed{\int_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS}$$

Here $\hat{n}$ is the unit normal vector outward to the surface S.

Gradient, Divergence and Curl

Examples: GRADIENT

1. If $\hat{A} = x^2 yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k},\ \vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, then value of $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)$ at $(1,0,-2)$ is equal to ?

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2 yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix} = \left(2x^3z^3 - xyz^2\right)\hat{i} + \left(2xz^3 + x^4yz\right)\hat{j} + \left(x^2y^2z + 4xz^4\right)\hat{k}$$

$$\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right) = -xz^2\hat{i} + x^4z\hat{j} + 2x^2yz\hat{k}, \qquad \frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right) = -z^2\hat{i} + 4x^3z\hat{j} + 4xyz\hat{k}$$

So, at $(1,0,-2)$, $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right) = -4\hat{i} - 8\hat{j}$

2. If $f(x,y,z) = 3x^2y - y^3z^2$, then grad *f* and the point $(1,-2,-1)$ is equal to ?

$$f = 3x^2y - y^3z^2; \quad \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 6xy\hat{i} + \left(3x^2 - 3y^2z^2\right)\hat{j} - 2y^3z\hat{k}$$

At $(1,-2,-1)$, $\nabla f = -12\hat{i} - 9\hat{j} - 16\hat{k}$

3. The gradient of $f(r)$, is equal to?

$$\nabla f(r) = \sum \hat{i}\frac{\partial}{\partial x}\left(f(r)\right) = \sum \hat{i} f'(r)\frac{\partial r}{\partial x} = \sum \hat{i} f'(r)\frac{x}{r} = \frac{f'(r)}{r}\sum \hat{i}x = \frac{f'(r)}{r}\vec{r}$$

Here summation is just representing next terms in symmetry *with* $\hat{j}$ *and* $\hat{k}$

4. $\nabla f(r)\times\vec{r}$ is equal to?

$\nabla f(r) = \frac{f'(r)}{r}\vec{r}$ [as solved in previous question]

$\nabla f(r)\times\vec{r} = 0$

5. $\nabla\left(\frac{1}{r}\right)$ is equal to?

$$\nabla\left(\frac{1}{r}\right) = \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} = \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{x}{r} = -\frac{1}{r^3}\sum \hat{i}x = -\frac{\vec{r}}{r^3}$$

6. $\nabla \log r$ is equal to?

$$\nabla \log r = \sum \hat{i} \frac{\partial}{\partial x} \log r = \sum \hat{i} \frac{1}{r} \cdot \frac{\partial r}{\partial x} = \frac{1}{r^2} \sum \hat{i}x = \frac{\vec{r}}{r^2}$$

7. ∇r^n is equal to ?

$$\nabla r^n = \sum \hat{i} \frac{\partial}{\partial x} r^n = \sum \hat{i} n r^{n-1} \frac{\partial r}{\partial x} = n r^{n-2} \sum \hat{i}x = n r^{n-2} \vec{r}$$

8. If $\vec{a}$ is constant vector & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then grad $(\vec{r} \cdot \vec{a})$ is equal to ?

$$\nabla(\vec{r} \cdot \vec{a}) = \sum \hat{i} \frac{\partial}{\partial x}(\vec{r} \cdot \vec{a}) = \sum \hat{i}\left(\frac{\partial \vec{r}}{\partial x} \cdot \vec{a}\right) = \sum \hat{i}\left(\hat{i} \cdot \vec{a}\right) = \vec{a}$$

9. Let $\vec{a}$ & $\vec{b}$ are constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ grad$\left[\vec{r}\,\vec{a}\,\vec{b}\right]$ is equal to ?

$$\text{grad}\left[\vec{r}\,\vec{a}\,\vec{b}\right] = \sum \hat{i} \frac{\partial}{\partial x}\left(\vec{r} \cdot \left(\vec{a} \times \vec{b}\right)\right) = \sum \hat{i}\left(\frac{\partial \vec{r}}{\partial x} \cdot \left(\vec{a} \times \vec{b}\right)\right) = \sum \hat{i}\left(\hat{i} \cdot \left(\vec{a} \times \vec{b}\right)\right) = \vec{a} \times \vec{b}$$

10. If $\vec{a}$ is a constant vector, ϕ is scalar field $(\vec{a} \cdot \nabla)\phi$ is equal to?

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a} \cdot \nabla = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$$

$$(\vec{a} \cdot \nabla)\phi = a_1 \frac{\partial \phi}{\partial x} + a_2 \frac{\partial \phi}{\partial y} + a_3 \frac{\partial \phi}{\partial z}$$

11. If $\vec{a}$ is constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} (\vec{a} \cdot \nabla)\vec{r}$ is equal to ?

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\quad \vec{a} \cdot \nabla = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$

$$(\vec{a} \cdot \nabla)\vec{r} = \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right)\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

12. The unit normal vector to the level surface $x^2 + y^2 - z = 4$ at point $(1,1,-2)$ is?

Normal vector lies in direction of ∇f . So, $\hat{n} = \frac{\nabla f}{|\nabla f|}$

$$f = x^2 + y^2 - z\,; \quad \nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

At $(1,1,-2)$; $\nabla f = 2\hat{i}+2\hat{j}-\hat{k}$, $|\nabla f| = \sqrt{9} = 3$

So, $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2\hat{i}+2\hat{j}-\hat{k}}{3} = \dfrac{2}{3}\hat{i}+\dfrac{2}{3}\hat{j}-\dfrac{1}{3}\hat{k}$

13. The directional derivative of $f(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of vector $2\hat{i}-\hat{j}-2\hat{k}$ is?

$\nabla f = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$

At $(1,-2,-1)$, $\nabla f = 8\hat{i}-\hat{j}-10\hat{k}$

So, directional derivative of *f* in direction of $2\hat{i}-\hat{j}-2\hat{k}$ is equal to

$\nabla f \cdot \hat{a} = \dfrac{1}{3}(8\hat{i}-\hat{j}-10\hat{k})\cdot(2\hat{i}-\hat{j}-2\hat{k}) = \dfrac{37}{3}$

14. The point P closest to origin on the plane $2x+y-z-5=0$ is ?

Closest point will be foot of perpendicular from origin

$S = 2x+y-z-5=0$; $\hat{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$

Coordinate of $P = \left(\dfrac{2}{\sqrt{6}}r, \dfrac{1}{\sqrt{6}}r, \dfrac{-1}{\sqrt{6}}r\right)$. It lies on S So, $r = \dfrac{5}{\sqrt{6}}$

Hence, $P = \left(\dfrac{5}{3}, \dfrac{5}{6}, -\dfrac{5}{6}\right)$

15. The temperature T at a surface is given by $T = x^2+y^2-z$. In which direction a mosquito at the point $(4,4,2)$ on the surface will fly so that it cools fastest?

$T = x^2+y^2-z$

Direction of fastest cooling will lie in direction opposite to the direction of gradient i.e. $-\nabla T$

$\nabla T = 2x\hat{i}+2y\hat{j}-\hat{k}$

$= 8\hat{i}+8\hat{j}-\hat{k}$

16. The scalar function *f* which corresponds to $\vec{V} = \nabla f$; where $\vec{V} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ is ?

$$f=\sqrt{x^2+y^2+z^2}+c$$

$$\nabla f=\frac{\vec{r}}{r}$$

17. One of the point at which the derivative of the function $f(x,y)=x^2-xy-y+y^2$ vanishes along the direction $\frac{\hat{i}+\sqrt{3}\hat{j}}{2}$ is ?

$$\nabla f=(2x-y)\hat{i}-(x+1-2y)\hat{j}$$

Directional derivative in direction given by $\frac{\hat{i}+\sqrt{3}\hat{j}}{2}$

$$=\frac{1}{2}(2x-y)-\frac{\sqrt{3}}{2}(x+1-2y)=\frac{2-\sqrt{3}}{2}x-\frac{\left(1-2\sqrt{3}\right)}{2}y-\frac{\sqrt{3}}{2}$$

It becomes zero at $\left(-1,\frac{2}{2\sqrt{3}-1}\right)$

18. Which of the following is a unit normal vector to the surface $z=xy$ at $P(2,-1,-1)$?

The surface is $f=xy-z=0$

$$\nabla f=y\hat{i}+x\hat{j}-\hat{k}=-\hat{i}+x\hat{j}-\hat{k}$$

$$\hat{n}=\frac{\nabla f}{|\nabla f|}=\frac{\hat{i}-2\hat{j}+\hat{k}}{\sqrt{6}}$$

19. Let $f(x,y)=\ln\sqrt{x+y}$ and $g(x,y)=\sqrt{x+y}$. Then the value of $\nabla^2(fg)$ at $(1,0)$?

$$f=\ln(x+y)^{1/2},g=\sqrt{x+y}$$

$$fg=\sqrt{x+y}\ln\sqrt{x+y}$$

$$\nabla^2 fg=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)fg$$

$$\frac{\partial}{\partial x}(fg)=\frac{1}{2\sqrt{x+y}}\ln\sqrt{x+y}+\sqrt{x+y}\frac{1}{\sqrt{x+y}}\cdot\frac{1}{2\sqrt{x+y}}$$

$$\frac{\partial^2 fg}{\partial x^2} = -\frac{1}{4}(x+y)^{-3/2}\ln(x+y) + \frac{1}{2(x+y)}\cdot\frac{1}{2\sqrt{x+y}} - \frac{1}{4}(x+y)^{-3/2}$$

$$\nabla^2 fg = 0$$

20. The spheres $x^2+y^2+z^2=1$ and $x^2+\left(y-\sqrt{3}\right)^2+z^2=4$ intersect at an angle?

$$x^2+y^2+z^2=1$$

$$x^2+y^2+z^2-2\sqrt{3}y=1$$

They intersect at plane $y=0$

$(0,0,1)$ is one point of intersection which is lying on the both sphere.

Let us find normal vector at this point and find angle between them

$$\hat{n}_i = x\hat{i}+y\hat{j}+z\hat{k}\,,\quad \hat{n}_2 = \frac{x\hat{i}+\left(y-\sqrt{3}\right)\hat{j}+z\hat{k}}{2}$$

$\cos\theta = \hat{n}_1\cdot\hat{n}_2 = \frac{1}{2}$ at point $(0,0,1)$; $\quad\theta=\pi/3$

21. For what values of *a* and *b*, the directional derivative of $u(x,y,z)=ax^2yz+bxy^2z$ at $(1,1,1)$ along $\hat{i}+\hat{j}-2\hat{k}$ is $\sqrt{6}$ and along $\hat{i}-\hat{j}+2\hat{k}$ is $3\sqrt{6}$?

$$\nabla u = \left(2axyz+by^2z\right)\hat{i}+\left(ax^2z+2bxyz\right)\hat{j}+\left(ax^2y+bxy^2\right)\hat{k}$$

The directional derivative of $u(x,y,z)$ along $\left(\hat{i}+\hat{j}-2\hat{k}\right)$ at $(1,1,1)$

$$(2a+b)\hat{i}+(a+2b)\hat{j}+(a+b)\hat{k}\cdot\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}(2a+b+a+2b-2a-2b) = \frac{a+b}{\sqrt{6}} = \sqrt{6} \text{ (Given)}$$

So, $a+b=6$

The directional derivative of $u(x,y,z)$ along $\left(\hat{i}-\hat{j}+2\hat{k}\right)$ at $(1,1,1)$

$$=\left((2a+b)\hat{i}+(a+2b)\hat{j}+(a+b)\hat{k}\right)\cdot\frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}(3a+b) = 3\sqrt{6} \text{ (Given)}$$

$3a+b=18$ ....(2)

Solving (1) & (2); $a=6, b=0$

22. Find the directional derivative of $f = x^2yz^3$ along $x = e^{-t}, y = 1+2\sin t, z = t-\cos t$ at $t=0$.

Solution. $\nabla f = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$

For $t=0, x=e^{-t}=1$

$y = 1+2\sin t = 1$

$z = t-\cos t = -1$

So, at $(1,1,-1)$ $\nabla f = -2\hat{i} - \hat{j} + 3\hat{k}$

The curve is described by vector

$\vec{r} = e^{-t}\hat{i} + (1+2\sin t)\hat{j} + (t-\cos t)\hat{k}$

$\vec{t} = \dfrac{d\vec{r}}{dt} = -e^{-t}\hat{i} + 2\cos t\vec{j} + (1+\sin t)\hat{k}$; the tangent vector (formula we read in chapter Curvature & Torsion)

At $t=0$ $\vec{t} = -\hat{i} + 2\hat{j} + \hat{k}$

Unit vector along tangent, ; $\hat{t} = \dfrac{-\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}}$

Directional derivative along the curve at $t=0 = \nabla f \cdot \hat{t} = (-2\hat{i} - \hat{j} + 3\hat{k}) \cdot \dfrac{(-\hat{i}+2\hat{j}+\hat{k})}{\sqrt{6}} = \dfrac{3}{\sqrt{6}} = \sqrt{\dfrac{3}{2}}$

23. If $\vec{r}_1$ and $\vec{r}_2$ are the vector joining the fixed point. $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ respectively to a variable point $P(x, y, z)$ then find the values of grad$(\vec{r}_1 \cdot \vec{r}_2)$ & $(\vec{r}_1 \times \vec{r}_2)$.

Solution. The vector $\vec{AP} = \vec{r}_1$ = position vector of P-position vector of A

$= (x\hat{i} + y\hat{j} + 2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = (x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}$

Similarly, $\vec{BP} = \vec{r}_1 = (x-x_2)\hat{i} + (y-y_2)\hat{j} + (z-z_2)\hat{k}$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}$$

$$=\left[(y-y_1)(z-z_2)-(y-y_2)(z-z_1)\right]\hat{i}+\left[(x-x_2)(z-z_1)-(x-x_1)(z-z_2)\right]\hat{j}$$
$$+\left[(y-y_1)(z-z_2)-(y-y_2)(z-z_1)\right]\hat{k}$$

$$=\left[y(z_1-z_2)+z(y_2-y_1)+(y_1z_2-y_2z_1)\right]\hat{i}+\left[z(x_1-x_2)+x(z_2-z_1)+(z_1x_2-z_2x_1)\right]\hat{j}$$

$$+\left[x(y_1-y_2)+y(x_2-x_1)+(x_1y_2-x_2y_1)\right]\hat{k}$$

$$\vec{r}_1\cdot\vec{r}_2=(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$$

$$\nabla(\vec{r}_1\cdot\vec{r}_2)=\sum\hat{i}\frac{\partial}{\partial x}(\vec{r}_1\cdot\vec{r}_2)=\sum\hat{i}(2x-x_1-x_2)=\sum\hat{i}\left[(x-x_1)+(x-x_2)\right]$$
$$=\sum\hat{i}(x-x_1)+\sum\hat{i}(x-x_2)=\vec{r}_1+\vec{r}_2$$

24. Find the equation of tangent plane and normal to the surface $2xz^2-3xy+4x=1$ at the point $(1,1,2)$

Solution. $\nabla f=\sum\hat{i}\frac{\partial}{\partial x}f=(2z^2-3y+4)\hat{i}-3x\hat{j}+4xz\hat{k}$; At $(1,1,2)$ $\nabla f=9\hat{i}-3\hat{j}+8\hat{k}$

Let $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$ is a position vector of any arbitrary point (x,y,z) on the tangent plane at point P.

The position vector of point P is $\vec{r}=\hat{i}+\hat{j}+2\hat{k}$

Equation of tangent plane at point P is

$$(\vec{R}-\vec{r}).\text{grad } f=0\Rightarrow(x-1)\frac{\partial f}{\partial x}+(y-1)\frac{\partial f}{\partial y}+(z-2)\frac{\partial f}{\partial z}=0$$

$$\Rightarrow 9(x-1)-3(y-1)+8(z-2)=0$$

$$9x-3y+8z=22$$

Equation of normal to the surface at point $(1,1,2)$ is

$$\frac{x-1}{\frac{\partial f}{\partial x}}=\frac{y-1}{\frac{\partial f}{\partial y}}=\frac{z-2}{\frac{\partial f}{\partial z}};\qquad \text{So, }\frac{x-1}{9}=\frac{y-1}{-3}=\frac{z-2}{8}$$

25. Find the equation of the tangent plane and normal to the surface $xyz=2$ at the point $(1,2,1)$.

Solution. At point $(1,2,1)$ $\nabla f=2\hat{i}+\hat{j}+2\hat{k}$

Let $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$ be the position vector of an arbitrary point (x,y,z) on the tangent plane.

Position vector of point of contact $(1,2,1)$; $\vec{r}=\hat{i}+2\hat{j}+\hat{k}$

Equation of tangent plane is $\left(\vec{R}-\vec{r}\right)\cdot\nabla f=0 \Rightarrow (x-1)\frac{\partial f}{\partial x}+(y-2)\frac{\partial f}{\partial y}+(z-1)\frac{\partial f}{\partial z}=0$

$\Rightarrow 2(x-1)+(y-2)+2(z-1)=6;\ 2x+y+2z=6$

Equation of normal to the surface at point $(1,2,1)$

$\frac{x-1}{\frac{\partial f}{\partial x}}=\frac{y-2}{\frac{\partial f}{\partial y}}=\frac{z-1}{\frac{\partial f}{\partial z}}$. So, $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{2}$

26. Give the curve $x^2+y^2+z^2=1, x+y+z=1$ (intersection of two surfaces) find the equation of the tangent line at the point $(1,0,0)$.

Solution At the point ; $(1,0,0)$ $\nabla S_1=2\hat{i}$, $\nabla S_2=\hat{i}+\hat{j}+\hat{k}$

The normal vector to surface S_1 & S_2 are given by $\hat{n}_1=\frac{\nabla S_1}{|\nabla S_1|}=\hat{i}$, $\hat{n}_2=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Tangent to the curve of intersection will be perpendicular to both $\hat{n}_1$ & $\hat{n}_2$ i.e. it lies in the direction of $\hat{n}_1\times\hat{n}_2$ i.e. $\hat{i}\times\frac{\left(\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{3}}=-\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$

So, equation of tangent passing through $(1,0,0)$ & parallel to vector $-\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$ is

$\frac{x-1}{0}=\frac{y-0}{-1/\sqrt{3}}=\frac{z-0}{1/\sqrt{3}}$

Assignment-1

1. Find the directional derivative of the function $f=x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of line PQ where Q is the point $(5,0,4)$.

Hint. At $(1,2,3)$, $\nabla f=2\hat{i}-4\hat{j}+12\hat{k}$

Now, vector $\overrightarrow{PQ}$ = position vector Q-position vector of P $=\left(5\hat{i}+4\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=4\hat{i}-2\hat{j}+\hat{k}$

Unit vector in direction of $\overrightarrow{PQ}$, $\hat{a}=\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{16+4+1}}=\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{21}}$

So, directional derivative of *f* in the direction of $\hat{a}=\nabla f\cdot\hat{a}$

$$=\left(2\hat{i}-4\hat{j}+12\hat{k}\right)\cdot\frac{\left(4\hat{i}-2\hat{j}+\hat{k}\right)}{\sqrt{21}}=\frac{28}{\sqrt{21}}=\frac{4}{3}\sqrt{21}$$

2. What is the greatest rate of increase of $u=xyz^2$ at the point $(1,0,3)$?

Solution. At $(1,0,3)$, $\nabla u=9\hat{j}$;So, maximum value of directional derivative

$=\nabla u\cdot\hat{a}$ with $\hat{a}$ being unit vector parallel to ∇u $=|\nabla u|=9$

3. Find the directional derivative of

(i) $4xz^3-3x^2y^2z^2$ at $(2,-1,2)$ along *z* axis.

(ii) $x^2yz+4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$.

Solution.

(i) $f=4xz^3-3x^2y^2z^2$

$\nabla f=\left(4z^3-6xy^2z^2\right)\hat{i}-6x^2yz^2\hat{j}+\left(12xz^2-6x^2y^2z\right)\hat{k}$

At $(2,-1,2)$, $\nabla f=-16\hat{i}+96\hat{j}+48\hat{k}$

Along *z* axis, the directional derivative along *z* axis; $=\nabla f\cdot\hat{k}=48$

(ii) $f=x^2yz+4xz^2;\nabla f=\left(2xyz+4z^2\right)\hat{i}+x^2z\hat{j}+\left(x^2y+8zx\right)\hat{k}$; At $(1,-2,1)$, $\nabla f=\hat{j}+6\hat{k}$

Unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$; $\hat{a}=\frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{9}}=\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}$

Directional derivative in direction of $2\hat{i}-\hat{j}-2\hat{k}$; $=\nabla f\cdot\hat{a}=\left(j+6\hat{k}\right)\cdot\left(\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}\right)=-\frac{13}{3}$

4. Find the directional derivative of $f(x,y)=x^2y^3-xy$ at the point $(2,1)$ in the directional of a unit vector which makes an angle of $\pi/3$ with *x* axis

Solution. At $(2,1)$, $\nabla f=5\hat{i}+14\hat{j}$

Unit vector making an angle of $\pi/3$ with *x* axis; $\hat{a}=\cos\frac{\pi}{3}\hat{i}+\sin\frac{\pi}{3}\hat{j}=\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{3}\hat{j}$

So, directional derivative of *f* in the direction of unit vector making angle of $\frac{\pi}{3}$ with the *x* axis

$$= \nabla f \cdot \hat{a} = \left(5\hat{i} + 14\hat{j}\right)\cdot\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = \frac{5+14\sqrt{3}}{2}$$

5. Find the constants *a* and *b* so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$.

Solution. Two surface S_1 & S_2 are orthogonal

So, $\hat{n}_1 \cdot \hat{n}_2 = 0$; $\dfrac{\nabla S_1}{|\nabla S_1|} \cdot \dfrac{\nabla S_2}{|\nabla S_2|} = 0$

$$\nabla S_1 \cdot \nabla S_2 = 0 \Rightarrow \left((a-2)\hat{i} - 2b\hat{j} + b\hat{k}\right)\cdot\left(-8\hat{i} + 4\hat{j} + 12\hat{k}\right) = 0$$

$$\Rightarrow -8(a-2) - 8b + 12b = 0 \Rightarrow -8a + 4b = -16$$

Point $(1,-1,2)$ lies on S_1, So, $a + 2b = a + 2 \Rightarrow b = 1$ So, $a = \dfrac{5}{2}$

Gradient, Divergence and Curl

EXAMPLES

1. $div\,\vec{r}$ equal to ?

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \vec{r} = \sum \hat{i} \cdot \frac{\partial}{\partial x}\vec{r} = \sum \hat{i}\cdot\hat{i} = \sum 1 = 3$$

2. curl $\vec{r}$ is equal to?

$$\text{curl } \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

3. The value of constant *a* for which the vector $\vec{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal is?

Vector $\vec{f}$ is solenoidal if div $\vec{f} = 0$

$$\text{div } \vec{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0 \Rightarrow 1+1+a = 0;\quad a = -2$$

4. If $\vec{a}$ is a constant vector, then $\nabla\cdot(\vec{r}\times\vec{a})$ is equal to ?

div $\left(\vec{r}\times\vec{a}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\vec{r}\times\vec{a}\right)=\sum\hat{i}\cdot\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)=\sum\hat{i}\cdot\left(\hat{i}\times a\right)=0$

5. If $\vec{a}$ is a constant vector, curl $\left(\vec{r}\times\vec{a}\right)$ is equal to?

curl $\left(\vec{r}\times\vec{a}\right)=\sum\hat{i}\times\frac{\partial}{\partial x}\left(\vec{r}\times\vec{a}\right)=\sum\hat{i}\times\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)=\sum\hat{i}\times\left(\hat{i}\times a\right)=\sum\left[\left(\hat{i}\cdot\vec{a}\right)\hat{i}-\left(\hat{i}\cdot\hat{i}\right)\vec{a}\right]$

$=\sum\left(\hat{i}\cdot\vec{a}\right)\hat{i}-\sum\vec{a}=\vec{a}-3\vec{a}=-2\vec{a}$

6. If $\vec{f}=e^{xyz}\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ the curl $\vec{f}$ at $(1,1,1)$ equal to ?

$$\nabla\times\vec{f}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^{xyz} & 2e^{xyz} & 3e^{xyz}\end{vmatrix}=e^{xyz}\left(3xz-2xy\right)\hat{i}+e^{xyz}\left(xy-yz\right)\hat{j}+e^{xyz}\left(2yz-xz\right)\hat{k}$$

At $(1,1,1)$,; $\nabla\times\vec{f}=e\left(\hat{i}\times\hat{k}\right)$

7. If $\vec{f}=xy^2\hat{i}+2x^2yz\hat{j}-3yz^2\hat{k}$, then value of div $\vec{f}$ at $(1,1,1)$ is equal to ?

div $\vec{f}=\frac{\partial}{\partial x}\left(xy^2\right)+\frac{\partial}{\partial y}\left(2x^2yz\right)-\frac{\partial}{\partial z}\left(3yz^2\right)=y^2+2x^2z-6yz$; At $(1,1,1)$, div $\vec{f}=-3$

8. If $\vec{f}=\left(x^2-y^2\right)\hat{i}+2xy\hat{j}+\left(y^2-xy\right)\hat{k}$, the curl $\vec{f}$ at $(1,1,1)$ is equal to?

$$\text{curl }\vec{f}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+1 & 1 & -x-y\end{vmatrix}=-\hat{i}+\hat{j}-\hat{k}$$

$$\vec{f}\cdot curl\ \vec{f}=\left((x+y+1)\hat{i}+\hat{j}+(-x-y)\hat{k}\right)\cdot\left(-\hat{i}+\hat{j}-\hat{k}\right)=-x-y-1+1+x+y=0$$

Assignment-2

1. Prove that div $\left(r^n\vec{r}\right)=\left(n+3\right)r^n$

Solution. div $=\left(r^n\vec{r}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(r^n\vec{r}\right)=\sum\hat{i}\cdot\left[nr^{n-1}\frac{\partial r}{\partial x}\vec{r}+r^n\frac{\partial\vec{r}}{\partial x}\right]$

$$= \sum \left[nr^{n-1}\frac{x}{r}\left(\hat{i}\cdot\hat{r}\right) + r^n\hat{i}\cdot\hat{i} \right]\left(\frac{\partial r}{\partial x} = \frac{x}{r}.\frac{\partial \vec{r}}{\partial x} = \hat{i}\right) = nr^{n-2} - \Sigma x^2 + r^n\Sigma 1 = nr^n + 3r^n = (n+3)r^n$$

2. Prove that $\nabla^2\left(r^n\vec{r}\right) = n(n+3)r^{n-2}\vec{r}$

Solution. $\nabla^2\left(r^n\vec{r}\right) = \nabla\left(\nabla\cdot\left(r^n\vec{r}\right)\right)$ (from previous example); $\nabla\cdot\left(r^n\vec{r}\right) = (n+3)r^n$

$$\nabla^2\left(r^n\vec{r}\right) = \nabla(n+3)r^n = (n+3)\sum \hat{i}\,\frac{\partial\left(r^n\right)}{\partial x} = (n+3)\sum nr^{n-1}\hat{i}\,\frac{\partial r}{\partial x} = n(n+3)r^{n-2}\Sigma x\hat{i}$$

$$= n(n+3)r^{n-2}\vec{r}$$

3. Prove that div $\left(\frac{\vec{r}}{r^3}\right) = 0$

Solution. div $\left(\frac{\vec{r}}{r^3}\right) = \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^3}\right) = \sum \hat{i}\cdot\left[\frac{1}{r^3}\frac{\partial \vec{r}}{\partial x} + \vec{r}\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right] = \sum \hat{i}\cdot\left[\frac{1}{r^3}\hat{i} + \vec{r}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right]$

$$= \frac{1}{r^3}\Sigma\hat{i}\cdot\hat{i} - \frac{3}{r^5}\Sigma\left(\hat{i}\cdot\vec{r}\right)x = \frac{1}{r^3}\Sigma 1 - \frac{3}{r^5}\Sigma x^2 = \frac{3}{r^3} - \frac{3}{r^5}\cdot r^2 = 0$$

4. Prove that div $\hat{e}_r = \frac{2}{r}$

Solution. $\nabla\cdot\hat{e}_r = \nabla\cdot\left(\frac{\vec{r}}{r}\right) = \sum \hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r}\right) = \sum \hat{i}\cdot\left(\frac{1}{r}\frac{\partial \vec{r}}{\partial x} + \vec{r}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right)$

$$= \sum \hat{i}\cdot\left(\frac{1}{r}\hat{i} + \vec{r}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right) = \sum\left(\frac{1}{r}\hat{i}\cdot\hat{i} - \frac{1}{r^2}\cdot\frac{x}{r}\left(\hat{i}\cdot\vec{r}\right)\right) = \frac{1}{r}\cdot\Sigma 1 - \frac{1}{r^3}\Sigma x^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

5. Prove that vector $f(r)\vec{r}$ is irrorational.

Solution. A vector function is said to be irrotational if its curl is zero

$$\nabla\times\left(f(r)\vec{r}\right) = \sum \hat{i}\times\frac{\partial}{\partial x}\left(f(r)\vec{r}\right) = \sum \hat{i}\times\left(f'(r)\frac{\partial r}{\partial x}\vec{r} + f(r)\frac{\partial \vec{r}}{\partial x}\right) = \sum \hat{i}\times\left(f'(r)\frac{x}{r}\vec{r} + f(r)\hat{i}\right)$$

$$= \frac{f'(r)}{r}\Sigma x\hat{i}\times\hat{r} + \Sigma f(r)\Sigma\hat{i}\times\hat{i} = \frac{f'(r)}{r}\vec{r}\times\vec{r} + f(r)\Sigma\hat{i}\times\hat{i} = \mathbf{0}$$

Since, curl of $f(r)\vec{r}$ is zero, hence $f(r)\vec{r}$ is irrotational.

6. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

Solution. $\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right)$

$$\nabla\left(\frac{1}{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\sum\hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)=\sum\hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right)=-\frac{1}{r^3}\Sigma x\hat{i}=-\frac{\vec{r}}{r^3}$$

$$\nabla\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right)=\nabla\cdot\left(-\frac{\vec{r}}{r^3}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(-\frac{\vec{r}}{r^3}\right)=-\sum\hat{i}\cdot\left(\frac{1}{r^3}\frac{\partial\vec{r}}{\partial x}+\vec{r}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right)$$

$$=-\sum\hat{i}\cdot\left(\frac{1}{r^3}\hat{i}+\vec{r}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right)=-\sum\left(\frac{1}{r^3}\left(\hat{i}\cdot\hat{i}\right)-\frac{3x}{r^5}\cdot\left(\hat{i}\cdot\vec{r}\right)\right)=-\frac{1}{r^3}\Sigma 1+\frac{3}{r^5}\Sigma x^2=-\frac{3}{r^3}+\frac{3}{r^5}\cdot r^2=0$$

7. Prove that div grad $r^n=n(n+1)r^{n-2}$

Solution. grad $r^n=\sum\hat{i}\frac{\partial}{\partial x}r^n=\sum\hat{i}nr^{n-1}\frac{\partial r}{\partial x}=\sum\hat{i}nr^{n-1}\frac{x}{r}=nr^{n-2}\Sigma x\hat{i}=nr^{n-2}\vec{r}$

div grad $r^n=\sum div\left(nr^{n-2}\vec{r}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(nr^{n-2}\vec{r}\right)=n\sum\hat{i}\cdot\left(r^{n-2}\frac{\partial\vec{r}}{\partial x}+\vec{r}\frac{\partial}{\partial x}\left(r^{n-2}\right)\right)$

$$=n\sum\hat{i}\cdot\left(r^{n-2}\hat{i}+\vec{r}\,(n-2)r^{n-3}\frac{\partial r}{\partial x}\right)=nr^{n-2}\Sigma\hat{i}\cdot\hat{i}+n\Sigma\hat{i}\cdot\left((n-2)r^{n-3}\frac{x}{r}\vec{r}\right)$$

$$=3nr^{n-2}+n(n-2)r^{n-4}\Sigma x\left(\hat{i}\cdot\vec{r}\right)=3nr^{n-2}+n(n-2)r^{n-1}\Sigma x^2=3nr^{n-2}+n(n-2)r^{n-2}=\left(n^2+n\right)r^{n-2}$$

8. Prove that $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi$

Solution. $\nabla^2(\phi\psi)=\nabla\cdot\left(\nabla(\phi\psi)\right)=\nabla\cdot\left(\psi\nabla\phi+\phi\nabla\psi\right)=\nabla\cdot\left(\psi\nabla\phi\right)+\nabla\cdot\left(\phi\nabla\psi\right)$

$=\psi\nabla^2\phi+2\nabla\phi\cdot\nabla\psi+\phi\nabla^2\psi$

9. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenaidal

Solution. $\vec{A}\times\vec{B}$ are irrotational . So, $\nabla\times\vec{A}=0$ & $\nabla\times\vec{B}=0$

Now, $\nabla\left(\vec{A}\cdot\vec{B}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)=\sum\left[\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]$

$$=\sum\left[\vec{B}\cdot\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)-\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)\right]=\vec{B}\cdot\sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}-\sum\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)=\vec{B}\cdot curl\,\vec{A}-\vec{A}\cdot\sum\hat{i}\times\frac{\partial\vec{B}}{\partial x}$$

$=\vec{B}\cdot curl\,\vec{A}-\vec{A}\,curl\,\vec{B}=0$

Since $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=0$. Hence, $\vec{A}\times\vec{B}$ is solenoidal.

10. If *f* and *g* are two scalar point function prove that div $\left(f\nabla g\right)= f\nabla^2 g+\nabla f\cdot\nabla g$.

Solution. We can use a vector identity; $\nabla\cdot\left(\phi\vec{f}\right)=\nabla\phi\cdot\vec{f}+\phi\nabla\cdot\vec{f}$ Where ϕ is a scalar function & $\vec{f}$ is a vector function . So, $\nabla\cdot\left(f\nabla g\right)=\nabla f\cdot\nabla g+f\nabla\cdot\left(\nabla g\right)=\nabla f\cdot\nabla g+f\nabla^2 g$

Other way.

$$f\nabla g = f\left(\sum \hat{i}\frac{\partial}{\partial x}g\right) = f\frac{\partial g}{\partial x}\hat{i}+f\frac{\partial g}{\partial y}\hat{j}+f\frac{\partial g}{\partial z}\hat{k}$$

$$\nabla\cdot\left(f\nabla g\right)=\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)\hat{i}+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$=\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}$$

$$=f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial x}\right)$$

$$=f\nabla^2 g+\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}\right)=f\nabla^2 g+\nabla f\cdot\nabla g$$

11. Prove that div $\left(\vec{A}\times\vec{r}\right)=\vec{r}\cdot curl\,\vec{A}$ when $\vec{A}$ is a constant vector

Solution. Using identity $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot curl\,\vec{A}-\vec{A}\cdot curl\,\vec{B}$

$\nabla\cdot\left(\vec{A}\times\vec{r}\right)=\vec{r}\cdot curl\,\vec{A}-\vec{A}\cdot curl\,\vec{r}=\vec{r}\cdot curl\,\vec{A}$ (as curl $\vec{r}=0$)

12. If $\vec{a}$ is a constant vector, prove that div $\left\{r^n\left(\vec{a}\times\vec{r}\right)\right\}=0$

Solution. $\nabla\cdot\left(r^n\left(\vec{a}\times\vec{r}\right)\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(r^n\left(\vec{a}\times\vec{r}\right)\right)=\sum\hat{i}\cdot\left(nr^{n-1}\frac{\partial r}{\partial x}\left(\vec{a}\times\vec{r}\right)+r^n\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)\right)$

$$=nr^{n-1}\left(\Sigma xi\right)\cdot\left(\vec{a}\times\vec{r}\right)+r^n\Sigma\hat{i}\cdot\left(\vec{a}\times\hat{i}\right)=nr^{n-2}\vec{r}\cdot\left(\vec{a}\times\vec{r}\right)+r^n\Sigma\hat{i}\cdot\left(\vec{a}\times\hat{i}\right)=0$$

13. Prove that

Solution. $\nabla\cdot\left(\phi\nabla\psi-\psi\nabla\phi\right)=\phi\nabla^2\psi-\psi\nabla^2\phi=\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi$

$$\nabla\cdot\left(\psi\nabla\phi\right)=\psi\nabla\cdot\left(\nabla\phi\right)+\nabla\psi\cdot\nabla\phi=\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi$$

$$\nabla\cdot\left(\phi\nabla\psi-\psi\nabla\phi\right)=\left(\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi\right)-\left(\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi\right)=\phi\nabla^2\psi-\psi\nabla^2\phi$$

14. If $\vec{a}$ and $\vec{b}$ are constant vectors, prove that

(i) div $\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=-2\vec{b}\cdot\vec{a}$ (ii) curl $\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\vec{b}\times\vec{a}$

Solution. (i) div $\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\nabla\cdot\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\cdot\left[\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)\times\vec{b}\right]$

$$=\sum\hat{i}\cdot\left[(\hat{i}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\cdot\left[(\hat{i}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\hat{i}\right]=\sum\left[(\hat{i}\cdot\vec{b})(\hat{i}\cdot\vec{a})-(\vec{a}\cdot\vec{b})(\hat{i}\cdot\hat{i})\right]$$

$$=\sum a_x b_x-(\vec{a}\cdot\vec{b})\Sigma 1=\vec{a}\cdot\vec{b}-3(\vec{a}\cdot\vec{b})=-2\vec{a}\cdot\vec{b}=-2\vec{b}\cdot\vec{a}$$

(ii) curl $\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\times\frac{\partial}{\partial x}\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\times\left[\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)\times\vec{b}\right]=\sum\hat{i}\times\left[(\hat{i}\times\vec{a})\times\vec{b}\right]$

$$=\sum\hat{i}\times\left[(\hat{i}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\hat{i}\right]=\sum(\hat{i}\cdot\vec{b})(i\times\vec{a})-(\vec{a}\cdot\vec{b})(\hat{i}\times\hat{i})=\sum b_x\left(a_y\hat{k}-a_z\hat{j}\right)$$

$$=\left(b_xa_y\hat{k}-b_xa_z\hat{j}\right)+\left(b_ya_z\hat{i}-b_ya_x\hat{k}\right)+\left(b_za_x\hat{j}-b_za_y\hat{i}\right)$$

$$=\left(b_ya_z-b_za_y\right)\hat{i}+\left(b_za_x-b_xa_z\right)\hat{j}+\left(b_xa_y-b_ya_x\right)\hat{k}=\vec{b}\times\vec{a}$$

15. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$. If a scalar field φ and a vector field $\vec{u}$ satisfy $\nabla\varphi=\nabla\times\vec{u}+f(r)\vec{r}$ where *f* is an arbitrary differentiable function, then show that $\nabla^2\varphi=rf'(r)+3f(r)$.

Solution. $\nabla\phi=\nabla\times\vec{u}+f(r)\vec{r}$;

$$\nabla^2\phi=\nabla\cdot\nabla\phi=\nabla\cdot(\nabla\times\vec{u})+\nabla\cdot\left(f(r)\vec{r}\right)=0+\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(f(r)\vec{r}\right)=\sum\left(\hat{i}\cdot\vec{r}\right)f'(r)\frac{x}{r}+\sum\hat{i}f(r)\cdot\frac{\partial\vec{r}}{\partial x}$$

$$=\frac{f'(r)}{r}\sum x^2+f(r)\sum\hat{i}\cdot\hat{i}=rf'(r)+3f(r)$$

16. If $\vec{r}$ is the position vector of the point (x,y,z) w.r.t. origin. Prove that ; $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$

Find $f(r)$ such that $\nabla^2 f(r)=0$.

Solution. $\nabla^2 f(r)=\nabla\cdot\nabla f(r)$

$$\nabla f(r)=\sum\hat{i}\frac{\partial}{\partial x}f(r)=\sum\hat{i}f'(r)\frac{\partial r}{\partial x}=\sum\hat{i}\,f'(r)\frac{x}{r}=\frac{f'(r)}{r}\vec{r}$$

$$\nabla^2 f(r) = \nabla \cdot \left(\frac{f'(r)}{r} \vec{r} \right) = \sum \hat{i} \cdot \frac{\partial}{\partial x} \left(\frac{1}{r} f'(r) \vec{r} \right)$$

$$= \sum \hat{i} \cdot \left[\frac{\partial}{\partial x} \left(\frac{1}{r} \right) f'(r) \vec{r} + \frac{1}{r} \frac{\partial}{\partial x} \left(f'(r) \right) \vec{r} + \frac{1}{r} f'(r) \frac{\partial \vec{r}}{\partial x} \right]$$

$$= \sum \hat{i} \cdot \left[-\frac{1}{r^2} \cdot \frac{x}{r} f'(r) \vec{r} + \frac{1}{r} f''(r) \frac{\partial r}{\partial x} \vec{r} + \frac{1}{r} f'(r) \hat{i} \right]$$

$$= \sum \left[-\frac{f'(r)}{r^3} \cdot x \left(\hat{i} \cdot \vec{r} \right) + \frac{1}{r} f''(r) \frac{x}{r} \cdot \left(\hat{i} \cdot \vec{r} \right) + \frac{1}{r} f'(r) \hat{i} \cdot \hat{i} \right]$$

$$= -\frac{f'(r)}{r^3} \Sigma x^2 + \frac{f''(r)}{r^2} \Sigma x^2 + \frac{1}{r} f'(r) \Sigma 1 = -\frac{f'(r)}{r} + f''(r) + \frac{3}{r} f'(r) = f''(r) + \frac{2}{r} f'(r)$$

Now, let us find $f(r)$ such that $\nabla^2 f(r) = 0$

Let $g(r) = f'(r)$

Now, $\nabla^2 f(r) = 0 \Rightarrow f''(r) + \frac{2}{r} f'(r) = 0$

$$\Rightarrow g'(r) + \frac{2}{r} g(r) = 0 \Rightarrow \frac{dg}{dr} + \frac{2}{r} g = 0 \Rightarrow \frac{dg}{g} + 2 \frac{dr}{r} = 0$$

Integrating ; $\int \frac{dg}{g} + 2 \int \frac{dr}{r}$ = constant $\Rightarrow gr^2 = C_1 \because g(r) = \frac{C_1}{r^2}$

$$\frac{df}{dr} = \frac{C_1}{r^2}; \quad f = \int \frac{C_1}{r^2} dr + C_2; \quad f(r) = -\frac{C_1}{r} + C_2$$

PREVIOUS YEARS QUESTIONS ANALYSIS

INTRODUCTION: VECTOR ANALYSIS

Q1. Prove that the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ can form the sides of a triangle. Find the lengths of the medians of the triangle. **[5b UPSC CSE 2016]**

Q2. Prove that $\bar{a} \times \left(\bar{b} \times \bar{c} \right) = \left(\bar{a} \times \bar{b} \right) \times \bar{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$. **[8c 2016 IFoS]**

Q3. For three vectors show that: $\bar{a} \times \left(\bar{b} \times \bar{c} \right) + \bar{b} \times \left(\bar{c} \times \bar{a} \right) + \bar{c} \times \left(\bar{a} \times \bar{b} \right) = 0$. **[5e 2014 IFoS]**

VECTOR DIFFERENTIAL CALCULUS

Q1. The position vector of a moving point at time t is $\bar{r} = \sin t\hat{i} + \cos 2t\hat{j} + \left(t^2 + 2t\right)\hat{k}$. Find the components of acceleration $\bar{a}$ in the directions parallel to the velocity vector $\bar{v}$ and perpendicular to the plane of $\bar{r}$ and $\bar{v}$ at time $t = 0$. **[5e UPSC CSE 2017]**

Q2. If $\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$, $\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$

find the value of $\frac{\partial^2}{\partial x \partial y}\left(\vec{A} \times \vec{B}\right)$ at $(1, 0, -2)$. **[5e UPSC CSE 2012]**

Q3. For two vectors $\bar{a}$ and $\bar{b}$ given respectively by $\bar{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\bar{b} = \sin t\hat{i} - \cos t\hat{j}$ determine: (i) $\frac{d}{dt}\left(\bar{a} \cdot \bar{b}\right)$ and (ii) $\frac{d}{dt}\left(\bar{a} \times \bar{b}\right)$ **[5e UPSC CSE 2011]**

Q4. The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is $\vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}$. At time $t = 1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. **[8d 2011 IFoS]**

GRADIENT, DIRECTIONAL DERIVATIVES

Q5(e) Show that $\nabla^2\left[\nabla \cdot \left(\frac{\vec{r}}{r}\right)\right] = \frac{2}{r^4}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. **UPSC CSE 2021**

Q5(e) Determine constants a, b, c so that the directional derivative of $\phi(x, y, z) = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 88 in a direction parallel to *z*-axis. **IFoS 2022**

Q1. Prove that for a vector $\vec{a}$, $\nabla\left(\vec{a} \cdot \vec{r}\right) = \vec{a}$; where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$. Is there any restriction on $\vec{a}$? Further, show that $\vec{a} \cdot \nabla\left(\vec{b} \cdot \nabla\frac{1}{r}\right) = \frac{3\left(\vec{a} \cdot \vec{r}\right)\left(\vec{b} \cdot \vec{r}\right)}{r^5} - \frac{\vec{a} \cdot \vec{b}}{r^3}$.

Give an example to verify the above. **[5e 2020 IFoS]**

Q2. Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1,1,1)$. **[5e UPSC CSE 2019]**

Q3. Find the angle between the tangent at a general point of the curve whose equations are $x = 3t, y = 3t^2, z = 3t^3$ and the line $y = z - x = 0$. **[5b UPSC CSE 2018]**

Q4. Find $f(r)$ such that $\nabla f = \frac{\vec{r}}{r^5}$ and $f(1)=0$. **[8a UPSC CSE 2016]**

Q5. Find the angle between the surfaces $x^2+y^2+z^2-9=0$ and $z=x^2+y^2-3$ at $(2,-1,2)$.

[5e UPSC CSE 2015]

Q6. Find the value of λ and μ so that the surfaces $\lambda x^2-\mu yz=(\lambda+2)x$ and $4x^2y+z^3=4$ may intersect orthogonally at $(1,-1,2)$. **[6c UPSC CSE 2015]**

Q7. A curve in space is defined by the vector equation $\vec{r}=t^2\hat{i}+2t\hat{j}-t^3\hat{k}$. Determine the angle between the tangents to this curve at the points $t=+1$ and $t=-1$. **[8b UPSC CSE 2013]**

Q8. If $u=x+y+z$, $v=x^2+y^2+z^2$, $w=yz+zx+xy$ prove that grad u, grad v and grad w are coplaner. **[5e 2012 IFoS]**

Q9. Examine whether the vectors ∇_u, ∇_v and ∇_w are coplaner, where u, v and w are the scalar functions defined by: $u=x+y+z$, $v=x^2+y^2+z^2$ and $w=yz+zx+xy$. **[8a UPSC CSE 2011]**

Q10. Find the directional derivative of $f(x,y)=x^2y^3+xy$ at the point $(2,1)$ in the direction of a unit vector which makes an angle of $\pi/3$ with the *x*-axis. **[1e UPSC CSE 2010]**

Q11. Find the directional derivation of $\vec{V}^2$, where, $\vec{V}=xy^2\vec{i}+zy^2\vec{j}+xz^2\vec{k}$ at the point $(2,0,3)$ in the direction of the outward normal to the surface $x^2+y^2+z^2=14$ at the point $(3,2,1)$.

[5f 2010 IFoS]

Q12. Find the directional derivative of -

(i) $4xz^3-3x^2y^2z^2$ at $(2,-1,2)$ along *z*-axis;

(ii) $x^2yz+4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$. **[5f UPSC CSE 2009**

DIVERGENCE

Q8© For a scalar point function ϕ and vector point function f, prove the identity $\nabla.(\phi\vec{f})=\nabla\phi.\vec{f}+\phi(\nabla.\vec{f})$. Also find the value of $\nabla.\left(\frac{f(r)}{r}\vec{r}\right)$ and then verify stated identity.

UPSC CSE 2023 (15)

Q1. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $f(r)$ is differentiable, show that $\text{div}\left[f(r)\vec{r}\right]=rf'(r)+3f(r)$.

Hence or otherwise show that $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right)=0$. **[5e 2018 IFoS]**

Q2. Calculate $\nabla^2\left(r^n\right)$ and find its expression in terms of *r* and *n, r* being the distance of any point (x, y, z) from the origin, *n* being a constant and ∇^2 being the Laplace operator.

[8a UPSC CSE 2013]

Q3. Prove that $\operatorname{div}\left(f\vec{V}\right)=f\left(\operatorname{div}\vec{V}\right)+\left(\operatorname{grad} f\right)\cdot\vec{V}$ where *f* is a scalar function.

[6c UPSC CSE 2010]

Q4. Show that, $\nabla^2 f(r)=\left(\frac{2}{r}\right)f'(r)+f''(r)$, where $r=\sqrt{x^2+y^2+z^2}$. **[4. 8a 2010 IFoS]**

Q5. Show that $\operatorname{div}\left(\operatorname{grad} r^n\right)=n(n+1)r^{n-2}$ where $r=\sqrt{x^2+y^2+z^2}$. **[5e UPSC CSE 2009]**

CURL

Q5 (e) If $\vec{a}=\sin\theta\hat{i}+\cos\theta\hat{i}+\theta k$, $\vec{b}=\cos\theta\hat{i}-\sin\theta\hat{i}-3k$, $\vec{c}=2\hat{i}+3\hat{i}-3k$ then find the values of the derivative of the vector function $\vec{a}\times(\vec{b}\times\vec{c})$ w.r.t θ at $\theta=\frac{\pi}{2}$ and $\theta=\pi$.

UPSC CSE 2023

Q5(e) Show that $\vec{A}=\left(6xy+z^3\right)\hat{i}+\left(3x^2-z\right)\hat{j}+\left(3xz^2-y\right)\hat{k}$ is irrotational. Also find ϕ such that $\vec{A}=\nabla\phi$. **UPSC CSE 2022**

Q7.(a) Derive vector identity for divergence of cross product of two vector point functions. Given a relation between linear and angular velocity as $\vec{v}=\vec{\omega}\times\vec{r}$.

If $\vec{\omega}$ is constant, then show that (i) curl $\vec{v}=2\vec{\omega}$ (ii) div $\vec{v}=0$.

(b) Given that $y_1=x^2$ is a solution of the differential equation. **IFoS 2022**

Q5(e) If $\vec{F}=\left(y\frac{\partial\phi}{\partial z}-z\frac{\partial\phi}{\partial y}\right)\hat{i}+\left(z\frac{\partial\phi}{\partial x}-x\frac{\partial\phi}{\partial z}\right)\hat{j}+\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)\hat{k}$, then prove that

$\vec{F}-\left(\vec{r}\times\nabla\phi\right)=\vec{F}\cdot\vec{r}=\vec{F}\cdot\nabla\phi=0$. **IFoS 2021**

Q8. $\vec{F}$ being a vector, prove that $\operatorname{curl}\operatorname{curl}\vec{F}=\operatorname{grad}\operatorname{div}\vec{F}-\nabla^2\vec{F}$ where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

[5c 2013 IFoS]

Q9. If *u* and *v* are two scalar fields and $\vec{f}$ is a vector field, such that $u\vec{f} = \text{grad}\, v$, find the value of $\vec{f} \cdot \text{curl}\, \vec{f}$. **[5f UPSC CSE 2011]**

Q10.If $\vec{r}$ be the position vector of a point, find the value(s) of *n* for which the vector $r^n\vec{r}$ is (i) irrotational, (ii) solenoidal. **[8c UPSC CSE 2011]**

Q11. Prove the vector identity:

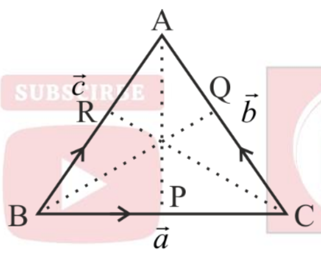
$\text{curl}\left(\vec{f} \times \vec{g}\right) = \vec{f}\ \text{div}\,\vec{g} - \vec{g}\,\text{div}\,\vec{f} + \left(\vec{g} \cdot \nabla\right)\vec{f} - \left(\vec{f} \cdot \nabla\right)\vec{g}$ and verify it for the vectors $\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$ and $\vec{g} = y\hat{i} + z\hat{k}$. **[8b 2011 IFoS]**

PYQs Analysis: Vector Analysis

1. Given $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ there.

$$\vec{b} + \vec{c} = \left(-i + 3\hat{j} + 4\hat{k}\right) + \left(4\hat{i} - 2\hat{j} - 6\hat{k}\right) = 3\hat{i} + \hat{j} - 2\hat{k} = \vec{a}$$

$\therefore\ \vec{a}, \vec{b}, \vec{c}$ form the side of triangles.



Let AP, BQ, CR he the medians of ΔABC.

Now, using triangle law of vector addition,

In ΔBQC,

$$\vec{a} + \frac{\vec{b}}{2} = \overrightarrow{BQ}$$

$$\overrightarrow{BQ} = \left(3 - \frac{1}{2}\right)\hat{i} + \left(1 + \frac{3}{2}\right)\hat{j} + (-2 + 2)\hat{k}$$

$$\overrightarrow{BQ} = \frac{5}{2}\hat{i} + \frac{5}{2}\hat{j} \qquad \therefore |BQ| = \sqrt{\frac{50}{4}} = \frac{5}{2}\sqrt{2}.$$

In ΔARC,

$$\overrightarrow{CR} + \overrightarrow{AC} = \overrightarrow{AR}$$

$$\overrightarrow{CR} = \left(2\hat{i} - \hat{j} - 3\hat{k}\right) - \vec{b}$$

$$\overrightarrow{CR} = 3\hat{i} - 4\hat{j} - 7\hat{k} \qquad |CR| = \sqrt{9 + 16 + 49} = \sqrt{74}$$

In ΔAPC,

$\overrightarrow{PC}+\overrightarrow{AC}=\overrightarrow{AP}$

$$\left(\frac{3}{2}-1\right)\hat{i}+\left(\frac{1}{2}+3\right)\hat{j}+3\hat{k}=\overrightarrow{AP}$$

$\overrightarrow{AP}=\frac{1}{2}\hat{i}+\frac{7}{2}\hat{j}+3\hat{k}.$ $\qquad\qquad \left|\overrightarrow{AP}\right|=\sqrt{\frac{50}{4}+9}=\frac{1}{2}\sqrt{86}$

2. Given *b* = 0 or *c* is collinear with '*a*' or, $\vec{b}$ is perpendicular both $\vec{a}$ & $\vec{c}$

To Prove: $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$

(1) If $\vec{b}=0$, then

$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=0$ & $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=0$

(2) If $\vec{c}=m\vec{a}$ then

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{a}\times\left(\vec{b}\times\left(m\vec{a}\right)\right)=\left\{m\left(\vec{a}.\vec{a}\right)\right\}\vec{b}-\left(\vec{a}.\vec{b}\right)m\vec{a}=m\left[|\vec{a}|^2\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{a}\right]$$

Now, $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=-\left\{\vec{c}\times\left(\vec{a}\times\vec{b}\right)\right\}=-\left\{m\vec{a}\times\left(\vec{a}\times\vec{b}\right)\right\}=-m\left[\left(\vec{a}.\vec{b}\right)\vec{a}-\left(\vec{a}.\vec{a}\right)\vec{b}\right]=m\left[|\vec{a}|^2\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{a}\right]$

$\therefore\ \vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$

(3) If $\vec{b}.\vec{a}=0$ & $\vec{b}.\vec{c}=0$

Now, $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}=\left(\vec{a}.\vec{c}\right)\vec{b}$

Also, $\left(\vec{a}\times\vec{b}\right)\times\vec{c}=-\left\{\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\vec{b}\right\}=\left(\vec{c}.\vec{a}\right)\vec{b}$

$\therefore\ \vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$

Similarly, if $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$

$\Rightarrow \left(a.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}=-\left\{\left(\vec{c}.\vec{b}\right)\vec{a}-\left(\vec{c}.\vec{a}\right)\right\}\vec{b}$

$\Rightarrow -\left(\vec{a}.\vec{b}\right)\vec{c}=-\left(\vec{c}.\vec{b}\right)\vec{a}$

$\Rightarrow \left(\vec{a}.\vec{b}\right)\vec{c}-\left(\vec{c}.\vec{b}\right)\vec{a}=0$

$\Rightarrow -\left\{\vec{b}\times\left(\vec{a}\times\vec{c}\right)\right\}=0$

$\Rightarrow \vec{b}\times\left(\vec{a}\times\vec{c}\right)=0$

$\therefore$ Either $\vec{b}=0$ or $\vec{a}\times\vec{c}=0$ i.e., $\vec{a}$ & $\vec{b}$ are collinear or, $\vec{b}.\vec{c} = \vec{a}.\vec{b} = 0$

3. $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\left(\vec{a}\times\vec{b}\right) = 0$

VECTOR CALCULUS

1. Position vector $\vec{r} = \sin t\hat{i} + \cos 2t\hat{j} + \left(t^2 + 2t\right)\hat{k}$

$$\therefore\ \vec{v} = \frac{d\vec{r}}{dt} = \cos t\hat{i} - 2\sin 2t\hat{j} + \left(2t+2\right)\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt} = -\sin t\hat{i} - 4\cos 2t\hat{j} + 2\hat{k}$$

At $t = 0$, $\vec{r} = \hat{j}$

$\vec{v} = \hat{i} + 2\hat{k}$

$\vec{a} = -4\hat{j} + 2\hat{k}$

Required equation in the direction of $\vec{v} = \dfrac{\left(\vec{a}.\vec{v}\right)}{|\vec{v}|}\times\dfrac{\vec{v}}{|\vec{v}|}$

Required equation in the direction of $\vec{v} = \dfrac{4}{\sqrt{5}}\times\dfrac{\vec{v}}{\sqrt{5}} = \dfrac{4}{5}\left(\hat{i}+2\hat{k}\right)$

An vector $\perp v$ to $\vec{r}$ and $\vec{v}$ is, $\vec{c} = \vec{r}\times\vec{v}$; $\quad \left(\vec{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\hat{i} - \hat{k}$

$\therefore$ Required equation in the direction of $\vec{c}$ is.

$$\frac{\vec{c}.\vec{a}}{|\vec{c}|}.\frac{\vec{c}}{|\vec{c}|} = \frac{-2}{\sqrt{5}}\times\frac{\vec{c}}{\sqrt{5}} = \frac{-2}{5}\left(2\hat{i}-\hat{k}\right)$$

2. $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

$\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix} = \left(2x^3z^3 - xyz^2\right)\hat{i} + \left(2xz^3 + x^4yz\right)\hat{j} + \left(x^2y^2z + 4xz^4\right)\hat{k}$$

Now,

$$\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right) = -xz^2\hat{i} + x^4z\hat{j} + 2x^2yz\hat{k}$$

$$\frac{\partial}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right)=-z^2\hat{i}++4x^3z\hat{j}+4xyz\hat{k}$$

$$\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right)\Bigg|_{(1,0,-2)}=-4\hat{i}-8\hat{j}$$

3. $\vec{a}=5t^2\hat{i}+t\hat{j}-t^3\hat{k},\vec{b}=\sin\hat{i}-\cos t\hat{j}$

(1) $\vec{a}.\vec{b}=5t^2\sin t-t\cos t$

$$\therefore\ \frac{d}{dt}\left(\vec{a}.\vec{b}\right)=5\left[t^2.\cos t+2t\sin t\right]-\left[t\left(-\sin t\right)+\cos t\right]=\left(5t^2-1\right)\cos t+\left(10t+t\right)\sin t$$

(2) $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ st^2 & t & -t^3\\ \sin t & -\cos t & 0\end{vmatrix}=-t^3\cos t\hat{i}-t^3\sin t\hat{j}-\left(5t^2\cos t+t\sin t\right)\hat{k}$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)-\left[\left\{t^3\left(-\sin t\right)+3t^2\cos t\right\}\hat{i}+\left(t^3\cos t+3t^2\sin t\right)\hat{j}.\left[5\left\{t^2\left(-\sin t\right)+2t\cos t\right\}+t.\cos t+\sin t\right]\hat{k}\right]$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)=t^3\sin t-3t^2\cos t\hat{i}-t^2\left(t\cos\theta+3\sin\theta t\right)\hat{j}.-\left\{\left(-5t^2+1\right)\sin t+11t\cos t\right\}\hat{k}\ .$$

4. $\vec{r}=\left(t^2-2t\right)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}$ & m = 2 units

$$\vec{v}=\frac{d\vec{r}}{dt}=\left(2t-2\right)\hat{i}+t\hat{j}+t\hat{k}$$

Now, K.E = $\frac{1}{2}m\left|\vec{v}\right|^2=\frac{1}{2}\times 2\times\left(\sqrt{\left(2t-2\right)^2+t^2+t^2}\right]^2$

$=1\times\left(\sqrt{2}\right)^2$ $\qquad$ $\left[\text{At } t=1\right]$

K.E = 2 units

At t = 1, $\vec{v}=\hat{j}+\hat{k}$

$$\vec{r}=-\hat{i}+\frac{3}{2}\hat{j}+\frac{1}{2}\hat{k}$$

Now, Angular Momentum $\left(\vec{L}\right)=\vec{r}+m\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 2 & 2\end{vmatrix}=2\hat{i}+2\hat{j}-2\hat{k}$ units.

Now $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times m\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times (m\vec{v})$

$$\frac{d\vec{L}}{dt} = \vec{r} \times m\frac{d\vec{v}}{dt} \qquad \left[\because \ \vec{v} = \frac{d\vec{r}}{dt} \quad \text{so, } \vec{v} \times (m\vec{v}) = 0\right]$$

$$\frac{d\vec{v}}{dt} = 2\hat{i} + \hat{j} + \hat{k}$$

Now, $\frac{dL}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 4 & 2 & 2 \end{vmatrix} = 2\hat{i} + 4\hat{j} - 8\hat{k}$ units

Now, Moment of resultant force is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \vec{r} \times m\frac{d\vec{v}}{dt}$$

$$\tau\Big|_{t=1} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 4 & 2 & 2 \end{vmatrix}$$

$\tau\Big|_{t=1} = 2\hat{i} + 4\hat{j} - 8\hat{k}$ units

5 (c) CSE 2021

$\nabla^2\left(\nabla \cdot \left(\frac{\vec{r}}{r^2}\right)\right) = ?$

$\nabla^2\left(\nabla \cdot \left(\frac{\vec{r}}{r^2}\right)\right) = \nabla^2\left[\nabla\left(r^{-2}\right) \cdot \vec{r} + r^{-2}\nabla\vec{r}\right]$

$= \nabla^2\left[\left((-2)r^{-3} \times \frac{\vec{r}}{r}\right) \cdot \vec{r} + r^{-2} \times 3\right]$ {using $\nabla(f(r)) = f'(r) \cdot \frac{\vec{r}}{r}$ & $\nabla\vec{r} = 3$}

$= \nabla^2\left[-2r^{-2} + 3r^{-2}\right] = \nabla^2\left(\frac{1}{r^2}\right) = \nabla \cdot \nabla\ (r^{-2}) = \nabla \cdot \left\{(-2)r^{-3} \cdot \frac{\vec{r}}{r}\right\}$ [using the same as above]

$= (-2)\ \nabla \cdot \left\{r^{-4}\,\vec{r}\right\} = -2\left[\nabla\left(r^{-4}\right) \cdot \vec{r} + r^{-4}\nabla \cdot \vec{r}\right]$ $\left\{\begin{aligned} &\because \nabla \cdot \left(\phi\vec{f}\right) \\ &= (\nabla\phi) \cdot \vec{f} + \phi\left(\nabla \cdot \vec{f}\right) \end{aligned}\right\}$

$= -2\left[\left(-4r^{-5} \times \frac{\vec{r}}{r}\right) \cdot \vec{r} + 3r^{-4}\right] = -2\left[\frac{-4}{r^4} + \frac{3}{r^4}\right] = -2 \times \frac{-1}{r^4}$

$$\nabla^2\left(\nabla \cdot \frac{\vec{r}}{r^2}\right) = \frac{2}{r^4}$$

5(e) IFOS 2021

$\varphi (x, y, z) = axy^2 + byz + cz^2x^3$

Directional Derivative of φ is given by grad φ

$\therefore \quad \Delta\varphi = (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$

Now, $\quad (\nabla \varphi)_{(1,\,2,\,-1)} = (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$

The maximum magnitude is in a direction parallel to z-axis so,

$4a + 3c = 0 \quad \ldots(1)$

$4a - b = 0 \quad \ldots(2)$

And, max. magnitude is 64.

$\therefore \quad 2b - 2c = 64$

$\Rightarrow \quad b - c = 32 \quad \ldots(3)$

Now, from (2) + (3),

$4a - c = 32$

$c = 4a - 32$

Using (c) in (1),

$4a + 12a - 96 = 0$

$16a = 96$

$\boxed{a = 6}$

$\therefore \quad \boxed{c = -8} \;\&\; \boxed{b = 24}$

Q1. 2020 IFOS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \quad \& \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad \vec{a}\cdot\vec{r} = xa_1 + ya_2 + za_3$

$\therefore \nabla\left(\vec{a}\cdot\vec{r}\right) = \frac{\partial}{\partial x}(a_1x + a_2y + a_3z)\,\hat{i} + \frac{\partial}{\partial y}(a_1x + a_2y + a_3z)\,\hat{k} + \frac{\partial}{\partial z}(a_1x + a_2y + a_3z) \ldots(1)$

If $\nabla\left(\vec{a}\cdot\vec{r}\right) = \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then

From (1) we can see that it is possible only when a_1, a_2, a_3 are all constant values, that not includes any variable x, y or z

Now, $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\cdot\frac{x}{r}\hat{i} + \frac{-1}{r^2}\cdot\frac{y}{r}\hat{j} + \frac{-1}{r^2}\cdot\frac{z}{r}\hat{k} = \frac{-1}{r^3}\times\vec{r}$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

$$\therefore \left\{\vec{b}\cdot\nabla\left(\frac{1}{r}\right)\right\} = -\frac{1}{r^3}\left(\vec{b}\cdot\vec{r}\right)$$

$$\nabla\left(\vec{b}\cdot\nabla\left(\frac{1}{r}\right)\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{-1}{r^3}\cdot\left(\vec{b}\cdot\vec{r}\right)\right)$$

$$= -\left[\frac{1}{r^3}\frac{\partial}{\partial x}\left(\vec{b}\cdot\vec{r}\right)\hat{i}+\left(b\cdot\vec{r}\right)\cdot\frac{-3}{r^4}\cdot\frac{x}{r}\hat{i}+\frac{1}{r^3}\frac{\partial}{\partial y}\left(\vec{b}\cdot\vec{r}\right)\hat{j}\right.$$

$$\left.+\left(\vec{b}\cdot\vec{r}\right)\cdot\frac{-3}{r^4}\cdot\frac{y}{r}\hat{j}+\frac{1}{r^3}\frac{\partial}{\partial z}\left(\vec{b}\cdot\vec{r}\right)\hat{k}+\left(\vec{b}\cdot\vec{r}\right)\cdot\frac{-3}{r^4-1}\cdot\frac{z}{r^2}\hat{k}\right]$$

$$= \frac{3}{r^5}\left(\vec{b}\cdot\vec{r}\right)\left\{x\hat{i}+y\hat{j}+z\hat{k}\right\}-\frac{1}{r^3}\nabla\left(\vec{b}\cdot\vec{r}\right)$$

$$= \frac{3}{r^5}\left(\vec{b}\cdot\vec{r}\right)\vec{r}-\frac{1}{r^3}\cdot\vec{b} \qquad \left\{\because \nabla\left(\vec{b}\cdot\vec{r}\right)=\vec{b}\right\}$$

$$\nabla\left(\vec{b}\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{3}{r^5}\left(\vec{b}\cdot\vec{r}\right)\vec{r}-\frac{1}{r^3}\vec{b}$$

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{3}{r^5}\left(\vec{b}\cdot\vec{r}\right)\left(\vec{a}\cdot\vec{r}\right)-\frac{+1}{r^3}\left(\vec{b}\cdot\vec{a}\right)$$

Let if $\vec{a}=\hat{i}$ & $\vec{b}=\hat{j}$

$$\therefore \vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\left(\frac{1}{r}\right)\right) = \hat{i}\cdot\nabla\left(\hat{j}\cdot\left(\frac{-\vec{r}}{r^3}\right)\right) = \hat{i}\cdot\nabla\left(\frac{-y}{r^3}\right) = \hat{i}\cdot\left\{-y\cdot\frac{-3}{r^4}\cdot\frac{x}{r}\hat{i}\right\} = \frac{+3xy}{r^5}$$

Now, $\frac{3}{r^5}\left(\vec{b}\cdot\vec{r}\right)\left(\vec{a}\cdot\vec{r}\right)-\frac{1}{r^3}\left(\vec{b}\cdot\vec{a}\right) = \frac{3}{r^5}(y)(x)-\frac{1}{r^3}\times 0 = \frac{3xy}{r^5}$

$\because$ LHS = RHS $\therefore$ Verified.

Q2 CSE 2019

$\varphi(x, y, z) = xy^2 + yz^2 = zx^2$

Now, D.D. = $\nabla\varphi$

D.D. = $(y^2 + 2zx)\,\hat{i} + (2xy + z^2)\,\hat{j} + (2yz + x^2)\,\hat{k}$

$(\text{D.D.})_{(1,1,1)} = 3\hat{i}+3\hat{j}+3\hat{k}$ ...(1)

Now, for the given curve,

$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2t,\ \frac{dz}{dt} = 3t^2$

Now, at point (1, 1, 1) the curve gives the value of $t = 1$.

$\therefore$ Now, vector along the tangent to the curve is given by,

$$\vec{a} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{a} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \qquad \text{[at } t = 1\text{]}$$

Now, Directional derivative in the direction of $\vec{a}$ is given by ,

$$= (\nabla \varphi)_{(1,\,1,\,1)} \,.\, \hat{a} = \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right)\cdot\frac{\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

2018 CSE

3. From the given curve,

$$\frac{dx}{dt} = 3,\ \frac{dy}{dt} = 6t,\ \frac{dz}{dt} = 9t^2.$$

$\therefore$ vector along the tangent to the curve is $\vec{a} = 3\hat{i} + 6t\hat{j} + 9t^2\hat{k}$

For the line,

$$y = z - x = 0 \ \therefore \qquad y = 0,\ z = x \ \therefore \quad \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

For the line, the dr's in vector form is, $\vec{b} = \hat{i} + \hat{k}$

Now, if θ is the angle between the tangent & the line, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{3+0+9t^2}{\sqrt{9+36t^2+81t^4}\times\sqrt{2}} = \frac{3+9t^2}{3\sqrt{2}\sqrt{1+4t^2+9t^4}}$$

$$\cos\theta = \frac{1+3t^2}{\sqrt{2}\,\sqrt{9t^4+4t^2+1}}\,;\ \theta = \cos^{-1}\left\{\frac{1+3t^2}{\sqrt{2}\,\sqrt{9t^4+4t^2+1}}\right\}$$

4. CSE 2016

Given, $\nabla f(r) = \dfrac{\vec{r}}{r^5}$ & $f(1) = 1$

$$\nabla f(r) = f'(r)\cdot\frac{\partial r}{\partial x} + \hat{i} + f'(r)\cdot\frac{\partial r}{\partial y}\hat{j} + f'(r)\cdot\frac{\partial r}{\partial r}\hat{k} = f'(r)\cdot\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right]$$

$$\nabla f(r) = \frac{f'(r)}{r}\vec{r} \Rightarrow \frac{\vec{r}}{r^5} = \frac{f'(r)}{r}\vec{r} \ \text{[given]} \Rightarrow f'(r) = r^{-4}$$

On integrating w.r.t. r,

$f(r) = \dfrac{-1}{3r^3} + c$, where c is some integration constant

Now, $\because f(1) = 0$

$\therefore$ Now, vector along the tangent to the curve is given by

$\therefore \ \frac{-1}{3c} + c = 0$; $c = \frac{1}{3}$ $\quad \therefore f(r) = \frac{-1}{3r^3} + \frac{1}{3}$

5. CSE 2015

$\varphi_1(x, y, z) = x^2 + y^2 + z^2 - 9$, $\varphi_2(x, y, z) = x^2 + y^2 - 3 - z$

Now, $\nabla \varphi_1 = 2x\hat{i} + 2\hat{j} + 2z\hat{k}$ & $\nabla \varphi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$(\nabla \varphi_1)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$ $\quad (\nabla \varphi_2)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$

Now, both $\nabla \varphi_1$ & $\nabla \varphi_2$ are normals to the given surfaces so, the angle between the two is given by

$$\cos\theta = \frac{(\nabla\phi_1)_{2,-1,2} \cdot (\nabla\phi_2)_{2,-1,2}}{\left|(\nabla\phi_1)_{(2,-1,2)}\right|\left|(\nabla\phi_2)_{2,-1,2}\right|}$$

$$\cos\theta = \frac{16+4-4}{\sqrt{36}\sqrt{21}};\ \cos\theta = \frac{16}{\sqrt{21}\times 6};\quad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

CSE 2015 Q6

Let $\varphi_1(x, y, z) = \lambda x^2 - \mu yz - (\lambda + 2)x = 0$

$\varphi_2(x, y, z) = 4x^2y + z^3 - 4 = 0$

Now, grad $\varphi_1 = \nabla \varphi_1 = \{2\lambda x - (\lambda + 2)\}\ \hat{i} + (-\mu z)\ \hat{j} - \mu y\,\hat{k}$

grad $\varphi_2 = \nabla \varphi_2 = 8xy\,\hat{i} + 4x^2\ \hat{j} - 3z^2\,\hat{k}$

Now, angle between φ_1 & φ_2 at (1, –1, 2) is given

$$\cos\theta = \left.\frac{grad\phi_1 \cdot grad\phi_2}{|grad\phi_1||grad\phi_2|}\right|_{(1,-1,2)}$$

$$\cos 90° = \frac{\left((2\lambda-\lambda-2)-2\mu\hat{j}+\mu\hat{k}\right)\cdot\left(-8\hat{i}+4\hat{j}+12\hat{k}\right)}{\sqrt{(\lambda-2)^2+(-2\mu)^2+\mu^2}\sqrt{64+16+144}}$$

$\Rightarrow 0 = -8(\lambda - 2) - 8\mu + 12\mu \Rightarrow -2\lambda + \mu = 0 \Rightarrow\ 2\lambda - \mu = 4$ ...(1)

Also point (1, –1, 2), lies on $\varphi_1(x, y, z)$

$\therefore \lambda + 2\mu - \lambda - 2 = 0$ $\boxed{\mu = 1}$

$\therefore$ from (1), $\boxed{\lambda = \frac{5}{2}}$

7. 2013 CSE

$\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$

Now, $\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=1} = \vec{a} = 2\hat{i} + 2\hat{j} - 3\hat{k}$$

$$\left.\frac{dr}{dt}\right|_{t=-1} = \vec{b} = -2\hat{i} + 2\hat{j} - 3\hat{k}$$

Now, the angle between the tangents $\vec{a}$ & $\vec{b}$ is,

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$; $\cos\theta = \dfrac{-4+4+9}{\sqrt{4+4+9}\sqrt{4+4+9}}$; $\cos\theta = \dfrac{9}{17}$; $\theta = \cos^{-1}\left(\dfrac{9}{17}\right)$

8. 2012 IFOS

$u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$

Now, grad $u = \hat{i} + \hat{j} + \hat{k}$

grad $v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

grad w = $(z + y)\,\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}$

Now,

$$\text{grad } v \times \text{grad } w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x & 2y & 2z \\ z+y & z+x & x+y \end{vmatrix}$$

grad v × grad w = $2[(xy + y^2 - z^2 - xz)\,\hat{i} + (z^2 + zy - x^2 - xy)\,\hat{j} + (xz + x^2 - yz - y^2)\,\hat{k}\,]$

Now, grad u . (grad v × grad w) = $2[xy + y^2 - z^2 - xz + z^2 + zy - x^2 - xy + xz + x^2 - yz - y^2] = 0$

Now, ∴ grad u, grad v & grad w are coplanar.

10 CSE 2010

$f(x, y) = x^2 y^3 + xy$

Directional Derivative at (2, 1) is given by $(\nabla f)_{(2,\,1)}$

∴ $\Delta f = (2xy^3 + y)\,\hat{i} + (3x^2 + y^2 + x)\,\hat{j}$

$(\Delta f)_{(2,\,1)} = 5\hat{i} + 14\hat{j}$

Now, A unit vector which makes an angle $\dfrac{\pi}{3}$ with x – axis is

$\vec{a} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$; $\vec{a} = \cos\dfrac{\pi}{3}\hat{i} + \sin\dfrac{\pi}{3}\hat{j}$;; $\vec{a} = \dfrac{1}{2}\hat{i} + \dfrac{\sqrt{3}}{2}\hat{j}$

∴ D.D. in the direction of unit vector ($\vec{a}$) is= $(\nabla)_{(2,1)}\cdot\vec{a} = (5\hat{i} + 14\hat{j})\cdot\left(\dfrac{1}{2}\hat{i} + \dfrac{\sqrt{3}}{2}\hat{j}\right) = \dfrac{5}{2} + 7\sqrt{3}$

∴ Now, vector along the tangent to the curve is

11. IFOS 2010

$\vec{V} = xy^2\,\hat{i} + zy^2\,\hat{j} + xz^2\,\hat{k}$

$\vec{V}^2 = x^2y^4 + z^2y^4 + x^2z^4$

$\therefore$ D.D. *of* $\vec{V}^2 = \nabla\,\vec{V}^2 = (2xy^4 + 2xz^4)\,\hat{i} + (4x^2y^3 + 4z^2y^3)\,\hat{j} + (2zy^4 + 4z^3x^2)\,\hat{k}$

D.D. of $\vec{V}^2$ at (2, 0, 3) = $324\,\hat{i} + 48\,\hat{k}$ ...(1)

Now, Let $\varphi_1 = x^2 + y^2 + z^2 - 14$

Now, normal to the surface φ_1, is given by

$\nabla\varphi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$(\nabla\varphi_1)_{(3,2,1)} = 2\left[3\hat{i} + 2\hat{j} + \hat{k}\right] = \vec{a}$ (say)

$\therefore$ Req D.D. of $\vec{V}^2$ in the directional $\vec{a}$ is given by

$$\left(324\hat{i} + 48\hat{k}\right)\cdot\frac{\left(6\hat{i} + 4\hat{j} + 2\hat{k}\right)}{\sqrt{36+16+4}} = \frac{324\times 6 + 16\times 27\times 2}{2\sqrt{9+4+1}} = \frac{972+432}{\sqrt{14}} = \frac{1404}{\sqrt{14}}$$

12. CSE 2009

I. Let $\varphi_1 = 4z^3x - 3x^2y^2z^2$

$\nabla\varphi_1 = (4z^3 - 6xy^2z^2)\,\hat{i} + -6x^2yz^2\,\hat{j} + (12z^2x - 6x^2y^2z)\,\hat{k}$

$(\nabla\varphi_1)_{(2,-1,2)} = (32 - 48)\,\hat{i} + 96\,\hat{j} + (96 - 48)\,\hat{k}$

$(\nabla\varphi)_{(2,-1,2)} = -16\,\hat{i} + 96\,\hat{j} + 48\,\hat{k}$

Now, D.D. of φ in the direction of *z*-axis is = $(\nabla\varphi_1)_{(2,-12)} = 48$

II. Let $\varphi_2 = x^2yz + 4xz^2$

$\Delta\varphi_2 = (2xyz + 4z^2)\,\hat{i} + x^2z\,\hat{j} + (x^2y + 8xz)\,\hat{k}$

$(\nabla\varphi_2)_{(1,-2,1)} = (-4 + 4)\,\hat{i} + \hat{j} + (-2 + 8)\,\hat{k}$

$(\nabla\varphi_2)_{(1,-2,1)} = \hat{j} + 6\,\hat{k}$

Now, D.D. of φ_2 in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$(\nabla\phi_2)_{(1,-2,1)}\cdot\frac{\left(2\hat{i} - \hat{j} - 2\hat{k}\right)}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$\frac{\left(\hat{j} + 6\hat{k}\right)\cdot\left(2\hat{i} - \hat{j} - 2\hat{k}\right)}{3} = \frac{-13}{3}$$

$\therefore$ Now, vector along the tangent to the curve is

DIVERGENCE

8.c FIOS 2015

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore\ \nabla\cdot\left(\phi\vec{f}\right) = \nabla\cdot\left(\phi f_1\hat{i}+\phi f_2\hat{j}+\phi f_3\hat{k}\right) = \frac{\partial}{\partial x}(\phi f_1)+\frac{\partial}{\partial y}(\phi f_2)+\frac{\partial}{\partial_2}(\phi f_3)$$

$$= \phi\frac{\partial f_1}{\partial x}+\frac{\partial\phi}{\partial x}f_1+\phi\frac{\partial f_2}{\partial y}+\frac{\partial\phi}{\partial y}f_2+\frac{\partial f_3}{\partial z}+\frac{\partial\phi}{\partial z}f_3 = \left(\frac{\partial\phi}{\partial x}f_1+\frac{\partial\phi}{\partial y}f_2+\frac{\partial\phi}{\partial z}f_3\right)+\phi\left\{\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right\}$$

$$= \nabla\phi\cdot\left(f_1\hat{i}+f_2\hat{j}+f_3\hat{k}\right)+\phi\nabla\cdot\vec{f}_x$$

$$\nabla\cdot\left(\phi\vec{f}\right) = (\nabla\phi)\cdot\vec{f}+\phi\left(\nabla\cdot\vec{f}\right)\ \text{...(1)}$$

$$\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = \nabla\cdot\left(\frac{x}{r}f(r)\cdot\hat{i}+\frac{y}{r}f(r)\hat{j}+\frac{z}{r}f(r)\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(f(x)\cdot\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(f(r)\cdot\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(f(r)\cdot\frac{z}{r}\right)$$

$$= f(r)\times\left[\frac{1}{r}+\frac{-x}{r^2}\times\frac{x}{r}\right]+\frac{x^2}{r^2}f'(r)+f(r)\left[\frac{1}{r}-\frac{y^2}{r^3}\right]$$

$$+\frac{y^2}{r^2}f'(r)+f(r)\left[\frac{1}{r}-\frac{z^2}{r^3}\right]+\frac{z^2}{r^2}f'(r)$$

$$= f(r)\left[\frac{3}{r}-\frac{r^2}{r^3}\right]+f'(r)\times\frac{1}{r^2}(x^2+y^2+z^2)$$

$$\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = f'(r)+f(r)\frac{2}{r}$$

To verify stated:-

From (1),

$$\nabla\cdot\left(\phi\vec{f}\right)=(\nabla\phi)\cdot\vec{f}+\phi\left(\nabla\cdot\vec{f}\right)\ \text{...(A)}$$

Here $\varphi = \frac{f(r)}{r}$, $\vec{f} = \vec{r}$

$$\therefore\ (\nabla\varphi)\cdot\vec{f} = \left(\nabla\frac{f(r)}{r}\right)\cdot\vec{r}$$

$$= \left\{\frac{\partial}{\partial x}\left(\frac{f(r)}{r}\right)\hat{i}+\frac{\partial}{\partial y}\left(\frac{f(r)}{r}\right)\hat{j}+\frac{\partial}{\partial y}\left(\frac{f(r)}{r}\right)\hat{k}\right\}\cdot\vec{r}$$

$$= \left[\left\{\frac{1}{r}\cdot f'(r).\frac{x}{r}+f(r)\cdot\frac{-1}{r^2}\times\frac{x}{r}\right\}\cdot\hat{i}+\left\{\frac{y}{r^2}f'(r)-f(r)\cdot\frac{y}{r^3}\right\}\hat{j}\right.$$

$$+ \left\{\frac{z}{r^2} f'(r) - f(r)\frac{z}{r^3}\right\}\hat{k}\Bigg]\cdot\vec{r} = \left\{\frac{f'(r)}{r^2}\vec{r} - \frac{f(r)}{r^3}\right\}\cdot\vec{r} = \frac{f'(r)}{r^2}\times r^2 - \frac{f(r)}{r^3}\cdot r^2$$

$$(\nabla\varphi)\cdot\vec{f} = f'(r) - \frac{f(r)}{r}$$

Now, $\phi\left(\nabla\cdot\vec{f}\right) = \frac{f(r)}{r}\left[\nabla\cdot\vec{r}\right] = \frac{f(r)}{r}\,[1 + 1 + 1]$

$$\phi\left(\nabla\cdot\vec{f}\right) = \frac{3}{r}f(r)$$

Now, $\therefore\ (\nabla\varphi)\cdot\vec{f} + \varphi\left(\nabla\cdot\vec{f}\right) = f'(r) + \frac{2}{r}f(r)$

$\Rightarrow\ \nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = f'(r) + \frac{2}{r}f(r)$ Verified

1. 5e 2018

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{div}\left[f(r)\vec{r}\right] = \nabla\cdot\left\{xf(r)\hat{i} + yf(r)\hat{j} + zf(r)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}\left(xf(r) + \frac{\partial}{\partial y}yf(r) + \frac{\partial}{\partial z}zf(r)\right)$$

$$= x\cdot f'(r)\frac{x}{r} + f(r) + \frac{y^2}{r}f'(r) + f(r) + \frac{z^2}{r}f'(r) + f(r)$$

$$= \frac{f'(r)}{r}\{x^2 + y^2 + z^2\} + 3f(r)$$

$$= \frac{f'(r)}{r}\times r^2 + 3f(r)$$

$$\text{div}\left[f(r)\vec{r}\right] = r\,f'(r) + 3f(r)$$

Now, $\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = \nabla\cdot\left(r^{-3}\vec{r}\right) = (\nabla r^{-3})\cdot\vec{r} + r^{-3}(\nabla\cdot\vec{r}) = \left\{-3r^{-4}\frac{\vec{r}}{r}\right\}\cdot\vec{r} + r^{-3}\times 3$

$= -3r^{-5}\left(\vec{r}\cdot\vec{r}\right) + 3r^{-3} = -3r^{-5}\cdot r^2 + 3r^{-3}$; $\quad \nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = 0$

2. 2013 CSE

$$\nabla^2(r^n) = \nabla\cdot\nabla(r^n) = \nabla\cdot\left\{nr^{n-1}\cdot\frac{\vec{r}}{r}\right\} = \nabla\cdot\left\{nr^{n-2}\,\vec{r}\right\} = n\nabla\cdot\left\{r^{n-2}\,\vec{r}\right\} = n\left[\left(\nabla r^{n-2}\right)\cdot\vec{r} + r^{n-2}\left(\nabla\cdot\vec{r}\right)\right]$$

$$= n\left[\left\{(n-2)r^{n-3}\times\frac{\vec{r}}{r}\right\}\cdot\vec{r} + 3r^{n-2}\right] = n\left[(n-2)r^{n-4}r^2 + 3r^{n-2}\right] = n[(n-2)\,r^{n-2} + 3r^{n-2}] = nr^{n-2}\,[n - 2 + 3]$$

$$\nabla^2(r^n) = n(n+1)\,r^{n-2}$$

Q4. 2010 IFOS

$$\nabla^2 f(r) = \nabla\cdot\nabla\,(f(r)) = \nabla\cdot\left\{f'(r)\cdot\frac{\vec{r}}{r}\right\} = \nabla\cdot\left\{\frac{f'(r)}{r}\cdot\vec{r}\right\} = \nabla\cdot\left\{\frac{f'(r)}{r}\right\}\cdot\vec{r} + \frac{f'(r)}{r}(\nabla\cdot\vec{r})$$

$$= \left[\left\{\frac{1}{r}f''(r) + f'(r)\times\frac{-1}{r^2}\right\}\times\frac{\vec{r}}{r}\right]\cdot\vec{r} + \frac{3}{r}f''(r)$$

{using $\nabla f(r) = f'(r)\times\frac{\vec{r}}{r}$}

$$\nabla^2 f(r) = \left\{\frac{1}{r}f''(r) + f'(r)\frac{-1}{r^2}\right\}\times\frac{r^2}{r} + \frac{3}{r}f'(r) = f''(r) - \frac{f'(r)}{r} + \frac{3}{r}f'(r)$$

$$\nabla^2 f(r) = f''(r) + \frac{2}{r}f(r)$$

CURL

7(a). $$\nabla.(\vec{u}\times\vec{v}) = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\vec{u}\times\vec{v}) = \hat{i}.\frac{\partial}{\partial x}(\vec{u}\times\vec{v}) + \hat{j}.\frac{\partial}{\partial y}(\vec{u}\times\vec{v}) + \hat{k}.\frac{\partial}{\partial z}(\vec{u}\times\vec{v})$$

$$= \hat{i}.\left\{\vec{u}\times\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial x}\times\vec{v}\right\} + \hat{j}.\left\{\vec{u}\times\frac{\partial\vec{v}}{\partial y} + \frac{\partial\vec{u}}{\partial y}\times\vec{v}\right\} + \hat{k}.\left\{\vec{u}\times\frac{\partial\vec{v}}{\partial z} + \frac{\partial\vec{u}}{\partial z}\times\vec{v}\right\}$$

$$= \left[\hat{i}\vec{u}\frac{\partial\vec{v}}{\partial x}\right] + \left[\hat{i}\frac{\partial\vec{u}}{\partial x}\vec{v}\right] + \left[\hat{j}\vec{u}\frac{\partial\vec{v}}{\partial y}\right] + \left[\hat{j}\frac{\partial u}{\partial y}\vec{v}\right] + \left[\hat{k}\vec{u}\frac{\partial v}{\partial z}\right] + \left[\hat{k}\frac{\partial\vec{u}}{\partial z}\vec{v}\right]$$

$$= \left[\vec{v}\,\hat{i}\frac{\partial\vec{u}}{\partial x}\right] + \left[\vec{v}\,\hat{j}\frac{\partial u}{\partial y}\right] + \left[\vec{v}\,\hat{k}\frac{\partial\vec{u}}{\partial z}\right] - \left[\vec{u}\,\hat{i}\frac{\partial\vec{v}}{\partial x}\right] - \left[\vec{u}\,\hat{j}\frac{\partial\vec{v}}{\partial y}\right] - \left[\vec{u}\,\hat{k}\frac{\partial\vec{v}}{\partial z}\right]$$

$$\left(\vec{f}\times\vec{g}\right) = \vec{f}div\,\vec{g} - \vec{g}div\,\vec{f} + (\vec{g}.\nabla)\vec{f} - (\vec{f}.\nabla)\vec{g}$$

$$= \vec{v}.\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{v} - \vec{u}.\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\times\vec{v}$$

$$\nabla.(\vec{u}\times\vec{v}) = \vec{v}.curl\ \vec{u} - \vec{u}.curl\,\vec{v}$$

Let $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_{31}\hat{k}$

$$\therefore\ \vec{v} = \vec{w}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zw_2 - yw_3 & xw_3 - zw_1 & xw_1 - zw_2 \end{vmatrix}$$

$$= 2w_1\hat{i} + 2w_2\hat{j} + 2w_3\hat{k}$$

Curl $\vec{v}$ = 2w

(ii) div $\vec{v} = \nabla.\vec{v} = \frac{\partial}{\partial x}(zw_2 - yw_3) + \frac{\partial}{\partial y}(xw_3 - zw_1) + \frac{\partial}{\partial z}(zw_1 - xw_2)$

div $\vec{v} = 0$

5(e). $\vec{F} = \left(y\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial y} \right)\hat{i} + \left(z\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial z} \right)\hat{j} + \left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \right)\hat{k}$

Now,

$$\vec{r} \times \nabla\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \hat{i}\left\{ y\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial y} \right\} + \hat{j}\left\{ z\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial z} \right\} + \hat{k}\left\{ x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \right\}$$

$$\vec{r}\nabla\phi = \vec{F} \Rightarrow \vec{F} - (\vec{r} \times \nabla\phi) = 0$$

Now, $\vec{F}.\vec{r} = xy\frac{\partial\phi}{\partial z} - xz\frac{\partial\phi}{\partial y} + yz\frac{\partial\phi}{\partial x} - xy\frac{\partial\phi}{\partial z} + xz\frac{\partial\phi}{\partial y} - yz\frac{\partial\phi}{\partial x}$

$$\vec{F}.\vec{r} = 0$$

$$\vec{F}.\nabla\phi = \frac{\partial\phi}{\partial x}y\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial x}z\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial y}z\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y}\times\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial z}x\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial z}y\frac{\partial\phi}{\partial x}$$

$$\vec{F}.\nabla\phi = 0 \therefore \vec{F} - (\vec{r} \times \nabla\phi) = \vec{F}.\vec{r} = \vec{F}.\nabla\phi = 0$$

Q2. $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Curl (curl $\vec{v}$) = $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla.\vec{v}) - (\nabla.\nabla)\vec{v}$ [Using $\vec{a} \times (\vec{b} \times c) = \vec{b}(\vec{a}.\vec{c}) - (\vec{a}.\vec{b})$]

$$= \nabla(div) - \nabla^2\vec{v}$$

Curl (curl $\vec{v}$) = grad (div $\vec{v}$) = $-\nabla^2\vec{\nabla}$

8. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left\{ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right\} + \hat{j}\left\{ \frac{\partial}{\partial z}F_1 - \frac{\partial}{\partial x}F_3 \right\} + \hat{k}\left\{ \frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 \right\}$$

$$\text{Curl (curl } \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial}{\partial x}F_3 & \frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 \end{vmatrix}$$

$$=\hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right\}+$$

$$+\hat{j}\left\{\frac{\partial^2 F_3}{\partial y\partial z}-\frac{\partial^2 F_2}{\partial z^2}-\left(\frac{\partial^2 F_2}{\partial x^2}-\frac{\partial^2 F_1}{\partial x\partial y}\right)\right\}+\hat{k}\left\{\frac{\partial^2 F_1}{\partial x\partial z}-\frac{\partial^2 F_3}{\partial x^2}-\left(\frac{\partial^2 F_3}{\partial y^2}-\frac{\partial^2 F_2}{\partial y\partial z}\right)\right\}$$

curl (curl $\vec{F}$) $=\hat{i}\left\{\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}\right\}\hat{j}\left\{\frac{\partial^2 F_2}{\partial y\partial z}+\frac{\partial^2 F_1}{\partial x\partial y}\right\}+\hat{k}\left\{\frac{\partial^2 F_1}{\partial x\partial z}+\frac{\partial^2 F_2}{\partial y\partial z}\right\}$

$$-\left[\hat{i}\left\{\frac{\partial^2 F_1}{\partial^2 y}+\frac{\partial^2 F_1}{\partial z^2}\right\}\hat{j}\left\{\frac{\partial^2 F_2}{\partial z^2}+\frac{\partial^2 F_2}{\partial x^2}\right\}+\hat{k}\left\{\frac{\partial^2 F_3}{\partial x^2}+\frac{\partial^2 F_3}{\partial y^2}\right\}\right]$$

$$i\frac{\partial}{\partial x}\left[\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right]+\hat{j}\frac{\partial}{\partial y}\left[\frac{\partial F_3}{\partial z}+\frac{\partial F_1}{\partial x}\right]+\hat{k}\frac{\partial}{\partial z}\left[\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right]$$

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right)$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{k}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}.\left(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right)-\nabla^2\vec{F}$$

$$=\nabla\left(div\vec{F}\right)-\nabla^2\vec{F}$$

Curl (curl $\vec{F}$) = grad (div $\vec{F}$) – $\nabla^2\vec{F}$

9. Given $u\vec{f}$ = grad v.

$\therefore$ curl $\left(u\vec{f}\right)=0\Rightarrow\nabla\times\left(u\vec{f}\right)=0$

$\Rightarrow\left(\nabla\times\vec{f}\,'\right)u+\nabla a\times\vec{f}=0$ [using $\nabla\times\left(\phi\vec{A}\right)$ = $\left(\nabla\phi\right)\times\vec{A}+\phi\left(\nabla\times\vec{A}\right)$]

$\Rightarrow u\vec{f}.\left(\nabla\times\vec{f}\right)+\vec{f}.\left(\nabla+\vec{f}\right)=0$

$\Rightarrow u\vec{f}$ curl $\vec{f}+0=0$ $\left[\because\vec{a}.\left(\vec{b}\times\vec{a}\right)=0\right]$

$\Rightarrow\vec{f}$ curl $\vec{f}$ = 0

10. (1) $\nabla\times\left(r^n\vec{r}\right)=\left(\nabla r^n\right)\times\vec{r}+r^n\left(\nabla\times\vec{r}\right)$

$$=\left(nr^{n-1}\frac{\vec{r}}{r}\right)\times\vec{r}+r^n\times\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\x&y&z\end{vmatrix}=nr^{n-1}\left(\vec{r}\times\vec{r}\right)+r^n\times0=0+0$$

curl $\left(r^n\vec{r}\right)=0\therefore$ $r^n\vec{r}$ is irrotational for any antitreaty value of n

(2) $\nabla.\left(r^n\vec{r}\right)=\nabla r^n.\vec{r}+r^n\left(\nabla.\vec{r}\right)=r\left\{nr^{n-1}.\frac{\vec{r}}{|r|}\right\}.\vec{r}+r^n\times 3=nr^{n-2}\left(\vec{r}.\vec{r}\right)+3r^n=nr^n+3r^n$

$\nabla.\left(r^n\vec{r}\right)=\left(n+3\right)r^n\ \therefore\ r^n\vec{r}$ is solo dial only if n = -3.

11. Let $\vec{f}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$, $\quad \vec{g}=g_1\hat{i}+g_2\hat{j}+g_3\hat{k}$

Now,

$$f\times\vec{g}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ f_1 & f_2 & f_3\\ g_1 & g_2 & g_3\end{vmatrix}=\left(f_2g_3-f_3g_2\right)\hat{i}+\left(f_3g_1-f_1g_3\right)\hat{j}+\left(f_1g_2-f_2g_1\right)\hat{k}$$

$$\text{curl}\left(\vec{f}\times\vec{g}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_2g_3-f_3g_2 & f_ag_2-f_1g_3 & f_1g_2-f_2g_1\end{vmatrix}$$

$$=\hat{i}\left\{f_1\frac{\partial g_2}{\partial y}+\frac{\partial f_1}{\partial y}g_2-f_2\frac{\partial g_1}{\partial y}-\frac{\partial f_2}{\partial y}g_1\right\}-f_3\frac{\partial g_1}{\partial z}-\frac{\partial g_3}{\partial z}g_1+f_1\frac{\partial g_3}{\partial z}+\frac{\partial f_1}{\partial z}g_3 \qquad \text{.....(A)}$$

$+\hat{j}\{-----\}++\hat{k}\{-----\}$

Now, calculating the $\hat{i}$ compliment of RHS, we get

$f_1\left(\nabla.\vec{g}\right)-g_1\left(\nabla.\vec{f}\right)+\left(\vec{g}.\nabla\right)f_1-\left(\vec{f}.\nabla\right)g_1$

$$=f_1\left\{\frac{\partial g_1}{\partial x}+\frac{\partial g_2}{\partial y}+\frac{\partial g_3}{\partial z}\right\}-g_1\left\{\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right\}+\left\{g_1\frac{\partial f_1}{\partial x}+g_2\frac{\partial f_1}{\partial y}+g_3\frac{\partial f_1}{\partial z}\right\}-\left(f_1\frac{\partial g_1}{\partial x}+f_2\frac{\partial g_1}{\partial y}+f_3\frac{\partial g_1}{\partial z}\right)$$

$$=f_1\frac{\partial g_2}{\partial y}+f_1\frac{\partial g_3}{\partial z}-\frac{\partial f_2}{\partial y}g_1-\frac{\partial f_3}{\partial z}g_1+\frac{\partial f_1}{\partial y}g_2+\frac{\partial f_1}{\partial z}g_3$$

$$=f_1\frac{\partial g_2}{\partial y}+f_1\frac{\partial g_3}{\partial z}-\frac{\partial f_2}{\partial y}g_1-\frac{\partial f_3}{\partial z}g_1+\frac{\partial f_1}{\partial y}g_2+\frac{\partial f_1}{\partial z}g_3-f_2\frac{\partial g_1}{\partial y}-f_3\frac{\partial g_1}{\partial y}-f_3\frac{\partial g_1}{\partial z} \qquad \text{.....(B)}$$

From (A) & (B)

The coefficients of $\hat{i}$ are same for both LHS & RHS

Similarly, the coeff. of $\hat{j}$ & $\hat{k}$ can also be shown equal.

$\therefore$ The identity is proved.

Now, $\vec{f}=x\hat{i}+z\hat{j}+y\hat{k}$, $\qquad \vec{g}=y\hat{i}+z\hat{k}$.

$$\vec{f}_1\times\vec{g}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & z & y\\ y & 0 & z\end{vmatrix}=z^2\hat{i}\left(y^2-xz\right)\hat{j}-yz\hat{k}$$

$$\therefore \text{ curl } \left(\vec{f} \times \vec{g}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 - xz & -yz \end{vmatrix}$$

curl $\left(\vec{f} \times \vec{g}\right) = (x - z)\hat{i} + 2z\hat{j} + (-z)\hat{k}$ .....(A)

div $\vec{g} = 1,$ div $\vec{f} = 1$

$$\left(\vec{g}.\vec{\nabla}\right)\vec{f} = \left(y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right)\left(x\hat{i} + z\hat{j} + y\hat{k}\right) = y\hat{i} + z\hat{j}$$

$$\left(\vec{f}.\vec{\nabla}\right)\vec{g} = \left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\left(y\hat{i} + z\hat{k}\right) = z\hat{i} + y\hat{k}$$

$\therefore$ RHS = $\vec{f}.1 - \vec{g}.1 + y\hat{i} + z\hat{j} - z\hat{i} - y\hat{k}$

$= x\hat{i} + z\hat{j} + y\hat{k} - y\hat{i} - z\hat{k} + y\hat{i} + z\hat{j} - z\hat{i} - y\hat{k}$

RHS = $(x - z)\hat{i} + 2z\hat{j} - z\hat{k}$ .....(B)

From (A) & (B):

curl $\left(\vec{f} \times \vec{g}\right) = \vec{f} div\, \vec{g} - \vec{g} div\, \vec{f} + (\vec{g}.\nabla)\vec{f} - \left(\vec{f}.\nabla\right)\vec{g}$

Line Integral & Green's Theorem

Part-1

Ex 1. Show that $\vec{F}=\left(2xy+z^3\right)\hat{i}+x^2\hat{j}+3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$. **[6c 2018 IFoS]**

Sol. curl $\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+z^3 & x^2 & 3xz^2\end{vmatrix}=\hat{i}(0-0)+\left(3z^2-3z^2\right)\hat{j}+(2x-2x)\hat{k}$

Curl $\vec{F}=\vec{0}\;\therefore\;\vec{F}$ is a conservative force **.** So $\exists$ a scalar potential ϕ s.t $\vec{F}$ = grad ϕ

$$\Rightarrow\left(2xy+z^3\right)\hat{i}+x^2\hat{j}+3xz^2\hat{k}=\frac{\partial\phi}{\partial y}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$\therefore\;\dfrac{\partial\phi}{\partial x}=2xy+z^3$ .....(1), $\quad\dfrac{\partial\phi}{\partial y}=3x^2$ .....(2), $\quad\dfrac{\partial\phi}{\partial z}=3xz^2$ .....(3)

On integrating (1), (2) & (3) w.r.t *x*, *y* & *z* respectively we get

$$\left.\begin{aligned}\phi(x,y,z)&=x^2y+z^3x+f(y,z)\\ \phi(x,y,z)&=x^2y+g(x,z)\\ \phi(x,y,z)&=xz^3+h(x,y)\end{aligned}\right\}$$ where *f*(*x*, *y*, *z*), *g*(*x*, *z*) & *h*(*x*, *y*) are into constants

Appropriately choosing the values as, $f(y,z)=0,\;\; g(x,z)=xz^3,\;\; h(x,y)=x^2y$

We have, $\phi(x,y,z)=x^2y+xz^3$ .....(A)

Now, Work done by a particle of unit from P(1, -2, 1) to Q(3, 1, 4) is given by

$$W=\int_P^Q\vec{F}.d\vec{r}\quad;\quad W=\int_P^Q(\nabla\phi).d\vec{r}$$

$$W=\int_{(1,-2,1)}^{(3,1,4)}\phi=\left[\phi(x,y,z)\right]_{1,-2,1}^{3,1,4}=\phi(3,1,4)-\phi(1,-2,1)=9+3\times64-\{-2+1\}$$ = 202 units.

Ex 2. For the vector $\bar{A}=\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\bar{A}=\nabla\phi$.

$$\vec{A}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$$

$$\text{Curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$$

$$= i\left\{\frac{-2zy}{\left(x^2+y^2+z^2\right)^2} + \frac{2zy}{\left(x^2+y^2+z^2\right)^2}\right\} + \hat{j}\left\{\frac{-2zx}{\left(x^2+y^2+z^2\right)^2} + \frac{2xz}{\left(x^2+y^2+z^2\right)^2}\right\}$$

$$+\hat{k}\left\{\frac{-2xy}{\left(x^2+y^2+z^2\right)^2} + \frac{2xy}{\left(x^2+y^2+z^2\right)^2}\right\}$$

Curl $\vec{A} = \vec{0} \therefore \vec{A}$ is irrotational. So, $\exists$ a scalar potential ϕ s.t $\vec{A} = \nabla\phi$

$\therefore \frac{\partial\phi}{\partial x} = \frac{x}{x^2+y^2+z^2}$ .....(1) ; $\frac{\partial\phi}{\partial y} = \frac{y}{x^2+y^2+z^2}$ .....(2) ; $\frac{\partial\phi}{\partial z} = \frac{z}{x^2+y^2+z^2}$ .....(3)

Integrating (1), (2) & (3) w.r.t *x*, *y* & *z* respectively

$$\left.\begin{aligned} \phi(x,y,z) &= \frac{1}{2}\log\left(x^2+y^2+z^2\right) + f(y,z) \\ \phi(x,y,z) &= \frac{1}{2}\log\left(x^2+y^2+z^2\right) + g(x,y) \\ \phi(x,y,z) &- \frac{1}{2}\log\left(x^2+y^2+z^2\right) + h(x,y) \end{aligned}\right\}$$ where *f*(*y*, *z*), *g*(*x*, *z*) & *h*(*x*, *y*) are integ. constants

Taking Appropriate values of $f(y,z) = g(x,z) = h(x,y) = 0 \ \therefore \ \phi(x,y,z) = \frac{1}{2}\log\left(x^2+y^2+z^2\right)$

Q1. For what value of a, b, c is the vector field $\bar{V} = (4x - 3y + az)\hat{i} + (bx + 3y + 5z)\hat{j} + (4x + cy + 3z)\hat{k}$ irrotational? Hence, express $\bar{V}$ as the gradient of a scalar function ϕ. Determine ϕ.

Hint: Put curl $\vec{V} = (c-5)\hat{i} - (4-a)\hat{j} + (b+3)\hat{k}$ =0, get $c = 5,\ a = 4,\ b = -3$.

$\therefore \vec{V} = (-4x - 3y + 4z)\hat{i} + (-3x + 3y + 5z)\hat{j} + (4x + 5y + 3z)\hat{k}$

$\vec{V} = \nabla\phi \Rightarrow \frac{\partial\phi}{\partial x} = (-4x - 3y + 4z), \frac{\partial\phi}{\partial y} = (-3x + 3y + 5z), \frac{\partial\phi}{\partial z} = (4x + 5y + 3z)$

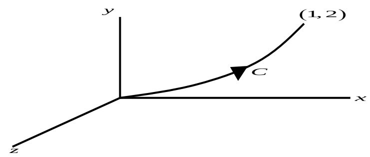
On integrating; and comparing three expressions for ϕ after taking integration constants appropriately as

$f(y,z) = \frac{3}{2}y^2 + 5yz + \frac{3}{2}z^2,\ g(z,x) = -2x^2 + 4xz + \frac{3}{2}z^2,\ h(x,y) = -2x^2 - 3xy + \frac{3}{2}y^2$, we get

Scalar potential as; $\phi = -2x^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 - 3xy + 4xz + 5yz$

Part 2: LINE INTEGRAL

Ex 1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ determine the value of $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0,0)$ to $(1,2)$.



Solution.

The curve lies in xy plane, so $z = 0$. z can never be taken as independent variable z is a dependent variable. Now, out of x and y, and one variable can be taken as independent.

- Suppose x is taken as independent variable

$y = 2x^2, dy = 4xdx$, $\vec{F}\cdot d\vec{r} = 3xydx - y^2dy = 6x^3dx - 4x^4\cdot 4xdx = \left(6x^3 - 16x^5\right)dx$

So, $\int_C \vec{f}\cdot d\vec{r} = \int_0^1 \left(6x^3 - 16x^5\right)dx = 6\left.\frac{x^4}{4}\right|_0^1 - 16\left.\frac{x^6}{6}\right|_0^1 = -\frac{7}{6}$

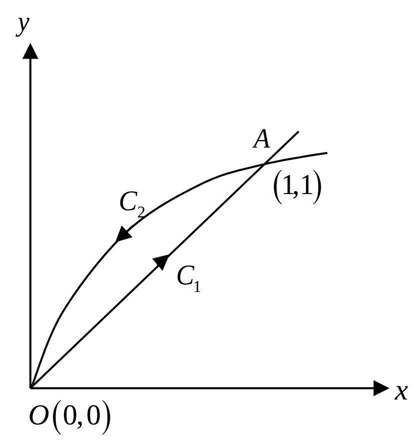
- If y is taken as independent variable then x can be expressed in terms of y as $x = \sqrt{\frac{y}{2}}$

$dx = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y}}dy$. So, $\vec{f}\cdot d\vec{r} = 3xydx - y^2dy = 3y\sqrt{\frac{y}{2}}\cdot\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y}}dy - y^2dy = \left(\frac{3}{4}y - y^2\right)dy$

So, $\int \vec{f}\cdot d\vec{r}$ reduces to a definite integral $= \int_0^2\left(\frac{3}{4}y - y^2\right)dy = \frac{3}{8}y^2 - \left.\frac{y^3}{3}\right|_0^2 = -\frac{7}{6}$

Ex 2. Find the value of $\int_C \left[\left(x + y^2\right)dx + \left(x^2 - y\right)dy\right]$ taken in the counter-clockwise sense along the closed curve C formed by straight line $y = x$ and curve $y^3 = x^2$.

Solution.



The curve C consists of chord OA and curved part AO as shown in figure.

Equation of OA is $y = x$ and curved part is $y^3 = x^2$.

Along chord OA, x can be taken as independent variable and $y = x$.

$\vec{F}\cdot d\vec{r} = \left(x + y^2\right)dx + \left(x^2 - y\right)dy = \left(x + x^2\right)dx + \left(x^2 - x\right)dx = 2x^2dx$

Along OA, x varies from 0 to 1. On curved part AO, let y be taken as independent variable & dependent variable x can be put as ; $x = y^{3/2}, dx = \frac{3}{2} y^{1/2} dy$.

$$d\vec{r} = \left(x + y^2\right)dx + \left(x^2 - y\right)dy = \left(y^{3/2} + y^2\right)\frac{3}{2}y^{1/2}dy + \left(y^3 - y\right)dy = \left(y^3 + \frac{3}{2}y^{5/2} + \frac{3}{2}y^2 - y\right)dy$$

y varies from 1 to 0.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 2x^2 dx + \int_1^0 \left(y^3 + \frac{3}{2}y^{5/2} + \frac{3}{2}y^2 - y\right)dy$$

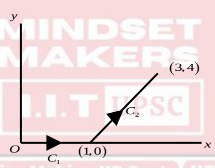
$$= \frac{2}{3}x^3\Big|_0^1 + \frac{1}{4}y^4 + \frac{3}{7}y^{7/2} + \frac{1}{2}y^3 - \frac{1}{2}y^2\Big|_1^0 = -\frac{1}{84}$$

Note: If the integral is carried out in clockwise direction. The answer will differ only in sign.

$\oint_C \vec{F}\cdot d\vec{r}$ in clockwise direction $= \frac{1}{84}$.

Ex 3. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x+2y)\hat{i} + (2y-x)\hat{j}$ and C is curve in xy plane consisting of the straight lines from $(0,0)$ to $(1,0)$ and then to $(3,4)$.

Solution.



C_1 is straight line from $(0,0)$ to $(1,0)$ i.e. $y=0$, C_2 is straight line from $(1,0)$ to $(3,4)$.

i.e. $y - 0 = \left(\frac{4-0}{3-1}\right)\cdot(x-1)$ or, $y = 2x - 2$

So, along $C_1, y = 0, dy = 0$ (x is an independent variable); $\vec{F}\cdot d\vec{r} = xdx$

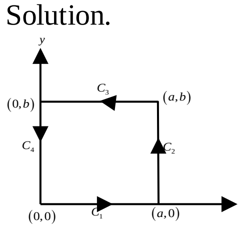
Along $C_2; y = 2x-2, dy = 2dx$ (let us take x as indepe.) $\vec{F}\cdot d\vec{r} = (x+2y)dx + (2y-x)dy$

on C_2, $\vec{F}\cdot d\vec{r} = \left(x + 2(2x-2)\right)dx + \left(2(2x-2) - x\right)\cdot 2dx = (11x - 12)dx$

So, $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 xdx + \int_1^3 (11x-12)\,dx = \frac{x^2}{2}\Big|_0^1 + \left(\frac{11}{2}x^2 - 12x\right)\Big|_1^3 = 20.5$

Ex 4. Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}$, where curve C is a rectangle in the xy plane bounded by $y = 0, x = a, y = b, x = 0$.

Solution.



The curve C as shown in figure consists of four pieces of smooth curves C_1, C_2, C_3 & C_4.

$\vec{F}\cdot d\vec{r} = \left(x^2+y^2\right)dx - 2xydy$

On $C_1, y=0, dy=0, \vec{F}\cdot d\vec{r} = x^2dx$, On $C_2, x=a, dx=0, \vec{F}\cdot d\vec{r} = -2aydy$

On $C_3, y=b, dy=0, \vec{F}\cdot d\vec{r} = \left(x^2+b^2\right)dx$, On $C_4, x=0, dx=0, \vec{F}\cdot d\vec{r} = 0$

$$\oint \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} + \int_{C_4}\vec{F}\cdot d\vec{r} = \int_0^a x^2dx + \int_0^b -2aydy + \int_a^0 \left(x^2+b^2\right)dx + \int_b^0 0\cdot dy$$

$$= \frac{x^3}{3}\Big|_0^a + \left[-ay^2\right]_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^0 = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2$$

Ex 5. Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1, y = 2t^2, z = t^3$ from $t=1$ to $t=2$.

Solution.

On curve C, the coordinates x, y, z are expressed in terms of parameter t.

$x = t+1, dx = 2tdt$; $y = 2t^2, dy = 4tdt$; $z = t, dz = 3t^2dt$; t varies from $t=1$ to $t=2$.

$\vec{F}\cdot d\vec{r} = 3xydx - 5zdy + 10xdz = 3\left(t^2+1\right)\cdot 2t^2\cdot 2tdt - 5t^3 4tdt + 10\left(t^2+1\right)\cdot 3t^2dt$

$= \left(12t^5 + 10t^4 + 12t^3 + 30t^2\right)dt$

So, $W = \int_C \vec{F}\cdot d\vec{r} = \int_1^2 \left(12t^5+10t^4+12t^3+30t^2\right)dt = \left(12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right)\Bigg|_1^2 = 303$

Ex 6. If $\vec{F} = \left(3x^2+6y\right)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int \vec{F}\cdot d\vec{r}$ where C is a straight line joining $(0,0,0)$ to $(1,1,1)$.

Solution. Equation of straight line joining $(0,0,0)$ to $(1,1,1)$ is given by $\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$, where t is parameter.

In parametric form equation of curve is given by

$x = t \Rightarrow dx = dt$; $y = t \Rightarrow dy = dt$; $z = t \Rightarrow dz = dt$; t varies from 0 to 1.

$\vec{F}\cdot d\vec{r} = \left(3x^2+6y\right)dx - 14yzdy + 20xz^2dz = \left(20t^3 - 11t^2 + 6t\right)dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(20t^3 - 11t^2 + 6t\right)dt = \left(5t^4 - \frac{11}{3}t^3 + 3t^2\right)\Bigg|_0^1 = \frac{13}{3}$$

Ex 7. Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \frac{y^2}{x^2+y^2}\hat{i} - \frac{x^2}{x^2+y^2}\hat{j}$, where C is the semi-circle $r = \sqrt{a^2-x^2}$.

Solution. The curve C is the semi-circle; $y = \sqrt{a^2-x^2}$

Parametric form: $x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta$; $y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$; θ varies from 0 to π

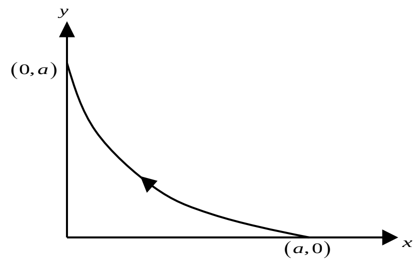
$$\vec{F}\cdot d\vec{r} = \frac{y^2dx - x^2dy}{x^2+y^2} = \frac{a^2\sin^2\theta(-a\sin\theta)d\theta - \left(a^2\cos^2\theta\right)\cdot a\cos\theta d\theta}{a^2} = -a\left(\sin^3\theta + \cos^3\theta\right)d\theta$$

$$\int_C \vec{F}\cdot d\vec{r} = -a\int_0^{\pi}\left(\sin^3\theta+\cos^3\theta\right)d\theta = -a\int_0^{\pi}\sin^3\theta d\theta - a\int_0^{\pi}\cos^3\theta d\theta$$

$$= -2a\int_0^{\pi/2}\sin^3\theta d\theta - 0\left(\text{Since, }\int_0^{\pi}\cos^3\theta d\theta = 0\right) = -2a\frac{\sqrt{2}\sqrt{1/2}}{2\sqrt{5/2}} = -\frac{4a}{3}$$

Ex 8. Evaluate $\int_C \frac{x^2dy - y^2dx}{x^{5/3}+y^{5/3}}$ where C is the quarter of the astroid $x = a\cos^3 t, y = a\sin^3 t$ from the point $(a,0)$ to the point $(0,a)$.

Solution.



$x = a\cos^3 t \Rightarrow dx = -3a\cos^2 t\sin t dt$; $y = a\sin^3 t \Rightarrow dy = 3a\sin^2 t\cos t dt$

(x, y) varies from $(a,0)$ to $(0,a)$; So, t varies from 0 to $\pi/2$.

The integrand $\frac{x^2dy - y^2dx}{x^{5/3}+y^{5/3}} = \frac{a^2\cos^6 t\left(3a\sin^2 t\cos t\right)dt - \left(a^2\sin^6 t\right)\cdot\left(-3a\cos^3 t\sin t\right)dt}{a^{5/3}\left(\cos^5 t+\sin^5 t\right)}$

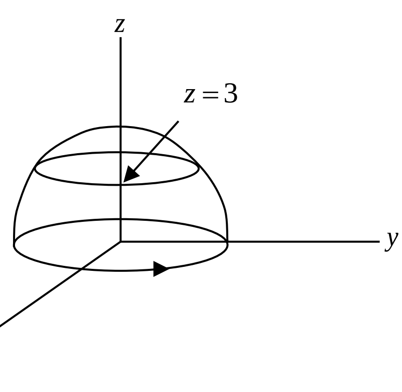
$= 3a^{4/3}\sin^3 t\cos^2 t dt$

The line integral reduces to

$$3a^{4/3}\int_0^{\pi/2}\sin^2 t\cos^2 t dt = 3a^{4/3}\frac{\lceil 3/2\,\lceil 3/2}{2\lceil 3} = \frac{3\pi a^{4/3}}{16}$$

Ex 9. Find the circulation of the field $\vec{F} = -x^2y\hat{i} + xy^2\hat{j} + \left(y^3 - x^3\right)\hat{k}$ around the curve C, where C is the intersection of the sphere $x^2+y^2+z^2 = 25$ and the plane $z = 3$. The orientation of the curve C is counterclockwise when viewed from above.

Solution.



$\vec{F} = -x^2y\hat{i} + xy^2\hat{j} + \left(y^3 - x^3\right)\hat{k}$

C is the curve of intersection of surfaces; $x^2+y^2+z^2 = 25, z = 3$. So, $x^2+y^2 = 16$

$$\vec F\cdot d\vec r = x^2ydx + xy^2dy + \left(y^3 - x^3\right)dz$$

For curve C, $z=3, dz=0$. So $\oint \vec F\cdot d\vec r = \int -x^2ydx + xy^2dy$

Let $x=4\cos\theta, y=4\sin\theta$

$$\oint \vec F\cdot d\vec r = \int_0^{2\pi}\left(256\cos^2\theta\sin^2\theta d\theta + 256\cos^2\theta\sin^2\theta\right)d\theta = 512\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta$$

$$= 512\times 4\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta \; = 2048\frac{\Gamma\left(3/2\right)\Gamma\left(3/2\right)}{2\Gamma 3} = 128\pi$$

Ex 10. If $\phi = 2x^2yz, \vec F = xy\hat i - z^2y\hat j + x^2\hat k$ and C is the curve $x=2t, y=t^2, z=t^3$ from $t=0$ and $t=1$. Evaluate the line integrals (a) $\int_C \phi d\vec r$ (b) $\int_C \vec F\times d\vec r$.

Solution. (a) Along $C, \phi = 2x^2yz = 2(2t)^2\cdot t^2\cdot t^3 = 8t^7$, $\vec r = 2t\hat i + t^2\hat j + t^3\hat k$; $d\vec r = \left(2\hat i + 2t\hat j + 3t^2\hat k\right)dt$

$$\int_C \phi d\vec r = \int_0^1 8t^7\left(2\hat i + 2t\hat j + 3t^2\hat k\right)dt = \hat i\int_0^1 16t^7dt + \hat j\int_0^1 16t^8dt + \hat k\int_0^1 24t^9dt = 2\hat i + \frac{16}{9}\hat j + \frac{12}{5}\hat k$$

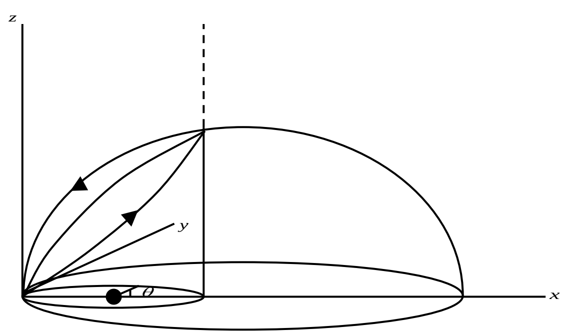
(b) Along C, $\vec F = xy\hat i - z^2y\hat j + x^2\hat k = 2t^3\hat i - t^8\hat j + 4t^2\hat k$

$$\vec F\times d\vec r = \begin{vmatrix}\hat i & \hat j & \hat k\\ 2t^3 & -t^8 & 4t^2\\ 2 & 2t & 3t^2\end{vmatrix} = \left(-3t^{10} - 8t^3\right)\hat i + \left(8t^2 - 6t^3\right)\hat j + \left(4t^4 + 2t^8\right)\hat k$$

$$\int_C \vec F\times d\vec r = \hat i\int_0^1\left(-3t^{10} - 8t^3\right)dt + \hat j\int_0^1\left(8t^2 - 6t^3\right)dt + \hat k\int_0^1\left(4t^4 + 2t^8\right)dt = -\frac{47}{11}\hat i + \frac{5}{3}\hat j + \frac{46}{45}\hat k$$

Ex 10. Evaluate $\int_C\left(y^2+z^2\right)dx + \left(z^2+x^2\right)dy + \left(x^2+y^2\right)dz$ where C is the part for which $z\ge 0$ of the intersection of the surfaces $x^2+y^2+z^2=4x, x^2+y^2=2x$ and curve beings at the origin and runs at first in the positive octant.

Solution.



The C is the intersection of the two surfaces $(x-1)^2 + y^2 = 1$ (Cylinder) & $z^2 = 2x$ (Parabolic cylinder)

The parametric equation of C is given as

$$x = 1+\cos\theta = 2\cos^2\theta/2;\; dx = -2\sin\theta/2\cos\theta/2\, d\theta$$

$$y=\sin\theta=2\sin{}^{\theta}\!/_{2}\cos{}^{\theta}\!/_{2};\ dy=\cos\theta d\theta,\ z=\sqrt{2(1+\cos\theta)}=2\cos{}^{\theta}\!/_{2};\ dz=-\sin{}^{\theta}\!/_{2}\,d\theta$$

$$\left(y^2+z^2\right)dx+\left(z^2+x^2\right)dy+\left(x^2+y^2\right)dz$$

$$=\left(4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}+4\cos^2\frac{\theta}{2}\right)\left(-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta\right)+\left(4\cos^2\frac{\theta}{2}+4\cos^4\frac{\theta}{2}\right)\cos\theta d\theta$$

$$+\left(4\cos^4\frac{\theta}{2}+4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)d\theta$$

So, the line integral becomes

$$\int_{-\pi}^{\pi}\left(y^2+z^2\right)dx+\left(z^2+x^2\right)dy+\left(x^2+y^2\right)dz$$

$$=-\int_{-\pi}^{\pi}4\left(\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)\sin\theta d\theta+4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\cos\theta d\theta-4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

The first and third integral vanishes since, the integrand is an odd function

So, integral reduces to $I=4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\cos\theta d\theta=\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}\right)\left(2+2\cos^2\frac{\theta}{2}\right)\cos\theta d\theta$

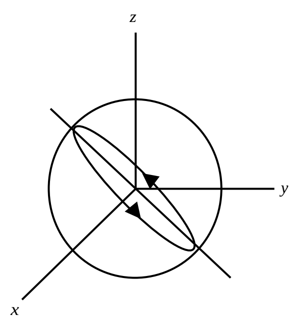
$$=\int_{-\pi}^{\pi}(1+\cos\theta)\cdot(3+\cos\theta)\cos\theta d\theta=\int_{-\pi}^{\pi}\cos^3\theta+4\cos^2\theta+3\cos\theta d\theta$$

$$=\int_{-\pi}^{\pi}\cos^3\theta d\theta+4\int_{-\pi}^{\pi}\cos^2\theta d\theta+3\int_{-\pi}^{\pi}\cos\theta d\theta=2\int_{0}^{\pi}\cos^3\theta d\theta+16\int_{0}^{\pi/2}\cos^2\theta d\theta+6\int_{0}^{\pi/2}\cos\theta d\theta$$

$$=0+16\cdot\frac{\pi}{4}+0=4\pi$$

Ex 11. Find the integral $\int_C (y+z)dx+(z+x)dy+(x+y)dz$ where C is the circle $x^2+y^2+z^2=a^2, x+y+z=0$.

Solution.



$$\oint(y+z)dx+(z+x)dy+(x+y)dz=\oint ydx+zdx+zdy+xdy+xdz+ydz=\oint d\left(xy+yz+zx\right)=0$$

The integral is an exact differential. i.e. $\vec{F}=\vec{\nabla}\phi$. So, $\oint\vec{F}\cdot d\vec{r}=0$.

Ex 12. Evaluate $\int_C x^2y^3dx+dy+zdz$ where C is the circle $x^2+y^2=R^2, z=0$.

Solution. $x=R\cos\theta, dx=R\sin\theta d\theta;\ y=R\sin\theta, dy=R\cos\theta d\theta$

$$I = \oint \left(x^2 y^3 dx + dy + zdz\right) = \int_0^{2\pi} R^2\cos^2\theta \cdot R^3 \sin^3\theta\left(-R\sin\theta\right)d\theta + \oint d\left(y + \frac{z^2}{2}\right)$$

$$= -R^6\int_0^{2\pi}\cos^2\theta\sin^4\theta d\theta + 0 = -4R^6\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta = -4R^6\frac{\left|\frac{5}{2}\right|\frac{3}{2}}{2\left|4\right.} = -\frac{\pi R^6}{8}$$

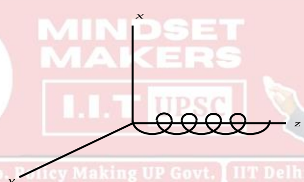
Ex 13. Evaluate $\int_C \vec{A}\cdot d\vec{r}$ along the curve $x^2 + y^2 = 1, z = 1$ from $(0,1,1)$ to $(1,0,1)$ if $\vec{A} = (yz + 2x)\hat{i} + xz\hat{j} + (xy + 2z)\hat{k}$.

Solution. The curve C is the circle of radius 1 with the centre at $(0,0,1)$ lying in a plane parallel to xy plane. $\vec{F}\cdot d\vec{r} = (yz + 2x)dx + xzdy + (xy + 2z)dz = d\left(xyz + x^2 + z^2\right)$

$\vec{F}\cdot d\vec{r}$ is an exact differential. So, line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of curve joining initial and final points; $\int_C \vec{F}\cdot d\vec{r} = \int d\left(xyz + x^2 + z^2\right) = \left[xyz + x^2 + z^2\right]_{(0,1,1)}^{(1,0,1)} = 1$

Ex 14. Evaluate $\int_C yzdx + zxdy + xydz$ where C is the arc of curve $x = b\cos t, y = b\sin t, z = \frac{at}{2\pi}$ from the point it intersects $z = 0$ to the point it intersects $z = a$.

Solution.



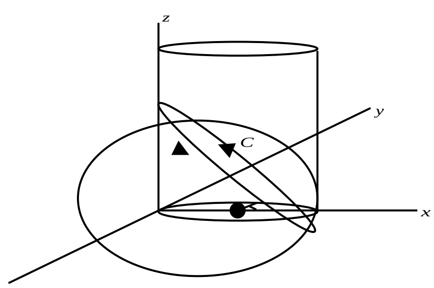
The curve C is a spiral given by $x = b\cos t, y = b\sin t, z = \frac{at}{2\pi}$

Since, z varies from $z = 0$ to $z = a$, hence, t varies from 0 to 2π

The line integral $\int_C (yzdx + zxdy + xydz) = \int_C d(xyz) = [xyz] = \left[\frac{ab^2}{2\pi} t\sin t\cos t\right]_0^{2\pi} = 0$

Ex 15. Evaluate $\int_C y^2dx + z^2dy + x^2dz$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax\,(a > 0, z \ge 0)$ integrated anticlockwise when viewed from the origin.

Solution.



C is the curve of intersection of $x^2 + y^2 = ax \Rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}, x^2 + y^2 + z^2 = a^2$

$\Rightarrow z^2 + ax = a^2 \Rightarrow z^2 = -a(x-a)$

Let $x = \frac{a}{2} + \frac{a}{2}\cos\theta \quad \Rightarrow dx = -\frac{a}{2}\sin\theta d\theta$; $y = \frac{a}{2}\sin\theta \quad \Rightarrow dy = \frac{a}{2}\cos\theta d\theta$

$z^2 = a(a-x) = a\left(\frac{a}{2} - \frac{a}{2}\cos\theta\right) = a^2\sin^2\frac{\theta}{2}$. So, $z = a\sin\frac{\theta}{2} \Rightarrow dz = \frac{a}{2}\cos\frac{\theta}{2}d\theta$; θ varies from 0 to 2π.

The line integral $I = \int y^2 dx + z^2 dy + x^2 dz$

$$= \int_0^{2\pi} \frac{a^2}{4}\sin^2\theta\left(-\frac{a}{2}\sin\theta d\theta\right)d\theta + \int_0^{2\pi}\frac{a^2}{2}(1-\cos\theta)\frac{a}{2}\cos\theta d\theta + \int_0^{2\pi}\frac{a^2}{4}(1+\cos\theta)^2\frac{a}{2}\cos\theta/2\, d\theta$$

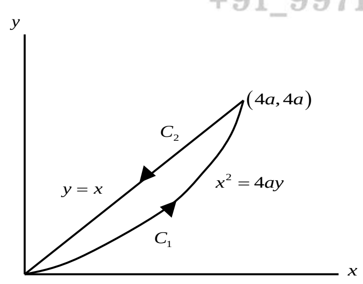
$$= 0 + \frac{a^3}{4}\int_0^{2\pi}\left(\cos\theta - \cos^2\theta\right)d\theta + \frac{a^3}{2}\int_0^{2\pi}\cos^5\theta/2\, d\theta = \frac{a^3}{2}\int_0^{\pi}\cos\theta d\theta - \frac{a^3}{2}\int_0^{\pi}\cos^2\theta d\theta + \frac{a^3}{2}\int_0^{2\pi}\cos^5\frac{\theta}{2}d\theta$$

$$= -a^3\int_0^{\pi/2}\cos^2\theta d\theta + a^3\int_0^{\pi}\cos^5\phi d\phi \quad (\phi = 0/2) = -\frac{a^3\pi}{4} + 0 = -\frac{a^3\pi}{4}$$

Part-3: Green's Theorem

Ex 1. Verify Green's theorem in the plane for $\oint_C \left(xy + x^2\right)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $x^2 = 4ay$.

Solution.



• Here $Mdx + Ndy = \left(xy + x^2\right)dx + x^2 dy$

$M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x, \quad N = x^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

Let us first evaluate the double integral over Region R bounded by $x^2 = 4ay$ (curve C_1) & $y = x$ (curve C_2) as

$$\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \int_0^{4a}\int_{x^2/4a}^{x} xdydx = \int_0^{4a} x\left(x - \frac{x^2}{4a}\right)dx = \frac{x^3}{3} - \frac{x^4}{16a}\Bigg|_0^{4a} = \frac{16a^3}{3}$$

• Now let us evaluate the line integral $\oint Mdx + Ndy$ on closed curve C.

The curve C is a piecewise smooth curve consisting of C_1 and C_2.

- On $C_1, y=\dfrac{x^2}{4a}$, $dy=\dfrac{x}{2a}dx$

$$Mdx+Ndy=\left(xy+x^2\right)dx+x^2dy=\left(\frac{x^3}{4a}+x^2\right)dx+x^2\frac{x}{2a}dx=\left(\frac{3}{4}\cdot\frac{x^3}{a}+x^2\right)dx$$

x varies from 0 to $4a$ on C_1.

So, $\displaystyle\int_{C_1} Mdx+Ndy=\int_0^{4a}\left(\frac{3x^3}{4a}+x^2\right)dx=\frac{3}{16a}x^4+\frac{x^3}{3}\Big|_0^{4a}=8a^3+\frac{64a^3}{3}=\frac{208a^3}{3}$

- On C_2, $y=x, dy=dx$

$$Mdx+Ndy=\left(xy+x^2\right)dx+x^2dy=3x^2dx$$

x varies from $4a$ to 0.

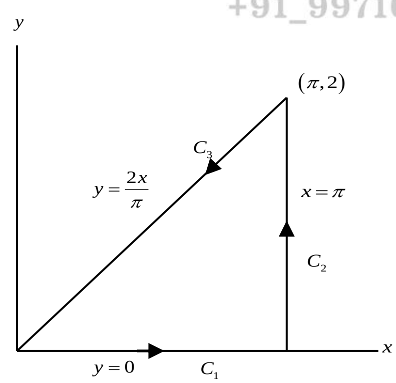
So, $\displaystyle\int_{C_2} Mdx+Ndy=\int_{4a}^{0}3x^2dx=x^3\Big|_{4a}^{0}=-64a^3$

- So, $\displaystyle\int_C Mdx+Ndy=\int_{C_1} Mdx+Ndy+\int_{C_2} Mdx+Ndy=\frac{208}{3}a^3-64a^3=\frac{16}{3}a^3$

Since, $\displaystyle\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\oint_C Mdx+Ndy$; So, Green's theorem is verified.

Ex 2. Apply Green's theorem in the plane to evaluate $\displaystyle\oint_C\left\{(y-\sin x)dx+\cos x dy\right\}$ where C is the triangle enclosed by the lines $y=0, x=\pi, \pi y=2x$.

Solution.



Here, $Mdx+Ndy=(y-\sin x)dx+\cos x dy$

So, $M=y-\sin x, \dfrac{\partial M}{\partial y}=1, \qquad N=\cos x, \dfrac{\partial N}{\partial x}=-\sin x$

According to Green's theorem, $\displaystyle\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

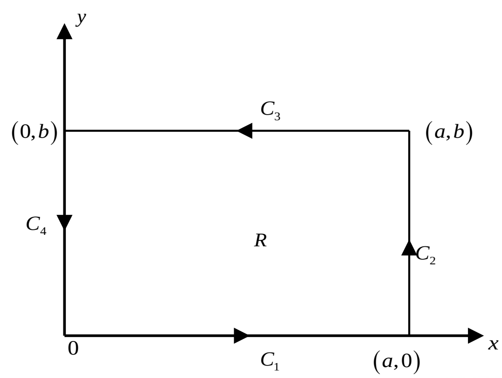
where R is the region enclosed by the piece wise smooth curve C consisting of curve $C_1(y=0)$, curve $C_2(x=\pi)$ curve $C_3(\pi y=2x)$ as shown in Figure.

So, $\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\int_{0}^{2}\int_{\pi y/2}^{\pi}(-\sin x-1)\,dxdy=\int_{0}^{2}[\cos x-x]_{\pi y/2}^{\pi}\,dy$

$$=\int_{0}^{2}\left(-1-\pi-\cos\frac{\pi y}{2}+\frac{\pi y}{2}\right)dy=-(1+\pi)y-\frac{2}{\pi}\sin\frac{\pi y}{2}+\frac{\pi y^2}{4}\Bigg|_{0}^{4}=-2-\pi$$

Ex 3. If $\vec{F}=\left(x^2-y^2\right)\hat{i}+2xy\hat{j}$ and $\vec{r}=x\hat{i}+y\hat{j}$, find the value of $\oint\left(x^2-y^2\right)dx+2xydy$ around the rectangular boundary $x=0, x=a, y=0$ and $y=b$.

Solution.



Here the curve C is a piecewise smooth curve consisting of $C_1(y=0), C_2(x=a), C_3(y=b)$ & $C_4(x=0)$.

The region bounded by C is shown in figure.

$\oint\vec{F}\cdot d\vec{r}=\oint\left(x^2-y^2\right)dx+2xydy=\oint Mdx+Ndy$

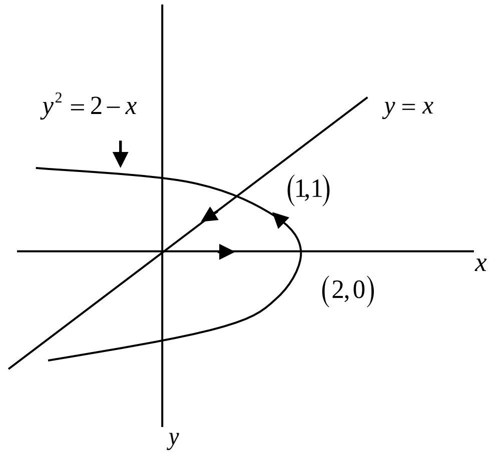
Here, $M=x^2-y^2$, $\frac{\partial M}{\partial y}=-2y$, $N=2xy$, $\frac{\partial N}{\partial x}=2y$

Applying Green's theorem

$$\oint Mdx+Ndy=\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=4\int_{0}^{b}\int_{0}^{a}ydxdy=4a\int_{0}^{b}ydy=2ab^2$$

Ex 4. Use Green's theorem to evaluate the integral $\oint_C x^2dx+\left(x+y^2\right)dy$, where C is the closed curve given by $y=0, y=x$ and $y^2=2-x$ in the first quadrant, oriented counter clockwise.

Solution.



The given integral is

$\oint x^2 dx + (x + y^2) dy = \oint M dx + N dy$; So, $M = x^2$; $N = x + y^2$

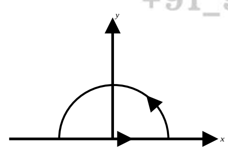
According to Green's theorem, $\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$

(R is the region of integration as shown in Figure)

So, $\oint x^2 dx + (x + y^2) dy = \iint dxdy = \int_0^1 \int_y^{2-y^2} dxdy = \int_0^1 (2 - y^2 - y) dy = \left[2y - \frac{y^3}{3} - \frac{y^2}{2} \right]_0^1 = \frac{7}{6}$

Ex 5. Let $\vec{F} = (x^2 - xy^2)\hat{i} + y^2 \hat{j}$. Using Green's theorem, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the positively oriented closed curve which is the boundary of the region enclosed by the x-axis and the semi-circle $y = \sqrt{1 - x^2}$ in the upper half plane.

Solution.



$\vec{F} = (x^2 - xy^2)\hat{i} + y^2 \hat{j}$

So, $\vec{F} \cdot d\vec{r} = (x^2 - xy^2)\hat{i} + y^2 \hat{j}$

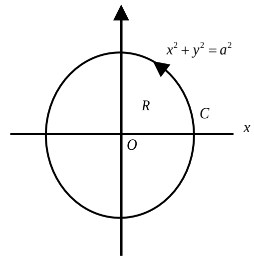
According to Green's theorem

$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$ (R is the region of integration shown in Figure)

$$= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} 2xy dy dx = \int_{-1}^{1} x \left[y^2 \right]_0^{\sqrt{1-x^2}} dx = \int_{-1}^{1} x(1 - x^2) dx = 0 \left(\int_{-a}^{a} f(x) = 0 \text{ if } f(x) \text{ is odd function} \right)$$

Ex 6. Evaluate by Green's theorem $\oint (\cos x \sin y - xy) dx + \sin x \cos y dy$ where C is the circle $x^2 + y^2 = a^2$.

Solution.



The given integral is

$$\oint_C (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$$

Where curve C is a circle of radius a and centered at origin enclosing region R as shown in Figure.

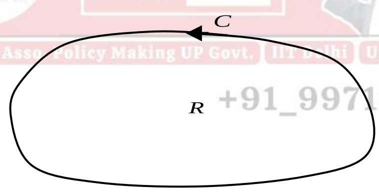
Here $M = \cos x \sin y - xy \Rightarrow \dfrac{\partial M}{\partial y} = \cos x \cos y - x$; $N = \sin x \cos y \Rightarrow \dfrac{\partial N}{\partial x} = \cos x \cos y$

Using Green's theorem, $\oint Mdx + Ndy = \iint_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right) dxdy$

$$= \iint xdxdy = \int_0^a \int_0^{2\pi} r\cos\theta r d\theta dr = \int_0^a r^2 \left[\sin\theta\right]_0^{2\pi} dr = 0$$

Ex 7. Show that the area bounded by a simple closed curve C is given by $\dfrac{1}{2}\oint xdy - ydx$. Hence find the area of the ellipse $x = a\cos\theta, y = b\sin\theta$.

Solution.



According to Green's theorem, if R is a plane region bounded by a simple closed curve C as shown in Figure according to the Green's Theorem

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint Mdx + Ndy$$

Let us put $M = -y/2, N = x/2$

So, $\dfrac{1}{2}\oint xdy - ydx = \iint \dfrac{\partial}{\partial x}\left(\dfrac{x}{2}\right) - \dfrac{\partial}{\partial y}\left(-\dfrac{y}{2}\right) dxdy = \iint dxdy$ = Area of region R bounded by C.

So, area of region bounded by simple closed curve C is given by

$$\frac{1}{2}\oint xdy - ydx$$

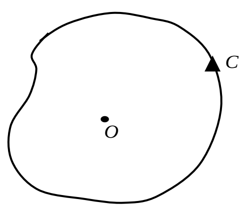
For as ellipse, $x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta$, $y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$

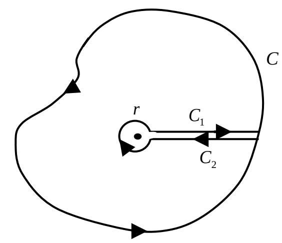
$$xdy - ydx = a\cos\theta b\cos\theta - b\sin\theta(-a\sin\theta)d\theta = abd\theta$$

So, area of region bounded by ellipse $= \dfrac{1}{2}\oint xdy - ydx = \dfrac{1}{2}\int_0^{2\pi} abd\theta = \dfrac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab$

Ex 8. Evaluate the line integral $\oint_C \frac{xdy - ydx}{x^2 + y^2}$ taken in the positive direction over any closed continuous curve C with the origin inside it.

Solution.





The given integral is

$$\oint \frac{xdy - ydx}{x^2 + y^2} = \oint Mdx + Ndy \quad \text{Here, } M = \frac{-y}{x^2 + y^2}, N = \frac{x}{x^2 + y^2}$$

Since, M & N are not continuous at origin O. Hence, Green's theorem will not hold good for the given curve C.

Let us enclose the origin by a circle Γ of radius $\in$

Consider the region R enclosed by curve C' made of C, C_2, Γ, C_1.

M and N are continuous function of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right) - \frac{\left(x^2 + y^2\right) - y \cdot 2y}{\left(x^2 + y^2\right)^2} = -\frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{\left(x^2 + y^2\right) - x \cdot 2x}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

So, line integral

$$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\Rightarrow \oint_C Mdx + Ndy = \int_C Mdx + Ndy + \int_{C_2} Mdx + Ndy + \int_\Gamma Mdx + Ndy + \int_{C_1} Mdx + Ndy = 0$$

But $\oint_{C_1} Mdx + Ndy = -\int_{C_2} Mdx + Ndy \Rightarrow \oint_C Mdx + Ndy = \int_C Mdx + Ndy + \int_\Gamma Mdx + Ndy = 0$

So, $\int_C Mdx + Ndy = -\int_\Gamma Mdx + Ndy$ ....(1)

In the figure curve Γ is oriented in negative direction.

On the curve Γ, $x = \in \cos\theta \Rightarrow dx = -\in \sin\theta d\theta$

$y = \in \sin\theta \Rightarrow dy = \in \cos\theta d\theta$

θ varies from 2π to 0.

$$\int_\Gamma \frac{xdy - ydx}{x^2 + y^2} = \int_{2\pi}^{0} \frac{\in \cos\theta \cdot \in \cos\theta d\theta - \in \sin\theta\left(-\in \sin\theta\right) d\theta}{\in^2} = \int_{2\pi}^{0} d\theta = -2\pi$$

So, from (1) $\int\limits_C Mdx + Ndy = -\int\limits_\Gamma Mdx + Ndy = 2\pi$

Ex 9. Using the line integral, compute the area of the loop of Descarte's folium $x^3 + y^3 = 3xy$.
Solution.

Putting $y = tx$ in the equation of folium $x^3 + y^3 = 3xy$; $\quad x = \dfrac{3t}{1+t^3}; y = \dfrac{3t^2}{1+t^3}$

Let $t = \dfrac{y}{x} = \tan\theta$ where θ varies from 0 to $\pi/2$. So, t varies from 0 to ∞.

$$dx = \frac{3(1-2t^3)}{(1+t^3)^2}dt, \qquad dy = \frac{3(2t-t^4)}{(1+t^3)^2}dt$$

Area of loop $A = \dfrac{1}{2}\int\limits_C xdy - ydx = \dfrac{9}{2}\int\limits_0^\infty \dfrac{t^2dt}{(1+t^3)^2} = \dfrac{3}{2}$

Ex 10. Evaluate $\oint\limits_C \dfrac{xdy - ydx}{x^2+4y^2}$ round the circle $x^2 + y^2 = a^2$ in the positive direction using Green's theorem.

Solution. The given line integral is $\oint\limits_C \dfrac{xdy - ydx}{x^2+4y^2} = \oint Mdx + Ndy$

Comparing the two integrals,

$$M = -\frac{y}{x^2+4y^2}, \frac{\partial M}{\partial y} = -\frac{x^2-4y^2}{(x^2+4y^2)^2} = \frac{-x^2+4y^2}{(x^2+4y^2)^2}$$

$$N = \frac{x}{x^2+4y^2}, \frac{\partial N}{\partial x} = \frac{-x^2+4y^2}{(x^2+4y^2)^2}$$

The curve C is the circle of radius a. R is the region enclosed by the circle $x^2 + y^2 = a^2$. M and N are not continuous at origin. So, the Green's theorem will not hold good for the given line integral. Proceeds similarly as done earlier.

$$\int\limits_C Mdx + Ndy = -\int\limits_\Gamma Mdx + Ndy = -\int\limits_\Gamma \frac{xdy - ydx}{x^2+4y^2} = -\int\limits_{2\pi}^{0} \frac{\in\cos\theta \in\cos\theta - \in\sin\theta(-\in\sin\theta)d\theta}{\in^2\cos^2\theta + 4\in^2\sin^2\theta}$$

(put $x = \in\cos\theta, y = \in\sin\theta$)

$$= \int\limits_0^{2\pi} \frac{1}{\cos^2\theta + 4\sin^2\theta}d\theta = \int\limits_0^{2\pi} \frac{\sec^2\theta}{1+4\tan^2\theta}d\theta$$

$$= 2\int\limits_0^{\pi} \frac{\sec^2\theta}{1+4\tan^2\theta}d\theta = 4\int\limits_0^{\pi/2} \frac{\sec^2\theta}{1+4\tan^2\theta}d\theta = 4\int\limits_0^\infty \frac{dt}{1+4t^2} = 4\frac{1}{2}\cdot\tan^{-1}2t\Big|_0^\infty = \pi$$

PREVIOUS YEARS QUESTIONS ANALYSIS: **Application of curl and vector integration**

Q1. For what value of a, b, c is the vector field $\vec{V} = (4x - 3y + az)\hat{i} + (bx + 3y + 5z)\hat{j} + (4x + cy + 3z)\hat{k}$ irrotational? Hence, express $\vec{V}$ as the gradient of a scalar function ϕ. Determine ϕ. **[5c UPSC CSE 2020]**

Hint: Refer example 3 page 3 part-1

Q2. Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Show that curl $(\text{curl}\,\vec{v}) = \text{grad}(\text{div}\,\vec{v}) - \nabla^2\vec{v}$. **[8a UPSC CSE 2018]**

Hint: refer examples solved for Gradient, div and curl.

Q3. Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$. **[6c 2018 IFoS] & [8a 2010 IFoS]**

Hint: Refer example 1 part-1 page 1

Q4. For what values of the constants a, b and c the vector $\vec{V} = (x + y + az)\hat{i} + (bx + 2y - z)\hat{j} + (-x + cy + 2z)\hat{k}$ is irrotational. Find the divergence in cylindrical coordinates of this vector with these values. **[5d UPSC CSE 2017]**

Hint: Refer example 3 page 2 part-1

Q5. A vector field is given by $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Verify that the field $\vec{F}$ is irrorational or not. Find the scalar potential. **[7c UPSC CSE 2015]**

Hint: take help from example 1 page 1 part-1

Q6. Examine if the vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational. If so, find the scalar potential ϕ such that $\vec{F} = \text{grad}\,\phi$. **[6d 2015 IFoS]**

Hint: take help from example 1 page 1 part-1

Q7. For the vector $\bar{A} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2 + y^2 + z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\bar{A} = \nabla\phi$. **[6d 2014 IFoS]**

Hint: Refer example 2 part-1 page 1

Q12. Show that the vector field defined by the vector function $\vec{V} = xyz(yz\vec{i} + xz\vec{j} + xy\vec{k})$ is conservative. **[1f UPSC CSE 2010] Hint:** take help from example 1 page 1 part-1

Line integral

Q1. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is an arbitrary closed curve in the *xy*-plane and $\vec{F} = \dfrac{y\hat{i} + x\hat{j}}{x^2 + y^2}$.

UPSC 6(c) **CSE 2021**

Q2. For the vector function $\bar{A}$, where $\bar{A}=\left(3x^2+6y\right)\hat{i}-14yz\hat{j}-14yz\hat{j}+20xz^2\hat{k}$, calculate $\int_C \bar{A}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths:

(i) $x=t, y=t^2, z=t^3$

(ii) Straight lines joining $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

(iii) Straight line joining $(0,0,0)$ to $(1,1,1)$

Is the result same in all the cases? Explain the reason. **[6b UPSC CSE 2020]**

Q3. Find the circulation of $\vec{F}$ round the curve C, where $\vec{F}=\left(2x+y^2\right)\hat{i}+\left(3y-4x\right)\hat{j}$ and C is the curve $y=x^2$ from $(0,0)$ to $(1,1)$ and the curve $y^2=x$ from $(1,1)$ to $(0,0)$. **[6b UPSC CSE 2019]**

Q4. Evaluate $\int_{(0,0)}^{(2,1)}\left(10x^4-2xy^3\right)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$. **[5e 2019 IFoS]**

Q5. Evaluate $\int_C e^{-x}\left(\sin y dx+\cos y dy\right)$, where C is the rectangle with vertices $(0,0),(\pi,0),\left(\pi,\frac{\pi}{2}\right),\left(0,\frac{\pi}{2}\right)$. **[8c UPSC CSE 2015]**

GREEN'S THEOREM

Q1. Verify Green's theorem in the plane for $\oint_C\left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy$, where C is the boundary curve of the region defined by $x=0, y=0$, $x+y=1$.[6c **UPSC CSE 2022]**

Q2. Let $\vec{F}=xy^2\vec{i}+\left(y+x\right)\vec{j}$. Integrate $\left(\nabla\times\vec{F}\right)\cdot\vec{k}$ over the region in the first quadrant bounded by the curves $y=x^2$ and $y=x$ using Green's theorem. **[8c UPSE CSE 2018]**

Q3. Using Green's theorem, evaluate the $\int_C F\left(\bar{r}\right)\cdot d\bar{r}$ counterclockwise where $F\left(\bar{r}\right)=\left(x^2+y^2\right)\hat{i}+\left(x^2-y^2\right)\hat{j}$ and $d\bar{r}=dx\hat{i}+dy\hat{j}$ and the curve C is the boundary of the region $R=\left\{(x,y)\middle|1\le y\le 2-x^2\right\}$. **[8c UPSE CSE 2017]**

Q4. Verify Green's theorem in the plane for $\oint_C\left[\left(xy+y^2\right)dx+x^2dy\right]$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$. **[8b UPSE CSE 2013]**

Q5. Find the value of the line integral over a circular path given by $x^2+y^2=a^2, z=0$, where the vector field, $\vec{F}=\left(\sin y\right)\vec{i}+x\left(1+\cos y\right)\vec{j}$.**[8b 2012 IFoS]**

Q6. Verify Green's theorem in the plane for $\oint_C\left[\left(3x^2-8y^2\right)dx+\left(4y-6xy\right)dy\right]$, where C is the boundary of the region enclosed by the curves $y=\sqrt{x}$ and $y=x^2$. **[8c 2011 IFoS]**

Q7. Verify Green's theorem for $e^{-x}\sin y dx + e^{-x}\cos y dy$ the path of integration being the boundary of the square whose vertices are $(0,0),(\pi/2,0),(\pi/2,\pi/2)$ and $(0,\pi/2)$. **[8c UPSE CSE 2010]**

Q8. Use Green's theorem in a plane to evaluate the integral, $\int_C \left[\left(2x^2-y^2\right)dx+\left(x^2+y^2\right)dy\right]$, where C is the boundary of the surface in the *xy*-plane enclosed by, $y=0$ and the semi-circle $y=\sqrt{1-x^2}$. **[8b 2012 IFoS]**

Q9. If $\vec{A}=2y\vec{i}-z\vec{j}-x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4, z=6$, evaluate the surface integral, $\iint_S \vec{A}\cdot\hat{n}\overrightarrow{dS}$. **[8c 2010 IFoS]**

Q10. Find the work done in moving the particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1, z=0$ under the field of force given by $\vec{F}=(2x-y+z)\hat{i}+\left(x+y-z^2\right)\hat{j}+(3x-2y+4z)\hat{k}$. **[8a UPSE CSE 2009]**

LINE INTEGRAL

1. Hint: $\because$ The given curve C is closed & bounded. Also, $\vec{F}=\frac{y\hat{i}+x\hat{j}}{x^2+y^2} \Rightarrow P=\frac{y}{x^2+y^2}$ & $Q=\frac{x}{x^2+y^2}$

P & Q are continuous & posses partial derivatives

So, here Green's theorem in xy-plane is applicable; $\int_c Pdx+Qdy=\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

$\frac{\partial Q}{\partial x}=\frac{\left(x^2+y^2\right)-x.2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$, $\frac{\partial P}{\partial y}=\frac{\left(x^2+y^2\right)-y.2y}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$; use in above integral.

2. **Hint: (i)** $\int \vec{A}.d\vec{r}=\int_{t=0}^{1}(3t^2+6t^2)dt-14(t^2)(t^3)d(t^2)+20(t)(t^3)^2d(t^3)=\int_0^1 9t^2dt-28t^6dt+60t^9dt=5$

(ii) $y=0, z=0, dy=0, dz=0; x=0 \text{ to } 1.$

$\int \vec{A}.d\vec{r}=\int_{x=0}^{1}(3x^2+6(0))dx-14(0)(0)(0)+20(x)(0^2)^2(0)=1$

For $(1,0,0)$ *to* $(1,1,0); x=1, z=0, dx=0, dz=0; y=0 \text{ to } 1$; $\int \vec{A}.d\vec{r}=0$

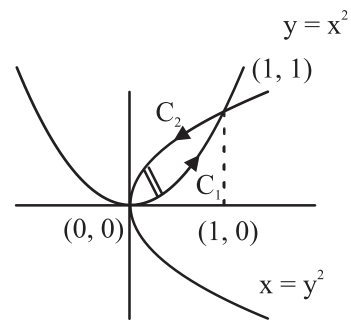
For $(1,1,0)$ *to* $(1,1,1); x=1, y=1, dx=0, dy=0; z=0 \text{ to } 1$; $\int \vec{A}.d\vec{r}=20/3$

Therefore, total $\int \vec{A}.d\vec{r}=1+0+20/3=23/3$

(iii) in parametric form; x=t, y=t, z=t and then solve $\int \vec{A}.d\vec{r}=13/3$

3. $\vec{F}=\left(2x+y^2\right)\hat{i}+(3y-4x)\hat{j}$

Now, the given region R is closed & bounded by the curve $C = C_1 + C_2$



And here, $\vec{F} = P\hat{i} + Q\hat{j}$, when

$\vec{P} = 2x + y^2$, $\quad Q = 3y - 4x$

i.e., $\int_c P dx + dy = \int_R \left(\frac{\partial P}{\partial y} - \frac{\partial P}{\partial y} \right) dx$

$\therefore \quad \frac{\partial P}{\partial y} = 2y$, $\frac{\partial Q}{\partial x} = -4$

Now, $\quad \int_c (2x + y^2)\hat{i} + (3y - 4x)\hat{j} = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-4 - 2y) dx dy = -\int_{x=0}^{1} \left[+4y + y^2 \right]_{x^2}^{\sqrt{x}} dx$

$= -\int_{x=0}^{1} \left\{ 4\sqrt{x} + x - \left(4x^2 + x^4 \right) \right\} dx = = -\left[4 \times \frac{2}{3} x^{\frac{3}{2}} + \frac{x^2}{2} - 4\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = -\left[\frac{8}{3} + \frac{1}{2} - \frac{4}{3} - \frac{1}{5} \right]$

$= -\left[\frac{80 + 15 - 40 - 6}{30} \right] = -\frac{49}{30}$

4. Question ; $\int_c P dx + Q dy$ we get, $P = 10x^4 - 2xy^3$ & $Q = -3x^2y^2 dy \therefore$ $\vec{F} = \left(10x^4 - 2xy^3 \right)\hat{i} + \left(-3x^2y^2 \right)\hat{j}$

$\because$ Circle $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 10x^4 - 2xy^3 & -3x^2y^2 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \hat{k}\left(-6xy^2 + 6xy^2 \right) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

So, $\vec{F} = \nabla\phi$; where ϕ is scalar potential $\quad \therefore \frac{\partial \phi}{\partial x} = 10x^4 - 2xy^3$ & $\frac{\partial \phi}{\partial y} = -3x^2y^2$,

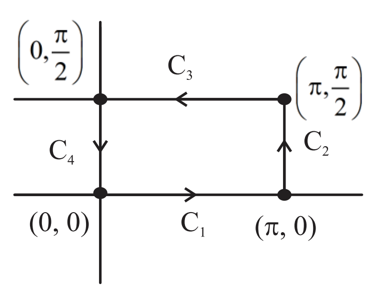
$\therefore \quad \phi(x, y) = 2x^5 - x^2y^3 + f(y)$ & $\phi(x, y) = -x^2y^3 + g(x)$,

$\therefore \quad$ After taking suitable values of constants we get, $\phi(x, y) = 2x^5 - x^2y^3$

$\therefore \quad \int_{(0,0)}^{(2,1)} \vec{F}.d\vec{r} = \int_{(0,0)}^{(2,1)} d\phi = [\phi(x, y)]_{(0,0)}^{(2,1)} = 64 - 4 = 60$

5. $I = \int_c e^{-x} (\sin y \, dx + \cos y \, dy)$

Here, the region R is bounded by the closed curve $C = C_1 + C_2 + C_3 + C_4$



Here, $\vec{F} = P\hat{i} + Q\hat{j}$, where

$P = e^{-x} \sin y$ $\quad$ & $Q = e^{-x} \cos y$; which are continuous & possesses partial derivatives

So, here Green's theorem in xy plane is applicable

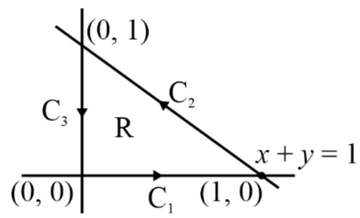
$\therefore \int_c P \, dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$

$$\therefore \ \frac{\partial Q}{\partial x} = -e^{-x}\cos y \ \& \ \frac{\partial P}{\partial y} = -e^{-x}\cos y$$

$$\therefore \ \int_c e^{-x}(\sin y\,dx + \cos y\,dy) = \int_{\pi=0}^{\pi}\int_{y=0}^{\pi/2} -2e^{-x}\cos y \ dxdy = -2\int_{\pi=0}^{\pi} e^{-x}\left[\sin y\right]_0^{\pi/2} dx = -2\left[-e^{-x}\right]_0^{\pi} = 2\left(e^{-\pi}-1\right)$$

GREEN'S THEOREM

1. $I = \oint_c \left(3x^2 - 8y^2\right)dx + 4y - 6xy)dy$,



where c is bounded by x=0, y= 0 & x+y = 1

From integral we have, P = $3x^2 - 8y^2$, Q = $4y - 6xy$

As, the curve C = $C_1 + C_2 + C_3$ is closed

Enclosing the region R & also P & Q are continuous f. & possess partial derivatives.

So, green theorem in plane is applicable here,

i.e., $I = \oint_c P dx + Q dy = \iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$ ……………(A)

Now, For C_1: y = 0, dy = 0; x = 0 to 1

For C_2: $y = 1 - x$ $dy = -dx$; x = 1 to 0

For C3: x = 0, dx = 0; y = 1 to 0

$$Pdx + Qdy = \int_{x=0}^{1} 3x^2 dx + \int_{x=1}^{0}\left\{3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)\right\} + \int_{y=1}^{0} 4y\,dy$$

$$= \left[x^3\right]_0^1 + \int_{x=0}^{1}\left\{3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2\right\}dx + \left[2y^2\right]_1^0 = 1 + \int_{x=0}^{1}\left(-11x^2 + 26x - 12\right)dx - 2$$

$$= -1 + \left[-11\frac{x^3}{3} + 13x^2 - 12x\right]_1^0 = -1 + \frac{11}{3} - 13 + 12 = -2 + \frac{11}{3} = \frac{5}{3}$$

Now,

$$\iint_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_{x=0}^{1}\int_{y=0}^{1-x}\left(-6y + 16y\right)dxdy = 10\int_{x=0}^{1}\left[\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_{x=0}^{1}(1-x)^2 dx = 5\left[\frac{(1-x)^2}{-3}\right]_0^1$$

$= \frac{-5}{3}[0-1] = \frac{5}{3}$ ∴ Green's Theorem is verified.

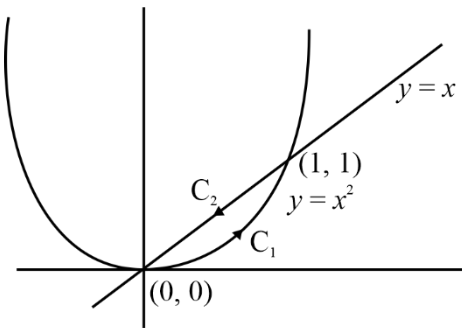
2.

$$\vec{F} = xy^2\hat{i} + (y+x)\hat{j} \ ; \ \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y+x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (1-2xy)\hat{k}$$

Curl $\vec{F} = (1-2xy)\hat{k}$ $\therefore$ (Curl $\vec{F}$).$\hat{k}$ $= 1 - 2xy$

Now, By Green's Theorem, $\iint\left(\nabla\times\vec{F}\right).\hat{k}dxdy = \int_{c} Pdx + Qdy$

$$\iint\left\{\frac{\partial\left(y=x\right)}{dx}-\frac{\partial\left(xy^{2}\right)}{\partial y}\right\}dxdy = \iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$



$\therefore$ Q = $y + x$ & P = xy^2

Now, the curve C = $C_1 + C_2$

$$\therefore \oint_{c} Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy \quad \text{-----(A)}$$

Now, For C_1 $y = x^2 \Rightarrow dy = 2xdx$; x = 0 to 1

$$\int_{C_1} Pdx + Qdy = \int_{x=0}^{1} x\times x^{4}dx+\left(x^{2}+x\right).2xdx$$

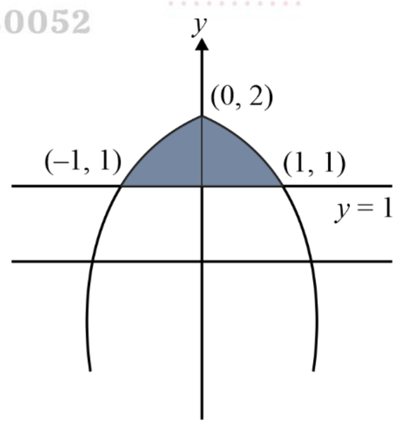
$$= \left[\frac{x^6}{6}+2\frac{x^4}{4}+2\frac{x^3}{3}\right]_0^1 = \frac{1}{6}+\frac{1}{2}+\frac{2}{3} = \frac{1+3+4}{6} = \frac{8}{6} = \frac{4}{3}$$

For C_2 : $y = x$, $dy = dx$; x: 1 to 0

$$\int_{C_2} Pdx + Qdy = \int_{x=1}^{0} x^{3}dx+2xdx = \left[\frac{x^4}{4}+x^2\right]_1^0 = \frac{-1}{4}-1 = \frac{-5}{4}$$

$$\therefore \oint_{c} Pdx + Qdy = \frac{4}{3}-\frac{5}{4} = \frac{16-15}{12} = \frac{1}{12}$$

3.



$\vec{F}$ -= $\left(x^2+y^2\right)\hat{i}+\left(x^2-y^2\right)\hat{j}$ & $d\vec{r} = dx\hat{i}+dy\hat{j}$;

C is boundary of region R = $\{(x, y) : 1 \le y \le 2 - x^2\}$

As, the region is closed & bounded so,

Here Green's Theorem is applicable

Also, here, P = $x^2 + y^2$ & Q = $x^2 - y^2$

Which are cont. f & possess partial derivative.

i.e, $\oint_{c} Pdx + Qdy = \int_{x}\int_{y}\left(\frac{\partial Q}{dx}-\frac{\partial P}{dy}\right)dxdy = 2\int_{y=1}^{2}\int_{x=-\sqrt{2-y}}^{\sqrt{2-y}}\left(x-y\right)dxdy$

$$=2\int_{y=1}^{2}\left[\frac{x^2}{2}-yx\right]_{x=-\sqrt{2-y}}^{\sqrt{2-y}}dy = 2\int_{y=1}^{2}\left[\frac{(2-y)}{2}-y\sqrt{2-y}-\frac{(2-y)}{2}-y\sqrt{2-y}\right]dy = -4\int_{y=1}^{2}y\sqrt{2-y}\,dy$$

Put t = $y - 1$; $dt = dy$

$= -4\int_0^1 (t+1)\sqrt{1-t}\ dt$

Put t = Sin²θ; dt = 2sinθ Cosθdθ,

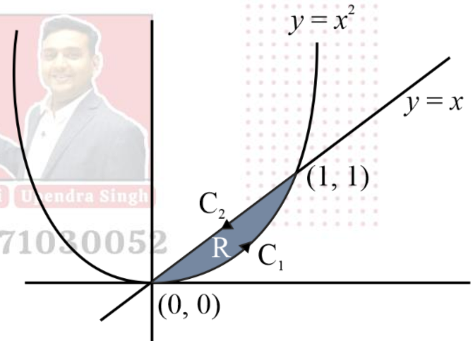
$= -4\int_{\theta=0}^{\pi/2} \left(1+\sin^2\theta\right)\cos\theta.2\sin\theta\cos\theta d\theta$

$$= -8\left[\int_{\theta=0}^{\pi/2}\sin\theta\cos^2\theta d\theta + \int_{\theta=0}^{\pi/2}\sin^3\theta\cos^2\theta d\theta\right] = -8\left[\frac{\sqrt{1}\sqrt{\frac{3}{2}}}{2\sqrt{\frac{5}{2}}} + \frac{\sqrt{2}\sqrt{\frac{3}{2}}}{2\sqrt{\frac{7}{2}}}\right] = -4\left[\frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\frac{3}{2}\times\frac{1}{2}\sqrt{\frac{1}{2}}} + \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\frac{1}{2}}}\right]$$

$= -4\left[\frac{2}{3}+\frac{4}{15}\right] = -4\times\frac{2}{3}\left[1+\frac{2}{5}\right]$

$\oint_c \mathrm{P}dx+\mathrm{Q}dy = \frac{-8}{3}\times\frac{7}{5} = \frac{-56}{15}$

4. I = $\oint\left(xy+y^2\right)dx+x^2dy$; c is the bounded region between $y = x$ & $y = x^2$



As, the region R is closed & bounded, & here P = $xy + y^2$ & Q = x^2; which are cont. f & posses partial derivative.

So, Green's Theorem is applicable

i.e. $\oint_c \mathrm{P}dx+\mathrm{Q}dy = \iint_R\left(\frac{\partial \mathrm{Q}}{\partial x}-\frac{\partial \mathrm{P}}{\partial y}\right)dxdy$ …………(A)

For C_1: $y = x^2$ → $dy - 2xdx$; x : 0 to 1

For C_2: $y = x$, $dy = dx$; x : 1 to 0.

$\therefore \oint_C \mathrm{P}dx+\mathrm{Q}dy = \int_{C_1}\mathrm{P}dx+\mathrm{Q}dy + \int_{C_2}\mathrm{P}dx+\mathrm{Q}dy$

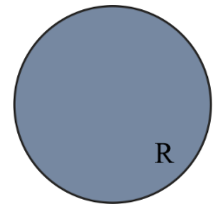
$$= \int_{x=0}^{1}\left\{\left(x^3+x^4\right)dx+x^2.2xdx\right\}+\int_{x=1}^{0}\left(2x^2dx+x^2dx\right) = \left[\frac{x^4}{4}+\frac{x^5}{5}+2\frac{x^4}{4}\right]_0^1+\left[2\frac{x^3}{3}+\frac{x^3}{3}\right]_1^0 = \frac{3}{4}+\frac{1}{5}+(-1) = \frac{-1}{20}$$

$$\text{Now, }\iint_R\left(\frac{\partial \mathrm{Q}}{\partial x}-\frac{\partial \mathrm{P}}{\partial y}\right)dxdy = \int_{x=0}^{1}\int_{y=x^2}^{x}2x-(x+2y)dxdy = \int_{x=0}^{1}\int_{y=x^2}^{x}2x-(x-2y)dxdy = \int_{x=0}^{1}\left[xy-y^2\right]_{y=x^2}^{x}dx$$

$$= \int_{x=0}^{1}\left\{x^2-x^2-\left(x^3-x^4\right)\right\}dx = \left[\frac{x^5}{5}-\frac{x^4}{4}\right]_0^1 = \frac{1}{5}-\frac{1}{4} = \frac{-5+4}{20} = \frac{-1}{20} \quad \therefore \oint_P \mathrm{P}dx+\mathrm{Q}dy = \iint_R\left(\frac{\partial \mathrm{Q}}{\partial x}-\frac{\partial \mathrm{P}}{\partial y}\right)dxdy$$

5. $\vec{F} = \sin y\hat{i}+x\left(1+\cos y\right)\hat{j}$: S is $x^2 + y^2 = a^2$ & $z = 0$

Clearly, the region R is closed & bounded



& here, P = sin *y*, Q = *x* (1 + cos *y*); which are continuous & possess partial derivative

∴ Here, Green's Theorem is applicable

i.e., $\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$$\oint Pdx + Qdy = \iint_R \left((1+\cos y) - \cos y\right) dxdy = \iint_R dxdy = \pi \times a^2$$

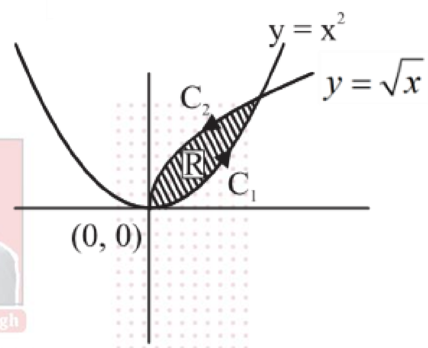
$$\int \sin y dx + x(1+\cos x) dy = \pi a^2$$

6. $I = \oint \left\{ (3x^2 - 8y^2) dx + (4y - 6xy) dy \right\}$; where C is the boundary between y = $\sqrt{x}$ & $y = x^2$

As the region R is closed & bounded by C = $C_1 + C_2$ & here

P = $3x^2 - 8y^2$, Q = $4y - 6xy$; which are cont. & possess partial derivative

∴ By Green's Theorem,



$$\oint_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \text{ \_\_\_\_\_\_\_\_\_\_ (A)}$$

Now, For C_1; $y = x^2$; $dy = 2xdx$; $x : 0$ to 1

For C_2; y = $\sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}$ dx; $x : 1$ to 0.

$$\therefore \oint_C Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy$$

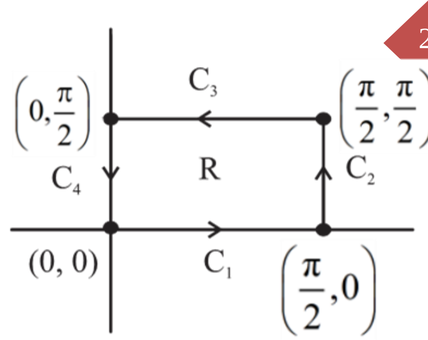
$$\oint_C Pdx + Qdy = \int_{x=0}^{1} \{(3x^2 - 8x^4)dx + (4x^2 - 6x^3)2xdx\} + \int_{x=1}^{0} \left\{ (3x^2 - 8x)dx + (4\sqrt{x} - 6x\sqrt{x}) \frac{1}{2\sqrt{x}} dx \right\}$$

$$= \int_{x=0}^{1} (-20x^4 + 8x^3 + 3x^2) dx + \int_{x=1}^{0} (3x^2 - 11x + 2) dx = \left[-4x^5 + 2x^4 + x^3\right]_0^1 + \left[x^3 - \frac{11x^2}{2} + 2x\right]_1^0$$

$$= -4 + 2 + 1 + \left\{0 - \left(1 - \frac{11}{2} + 2\right)\right\} = -1 - 3 + \frac{11}{2} = \frac{-2 - 6 + 11}{2} = 3/2$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y) dxdy = 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 5 \int_{x=0}^{1} (x - x^4) dx = 5 \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3}{2}$$

$\therefore$ Green's Theorem verified.



7.Given $I=\iint_R e^{-x}\sin y\,dx+e^{-x}\cos y\,dy$; the region is the square with vertices

(0, 0), $\left(\frac{\pi}{2},0\right)$, $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ & $\left(0,\frac{\pi}{2}\right)$

As, the region R is bounded by the curve

C = $C_1 + C_2 + C_3 + C_4$ & here $P=e^{-x}\sin y$ & $Q=e^{-x}\cos y$

So, here by Green's Theorem,

$$\oint_C Pdx+Qdy=\int_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy \qquad \text{.....(A)}$$

For C_1: $\quad y=0,\quad dy=0;\qquad x:0 \quad \text{to} \quad \frac{\pi}{2}$

For C_2: $\quad x=\frac{\pi}{2},\quad dx=0;\qquad y:0 \quad \text{to} \quad \frac{\pi}{2}$

For C_3: $\quad y=\frac{\pi}{2},\quad dy=0;\qquad x:\frac{\pi}{2} \quad \text{to} \quad 0$

For C_4: $\quad x=0,\quad dx=0;\qquad y:\frac{\pi}{2} \quad \text{to} \quad 0$

$$\therefore \oint Pdx+Qdy=\int_{x=0}^{\pi/2}0dx+\int_{y=0}^{\pi/2}e^{-\pi/2}\cos ydy+\int_{x=\pi/2}^{0}e^{-x}dx+\int_{y=\pi/2}^{0}\cos ydx$$

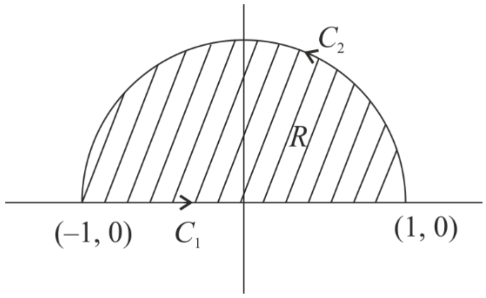
$$=e^{-\frac{\pi}{2}}\left[+\sin y\right]_0^{\pi/2}+\left[-e^{-x}\right]_{\pi/2}^{x=\pi/2}+\left[\sin y\right]_{\pi/2}^{0}=e^{-\frac{\pi}{2}}-\left[1-e^{-\pi/2}\right]+(0-1)=2e^{-\frac{\pi}{2}}-2=2\left(e^{-\pi/2}-1\right)$$

Now, $$\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2}\left\{-e^{-x}\cos y-e^{-x}\cos y\right\}dxdy$$

$$=-2\int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2}e^{-x}\cos ydxdy=-2\int_{y=0}^{\pi/2}\cos y\left[-e^{-x}\right]_0^{\pi/2}=2\int_{y=0}^{\pi/2}\left(e^{-\pi/2}-1\right)\cos y=2\left(e^{-\pi/2}-1\right)$$

$$\therefore \oint_c Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

8. $I=\int_c\left(2x^2-y^2\right)dx+\left(x^2+y^2\right)dy$; C is the boundary in xy plane $y=\sqrt{1-x^2}$ & $y=0$



Clearly the region R is closed & bounded by curve $c = c_1 + c_2$ & here $P = 2x^2 - y^2$ & $Q = x^2 + y^2$; which are cont. & posses partial derivative So, here Green's Theorem is applicable here,

i.e., $\oint_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R (2x + 2y) dxdy$

$$\oint_c Pdx + Qdy = 2\iint_R (x+y)\,dxdy = 2\int_{r=0}^{1}\int_{\theta=0}^{\pi} r(\cos\theta + \sin\theta) \times r dr d\theta$$

$$= \frac{2}{3}\int_{\theta=0}^{\pi} (\cos\theta + \sin\theta)\, d\theta = \frac{2}{3}\int_{\theta=0}^{\pi} \sin\theta\, d\theta \qquad \left[\because \int_{\theta=0}^{\pi} \cos\theta = 0\right]$$

$$= \frac{2}{3}[-\cos\theta]_0^{\pi} = \frac{-2}{3}[-1-1] = \frac{4}{3}$$

9. $\vec{A} = 2y\hat{i} - z\hat{j} - x^2\hat{k}$; S ; $y^2 = 8x \Rightarrow 8x - y^2 = 0$ & bounded by planes $y = 4$ & $z = 6$,

Here $\hat{n} = \frac{\nabla S}{|\nabla S|}$; $\hat{n} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}}$ And $dS = \frac{dydz}{|\hat{i}.\hat{n}|} = \frac{dydz}{\frac{8}{\sqrt{84 + 4y^2}}}$

$$\therefore \iint_s \vec{A}.\hat{n}dS = \int_y\int_z (2y\hat{i} - z\hat{j} - x^2\hat{k}), \frac{(8\hat{i} - 2y\hat{j})}{\sqrt{64 + 4y^2}} \times \frac{dydz}{\frac{8}{\sqrt{64 + 4y^2}}}$$

$$= \int_{y=0}^{4}\int_{z=0}^{6} (2y\hat{i} - z\hat{j} - x^2\hat{k}).\frac{(4\hat{i} - 4\hat{j})}{\sqrt{16 + y^2}} \times \frac{dydz}{\frac{4}{\sqrt{16 + y^2}}} = \frac{1}{4}\int_{y=0}^{4}\int_{z=0}^{6} (8y + yz)dydz$$

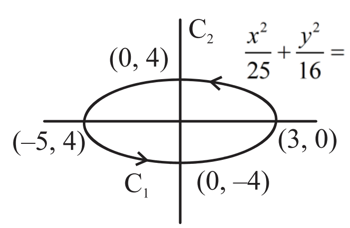
$$= \frac{1}{4}\int_{z=0}^{6} (8 + z) \times \left[\frac{y^2}{z}\right]_0^4 dz = \frac{8}{4}\int_{z=0}^{6} (8 + z)dz = 2\left[8z + \frac{z^2}{2}\right]_0^6 = 2[48 + 18] = 132 \text{ units}$$

10. $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$

The given ellipse is, $\frac{x^2}{25} + \frac{y^2}{16} = 1$, z = 0

As, the given surface is closed & bounded by the c = $c_1 + c_2$

So, that Stoke's theorem is applicable



i.e., $\oint_c \vec{F}.\overrightarrow{dr} = \iint_s cuul\, F.\hat{n}ds$ __________ (i)

Now,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y + z & x + y - z^2 & 3x - 2y + 4z \end{vmatrix} = \hat{i}\{-2 - 2z\} + \hat{j}\{1 - 3\} + \hat{k}\{1 + 1\}$$

$$\text{Curl } \vec{F} = -2(1+z)\hat{i} - 2\hat{j} + 2\hat{k}$$

Hence, $\hat{n} = \hat{k}$ $\therefore$ dS $= \dfrac{dzdy}{|\hat{n}.\hat{k}|} = dxdy$

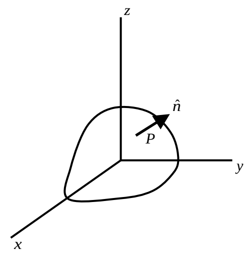
$\therefore$ from (1),

$\oint \vec{F}.\overrightarrow{dr} = \int_x \int_y 2dxdy$ = 2 × Area of ellipse= 2 × π × 5 × 4= 40 π [$\therefore$ Area of ellipse = π ab]

Surface Integral

Ex 1. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ which lies in the first octant.

Solution.



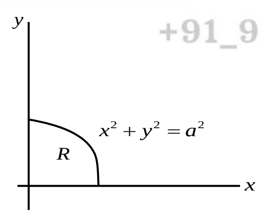
The sphere belongs to a family of level surface given by $S = x^2 + y^2 + z^2 = c$

So, the unit vector $\hat{n}$ at any point P is given by; $\hat{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$

$$\hat{n}\cdot\hat{k} = \frac{z}{a}$$

$$\vec{F}\cdot\hat{n} = \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)\cdot\frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{a} = \frac{3xyz}{a}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{a}{z}dxdy$$



$$\vec{F}\cdot\hat{n}dS = \frac{3xyz}{a}\cdot\frac{a}{z}\cdot dxdy = 3xydxdy$$

$\int_S \vec{F}\cdot\hat{n}dS = 3\iint_R xy\,dx\,dy$ (The region of integration of double integration given by R)

$$= 3\int_0^{\pi/2}\int_0^a r^3\cos\theta\sin\theta dr\,d\theta = 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \cos\theta\sin\theta d\theta = \frac{3a^4}{4}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

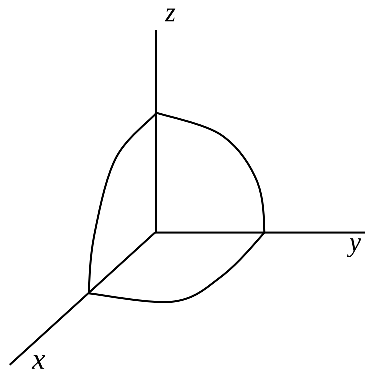
$$= \frac{3a^4}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3}{8}a^4$$

Ex 2. Evaluate

$$I = \iint xdydz + dzdx + xz^2dxdy$$

where S is the part of sphere $x^2+y^2+z^2=a^2$ in the first octant.

Solution.



S is the part of sphere $x^2+y^2+z^2=a^2$ lying the first octant as shown in fig.

S belongs to family of level surface given by $S: x^2+y^2+z^2$ = constant

Outward drawn unit normal vector to S,

$$\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$|\hat{n}\cdot\hat{k}|=\frac{z}{a}$$

$$dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\frac{a}{z}\cdot dxdy$$

The given integral can be written as

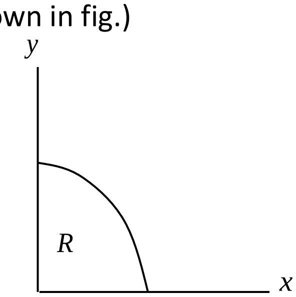
$$\iint xdydz+dzdx+xz^2dxdy=\int\left(x\hat{i}+\hat{j}+xz^2\hat{k}\right)\cdot\hat{n}dS=\int\vec{F}\cdot\hat{n}\,dS$$

Where $\vec{F}=x\hat{i}+\hat{j}+xz^2\hat{k}$

$$\vec{F}\cdot\hat{n}=\left(x\hat{i}+\hat{j}+xz^2\hat{k}\right)\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)=\frac{1}{a}\left(x^2+y+xz^3\right)$$

$$\int_S\vec{F}\cdot\hat{n}\,dS=\iint\frac{\left(x^2+y+xz^3\right)}{z}dxdy=\iint_R\left[\frac{x^2+y}{\sqrt{a^2-x^2-y^2}}+x\left(a^2-x^2-y^2\right)\right]dxdy$$

(R is the region of integration as shown in fig.)



$$=\iint_R\frac{x^2dxdy}{\sqrt{a^2-x^2-y^2}}+\iint_R\frac{y}{\sqrt{a^2-x^2-y^2}}dydx+\iint_R x\left(a^2-x^2-y^2\right)dxdy$$

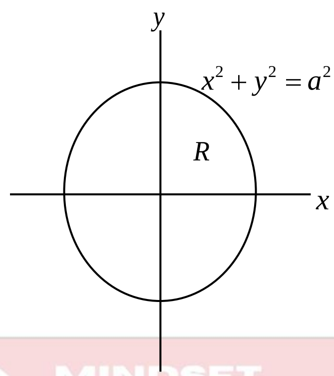
$$=\int_0^a\int_0^{\pi/2}\frac{r^3\cos^2\theta}{\sqrt{a^2-r^2}}d\theta dr+\int_0^a\int_0^{\pi/2}\frac{r^2\sin\theta}{\sqrt{a^2-r^2}}d\theta dr+\int_0^a\int_0^{\pi/2}r^2\left(a^2-r^2\right)\cos\theta d\theta dr$$

$$=\frac{\pi}{4}\int_0^a\frac{r^3}{\sqrt{a^2-r^2}}dr+\int_0^a\frac{r^2}{\sqrt{a^2-r^2}}dr+\int_0^a\left(a^2r^2-r^4\right)dr$$

$$=\frac{\pi a^3}{6}+\frac{a^2\pi}{4}+\left(a^2\frac{r^3}{3}-\frac{r^5}{5}\right)_0^a=\frac{\pi a^3}{6}+\frac{\pi a^2}{4}+\frac{2a^5}{15}$$

Ex 3. Evaluate the integral $\int_S x^2y^2dS$ where S is the hemisphere $z=\sqrt{a^2-x^2-y^2}$.

Solution.



S is the surface of hemisphere $z=\sqrt{a^2-x^2-y^2}$.

An outward drawn unit normal vector is S.

$$\hat{n}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$dS=\frac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|}=\frac{dxdy}{z}$$

$$\int_S x^2y^2dS=\iint_R\frac{x^2y^2}{z}dxdy$$

(R is the region of double integration as shown in fig.)

$$=\iint_R\frac{x^2y^2}{\sqrt{a^2-x^2-y^2}}\cdot dxdy=\int_0^a\int_0^{2\pi}\frac{r^5\sin^2\theta\cos^2\theta}{\sqrt{a^2-r^2}}d\theta dr$$

$$=4\int_0^a\int_0^{\pi/2}\frac{r^5}{\sqrt{a^2-r^2}}\sin^2\theta\cos^2\theta d\theta dr$$

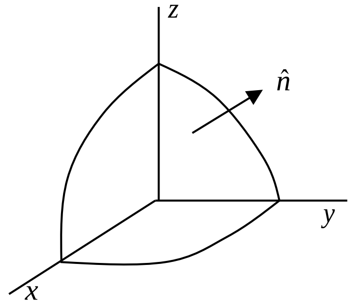
$$=4\int_0^a\frac{r^5}{\sqrt{a^2-r^2}}\frac{\Gamma\left(3/2\right)\Gamma\left(3/2\right)}{2\Gamma 3}dr$$

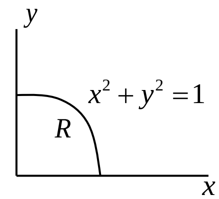
$$=\frac{\pi}{4}\int_0^1\frac{r^5}{\sqrt{a^2-r^2}}dr$$

$$=\frac{2}{15}\pi a^6$$

Ex 4. Evaluate $\int x\,dS$ where S is the portion of the sphere $x^2+y^2+z^2=1$ lying in the first octant.

Solution.





S is the surface of sphere lying in the first octant as shown in fig. and belongs to family of level surface $S: x^2+y^2+z^2$ = constant.

An outward drawn unit normal vector to S.

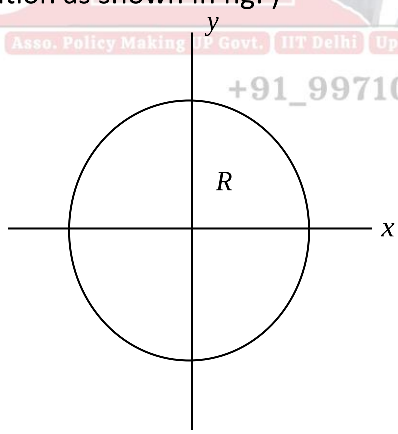
$$\hat{n}=\frac{\nabla S}{|\nabla S|}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\hat{n}\cdot\hat{k}=z$$

$$dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\frac{1}{z}dxdy=\frac{1}{\sqrt{1-x^2-y^2}}dxdy$$

$$\int_S x\,dS=\iint_R \frac{x}{\sqrt{1-x^2-y^2}}$$

(R is the region of double integration as shown in fig.)



$$=\int_0^1\int_0^{\pi/2}\frac{r^2\cos\theta}{\sqrt{1-r^2}}d\theta dr$$

$$=\int_0^1\frac{r^2}{\sqrt{1-r^2}}dr=\frac{\pi}{4}$$

Ex 5. Evaluate the integral $\int_S \sqrt{1-x^2-y^2}\,dS$ where S is the hemisphere $z=\sqrt{1-x^2-y^2}$.

Solution.

S is the surface of hemisphere $z=\sqrt{1-x^2-y^2}$

An outward drawn unit normal vector to S

$\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{dxdy}{z}$$

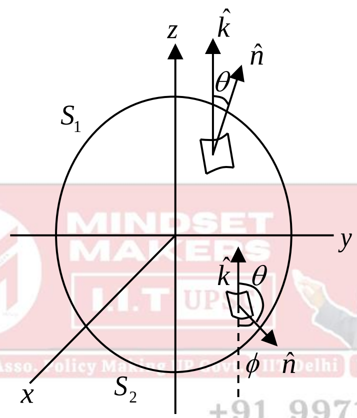
$$\int_S \sqrt{1-x^2-y^2}\,dS = \iint_R \sqrt{1-x^2-y^2}\,\frac{dxdy}{z} \quad \left(z=\sqrt{1-x^2-y^2}\right)$$

$= \iint_R dxdy$ = Area of region R

(R is the region of double integration as shown in fig.)

Ex 6. Evaluate the surface integral $\iint z\cos\theta dS$ over the surface or sphere $x^2+y^2+z^2=a^2$ where θ is the inclination of normal at any point of the sphere with the z axis.

Solution.



S is the surface of sphere consisting of

upper hemisphere $S_1 : z = \sqrt{a^2-x^2-y^2}$ and

lower hemisphere $S_2 : z = \sqrt{a^2-x^2-y^2}$ as shown in fig.

• Over $S_1, dS\cos\theta = dxdy, z = \sqrt{a^2-x^2-y^2}$

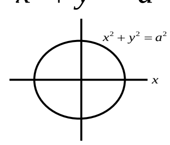
$z\cos\theta\, dS = \sqrt{a^2-x^2-y^2}\,dxdy$

• Over $S_2, dS\cos\theta = dS\cos(\pi-\phi) = -dS\cos\phi = -dxdy$

$z = -\sqrt{a^2-x^2-y^2}$

$z\cos\theta dS = \sqrt{a^2-x^2-y^2}\,dxdy$

Since projection of S_1 and S_2 is same i.e. $x^2+y^2=a^2$



$$\int_{S_1} z\cos\theta dS = \int_{S_2} z\cos\theta dS$$

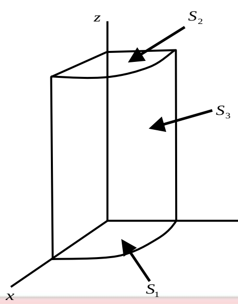
So, $\int_S z\cos\theta dS = \int_{S_1} z\cos\theta\, dS + \int_{S_2} z\cos\theta\, dS$

$= 2\iint_R \sqrt{a^2 - x^2 - y^2}\, dxdy$ (R is the region of integration as shown in fig.)

$= 2\int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2}\, r d\theta\, dr = 4\pi \int_0^a \sqrt{a^2 - r^2}\, r\, dr = \dfrac{4\pi}{3} a^3$

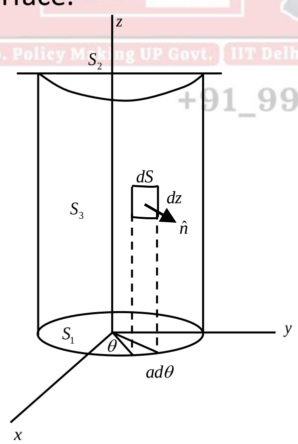
Ex 7. Evaluate $\int_S \vec{F} \cdot \hat{n} dS$ where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = a^2$ along with the bases included in the first octant between $z = 0$ & $z = b$.

Solution.



The cylinder is a piecewise smooth surface consisting of S_1, S_2 and S_3 where S_1 is lower base $z = 0$, S_2 is upper base $z = b, S_3$ is the curved surface of cylinder, as shown in figures.

$\hat{n}$ is an outward drwan normal to surface.



$\int_S \vec{F} \cdot \hat{n}\, dS = \int_{S_1} \vec{F} \cdot \hat{n}\, dS + \int_{S_2} \vec{F} \cdot \hat{n}\, dS + \int_{S_4} \vec{F} \cdot \hat{n}\, dS$

• On $S_1, \hat{n} = -\hat{k}, z = 0, dS = dxdy$

$\vec{F} \cdot \hat{n} = \vec{F} \cdot (-k) = 3y^2 z = 0$ (as $z = 0$ on S_1)

So, $\int_{S_4} \vec{F} \cdot \hat{n}\, dS = 0$

• On $S_2, \hat{n} = \hat{k}, z = b, dS = dxdy$

$\vec{F} \cdot \hat{n} = 3y^2 z = -3by^2$

So, $\int_{S_4} \vec{F}\cdot\hat{n}\,dS = -3b\iint y^2 dxdy$

$$= -3b\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = -3b\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin^2\theta\,d\theta = -\frac{3}{4}ba^4\int_0^{\pi/2}\sin^2\theta\,d\theta = -\frac{3}{16}\pi a^4 b$$

- The curved surface S_3 belongs to family of level surface $S \equiv x^2 + y^2 =$ constant

The unit normal vector to the surface S_3 is given by $= \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i}+y\hat{j}}{a}$

- For S_3, $\vec{F}\cdot\hat{n} = \left(z\hat{i}+x\hat{j}-3y^2z\hat{k}\right)\cdot\frac{\left(x\hat{i}+y\hat{j}\right)}{a} = \frac{1}{a}(zx+xy)$

- $dS = a\,d\theta\,dz$
- On $S_3, x = a\cos\theta, y = a\sin\theta$

So, $\vec{F}\cdot\hat{n} = \frac{1}{a}\left[az\cos\theta + a^2\sin\theta\cos\theta\right] = z\cos\theta + a\sin\theta\cos\theta$

The surface integral becomes

$$\int_{S_3}\vec{F}\cdot\hat{n}\,dS = \int_0^b\int_0^{\pi/2}(z\cos\theta + a\sin\theta\cos\theta)\,a\,d\theta dz$$

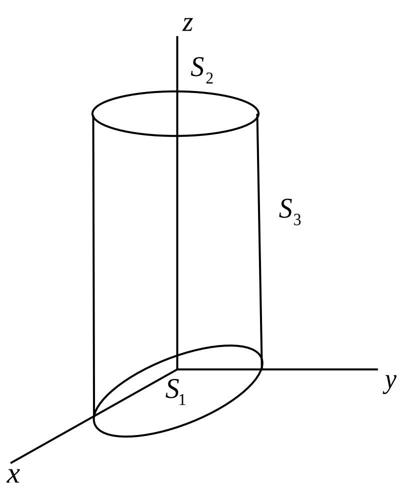
$$= a\int_0^b\left[z\sin\theta - \frac{a}{4}\cos 2\theta\right]dz = a\int_0^b\left(z+\frac{a}{2}\right)dz = a\left[\frac{z^2}{2}+\frac{a}{2}z\right]_0^b = \frac{ab}{2}(a+b)$$

- $\int_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS + \int_{S_3}\vec{F}\cdot\hat{n}\,dS = -\frac{3}{18}\pi a^4 b + \frac{ab}{2}(a+b)$

Ex 8. Evaluate $\int_S (x+y+z)(ax+by+cz)\,dS$ where S is the surface of region $x^2+y^2 \le 1, 0 \le z \le 1$

.

Solution.



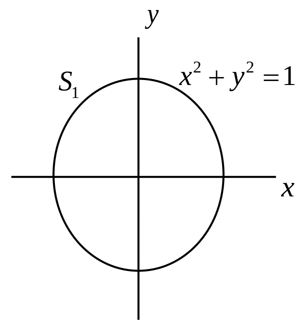
S is the surface bounding the region $x^2+y^2 \le 1$ & $0 \le z \le 1$

S is a piece wise smooth surface consisting of

S_1 : lower base $z=0$

S_2 : upper base $z=1$

S_3 : curved surface of cylinder, $x^2+y^2=1$ as shown in fig.



On $S_1 : z=0, dS = dxdy$

$$\int_{S_1} (x+y+z)(ax+by+cz)\,dS$$

$$=\iint (x+y)(ax+by)\,dxdy$$

$$=\iint \left(ax^2+(a+b)xy+by^2\right)dxdy$$

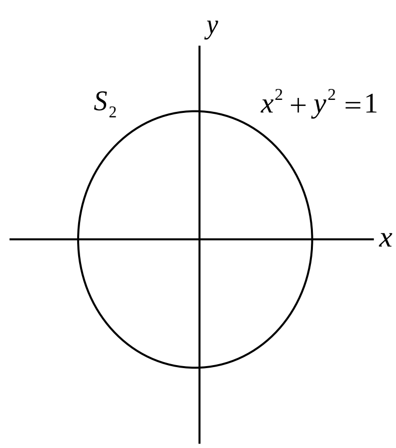
$$=\int_0^{2\pi}\int_0^1 \left(ar^2\cos^2\theta+(a+b)r^2\sin\theta\cos\theta+br^2\sin^2\theta\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(a\cos^2\theta+(a+b)\sin\theta\cos\theta+b\sin^2\theta\right)\cdot\left.\frac{r^4}{4}\right|_0^1 d\theta$$

$$=\frac{1}{4}\cdot\int_0^{2\pi}\left(a\cos^2\theta+b\sin^2\theta+(a+b)\sin\theta\cos\theta\,d\theta\right)$$

$$=\frac{1}{4}a\int_0^{2\pi}\cos^2\theta d\theta+\frac{b}{4}\cdot\int_0^{2\pi}\cos^2\theta d\theta+\frac{(a+b)}{4}\cdot\int_0^{2\pi}\sin\theta\cos\theta d\theta$$

$$=(a+b)\frac{\pi}{4}$$



On $S_2 : z = 1, dS = dxdy$

$$\int_{S_2} (x+y+z)(ax+by+cz)\,dS$$

$$= \iint (x+y+z)(ax+by+c)\,dxdy$$

$$= \iint (x+y)(ax+by)\,dxdy + \int_{-1}^{1} \int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} (a+c)\,x\,dxdy + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} (b+c)\,y\,dydx + c\iint dxdy$$

$$= (a+b)\frac{\pi}{4} + c\pi$$

On $S_3 : x = \cos\theta, y = \sin\theta, dS = d\theta\,dz$,

$$\int_{S_3} (x+y+z)(ax+by+cz)\,dS$$

$$= \int_0^1 \int_0^{2\pi} (\cos\theta + \sin\theta + z)(a\cos\theta + b\sin\theta + cz)\,d\theta dz$$

$$= \int_0^1 \left((a+b)\pi + 2\pi c z^2\right) dz$$

$$= (a+b)\pi + \frac{2c\pi}{3}$$

$$\int_S (x+y+z)(ax+by+cz)\,dS$$

$$= \int_{S_1} + \int_{S_2} + \int_{S_3}$$

$$= (a+b)\frac{\pi}{4} + (a+b)\frac{\pi}{4} + c\pi + (a+b)\pi + \frac{2\pi c}{3}$$

$$= \frac{3}{2}(a+b)\pi + \frac{5c\pi}{3}$$

Ex 9. Evaluate $\int_S x\,dS$ where S is the entire surface of solid bounded by the cylinder $x^2 + y^2 = a^2$ and $z = 0$, $z = x + 2$.

Solution.

S is piece wise smooth surface consisting of

S_1 : Base of cylinder, $z = 0$

S_2 : roof of cylinder, $z = x + 2$

S_3 : curved surface of cylinder $x^2 + y^2 = a^2$

On S_1 , $dS = dxdy$

$$\int_{S_1} x\,dS = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x\,dxdy = 0$$

S_2 belongs to family of level surface given by $S_2 : z - x$ = constant.
So, outwards drawn unit normal to S_2

$$\hat{n} = \frac{-\hat{i} + \hat{k}}{\sqrt{2}}$$

On S_2, $dS = \dfrac{dxdy}{|\hat{n} \cdot \hat{k}|} = \sqrt{2}\, dxdy$

So, $\displaystyle\int_{S_2} x\, dS = \sqrt{2} \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0$

On S_3, $dS = ad\theta dz,\ x = a\cos\theta,\ y = a\sin\theta$

z varies from 0 to $x+2$ i.e. 0 to $2 + a\cos\theta$

$$\int_{S_3} x\, dS = \int_0^{2\pi} \int_0^{2+a\cos\theta} a\cos\theta a\, dzd\theta$$

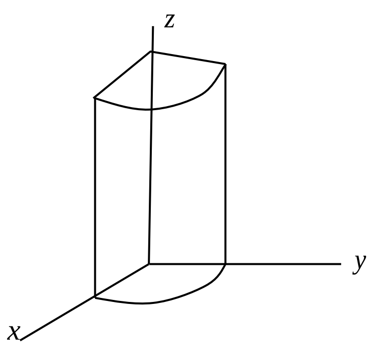
$$= a^2 \int_0^{2\pi} \cos\theta \cdot (2 + a\cos\theta)\, d\theta$$

$$= 2a^2 \int_0^{2\pi} \cos\theta d\theta + a^3 \int_0^{2\pi} \cos^2\theta d\theta$$

$$= \pi a^3 \left(\int_0^{2\pi} \cos\theta d\theta = 0 \right)$$

So, $\displaystyle\oiint_S \vec{F} \cdot \hat{n}\, dS = \int_{S_1} \vec{F} \cdot \hat{n}\, dS + \int_{S_2} \vec{F} \cdot \hat{n}\, dS + \int_{S_3} \vec{F} \cdot \hat{n}\, dS = \pi a^3$

Ex 9. Find the value of surface integral $\iint yzdxdy + xzdydz + xydxdz$ where S is the outer side of the surface formed by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0, y = 0, z = 0$ & $z = 2$.
Solution.



S is a piece wisesmooth surface bounded by $S_1 : x = 0, S_2 : y = 0, S_3 : z = 0$ & $S_4 : x^2 + y^2 = 4$.

$$\iint yzdxdy + xzdydz + xydxdz = \oiint_S \left(xz\hat{i} + xy\hat{j} + yz\hat{k} \right) \cdot \hat{n}dS$$

$$= \oiint \vec{F} \cdot \hat{n}dS$$

On $S_1, \hat{n} = -\hat{i}, dS = dydz, x = 0, \vec{F} \cdot \hat{n} = xz = 0$

So, $\displaystyle\int_{S_1} \vec{F} \cdot \hat{n}\, dS = 0$

On $S_2, y=0, \hat{n}=-\hat{j}, dS=dxdz, \vec{F}\cdot\hat{n}=xy=0$

So, $\int_{S_2} \vec{F}\cdot\hat{n}\,dS=0$

On $S_3, z=0, \hat{n}=-\hat{k}, dS=dxdy, \vec{F}\cdot\hat{n}=yz=0$

So, $\int_{S_3} \vec{F}\cdot\hat{n}\,dS=0$

On $S_4, x^2+y^2=4, \hat{n}=\dfrac{x\hat{i}+y\hat{j}}{2}, x=2\cos\theta, y=2\sin\theta$

So, $\vec{F}\cdot\hat{n}=\dfrac{x^2z+xy^2}{2}=\dfrac{4z\cos^2\theta+8\cos\theta\sin^2\theta}{2}$

$=2z\cos^2\theta+4\cos\theta\sin^2\theta$

$dS=2d\theta dz$

$\int_{S_4}\vec{F}\cdot\hat{n}\,dS=\iint\left(2z\cos^2\theta+4\cos\theta\sin^2\theta\right)2d\theta dz$

$=4\int_0^2\int_0^{\pi/2}\left(z\cos^2\theta+2\cos\theta\sin^2\theta\right)d\theta dz$

$=4\int_0^2\left(2\cdot\dfrac{\pi}{4}+\dfrac{2}{3}\right)dz$

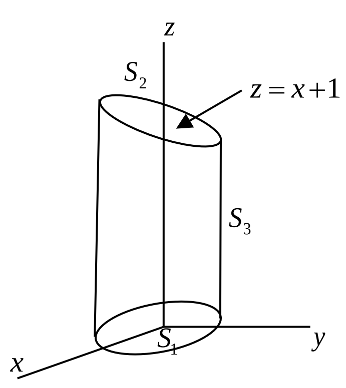
$=4\left[\dfrac{\pi}{8}z^2+\dfrac{2}{3}z\right]_0^2$

$=4\left(\dfrac{\pi}{2}+\dfrac{4}{3}\right)$

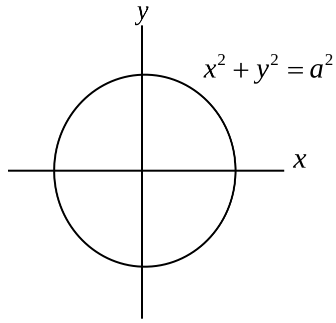
So, $\oint_S \vec{F}\cdot\hat{n}\,dS=4\left(\dfrac{\pi}{2}+\dfrac{4}{3}\right)$

Ex 10. Evaluate $\oint \vec{F}\cdot\hat{n}\,dS$ where S is the entire surface of the solid formed by $x^2+y^2=a^2, z=x+1$ and $\hat{n}$ is the outward drawn unit normal and the vector function $\vec{F}=2x\hat{i}-3y\hat{j}+z\hat{k}$.

Solution.



S is the pricewise smooth surface consisting of $S_1 : z = 0$, $S_2 : z = x+1$ and $S_3 : x^2 + y^2 = a^2$ (curved surface) as shown in fig.



On $S_1, z = 0,\ \hat{n} = -\hat{k},\ \vec{F}\cdot\hat{n} = -z = 0$

So, $\int_{S_1} \vec{F}\cdot\hat{n}dS = 0$

On $S_2, z = x+2,\ \hat{n} = \dfrac{-\hat{i}+\hat{k}}{\sqrt{2}}$ (as done in previous question)

$$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{2}}(-2x+z) = \frac{1}{\sqrt{2}}(-x-1)$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{2}dxdy$$

$$\vec{F}\cdot\hat{n}dS = (1-x)dxdy$$

$$\int_{S_2} \vec{F}\cdot\hat{n}dS = \iint_R (1-x)dxdy$$

$$= \iint dxdy - \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy$$

(R is the region of double integration as shown in fig.)

$$= \iint dxdy \quad \left(\text{as} \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0\right)$$

$$= \pi a^2$$

S_3 belong to family of level surface $S_3 : x^2 + y^2$ = constant.

Outward drawn unit normal vector.

$$\hat{n} = \frac{\nabla S_3}{|\nabla S_3|} = \frac{x\hat{i}+y\hat{j}}{a}$$

On S_3 $\vec{F}\cdot\hat{n} = \dfrac{1}{a}(2x^2-3y^2)$, $x = a\cos\theta, y = a\sin\theta$

$$dS = ad\theta\, dz$$

$$\vec{F}\cdot\hat{n}dS = (2x^2-3y^2)ad\theta\, dz$$

$= a^3\left(2\cos^2\theta - 3\sin^2\theta\right)dzd\theta$

z varies from 0 to $x+1$, i.e. 0 to $1+a\cos\theta$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^{(1+a\cos\theta)} a^3\left(2\cos^2\theta - 3\sin^2\theta\right)dzd\theta$$

$$= a^3\int_0^{2\pi}\left(2\cos^2\theta - 3\sin^2\theta\right)d\theta - a^4\int_0^{2\pi}\left(2\cos^2\theta - 3\sin^2\theta\right)\cos\theta d\theta$$

$= -\pi a^3$

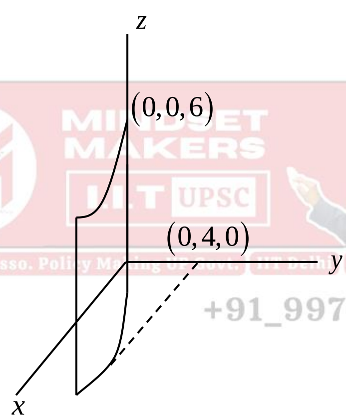
So, $\oint\!\!\oint_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS + \int_{S_3}\vec{F}\cdot\hat{n}\,dS$

$= 0 + \pi a^2 - \pi a^3$

$= \pi a^2(1-a)$

Ex 11. If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 4x$ in the first octant bounded by the planes $y = 4$ and $z = 6$ then evaluate $\int_S \vec{F}\cdot\hat{n}dS$.

Solution.



The parabolic surface as shown in fig. belong to family of level surface $S = 4x - y^2$ = constant.

The unit normal vector to the parabolic cylinder is given by $\hat{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{2\hat{i} - y\hat{j}}{\sqrt{y^2+4}}$

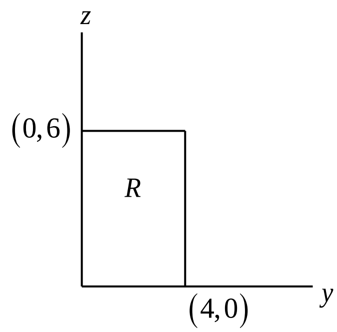
$$\vec{F}\cdot\hat{n} = \left(2y\hat{i} - z\hat{j} + x^2\hat{k}\right)\cdot\frac{\left(2\hat{i} - y\hat{j}\right)}{\sqrt{y^2+4}} = \frac{4y+4z}{\sqrt{y^2+4}}$$

$$\hat{n}\cdot\hat{i} = \frac{2}{\sqrt{y^2+4}}$$

$$dS = \frac{dydz}{\left|\hat{n}\cdot\hat{i}\right|} = \frac{1}{2}\sqrt{y^2+4}dydz$$

$$\vec{F}\cdot\hat{n}dS = \frac{1}{2}(4y+yz)dydz$$

So, the surface integral reduces to double integral whose region of integration R is given in fig.



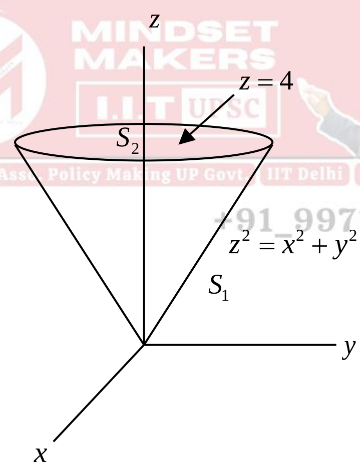
$$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{2}\iint_R (4y + yz)\,dydz$$

Region R is the projection of parabolic cylinder on yz plane

$$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{2}\int_0^6\int_0^4 (4y + yz)\,dydz = \frac{1}{2}\int_0^6 2y^2 + \frac{y^2 z}{2}\Big|_0^4 dz = \int_0^6 (16 + 4z)\,dz = 16z + 2z^2\Big|_0^6 = 168$$

Ex 12. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ over the entire surface of the region above xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$ if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Solution.



The conical surface S, as shown in the fig. belongs to a family of level surface given by $S = x^2 + y^2 - z^2$ = constant.

The unit normal vector to cone is given by $\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\vec{F}\cdot\hat{n} = \left(4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}\right)\cdot\frac{\left(x\hat{i} + y\hat{j} - z\hat{k}\right)}{\sqrt{x^2 + y^2 + z^2}} = \frac{4x^2 z + xy^2 z^2 - 3z^2}{\sqrt{x^2 + y^2 + z^2}}$$

$$\hat{n}\cdot\hat{k} = \frac{-z}{\sqrt{x^2 + y^2 + z^2}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{\sqrt{x^2+y^2+z^2}}{z}dxdy$$

$$\vec{F}\cdot\hat{n}dS = \frac{1}{z}\left(4x^2z + xy^2z^2 - 3z^2\right)dxdy$$

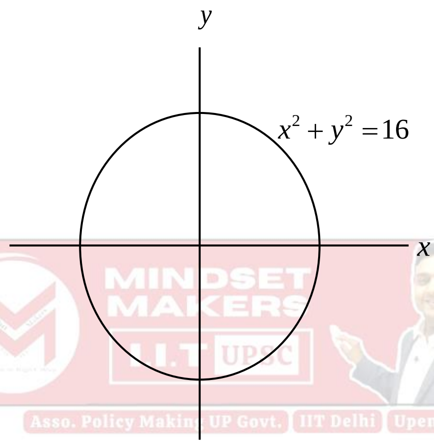
$$= \left(4x^2 + xy^2z - 3z\right)dxdy$$

$$= \left(4x^2 + xy^2z - 3z\right)dxdy$$

$$= \left(4x^2 + xy^2\sqrt{x^2+y^2} - 3\sqrt{x^2+y^2}\right)dxdy$$

So, $\int\limits_S \vec{F}\cdot\hat{n}\,dS = \iint\limits_R \left(4x^2 + xy^2\sqrt{x^2+y^2} - 3\sqrt{x^2+y^2}\right)dxdy$

(R is the region of integration given by projection of cone on xy plane as shown in fig.



$$= 4\iint x^2dxdy - 3\iint\sqrt{x^2+y^2}dxdy; \quad \because \int\limits_{-4}^{4}\int\limits_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} xy^2\sqrt{x^2+y^2}dxdy$$

$$= 4\int\limits_0^{2\pi}\int\limits_0^4 r^3\cos^2\theta drd\theta - 3\int\limits_0^{2\pi}\int\limits_0^4 r^2drd\theta = 4\int\limits_0^{2\pi}\left.\frac{r^4}{4}\right|_0^4\cos^2\theta d\theta - 3\int\limits_0^{2\pi}\left.\frac{r^3}{3}\right|_0^4 d\theta$$

$$= 256\int\limits_0^{2\pi}\cos^2\theta d\theta - 64\int\limits_0^{2\pi}d\theta = 256\pi - 128\pi = 128\pi$$

• On S_2, $\hat{n} = \hat{k},\ dS = dxdy$

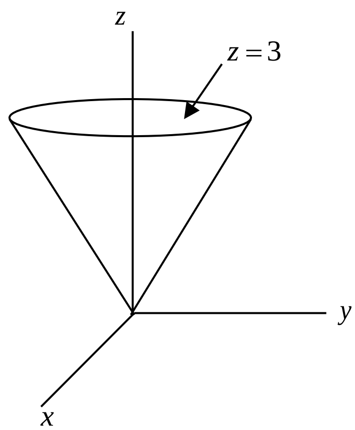
$$\vec{F}\cdot\hat{n} = 3z = 12$$

$$\int\limits_{S_2}\vec{F}\cdot\hat{n}\,dS = 12\iint dxdy = 192\pi$$

• So, $\int\limits_S \vec{F}\cdot\hat{n}\,dS = \int\limits_{S_1}\vec{F}\cdot\hat{n}\,dS + \int\limits_{S_2}\vec{F}\cdot\hat{n}\,dS = 128\pi + 192\pi = 320\pi$

Ex 13. Evaluate $\iint\left(x^2+y^2\right)dS$ where S is the surface of the cone $z^2 = x^2+y^2$ bounded by $z=0$ & $z=3$.

Solution.



Upper part of a cone is given by

$z=\sqrt{x^2+y^2}$ as shown in fig.

It belongs to family of level surface given by

$S:\sqrt{x^2+y^2}-z$ = constant.

Outward drawn unit normal vector is given by

$$\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{\frac{x}{\sqrt{x^2+y^2}}\hat{i}+\frac{y}{\sqrt{x^2+y^2}}\hat{j}-\hat{k}}{\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}}$$

$$\left|\hat{n}\cdot\hat{k}\right|=\frac{1}{\sqrt{2}}$$

$$dS=\frac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|}=\sqrt{2}dxdy$$

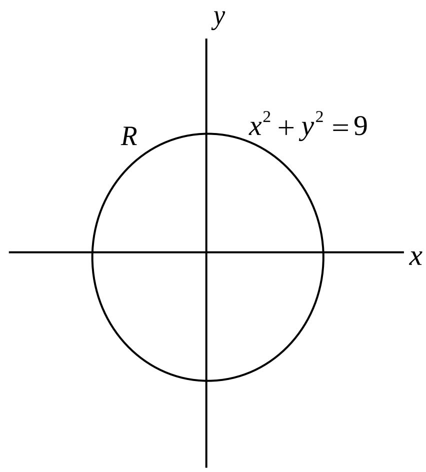
S is a piecewise smooth surface consisting of conical part

$S_1:\sqrt{x^2+y^2}-z=0$ and $S_2:z=3$ as shown in fig.

• On S_1, $dS=\sqrt{2}dxdy$

So, $\int\limits_{S_1}\left(x^2+y^2\right)dS=\iint\limits_{R}\left(x^2+y^2\right)\sqrt{2}dxdy$

The region of double integration R is projection of cone $x^2+y^2=z^2$ on the xy plane as shown in fig.



$$=\sqrt{2}\int_0^{2\pi}\int_0^3 r^2\cdot r\,dr\,d\theta=\sqrt{2}\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^3 d\theta=\frac{81\sqrt{2}}{4}\int_0^{2\pi}d\theta=\frac{81\sqrt{2}}{2}\pi$$

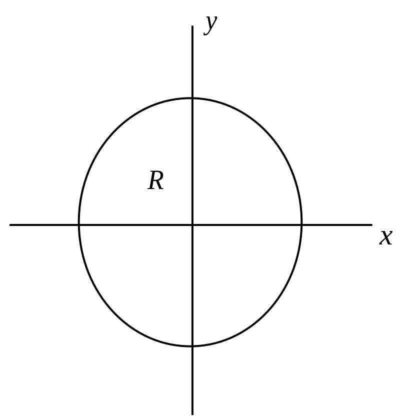
• On $S_2, z=3$, $dS=dxdy=rdrd\theta$

$$\int_{S_2}\left(x^2+y^2\right)dS=\int_0^{2\pi}\int_0^3 r^2 rd\theta dr=\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^3 d\theta=\frac{81}{4}\int_0^{2\pi}d\theta=\frac{81}{2}\pi$$

• So, $\int_S\left(x^2+y^2\right)dS=\int_{S_1}\left(x^2+y^2\right)dS+\int_{S_2}\left(x^2+y^2\right)dS=\frac{81\sqrt{2}}{2}\pi+\frac{81}{2}\pi=\frac{81}{2}\pi\left(\sqrt{2}+1\right).$

Ex 14. Evaluate the surface integral $\int_S \frac{dS}{r}$ where S is the portion of the surface of hyperbolic paraboloid $z=xy$ cut by the cylinder $x^2+y^2=1$ and r is the distance from a point on the surface to z axis.

Solution.



Surface of hyperbolic paraboloid belongs to the family of level surface $S: xy-z$ = constant.

The unit normal vector to surface is given by $\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{y\hat{i}+x\hat{j}-\hat{k}}{\sqrt{x^2+y^2+1}}$

$$\left|\hat{n}\cdot\hat{k}\right|=\frac{1}{\sqrt{x^2+y^2+1}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{x^2+y^2+1}\,dxdy$$

So, the surface integral reduces to a double integral

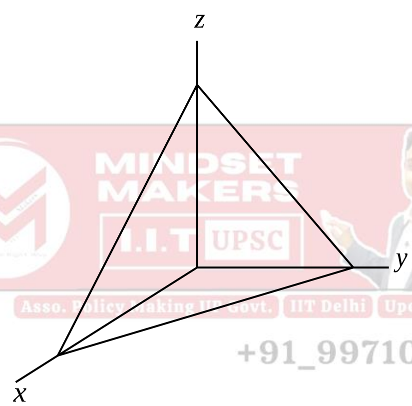
$$I = \int_S \frac{dS}{r} = \iint_R \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}\,dxdy$$

where R is the region of the integration of double integral as shown in fig. which is projection of surface on xy plane.

$$I = \iint_R \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}\,dxdy = 2\pi\left[\frac{r}{2}\sqrt{1+r^2}+\frac{1}{2}\log\left(r+\sqrt{1+r^2}\right)\right]_0^1 = \pi\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$$

Ex 15. Evaluate $\int_S xyz\,dS$ over the portion of $x+y+z=a,\ a>0$, lying in the first octant.

Solution.



S is the surface given by $x+y+z=a$ in the first octant. It belong to family of level surface given by $S: x+y+z$ = constant as shown in fig.

Unit normal vector to the surface S

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

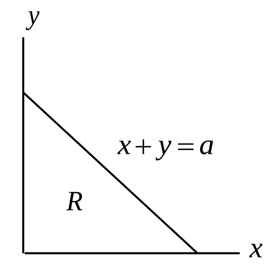
$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{3}}$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{3}\,dydx$$

So, $\int_S xyz\,dS = \iint_R xyz\sqrt{3}\,dydx$

$$= \sqrt{3}\int_0^a\int_0^{a-x} xy(a-x-y)\,dydx$$

(R is the region of double integration as shown in fig.)



$$=\sqrt{3}\int_0^a x(a-x)\frac{y^2}{2}-\frac{xy^3}{3}\Big|_0^{a-x}\,dx$$

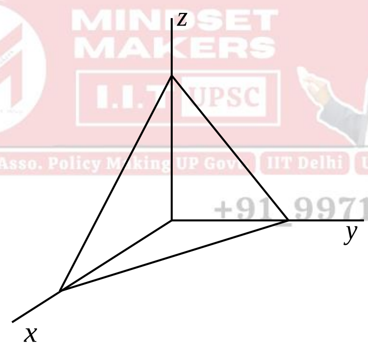
$$(z=a-x-y)$$

$$=\sqrt{3}\int_0^a \frac{x}{2}(a-x)^3-\frac{x}{3}(a-x)^3\,dx$$

$$=\frac{\sqrt{3}}{6}\int_0^a x(a-x)^3\,dx=\frac{1}{40\sqrt{3}}a^5$$

Ex 16. Evaluate $\iint_y (xz\,dx\,dy+xy\,dy\,dz+yz\,dz\,dx)$ where S is the outer side of the pyramid formed by the planer $x=0, y=0, z=0$ and $x+y+z=a$.

Solution.



S is the piece wise smooth surface formed by

$S_1: x=0.\ S_2: y=0,\ S_3: z=0, S_4=x+y+z=a$ as shown in fig.

$$\iint_S xzdxdy+xydydz+yzdzdx=\iint\left(xy\,\hat{i}+yz\,\hat{j}+xz\,\hat{k}\right)\cdot\left(dydz\,\hat{i}+dzdx\,\hat{j}+dxdy\,\hat{k}\right)$$

$$=\int_S\left(xy\,\hat{i}+yz\,\hat{j}+xz\,\hat{k}\right)\cdot\hat{n}\,dS$$

$$\vec{F}=xy\,\hat{i}+yz\,\hat{j}+xz\,\hat{k}$$

Here, $\oint_S \vec{F}\cdot\hat{n}\,dS=\int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS+\int_{S_3}\vec{F}\cdot\hat{n}\,dS+\int_{S_4}\vec{F}\cdot\hat{n}\,dS$

On $S_1: x=0, \hat{n}=-\hat{i}, \vec{F}\cdot\hat{n}=-xy=0$

$$\int_{S_1}\vec{F}\cdot\hat{n}\,dS=0$$

On $S_2: y=0, \hat{n}=-\hat{j}, \vec{F}\cdot\hat{n}=-yz=0$

$$\int_{S_2} \vec{F}\cdot\hat{n}\,dS = 0$$

On $S_3 : z = 0, \hat{n} = -\hat{k}, \vec{F}\cdot\hat{n} = -xz = 0$

$$\int_{S_3} \vec{F}\cdot\hat{n} = 0$$

S_4 belongs to family of level surface

$S_4 : x + y + z$ = constant

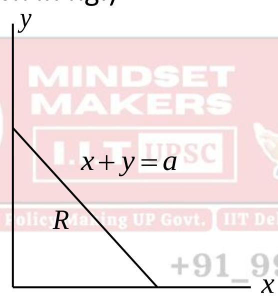
$$\hat{n} = \frac{\nabla S_4}{|\nabla S_4|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{3}}(xy + yz + zx)$$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{3}\,dx\,dy$$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,dS = \iint_R (xy + yz + zx)dydx$$

(R is the region of integration as shown in fig.)



$$= \iint\left[xy + (x+y)(a-x-y)\right]dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left(ax + ay - x^2 - y^2 - xy\right)dydx$$

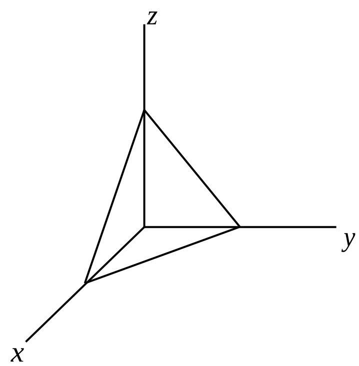
$$= \int_0^a axy + \frac{ay^2}{2} - x^2y^2 - \frac{y^3}{3} - \frac{xy^3}{2}\Bigg|_0^{a-x} dx$$

$$= \int_0^a a^2x - 2ax^2 + x^3 + \frac{1}{6}(a-x)^3\,dx$$

$$= \frac{1}{8}a^4$$

Ex 17. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $x + 2y + 3z = 6$ in the first octant.

Solution.



The plane belongs to the family of level surface given by $S = x + 2y + 3z$ = constant

A unit vector normal to the surface is given by $\hat{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$

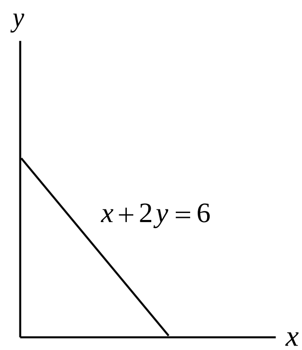
$$\vec{F} \cdot \hat{n} = \frac{1}{\sqrt{14}}\left[\left(x + y^2\right) - 4x + 6yz\right] = \frac{1}{\sqrt{14}}\left[x + y^2 - 4x + 2y(6 - x - 2y)\right]\left(z = \frac{1}{3}(6 - x - 2y)\right)$$

$$= \frac{1}{\sqrt{14}}\left(12y - 3x - 3y^2 - 2xy\right)$$

$$dS = \frac{dxdy}{\left|\hat{n} \cdot \hat{k}\right|} = \frac{\sqrt{14}}{3} \cdot dxdy$$

$$\vec{F} \cdot \hat{n} dS = \frac{1}{3}\left(12y - 3x - 3y^2 - 2xy\right) dxdy$$

So, $\int_S \vec{F} \cdot \hat{n}\, dS = \dfrac{1}{3}\int_0^6 \int_0^{\frac{6-x}{2}} \left(2y - 3x - 3y^2 - 2xy\right) dydx$ (The region of double integration is shown in figure)

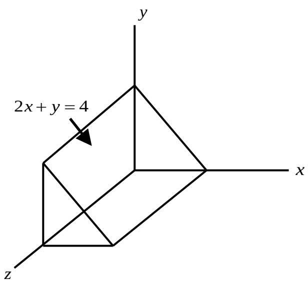


$$= \frac{1}{3}\int_0^6 \left[6y^2 - 3xy - y^3 - xy^2\right]_0^{\frac{6-x}{2}} \cdot dx$$

$$= \frac{1}{3}\int_0^6 \left(-\frac{x^3}{8} + \frac{15}{4}x^2 - \frac{45}{2}x + 27\right) dx = \frac{1}{3}\left[-\frac{x^4}{32} + \frac{5x^3}{4} - \frac{45}{4}x^2 + 27x\right]_0^6 = 4.5$$

Ex 18. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x + y = 4$ in the first octant cut off by the plane $z = 4$.

Solution.



The surface of the plane $2x + y = 4$ belongs to family of level surface $S = 2x + y$ = constant.

A unit vector normal to the surface

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}}$$

The integral $\vec{F}\cdot\hat{n} = \left(y\hat{i} + 2x\hat{j} - z\hat{k}\right)\cdot\left(\dfrac{2\hat{i} + \hat{j}}{\sqrt{5}}\right)$

$$= \frac{1}{\sqrt{5}}(2y + 2x) = \frac{2}{\sqrt{5}}(x + y)$$

$$\hat{n}\cdot\hat{j} = \frac{1}{\sqrt{5}}\left(2i + \hat{j}\right)\cdot\hat{j} = \frac{1}{\sqrt{5}}$$

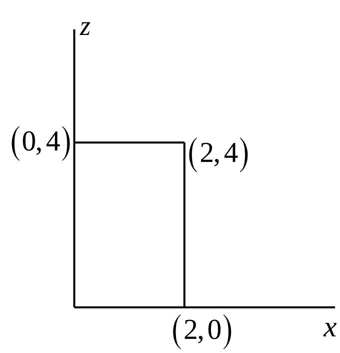
Now, taking projection of the surface on xz plane as shown in figure

$$dS = \frac{dxdz}{\left|\hat{n}\cdot\hat{j}\right|} = \sqrt{5}dxdz$$

$$\vec{F}\cdot\hat{n}dS = \frac{2}{\sqrt{5}}(x + y)\sqrt{5}dxdz = 2(x + y)dxdz$$

$= 2(x + 4 - 2x)dxdz$ ($y = 4 - 2x$ from the equation of surface)

$$= 2(4 - x)dxdz$$

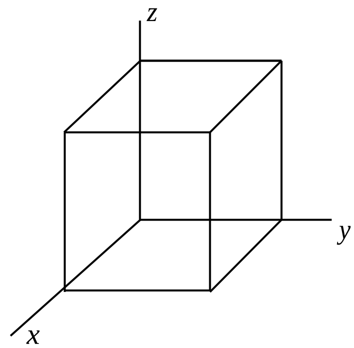


So, Surface integral becomes

$$\int_S \vec{F}\cdot\hat{n}\,dS = 2\int_0^4\int_0^2 (4 - x)\,dxdz = 2\int_0^4 \left. 4x - \frac{x^2}{2}\right|_0^2 dz = 12\int_0^4 dz = 48$$

Ex 19. Evaluate the surface integral $\iint \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot \hat{n}\, dS$ where S is the positive side of the cube formed by the plane $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$.

Solution.



S is piece wise smooth surface consisting of

$S_1 : x = 0, S_2 : y = 0, S_3 : z = 0; S_4 : x = 1, S_5 : y = 1, S_6 : z = 1$ as shown in fig.

On $S_1 : x = 0, dS = dydz, \hat{n} = -\hat{i}, \vec{F}\cdot\hat{n} = -x = 0$

$$\int_{S_1} \vec{F}\cdot\hat{n}\, dS = 0$$

On $S_2 : y = 0, dS = dxdz, \hat{n} = -\hat{j}, \vec{F}\cdot\hat{n} = -y = 0$

$$\int_{S_2} \vec{F}\cdot\hat{n}\, dS = 0$$

On $S_3 : z = 0, dS = dxdy, \hat{n} = -\hat{k}, \vec{F}\cdot\hat{n} = -z = 0$

$$\int_{S_3} \vec{F}\cdot\hat{n}\, dS = 0$$

On $S_4 : x = 1, dS = dydz, \hat{n} = \hat{i}, \vec{F}\cdot\hat{n} = x = 1$

$$\int_{S_4} \vec{F}\cdot\hat{n}\, dS = \iint dydz = 1$$

On $S_5 : y = 1, dS = dxdz, \hat{n} = \hat{j}, \vec{F}\cdot\hat{n} = y = 1$

$$\int_{S_5} \vec{F}\cdot\hat{n}\, dS = \iint dxdz = 1$$

On $S_6 : z = 1, dS = dxdy, \hat{n} = \hat{k}, \vec{F}\cdot\hat{n} = z = 1$

$$\int_{S_6} \vec{F}\cdot\hat{n}\, dS = \iint dxdy = 1$$

So, $\iint \vec{F}\cdot\hat{n}dS = \int_{S_1} \vec{F}\cdot\hat{n}dS + \int_{S_2} \vec{F}\cdot\hat{n}dS + \int_{S_3} \vec{F}\cdot\hat{n}dS + \int_{S_4} \vec{F}\cdot\hat{n}dS + \int_{S_5} \vec{F}\cdot\hat{n}dS + \int_{S_6} \vec{F}\cdot\hat{n}dS$

Ex 20. Evaluate $\int_S (x\cos\alpha + y\cos\beta + z\cos\gamma)dS$ where $\cos\alpha, \cos\beta, \cos\gamma$ are directional cosines of the outward drawn normal to the surfaces where S is the outer surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lying above the xy plane.

Solution.

S is the outer surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lying above the xy plane.

An outward drawn unit normal vector to S is given as

$$\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$

$$dS = \frac{dydz}{|\hat{n}\cdot\hat{i}|} = \frac{dydz}{\cos\alpha} \Rightarrow dydz = dS\cos\alpha$$

Similarly, $dxdy = dS\cos\gamma$

$$dxdz = dS\cos\beta$$

$$I = \int_S (x\cos\alpha + y\cos\beta + z\cos\gamma)dS = \iint xdydz + ydxdz + zdxdy$$

$$\left(\iint xdydz = \iint ydxdz = \iint zdxdy = \text{volume of ellipsoid in the above } xy \text{ plane} = \frac{2\pi}{3}abc\right)$$

So, $\int_S (x\cos\beta + y\cos\beta + z\cos\gamma)dS = 3\times\frac{2\pi}{3}abc = 2\pi abc$

Gauss Divergence Theorem

Few examples to understand GDT by taking different possible functions. It'll help us in expressing mathematical language.

Ex 1. Green's Theorem. Let ϕ and ψ are scalar point function which together with their derivatives in any direction are uniform and continuous within the region V bounded by closed surface S then

$$\oiint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{n}\,dS = \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)d\tau$$

Proof: By Gauss Divergence theorem

$$\oiint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau$$

Let $\vec{F} = \phi\nabla\psi - \psi\nabla\phi$

$\nabla\cdot\vec{F} = \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\nabla\phi)$

$= (\phi\nabla^2\psi + \nabla\phi.\nabla\psi) - (\psi\nabla^2\phi + \nabla\psi.\nabla\phi)$

$= \phi\nabla^2\psi - \psi\nabla^2\phi$

So, $\oiint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{n}\,dS = \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)d\tau$ ....(1)

Extra: Since, $\nabla\psi = \frac{\partial\psi}{\partial n}\hat{n}$

$\nabla\phi = \frac{\partial\phi}{\partial n}\hat{n}$

So, (1) can be written as

$\oiint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)d\tau$ ....(2)

Note: Harmonic function: A scalar function ϕ is said to be harmonic function if it satisfies Laplace's equation $\nabla^2\phi = 0$

If ϕ and ψ both are harmonic, i.e. $\nabla^2\phi = \nabla^2\psi = 0$ equation (2) reduces to

$$\oiint \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = 0.$$

Ex 2. Prove that

$$\int_V \nabla\phi d\tau = \int_S \phi\hat{n}\,dS$$

Proof: Let $\vec{F} = \phi\vec{C}$ where $\vec{C}$ is any arbitrary constant non zero vector(just to get an expression like to get the target)

$\nabla\cdot\vec{F} = \nabla\phi\cdot\vec{C} + \phi\nabla\cdot\vec{C}$

$= \nabla\phi\cdot\vec{C} \quad \left(\text{as}\,\nabla\cdot\vec{C} = 0\right)$

Applying Divergence theorem

$$\oiint_S \vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{F}\,d\tau$$

Where S is bounding surface of V.

$$\oiint \phi\vec{C}\cdot\hat{n}dS = \int \nabla\cdot\left(\phi\vec{C}\right)d\tau$$

$$\Rightarrow \quad \vec{C}\cdot\oiint \phi\hat{n}dS = \vec{C}\cdot\int\nabla\phi\,d\tau$$

$$\Rightarrow \quad \vec{C}\cdot\left[\int\nabla\phi\,d\tau - \oiint\phi\hat{n}\,dS\right] = 0$$

Since, $\vec{C}\cdot\left[\int\nabla\phi\,d\tau - \oiint\phi\hat{n}\,dS\right]$ is zero for any arbitrary non-zero vector $\vec{C}$.

So, $\int\nabla\phi\,d\tau - \oiint\phi\hat{n}\,dS = 0$

Hence, $\int\nabla\phi\,d\tau = \oiint\phi\hat{n}\,dS$

Ex 3. Prove that $\int\nabla\times\vec{g}\,d\tau = \oiint\hat{n}\times\vec{g}\,dS$.

Proof: Let $\vec{F} = \vec{g}\times\vec{C}$ where C is any arbitrary non-zero vector.

$$\nabla\cdot\vec{F} = \nabla\cdot\left(\vec{g}\times\vec{C}\right) = \vec{C}\cdot\text{curl}\,\vec{g} - \vec{g}\cdot\text{curl}\,\vec{C}$$

$$= \vec{C}\cdot\text{curl}\,\vec{g} \quad \left(\because \quad \text{curl}\,\vec{C} = 0\right)$$

Applying Divergence theorem

$$\oiint\vec{F}\cdot\hat{n}\,dS = \int\nabla\cdot\vec{F}\,d\tau$$

$$\Rightarrow \oiint\vec{g}\times\vec{C}\cdot\hat{n}\,dS = \int\vec{C}\cdot\text{curl}\,\vec{g}\,d\tau$$

$$\Rightarrow \oiint\left(\hat{n}\times\vec{g}\right)\cdot\vec{C}\,dS = \int\vec{C}\cdot\text{curl}\,\vec{g}\,d\tau \left(\because \left(\vec{A}\times\vec{B}\right)\cdot\vec{C} = \left(\vec{C}\times\vec{A}\right)\cdot\vec{B}\right)$$

$$\Rightarrow \quad \vec{C}\cdot\left[\int\text{curl}\,\vec{g}\,d\tau - \oiint\hat{n}\times\vec{g}\,dS\right] = 0$$

Since, $\vec{C}\cdot\left[\int\text{curl}\,\vec{g}\,d\tau - \oiint\hat{n}\times\vec{g}\,dS\right]$ is zero for any arbitrary non-zero vector $\vec{C}$,

So, $\int\text{curl}\,\vec{g}\,d\tau - \oiint\hat{n}\times\vec{g}\,dS = 0$

So, $\int_V\nabla\times\vec{g}\,d\tau = \oiint_S\hat{n}\times\vec{g}\,dS$

Ex4. Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the inward unit normal vector to S. Then $\oiint_S\vec{F}\cdot\hat{n}dS$ is equal to?

$$\oiint\vec{F}\cdot\hat{n}dS = -\oiint\vec{F}\cdot n'\,dS$$

Where $\hat{n}'$ is outward drawn unit normal vector to S i.e. $\hat{n} = -\hat{n}'$

$= -\int\nabla\cdot\vec{F}\,d\tau$ (Gauss Divergence theorem) $= -6\times$volume of sphere (Since, $\nabla\cdot\vec{F} = 6$) $= -8\pi$

Ex 5. Let S be a closed surface for which $\iint_S \vec{r}\cdot\hat{n}\,d\sigma = 1$. Then the volume enclosed by the surface is ?

$$\oiint \vec{r}\cdot\hat{n}dS = 1$$

$\Rightarrow \int \nabla\cdot\vec{r}d\tau = 1$ (Using Gauss Divergence theorem)

$\Rightarrow 3\int d\tau = 1$ (Since, $\nabla\cdot\vec{r} = 3$)

Volume $V = \int d\tau = \dfrac{1}{3}$

Ex 6. Let $\left\{(x,y,z)\in\mathbf{R}^3 : \frac{1}{4} \le x^2+y^2+z^2 \le 1\right\}$ and $\vec{F} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\left(x^2+y^2+z^2\right)^2}$ for $(x,y,z)\in V$. Let $\hat{n}$ denote the outward unit normal vector to the boundary of V and S denotes the part $\left\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+z^2 = \frac{1}{4}\right\}$ of the boundary of V. Then $\int_S \vec{F}\cdot\hat{n}dS$ is equal to?

Sol. Outward unit normal to boundary of V.

$$\hat{n} = -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{1/2} = -2\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$\int \vec{F}\cdot\hat{n}dS = -2\int \frac{\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{\left(x^2+y^2+z^2\right)^2}\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)dS = -2\int \frac{1}{x^2+y^2+z^2}dS$$

$$= -8\int dS \quad \left(\text{Since}, x^2+y^2+z^2 = \frac{1}{4}\text{ on S}\right) = -8\times 4\pi\cdot\frac{1}{4} = -8\pi$$

Ex 7. The value of the integral $\oiint_S \vec{F}\cdot\hat{n}dS$, where $\vec{F} = 3x\hat{i}+2y\hat{j}+z\hat{k}$ and S is the closed surface given by the planes $x=0, x=1, y=0, y=2, z=0$ and $z=3$ is ?

By divergence theorem

$$\oiint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau = 6\int_0^1\int_0^2\int_0^3 dx\,dy\,dz = 36$$

Ex 8. For any closed surface S, the surface integral $\oiint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS$ is equal to?

By divergence theorem

$$\oiint \operatorname{curl}\vec{F}\cdot\hat{n}dS = \int \operatorname{div}\left(\operatorname{curl}\vec{F}\right)d\tau = 0 \text{ since, } \nabla\cdot\left(\nabla\times\vec{F}\right) = 0$$

Ex 9. For any closed surface S, the integral $\oiint_S \vec{r}\cdot\hat{n}\,dS$ is equal to ?

By Divergence theorem

$$\oiint \vec{r}\cdot\hat{n}\,dS = \int \nabla\cdot\vec{r}d\tau = \int 3d\tau \quad (\nabla\cdot\vec{r} = 3) = 3\times\text{volume enclosed by surface S} = 3V$$

Ex 10-. If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}, a, b, c$ are constants, then the integral $\oint\!\!\oint_S \vec{F}\cdot\hat{n}\,dS$, S as a sphere of radius r is equal to?

By Divergence theorem

$$\oint\!\!\oint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

Where V is bounding surface of volume S.

Let $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

$$\oint\!\!\oint \vec{F}\cdot\hat{n}\,dS = \oint\!\!\oint \left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)\cdot\hat{n}dS$$

$= \int \nabla\cdot\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)d\tau$ (By Gauss divergence theorem)

$= (a+b+c)\int d\tau = (a+b+c)\times$ volume of sphere of radius $r = (a+b+c)\frac{4}{3}\pi r^3$

Ex 11. If $\hat{n}$ is the outward drawn unit normal vector to S then the integral $\int_V div\,\hat{n}\,d\tau$ is equal to ?

By Divergence theorem

$$\int \nabla\cdot\vec{F}d\tau = \oint\!\!\oint \vec{F}\cdot\hat{n}dS$$

So, $\int_V \nabla\cdot\hat{n}d\tau = \oint\!\!\oint_S \hat{n}\cdot\hat{n}dS = \oint\!\!\oint_S dS = S$.

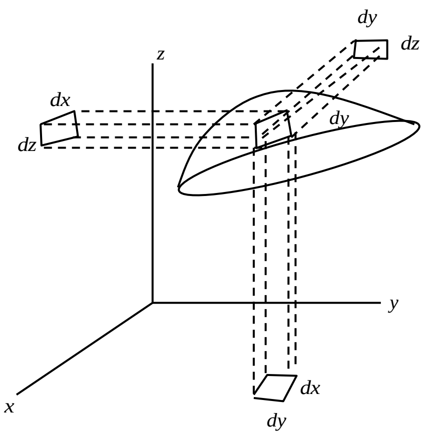
Ex 12. Let S be the surface of the cube bounded by $x=-1, y=-1, z=-1, x=1, y=1, z=1$. The integral $\oint\!\!\oint \vec{r}\cdot\hat{n}\,dS$ is equal to ?

Using Divergence theorem

$$\oint\!\!\oint \vec{r}\cdot\hat{n}\,dS = \int \nabla\cdot\vec{r}d\tau = 3\int d\tau = 3\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dxdydz = 3\times 8\int_0^1\int_0^1\int_0^1 dxdydz = 24$$

Examples: for CSE & IFoS

Ex1. S be the surface of sphere $x^2+y^2+z^2=9$. The integral $\iint_S \left[(x+z)\,dydz+(y+z)\,dzdx+(x+y)\,dxdy\right]$ is equal to?



The surface element

$$\hat{n}dS = dydz\,\hat{i} + dxdz\,\hat{j} + dxdy\,\hat{k}$$

So, $\iint\left[(x+z)\,dydz+(y+z)\,dzdx+(x+y)\,dxdy\right]$

$$=\iint\left[(x+z)\hat{i}+(y+z)\,\hat{j}+(x+y)\hat{k}\right]\cdot\left[dydz\,\hat{i}+dzdx\,\hat{j}+dxdy\,\hat{k}\right]$$

$$=\oint\left((x+z)\hat{i}+(y+z)\,\hat{j}+(x+y)\hat{k}\cdot\hat{n}\,dS\right)=\int\nabla\cdot\left((x+z)\hat{i}+(y+z)\,\hat{j}+(x+y)\hat{k}\right)d\tau=\int 2d\tau$$

$$=2\times\frac{4}{3}\cdot\pi(3)^3=72\pi$$

Ex 2. Use divergence theorem to evaluate $\iint_S x^3dydz+x^2ydzdx+x^2zdxdy$ where S is the sphere $x^2+y^2+z^2=1$.

Solution.

$$I\iint x^3dydz+x^2ydzdx+x^2zdxdy=\oint\left(x^3\hat{i}+x^2y\hat{j}+x^2z\hat{k}\right)\cdot\hat{n}dS$$

$=\int\nabla\cdot\left(x^3\hat{i}+x^2y\hat{j}+x^2z\hat{k}\right)d\tau$ (By Gauss Divergence theorem) $=5\iiint x^2dxdydz$

$$=5\iiint r^2\sin^2\theta\cos^2\phi\cdot r^2\sin\theta drd\theta d\phi=5\int_0^1\int_0^{\pi}\int_0^{2\pi}r^4\sin^3\theta\cos^2\phi\cdot d\phi d\theta dr$$

$$=5\pi\int_0^1\int_0^{\pi}r^4\sin^3\theta d\theta dr=10\pi\int_0^1\int_0^{\pi/2}r^4\sin^3\theta d\theta dr=10\pi\cdot\frac{2}{3}\int_0^1 r^4dr=\frac{4}{3}\pi$$

$$\left[\int_0^{\pi/2}\sin^3\theta d\theta=\frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{2}}=\frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}=\frac{2}{3}\right]$$

Ex 3. Using divergence theorem, evaluate $\oiint\limits_S \vec{A}\cdot\hat{n}dS$ where $A = x^3\hat{i} + y^3\hat{j} + z^3k$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution.

Using divergence theorem

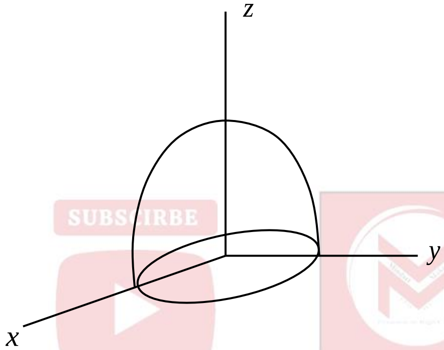
$$\oiint_S \vec{A}\cdot\hat{n}dS = \int\limits_V \nabla\cdot\vec{A}d\tau = 3\int\left(x^2+y^2+z^2\right)d\tau = 3\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{a} r^2r^2\sin\theta drd\theta d\phi$$

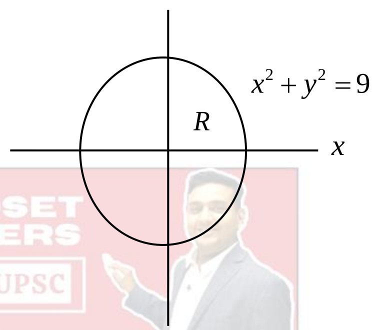
$$= 3\int_0^{2\pi}\int_0^{\pi}\sin\theta\left[\frac{r^5}{5}\right]_0^a d\theta d\phi = \frac{3}{5}a^5\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = \frac{3}{5}a^5\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi}d\phi = \frac{6}{5}a^5\int_0^{2\pi}d\phi = \frac{12}{5}\pi a^5$$

Ex 4. Evaluate $\oiint\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane bounded by this plane.

Solution.





By divergence theorem

$$\oiint\left(y^2z\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}dS = \int\nabla\cdot\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot d\tau = \iiint 2zy^2dxdydz$$

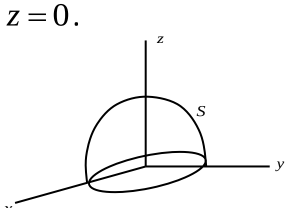
$$= \iiint 2r\cos\theta\cdot r^2\sin^2\theta\sin^2\phi\cdot r^2\sin\theta drd\theta d\phi = 2\int\limits_0^{a}\int\limits_0^{\pi/2}\int\limits_0^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi d\phi d\theta dr$$

$$= 2\pi\int_0^a\int_0^{\pi/2} r^5\sin^3\theta\cos\theta dr = 2\pi\int_0^a r^5\left.\frac{\sin^4\theta}{4}\right|_0^{\pi/2} dr = \frac{\pi}{2}\int_0^a r^5dr = \frac{1}{12}\pi a^6$$

Ex 5. Evaluate by divergence theorem the integral

$$\iint_3 xz^2dydz + \left(x^2y - z^3\right)dzdx + \left(2xy + y^2z\right)dxdy$$

Where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$.



The surface is shown in Figure

$$\hat{n}\, dS = dydz\hat{i} + dxdz\hat{j} + dxdy\hat{k} = \iint_S xz^2 dydz + \left(x^2y - z^3\right)dzdx + \left(2xy + y^2z\right)dxdy$$

$$= \oiint_S \left(xz^2\hat{i} + \left(x^2y - z^3\right)\hat{j} + \left(2xy + y^2z\right)\hat{k}\right)\cdot \hat{n}dS$$

S is the surface of hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ as shown in Figure.

$$\int_V \nabla\cdot\left(xz^2\hat{i} + \left(x^2y - z^3\right)\hat{j} + \left(2xy + y^2z\right)\hat{k}\right)d\tau$$

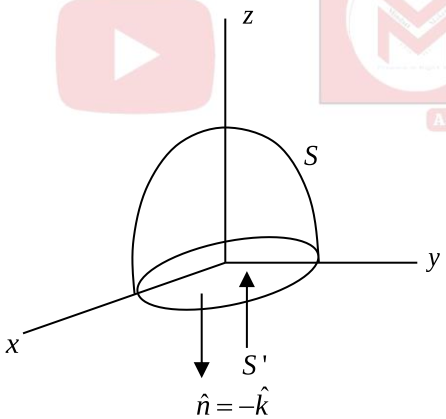
(By Gauss Divergence theorem $\oiint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot F d\tau$)

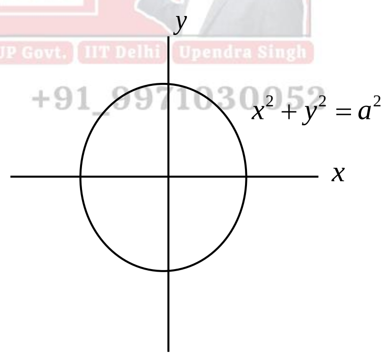
$$= \iiint\left(z^2 + x^2 + y^2\right)dxdydz = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a r^2\cdot r^2\sin\theta dr d\theta d\phi = \int_0^{2\pi}\int_0^{\pi/2}\left.\frac{r^5}{5}\right|_0^a \sin\theta d\theta d\phi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi/2}\sin\theta d\theta d\phi = \frac{a^5}{5}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/2} d\phi = \frac{a^5}{5}\int_0^{2\pi} d\phi = \frac{2\pi a^5}{5}$$

Ex 6. If $\vec{F} = \left(x^2 + y - 4\right)\hat{i} + 3xy\,\hat{j} + \left(2xz + z^2\right)\hat{k}$. Evaluate $\int_S \left(\nabla\times\vec{F}\right)\cdot\hat{n}dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above xy plane.

Solution.





The surface S is sphere $x^2 + y^2 + z^2 = a^2$ above xy plane as shown in Figure.

So, S is a open surface. But, Gauss theorem applies only to surface integral on closed surface. Had the surface S been closed, the integral $\int_S \nabla\times\vec{F}\cdot\hat{n}dS$ would have been zero because

$$\int_S \nabla\times\vec{F}\cdot\hat{n}dS = \int\nabla\cdot\left(\nabla\times\vec{F}\right)d\tau = 0$$

Since, divergence of curl $\vec{F}$ will be zero.

S is open surface. Here we will make use of the fact that $\int\nabla\times\vec{F}\cdot\hat{n}dS$ over the closed surface will be zero.

Consider a closed piecewise smooth surface S_2 consisting of spherical surface $S : x^2 + y^2 + z^2 = a^2$ and $S' : z = 0$ enclosing a volume V.

$$\oiint_{S_2}\left(\nabla\times\vec{F}\right).\hat{n}dS=\int_V\nabla\cdot\left(\nabla\times\vec{F}\right)dV=0\Rightarrow\int_S\nabla\times\vec{F}\cdot\hat{n}dS+\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS=0\because\ \nabla\cdot\left(\nabla\times\vec{F}\right)=0$$

$$\Rightarrow\int_S\nabla\times\vec{F}\cdot\hat{n}dS=-\int_{S'}\nabla\times\vec{F}\cdot\hat{n}dS$$

And On $S', z=0, dS=dxdy, \hat{n}=-\hat{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}$$

$$=-2z\hat{j}+\left(3y-1\right)\hat{k}$$

So, $\nabla\times\vec{F}\cdot\hat{n}=\left(-2z\hat{j}+\left(3y-1\right)\hat{k}\right)\cdot\left(-\hat{k}\right)=-\left(3y-1\right)$

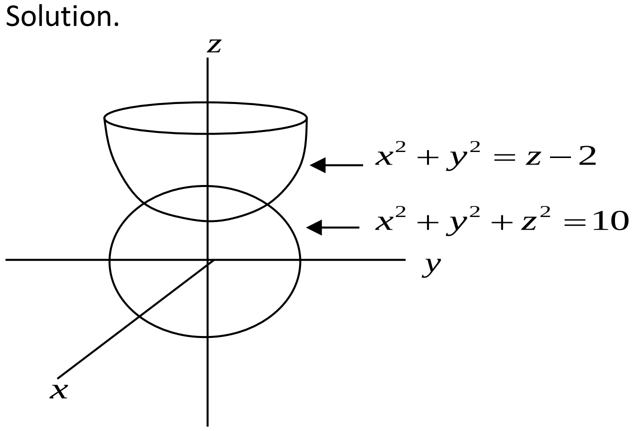
So, $\int_S\nabla\times\vec{F}\cdot\hat{n}dS=-\int_S\nabla\times\vec{F}\cdot\hat{n}dS=\iint\left(3y-1\right)dydx$

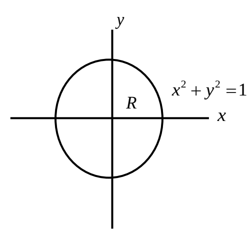
$$=3\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}y\,dydx-\iint dydx=0 - \text{Area of base}=-\pi a^2$$

Note: In this problem, we have converted as integral over a curved surface to integral over a plane surface.

Ex7. Using Gauss's divergence theorem, evaluate the integral $\int_S\vec{F}\cdot\hat{n}dS$, where $\vec{F}=4xz\hat{i}-y^2\hat{j}+4yz\hat{k}$, S is the surface of the solid bounded by the sphere $x^2+y^2+z^2=10$ and the paraboloid $x^2+y^2=z-2$, and $\hat{n}$ is the outward unit normal vector to S.

Solution.





$$\vec{F}=4xz\hat{i}-y^2\hat{j}+4yz\hat{k}$$

$$\nabla\cdot\vec{F}=4z+2y$$

Using Gauss divergence theorem

$$\oiint_S\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\vec{F}d\tau$$

$$= \iint \int_{x^2+y^2+2}^{\sqrt{10-x^2-y^2}} (4z+2)\,dzdydx \;\; (d\tau = dxdydz)$$

$$= 2\iint \left[z^2 + yz \right]_{x^2+y^2+2}^{\sqrt{10-x^2-y^2}} dxdy$$

$$= 2\iint \left(6 - 5\left(x^2+y^2\right) - \left(x^2+y^2\right)^2 + y\left(\sqrt{10-x^2-y^2} - x^2 - y^2 - 2\right)\right) dxdy$$

Surfaces bounding the volume are $x^2+y^2+z^2=10$ & $x^2+y^2=z-2$ as shown in Figure.

So, curve of intersection of surfaces is given as

$x^2+z-2=10 \Rightarrow z=3$

$$\left.\begin{array}{l} x^2+y^2=1 \\ z=3 \end{array}\right\} \text{Curve of intersection}$$

Putting $x = r\cos\theta, y = r\sin\theta, dxdy = rd\theta dr$

(r is the region of integration of double integration)

$$\oiint_S \vec{F}\cdot\hat{n}dS = 2\int_0^1\int_0^{2\pi}\left[\left(6-5r^2-r^4\right)+r\sin\theta\left(\sqrt{10-r^2}-\left(r^2+2\right)\right)\right] rd\theta dr$$

$$= 2\int_0^1\int_0^{2\pi}\left(6-5r^2-r^4\right)rd\theta dr + 2\int_0^1\int_0^{2\pi} r^2\left(\sqrt{10-r^2}-r^2-2\right)\sin\theta d\theta dr$$

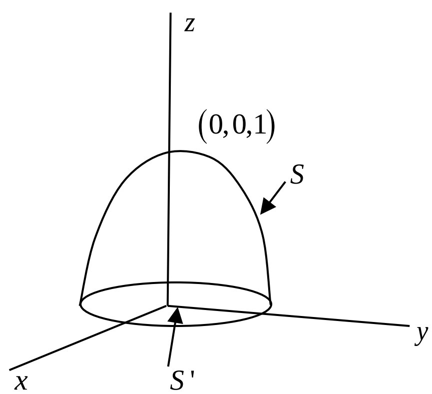
Now, $\int_0^{2\pi}\sin\theta d\theta = 0$ So, integral of second term

$$2\int_0^1\int_0^{2\pi} r^2\left(\sqrt{10-r^2}-r^2-2\right)\sin\theta d\theta dr = 0$$

$$\oiint \vec{F}\cdot\hat{n}dS = 4\pi\int_0^1\left[6r-5r^3-r^5\right]dr = 4\pi\left[3r^2-\frac{5r}{4}-\frac{r^6}{6}\right]_0^1 = \frac{19}{3}\pi$$

Ex 8. Let S be the surface $\left\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+2z=2, z\geq 0\right\}$, and let $\hat{n}$ be the outward unit normal to S. If $\vec{F} = y\hat{i} + xz\hat{j} + \left(x^2+y^2\right)\hat{k}$, then evaluate the integral $\int_S \vec{F}\cdot\hat{n}dS$.

Solution.



$S: x^2+y^2 = -2(z-1)$

is a paraboloid with vertex at $(0,0,1)$ as shown is Figure

$\vec{F} = y\hat{i} + xz\hat{j} + \left(x^2+y^2\right)\hat{k}; \quad \nabla\cdot\vec{F} = 0$

Consider a closed surface S which consists of two piecewise smooth surface S and S", where S' is base of Paraboloid and S is paraboloid

$$\oiint_{\Sigma} \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau = 0$$

$$\oiint_{\Sigma} \vec{F}\cdot\hat{n}dS = \int_S \vec{F}\cdot\hat{n}dS + \int_{S'} \vec{F}\cdot\hat{n}dS = 0$$

$$\int_S \vec{F}\cdot\hat{n}dS = -\int_{S''} \vec{F}\cdot\hat{n}dS$$

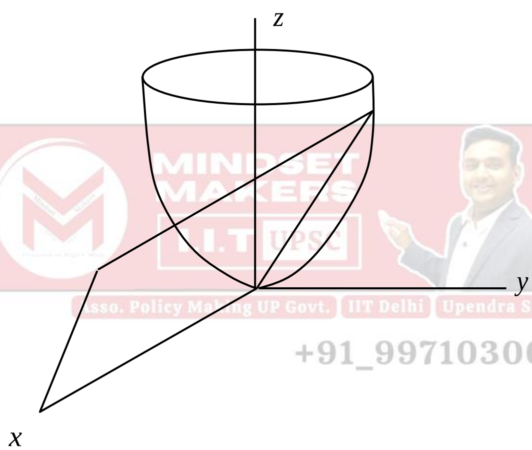
For $S', \hat{n} = -\hat{k} \quad dS = dxdy$

So, $\int_S \vec{F}\cdot\hat{n}dS = -\int_{S''} \vec{F}\cdot\hat{n}dS$

$$= -\iint\left(y\hat{i} + xz\hat{j} + \left(x^2+y^2\right)\hat{k}\right)\cdot\left(-\hat{k}\right)dxdy = \iint\left(x^2+y^2\right)dxdy = \int_0^{2\pi}\int_0^{\sqrt{2}} r^2 r\,d\theta\,dr = \int_0^{2\pi}\left|\frac{r^4}{4}\right|_0^{\sqrt{2}} d\theta = 2\pi$$

Ex 9. Use divergence theorem to evaluate $\oiint \vec{V}\cdot\hat{n}dS$ where $\vec{V} = x^2z\hat{i} + y\hat{j} - xz^2\hat{k}$ and is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

Solution.

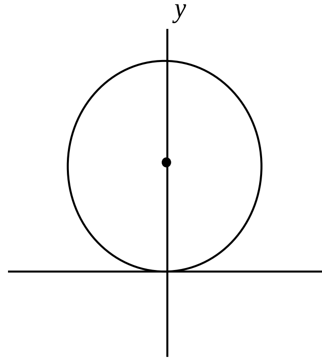


Applying Gauss divergence theorem

$\oiint \vec{V}\cdot\hat{n}dS = \int \nabla\cdot\vec{V}d\tau = \int d\tau$ (This region of volume integration is as shown in Figure)

$$= \iint \int_{x^2+y^2}^{4y} dzdxdy = \iint\left(4y - x^2 - y^2\right)dxdy$$

The region of integration of double integration in the projection of region V on xy plane as shown in Figure



$x^2+y^2=4y \Rightarrow x^2+(y-2)^2=4$

In polar form, $r=4\sin\theta$

$$I=\iint\left(4y-x^2-y^2\right)dxdy=\int_0^{\pi}\int_0^{4\sin\theta}\left(4r\sin\theta-r^2\right)rdrd\theta=\int_0^{\pi}\left[\frac{4}{3}r^3\sin\theta-\frac{r^4}{4}\right]_0^{4\sin\theta}d\theta$$

$$=\frac{64}{3}\int_0^{\pi}\sin^4\theta d\theta=\frac{128}{3}\int_0^{\pi/2}\sin^4\theta d\theta=\frac{128}{3}\cdot\frac{3\pi}{16}=8\pi \qquad \left[\int_0^{\pi/2}\sin^4\theta d\theta=\frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)}=\frac{\frac{3}{2}\cdot\frac{1}{2}\pi}{2\times2\times1}\right]$$

Ex 10. Evaluate by using Gauss divergence theorem

(i) $\oiint_S\left(a^2x^2+b^2y^2+c^2z^2\right)^{1/2}dS$

(ii) $\oiint_S\left(a^2x^2+b^2y^2+c^2z^2\right)^{-1/2}dS$

over the ellipsoid $ax^2+by^2+cz^2=1$.

Solution.

S is the ellipsoid belonging to family to level surface as shown in Figure.

$S: ax^2+by^2+cz^2$ = constant

The outward drawn unit normal vector $\hat{n}$ to S is given by $\hat{n}=\dfrac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$

(i) $\oiint\left(a^2x^2+b^2y^2+c^2z\right)^{1/2}dS=\oiint\vec{F}\cdot\hat{n}dS$

Comparing the integrals

$\vec{F}\cdot\hat{n}=\left(a^2x^2+b^2y^2+c^2z^2\right)^{1/2}$

$\vec{F}\cdot\dfrac{\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)}{a^2x^2+b^2y^2+c^2z^2}=\left(a^2x^2+b^2y^2+c^2z^2\right)^{1/2}$

$\vec{F}\cdot\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)=a^2x^2+b^2y^2+c^2z^2$

For using Gauss Divergence theorem, $\vec{F}$ should continuous and should have continuous partial derivatives in region V enclosed by ellipsoid S. The surface $\vec{F}$ can be taken as

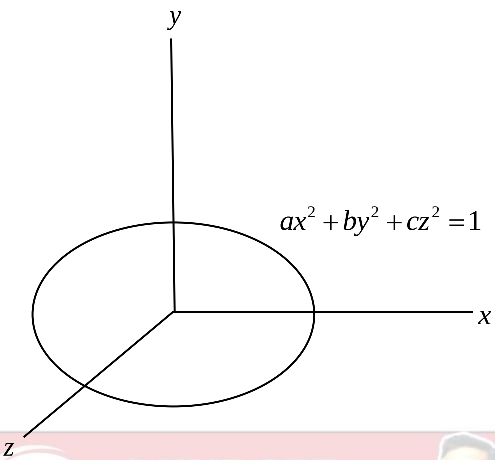
$\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

So, $\iint_S \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{1/2} dS = \iint_S \left(ax\hat{i} + by\hat{j} + c^2\hat{k}\right)\cdot \hat{n}dS = \int_V \nabla\cdot\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)d\tau$

According to Gauss Divergence theorem

$$\iint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau = (a+b+c)\int_V d\tau = (a+b+c)\times \text{volume of ellipsoid } = \frac{4\pi(a+b+c)}{3\sqrt{abc}}$$

(ii) $\iint \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{-1/2} dS = \iint \vec{F}\cdot\hat{n}dS$



Comparing the integral

$\vec{F}\cdot\hat{n} = \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{-1/2}$

$$\Rightarrow \vec{F}\cdot\frac{\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \frac{1}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} \Rightarrow \vec{F}\cdot\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right) = 1$$

The function $\vec{F}$ can be taken as

$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{F}\cdot\hat{n} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right) = ax^2 + by^2 + cz = 1$ (on $S, ax^2 + by^2 + cz^2 = 1$)

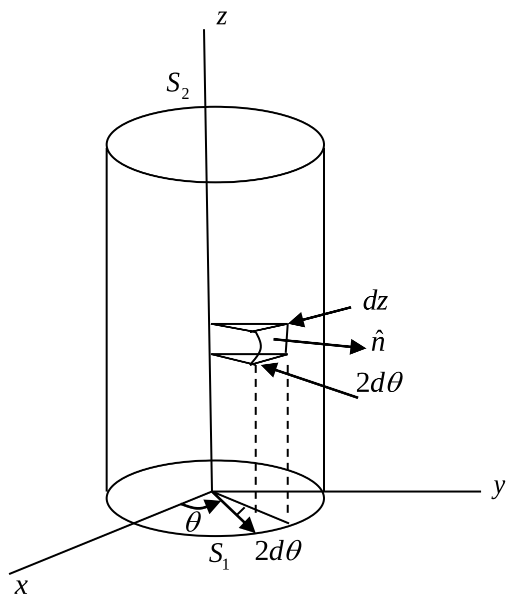
$$\iint_S \left(ax^2 + by^2 + cz\right)dS = \iint \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot\hat{n}dS = \int \nabla\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)d\tau = 3\int d\tau$$

$= 3\times$ volume of ellipsoid $= \dfrac{4\pi}{\sqrt{abc}}$

Note. While evaluating surface integration, we can incorporate the equation of surface.

Ex 11. Verify the divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$.

Solution.



Let us first calculate the volume integral

$\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$; $\qquad \nabla \cdot \vec{F} = (4 - 4y + 2z)$

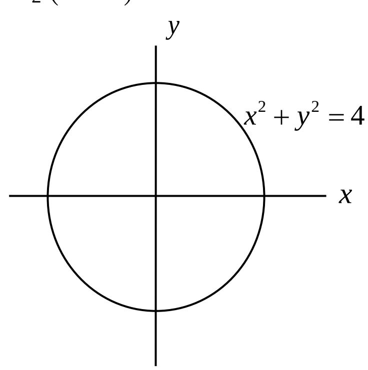
$$\int_0 \nabla \cdot \vec{F} d\tau = \iint \int_0^3 (4 - 4y + 2z)\, dz dy dx = \iint \left[(4-4y)z + z^2\right]_0^3 dy dx = \iint (21 - 12y)\, dy dx$$

The region of double integral is shown in Figure

$$\iint (21 - 12y)\, dy dx = 21 \iint dy dx - 12 \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} y dy dx = 21 \iint dy dx - 0 \qquad \left(\because \int_{-a}^{a} f(x) dx = 0 \text{ if } f \text{ in odd} \right)$$

$= 84\pi$

This volume V is bounded by the surface S which is a piecewise smooth surface consisting of lower base $S_1(z=0)$, upper base $S_2(z=3)$ and curved surface $S_3(x^2+y^2=4)$.



On $S_1, z = 0, dS = dxdy, \hat{n} = -\hat{k}, \vec{F} \cdot \hat{n} = 0$

$\int_{S_1} \vec{F} \cdot \hat{n} dS = 0$

On $S_2, z = 3, dS = dxdy, \hat{n} = \hat{k}, \vec{F} \cdot \hat{n} = z^2 = 9$

$\int_{S_2} \vec{F} \cdot \hat{n} dS = 9 \int_{S_2} dS = 9 \times$ Area of circle of radius 2

$= 36\pi$

On $S_3 x = 2\cos\theta, y = 2\sin\theta$

Equation of S_3 belongs to family of level surface $S : x^2 + y^2$ =constant

An outward drawn unit normal vector

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j}}{2}$$

$$\vec{F} \cdot \hat{n} = \left(2x^2 - y^3\right) = 8\cos^2\theta - \sin^3\theta$$

$$dS = 2d\theta dz$$

$$\int_{\pi_3} \vec{F} \cdot \hat{n} dS = 16\int_0^{2\pi}\int_0^3 \left(\cos^2\theta - \sin^3\theta\right) dz d\theta = 48\int_0^{2\pi}\left(\cos^2\theta - \sin^3\theta\right) d\theta$$

$$= 48\int_0^{2\pi}\cos^2\theta d\theta - 48\int_0^{2\pi}\sin^3\theta d\theta \quad \left(\int_0^{2\pi}\sin^3\theta d\theta = 0\right) = 48\pi$$

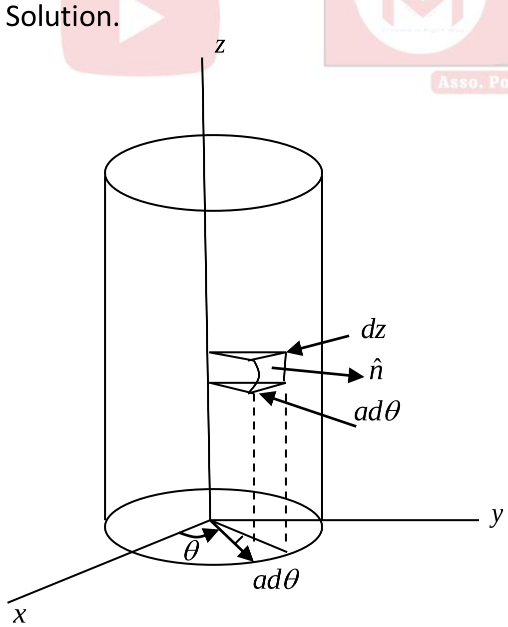
The surface integral over S

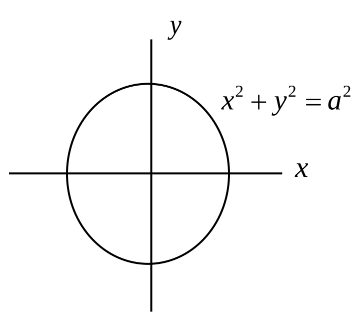
$$\oiint_S \vec{F} \cdot \hat{n} dS = \int_{S_1} \vec{F} \cdot \hat{n} dS + \int_{S_2} \vec{F} \cdot \hat{n} dS + \int_{S_3} \vec{F} \cdot \hat{n} dS = 0 + 36\pi + 48\pi = 84\pi$$

Hence, $\oiint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} d\tau$

Ex 12. Using Divergence theorem evaluate $I = \iint x^3 dydz + x^2 ydzdx + x^2 zdxdy$ where S is the closed surface bounded by the planes $z = 0, z = b$ and the cylinder $x^2 + y^2 = a^2$.

Solution.





$$I = \iint x^3 dydz + x^2 ydzdx + x^2 zdxdy = \oiint\left(x^3\hat{i} + x^2 y\hat{j} + x^2 z\hat{k}\right) \cdot \hat{n} dS = \int \nabla \cdot \left(x^3\hat{i} + x^2 y\hat{j} + x^2 z\hat{k}\right) d\tau$$

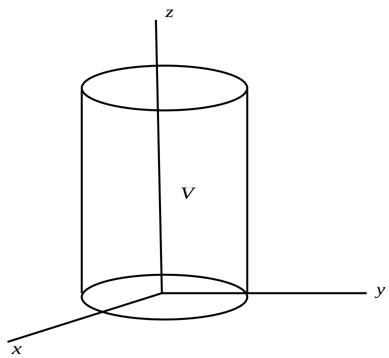
$$= 5\iint\int_0^b x^2 dzdxdy = 5b\iint_R x^2 dxdy$$

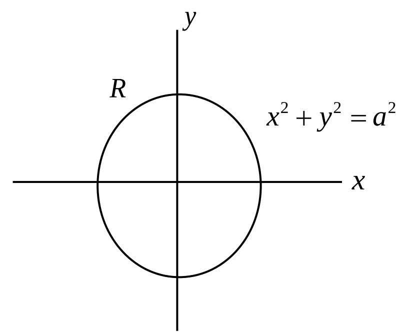
(This region of double integral R is given by projection cylinder on xy plane as shown in Figure)

$$= 5b\int_0^{2\pi}\int_0^a r^2\cos^2\theta rdrd\theta = 5b\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^a \cos^2\theta d\theta = \frac{5}{4}a^4b\int_0^{2\pi}\cos^2\theta d\theta = \frac{5}{4}\pi a^4 b$$

Ex 13. If $\vec{F} = x\hat{i} - y\hat{j} + \left(z^2 - 1\right)\hat{k}$ find the value of $\oiint \vec{F}\cdot\hat{n}dS$ where S in the closed surface bounded by the planes $z = 0, z = b$ and the cylinder $x^2 + y^2 = a^2$.

Solution.





By Gauss Divergence theorem

$\oiint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$

$\vec{F} = x\hat{i} - y\hat{j} + \left(z^2 - 1\right)\hat{k}$; $\quad \nabla\cdot\vec{F} = 2z$

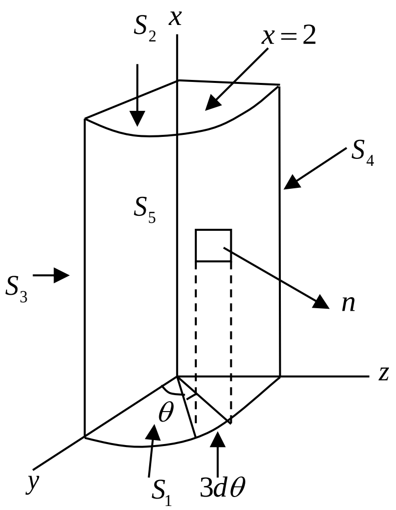
$\int \nabla\cdot Fd\tau = \iint\int_0^b 2zdzdxdy = \iint \left. z^2\right|_0^b dxdy = b^2\iint_R dxdy$

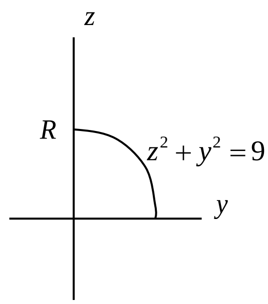
(The region of integration R is projection of volume region V on xy plane as shown in Figure)

$= b^2\times$ area of circle of radius $a = \pi a^2 b^2$

Ex 14. Verify divergence theorem for $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ & $x = 2$.

Solution.





Let us first find the volume integral $\int_V \nabla\cdot F d\tau$, V is the volume enclosed by surface $y^2+z^2=9$ & $x=2$ in first octant as shown in Figure.

$\vec{F}=2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k}$; $\quad\quad \nabla\cdot\vec{F}=4xy-2y+8z$

$$\int_V \nabla\cdot F d\tau=\iint\int_0^2(4xy-2y+8xz)dxdydz=\iint_R 2x^2y-2xy+4x^2z\Big|_0^2 dydz$$

(R is the projection of V is xy plane as shown in Figure).

$$=4\int_0^3\int_0^{\pi/2}(r\cos\theta+4r\sin\theta)rd\theta dr=4\int_0^3 r^2\left[\sin\theta-4\cos\theta\right]_0^{\pi/2}dr=20\int_0^3 r^2dr=180$$

Now, let us calculate the surface integral over S. S is a piecewise smooth surface consisting of $S_1(x=0), S_2(x=2), S_3(z=0), S_4(y=0), S_5(y^2+z^2=9)$

On $S_1, x=0, dS=dydz, \hat{n}=-\hat{i}, \vec{F}\cdot\hat{n}=0$

So, $\int_{S_1}\vec{F}\cdot\hat{n}dS=0$

On $S_2, x=2, dS=dydz, \hat{n}=\hat{i}, \vec{F}\cdot\hat{n}=8y$

So, $\int_{S_2}\vec{F}\cdot\hat{n}dS=8\iint ydydz=8\int_0^3\int_0^{\pi/2}r\cos\theta rd\theta dr=8\int_0^3 r^2dr=72$

On $S_3, z=0, dS=dxdy, \hat{n}=-\hat{k}, \vec{F}\cdot\hat{n}=0$

So, $\int_{S_3}\vec{F}\cdot\hat{n}dS=0$

On $S_4, y^2+z^2=9, dS=3d\theta dx$

$\hat{n}=\dfrac{y\hat{j}+z\hat{k}}{3}, \vec{F}\cdot\hat{n}=\dfrac{1}{3}\left(4xz^3-y^3\right)$

Let $y=3\cos\theta, z=3\sin\theta$

$\vec{F}\cdot\hat{n}=9\left(4x\sin^3\theta-\cos^3\theta\right)$

$\vec{F}\cdot\hat{n}dS=27\left(4x\sin^3\theta-\cos^3\theta\right)d\theta dx$

$$\int_{S_5} \vec{F}\cdot\hat{n}\,dS = 27\int_0^2\int_0^{\pi/2}\left(4x\sin^3\theta-\cos^3\theta\right)d\theta\,dx$$

$$\left[\int_0^{\pi/2}\sin^3\theta\,d\theta=\int_0^{\pi/2}\cos^3\theta\,d\theta=\frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{2}}=\frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}=\frac{2}{3}\right]$$

$$=18\int_0^2(4x-1)\,dx=18\left[2x^2-x\right]_0^2=108$$

So, surface integral $\oiint_S \vec{F}\cdot\hat{n}\,dS$ is give as

$$\oiint_S \vec{F}\cdot\hat{n}\,dS=\int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS+\int_{S_3}\vec{F}\cdot\hat{n}\,dS+\int_{S_4}\vec{F}\cdot\hat{n}\,dS+\int_{S_5}\vec{F}\cdot\hat{n}\,dS=0+72+0+0+108$$

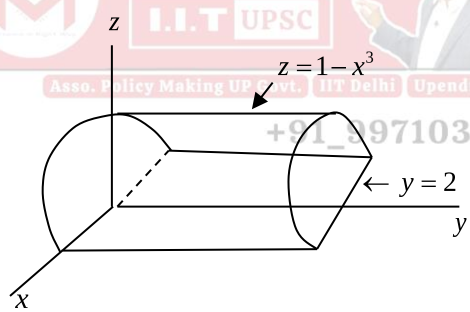
$$=180$$

So, $\oiint \vec{F}\cdot\hat{n}\,dS=\int\nabla\cdot\vec{F}\,d\tau$

Hence, Gauss divergence theorem is verified.

Ex 15. Let S be the boundary of the region consisting of the parabolic cylinder $z=1-x^2$ and the planes $y=0, y=2$ and $z=0$. Evaluate the integral $\oiint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F}=xy\hat{i}+\left(y^2+e^{xz^2}\right)\hat{j}+\sin(xy)\hat{k}$ and $\hat{n}$ is the outward drawn unit normal to S.

Solution.



The surface S is shown Figure

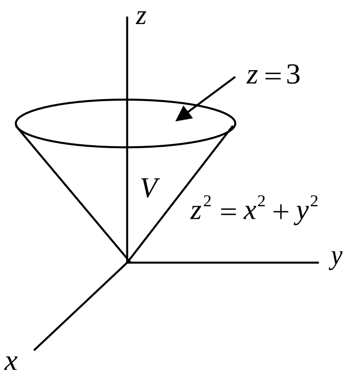
$$\oiint_S \vec{F}\cdot\hat{n}\,dS=\int_V\nabla\cdot\vec{F}\,d\tau=\iint\int_0^2 3y\,dx\,dz\,dy=3\iint\left.\frac{y^2}{2}\right|_0^2 dx\,dz$$

$$=6\int_{-1}^{1}\int_0^{1-x^2}dx\,dz=6\int_{-1}^{1}\left(1-x^2\right)dx=6\left[x-\frac{x^3}{3}\right]_{-1}^{1}=12\times\frac{2}{3}=8$$

Ex 16. Evaluate $\oiint \vec{F}\cdot\hat{n}\,dS$ over the entire surface of the region above the xy plane bounded by the cone $z^2=x^2+y^2$ and the plane $z=3$ if $\vec{F}=4xz\hat{i}+xyz^2\hat{j}+3z\hat{k}$.

Solution.

By Gauss Divergence



$$\oiint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$$

$$= \int_V \nabla\cdot\left(4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}\right)d\tau$$

(V is volume enclosed by cone $z^2 = x^2 + y^2$ and the plane $z = 3$ as shown in Figure)

$$= \iint \int_{\sqrt{x^2+y^2}}^{3} \left(4z + xz^2 + 3\right)dzdxdy = \iint_R 2z^2 + x\frac{z^3}{3} + 3z\Big|_{\sqrt{x^2+y^2}}^{3} dxdy$$

(The region of double integration R is projection of volume V on xy plane as shown Figure)

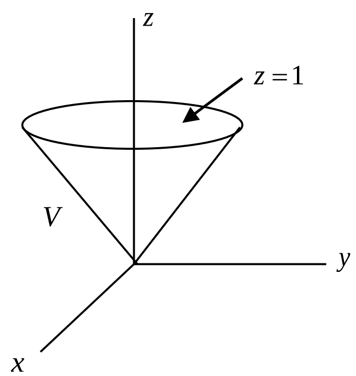
$$\iint\left[2\left(9 - x^2 - y^2\right) + \frac{x}{3}\left(27 - \left(x^2+y^2\right)^{3/2}\right) + 3\left(3 - \sqrt{x^2+y^2}\right)\right]dx\,dy$$

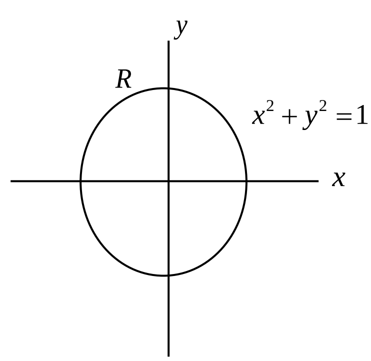
$$= \int_0^3\int_0^{2\pi}\left[\left(27 - 2r^2 - 3r\right) + \frac{1}{3}r\cos\theta\left(27 - r^3\right)\right]rd\theta\,dr = 2\pi\left[\frac{27}{2}r^2 - \frac{r^4}{2} - r^3\right]_0^3 = 108\pi$$

Ex 17. By using Gauss Divergence theorem, Evaluate $\oiint\left(x\hat{i} + y\hat{j} + z^2\hat{k}\right)\cdot\hat{n}\,dS$

where S is the closed surface bounded by cone $x^2 + y^2 = z^2$ and the plane $z = 1$.

Solution.





Using Gauss Divergence theorem

$$\oiint_S \vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{F}d\tau$$

$$\int_S\left(x\hat{i} + y\hat{j} + z^2\hat{k}\right)\cdot\hat{n}dS = \int_V \nabla\cdot\left(x\hat{i} + y\hat{j} + z^2k\right)d\tau = 2\iint\int_{\sqrt{x^2+y^2}}^{1}(z+1)\,dzdxdy$$

(V is volume enclosed by cone $x^2 + y^2 = z^2$ & the plane $z = 1$ as shown in Figure)

$$= 2\iint_R \frac{z^2}{2} + z\Big|_{\sqrt{x^2+y^2}}^{1} dxdy$$

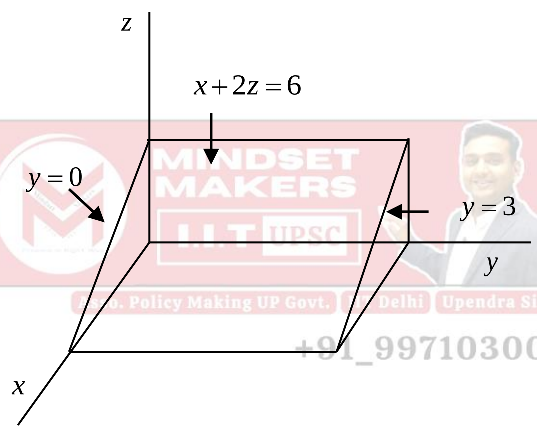
(The region of integration of double integral R is the projection of volume V on xy plane as shown in Figure)

$$= \iint (1 - x^2 - y^2) + 2\left(1 - \sqrt{x^2+y^2}\right) dxdy = \int_0^{2\pi}\int_0^1 (3 - 2r - r^2)\, r dr d\theta$$

$$= \int_0^{2\pi} \frac{3}{2}r^2 - \frac{2r^3}{3} - \frac{r^4}{4}\Big|_0^1 d\theta = \frac{7}{12}\int_0^{2\pi} d\theta = \frac{7\pi}{6}$$

Ex 18. Let W be the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + 2z = 6$. Let S be the boundary of this region. Using Gauss divergence theorem, evaluate $\int_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and $\hat{n}$ is the outward unit normal vector to S.

Solution.



Using Gauss Divergence theorem

$$\oiint \vec{F} \cdot \hat{n} dS = \int \nabla \cdot F d\tau$$

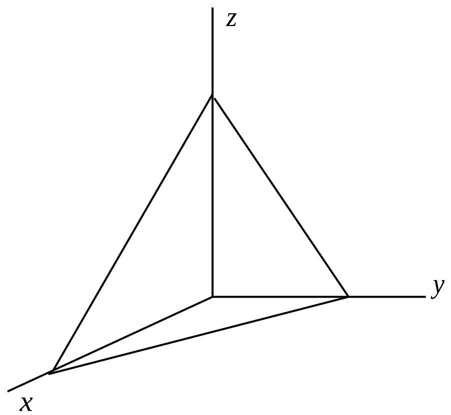
$$= \iiint (2y + z^2 + x)\, dxdydz = \iint \int_0^3 (x + 2y + z^2)\, dydxdz = \iint xy + y^2 + z^2 y\Big|_0^3 dxdz$$

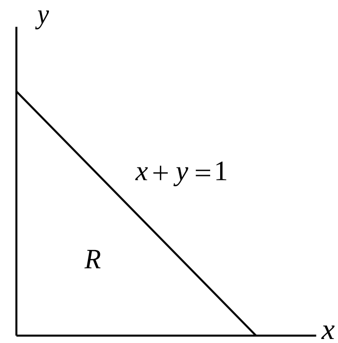
$$= \int_0^6 \int_0^{\frac{6-x}{2}} (3x + 3z^2 + 9)\, dzdx = \int_0^6 3xz + 9z + z^3\Big|_0^{\frac{6-x}{2}} dx = \frac{1}{8}\int_0^6 (-x^3 + 6x^2 - 72x + 512)\, dx$$

$$= \frac{1}{8}\left[-\frac{x^4}{4} + 2x^3 - 36x^2 + 512x\right]_0^6 = 235.5$$

Ex 19. Verify Gauss divergence theorem for $\vec{F} = xy\hat{i} + z^2\hat{j} + 2yz\hat{k}$ on the tetrahedron $x = y = z = 0, x + y + z = 1$

Solution.





• Let us find volume integral $\int_V \nabla\cdot\vec{F}d\tau$

V is the region bounded by $x=0, y=0, z=0$ and $x+y+z=1$ as shown in Figure

$\vec{F}=xy\hat{i}+z^2\hat{j}+2yz\hat{k}$; $\quad\nabla\cdot\vec{F}=3y$

$$\int\nabla\cdot\vec{F}d\tau=3\iint\int_0^{1-x-y}ydzdxdy=3\iint_R y(1-x-y)dxdy$$

Where R is the region of double integral obtained by taking projection of V on the xy plane as shown in Figure

$$=3\int_0^1\int_0^{1-x}\left(y(1-x)-y^2\right)dydx=3\int_0^1\left[(1-x)\frac{y^2}{2}-\frac{y^3}{3}\right]_0^{1-x}dx=\frac{1}{2}\int_0^1(1-x)^3dx=-\frac{1}{2}\frac{(1-x)^4}{4}\bigg|_0^1=\frac{1}{8}$$

• The volume V is bounded by surface S. S is a piecewise smooth surface consisting of $S_1(x=0)$,

$S_2(y=0), S_3(z=0), S_4(x+y+z=1)$

On $S_1, x=0, \hat{n}=-\hat{i}, dS=dydz, \vec{F}\cdot\hat{n}=0$

$$\int_{S_1}\vec{F}\cdot\hat{n}dS=0$$

On $S_2, y=0, dS=dxdz, \hat{n}=-\hat{j}, \vec{F}\cdot\hat{n}=-z^2$

$$\int_{S_2}\vec{F}\cdot\hat{n}dS=\int_0^1\int_0^{1-x}z^2dzdx=-\int_0^1\frac{z^3}{3}\bigg|_0^{(1-x)}dx=-\frac{1}{3}\int_0^1(1-x)^3dx=\frac{1}{12}(1-x)^4\bigg|_0^1=-\frac{1}{12}$$

On $S_3, z=0, dS=dxdy, \hat{n}=-\hat{k}, \vec{F}\cdot\hat{n}=0$

$$\int_{S_3}\vec{F}\cdot\hat{n}dS=0$$

On S_4, equation of S_4 belongs to family of level surface given by

$S: x+y+z$ = constant

Outward drawn unit normal to S_4

$$\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\vec{F}\cdot\hat{n}=\frac{1}{\sqrt{3}}\left(xy+z^2+2yz\right)=\frac{1}{\sqrt{3}}\left(xy+(1-x-y)^2+2y(1-x-y)\right)=\frac{1}{\sqrt{3}}\left(x^2-y^2+xy-2x+1\right)$$

$$dS=\frac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|}=\sqrt{3}dxdy$$

So, $\int_{S_4}\vec{F}\cdot\hat{n}dS=\int\limits_0^1\int\limits_0^{1-x}\left(x^2-y^2+xy-2x+1\right)dydx$

(The region of double integration is given by projection of V on xy plane as shown in Figure)

$$=\int_0^1\left(x^2-2x+1\right)y-\frac{y^3}{3}+\frac{xy^2}{2}\Bigg|_0^{1-x}dx=\int_0^1\left(\frac{2}{3}(1-x)^3+\frac{x(1-x)^2}{2}\right)dx$$

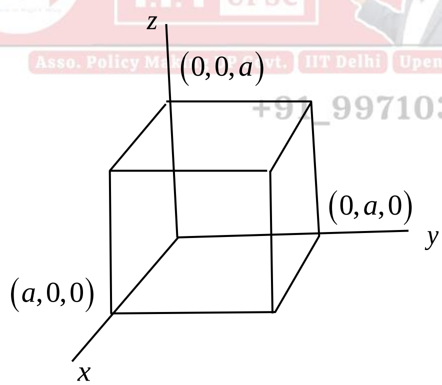
$$=-\frac{1}{6}(1-x)^4\Big|_0^1+\frac{1}{2}\left(\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}\right)\Bigg|_0^1=\frac{1}{6}+\frac{1}{2}\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right)=\frac{5}{24}$$

So, $\oiint_S\vec{F}\cdot\hat{n}dS=\int_{S_1}\vec{F}\cdot\hat{n}dS+\int_{S_2}\vec{F}\cdot\hat{n}dS+\int_{S_3}\vec{F}\cdot\hat{n}dS+\int_{S_4}\vec{F}\cdot\hat{n}dS=0+\left(-\frac{1}{12}\right)+0+\frac{5}{24}=\frac{1}{8}$

Hence, $\oiint_S\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\vec{F}d\tau$

Ex 20. Verify divergence theorem for $\vec{F}=4xz\hat{i}-y^2\hat{j}+yz\hat{k}$ taken over the cube bounded by $x=0, y=0$, $z=0, x=a, y=a, z=a$.

Solution.



Let us first find the volume integral

$$\int\limits_V\nabla\cdot\vec{F}d\tau=\int\limits_0^a\int\limits_0^a\int\limits_0^a(4z-y)\,dx\,dy\,dz$$

$$=\int\limits_0^a\int\limits_0^a(4z-y)[x]_0^a\,dy\,dz=a\int\limits_0^a 4yz-\frac{y^2}{2}\Bigg|_0^a dz=a^2\int\limits_0^a\left(4z-\frac{a}{2}\right)dz=a^2\left[2z^2-\frac{az}{2}\right]_0^a=\frac{3}{2}a^4$$

The region V is bounded by S. S is a piecewise smooth surface consisting of $S_1(x=0), S_2(x=a)$, $S_3(y=0), S_4(y=a), S_5(z=0), S_6(z=a)$

$$\oiint \vec{F}\cdot\hat{n}dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS + \int_{S_3}\vec{F}\cdot\hat{n}\,dS + \int_{S_4}\vec{F}\cdot\hat{n}dS + \int_{S_5}\vec{F}\cdot\hat{n}dS + \int_{S_6}\vec{F}\cdot\hat{n}dS \qquad ....(1)$$

On $S_1, x=0, \hat{n}=-\hat{i}, \vec{F}\cdot\hat{n}=0, dS=dydz$

So, $\int_{S_1}\vec{F}\cdot\hat{n}dS = 0$

On $S_2, x=a, \hat{n}=\hat{i}, \vec{F}\cdot\hat{n}=4az, dS=dydz$

So, $\int_{S_2}\vec{F}\cdot\hat{n}dS = \int_0^a\int_0^a 4az\,dydz = \int_0^a 4az[y]_0^a\,dz = 4a^2\int_0^a zdz = 4a^4$

On $S_3, y=0, \hat{n}=-\hat{j}, dS=dxdz, \vec{F}\cdot\hat{n}=0$

So, $\int_{S_3}\vec{F}\cdot\hat{n}dS = 0$

On $S_4, y=a, \hat{n}=\hat{j}, \vec{F}\cdot\hat{n}=-a^2, dS=dxdz$

So, $\int_{S_4}\vec{F}\cdot\hat{n}dS = -\int_0^a\int_0^a a^2dxdz = -a^4$

On $S_5, z=0, \hat{n}=-\hat{k}, \vec{F}\cdot\hat{n}=0, dS=dxdy$

So, $\int_{S_5}\vec{F}\cdot\hat{n}dS = 0$

On $S_6, z=a, \hat{n}=\hat{k}, \vec{F}\cdot\hat{n}=ay, dS=dxdy$

$\int_{S_6}\vec{F}\cdot\hat{n}dS = \int_0^a\int_0^a ay\,dxdy = \frac{a^4}{2}$

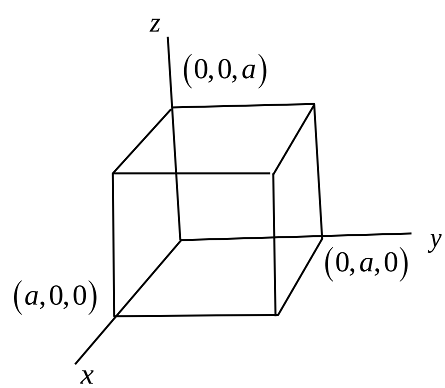
From (1)

$$\oiint \vec{F}\cdot\hat{n}dS = 0+2a^4+0-a^4+0+\frac{a^4}{2} = \frac{3a^4}{2}$$

Hence, $\oiint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$

Ex 21. Verify divergence theorem for $\vec{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$ taken over the rectangular parallelepiped $0\le x\le a, 0\le y\le a, 0\le z\le a$.

Solution.



Let us first calculate the volume integral

$$\vec{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}$$

$$\nabla \cdot \vec{F} = 2\left(x + y + z\right)$$

The volume integral

$$\int_V \nabla \cdot \vec{F} d\tau = 2\int_0^a \int_0^a \int_0^a \left(x + y + z\right) dxdydz = 2\int_0^a \int_0^a \left[\frac{x^2}{2} + x\left(y + z\right)\right]_0^a dydz$$

$$= 2a\int_0^a \int_0^a \left(\frac{a}{2} + \left(y + z\right)\right) dydz = 2a\int_0^a \frac{ay}{2} + \frac{y^2}{2} + zy\Big|_0^a dz = 2a^2\int_0^a \left(\frac{a}{2} + \frac{a}{2} + z\right) dz = 2a^2\left[az + \frac{z^2}{2}\right]_0^a = 3a^4$$

The surface S enclosing volume V consists of six pieces of smooth surfaces, $S_1\left(x=0\right), S_2\left(x=a\right), S_3\left(y=0\right), S_4\left(y=a\right), S_5\left(z=0\right), S_6\left(z=a\right)$.

$$\oiint_S \vec{F} \cdot \hat{n}\, dS = \int_{S_1} \vec{F} \cdot \hat{n}\, dS + \int_{S_2} \vec{F} \cdot \hat{n}\, dS + \int_{S_3} \vec{F} \cdot \hat{n}\, dS + \int_{S_4} \vec{F} \cdot \hat{n}\, dS + \int_{S_5} \vec{F} \cdot \hat{n}\, dS + \int_{S_6} \vec{F} \cdot \hat{n}\, dS$$

On $S_1, x = 0, \hat{n} = -\hat{i}, dS = dydz, \vec{F} \cdot \hat{n} = yz$

$$\int_{S_1} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^a yzdydz = \frac{a^4}{4}$$

On $S_2, x = a, \hat{n} = \hat{i}, dS = dydz, \vec{F} \cdot \hat{n} = \left(a^2 - yz\right)$

$$\int_{S_2} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^a \left(a^2 - yz\right) dydz = \int_0^a \int_0^a a^2 dydz - \int_0^a \int_0^a yzdydz = a^4 - \frac{a^4}{4} = \frac{3}{4}a^4$$

On $S_3, y = 0, \hat{n} = -\hat{j}, dS = dxdz, \vec{F} \cdot \hat{n} = zx$

$$\int_{S_3} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^a zxdxdz = \frac{a^4}{4}$$

On $S_4, y = a, \hat{n} = \hat{j}, dS = dxdz, \vec{F} \cdot \hat{n} = \left(a^2 - zx\right)$

$$\int_{S_4} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^a \left(a^2 - zx\right) = \frac{a^4}{4} = \frac{3}{4}a^4$$

On $S_5, z = 0, \hat{n} = -\hat{k}, dS = dxdy, \vec{F} \cdot \hat{n} = xy$

$$\int_{S_5} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^a xy\, dxdy = \frac{a^4}{4}$$

On $S_6, z = a, \hat{n} = \hat{k}, dS = dxdy, \vec{F} \cdot \hat{n} = a^2 - xy$

$$\int_{S_6} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^a \left(a^2 - xy\right) dxdy = \frac{3}{4}a^4$$

So, $\oiint \vec{F} \cdot \hat{n}\, dS = \frac{a^4}{4} + \frac{3a^4}{4} + \frac{a^4}{4} + \frac{3a^4}{4} + \frac{a^4}{4} + \frac{3a^4}{4} = 3a^{-1}$

Hence, $\oiint \vec{F}\cdot\hat{n}\,dS = \int \nabla\cdot F \cdot d\tau$

Ex 22. Evaluate $\iint x^2dydz + y^2dzdx + 2z(xy - x - y)dxdy$ where S is the surface of the cube $0 \le x \le a, 0 \le y \le a, 0 \le z \le a$

Solution.

$\hat{n}dS = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$x^2dydz + y^2dzdx + 2z(xy - x - y)dxdy = \left(x^2\hat{i} + y^2\hat{j} + 2z(xy - x - y)\hat{k}\right)\cdot\hat{n}dS$

So, $\iint x^2dydz + y^2dzdx + 2z(xy - x - y)dxdy = \int_S \left(x^2\hat{i} + y^2\hat{j} + 2z(xy - x - y)\hat{k}\right)\cdot\hat{n}\,dS$

$= \int \nabla\cdot\left(x^2\hat{i} + y^2\hat{j} + 2z(xy - x - y)\hat{k}\right)d\tau$ (By Gauss Divergence theorem)

$= 2\int_0^a\int_0^a xydxdy = \frac{a^2}{2}$

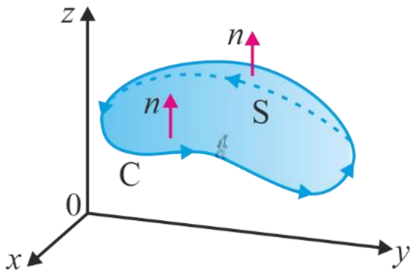
Stoke's Theorem

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

- Green's Theorem relates a double integral over a plane region *D* to a line integral around its plane boundary curve.
- Stokes' Theorem relates a surface integral over a surface *S* to a line integral around the boundary C curve of *S* (a space curve).

Oriented surface with unit normal vector $\hat{\mathbf{n}}$.

- The orientation of *S* induces the positive orientation of the boundary curve *C*.
- If you walk in the positive direction around *C* with your head pointing in the direction of $\hat{\mathbf{n}}$, the surface will always be on your left.



Let:

- *S* be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve *C* with positive orientation.
- *F* be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains *S*. **Then** $\int_C \vec{F}.dr = \int_S \text{curl}\vec{F}.\hat{n}dS$
- Stokes' Theorem becomes: in 2D plane $\int_C \vec{F}.dr = \iint_S \text{curl}\vec{F}.\hat{n}dS = \iint_S (\text{curl}\vec{F}).\hat{k}dA$; Thus,

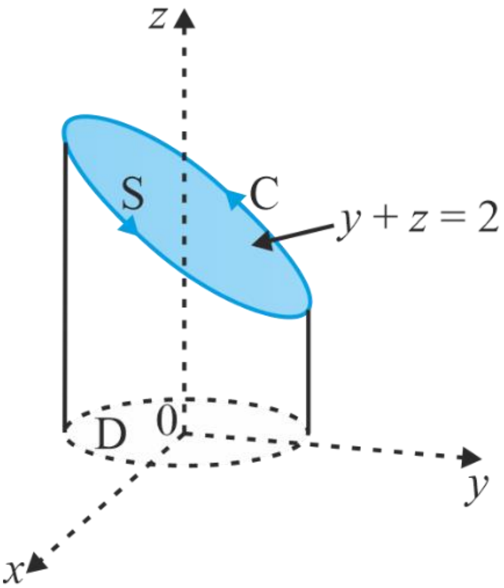
we see that Green's Theorem is really a special case of Stokes' Theorem.

Example 1: Evaluate $\int_C F.dr$, where: $F(x, y, z) = -y^2\, i + xj + z^2\, k$. *C* is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient *C* to be counterclockwise when viewed from above.)

Learnings: $\int_C \mathbf{F}.dr$ Could be evaluated directly, however, it's easier to use Stokes' Theorem.

We fist compute for $\mathbf{F}$ (x, y, z) We first compute for $\mathbf{F}$ $(x, y, z) = -y^2$ i + x j + z^2 k:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\mathbf{k}$$



There are many surfaces with boundary C.

- The most convenient choice, though, is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, C has the induced positive orientation.

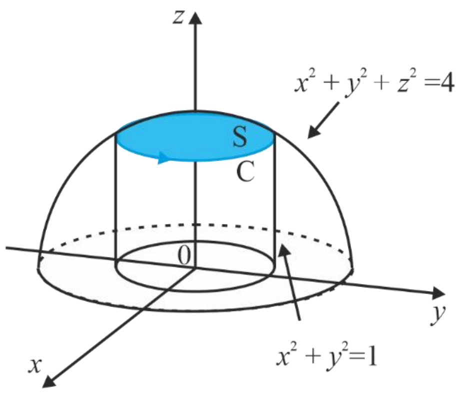
The projection D of S on the *xy*-plane is the disk $x^2 + y^2 \leq 1$.

$$\int_C \mathbf{F}.d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}.d\mathbf{S} = \iint_D (1+2y)\,dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}+\frac{2}{3}\sin\theta\right)d\theta$$

$$= \frac{1}{2}(2\pi) + 0 = \pi$$

Another example: Use Stokes' Theorem to compute $\iint_S \operatorname{curl}\mathbf{F}.d\mathbf{S}$ where:

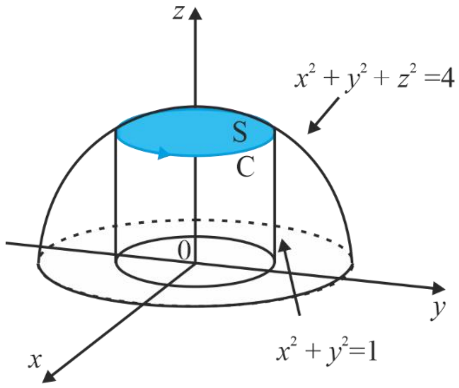
- $\mathbf{F}(x, y, z) = xz\,\mathrm{i} + yz\,\mathrm{j} + xy\,\mathrm{k}$
- S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the *xy*-plane.



To find the boundary curve C, we solve: $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$

- Subtracting, we get $z^2 = 3$, and (since $z > 0$), $z = \sqrt{3}$

- So, C is the circle given by: $x^2 + y^2 = 1,\ z = \sqrt{3}$



A vector equation of C is:

r(*t*) = cos *t* i + sin *t* j + $\sqrt{3}$ k **0 ≤ *t* ≤ 2π**

- Therefore, r′(*t*) = – sin *t* i + cos *t* j

Also, we have:

$$\mathrm{F}\left(\mathrm{r}(t)\right) = \sqrt{3}\cos t\,\mathrm{i} + \sqrt{3}\sin t\,\mathrm{j} + \cos t \sin t\,\mathrm{k}$$

Thus, by Stokes' Theorem,

$$\iint_S \mathrm{curl}\,\mathbf{F}.d\mathrm{S} = \int_C \mathbf{F}.d\mathbf{r} = \int_0^{2\pi} \mathrm{F}\left(\mathrm{r}(t)\right).\mathrm{r}'(t)\,dt = \int_0^{2\pi}\left(-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t\right)dt = \sqrt{3}\int_0^{2\pi} 0\,dt = 0$$

❖ Note that, in Example 2, we computed a surface integral simply by knowing the values of **F** on the boundary curve C.

❖ This means that:

- If we have another oriented surface with the same boundary curve C, we get exactly the same value for the surface integral!

❖ In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \mathrm{curl}\,\mathbf{F}.d\mathbf{S} = \int_C \mathbf{F}.d\mathbf{r} = \iint_{S_2} \mathrm{curl}\,\mathbf{F}.d\mathbf{S}$$

❖ This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

We now use Stokes' Theorem to throw some light on the meaning of the curl vector.

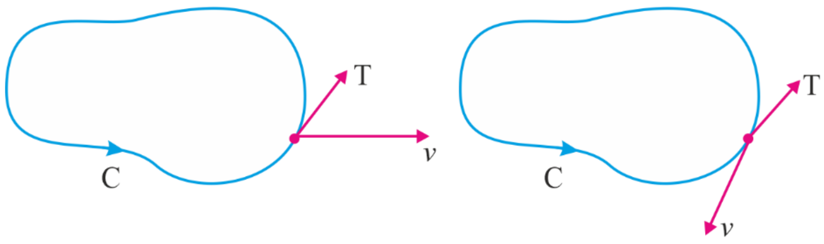
- Suppose that C is an oriented closed curve and v represents the velocity field in fluid flow.

Consider the line integral $\int_C \mathrm{v}.d\mathrm{r} = \int_C \mathrm{v}.\mathrm{T}\,ds$ **and recall that v · T is the component of v in the direction of the unit tangent vector T.**

- This means that the closer the direction of v is to the direction of T, the larger the value of **v · T**.

Thus, $\int_C \mathbf{v}.d\mathbf{r}$ **is a measure of the tendency of the fluid to move around C.**

- It is called the circulation of v around C.



<u>**Solved Examples**</u>

Based on definitions

1. The value of $\oint \vec{r}\cdot d\vec{r}$ is equal to

Using Stoke's law

$$\oint_C \vec{r}\cdot d\vec{r} = \iint_S \nabla\times\vec{r}\cdot\hat{n}\,dS = 0 \text{ as } \left(\nabla\times\vec{r} = 0\right)$$

2. The value of $\oint \phi\nabla\psi\cdot d\vec{r} + \oint \psi\nabla\phi\cdot d\vec{r}$

$$\oint_C \phi\nabla\psi\cdot d\vec{r} + \oint \psi\nabla\phi\cdot d\vec{r} = \oint_C \left(\phi\nabla\psi + \psi\nabla\phi\right)\cdot d\vec{r} = \int_C \nabla\left(\phi\psi\right)\cdot d\vec{r}$$

$= \int_S \nabla\times\left(\nabla\phi\psi\right)\cdot\hat{n}\,dS = 0$ (By Stoke's theorem $\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$)

(as Curl of gradient of scalar function = **0**)

3. The value of $\oint_C \phi\nabla\phi\cdot d\vec{r}$ for closed curve C is equal to

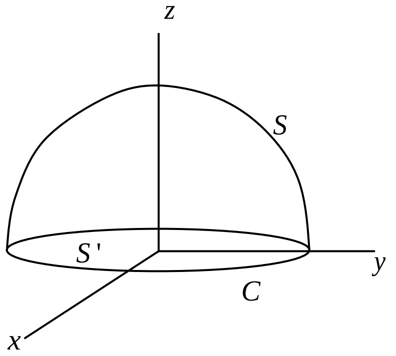
$\oint_C \phi\nabla\psi\cdot d\vec{r} = \int_S \nabla\times\left(\phi\nabla\phi\right)\cdot\hat{n}\,dS$ (Using Stoke's law)

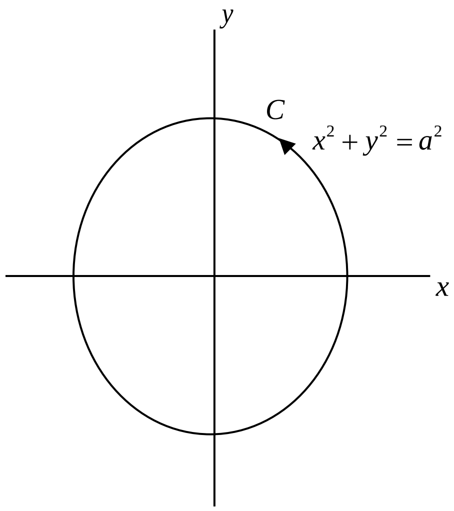
$= \int_S \left(\nabla\phi\times\nabla\phi + \phi V\times\nabla\phi\right)\cdot\hat{n}\,dS = 0$ (as $\nabla\phi\times\nabla\phi = 0$ & $\nabla\times\left(\nabla\phi\right) = 0$)

EXAMPLES FOR CSE & IFoS

Ex 1. Verify Stokes theorem for $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = a^2$ and C is its bounding curve.

Solution.





S is the surface of sphere $x^2 + y^2 + z^2 = a^2$ lying above xy plane and bounded by the circle $C: x^2 + y^2 = a^2$

On curve C, $x = a\cos\theta, y = a\sin\theta, z = 0$

$dx = -a\sin\theta d\theta, dy = a\cos\theta d\theta, dz = 0$

So, $\vec{F}\cdot d\vec{r} = ydx + zdy + xdz = ydx \quad (z = 0 \text{ on C})$

$$\oint \vec{F}\cdot d\vec{r} = \int ydx = -\int_0^{2\pi} a\sin\theta \cdot a\sin\theta\, d\theta = -a^2\int_0^{2\pi}\sin^2\theta d\theta = -\pi a^2$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Consider a closed piecewise smooth surface S_2 consisting of spherical surface $S: x^2 + y^2 + z^2 = a^2$ and $S': z = 0$ enclosing a volume V.

$$\oint_{S_2}\left(\nabla\times\vec{F}\right).\hat{n}dS = \int_V \nabla\cdot\left(\nabla\times\vec{F}\right)dV = 0 \Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}dS + \int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS = 0 \because \nabla\cdot\left(\nabla\times\vec{F}\right) = 0$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}dS = -\int_{S'}\nabla\times\vec{F}\cdot\hat{n}dS$$

On S', outward drawn unit normal $\hat{n} = -\hat{k}$

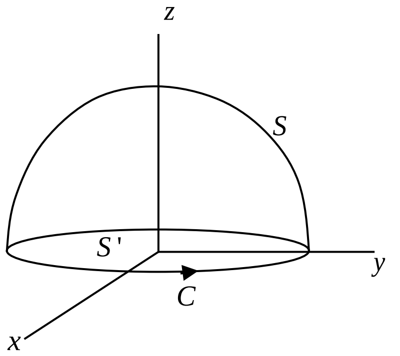
$$\nabla\times\vec{F}\cdot\hat{n} = \left(-\hat{i} - \hat{j} - \hat{k}\right)\cdot\left(-\hat{k}\right) = 1$$

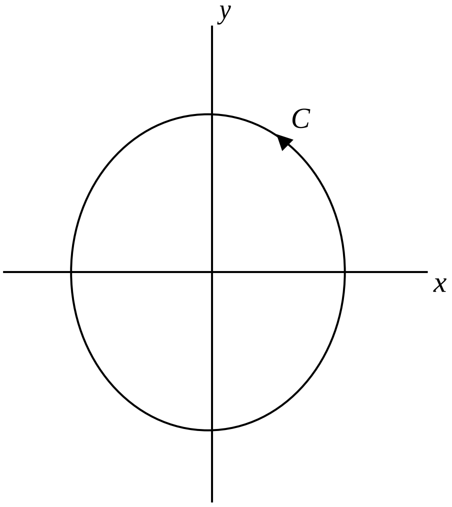
So, $\int_S \nabla\times\vec{F}\cdot\hat{n}dS = -\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'} dS = -\pi a^2$

Hence, $\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}dS$. Stoke's theorem is verified.

Ex 2. Verify Stoke's theorem for the function $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ where curve C is the unit circle in the xy plane bounding the hemisphere $z = \sqrt{1 - x^2 - y^2}$.

Solution.





The curve C is a unit circle $x^2 + y^2 = 1, z = 0$ bounding the surface S which is a hemisphere of unit radius given by $z = \sqrt{1 - x^2 - y^2}$

On C, $x = \cos\theta, y = \sin\theta, z = 0$

$dx = -\sin\theta d\theta, dy = \cos\theta d\theta, dz = 0$

$\vec{F} \cdot d\vec{r} = zdx + xdy + ydz = xdy$ on curve C

So, $\oint_C \vec{F} \cdot d\vec{r} = \int_C xdy = \int_0^{2\pi} \cos^2\theta d\theta = \pi$

Now, let us evaluate the surface integral $\int_S \nabla \times \vec{F} \cdot \hat{n}\, dS$.

Consider a closed piecewise smooth surface S_2 consisting of spherical surface $S : x^2 + y^2 + z^2 = 1$ and $S' : z = 0$ enclosing a volume V.

$$\oint_{S_2} (\nabla \times \vec{F}).\hat{n}dS = \int_V \nabla \cdot (\nabla \times \vec{F}) dV = 0 \Rightarrow \int_S \nabla \times \vec{F} \cdot \hat{n}dS + \int_{S'} \nabla \times \vec{F} \cdot \hat{n}\, dS = 0 \because \nabla \cdot (\nabla \times \vec{F}) = 0$$

$$\Rightarrow \int_S \nabla \times \vec{F} \cdot \hat{n}dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n}dS$$

On $S', \hat{n} = -\hat{k},\ dS = dxdy ,; \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$

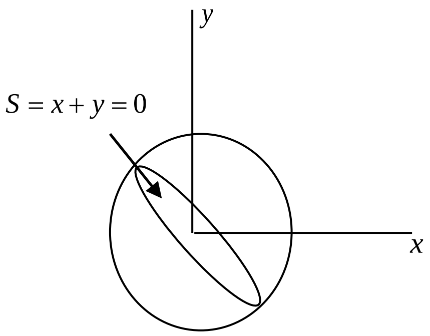
On S'; $\nabla \times \vec{F} \cdot \hat{n} = -1$

So, $\int_S \nabla \times \vec{F} \cdot \hat{n}\, dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n}\, dS = \int_{S'} dS$ = area of base S' $= \pi$

Hence, $\oint_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot \hat{n}\, dS$. Thus, Stoke's theorem is verified.

Ex 3. Using Stoke's theorem evaluate the line integral $\int_C \left(y\hat{i}+z\hat{j}+x\hat{k}\right)\cdot d\vec{r}$ where C is the intersection of $x^2+y^2+z^2=1$ and $x+y=0$ traversed in the clockwise direction when viewed from the point $(1,1,0)$.

Solution.



Using Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \int \nabla\times\vec{F}\cdot\hat{n}dS$$

$$\oint\left(y\hat{i}+z\hat{j}+x\hat{k}\right)\cdot d\vec{r} = \int \nabla\times\left(y\hat{i}+z\hat{j}+x\hat{k}\right)\cdot\hat{n}dS$$

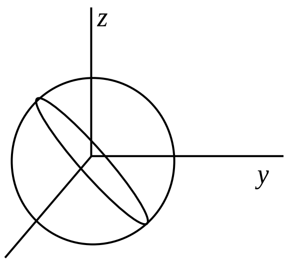
$\hat{n} = -\dfrac{\nabla S}{|\nabla S|} = -\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$; S is the surface of plane.

$$\nabla\times\left(y\hat{i}+z\hat{j}+x\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i}-\hat{j}-\hat{k}$$

$\int\left(-\hat{i}-\hat{j}-\hat{k}\right)\cdot\left(-\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}\right)dS = \sqrt{2}\int dS = \sqrt{2}\pi$; Note that the plane passes through the point (0,0,0) as x=0,y=0,z=0 satisfies given equation of plane. Also the center of given sphere is (0,0,0). So we find that given plane passes through the center of sphere and so the radius of circle will be equal to radius of given sphere. (intersection of plane and a sphere is a circle)

Ex 4. Evaluate using Stoke's theorem $\oint_C (y+z)dx+(z+x)dy+(x+y)dz$ where C is the circle $x^2+y^2+z^2=1, x+y+z=0$.

Solution.



The bounding curve C is the curve of intersection of sphere $x^2+y^2+z^2=1$ and plane $x+y+z=0$.

Let the surface S be a disc of radius 1 with centre at origin bounded by C

$$\oint \left((xy+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k} \right) \cdot d\vec{r}$$

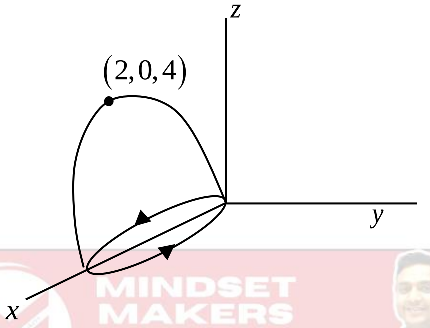
So, $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = 0$$

By Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot \hat{n}\, dS = 0$

Ex 5. Evaluate $\int_S \text{curl}\, \vec{A} \cdot \hat{n}\, dS$ where S is the open surface $x^2 + y^2 - 4x + 4z = 0, z > 0$ and $\vec{A} = (y^2 + z^2 - x^2)\hat{i} + (2z^2 + x^2 - y^2)\hat{j}$.

Solution.



S is the open surface $x^2 + y^2 - 4x + 4z = 0$ (paraboloid) above xy plane. The bounding curve C of the surface S is given by $x^2 + y^2 - 4x = 0 \Rightarrow (x-2)^2 + y^2 = 4$ i.e. circle of radius 2 with origin $(2,0)$ in xy plane.

On curve C, $x = 2 + 2\cos\theta, y = 2\sin\theta, z = 0$; $dx = -2\sin\theta d\theta, y = 2\cos\theta d\theta, dz = 0$

$$\vec{F} \cdot d\vec{r} = (y^2 + z^2 - z^2)dx + (2z^2 + x^2 - y^2)dy = (y^2 - x^2)dx + (x^2 - y^2)dy$$

$$= (4\sin^2\theta - 4 - 4\cos^2\theta - 8\cos\theta)(-2\sin\theta)d\theta + (4 + 4\cos^2\theta + 8\cos\theta - 4\sin^2)(2\cos\theta)d\theta$$

$$= 8\left[-\sin^3\theta + \sin\theta + \cos^2\theta\sin\theta + 2\cos\theta\sin\theta + \cos\theta + \cos^3 + \cos^3\theta + 2\cos^2\theta - \sin^2\theta\cos\theta \right] d\theta$$

$$= 8\left[\cos^3\theta - \sin^3\theta + \cos^2\theta\sin\theta - \sin^2\theta\cos\theta + 2\cos^2\theta + 2\cos\theta\sin\theta + \cos\theta + \sin\theta \right] d\theta$$

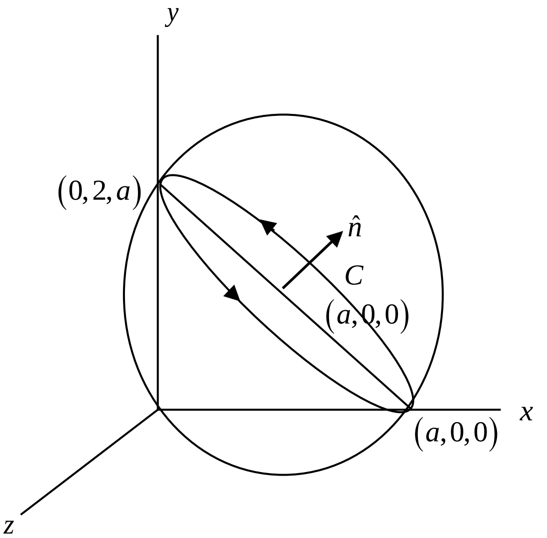
θ varies from 0 to 2π

$$\oint \vec{F} \cdot d\vec{r} = 8\int_0^{2\pi} \left(\cos^3\theta - \sin^3\theta + \cos^2\theta\sin\theta - \sin^2\theta\cos\theta + 2\cos^2\theta + 2\cos\theta\sin\theta + \cos\theta + \sin\theta \right) d\theta$$

$$= 16\int_0^{2\pi} \cos^2\theta d\theta = 16\pi$$

Ex 6. Apply Stoke's theorem to prove that $\int_C ydx + zdy + xdz = -2\sqrt{2}\pi a^2$ where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$ and begins at the point $(2a, 0, 0)$ and goes first below the xy plane.

Solution.



Curve C is the curve of intersection of sphere $x^2+y^2+z^2-2ax-2ay=0$ and plane $x+y=2a$ as shown in Fig.

The centre of sphere $(a,a,0)$ and radius $\sqrt{2}a$.

$$\left(x-a^2\right)+\left(y-a\right)^2+c^2=2a^2$$

The centre of sphere $(a,a,0)$ lies on the plane $x+y=2a$.

So, the curve of intersection is the greatest circle. Let the surface enclosed by greatest circle is a disc of radius $\sqrt{2}a$ as shown in Fig.

For given orientation of curve C, the normal to the surfaces S is $\hat{n}=\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$

$$\oint_C ydx+zdy+xdz=\oint\left(y\hat{i}+z\hat{j}+x\hat{k}\right)\cdot d\vec{r}=\oint\vec{F}\cdot d\vec{r}$$

So, $\vec{F}=y\hat{i}+z\hat{j}+x\hat{k}\,;\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y & z & x\end{vmatrix}=-\hat{i}-\hat{j}-\hat{k}$

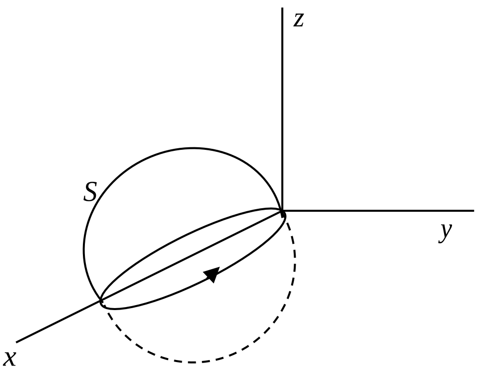
$$\nabla\times\vec{F}\cdot\hat{n}=-\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)=-\sqrt{2}$$

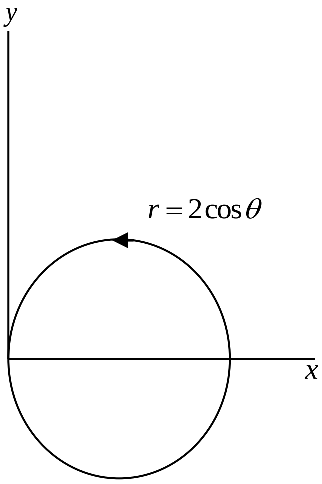
By Stoke's law

$\oint_C\vec{F}\cdot d\vec{r}=\int_S\nabla\times\vec{F}\cdot\hat{n}\,dS=-\sqrt{2}\int_S dS=-\sqrt{2}\times$ area of disc of radius $\sqrt{2}a=-2\sqrt{2}\pi a^2$

Ex 7. If $\vec{F}=\left(y^2+z^2-x^2\right)\hat{i}+\left(z^2+x^2-y^2\right)\hat{j}+\left(x^2+y^2-z^2\right)\hat{k}$, evaluate $\int_S\nabla\times\vec{F}\cdot\hat{n}\,dS$ taken over the portion of the surface $x^2+y^2+z^2-2x+z=0$ above the plane $z=0$ and verify Stoke's theorem.

Solution.





S is a part of sphere $x^2+y^2+z^2-2x+z=0$ i.e. $(x-1)^2+y^2+(z+½)^2=\frac{5}{4}$ of radius $\frac{\sqrt{5}}{2}$. It is bounded by circle $x^2+y^2-2x=0$ lying in xy plane.

Let us first evaluate the surface integral $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$. Consider a closed piecewise surface S_2 consisting of hemisphere S and its base in xy plane S'.

By Gauss divergence theorem

$$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS=\int\nabla\cdot(\nabla\times\vec{F})d\tau=0 \Rightarrow \int_S\nabla\times\vec{F}\cdot\hat{n}\,dS+\int_{S'}\nabla\times\hat{n}\,dS=0$$

So, $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS=-\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS$

$$\vec{F}=(y^2+z^2-x^2)\hat{i}+(z^2+x^2-y^2)\hat{j}+(x^2+y^2+z^2)\hat{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2\end{vmatrix}=2(y-z)\hat{i}+2(z-x)\hat{j}+2(x-y)\hat{k}$$

On $S', \hat{n}=-\hat{k}, \nabla\times\vec{F}\cdot\hat{n}=-2(x-y)$; $dS=dxdy$

So, $\int_S\nabla\times\vec{F}\cdot\hat{n}\,dS=-\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS=2\iint(x-y)\,dxdy=2\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}r(\cos\theta-\sin\theta)r\,dr\,d\theta$

$$=2\int_{-\pi/2}^{\pi/2}(\cos\theta-\sin\theta)\left[\frac{r^3}{3}\right]_0^{2\cos\theta}d\theta=\frac{16}{3}\int_{-\pi/2}^{\pi/2}(\cos\theta-\sin\theta)\cos^3\theta d\theta$$

$$=\frac{16}{3}\left[\int_{-\pi/2}^{\pi/2}\cos^4\theta d\theta-\int_{-\pi/2}^{\pi/2}\cos^3\theta\sin\theta d\theta\right]=\frac{32}{3}\int_0^{\pi/2}\cos^4\theta d\theta=\frac{32}{3}\cdot\frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{2\Gamma 3}=2\pi$$

Let us now evaluate the line integral $\oint_C \vec{F}\cdot d\vec{r}$

The curve C is a circle $(x-1)^2+y^2=1$ in xy plane.

$$x=1+\cos\theta, dx=-\sin\theta d\theta, y=\sin\theta, dy=\cos\theta d\theta, \; z=0, dz=0$$

$$\vec{F}\cdot d\vec{r}=(y^2+z^2-x^2)dx+(z^2+x^2-y^2)dy+(x^2+y^2-z^2)dz$$

$=\left(y^2-x^2\right)dx+\left(x^2-y^2\right)dy=\left(x^2-y^2\right)\left(dy-dx\right)$

$=\left(1+\cos^2\theta+2\cos\theta-\sin^2\theta\right)\cdot\left(\cos\theta+\sin\theta\right)d\theta$

$=\left(\cos^3\theta-\sin^3\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta+2\cos^2\theta+2\cos\theta\sin\theta+\cos\theta+\sin\theta\right)d\theta$

So,

$$\oint\vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left(\cos^3\theta-\sin^3\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta+2\cos^2\theta+2\cos\theta\sin\theta+\cos\theta+\sin\theta\right)d\theta$$

$$=2\int_0^{2\pi}\cos^2\theta d\theta=2\pi$$

Hence, $\oint\vec{F}\cdot d\vec{r}=\int\nabla\times\vec{F}\cdot\hat{n}\,dS$. So, Stoke's theorem is verified.

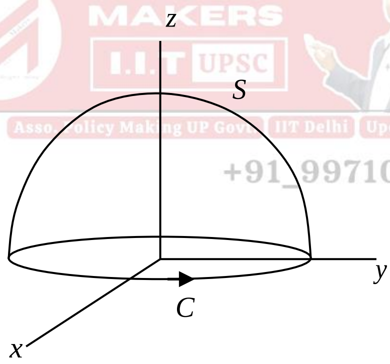
Ex8. By converting into a line integral, evaluate $\int_S\nabla\times\vec{F}\cdot\hat{n}\,dS$ where $\vec{F}=\left(x^2+y-4\right)\hat{i}+3xy\,\hat{j}+\left(2xz+z^3\right)\hat{k}$ and S is the surface of

(i) the hemisphere $x^2+y^2+z^2=a^2$ above xy plane.

(ii) the paraboloid $z=9-\left(x^2+y^2\right)$ above the xy plane.

Solution.

(i) S is the surface of hemisphere $x^2+y^2+z^2=a^2$ above the xy plane and bounded by the circle C $x^2+y^2=a^2$ in xy plane as shown in Fig.



On C, $x=a\cos\theta, y=a\sin\theta, z=0$

$dx=-a\sin\theta d\theta, dy=a\cos\theta dgq, dz=0$

$\vec{F}\cdot d\vec{r}=\left(x^2+y-4\right)dx+3xydy+\left(2xz+z^3\right)dz$

$=\left(a^2\cos^2\theta+a\sin\theta-4\right)\cdot\left(-a\sin\theta d\theta\right)+3a^2\cos\theta\sin\theta\left(a\cos\theta\right)d\theta$

$=\left(-a^2\cos^2\sin\theta-a^2\sin^2\theta+4a\sin\theta+3a^3\cos^2\theta\sin\theta\right)d\theta$

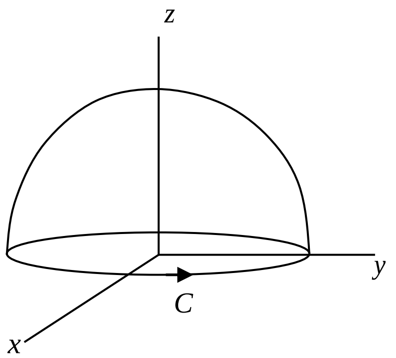
By Stoke's theorem

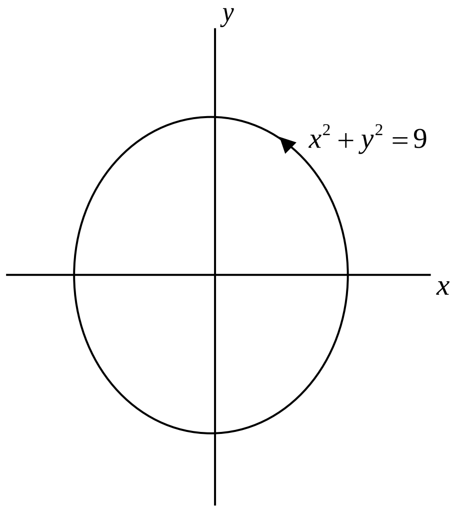
$$\int_S\nabla\times\vec{F}\cdot\hat{n}\,dS=\oint_C\vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left(-a^3\cos^2\theta\sin\theta-a^2\sin^2\theta+4a\sin\theta+3a^3\cos^2\theta\sin\theta\right)d\theta$$

$$=-a^2\int_0^{2\pi}\sin^2\theta d\theta=-\pi a^2$$

(ii) S is the surface of paraboloid above xy plane bounded by curve C in xy plane.

Bounding curve C is circle $x^2+y^2=9$ of radius 3 and centre at origin as shown in fig.





On C, $x=3\cos\theta, y=3\sin\theta, z=0$

$dx=-3\sin\theta d\theta, dy=3\cos\theta d\theta, dz=0$

$\vec{F}\cdot d\vec{r}=\left(x^2+y-4\right)dx+3xydy+\left(2xz+z^3\right)dz$

$=\left(9\cos^2\theta+3\sin\theta-4\right)\cdot\left(-3\sin\theta d\theta\right)+81\cos^2\theta\sin\theta d\theta+0$

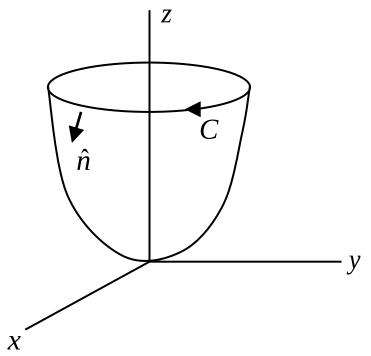
$=\left(54\cos^2\theta\sin\theta-9\sin^2\theta+12\sin\theta\right)d\theta$

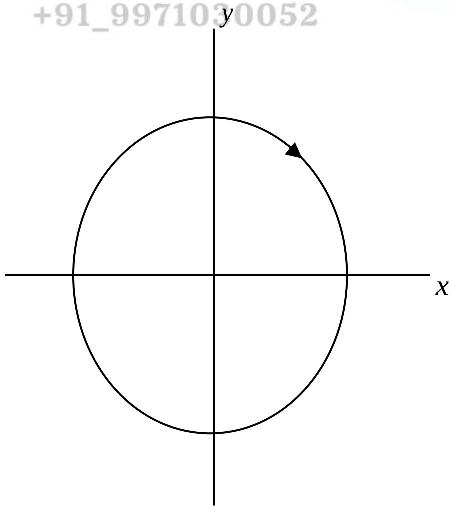
So, By Stoke's theorem

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS=\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left(54\cos^2\theta\sin\theta-9\sin^2\theta+12\sin\theta\right)d\theta=-9\int_0^{2\pi}\sin^2\theta d\theta=-9\pi$$

Ex 9. Verify Stoke's for the vector $\vec{F}=3y\hat{i}-xz\hat{j}+yz^2\hat{k}$ where S is the surface of the paraboloid $z=x^2+y^2$ bounded by $z=4$ and C is its boundary.

Solution.





S is the surface of paraboloid $z=x^2+y^2$ and bounded by $z=4$.

The bounding curve C will be a circle $x^2+y^2=4, z=4$.

For a given surface, if $\hat{n}$ is an outward drawn normal, then the corresponding orientation of curve will be clockwise if seen from above.

Let us first evaluate the line integral

On C, $x=2\cos\theta, y=2\sin\theta, z=4$

$dx=-2\sin\theta d\theta, dy=2\cos\theta d\theta, dz=0$

$\vec{F}\cdot d\vec{r} = 3ydx - xzdy + yz^2dz$

$= -12\theta\sin^2\theta d\theta - 16\cos^2\theta d\theta + 0$

$$\oint \vec{F}\cdot d\vec{r} = -12\int_{2\pi}^{0}\sin^2\theta d\theta - 16\int_{2\pi}^{0}\cos^2\theta d\theta$$

$= 28\pi$

Now, consider a closed piecewise smooth surface Σ consisting of parabolic part S and base S' at $(z=4)$.

So, by Gauss divergence theorem

$$\oint_{\Sigma}\nabla\times\vec{F}\cdot\hat{n}\,dS = \int\nabla\cdot\left(\nabla\times\vec{F}\right)d\tau = 0 \Rightarrow \int_{S}\nabla\times\vec{F}\cdot\hat{n}\,dS + \int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS = 0$$

So, $\int_{S}\nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS$

On $S', \hat{n} = \hat{k},\ dS = dxdy$; $\vec{F} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3y & -xz & yz^2\end{vmatrix} = \left(z^2+x\right)\hat{i} + 0\hat{j} - \left(z+3\right)\hat{k}$$

On S', $\nabla\times\vec{F}\cdot\hat{n} = -(z+3) = -7$ ($z=4$ on S)

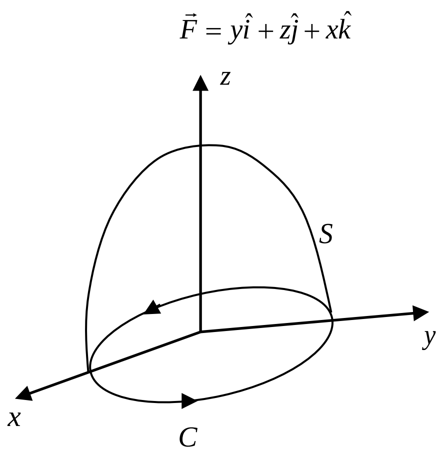
So, $\int_{S}\nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_{S'}\nabla\times\vec{F}\cdot\hat{n}\,dS = 7\int_{S'}dS = 7\times$ Area of base $= 28\pi$

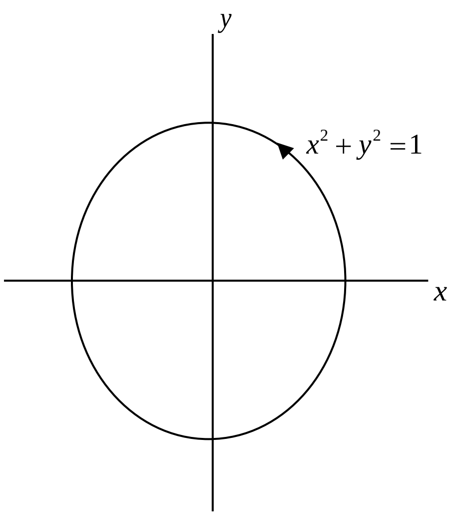
Since $\oint\vec{F}\cdot d\vec{r} = \int_{S}\nabla\times\vec{F}\cdot\hat{n}\,dS$

Hence, Stoke's theorem is verified.

Ex 10. Evaluate the surface integral $\int_{S}\nabla\times\vec{F}\cdot\hat{n}\,dS$ by transforming it into a line integral, S being that part of the surface of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$ and $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$.

Solution.





S is the surface of paraboloid $z = 1 - x^2 - y^2$ lying above xy plane and bounded by curve C, which is a circle $x^2 + y^2 = 1$ lying in xy plane

$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$\vec{F} \cdot d\vec{r} = ydx + zdy + xdz$

On C, $x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$

$y = \sin\theta \Rightarrow dy = \cos\theta d\theta$

$z = 0 \Rightarrow dz = 0$

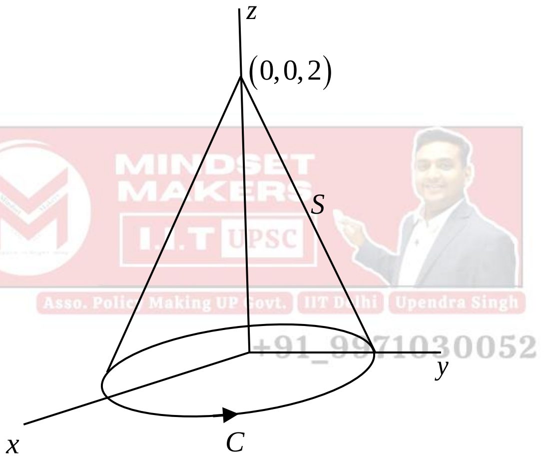
So, $\vec{F} \cdot d\vec{r} = -\sin^2\theta d\theta$

So, by Stoke's theorem

$$\int \nabla \times \vec{F} \cdot \hat{n}\, dS = \oint \vec{F} \cdot d\vec{r} = -\int_0^{2\pi} \sin^2\theta d\theta = -\pi$$

Ex 11. By converting into a line integral, evaluate $\int \nabla \times \vec{A} \cdot \hat{n}\, dS$ when $\vec{A} = (x - z)\hat{i} + \left(x^3 + yz\right)\hat{j} - 3xy^2\hat{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy plane.

Solution.



S is the surface of cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy plane. It is bounded by curve C in xy plane.

The curve C: is circle of radius 2 and centre at origin in xy plane.

On curve C, $x = 2\cos\theta, y = 2\sin\theta, z = 0$

$dx = -2\sin\theta\, d\theta, dy = 2\cos\theta d\theta, dz = 0$

$\vec{A} \cdot d\vec{r} = (x - z)dx + \left(x^3 + yz\right)dy - 3xy^2 dz = xdx + x^3 dy$ (as $z = 0, dz = 0$ on C)

$= -4\cos\theta\sin\theta d\theta + 16\cos^4\theta \cdot d\theta$

On C, θ varies from 0 to 2π

$$\oint \vec{A} \cdot d\vec{r} = -4\int_0^{2\pi} \cos\theta\sin\theta d\theta + 16\int_0^{2\pi} \cos^4\theta d\theta = 0 + 64\int_0^{\pi/2} \cos^4\theta d\theta = 64 \cdot \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} = 12\pi$$

Exam point: Working with Cylinder

Intersection of Plane and cylinder- We may have different scenarios depending on the nature of cut the cylinder by plane.

E.g. If Plane is at right angle to axis of cylinder, then it cuts into a circle.

If not above and utting the cylinder then it may be an ellipse kind of or a tilted circle kind of.

Similarly, we can think if plane is parallel to the axis of cylinder etc.

Mathematically we work as: *parameterization process*

Example : Intersection of $x^2 + y^2 = a^2$, $x + z = 1$.

$x = a.\cos t,\ y = a.\sin t;\quad z = 1 - a\cos t$. Here $0 \le t \le 2\pi$.

So, for LINE INTEGRAL: the boundary curve is : $(a\cos t, a\sin t, 1-\cos t); 0 \le t \le 2\pi$.

And for surface S; the projection on xy plane which is circle $x^2 + y^2 = a^2$

General

Example : Intersection of $x^2 + y^2 = 1, z = y^2$.

$x = 1.\cos t,\ y = 1.\sin t;\quad z = \sin^2 t$. Here $0 \le t \le 2\pi$.

So, for LINE INTEGRAL: the boundary curve is : $(\cos t, \sin t, \sin^2 t); 0 \le t \le 2\pi$.

And for surface S; the projection on xy plane which is circle $x^2 + y^2 = 1$

Beautiful instrument to verify Stoke's theorem for such intersecting scenarios:

So here the boundary curve C is a circle bounding the Lower base S_1 and Curved surface S_2 of cylinder. So if we include the plane surface S_3 with these two surfaces of cylinder, then we'll have a closed surface and so Gauss divergence theorem will be applicable and more beautifully, we can evaluate the surface integral on two surfaces S_1 & S_2 as

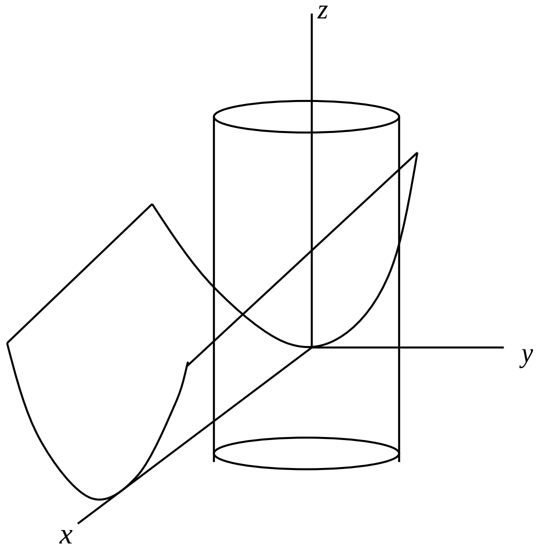
$$\int_{S_1} \mathbf{F}.\hat{n}ds_1 + \int_{S2} \mathbf{F}.\hat{n}ds_2 + \int_{S_3} \mathbf{F}.\hat{n}ds_3 = 0; \quad div(curl\ \mathbf{F}) = 0$$

$$\int_{S_1} \mathbf{F}.\hat{n}ds_1 + \int_{S2} \mathbf{F}.\hat{n}ds_2 = -\int_{S_3} \mathbf{F}.\hat{n}ds_3$$

. Now evaluating RHS integral is easier.

Ex 12. Evaluate by Stoke's theorem $\oint yzdx + xzdy + xydz$ where C is the curve of intersection of $x^2 + y^2 = 1, z = y^2$.

Solution.



The curve C is the curve of intersection of $x^2 + y^2 = 1$ and $x = y^2$ as shown in fig.

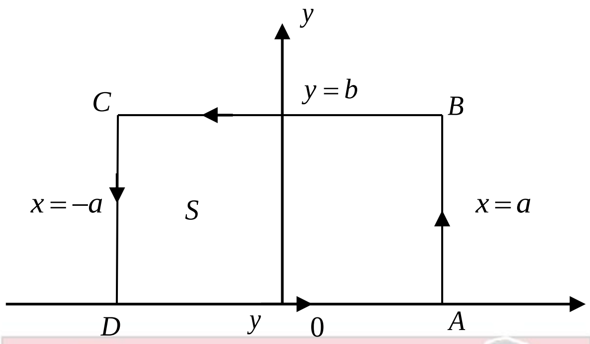
$$\oint yzdx + xzdy = \oint \left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right)\cdot d\vec{r}$$

where $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = 0$’By Stoke's theorem $\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = 0$ (as curl $\vec{F} = 0$)

Ex 13. Verify Stoke's theorem for $\vec{F} = \left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by $x = \pm a, y = 0, y = b$.

Solution.



C is piecewise smooth curve consisting of $y = 0, x = a, y = b$ & $x = -a$. The curve C encloses as plane surface lying in xy plane as shown in fig.

Let us first evaluate the surface integral $\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$.

Let us orient the curve in anticlockwise direction. With this orientation $\hat{n} = \hat{k}$.

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = -4y\hat{k}$$

$\nabla\times\vec{F}\cdot\hat{n} = -4y$

The surface element $dS = dxdy$

$$\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS = -4\int_{-a}^{a}\int_0^b y\,dydx = -4ab^2$$

Now, let us evaluate the line integral

$$\int_C \vec{F}\cdot dr = \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CD}\vec{F}\cdot d\vec{r} + \int_{DA}\vec{F}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = \left(x^2 + y^2\right)dx - 2xydy$$

On $AB, x = a, dx = 0, y$ varies from 0 to b ,; $\vec{F}\cdot d\vec{r} = -2aydy$; $\int_{AB}\vec{F}\cdot d\vec{r} = -2a\int_0^b y\,dy = -ab^2$

On BC, $y=b, dy=0, x$ varies from a to $-a$; $\vec{F}\cdot d\vec{r} = \left(x^2+b^2\right)dx$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^{-a}\left(x^2+b^2\right)dx = \left.\frac{x^3}{3}+b^2x\right|_a^{-a} = -\frac{2a^3}{3}-2ab^2$$

On $CD, x=-a, dx=0, y$ varies from b to 0.; $\vec{F}\cdot d\vec{r} = 2ay dy$; $\int_{CD} \vec{F}\cdot d\vec{r} = 2a\int_b^0 y\,dy = -ab^2$

On $DA, y=0, dy=0, x$ varies from $-a$ to a; $\vec{F}\cdot d\vec{r} = x^2dx$; $\int_{DA} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2dx = \frac{2a^3}{3}$

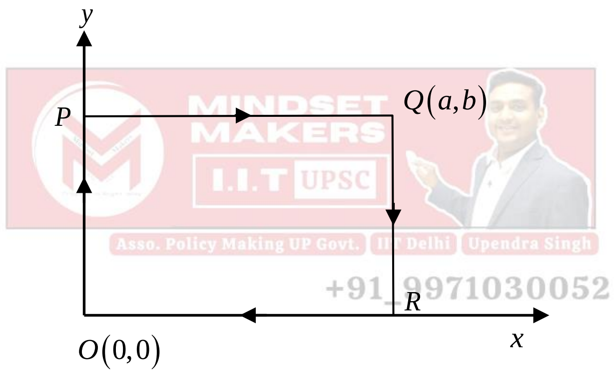
So,

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CD}\vec{F}\cdot d\vec{r} + \int_{DA}\vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$$

Hence, $\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,dS$

Ex 14. How much work is done when an object moves from $O\to P\to Q\to R\to O$ in a force field given by $\vec{F}(x,y) = \left(x^2-y^2\right)\hat{i} + 2xy\,\hat{j}$. Along the rectangular path shown in fig. Find the answer by evaluating the line integral and also using the Stokes' theorem.

Solution.



$$\vec{F} = \left(x^2-y^2\right)\hat{i} + 2xy\,\hat{j}\,;\ \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = (2y+2y)\hat{k} = 4y\hat{k} \qquad ....(1)$$

For $OP, x=0, dx=0$; $\int_{OP}\vec{F}\cdot d\vec{r} = 0$

For $PQ, y=b, dy=0$; $\int_{PQ}\vec{F}\cdot d\vec{r} = \int_0^a\left(x^2-b^2\right)dx = \left.\frac{x^3}{3}-b^2x\right|_0^a = \frac{a^3}{3}-ab^2$

For $QR, x=a, dx=0$; $\int_{QR}\vec{F}\cdot d\vec{r} = \int_b^0 2ay dy = \left.ay^2\right|_b^0 = -ab^2$

For $RO, y=0, dy=0$; $\int \vec{F}\cdot d\vec{r} = \int_a^0 x^2dx = -\frac{a^3}{3}$

Using (1) $\oint \vec{F}\cdot d\vec{r} = 0 + \left(\frac{a^3}{3}-ab^2\right) - ab^2 - \frac{a^3}{3} = -2ab^2$

Using Stoke's theorem

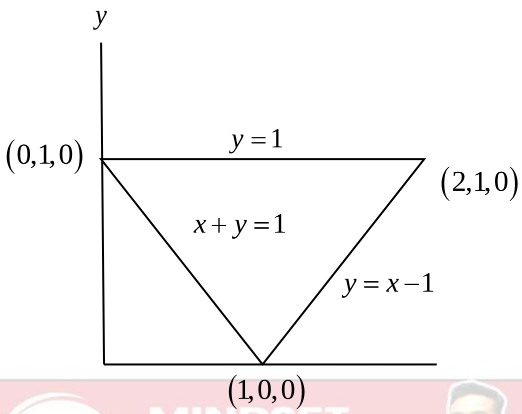
$$\oint \vec{F}\cdot d\vec{r} = \int \nabla\times\vec{F}\cdot\hat{n}\,dS$$

For given orientation of loop, $\hat{n} = -\hat{k}, dS = dxdy$

$$\int \nabla\times\vec{F}\cdot\hat{n}\,dS = -\int_0^a\int_0^b 4y\,dxdy = -2ab^2$$

Ex 15. Let C be the boundary of the triangle with vertices $(0,1,0),(1,0,0)$ and $(2,1,0)$. If $\vec{F}(x,y,z) = -y\hat{i} + y^2 z\hat{j} + zx\hat{k}$, then use Stoke's theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ when C is traversed counter clockwise when viewed from above.

Solution.



According to Stoke's Law

$$\oint \vec{F}\cdot d\vec{r} = \int_S \left(\nabla\times\vec{F}\right)\cdot\hat{n}\,dS = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & y^2 z & zx \end{vmatrix} = -y^2\hat{i} - z\hat{j} + \hat{k}$$

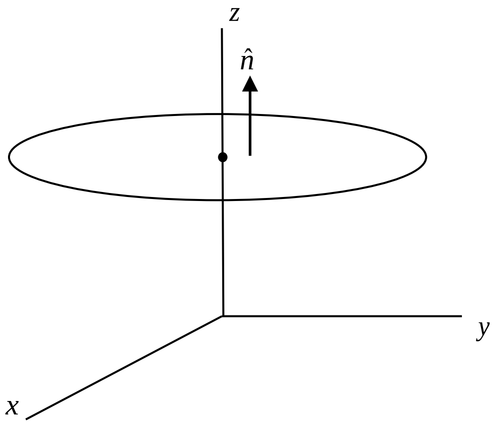
For the surface bounded by triangle, $\hat{n} = \hat{k}$.

$\left(\nabla\times\vec{F}\right)\cdot\hat{n} = 1, dS = dxdy$,

$$\int_S \left(\nabla\times\vec{F}\right)\cdot\hat{n}dS = \int dS = \int_0^1\int_{1-y}^{y+1} dxdy = \int_0^1 2ydy = 1$$

Ex 16. Evaluate by Stoke's theorem $\oint e^x dx + 2ydy - dz$ where C is the curve $x^2 + y^2 = 9$ & $z = 2$.

Solution.



The curve C is a circle of radius 3 units at a height 3 units from xy plane and having centre on the z axis. Let the surface enclosed by this curve is a disc of radius 3 as shown in figure. Students kindly note that the man with head in the direction of $\hat{n}$ and moving along the given orientation along the perifery should see the surface on his left. So, the direction of $\hat{n}$ and orientation should be matched accordingly

$$\oint e^x dx + 2y dy - dz = \oint \left(e^x \hat{i} + 2y\hat{j} - \hat{k}\right) \cdot d\vec{r}$$

So, $\vec{F} = e^x \hat{i} + 2y\hat{j} - \hat{k}$

Curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0$;By Stokes theorem $\oint \vec{F} \cdot d\vec{r} = \int_S \text{curl}\,\vec{F} \cdot \hat{n}\, dS = 0$ (Since, curl $\vec{F} = 0$)

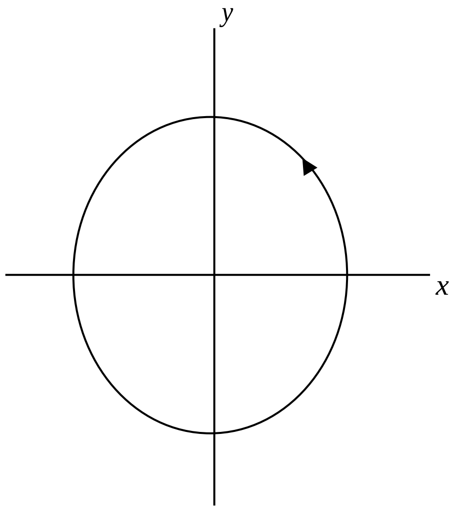
Ex 17. Use Stoke's theorem to evaluate the line integral $\oint_C x^2 y^3 dx + dy + zdz$ where C is the circle $x^2 + y^2 = 4, z = 0$.

Solution. The curve C is circle $x^2 + y^2 = 4$ in xy plane. Let the surface enclosed is a disc of radius 2 lying in xy plane and bounded by C.

For anticlockwise orientation, $\hat{n} = \hat{k}$

$$\oint x^2 y^3 dx + dy + z\,dz = \oint \left(x^2 y^3 \hat{i} + \hat{j} + z\hat{k}\right) \cdot d\vec{r} = \oint \vec{F} \cdot d\vec{r}$$

So, $\vec{F} = x^2 y^3 \hat{i} + \hat{j} + z\hat{k}$



$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y^3 & 1 & z\end{vmatrix}=-3x^2y^2\hat{k}$$

On S, $\hat{n}=\hat{k}, dS=dxdy, z=0$

$\nabla\times\vec{F}\cdot\hat{n}=-3x^2y^2$

So, by Stoke's theorem

$$\oint_C \vec{F}\cdot d\vec{r}=\int_S \nabla\times\vec{F}\cdot\hat{n}\,dS=-3\iint_S x^2y^2dxdy=-3\int_0^{2\pi}\int_0^{2} r^5\sin^2\theta\cos^2\theta\,drd\theta$$

$$=-3\int_0^{2\pi}\left[\frac{r^6}{6}\right]_0^2\sin^2\theta\cos^2\theta\,d\theta=-128\frac{\Gamma(3/2)\Gamma(3/2)}{2\cdot\Gamma(3)}=-8\pi 35.$$

GAUSS' DIVERGENCE THEOREM

Q1. Evaluate the integral $\iint_S \left(3y^2z^2\hat{i}+4z^2x^2\,j+z^2y^2\,k\right).n\,dS$ where *S* is the upper part of the surface $4x^2+4y^2+4z^2=1$ above the plane *z* = 0 and bounded by the *xy*-plane. Hence, verify Gauss-Divergence theorem. **[6b UPSC CSE 2023]**

Ans. Hint: Refer example 4 GDT page number 6; z=0 means xy plane.

Q2. Using Gauss' divergence theorem, evaluate , where $\vec{F}=x\hat{i}-y\hat{j}+\left(z^2-1\right)\hat{k}$ and S is the cylinder formed by the surfaces $z=0, z=1, x^2+y^2=4$. **[8c UPSC CSE 2022]**

Ans. Hint: Refer example 11 GDT page number 12.

Q3. Verify Gauss divergence theorem for $\vec{F}=2x^2y\hat{i}-y^2\hat{j}+4xz\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$. **UPSC CSE 2021**

Hint: $\iint_S \vec{F}\cdot\vec{n}\,dS=\int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} div\,\vec{F}\,dxdydz$

Q4. Given a portion of a circular disc of radius 7 units and of height 1.5 units such that $x,y,z\ge 0$. Verify Gauss Divergence Theorem for the vector field $\vec{f}=\left(z,x,3y^2z\right)$ over the surface of the above mentioned circular disc. **[7c 2020 IFoS]**

Hint: It's the cylinder of radius 7 and height 1.5. So we consider it as $z=0, z=1.5, x^2+y^2=7$.
Now **Refer example 11 GDT page number 12.**

Q5. State Gauss divergence theorem. Verify this theorem for $\vec{F}=4x\hat{i}-2y^2j+z^2\hat{k}$, taken over the region bounded by $x^2+y^2=4, z=0$ and $z=3$. **[8c UPSC CSE 2019]**

Hint: Refer example 11 GDT page number 12.

Q6. If S is the surface of the sphere $x^2+y^2+z^2=a^2$, then evaluate $\iint_S \left[(x+z)dydz+(y+z)dzdx+(x+y)dxdy\right]$ using Gauss' divergence theorem.

Hint: Refer example 1 GDT page number 5. **[6d UPSC CSE 2018]**

Q7. Evaluate the integral: $\iint_S \bar{F}\cdot\hat{n}ds$ where $\bar{F}=3xy^2\hat{i}+\left(yx^2-y^3\right)\hat{j}+3zx^2\hat{k}$ and S is a surface of the cylinder $y^2+z^2\le 4, -3\le x\le 3$, using divergence theorem. **[8c UPSC CSE 2017]**

Hint: Refer example 11 GDT page number 12. Only difference is the circle is in yz palne and height is according to x now. So while solving Surface integral for middle curved surface, we take projection of either *ds* on zx palne or on xy plane.

Note: If we solve by converting into cylindrical coordinates then

$ds = 4d\theta dx; \quad 0 \le \theta \le 2, \quad -3 \le x \le 3$

Way.2 Apply Gauss div theorem. Get

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \int_{x=-3}^{3}\int_{y=-2}^{2}\int_{z=-\sqrt{4-z^2}}^{\sqrt{4-z^2}} div\,\vec{F}\,dxdydz = 288\pi$$

Q8. If E be the solid bounded by the *xy* plane and the paraboloid $z = 4 - x^2 - y^2$, then $\iint_S \bar{F}\cdot dS$ where S is the surface bounding the volume E and $\bar{F} = \left(zx\sin yz + x^3\right)\hat{i} + \cos yz\hat{j} + \left(3zy^2 - e^{\lambda^2+y^2}\right)\hat{k}$. **[5e 2016 IFoS]**

Hint: Refer example 8 GDT page number 9. Here given paraboloid is $S: x^2 + y^2 = -(z-4)$.

Hint: Apply gauss div theorem $\iint_S \bar{F}\cdot dS = 5\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{4-r^2} 3r^2 r dr d\theta dz = 32\pi$

Q9. Using divergence theorem, evaluate $\iint_S \left(x^3dydz + x^2ydzdx + x^2zdydx\right)$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. **[7b 2015 IFoS]**

Hint: Refer example 1 GDT page number 5.

Apply gauss div theorem $\iint_S \bar{F}\cdot dS = 5\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (r\sin\theta\cos\phi)^2.r^2\sin\theta dr d\theta d\phi = 2\pi/3$

Q10. Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0, z = 3$. **[8c 2014 IFoS]**

Hint: Refer example 11 GDT page number 12.

Q11. By using Divergence Theorem of Gauss, evaluate the surface integral $\iint\left(a^2x^2 + b^2y^2 + c^2z^2\right)^{-\frac{1}{2}} dS$, where S is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1, a, b$ and c being all positive constants. **[8c UPSC CSE 2013]**

Hint: Refer example 10 GDT page number 11.

Q12. Evaluate $\int_S \vec{F}\cdot d\vec{s}$, where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and s is the surface bounding the region $x^2 + y^2 = 4, z = 0$ and $z = 3$. **[6b 2013 IFoS]**

Hint: Refer example 11 GDT page number 12.

Q13. Verify the Divergence theorem for the vector function $\vec{F} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - xz\right)\vec{j} + \left(z^2 - xy\right)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. **[8b 2013 IFoS]**

Hint: Refer example 21 GDT page number 22.

Q14. Verify Gauss' Divergence Theorem for the vector $\vec{v} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ taken over the cube $0 \le x, y, z \le 1$. **[8d UPSC CSE 2011]**

Hint: Refer example 20 GDT page number 21. Take a=1 here.

Q15. Use the divergence theorem to evaluate $\iint_S \vec{V} \cdot \vec{n}\, dA$ where $\vec{V} = x^2 z\vec{i} + y\vec{j} - xz^2\vec{k}$ and S is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

Hint: Refer example 9 GDT page number 10. **[7c UPSC CSE 2010]**

Q16. Use divergence theorem to evaluate, $\iint_S \left(x^3 dy\, dz + x^2 y dz\, dx + x^2 z\, dy\, dx\right)$ where S is the sphere, $x^2 + y^2 + z^2 = 1$. **[8b 2010 IFoS]**

Hint: Refer example 11 GDT page number 12.

Q17. Using divergence theorem, evaluate $\iint_S \vec{A} \cdot d\vec{S}$ where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. **[8b UPSC CSE 2009]**

Hint: Refer example 11 GDT page number 12.

STOKE'S THEOREM

Q1. Given that C is a curve of the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 2$ and C is described counterclockwise. Verify Stokes' theorem for the line integral $\int_C -y^3 dx + x^3 dy - z^3 dz$. [6c**IFoS 2022]**

Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page.

Q2. Using Stokes' theorem, evaluate $\iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\, dS$, where $\vec{F} = \left(x^2 + y - 4\right)\hat{i} + 3xy\hat{j} + \left(2xy + z^2\right)\hat{k}$ and S is the surface of the paraboloid $z = 4 - \left(x^2 + y^2\right)$ above the *xy*-plane. Here, $\hat{n}$ is the unit outward normal vector on S. [8c**UPSC CSE 2021]**

Hint: Refer example 5 page 8

Parameterize:

$x = 2\cos\theta, y = 2\sin\theta, z = 0.$

$$\int_S curl\vec{F}.\hat{n}dS = \int_C \vec{F}.d\vec{r} = \int_{\theta=0}^{2\pi} (4\cos^2\theta + 2\sin\theta - 4)(-2\sin\theta d\theta) + 12\cos\theta\sin\theta(2\cos\theta d\theta) + 0 = -4\pi$$

Q3. Verify Stores' theorem for $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface: $x + y + z = 1$ lying in the first octant. [7a**UPSC CSE 2022]**

Hint: find $\hat{n}$ for the given plane's equation by gradient method. Then take $ds = \dfrac{dxdy}{|\hat{k}.\hat{n}|}$. Limits of x and y take from x=0, y=0 and from x+y=1. For line integral, work for three segments, xy plane i.e. z=0, yz palne i.e. x=0, zx plane i.e. y=0.

$$\int_S curl\vec{F}.\hat{n}dS = \int_{x=0}^{1}\int_{y=0}^{1-x} 2(y-(1-x-y)\frac{dxdy}{\left|\frac{1}{\sqrt{3}}\right|} = 0$$

Q4. Verify the Stokes' theorem for the vector field $\bar{F} = xy\hat{i} + yz\hat{j} + xz\hat{k}$ on the surface S which is the part of the cylinder $z^2 = 1 - x^2$ for $0 \le x \le 1, -2 \le y \le 2$; S is oriented upwards.

Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page.

[7a UPSC CSE 2020]

Q5. Evaluate the surface integral $\iint_S \nabla \times \bar{F} \cdot \hat{n}dS$ for $\bar{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the *xy*-plane. **[8b UPSC CSE 2020]**

Hint: Refer example 1, 7 page 5,9.

Q6. Evaluate by Stokes' theorem $\oint_C e^x dx + 2ydy - dz$, where C is the curve $x^2 + y^2 = 4, z = 2$.

Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page.

[8c UPSC CSE 2019]

$curl\vec{F} = \mathbf{0}$. Ans. I=0

Q7. Verify Stokes's theorem for $\bar{V} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **[6c 2019 IFoS]**

Hint: Refer example 1, 7 page 5,9.

$$\int_C \vec{F}.d\vec{r} = \pi = \int_S \vec{F}.\hat{n}dS$$

Q8. Evaluate the line integral $\int_C -y^3dx + x^3dy + z^3dz$ using Stokes's theorem. Here C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. The orientation on C corresponds to counterclockwise motion in the *xy*-plane. **[8b UPSC CSE 2018]**

Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page.

$curl\vec{F} = 3(x^2 + y^2)\hat{k}, \hat{n} = \dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}; curl\vec{F}.\hat{n} = \sqrt{3}(x^2 + y^2)$. $I = \iint_S \sqrt{3}(x^2 + y^2)\dfrac{dxdy}{1/\sqrt{3}} = \dfrac{3}{2}\pi$

Q9. Using Stoke's theorem evaluate

$\oint_C [(x + y)dx + (2x - z)dy + (y + z)dz]$, where C is the boundary of the triangle with vertices at $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$. **[6c 2017 IFoS].**

Observe: Here it same as question 3. Only we need to do is: writing equation of plane passing through given three points. It is $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$.

Hint: find $\hat{n}$ for the given plane's equation by gradient method. Then take $ds=\frac{dxdy}{|\hat{k}.\hat{n}|}$.

Area of triangle; $\Delta^2=\Delta_x{}^2+\Delta_y{}^2+\Delta_z{}^2=(\frac{1}{2}.3.6)^2+(\frac{1}{2}.2.6)^2+(\frac{1}{2}.2.3)^2 \Rightarrow \Delta=3\sqrt{14}$; from 3D.

$curl\vec{F}.\hat{n}=\frac{7}{\sqrt{14}}$; $\int_C \vec{F}.d\vec{r}=\frac{7}{\sqrt{14}}.3\sqrt{14}=21$; on applying stoke's theorem.

Q10. Evaluate $\iint_S \left(\nabla\times\vec{f}\right)\cdot\hat{n}dS$, where S is the surface of the cone, $z=2-\sqrt{x^2+y^2}$ above xy-plane and $\vec{f}=(x-z)\hat{i}+\left(x^3+yz\right)\hat{j}-3xy^2\hat{k}$. **[7d 2017 IFoS]**

Hint: Refer example 2 page 6.

Q11. Prove that $\oint_C fd\vec{r}=\iint_S d\vec{S}\times\nabla f$. **[8b UPSC CSE 2016]**

Hint: refer definition based examples given on page 4.

Q12. Evaluate $\iint_S \left(\nabla\times\vec{f}\right)\cdot\hat{n}dS$ for $\vec{f}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by its projection on the xy plane. **[6d 2016 IFoS]**

Hint: Refer example 2 page 6.

Q13. State Stokes' theorem. Verify the Stokes' theorem for the function $\overline{f}=x\hat{i}+z\hat{j}+2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z=x$ and the cylinder $x^2+y^2=1$ and S is the surface inside the intersected one. **[7a 2016 IFoS]**

Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page.

$curl\vec{F}=\hat{i}$, for the plane, $x-z=0$, $\hat{n}=\frac{1}{\sqrt{2}}\hat{i}+0.\hat{j}-\frac{1}{\sqrt{2}}\hat{k}$; for downward, we take negative sign,

so $-\hat{n}.\int_S curl\vec{F}.\hat{n}dS=\iint_D \frac{1}{\sqrt{2}}\frac{dxdy}{|\hat{n}.\hat{k}|}=-\iint_D dxdy=-\pi$ ..(i)

For line integral, parametrize, $x=\cos\theta, y=\sin\theta; z=\cos\theta;\quad 0\le\theta\le 2\pi$

$\int_C \vec{F}.d\vec{r}=\int_{\theta=0}^{2\pi}(-\frac{1}{2}\sin\theta+\frac{3}{2}\cos 2\theta-\frac{1}{2})d\theta=-\pi$ ..(ii)

Q14. If $\vec{F}=y\hat{i}+(x-2xz)\hat{j}-xy\hat{k}$, evaluate $\iint_S \left(\nabla\times\vec{F}\right)\cdot\hat{n}dS$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **[8b 2015 IFoS]**

Hint: Refer example 1, 7 page 5,9.

Q15. Evaluate by Stokes' theorem $\int_{\Gamma}(y\,dx+z\,dy+x\,dz)$ where Γ is the curve given by $x^2+y^2+z^2-2ax-2ay=0, x+y=2a$ starting from $(2a,0,0)$ and then going below the z-plane. **[6c UPSC CSE 2014]**
Hint: Refer example 6 page 8.

Q16. Evaluate $\iint_S \nabla\times\bar{A}\cdot\vec{n}dS$ for $\bar{A}=(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}$ and S is the surface of hemisphere $x^2+y^2+z^2=16$ above xy plane. **[7b 2014 IFoS]**
Hint: Refer example 1, 7 page 5,9.

Q17. Use Stokes' theorem to evaluate the line integral $\int_C(-y^3dx+x^3dy-z^3dz)$, where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$. **[8d UPSC CSE 2013]**
Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page.

Q18. If $\vec{F}=y\vec{i}+(x-2xz)\vec{j}-xy\vec{k}$, evaluate $\iint_S(\vec{\nabla}\times\vec{F})\cdot\vec{n}d\vec{s}$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **[8c UPSC CSE 2012]**
Hint: Refer example 1, 7 page 5,9.

Q19. Find the value of $\iint_S(\vec{\nabla}\times\vec{F})\cdot d\vec{s}$ taken over the upper portion of the surface $x^2+y^2-2ax+az=0$ and the bounding curve lies in the plane $z=0$, when $\vec{F}=(y^2+z^2-x)\vec{i}+(z^2+x^2-y^2)\vec{j}+(x^2+y^2-z^2)\vec{k}$. **[6b 2012 IFoS]**
Hint: Refer example 6 page 8.

Q20. If $\bar{u}=4y\hat{i}+x\hat{j}+2z\hat{k}$, calculate the double integral $\iint(\nabla\times\bar{u})\cdot d\vec{s}$ over the hemisphere given by $x^2+y^2+z^2=a^2,\ z\geq 0$. **[8b UPSC CSE 2011]**
Hint: Refer example 1, 7 page 5,9.

Q21. Evaluate the line integral $\oint_C(\sin x dx+y^2dy-dz)$, where C is the circle $x^2+y^2=16, z=3$, by using Stokes' theorem. **[5e 2011 IFoS]**
Hint: Cylinder and plane. Refer example 1 in the explanation of Stoke's theorem on first page. It's simple now. As only to evaluate surface integral on circular disk. Because z= constant, so its intersection with cylinder will be just the circular disk $x^2+y^2\leq 16$. Now evaluate as we did in surface integral for upper surface S_2.

Q22. Find the value of $\iint_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ taken over the upper portion of the surface $x^2+y^2-2ax+az=0$ and the bounding curve lies in the plane $z=0$, when

$\vec{F} = \left(y^2 + z^2 - x^2\right)\hat{i} + \left(z^2 + x^2 - y^2\right)\hat{j} + \left(x^2 + y^2 - z^2\right)\hat{k}$. **[8c UPSC CSE 2009]**

Hint: Refer example 6 page 8.

CHAPTER: CURVES IN SPACE, $\vec{T}\,\vec{N}\,\vec{B}$ FRAME, CURVATURE & TORSION

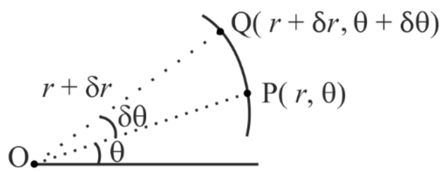
- **Curve in a Space:** A curve in space in described by a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where *x*, *y*, *z* may be some function of parameter '*t*',

$$\left.\begin{aligned} x &= f(t), y = g(t) \\ z &= h(t) \\ r &= x\hat{i} + y\hat{j} + z\hat{k} \end{aligned}\right\}$$

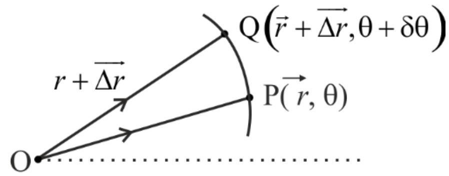
Tangent Line : It is the *best* tool tool to get the approximation of *curve* at any particular point.

E.g. $x = t$, $y = t^2$, $z = 2t$; a curve

Point on Curve:

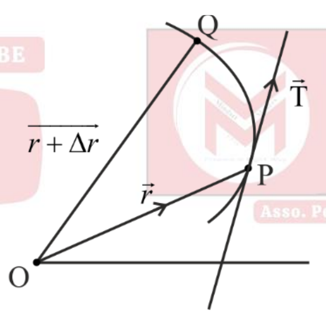


Differential Calculus



Vector Calculus

Tangent at point:-



When the point Q approaches to *P*, chord *PQ* or the arc *PQ* may be treated as a straight line

Let *A* be fixed point on curve s.t *AP* (arc length) = *s*, *AQ* (arc length) = $s + \delta s$

By definition, Tangent vector = $\lim\limits_{\delta s \to 0} \dfrac{\delta \vec{r}}{\delta s}$

Mathematical explanation about tangent vector at point P.

$$\vec{T} = \lim_{\delta s \to 0} \frac{\delta \vec{r}}{\delta s} = \lim_{\substack{\delta s \to 0 \\ \delta t \to 0}} \frac{\delta \vec{r}}{\delta t} \times \frac{\delta t}{\delta s}; \qquad \text{i.e. } \vec{T} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds}$$

Unit vector along the tangent to given curve

Taking the reference from previous discussion.

$\because\ \overrightarrow{OP} = \vec{r}$, $\overrightarrow{OQ} = \overrightarrow{r + \delta r}$, $\overrightarrow{PQ} = \overrightarrow{\delta r}$

Now, unit vector along the chord $PQ = \frac{\delta\vec{r}}{|\delta\vec{r}|} = \frac{\delta\vec{r}}{\delta s}.\frac{\delta s}{|\delta\vec{r}|}$

Let Q $\to$ P, then chord PQ be the tangent at P

$\therefore$ Unit vector along the tangent at P

$$= \lim_{Q\to P}\frac{\delta\vec{r}}{\delta s}\frac{\delta s}{|\delta\vec{r}|} = \lim_{Q\to P}\frac{\delta\vec{r}}{\delta s}\left(\frac{arc\ PQ}{chord\ PQ}\right)$$

$$\hat{t} = \frac{d\vec{r}}{ds};\ \left\{\begin{matrix}\because Q\to P \\ \frac{arc\ PQ}{chord\ PQ}\to 1\end{matrix}\right\};\ \hat{t} = \frac{d\vec{r}}{dt}.\frac{dt}{ds} \ \ldots \text{(i)} \quad \Rightarrow |\hat{t}| = \left|\frac{d\vec{r}}{dt}\right|.\left|\frac{dt}{ds}\right| \Rightarrow 1 = \left|\frac{d\vec{r}}{dt}\right|.\left|\frac{dt}{ds}\right| \quad \Rightarrow \left|\frac{ds}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$$

Exam point (1): CSE & IFoS

$$\hat{t} = \frac{d\vec{r}}{ds},\ \hat{t} = \frac{\left(\frac{d\vec{r}}{dt}\right)}{\left|\frac{d\vec{r}}{-dt}\right|}$$

Ex. Find a unit vector along the tangent to the given curve $\vec{r} = au\sin u\hat{i} + a(1-\cos u)\hat{i} + bu\hat{k}$

Sol. Given ; $\vec{r} = au\sin u\hat{i} + a(1-\cos u)\hat{i} + bu\hat{k}$

Hence x = a u sinu, y = a(1 – cos u)*, z = bu,* when *u* is the parameter

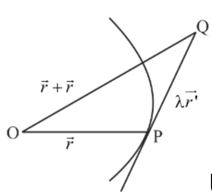
Required unit vector along the tangent; $\hat{t} = \frac{\frac{d\vec{r}}{du}}{\left|\frac{d\vec{r}}{du}\right|}$

□ $\frac{d\vec{r}}{du}$ = *a*(*u cos u +sin u*) $\hat{i}$ + *a sin u* $\hat{j}$ + *b* $\hat{k}$

$$\left|\frac{d\vec{r}}{du}\right| = \sqrt{a^2(u\cos u+\sin u)^2 + a^2\sin^2 u + b^2} = \sqrt{a^2u^2\cos^2 u + 2a^2\sin^2 u + b^2 + u\sin 2u}$$

$$\square\hat{t} = \frac{a(u\cos u+\sin u) + a\sin u\hat{j} + b\hat{k}}{\sqrt{a^2u^2\cos^2 u + 2a^2\sin^2 u + u\sin 2u}}$$

Supporting stuff to study about curvature & Torsion: (Not directly Question required/asked)



Unit vector along tangent at P; $\hat{t} = \frac{d\vec{r}}{ds}$.It is also denoted by, $\vec{r}\,'$

Let $\vec{R}$ be position vector of Q on the tangent line at P.

(tangent line parallel to $\hat{t}$ or $\vec{r}\,'$); $\overrightarrow{PQ} = \left|\overrightarrow{PQ}\right|\vec{r}\,' = \lambda \vec{r}\,'$, when λ is source parameter.

$\because \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$; $\vec{R} = \vec{r} + \lambda \vec{r}\,'$, which is the required equation of tangent line at P.

Equation of tangent line at P

- $R = X\hat{i} + Y\hat{j} + Z\hat{k}$; R(X,Y, Z); variable point on the tangent line.
- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; Point P(x, y, z)
- $\vec{r}\,' = \frac{dr}{ds}$ {s$\rightarrow$ arc length}

$$\therefore\ X\hat{i} + Y\hat{j} + Z\hat{k} = x\hat{i} + y\hat{j} + z\hat{k} + \lambda\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right) \Rightarrow X - x = \lambda\frac{dx}{ds},\quad Y - y = \lambda\frac{dy}{ds},\quad Z - z = \lambda\frac{dz}{ds}$$

$$\frac{X-x}{\left(\frac{dx}{ds}\right)} = \frac{Y-y}{\left(\frac{dy}{ds}\right)} = \frac{Z-z}{\left(\frac{dz}{ds}\right)} = \lambda;$$ is the tangent line at P on curve $\vec{r}$.

Note:(1) If the curve is given is parameter *t* i.e., $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where $x = f(t), y = g(t), z = h(t)$

$$\frac{X-x}{\left(\frac{dx}{dt}\right)} = \frac{Y-y}{\left(\frac{dy}{dt}\right)} = \frac{Z-z}{\left(\frac{dz}{dt}\right)} = \mu$$

Note (2) If the given curve is intersection of $f_1(x,y,z) = 0$ & $f_2(x,y,z) = 0$, then finding equation of tangent line at $P(x,y,z)$ on the curve.

$$\because\ f_1(x,y,z) = 0 \Rightarrow \frac{df_1}{ds} = 0 \Rightarrow \frac{\partial f_1}{\partial x}\frac{dx}{ds} + \frac{\partial f_1}{\partial y}\frac{dy}{ds} + \frac{\partial f_1}{\partial z}\frac{dz}{ds} = 0 \qquad ....(1)$$

Also, $$f_2(x,y,z) = 0 \Rightarrow \frac{df_2}{ds} = 0 \Rightarrow \frac{\partial f_2}{\partial x}\frac{dx}{ds} + \frac{\partial f_2}{\partial y}\frac{dy}{ds} + \frac{\partial f_2}{\partial z}\frac{dz}{ds} = 0 \qquad .....(2)$$

Solving (1) & (2) for $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$;
$$\frac{\left(\frac{dx}{ds}\right)}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}} = \frac{\left(\frac{dy}{ds}\right)}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}} = \frac{\left(\frac{dz}{ds}\right)}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}} = k$$

Let $\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y} = A$, $\quad \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z} = B \quad \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} = C$

$\therefore$ Req. equation of tangent line is {using exam point}

$$\frac{X-x}{A} = \frac{Y-y}{B} = \frac{Z-z}{C}$$

Ex-1. Show that the tangent at any point of the curve whose equation referred to rectangular axis are $x = 3t$, $y = 3t^2$, $z = 2t^3$ makes a constant angle with the line $y = z - x = 0$

Ex-2. Find the equation of tangent line at the point $t = 1$ to the curve $x = 1 + t, y = -t^2, z = 1 + t^2$

$\because$ given curve $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x = 1 + t, \qquad y = -t^2, \qquad z = 1 + t^2$

$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -2t, \frac{dz}{dt} = 2t$

At $t = 1$; $x = 2$, $y = -1$, $z = 2$

At $t = 1$; $\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -2, \frac{dz}{dt} = -2$

So, Required equation of tangent line at point ($t = 1$) i.e., $P(2, -1, 2)$; $\frac{X - x}{\left(\frac{dx}{dt}\right)} - \frac{Y - y}{\left(\frac{dy}{dt}\right)} = \frac{Z - z}{\left(\frac{dz}{dt}\right)}$

$$\frac{X - 2}{1} = \frac{Y - (-1)}{-2} = \frac{Z - 2}{-2}$$

Revising from **Analytical geometry:**

$\frac{x - \alpha_1}{l_1} = \frac{x - \beta_1}{m_1} = \frac{z - \gamma_1}{n_1}$ .....(1); $\frac{x - \alpha_2}{l_2} = \frac{x - \beta_2}{m_2} = \frac{z - \gamma_2}{n_2}$ .....(2)

Angle between lines (1) & (2) so given by; $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

Direction cosines (d.c.s) *l*, *m*, *n* Direction ratios (d.r.s) *a*, *b*, *c*

If *a*, *b*, *c* are given then, $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

So, Answer to our question; For tangent line at some point on given curve

$\frac{dx}{dt} = 3, \frac{dy}{dt} = 6t, \frac{dz}{dt} = 6t^2$

Equation of tangent line at any point (α, β, γ)

$\frac{x - \alpha}{\frac{dx}{dt}} = \frac{y - \beta}{\left(\frac{dy}{dt}\right)} = \frac{y - \gamma}{\left(\frac{dz}{dt}\right)}$; $\frac{x - \alpha}{3} = \frac{x - \beta}{6t} = \frac{z - \alpha}{t^2}$ .....(1)

Level (2): Another example

$y = z - x = 0$; $y = 0, z = x \Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$ .....(2)

$\therefore$ From (1); $l_1 = \dfrac{3}{\sqrt{3^2 + 36t^2 + 36t^4}} = \dfrac{3}{3\sqrt{1+4t^2+4t^4}} = \dfrac{1}{\left(1+2t^2\right)}$

$m_1 = \dfrac{6t}{3\left(1+2t^2\right)} = \dfrac{2t}{\left(1+2t^2\right)}$, $n_1 = \dfrac{6t^2}{3\left(1+2t^2\right)} = \dfrac{2t^2}{\left(1+2t^2\right)}$

From (2), $l_2 = \dfrac{1}{\sqrt{2}}$, $m_2 = 0$, $n_2 = \dfrac{1}{\sqrt{2}}$

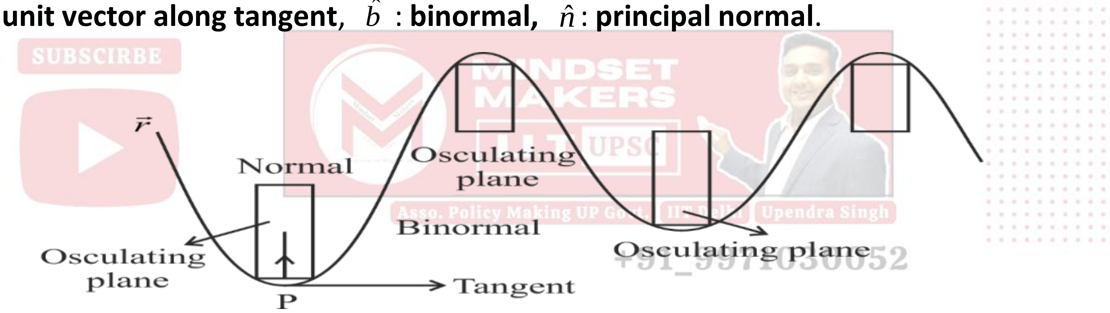
Now, $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$; $\cos\theta = \dfrac{1}{\sqrt{2}\left(1+2t^2\right)} + 0 + \dfrac{2t^2}{\sqrt{2}\left(1+2t^2\right)} \Rightarrow \cos\theta = \dfrac{\left(1+2t^2\right)}{\sqrt{2}\left(1+2t^2\right)}$ $\cos\theta = \dfrac{1}{\sqrt{2}}$

θ = 45°, which does not depend on t. $\therefore$ Angle between lines is constant.

Osculating planes equation at some point P on source given curve $\vec{r}$

Osculating Plane: A plane which touches at each point of given curve.

$\hat{t}$: **unit vector along tangent,** $\hat{b}$: **binormal,** $\hat{n}$: **principal normal.**

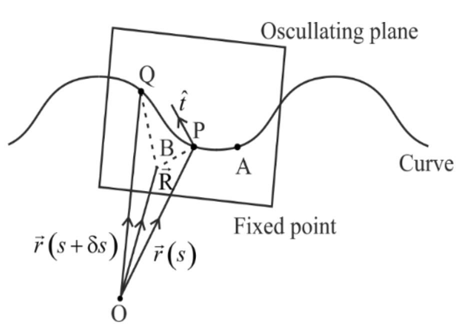


Mindmap: Categories of Questions

$\rightarrow$ Equation of osculating plane$\rightarrow$ Equation of binormal$\rightarrow$ Torsion & curvature

$\rightarrow$ Frenet serret (FS) formula (Proof type) Easy but after understanding

Osculating Plane:



B: arbitrary point on required osculating plane, $\hat{t}$: unit vector along tangent arc AP = s, arc $AQ = s+\delta s$, $\overrightarrow{OP} = \vec{r}(s)$, $\overrightarrow{OQ} = \vec{r}(s+\delta s)$

$\because$ $\overrightarrow{PB}, \hat{t}, \overrightarrow{PQ}$ are coplanar $\Rightarrow$ Their scalar triple product must be zero $\Rightarrow \left[\overrightarrow{PB}\,\hat{t}\,\overrightarrow{PQ}\right] = 0$ .....(1)

$$\because\ \overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \vec{R} - \vec{r}(s), \qquad \hat{t} = \frac{d\vec{r}}{ds} = \vec{r}\,', \qquad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}(s+\delta s) - \vec{r}$$

From (1), we have, $\left[\vec{R} - \vec{r}(s), \vec{r}\,'(s), (\vec{r}(s+\delta s) - \vec{r}(s))\right] = 0$

Note: $\because\ \vec{r}(s+\delta s) = \vec{r}(s) + \delta s\,\vec{r}\,'(s) + \frac{(\delta s)^2}{2!}\vec{r}\,''(s) + ---$

$$\vec{r}(s+\delta s) = \vec{r}(s) + \delta s\,\vec{r}\,'(s) + \frac{(\delta s)^2}{2!}\vec{r}\,''(s)$$ {Neglecting higher power of □(s)}

Now, $\left[\vec{R} - \vec{r}(s), \vec{r}\,'(s), \vec{r}(s+\delta s) - \vec{r}(s)\right] = 0 \Rightarrow \left(\vec{R} - \vec{r}(s)\right).\left\{\vec{r}\,'(s) \times \left(\delta s\,\vec{r}\,'(s) + \frac{(\delta s)^2}{2!}\vec{r}\,''(s)\right)\right\} = 0$

$$\Rightarrow \left(\vec{R} - \vec{r}(s)\right).\left\{0 + \frac{(\delta s)^2}{2!}\vec{r}\,'(s) + \vec{r}\,''(s)\right\} = 0 \Rightarrow \left(\vec{R} - \vec{r}(s)\right).\frac{(\delta s)^2}{2!}\left(\vec{r}\,'(s) \times \vec{r}\,''(s)\right) = 0$$

$$\Rightarrow \vec{R} - \vec{r}(s).\left(\vec{r}(s) \times \vec{r}\,''(s)\right) = 0 \ \Rightarrow \left[\vec{R} - \vec{r}(s),\ \vec{r}\,'(s),\ \vec{r}\,''(s)\right] = 0$$

Exam points:

1- Equation of osculating plane if given curve is $\vec{r}(s)$ is given by,

$\left[\vec{R} - \vec{r}(s), \vec{r}\,'(s), \vec{r}\,''(s)\right] = 0$ can be written in deferent form.

2- If given curve $\vec{r}$ is in parameter t then equation of osculating plane is

$$\begin{vmatrix} x - x(t) & y - y(t) & z - z(t) \\ \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \\ \ddot{x}(t) & \ddot{y}(t) & \ddot{z}(t) \end{vmatrix} = 0$$

Ex- Find the equation of osculating plane at a general point on a cubic centre given by $\vec{r} = (t, t^2, t^3) \because\ \vec{r} = x\hat{i} + y\hat{j}z + z\hat{k} = t\hat{i} + t^2 j + t^3\hat{k}$ $\therefore$ $x = t$, $y = t^2$, $z = t^3$

$\therefore$ Required equation of osculating plane at (x, y, z) = (t, t^2, t^3) is given by

$$\begin{vmatrix} x - t & y - t^2 & z - t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 0 \Rightarrow (x-t)6t^2 - 6t\left(y - t^2\right) + 2\left(z - t^3\right) = 0$$

Normal: Let if $f(x, y, z) = 0$ is given surface then equation of normal at any point $p(x, y, z)$; $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by, $\boxed{(\vec{R}-\vec{r}).Grad\ f = 0}$.

Normal plane: Plane through P and perpendicular to the tangent line at P is called normal plane at P is the given curve. Given by $\boxed{(\vec{R}-\vec{r}).\vec{r}' = 0} \Rightarrow \boxed{(\vec{R}-\vec{r}).\hat{t} = 0}$

Cartesian form:

1. $(X-x)x' + (Y-y)y' + (Z-z)z' = 0$; in case of parameter 's'.

2. $(X-x)\dot{x} + (Y-y)\dot{y} + (Z-z)\dot{z} = 0$; in case of parameter 't'.

Note: Normal plane is perpendicular to osculating plane.

Exam point:- Equation of osculating plane at a point on the curve of intersection of two surface

$f_1(x,y,z) = 0$ & $f_2(x,y,z) = 0$ is given by, $\boxed{\dfrac{(\vec{R}-\vec{r}).(grad\ f_1)}{|\dot{\vec{r}}.(grad\ f_1)|} = \dfrac{(\vec{R}-\vec{r}).(grad\ f_2)}{|\dot{\vec{r}}.(grad\ f_2)|}}$

Two special Normals:

- **Principal Normal:** The normal which is in the osculating plane at a point on the curve.
- **Bi Normal:** The normal which is perpendicular to osculating plane at a point. **The unit vector along binormal is denoted by $\hat{b}$**
- **Direction (Direction ratios drs)** of binormal:-

  $\because$ Binormal is perpendicular to osculating plane and osculating plane is perpendicular to the vector $\dot{\vec{r}} \times \ddot{\vec{r}}$. So, Binormal is parallel to $\dot{\vec{r}} \times \ddot{\vec{r}}$ $\therefore$ d.r.s $\dot{y}\ddot{z} - \dot{z}\ddot{y}, \dot{z}\ddot{x} - \dot{x}\ddot{z}, \dot{x}\ddot{y} - \dot{y}\ddot{x}$

- **Dr's of Principal Normal:** $\because$ Principal Normal is perpendicular to tangent and binormal, therefore principal normal is parallel to cross product $\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})$.

  $\therefore$ Required dr's may be found from above cross product expression.

Curvature

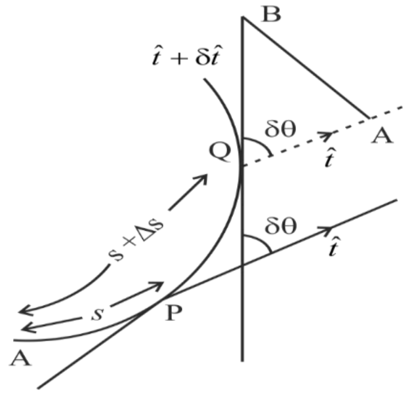
Curvature of a curve at some point P is defined as the rate of rotation of curve (i.e. tangents at the point P & at a point Q which is closer to P will be giving idea about it).

If the angle between tangents at P & Q is $\delta\theta$, then the curvature at the point P is given by

$$\underset{(\text{Kappa})}{\kappa} = \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}$$

Exampoint: (1) $\kappa = \dfrac{d\theta}{ds}$ (2) $\dfrac{1}{\kappa} = \rho$ is called radius of curvature at that point.

To find the expression for κ **:-**



$\because \left|\overrightarrow{QA}\right|=1, \left|\overrightarrow{QB}\right|=1$ $\{\because$ unit velocity along tangent$\}$

$$\Rightarrow \left|\overrightarrow{QA}\times\overrightarrow{QB}\right| = 1.1.\sin\delta\theta \Rightarrow \left|\hat{t}\times\left(\hat{t}+\delta\hat{t}\right)\right| = \sin\delta\theta \Rightarrow \left|\hat{t}\times\delta\hat{t}\right| = \sin\delta\theta \quad \left\{\because \hat{t}\times\hat{t}=0\right\}$$

$$\Rightarrow \left|\hat{t}\times\frac{\delta\hat{t}}{ds}\right| = \frac{\sin\delta\theta}{\delta s} \Rightarrow \left|\hat{t}\times\frac{\delta\hat{t}}{ds}\right| = \frac{\sin\delta\theta}{\delta\theta}\times\frac{\delta\theta}{\delta s}$$

Now, proceeding as Q $\to$ P, i.e., $\delta\theta\to 0$; $\Rightarrow \left|\hat{t}\times\frac{\delta\hat{t}}{\delta s}\right| = \lim_{\delta s\to 0} 1.\frac{\delta\theta}{\delta s} \Rightarrow \kappa = \left|\hat{t}\times\frac{\delta\hat{t}}{\delta s}\right|$ .....(1)

Now, $\because$ $\hat{t}$ is unit vector $\therefore$ $\hat{t}.\hat{t}=1$; Differentiating w.r.t s; $\hat{t}.\frac{d\hat{t}}{ds}+\frac{d\hat{t}}{ds}.\hat{t}=0 \Rightarrow 2\hat{t}.\frac{d\hat{t}}{ds}=0 \Rightarrow \hat{t}.\frac{d\hat{t}}{ds}=0$

$\therefore$ $\hat{t}$ is perpendicular to $\frac{d\hat{t}}{ds}=\hat{t}'$

From (1); $\kappa = \left|\hat{t}\right|\left|\frac{d\hat{t}}{ds}\right|\sin\frac{\pi}{2} \Rightarrow \kappa = \left|\frac{d\hat{t}}{ds}\right| \Rightarrow \kappa = \left|\frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)\right|$ $\left\{\because \hat{t}=\frac{d\vec{r}}{ds}=\vec{r}'\right\}$

$$\kappa = \left|\frac{d^2\vec{r}}{ds^2}\right| \Rightarrow \boxed{\kappa = \vec{r}''}$$

•Torsion of a curve at some point on curve: Def- Arc rate of rotation of binormal vector at point.

Denote by τ and $\frac{1}{\tau}$ is called the radius of torsion. $\tau = \lim_{\delta s\to 0}\frac{\delta\phi}{\delta s} = \frac{d\phi}{ds}$

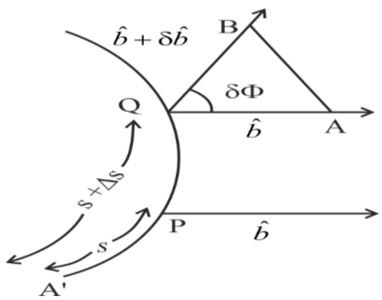
Expression for Torsion: $\left|\overrightarrow{QA}\times\overrightarrow{QB}\right| = 1.1\sin\delta\phi \Rightarrow \left|\hat{b}\times\frac{\delta\hat{b}}{\delta s}\right| = \frac{\sin\delta\phi}{\delta\phi}.\frac{\delta\phi}{\delta s}$

As Q $\to$ P, $\left|\hat{b}\times\frac{\delta\hat{b}}{\delta s}\right| = \lim_{\delta s\to 0}\frac{\delta\phi}{\delta s} \Rightarrow \left|\hat{b}\times\frac{\delta\hat{b}}{\delta s}\right| = \tau$ .....(1)

$\because$ $\hat{b}.\hat{b}=1$. So on Differentiating w.r.t. s; $\hat{b}.\frac{d\hat{b}}{ds}=0 \Rightarrow$ $\hat{b}$ is perpendicular $\frac{d\hat{b}}{ds}$.

From (1), $\tau = \left| \hat{b} \times \frac{\delta \hat{b}}{\delta s} \right| \Rightarrow \quad \tau = \left| \hat{b} \right| \left| \frac{d\hat{b}}{ds} \right| \sin \frac{\pi}{2}$ $\left\{ \begin{array}{l} \because \; Q \to P, \text{ so, } \delta s \to 0 \\ \therefore \lim\limits_{\delta s \to 0} \frac{\delta \hat{b}}{\delta s} = \frac{d\hat{b}}{ds} \end{array} \right\}$

$$\boxed{\tau = \frac{d\hat{b}}{ds}}$$



- **Screw Curvature:-** Arc-rate of rotation at which principal normal changes the direction. It's magnitude is given by $\sqrt{\kappa^2 + \tau^2}$ i.e., $\frac{d\hat{r}}{ds}$ as $P(\vec{r})$ moves along the curve is called the screw curvature.

- **SERRET-FRENET Formulae:- (Proofs are asked Multiple times)**

(1) $\frac{d\hat{t}}{ds} = \kappa \hat{n}$ (2) $\frac{d\hat{b}}{ds} = -\tau \hat{n}$ (3) $\frac{d\hat{n}}{ds} = \tau \hat{b} - \kappa \hat{t}$

Proof(1) $\because \; \hat{t} = \frac{d\vec{r}}{ds}, \quad \hat{t} . \frac{d\hat{t}}{ds} = 0$ {$\because \; \hat{t}.\hat{t} = 1$, on difference w.r.t. s we get it}

$\Rightarrow \hat{t}.\hat{t}' = 0 \qquad \Rightarrow \hat{t}$ is $\perp$ to $\hat{t}'$

- equation of osculating plane.

$\left[\vec{R} - \vec{r} \quad \vec{r}' \quad \vec{r}'' \right] = 0$ i.e., $\left[\vec{R} - \vec{r}, \; \hat{t} \quad \hat{t}' \right] = 0$ {Condition of coplanar}

$\hat{t}'$ is in the osculating plane, which is perpendicular to binormal $\hat{b} \Rightarrow \hat{t}'$ is perpendicular to $\hat{t}$ and $\hat{b}$.

$\because \; \hat{t}'$ is parallel to $\hat{t} \times \hat{b} \Rightarrow \hat{t}'$ is collinear with $\hat{n}$

- $\left| \hat{t}' \right| = \left| \vec{r}'' \right| = \kappa \; \Rightarrow \hat{t}' = \pm \kappa \hat{n}$

We choose the direction of $\hat{n}$ such that curvature κ is always positive.

$\therefore \; \hat{t}' = \kappa \hat{n}$. $\frac{d\hat{t}}{ds} = \kappa \hat{n}$ **Remember**

(2) **Target:** $\frac{d\hat{b}}{ds} = -\tau\hat{n}$

Proof: $\because$ $\hat{t}$ and $\hat{b}$ are perpendicular $\therefore$ $\hat{t}.\hat{b} = 0$

Diff. w.r.t. s, $\hat{t}.\hat{b}' + \hat{t}'\hat{b} = 0 \Rightarrow \hat{t}.\hat{b}' + (\kappa\hat{n}).\hat{b} = 0$ $\left\{\because \hat{t}' = \frac{dt}{ds} = \kappa\hat{n}\right\}$

$\Rightarrow \hat{t}.\hat{b}' = 0$ .....(1) $\Rightarrow$ $\hat{b}'$ is perpendicular to $\hat{t}$,

Now, $\hat{b}.\hat{b} = 1$; Diff. w.r.t s, $2\left(\hat{b}.\frac{d\hat{b}}{ds}\right) = 0$ $\Rightarrow \hat{b}.\hat{b}' = 0$ $\therefore$ $\hat{b}$ is perpendicular to $\hat{b}'$ ......(2)

- From (1) & (2),

$\hat{b}'$ is normal to the plane courting $\hat{t}$ & $\hat{b}$ $\therefore$ $\hat{b}'$ is parallel to $\hat{b} \times \hat{t}$ i.e., $\hat{b}'$ is parallel to $\hat{n}$.

- $\because$ $\left|\hat{b}'\right| = \left|\frac{d\hat{b}}{ds}\right| = \tau$ $\therefore$ $\hat{b}' = \pm\tau\hat{n}$; $\frac{d\hat{b}}{ds} = \pm\tau\hat{n}$

By tradition, we take $\hat{b}'$ is taken opposite to $\hat{n}$ $\therefore \frac{d\hat{b}}{ds} = -\tau\hat{n}$ **Remember**

(3) **Target:** $\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{n}$

Proof:- We know that, $\hat{n} = \hat{b} \times \hat{t}$

Diff. w.r.t s, $\frac{d\hat{n}}{ds} = \hat{b} \times \frac{d\hat{t}}{ds} + \frac{d\hat{b}}{ds} \times \hat{t}$ = $\hat{b} \times (\kappa\hat{n}) + (-\tau\hat{n}) + \hat{t}$ = $\kappa(\hat{b} \times \hat{n}) - \tau(\hat{n} \times \hat{t})$ = $\kappa(-\hat{t}) - \tau(-\hat{b})$

$\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{n}$ **Remember**

Exam point:-

F-S Formulae can be written in matrix form.

$$\begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}$$; Called: $\hat{t}\ \hat{n}\ \hat{b}$ frame

Note:

Demand of CSE/IFoS
- Proof type things
- Numerical type Q. (Formula based)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$; representation of a curve in space with parameter t.

Formula: for $\vec{r}$: for τ & κ.

(1) If $\vec{r}$ is position vector of any point P one the curve, then.

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}.\frac{ds}{dt} = \vec{r}\,'\frac{ds}{dt} = \vec{r}\,'\dot{s} = \hat{t}\,\dot{s} \text{ ....(1)} \quad \therefore \ |\dot{\vec{r}}| = |\hat{t}\,\dot{s}| = \dot{s} \text{ .....(2)}$$

Diff. w.r.t t, $\ddot{\vec{r}} = \hat{t}\,\ddot{s} + \hat{t}\,'(\dot{s})^2 \Rightarrow \ddot{\vec{r}} = \hat{t}\,\ddot{s} + (\kappa\hat{n})(\dot{s})^2$ .....(3)

Taking cross product of (1) & (3), we have,

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \hat{t}\dot{s} \times \left\{\kappa\hat{n}(\dot{s})^2 + \hat{t}\ddot{s}\right\} \Rightarrow \dot{\vec{r}} \times \ddot{\vec{r}} = \kappa(\dot{s})^3(\hat{t} \times \hat{n}) \quad \left\{\because \ \hat{t} \times \hat{t} = \vec{0}\right\}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \kappa(\dot{s})^3\,\hat{b} \quad \text{.....(4)} \Rightarrow |\dot{\vec{r}} \times \ddot{\vec{r}}| = |\kappa(\dot{s})^3|.1 \qquad \left\{\because \ |\hat{b}| = 1\right\}$$

$$\boxed{\kappa = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}}|^3}} \qquad \left\{\dot{\vec{r}} = \dot{s} \ (\text{from (2)})\right\}$$

(2) Formula for Torsion τ :- Exam point: $\boxed{\tau = \dfrac{[\dot{\vec{r}} \ \ \ddot{\vec{r}} \ \ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^3}}$

Formula for κ & τ for $\vec{r}\,'$; **(1)** $\boxed{\kappa = |\vec{r}\,'' \times \vec{r}\,''|}$ **(2)** $\boxed{\tau = \dfrac{[\vec{r}\,' \ \ \vec{r}\,'' \ \ \vec{r}\,''']}{|\vec{r}\,' \times \vec{r}\,''|^2}}$

Explanation:- $\because \ \vec{r}\,' = \hat{t} \ \ \& \ \ \vec{r}\,'' = \kappa\hat{n} \quad \therefore \ \vec{r}\,' \times \vec{r}\,'' = \hat{t} \times \kappa\hat{n} = \hat{k}\hat{b} \quad \left\{\because \ \hat{t} \times \hat{n} = \hat{b}\right\}$

$$|\vec{r} \times \vec{r}\,''| = \kappa \,; \ \left[\because \ |\hat{b}| = 1\right]$$

$\because \ \vec{r}\,' = \hat{t} \ \ \& \ \vec{r}\,'' = \kappa\hat{n}$ .....(2)

Diff- equation (2) w.r.t s,

$$\vec{r}\,''' = \kappa\frac{d\hat{n}}{ds} + \frac{d\dot{\kappa}}{ds}\hat{n} \Rightarrow \vec{r}\,''' = \kappa\tau\hat{b} - \kappa^2\hat{t} + \kappa'\hat{n} \text{ .....(3)} \qquad \left[\because \ \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}\right]$$

Equation (1), (2) & (3) can be written as,

$$\vec{r}\,' = 1\hat{t} + 0.\hat{n} + 0.\hat{b}\,, \quad \vec{r}\,'' = 0\hat{t} + \kappa\hat{n} + 0\hat{b}\,, \quad \vec{r}\,''' = -\kappa^2\hat{t} + \kappa'\hat{n} + \kappa\tau\hat{b}$$

Writing triple product in determent for, we get,

$$\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]=\begin{vmatrix}1 & 0 & 0\\ 0 & \kappa & 0\\ -\kappa^2 & \kappa' & \kappa\tau\end{vmatrix}=\kappa^2\tau \Rightarrow \tau=\frac{\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]}{\kappa^2} \Rightarrow \boxed{\tau=\frac{\left[\vec{r}\ \vec{r}''\ \vec{r}'''\right]}{\left|\vec{r}\times\vec{r}''\right|^2}}$$

• **Radius of curvature (in polar form)** if $r = f(\theta)$ is given curve is given by $\rho=\dfrac{r^2+r'^2}{r^2+2r'^2-rr''}$

Subjective Examples

Ex-1: The necessary & sufficient condition for the curve to be a straight line is that the curvature $\kappa=0$ at all points on curve.

Just Fundamental:

Vector equation of straight line,

$\vec{r}=s\vec{a}+\vec{b}$, when $\vec{a}\ \&\ \vec{b}$ are constant vectors

Also, in symmetrical form,

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Ans: Let if curve is a straight

$\Leftrightarrow\ \therefore\ \vec{r}=s\vec{a}+\vec{b}$, when $\vec{a}\ \&\ \vec{b}$ are .... on diff. w.r.t. s,

$$\Leftrightarrow\ \frac{d\vec{r}}{ds}=\vec{a}=\vec{r}' \Leftrightarrow \frac{d^2\vec{r}}{ds^2}=\vec{0}=\vec{r}'' \Leftrightarrow\ \therefore\ \kappa=\left|\vec{r}'\times\vec{r}''\right|=\left|\vec{a}\times\vec{0}\right|=\left|\vec{0}\right|=0$$

Ex.2: The necessary & sufficient condition for a given curve to be a plane curve is that $\tau=0$ at all points of the curve.

Sol. Let if curve is place then we have to prove $\tau=0$

$\therefore$ By a plane curve means the tangents & normals at all points of the curve is in the plane of curve.

So, we can conclude that osculating plane at all points of the curve, is the plane of the curve

$\therefore\ \hat{b}$; the unit vector along binormal is constant.

$\hat{b}$ = constant $\Rightarrow \dfrac{d\hat{b}}{ds}=0\quad \therefore\ \tau=0$

Exampoint: Osculating plane at all points of the curve, is the plane of curve, then $\hat{b}$ is constant.

- **The condition is sufficient i.e.,**

If $\tau=0$ then we have to prove given curve is a plane curve.

- ❖ Let if $\tau = 0$ at all points of curve $\because \frac{d\hat{b}}{ds} = -\tau\hat{n} = 0 \therefore \hat{b}$ is constant vector
- ❖ $\frac{d}{ds}\left(\vec{r}.\hat{b}\right) = \frac{d\vec{r}}{ds}.\hat{b} + \vec{r}.\frac{d\hat{b}}{ds} = \hat{t}.\hat{b}$: {using (1)}

  $\frac{d}{ds}\left(\vec{r}.\hat{b}\right) = 0$ {$\because$ $\hat{t}$ & $\hat{b}$ are perpendicular i.e, $\hat{t}.\hat{b} = 0$}

$\Rightarrow$ $\vec{r}.\hat{b}$ = constant. So, $\vec{r}.\hat{b}$ is constant.

We know that, $\vec{r}.\hat{b}$ denotes the projection of $\vec{r}$ on $\vec{b}$ i.e., projection of position vector $\vec{r}$ on $\vec{b}$ is same at all points of curve . So, curve is a plane curve.

Ex.3: The necessary & sufficient for the curve to be a plane curve is $\left[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''\right]$ = 0

- Let the curve is plane curve. So, $\tau = 0$ at all points of curve.

  We know that, $\left[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''\right] = \kappa^2\tau \Rightarrow \left[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''\right] = 0$

- Let if $\left[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''\right] = 0$ at all points of curve, then we want to prove curve is a plane curve As, $\left[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''\right] = 0 \Rightarrow \kappa^2\tau = 0$; Either $\kappa = 0$ or $\tau = 0$

Note: if possible $\tau \neq 0$ at some point of curve then in the neighborhood of this point

$\tau \neq 0$ $\Rightarrow$ $\kappa = 0$ is the neighborhood of this point

$\therefore$ $\tau = 0$ on straight line contradict to assumption.

Q. Find the radius of curvature & radius of torsion of the helix $x = a\cos u, y = a\sin u, z = au\tan\alpha$

Ans: We know, $\kappa = \dfrac{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du}\right|}{\left|\frac{d\vec{r}}{du}\right|^3}$ .....(1) and $\tau = \dfrac{\left[\frac{d\vec{r}}{du} \quad \frac{d^2\vec{r}}{du} \quad \frac{d^3\vec{r}}{du}\right]}{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|^3}$ .....(2)

$\because$ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \vec{r} = a\cos u\hat{i} + a\sin uj + au\tan\alpha\hat{k}$

$\frac{d\vec{r}}{du} = -a\sin u\hat{i} + a\cos u\hat{j} + a\tan\alpha\hat{k}$, $\frac{d^2\vec{r}}{du^2} = -a\cos u\hat{i} - a\sin u\hat{j} + 0\hat{k}$, $\frac{d^3\vec{r}}{du^3} = a\cos u\hat{i} - a\cos u\hat{j}$

$$\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -a\sin u & a\cos u & a\tan\alpha\\ -a\cos u & -a\sin u & 0\end{vmatrix}$$

$$=i\left\{0+a^2\sin u\tan\alpha\right\}-\hat{j}\left\{0+a^2\cos u\tan\alpha\right\}+\hat{k}\left\{a^2\sin^2 u+a^2\cos^2 u\right\}$$

$$=a^2\sin u\tan\alpha\hat{i}-a^2\cos u\tan\alpha\hat{j}+a^2\hat{k}$$

$$\left[\frac{d\vec{r}}{du}\;\frac{d^2\vec{r}}{du^2}\;\frac{d^3\vec{r}}{du^3}\right]=\begin{vmatrix}-a\sin u & acous & a\tan\alpha\\ -a\cos u & -a\sin u & 0\\ a\sin u & -a\cos u & 0\end{vmatrix}=a\tan\alpha\left\{a^2\cos^2 u+a^2\sin^2 u\right\}=a^3\tan\alpha$$

$$\left|\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du^2}\right|=\sqrt{a^4\tan^2\alpha+a^4}=a^2\sec\alpha\,,\quad\left|\frac{d\vec{r}}{du}\right|=\sqrt{a^2\left(1+\tan^2\alpha\right)}=a\sec\alpha$$

$\therefore$ Radius of curvature = $\frac{1}{\kappa}=\frac{\left(a\sec\alpha\right)^3}{a^2\sec\alpha}=a^2\sec^2\alpha$

Radius of torsion = $\frac{1}{\tau}=\frac{\left(a^2\sec\alpha\right)^3}{a^3\tan a}=\frac{a^6\sec^2\alpha}{a^3\sin\alpha.\sec\alpha}=\frac{a^2}{\sin\alpha\cos^2\alpha}=a^2\cos ec\alpha\sec^2\alpha$

PREVIOUS YEARS QUESTIONS

CURVATURE & TORSION

Q7© If the tangent to a curve makes a constant angle θ with a fixed line, then prove that the ratio of radius of torsion to radius of curvature is proportional to $\tan\theta$. Further prove that if this ratio is constant, then the tangent makes a constant angle with a fixed direction. **UPSC CSE 2023** **(15)**

Q8(c) If a curve in a space is represented by $\vec{r}=\vec{r}(t)$, then derive expressions of its torsion and curvature in terms of $\dot{\vec{r}}$, $\ddot{\vec{r}}$ and $\dddot{\vec{r}}$. Find the curvature and torsion of the curve given by $\vec{r}=\left(at-a\sin t, a-a\cos t, bt\right)$. **IFoS 2022**

Q1. A tangent is drawn to a given curve at some point of constant. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{\left[25\kappa^2\tau^2\left(1+25\kappa^2\right)+\left\{\kappa+5\frac{d\kappa}{ds}+25\kappa^3\right\}\right]^{1/2}}{\left(1+25\kappa^2\right)^{3/2}}$$

Find the curvature and torsion of the curve $\vec{r}=t\hat{i}+t^2\hat{j}+t^3\hat{k}$. **[6c 2020 IFoS]**

Q2. Find the radius of curvature and radius of torsion of the helix $x=a\cos u$, $y=a\sin u, z=au\tan\alpha$. **[7b UPSC CSE 2019]**

Q3. Let $\bar{r}=\bar{r}(s)$ represent a space curve. Find $\frac{d^3\bar{r}}{ds^3}$ in terms of $\bar{T},\bar{N}$ and $\bar{B}$, where $\bar{T},\bar{N}$ and $\bar{B}$ represent tangent, principal normal and binormal respectively. Compute $\frac{d\bar{r}}{ds}\cdot\left(\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}\right)$ in terms of radius of curvature and the torsion. **[5d 2019 IFoS]**

Q4. Derive the Frenet-Serret formulae. Verify the same for the space curve $x=3\cos t, y=3\sin t, z=4t$. **[7c 2019 IFoS]**

Q5. Find the curvature and torsion of the curve $\vec{r}=a(u\sin u)\vec{i}+a(1-\cos u)\vec{j}+bu\vec{k}$.

[7b UPSC CSE 2018]

Q6. Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of) a circle. **[7d 2018 IFoS]**

Q7. For a curve lying on a sphere of radius a and such that the torsion is never 0, show that

$\left(\frac{1}{\kappa}\right)^2+\left(\frac{\kappa'}{\kappa^2\tau}\right)^2=a^2$ **[8c 2018 IFoS]**

Q8. Find the curvature vector and its magnitude at any point $\vec{r}=(\theta)$ of the curve $\bar{r}=(a\cos\theta, a\sin\theta, a\theta)$. Show that the locus of the feet of the perpendicular from the origin to the tangent is a curve that completely lies on the hyperboloid $x^2+y^2-z^2=a^2$.

[7a UPSC CSE 2017]

Q9. Find the curvature and torsion of the circular helix $\vec{r}=a(\cos\theta,\sin\theta,\theta\cot\beta)$, β is the constant angle at which it cuts its generators. **[8c 2017 IFoS]**

Q10. If the tangent to a curve makes a constant angle α, with a fixed lines, then prove that $\kappa \cos\alpha \pm \tau \sin\alpha = 0$. Conversely, if $\frac{\kappa}{\tau}$ is constant, then show that the tangent makes a constant angle with a fixed direction. **[8d 2017 IFoS]**

Q11. For the cardioid $r = a(1+\cos\theta)$, show that the square of the radius of curvature at any point (r,θ) is a proportional to *r*. Also find the radius of curvature if $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$. **[8d UPSC CSE 2016]**

Q12. Find the curvature and torsion of the curve $x = a\cos t, y = a\sin t, z = bt$.

[5c 2015 IFoS]

Q13. Find the curvature vector at any point of the curve $\bar{r}(t) = t\cos t\hat{i} + t\sin t\hat{j},\ 0 \le t \le 2\pi$. Give its magnitude also. **[5e UPSC CSE 2014]**

Q14. Show that the curve $\vec{x}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} + \left(\frac{1-t^2}{t}\right)\hat{k}$ lies in a plane. **[5e UPSC CSE 2013]**

Q15. Derive the Frenet-Serret formulae. Define the curvature and torsion for a space curve. Compute them for the space curve $x = t, y = t^2, z = \frac{2}{3}t^3$. Show that the curvature and torsion are equal for this curve. **[8a UPSC CSE 2012]**

Q16. Find the curvature, torsion and the relation between the arc length S and parameter u for the curve: $\vec{r} = \vec{r}(u) = 2\log_e u\hat{i} + 4u\hat{j} + (2u^2+1)\hat{k}$. **[8a 2011 IFoS]**

Q17. Find κ/τ for the curve

$\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + bt\vec{k}$. **[1c UPSC CSE 2010]**

Hints:

PYQ 1: refer example

PYQ 3: Theory done in class notes

PYQ 4: Theory + application like PYQ (2 & 3)

PYQ 5: Similar to PYQ 3

Verify the answer

$$\kappa = \frac{a\sqrt{b^2 + 2a^2(\cos u - 1)^2}}{\left[\sqrt{b^2 + 2a^2(1-\cos u)}\right]^3},\ \tau = \frac{-b}{\sqrt{b^2 + a^2(1-\cos u)^2}}$$

PYQ 6: Theoretical explanation (class notes)

PYQ 7: Hint: Let $x = x(s)$ lie on the sphere with centre y_0 and radius a. Then for all & $(x(s)-y_0).(x(s)-y_0)=a^2$

Differentiating $2(x-y_0).\dot{x}=0$ or $(x-y_0)\,.\,t=0$

Differentiating $(x-y_0).\dot{t}+\dot{x}.t=0$

Note it follows that $x\neq 0$ and $(x-y_0).x=\frac{-1}{x}$

Differentiating again $\dot{x}.n+(x-y_0).\dot{n}$

PYQ 8:- $\because\ \kappa=\frac{d\hat{t}}{ds}$, where $\hat{t}$ is the unit tangential vector.

$$\hat{t}=\frac{d\vec{r}}{ds}=\frac{d\vec{r}/d\theta}{ds/d\theta},\quad \Rightarrow\left|\frac{d\vec{r}}{d\theta}\right|=\left|\frac{ds}{d\theta}\right| \quad \because \frac{d\vec{r}}{ds} \text{ is unit vector}$$

$$\frac{dr}{d\theta}=a\sqrt{2},\ T=\frac{dr/d\theta}{ds/d\theta}$$

$$\therefore \quad \tau=\frac{1}{a\sqrt{2}}\left(-a\sin\theta\,\hat{i}+a\cos\theta\,j+ak\right)=\frac{-1}{\sqrt{2}}\left(\sin\theta\,\hat{i}-\cos\theta\,j-k\right)$$

$$\kappa=\frac{d\hat{t}/d\theta}{ds/d\theta}=-\frac{1}{2a}\left(\cos\theta\hat{i}+\sin\theta\,j\right)\therefore|\kappa|=\frac{1}{2a}$$

PYQ 9:- Similar calculation like PYQ2

Answer verification:

$$\kappa=\frac{1}{a\left(1+\cos^2\beta\right)},\tau=\frac{\cos\beta}{a\left(1+\cos^2\beta\right)}$$

PYQ 10: Theory

PYQ 11: Radius of curvature (in polar form) if $r=f(\theta)$ is given curve is given by

$$\rho=\frac{r^2+r'^2}{r^2+2r'^2-rr''}$$

$\because$ Given $r=a(1+\cos\theta)$; using this in above formula

$$\rho^2=\left(\frac{8}{3}a\right)a\left(1+\cos\theta\right)\Rightarrow\rho^2\ \alpha\ r$$

Now when $\theta=0,\ \rho=4a-/3$

When $\theta = \frac{\pi}{4}$, $\rho = \frac{2\sqrt{2}a}{3}\sqrt{1+\frac{1}{\sqrt{2}}}$

When $\theta = \frac{\pi}{2}$, $\rho = \frac{2\sqrt{2}a}{3}$

PYQ 12: Similar to PYQ (2)

Answer verification: $\kappa = a/\left(a^2+b^2\right)$

$\tau = b/\left(a^2+b^2\right)$

PYQ 13: Similar to PYQ (2)

Answer verification: $K = \frac{2+t^2}{\left(1+t^2\right)3/2}$

PYQ 14: Calculate τ by this formula

$\because \quad \tau = \frac{\left[x'(t)\,x''(t)\,x'''(t)\right]}{\|x'\times x''\|^2} = 0$

$\therefore$ Given curve is plane.

PYQ 15: Theory

PYQ 16: Similar to PYQ (2)

PYQ 17: Similar to PYQ (2)

Mentor's advice: Although less number of questions asked from this theoretical perspective from following segment in CSE & IFoS. But if students are interested and have time to read, they can follow these few pages.

Source: These pages have been taken from a research paper but content is managed according to demand of CSE & IFoS examination.

1. Introduction

- It is a well–known fact that a space curve is uniquely determined, up to a choice of coordinate system,
by specifying the curvature *k* and torsion τ as functions of its arc length *s*. The functions *k*(*s*) and τ(*s*), which describe the deviation of a curve from linearity and planarity, are known as the "natural" or "intrinsic" equations of a curve.

- In general, the curvature and torsion are independent, but certain "special" curves with distinctive geometrical properties correspond to the existence of relationships between them.

The simplest cases are the *helical curves*, identified by the proportionality condition τ(*s*) = *k*(*s*) = *c*, a constant.

Equivalently, the curve tangent $\hat{\mathbf{t}}$ maintains a constant angle $\Psi = \cot^{-1} c$ with a fixed direction in space, the axis of the helical curve.

If k and τ are both constant we have a circular helix, while a general helix corresponds to non–constant k and τ.

- A *slant helix* may be regarded as a variation on the general helix, in which the curve principal normal $\hat{\mathbf{n}}$ (rather than the tangent $\hat{\mathbf{t}}$) maintains a constant angle with a fixed direction in space. This incurs a more complicated relation between k, τ, and the derivative of the $\tau = k$ ratio. The slant helices encompass the general helices as the particular case where the $\tau = k$ ratio is a constant; a proper slant helix has a non–constant $\tau = k$ ratio.

The *rectifying curves* are identified by a torsion/curvature ratio that is a linear function of the arc length, rather than a constant, i.e., $\tau(s) = k(s) = as + b$ where $a \neq 0$ and b are constants. A rectifying curve $\alpha(s)$ satisfies the condition $\langle \alpha(s), \hat{\mathbf{n}}(s) \rangle \equiv 0$, where $\hat{\mathbf{n}}$ (s) is the principal normal, i.e., at each point the position vector lies in the rectifying plane, spanned by tangent and binormal.

The *Salkowski curves* may be viewed as generalizations of the circular helix, since they exhibit a constant curvature but non–constant torsion. The Salkowski curves are proper slant helices.

The *spherical curves* (i.e., curves that lie on a sphere) are a further related category.

The identification of characterizations for helices, rectifying curves, slant helices, and spherical curves, and the study of their inter–relationships, are interesting basic problems in the theory of Frenet curves.

- An important concept associated with a <u>**unit–speed** Frenet curve $\alpha(s)$ is its centrode</u>

$\omega = \tau\, t + k\, b$, i.e., the

locus traced by the angular velocity vector, which determines the variation of the Frenet frame along $\alpha(s)$. The centrode has been employed to characterize rectifying curves.

2. Preliminaries

A unit–speed curve $\alpha(s) : I \rightarrow E^3$ is said to be a Frenet curve if $k(s) > 0$ at every point, and $\tau(s) \neq 0$. The Frenet frame $(t;\, n;\, b)$ consisting of the curve tangent, principal normal, and binormal satisfies the Frenet–Serret relations

$t' = k\,n,\; n' = -k\,t + \tau\,b,\; b' = -\tau\,n$……..(1)

where primes denote arc–length derivatives.

- A Frenet curve $\alpha(s)$ is a general helix if a fixed unit vector u exists, such that $\langle t(s), u \rangle = \cos\Psi$ for some fixed angle Ψ (the helix angle). The Lancret characterization states that a space curve $\alpha(s)$ is a general helix if and only if

$$\frac{\tau(s)}{k(s)} = c, \quad (2)$$

where $c = \cot \Psi$. When k and τ are both constant, $\alpha(s)$ is a circular helix.

• A curve $\alpha(s)$ is whose principal normal $n(s)$ makes a constant angle with a fixed unit vector is called a *slant helix*. It is known that $\alpha(s)$ is a slant helix if and only if its curvature and torsion satisfy

$$\frac{k^2(\tau/k)'}{\left(k^2+\tau^2\right)^{3/2}} = c \qquad (3);\ \text{for some constant } c.$$

• A *rectifying curve* $\alpha(s)$ satisfies $\langle \alpha(s), n(s) \rangle = 0$, i.e., the position vector $\alpha(s)$ always lies in the curve rectifying plane [4, 5]. It is known [4] that $\alpha(s)$ is a rectifying curve if and only if its torsion $\tau(s)$ and curvature $k(s)$ satisfy

$$\frac{\tau(s)}{k(s)} = as + b, \qquad (4);\ \text{where } \alpha \neq 0 \text{ and } b \text{ are constants.}$$

• A *spherical curve*, i.e., a curve that lies on a sphere of radius r with center at the origin, may be characterized [19] by the relation

$$(\rho'\sigma)' + \frac{\rho}{\sigma} = 0, \text{ where } \rho = \frac{1}{\kappa}, \sigma = \frac{1}{\tau}. \qquad (5)$$

It is known [4] that a Frenet curve $\alpha(s)$ is a rectifying curve if and only if a unit–speed spherical curve

$\gamma(s) : I \to S^2$ exists, such that

$\alpha(s) = a \sec(s + s_0)\, \gamma(s)$,

where S^2 is the unit sphere with center at the origin, and $a \neq 0$ and s_0 are constants. If $\{k, \tau, t, n, b\}$ is the Frenet–Serret apparatus of the rectifying curve $\alpha(s) : I \to E^3$ and k_γ is the curvature of the unit–speed curve $\gamma(s) : I \to S^2$, then we have [7]:

$$k = \frac{1}{a}\cos^3(s+s_0)\sqrt{\kappa_\gamma^2 - 1}, \quad \tau = \frac{1}{a}\cos^2(s+s_0)\sin(s+s_0)\sqrt{\kappa_\gamma^2 - 1}. \qquad (6)$$

The centrode of a unit–speed curve $\alpha(s)$ is defined by

$$\omega(s) = \tau(s)\, t(s) + k(s)\, b(s), \qquad (7)$$

i.e., it is the locus traced by the angular velocity vector (or *Darboux vector*) of the Frenet frame along $\alpha(s)$, which describes the variation of the frame vectors through the relations

$t' = \omega \times t,\ n' = \omega \times n,\ b' = \omega \times b,$

which are an alternative expression of equations (1). The centrode of a unit speed curve has been used to characterize rectifying curves [4, 5]. Also, the curve defined by

$$\omega_d(s) = \frac{\omega(s)}{k(s)}$$

is called the *dilated centrode*, and for a non–helical unit speed Frenet curve, it is shown in [7] that $\omega_d(s)$ is always a rectifying curve.

3. Characterizations of slant helices: ***Read only for learning the general procedure to address demand of questions in CSE & IFoS.***

In this section, some properties and characterizations of proper slant helices and Salkowski curves are derived. In particular, we will show that a unique general helix may be associated with each proper slant helix, and that the centrode of a Salkwoski curve is a proper slant helix. Let $\alpha(s) : I \to E^3$ be a unit–speed slant helix, with Frenet–Serret apparatus $\{k, \tau, t, n, b\}$. Then a fixed unit vector u and constant c exist, such that $\langle u, n(s)\rangle = c$, $s \in I$ [14].

For a proper slant helix, with $c \neq 0$, we show that no point $s_0 \in I$ exists, such that $\langle u, b(s_0)\rangle = 0$.

Differentiating $\langle u, n(s)\rangle = c$ and using (1) gives; $k\langle u, t(s)\rangle = \tau\langle u, b(s)\rangle$. (9)

If $\langle u, b(s_0)\rangle = 0$, this equation implies that $\langle u, t(s_0)\rangle = 0$, and consequently $u = \pm n(s_0)$ since u is a unit vector, so $c = \pm 1$. Writing $u = \langle u, t(s)\rangle t(s) \pm n(s) + \langle u, b(s)\rangle b(s)$

and taking the norm of both sides then gives; $1 = \sqrt{\langle u,t(s)\rangle^2 + 1 + \langle u,b(s)\rangle^2}$,

which can only be satisfied if $\langle u,t(s)\rangle \equiv 0$ and $\langle u,b(s)\rangle \equiv 0$, i.e., $u = \pm n(s)$.

Differentiating this and using nequations (1) gives $k(s) \equiv 0$ and $\tau(s) \equiv 0$, in contradiction with the assumption that $\alpha(s)$ is a proper slant helix. Hence, $\langle u,b(s)\rangle \neq 0$ for all $s \in I$, and equation (9) gives

$$\frac{\tau(s)}{k(s)} = \frac{\langle u,t(s)\rangle}{\langle u,b(s)\rangle}. \qquad (10)$$

Lemma 3.1. If $\alpha(s) : I \to E^3$ is a proper slant helix with the Frenet–Serret apparatus $\{k, \tau, t, n, b\}$ its unit axis vector u is given by

$$u = \frac{\sqrt{1-c^2}}{\sqrt{1+(\tau/k)^2}}(\tau/k)t + cn + \frac{\sqrt{1-c^2}}{\sqrt{1+(\tau/k)^2}}b. \qquad (11)$$

Proof: We have $u = \langle u,t\rangle t + cn + \langle u,b\rangle b$, (12)

which gives hu; $u = \langle u,t\rangle^2 + \langle u,b\rangle^2 = 1 - c^2$. From equation (10) we obtain

$$\frac{\tau^2 + k^2}{k^2} = \frac{\langle u,t\rangle^2 + \langle u,b\rangle^2}{\langle u,b\rangle^2} = \frac{1-c^2}{\langle u,b\rangle^2}$$

Since $\langle u, b(s)\rangle$ does not change sign on the connected interval $s \in I$, we may choose the direction of u that gives a positive value, and write $\langle u, b\rangle = \dfrac{\sqrt{1-c^2}}{\sqrt{1+(\tau/k)^2}}$.

Substituting this and (10) into (12) yields the stated form (11) of u.

Corollary 3.1. A unit–speed Frenet curve $\alpha(s) : I \to E^3$ is a proper slant helix if and only if its curvature $k(s)$ and torsion $\tau(s)$ satisfy

$$\left(\frac{(\tau/k)}{\sqrt{1+(\tau/k)^2}}\right)' = \frac{c}{\sqrt{1-c^2}}k, \quad \left(\frac{1}{\sqrt{1+(\tau/k)^2}}\right)' = \frac{c}{\sqrt{1-c^2}}\tau \qquad (13)$$

for some non–zero constant c.

Proof: Suppose the curve $\alpha(s)$ is a proper slant helix. Then differentiating (11) and equating components yields the relations (13).

Conversely, suppose that the two relations (13) hold for a unit–speed Frenet curve. Then the first relation gives

$$(\tau/k)\left(\frac{1}{\sqrt{1+(\tau/k)^2}}\right)' + \left(\frac{1}{\sqrt{1+(\tau/k)^2}}\right)(\tau/k)' = \frac{c}{\sqrt{1-c^2}}k$$

and substituting the second relation of (13) into the above yields

$$-\frac{c}{\sqrt{1-c^2}}\frac{\tau^2}{\kappa} + \left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}}\right)(\tau/\kappa)' = \frac{c}{\sqrt{1-c^2}}\kappa,$$

which reduces to $\dfrac{(\tau/\kappa)'}{\left(1+(\tau/\kappa)^2\right)^{3/2}} = \dfrac{c}{\sqrt{1-c^2}}\kappa.$

Since this is equivalent to equation (3), the curve is a proper slant helix.

Theorem 3.1.

- A unit–speed Frenet curve $\alpha(s) : I \to E^3$ with Frenet–Serret apparatus $\{\kappa, \tau, t, n, b\}$ is a proper slant helix if and only if

$$\tau/\kappa = \frac{f}{\sqrt{1-f^2}} \quad \text{where} \quad f = c\int \kappa\, ds \qquad (14)$$

and c is a non–zero constant.

Proof : Suppose the Frenet curve (s) satisfies the condition (14). Then we have

$$(\tau/\kappa)' = \frac{f'}{\left(1-f^2\right)^{3/2}} = \frac{c\kappa}{\left(1-f^2\right)^{3/2}} \quad \text{and} \quad 1+(\tau/\kappa)^2 = \frac{1}{1-f^2}$$

These equations give $\dfrac{(\tau/\kappa)'}{\left(1+(\tau/\kappa)^2\right)^{3/2}} = c\kappa$

- which with $c \neq 0$ is equivalent to the condition (3) for a proper slant helix.

Conversely, suppose $\alpha(s)$ is a proper slant helix. Then by Theorem A in [15], the indefinite integrals of κ and τ satisfy; $\left(\int \kappa ds\right)^2 + \left(\int \tau ds\right)^2 = \tan^2\theta,$ (15)

- where $0 < \theta < \frac{1}{2}\pi$ is the angle between $n(s)$ and the fixed direction u. From this, one can easily deduce the relations

$$\frac{\cos^2\theta}{\sin^2\theta}\left(\int \kappa ds\right)^2 < 1, \quad \frac{\kappa}{\tau} = -\frac{\int \tau ds}{\int \kappa ds} \quad . \qquad (16)$$

Now from (15) we obtain

$$1 + \frac{\left(\int \tau ds\right)^2}{\left(\int \kappa ds\right)^2} = \frac{\sin^2\theta}{\cos^2\theta\left(\int \kappa ds\right)^2},$$

and on using the second relation in (16), this becomes

$$1 + (\kappa/\tau)^2 = \frac{\sin^2\theta}{\cos^2\theta\left(\int \kappa ds\right)^2}, \text{ from which we obtain } (\tau/\kappa)^2 = \frac{\cos^2\theta\left(\int \kappa ds\right)^2}{\sin^2\theta - \cos^2\theta\left(\int \kappa ds\right)^2}$$

- This is equivalent to the stated condition (14) with $c = \pm\cot\theta$, and we note from (16) that $f^2 < 1$.

As a consequence of Theorem 3.1, and the fact that every Salkowski curve is a proper slant helix, we have the following characterization of Salkowski curves – essentially a result in [20].

Corollary 3.2.

- Corollary 3.2. A unit-speed Frenet curve $\alpha(s) : I \to E^3$ with curvature[1)] $\kappa = 1$ is a Salkowski curve if and only if its torsion is of the form

$$\tau(s) = \frac{cs}{\sqrt{1-c^2s^2}}; \text{ where } c \text{ is a non–zero constant.}$$

- It is interesting to observe, as the following theorem shows, that a unique general helix may be associated with each proper slant helix, such that the principal normal vector field of the slant helix coincides with the binormal vector field of the general helix.

Theorem 3.2.

Let $\alpha(s) : I \to E^3$ be a proper slant helix with axis vector u and Frenet–Serret apparatus $\{\kappa, \tau, t, n, b\}$

where $\kappa > 0$ and $\langle u, n \rangle = c$. Then a unique general helix $\beta(s) : I \to E^3$ exists with curvature

$c\sqrt{\tau^2+\kappa^2}\,/\sqrt{1-c^2}$, torsion $\sqrt{\tau^2+\kappa^2}$, and binormal vector field *n*.

Proof: We define the following unit vector fields $\mathbf{p}=\dfrac{(\tau/\kappa)\mathbf{t}+\mathbf{b}}{\sqrt{1+(\tau/\kappa)^2}},\quad \mathbf{q}=\dfrac{\mathbf{t}-(\tau/\kappa)\mathbf{b}}{\sqrt{1+(\tau/\kappa)^2}}$ (17)

along the curve α(*s*). Then one can easily verify that (*p*, *q*, *n*) is an oriented orthonormal frame along α(*s*), with **p** × **q** = *n*, **q** × *n* = **p**, *n* × **p** = **q**.

Differentiating equations (17), and using the relations (13) for a proper slant helix, we obtain

$$\mathbf{p}'=\frac{c}{\sqrt{1-c^2}}\sqrt{\tau^2+\kappa^2}\,\mathbf{q},\quad \mathbf{q}'=\sqrt{\tau^2+\kappa^2}\left(\mathbf{n}-\frac{c}{\sqrt{1-c^2}}\mathbf{p}\right), \qquad (18)$$

and we also have $\mathbf{n}'=-\kappa\mathbf{t}+\tau\mathbf{b}=-\sqrt{\tau^2+\kappa^2}\,\mathbf{q}$. (19)

Equations (18) – (19) indicate, by the existence theorem [19] for curves, that

$\left(\dfrac{c}{\sqrt{1-c^2}}\sqrt{\tau^2+\kappa^2},\sqrt{\tau^2+\kappa^2},\mathbf{p},\mathbf{q},\mathbf{n}\right)$; is the Frenet–Serret apparatus for a unique unit–speed curve β(*s*) : I → E^3, and that β(*s*) is a general helix.

1) The assumption κ = 1 is conventional in the study of Salkowski curves [20], and can be achieved for any curve of constant curvature by an appropriate scaling.

Remark 3.1.

For the example of a proper slant helix on page 161 of Izumiya–Takeuchi [14], we obtain the associated circular helix with constant curvature $\bar{\kappa}=b/\sqrt{a^2-b^2}$ and constant torsion $\bar{\tau}=\sqrt{a^2-b^2}$

Remark 3.2.

The Salkowski curves considered by Monterde [20] are proper slant helices, with curvature κ = 1 and torsion $\tau(s)=\dfrac{\pm s}{\sqrt{\tan^2\phi-s^2}}$,

- where ϕ is the constant angle made by the principal normal *n* with a fixed direction *u* (see Lemma 1 and Theorem 1 in [20]). Thus, setting *c* = cot ϕ, the curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of the general helix associated with a Salkowski curve are given by $\bar{\kappa}=\dfrac{c}{\sqrt{1-c^2}\sqrt{1-c^2s^2}},\ \bar{\tau}=\dfrac{1}{\sqrt{1-c^2s^2}}$.
- Every Salkowski curve is a proper slant helix, but there exist proper slant helices that are not Salkowski curves (for instance, the example given in [14]). The centrodes ω = τ *t* + κ *b* of Frenet curves are valuable in analyzing the kinematics of joints [13, 27], and it is of interest to ask whether the centrode of a proper slant helix is always a proper slant helix. The answer is negative, as illustrated by the example

$$\alpha(s)=-\frac{a^2-b^2}{2a}\left(\frac{\cos((a+b)s)}{(a+b)^2}+\frac{\cos((a-b)s)}{(a-b)^2},\frac{\sin((a+b)s)}{(a+b)^2}+\frac{\sin((a-b)s)}{(a-b)^2},\frac{2}{b\sqrt{a^2-b^2}}\cos bs\right),$$

in [14]. For $0 < b < a$, this is a unit–speed proper slant helix, with curvature and torsion

$$\kappa(s) = \sqrt{a^2 - b^2}\cos bs\,,\;\; \tau(s) = \sqrt{a^2 - b^2}\sin bs$$

The centrode $\omega = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ of this curve has parametric speed $v_\omega = |\omega'(s)|$, curvature κ_ω, and torsion τ_ω given by

$$v_\omega = b\sqrt{a^2 - b^2}\,,\quad \kappa_\omega = \frac{a}{b\sqrt{a^2 - b^2}}\,,\quad \tau_\omega = 0\,.$$

Thus, the centrode of $\alpha(s)$ is is an arc of a circle, and not a proper slant helix. On the other hand, one can show that the centrode of a Salkowski curve is a slant helix, as follows.

Theorem 3.3. A Salkowski curve $\alpha(s) : I \to E^3$ has a centrode $\omega = \tau\,\mathbf{t} + \kappa\,\mathbf{b}$ that is a proper slant helix, but is not a Salkowski curve.

Proof: The unit–speed Salkowski curve $\alpha(s)$ has curved and torsion given [20] by

$$\kappa(s) = 1, \qquad \tau(s) = \frac{\pm ms}{\sqrt{1 - m^2 s^2}}\,, \tag{20}$$

where $m \neq 0,\ \pm 1/\sqrt{3}$ is a real number, and the domain of $\alpha(s)$ is given by $|ms| < 1$. Thus, the centrode of the Salkowski curve is $\omega(s) = \frac{\pm ms}{\sqrt{1 - m^2 s^2}}\mathbf{t}(s) + \mathbf{b}(s)$,

from which we obtain

$$\omega'(s) = \frac{\pm m}{\left(1 - m^2 s^2\right)^{3/2}}\mathbf{t}(s)\,. \tag{21}$$

If s^ω is arc length along the centrode $\omega(s)$, its parametric speed v_ω is

$$v_\omega(s) = \frac{ds_\omega}{ds} = |\omega'(s)| = \frac{|m|}{\left(1 - m^2 s^2\right)^{3/2}} \tag{22}$$

and by the chain rule we have

$$\frac{d}{ds_\omega} = \frac{1}{v_\omega}\frac{d}{ds}\,. \tag{23}$$

From (21) we obtain the tangent to the centrode as

$$\mathbf{t}_\omega(s) = \frac{\omega'(s)}{|\omega'(s)|} = \pm\mathbf{t}(s)\,. \tag{24}$$

Its curvature κ_ω and principal normal n! are obtained using (22)–(23) from

$$\frac{dt_\omega}{ds_\omega} = \frac{\pm 1}{v_\omega}\frac{dt}{ds} = \kappa_\omega \mathbf{n}_\omega\,,$$

and since dt = ds = κ **n** with $\kappa(s)$ given by (20), we have

$$\kappa_\omega(s) = \frac{\left(1 - m^2 s^2\right)^{3/2}}{|m|}\,,\;\; \mathbf{n}_\omega(s) = \pm\mathbf{n}(s)\,. \tag{25}$$

Equations (24)–(25) give the centrode binormal vector as $\mathbf{b}_\omega \times \mathbf{t}_\omega \times \mathbf{n}_\omega = \mathbf{b}$. Since

$$\frac{\mathrm{d}\mathbf{b}_\omega}{\mathrm{d}s_\omega} = \frac{1}{v_\omega}\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\tau_\omega \mathbf{n}_\omega ,$$

and $\mathrm{d}\mathbf{b} \setminus \mathrm{d}s = -\tau\, \mathbf{n}$ where $\tau(s)$ is given by (20), we obtain the torsion of the centrode as

$$\tau_\omega(s) = \pm\left(1 - m^2 s^2\right)s \,. \qquad (26)$$

Since $\mathbf{n}_\omega(s) = \pm\mathbf{n}(s)$, the centrode is a slant helix. Moreover, it is a proper slant helix, since the ratio $\tau_\omega(s) / \kappa_\omega(s)$ is non–constant. The constant c in equation (3) can be found as follows. From (23) and (25)–(26),

we have

$$\frac{1}{\left(\kappa_\omega^2 + \tau_\omega^2\right)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}s_\omega}\frac{\tau_\omega}{\kappa_\omega} = \frac{1}{\left(\kappa_\omega^2 + \tau_\omega^2\right)^{3/2}}\frac{1}{v_\omega}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\tau_\omega}{\kappa_\omega} = \frac{\pm m^3}{\left(1 - m^2 s^2\right)^3} = \frac{\pm m}{\kappa_\omega^2} \,.$$

- Hence, the centrode of a Salkowski curve is a proper slant helix with constant $c = \pm\, m$ in in equation (3), and it is not a Salkowski curve since $\kappa_\omega \neq$ constant.

Remark 3.3.

The torsion/curvature ratio properties of general helices and rectifying curves indicate that they are mutually disjoint families of curves. It is not known whether a proper slant helix can also be a rectifying curve. However, Theorem 3.3 and the following Corollary show that the centrode of a Salkowski curve is both a proper slant helix and a rectifying curve.

Corollary 3.3.

The centrode $\omega = \tau\mathbf{t} + \kappa\mathbf{b}$ of a Salkowski curve $\alpha(s) : I \to E^3$ is a rectifying curve.

Proof: Using equations (25) and (26), we have

$$\frac{\tau_\omega(s)}{\kappa_\omega(s)} = \frac{|\,m\,|\, s}{\sqrt{1 - m^2 S^2}} \,.$$

Consequently, if s_ω is arc length along $\omega(s)$, using equation (23) we have

$\frac{\mathrm{d}}{\mathrm{d}s_\omega}\frac{\tau_\omega}{\kappa_\omega} = \frac{1}{v_\omega}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\tau_\omega}{\kappa_\omega} = 1$; so $\omega(s)$ is a general helix, since it satisfies (2) with non–constant τ_ω and κ_ω.

Moreover, integrating the above relation with respect to s_ω gives $\frac{\tau_\omega}{\kappa_\omega} = s_\omega + b$,

for some constant b, i.e., the centrode is a rectifying curve satisfying (4).

4. Associated circular helices of Frenet curves

Among all Frenet curves in E^3, the helices have a special stature due to their widespread applications in science and technology. In the present section, we highlight the importance and ubiquity of helices by showing that every Frenet curve is either a general helix, or else has a unique circular helix associated with it. We begin by proving this very general result.

Theorem 4.1.

Let $\alpha(s) : I \to E^3$ be a unit–speed Frenet curve of class C^k, $k \geq 4$ with Frenet Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Then $\alpha(s)$ is either a general helix, or there is a unique circular helix associated with it, defined by

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+(\tau/\kappa)^2}}, \frac{\tau/\kappa}{\sqrt{1+(\tau/\kappa)^2}}, \tan^{-1}(\tau/\kappa)\right) \qquad (27)$$

Proof: Suppose that $\alpha(s)$ is a Frenet curve that is not a general helix, i.e., $(\tau/\kappa)' \neq 0$. Then $\beta(s) : I \to E^3$ defined by (27) is a regular curve, with parametric speed $v_\beta(s) = \dfrac{ds_\beta}{ds} = \left|\boldsymbol{\beta}'(s)\right| = \dfrac{\left|(\tau/\kappa)'\right|}{1+(\tau/\kappa)^2}$,

where s_β is arc length along $\beta(s)$. Hence, using the Frenet–Serret relations, the Frenet–Serret apparatus of $\beta(s)$ can be computed as

$$\kappa_\beta = \tau_\beta = \frac{1}{\sqrt{2}}, \quad \mathbf{t}_\beta = \pm\frac{1}{\sqrt{2}}\left(\frac{-\tau/\kappa}{\sqrt{1+(\tau/\kappa)^2}}, \frac{1}{\sqrt{1+(\tau/\kappa)^2}}, 1\right),$$

$$\mathbf{n}_\beta = \frac{-(1, \tau/\kappa, 0)}{\sqrt{1+(\tau/\kappa)^2}}, \quad \mathbf{b}_\beta = \frac{\pm\left(\tau/\kappa, -1, \sqrt{1+(\tau/\kappa)^2}\right)}{\sqrt{2}\sqrt{1+(\tau/\kappa)^2}}.$$

Thus, $\beta(s)$ is a circular helix, since $\tau_\beta/\kappa_\beta = 1$. Hence, the unit speed Frenet curve $\alpha(s)$ is either a general helix, or there is a unique circular helix $\beta(s)$ defined by (27) associated with it.

Definition 4.1.

For a unit speed Frenet curve $\alpha(s) : I \to E^3$ of class C^k, $k \geq 4$ that is not a general helix, the unique circular helix $\beta(s)$ identified by (27) is called the associated circular helix of the Frenet curve $\alpha(s)$.

- In the remainder of this section, we use the circular helix associated with non–helical Frenet curves to formulate new characterizations for slant helices, Salkowski curves, spherical curves and rectifying curves.
- Note that a given proper slant helix $\alpha(s) : I \to E^3$ has two helices associated with it: the general helix identified in Theorem 3.2 and the associated circular helix (27). We now prove the following characterization for a proper slant helix.

Proposition 4.1. A unit–speed Frenet curve $(s) : I \to E^3$ of class C^k, $k \geq 4$ with Frenet–Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a proper slant helix if and only if the circular helix associated with it is given by

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\sqrt{1-f^2}, f, \sin^{-1} f\right),$$

Where $f = c \text{ f } \kappa\, ds$ and c is a non–zero constant.

Proof : Let $\alpha(s)$ be a unit-speed proper slant helix, which by the prof of theorem 3.1 satisfies Then substituting $\tau/\kappa = \tan\theta$ in equation (3) yields $\pm\,\theta' \cos\theta = c\,\kappa$. Absorbing the sign ambiguity into the constant c and integrating we find

$\sin\theta = c\int \kappa\, ds = f.$

Since $c \neq 0$, α(s) is not a general helix. The circular helix (27) associated with α(*s*) is thus given by

$$\beta(s) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \theta) = \frac{1}{\sqrt{2}}\left(\sqrt{1-f^2}, \sin^{-1} f\right).$$

Conversely, let the circular helix associated with the unit-speed Frenet curve $\alpha(s): I \to E^3$ be given by (28)

where $f = c\int \kappa ds, c \neq 0.$ then we have, $1+(\tau/\kappa)^2 = \dfrac{1}{1-f^2}$ and $f = \dfrac{\tau/\kappa}{\sqrt{1+(\tau/\kappa)^2}}$,

That is, $\tau/\tau = \dfrac{f}{\sqrt{1-f^2}}, where f = c\int \kappa\, ds,$ Which by theorem 3.1 shows that α(*s*) is a proper slant helix.

Recalling [20] that every Salkowski curve is a proper slant helix, we now find the constant *c* in equation (3). The curvature and torsion of a Salkowski curve α(*s*) are given by (20) with *m* = cot ϕ, where ϕ is the constant angle made by principal normal with a fixed direction and *s* is are length. Hence, for a unit-speed Salkowski curve, we obtain

$$(\tau/\kappa)' = \frac{\pm m}{(1-m^2s^2)^{3/2}} \; and \; 1+(\tau/\kappa)^2 = \frac{1}{1-m^2s^2}.$$

Thus, the equation (3) takes the form

$\dfrac{\kappa^2(\tau/\kappa)'}{(\tau^2+\kappa^2)^{3/2}} = \pm m$, And the constant is $c = \pm m$, this leads to the following characterization of Salkowski curves in terms of their associated circular helices.

Proposition 4.2. A unit-speed Frenet curve $\alpha(s): I \to E^3$ of class C^k, $k \geq 4$ with Frenet-Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a Salkowski curve if and only if the circular helix associated with it is given by

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\sqrt{1-m^2s^2}, \pm ms, \pm\sin^{-1}(ms)\right),$$ where $m \neq 0, \pm 1/\sqrt{3}$ is *a* non-zero constant.

Proof : Let α(*s*) be a unit-speed Salkowski curve with curvature and torsion given by (20). Since α(*s*) is a proper slant helix satisfying (3) with $c = \pm m$, its associated circular helix is given by equation (28) in Proposition 4.1 where $f = \pm m\int \kappa\, ds = \pm ms + b$. By the re-parametrization $s \to s - b/(\pm m)$, we obtain $f = \pm ms$ and then equation (28) reduces to the stated from (30).

Conversely, suppose the unit-speed Frenet curve α(s) has the curve (27) as its associated circular helix.

Setting $f = \pm ms = c\int ds,$ this becomes $\beta(s) = \dfrac{1}{\sqrt{2}}(\sqrt{1-f^2}, f, \sin^{-1} f),$

Which by proposition 4.1 indicates that α(s) is a proper slant helix with curvature κ = 1 and torsion τ satisfying (29) so that

$$1+\left(\frac{\tau}{1}\right)^2 \frac{1}{1-f^2}, \; i.e., \; \tau(s) = \frac{\pm ms}{\sqrt{1-m^2s^2}}.$$

Hence, $\alpha(s)$ is a Salkowski curve [20],

We consider next the circular helices associated with spherical curves.

Proposition 4.3. A non-helical unit-speed Frenet curve $\alpha(s) : I \to E^3$ of closs C^k, $k \geq 4$ with Frenet-Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a spherical curve on a sphere of radius c if and only if the circular helix associated with it is given by

$$\beta(s) = \frac{1}{2}\left(\frac{1}{\sqrt{1+f^2}}, \frac{f}{\sqrt{1+f^2}}, \tan^{-1} f \right),$$ Where $f = c\,\tau \cos\left(\int \tau\, ds\right)$ and c is a positive constant.

Proof: Suppose that $\alpha(s)$ is a non-helical unit-speed spherical curve that lies on a sphere of radius c. Then by integration of equation (5) we have $\frac{1}{\kappa^2} + \frac{\kappa'^2}{k^4 \tau^2} = c^2$,

Which gives $\frac{\kappa'}{\kappa\sqrt{\kappa^2 - 1/c^2}} = \pm c\tau$

And on integration this yields; $c\,\kappa = \pm \sec\left(\int \tau\, ds\right)$.

Absorbing the sign ambiguity into the constant c and setting f = τ/κ, this is equivalent to

$f = c\tau \cos\left(\int \tau ds\right)$.

Hence, the circular helix (27) associated with $\alpha(s)$ is given by

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+f^2}}, \frac{f}{\sqrt{1+f^2}}, \tan^{-1} f \right), \; f = c\tau \cos\left(\int \tau ds\right).$$

Conversely, suppose that the circular helix associated with $\alpha(s)$ is given by (31), where $f = c\tau \cos\left(\int \tau\, ds\right)$ which c a non-zero constant. Then the first component of $\beta(s)$ gives τ/κ = f, and consequently we have $\rho = c \cos\left(\int \tau\, ds\right)$.

Differentiation this twice yields

$(\rho'\sigma)' = -c\tau \cos\left(\int \tau\, ds\right)$,

And combining these two relations indicates satisfaction of equation (5), so that $\alpha(s)$ is spherical curve that lies on the sphere of radius c.

Finally, we consider the circular helices associated with rectifying curves. We first obtain the following result, characterizing rectifying curves in terms of their dilated centrodes $\omega_d(s)$ defined by (8).

Proposition 4.4. A unit-speed Frenet curve $\alpha(s) : I \to E^3$ with Frenet-Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a rectifying curve if and only if its position vector is given by $\alpha(s) = \frac{\omega_d(s)}{(\tau/\kappa)'}$,

Where $\omega_d(s)$ is the dilated centrode of $\alpha(s)$.

Proof: Suppose that the unit-speed curve $\alpha(s) : I \to E^3$ with Frenet-serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

Where a and $c \neq 0$ are constants. Differentiation this relation yields $\alpha'(s) = t + ((s + a)\kappa - c\tau)\, n = t$, since $\alpha(s)$ is unit speed. Hence, we have $\tau/\kappa = \frac{s+a}{c}$ and $(\tau/\kappa)' = \frac{1}{c}$.

Consequently, using equations (7)–(8) and (33), we have $\alpha(s) = c(\tau/\kappa)t + cb = \frac{\omega_d(s)}{(\tau/\kappa)'}$.

Conversely, if $\alpha(s)$ is of them form (32), we have $\langle \alpha(s), n(s) \rangle = 0$ for $s \in I$, since $\omega_d = (\tau/\kappa)\, \boldsymbol{t} + \mathbf{b}$, and thus $\alpha(s)$ is a rectifying curve.

Proposition 4.5. A unit-speed Frenet curve $\alpha(s) : I \to E^3$ of class C^k, $k \geq 4$ with Frenet-Serret apparatus $\{\kappa, \tau, \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a rectifying curve if and only if the circular helix associated with it is given by

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\frac{c}{\sqrt{c^2+s^2}}, \frac{s}{\sqrt{c^2+s^2}}, \tan^{-1}(s/c)\right),$$ Where c is a non-zero constant.

Proof: Suppose that $\alpha(s)$ is a unit-speed rectifying curve. Then by equation (4), we have

$$\frac{\tau(s)}{\kappa(s)} = as + b,$$

Where $a \neq 0$, b are constants, the re-parametrization $s \to s - b/a$ yields

$$\frac{\tau(s)}{\kappa(s)} = as$$

And since $\alpha(s)$ is not a general helix, its associated circular helix is given by Theorem 4.1 as

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+(as)^2}}, \frac{as}{\sqrt{1+(as)^2}}, \tan^{-1}(as)\right),$$

Which is the required form (34) with $c = a^{-1}$.

Conversely, suppose that the unit speed curve has the associated circular helix (34). Then from equation (27) we have $\frac{\tau}{\kappa} = \frac{s}{c}$,

i.e, the torsion/curvature ration of $\alpha(s)$ is a non-trivial linear function of arc length, and hence it is a rectifying curve.

Remark 4.1. Recall that there are essentially two ways to generate rectifying curves: through the dilated centrodes of a Frencet curve, and by the dilation of certain spherical curves. Note that for teach rectifying curve $\alpha(s)$, there is a unique unit-speed curve $\gamma(s)$ (excluding great circles) on the unit sphere S^2 with center at the origin [7] such that

$\alpha(s) = a \sec(s + s_0)\gamma(s)$,

Where $a \neq 0$ and s_0 are constants. However, this expression does not define a unit-speed curve – if s_α is are length along $\alpha(s)$, its parametric speed (assuming that $a > 0$) is

$$\upsilon_\alpha = \frac{ds_\alpha}{ds} = |\alpha'(s)| = a\sec^2(s+s_0),$$

Since $|\gamma(s)| = |\gamma(s)| = 1$ $\langle \gamma(s), \gamma'(s) \rangle = 0$. The curvature κ_α of $\alpha(s)$ are given by equation (6). Integration (35), the are length of $\alpha(s)$ is $s_\alpha = a\tan(s + s_0) + b$ for some constant b, and using the re-parameterization $s_\alpha \to s_\alpha - b$ and equation (6), we obtain

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{s}{a}.$$

Since α(s) is a rectifying curve, it is not a general helix, and its associated circular helix is thus obtained from (27) as

$$\beta(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+(s/a)^2}}, \frac{s/a}{\sqrt{1+(s/a)^2}}, \tan^{-1}(s/a) \right),$$

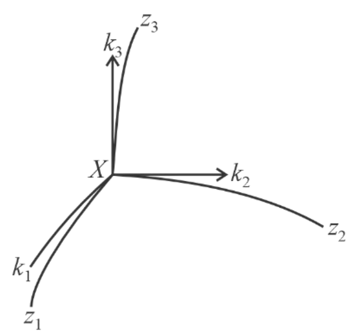
Which is in agreement with the expression as given in Proposition 4.5

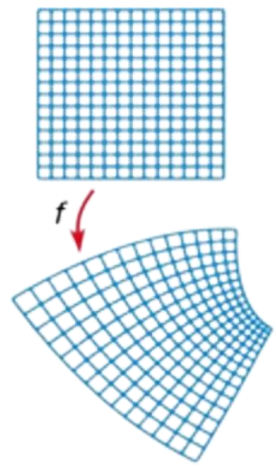
CURVILINEAR COORDINATES

Orthogonal curvilinear coordinate systems

Suppose that Cartesian coordinates (x, y, z) are expressed in terms of the new Coordinates (x_1, x_2, x_3) ; $x = x\,(x_1\, x_2, x_3)$, $y = y\,(x_1\, x_2, x_3)$, $z = z\,(x_1\, x_2, x_3)$

Where it is assumed that the correspondence is unique and that the inverse mapping exists.





A conformal map acting on a rectangular grid. note that the orthogonality of the curved grid is retained.

Figures above show Cartesian and orthogonal curvilinear coordinate systems and conformal mapping.

For example, circular cylindrical coordinates $(x_1, x_2, x_3) = (r, \theta, z)$; $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

i.e., at any point P, x_1 curve is a straight line, x_2 curve is a circle, and the x_3 curve is a straight line, i.e. $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} y/x$, $z = z$.

The position of a point P in space: $\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

- $\mathbf{R} = (r\cos\theta)\ \hat{\mathbf{i}} + (r\sin\theta)\ \hat{\mathbf{j}} + (z)\ \mathbf{k}$ for cylindrical coordinates
- A vector tangent to the x_1 curve is given by:

$\mathbf{R}_{x_1} = x_{x_1}\hat{\mathbf{i}} + y_{x_1}\hat{\mathbf{j}} + z_{x_1}\hat{\mathbf{k}}$ (Subscript denotes partial differentiation)

E.g. $\mathbf{R}_r = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$

Similarly, for x_2 and x_3; $\mathbf{R}_\theta = -r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}}$, $\mathbf{R}_z = \hat{\mathbf{k}}$

- So that the unit vectors tangent to the x_i curve are

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{R}_{x_1}}{h_1}, \hat{e}_2 \frac{\mathbf{R}_{x_2}}{h_2}, \hat{\mathbf{e}}_3 = \frac{\mathbf{R}_{x_3}}{h_3}$$

Where h_1 = $|\mathbf{R}_{x_1}|$ are called the metric coefficients or scale factors

E.g. for cylindrical coordinates: $h_r = 1, \quad h_\theta = r, \quad h_z = 1$

- The arc length a curve in any direction is given by

$ds^2 = d\mathbf{R}.d\mathbf{R} = h_1^2 dx_1{}^2 + h_2^2 dx_2^2 + h_3{}^2 dx_3^2$

Since $d\mathbf{R} = \mathbf{R}_{x_i} dx_i = h_i dx_i \, \hat{e}_i$, $\mathbf{R}_{x_i} = h_i \hat{\mathbf{e}}_i$ and x_i are orthogonal, i.e., $\hat{e}_i.\hat{e}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

On the surface x_1 = constant, the vector element of <u>surface area</u> is given by

$$d\mathbf{s}_1 = d\mathbf{R}_2 \times d\mathbf{R}_3 = h_2 dx_2 \hat{\mathbf{e}}_2 \times h_3 dx_3 \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 h_2 h_3 dx_2 dx_3$$

Where since x_i are orthogonal

$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1, \quad \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1$

and $-\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2, -\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3$ and $-\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1$ since **a** × **b** = –**b** × **a**

With similar results for x_2 and x_3 = constant

$$d\mathbf{s}_2 = d\mathbf{R}_3 \times d\mathbf{R}_1 = \hat{\mathbf{e}}_2 h_3 h_1 dx_3 dx_1$$

$$d\mathbf{s}_2 = d\mathbf{R}_3 \times d\mathbf{R}_1 = \hat{\mathbf{e}}_2 h_3 h_1 dx_3 dx_1$$

An element of volume is given by the triple product

$$\boxed{dV = d\mathbf{s_3}.d\mathbf{R}_3 = d\mathbf{R}_1 \times d\mathbf{R}_2.d\mathbf{R}_3 = (h_1 dx_1 \hat{e}_1 \times h_2 dx_2 \hat{e}_2).h_3 dx_3 \hat{e}_3 = h_1 h_2 h_3 dx_1 dx_2 dx_3}$$

$dV = J dx_1 dx_2 dx_3$; Where $J = h_1 h_2 h_3$ is the Jacobians of the transformation.

Gradient $\boxed{\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\hat{e}_3}$

By definition: $df = \nabla f . d\mathbf{R} = f_{x_i} dx_i$

If we temporarily write $\nabla f = \lambda_1 \hat{e}_1 + \lambda_2 \hat{e}_2 + \lambda_3 \hat{e}_3$ and using $d\mathbf{R} = R_{x_i} dx_i = h_i dx_i \hat{e}_i$

Then by comparison; $df = f_{x_i} dx_i = \lambda_i h_i dx_i \Rightarrow \lambda_i = \frac{1}{h_i}\frac{\partial f}{\partial x_i}$

$$\boxed{\nabla = \frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{e}_3}$$

Exam Point: $\boxed{\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z}$ **for cylindrical coordinates**

Note $\nabla_{x_i} = \frac{\hat{e}_i}{h_i} = R_{x_i}$

- By definition (curl(grad f) = **0**); $\nabla \times \nabla x_i = \nabla \times \frac{\hat{e}_i}{h_i} = \mathbf{0}$

Also $\frac{\hat{e}_1}{h_2 h_3} = \frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} = \nabla x_2 \times \nabla x_3$

- By definition (∇. ($\nabla f \times \nabla g$) = 0); $\nabla.\left(\frac{\hat{e}_1}{h_2 h_3}\right) = \nabla\left(\frac{\hat{e}_2}{h_3 h_1}\right) = \nabla.\left(\frac{\hat{e}_3}{h_1 h_2}\right) = 0$

Divergence

$$\boxed{\nabla.\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]}$$

Proof: $\nabla.\mathbf{F} = \nabla.(F_1\hat{e}_1) + \nabla.(F_2\hat{e}_2) + \nabla.(F_3\hat{e}_3)$

$\nabla.(F_1\hat{e}_1) = \nabla.\left[h_2 h_3 F_1\left(\frac{\hat{e}_1}{h_2 h_3}\right)\right]$ Using $\nabla.(\varphi u) = \varphi\nabla.u + u.\nabla\varphi$

$= \frac{\hat{e}_1}{h_2 h_3}.\nabla(h_2 h_3 F_1)$ using $\nabla.\left(\frac{\hat{e}_1}{h_2 h_3}\right) = \nabla.\left(\frac{\hat{e}_2}{h_3 h_1}\right) = \nabla\left(\frac{\hat{e}_3}{h_1 h_2}\right) = 0$

$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_1}(h_2 h_3 F_1)$

Treating the other terms in a similar manner, we get

$$\nabla.\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]$$

E.g. $\nabla.\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1) + \frac{\partial}{\partial \theta}(F_2) + \frac{\partial}{\partial z}(rF_3)\right]$

$\boxed{\nabla.\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(F_2) + \frac{\partial}{\partial z}(F_3)}$ for cylindrical coordinates.

Curl

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$\nabla \times \mathbf{F} = \nabla \times (F_1\hat{e}_1) + \nabla \times (F_2\hat{e}_2) + \nabla \times (F_3\hat{e}_3)$

$$\nabla\times(F_1\hat{e}_1)=\nabla\times\left[(h_1F_1)\left(\frac{\hat{e}_1}{h_1}\right)\right]$$

$$=\frac{\hat{e}_1}{h_1}\times\nabla(h_1F_1)$$ Using $\nabla\times(\phi\vec{u})=\phi\nabla\times\vec{u}+\nabla\phi\times\vec{u};(\nabla\phi\times\vec{u}=-\vec{u}\times\nabla\phi)$ and $\nabla\times\frac{\hat{e}_i}{h_i}=0$

$$=-\frac{\hat{e}_1}{h_1}\times\left[\frac{1}{h_1}\frac{\partial(h_1F_1)}{\partial x_1}\hat{e}_1+\frac{1}{h_2}\frac{\partial(h_1F_1)}{\partial x_2}\hat{e}_2+\frac{1}{h_3}\frac{\partial(h_1F_1)}{\partial x_3}\hat{e}_3\right]$$

$$=-\frac{\hat{e}_3}{h_1h_2}\frac{\partial}{\partial x_2}(h_1F_1)+\frac{\hat{e}_2}{h_3h_1}\frac{\partial}{\partial x_3}(h_1F_1)$$

$$=-\frac{1}{h_1h_2h_3}\left[h_2\hat{e}_2\frac{\partial}{\partial x_3}-h_3\hat{e}_3\frac{\partial}{\partial x_2}\right](h_1F_1)$$

So, $$\nabla\times F=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$$

E.g. $$\nabla\times\mathbf{F}=\frac{1}{r}\begin{vmatrix}\hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ F_1 & rF_2 & F_3\end{vmatrix}$$ for cylindrical coordinates.

Laplacian acting on a scalar

$$\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial x_3}\right)\right]$$

$$\nabla^2=\nabla\cdot\nabla=\nabla\cdot\frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1+\frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2+\frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\right)\right]$$

$$\nabla^2=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right]$$

E.g. $$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)+\frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)$$ for cylindrical coordinates.

Laplacian acting on a vector: $\nabla^2\mathbf{F}=\nabla(\nabla.\mathbf{F})-\nabla\times(\nabla\times\mathbf{F})$

Using $\nabla = \frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{\mathbf{e}}_3$

and $\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]$

$$\nabla(\nabla.\mathbf{F}) = \frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]\right]\hat{\mathbf{e}}_1$$

$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]\right]\hat{\mathbf{e}}_2$$

$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]\right]\hat{\mathbf{e}}_3$$

Using $\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$

$$\nabla \times (\nabla \times \mathbf{F}) = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 F_2) - \frac{\partial}{\partial x_2}(h_1 F_1)\right]\right) - \frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1 h_3}\left[\frac{\partial}{\partial x_3}(h_1 F_1) - \frac{\partial}{\partial x_1}(h_3 F_3)\right]\right)\right]\hat{\mathbf{e}}_1$$

$$+\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 F_3) - \frac{\partial}{\partial x_3}(h_2 F_2)\right]\right) - \frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 F_2) - \frac{\partial}{\partial x_2}(h_1 F_1)\right]\right)\right]\hat{\mathbf{e}}_2$$

$$+\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\left[\frac{\partial}{\partial x_3}(h_1 F_1) - \frac{\partial}{\partial x_1}(h_3 F_3)\right]\right) - \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 F_3) - \frac{\partial}{\partial x_3}(h_2 F_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

Combining those two terms gives $\nabla^2\mathbf{F} = \nabla(\nabla.\mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$

$$= \frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]\right]\hat{\mathbf{e}}_1$$

$$-\frac{1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 F_2) - \frac{\partial}{\partial x_2}(h_1 F_1)\right]\right) - \frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1 h_3}\left[\frac{\partial}{\partial x_3}(h_1 F_1) - \frac{\partial}{\partial x_1}(h_3 F_3)\right]\right)\right]\hat{\mathbf{e}}_1$$

$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 F_1) + \frac{\partial}{\partial x_2}(h_3 h_1 F_2) + \frac{\partial}{\partial x_3}(h_1 h_2 F_3)\right]\right]\hat{\mathbf{e}}_2$$

$$-\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 F_3) - \frac{\partial}{\partial x_3}(h_2 F_2)\right]\right) - \frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 F_2) - \frac{\partial}{\partial x_2}(h_1 F_1)\right]\right)\right]\hat{\mathbf{e}}_2$$

$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]\hat{\mathbf{e}}_3$$

$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

E.g. For cylindrical coordinates (r, θ, z), $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_z = 1$, and use the definition of Laplacian operator acting on a scalar $\nabla^2 f$

$$\nabla^2 f = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right]=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

$$=\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

$$\nabla^2\mathbf{F} = a\hat{\mathbf{e}}_r + b\hat{\mathbf{e}}_\theta + c\hat{\mathbf{e}}_z = \left(\nabla^2 F_1 - \frac{1}{r^2}F_1 - \frac{2}{r^2}\frac{\partial F_2}{\partial \theta}\right)\hat{\mathbf{e}}_r + \left(\nabla^2 F_2 - \frac{F_2}{r^2} + \frac{2}{r^2}\frac{\partial F_1}{\partial \theta}\right)\hat{\mathbf{e}}_\theta + \left(\nabla^2 F_3\right)\hat{\mathbf{e}}_z$$

Where $a = \frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]$

$$-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)\right]$$

$$=\frac{\partial}{\partial r}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial \theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2)-\frac{\partial}{\partial \theta}(F_1)\right]\right)-\frac{\partial}{\partial z}\left(r\left[\frac{\partial}{\partial z}(F_1)-\frac{\partial}{\partial r}(F_3)\right]\right)\right]$$

$$=\frac{\partial}{\partial r}\left[\frac{1}{r}\left[F_1 + r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial \theta}+r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}\left[F_2 + r\frac{\partial F_2}{\partial r}-\frac{\partial F_1}{\partial \theta}\right]\right)-\frac{\partial}{\partial z}\left(r\frac{\partial F_1}{\partial z}-r\frac{\partial F_3}{\partial r}\right)\right]$$

$$=\frac{\partial}{\partial r}\left[\frac{1}{r}F_1+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial \theta}+\frac{\partial F_3}{\partial z}\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}F_2+\frac{\partial F_2}{\partial r}-\frac{1}{r}\frac{\partial F_1}{\partial \theta}\right)-r\frac{\partial^2 F_1}{\partial z^2}+r\frac{\partial^2 F_3}{\partial z\partial r}\right]$$

$$=\left(\frac{-1}{r^2}F_1\right)+\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2 F_1}{\partial r^2}+\left(\frac{-1}{r^2}\frac{\partial F_2}{\partial \theta}\right)+\frac{1}{r}\frac{\partial^2 F_2}{\partial r\partial \theta}+\frac{\partial^2 F_3}{\partial r\partial z}$$

$$-\frac{1}{r}\left[\frac{1}{r}\frac{\partial F_2}{\partial\theta}+\frac{\partial^2 F_2}{\partial\theta\partial r}-\frac{1}{r}\frac{\partial^2 F_1}{\partial\theta^2}-r\frac{\partial^2 F_1}{\partial z^2}+r\frac{\partial^2 F_3}{\partial z\partial r}\right]$$

$$=\frac{-1}{r^2}F_1+\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2 F_1}{\partial r^2}-\frac{1}{r^2}\frac{\partial F_2}{\partial\theta}+\frac{1}{r}\frac{\partial^2 F_2}{\partial r\partial\theta}+\frac{\partial^2 F_3}{\partial r\partial z}$$

$$+\left[-\frac{1}{r^2}\frac{\partial F_2}{\partial\theta}-\frac{1}{r}\frac{\partial^2 F_2}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}-\frac{\partial^2 F_3}{\partial z\partial r}\right]$$

$$=\frac{-1}{r^2}F_1+\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2 F_1}{\partial r^2}-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}+\left[\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}\right]$$

$$=\left(\frac{1}{r}\frac{\partial F_1}{\partial r}+\frac{\partial^2 F_1}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}\right)-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$$=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial F_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F_1}{\partial\theta^2}+\frac{\partial^2 F_1}{\partial z^2}\right)-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}$$

$$\boxed{a=\nabla^2 F_1-\frac{1}{r^2}F_1-\frac{2}{r^2}\frac{\partial F_2}{\partial\theta}}$$

$$b=\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]$$

$$-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2F_2)-\frac{\partial}{\partial x_2}(h_1F_1)\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial\theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]\right]$$

$$-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\left[\frac{\partial}{\partial\theta}(F_3)-\frac{\partial}{\partial z}(rF_2)\right]\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2)-\frac{\partial}{\partial\theta}(F_1)\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\left[F_1+r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial\theta}+r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial F_3}{\partial\theta}-\frac{\partial F_2}{\partial z}\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\left[F_2+r\frac{\partial F_2}{\partial r}-\frac{\partial F_1}{\partial\theta}\right]\right)\right]$$

$$=\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}F_1+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial\theta}+\frac{\partial F_3}{\partial z}\right]$$

$$-\left[\frac{1}{r}\frac{\partial^2 F_3}{\partial z\partial\theta}-\frac{\partial^2 F_2}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}F_2+\frac{\partial F_2}{\partial r}-\frac{1}{r}\frac{\partial F_1}{\partial\theta}\right)\right]$$

$$=\frac{1}{r}\left[\frac{1}{r}\frac{\partial F_1}{\partial\theta}+\frac{\partial^2 F_1}{\partial\theta\partial r}+\frac{1}{r}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{\partial^2 F_3}{\partial\theta\partial z}\right]$$

$$-\left[\frac{1}{r}\frac{\partial^2 F_3}{\partial z\partial\theta}-\frac{\partial^2 F_2}{\partial z^2}-\left(\frac{-F_2}{r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}-\frac{-1}{r^2}\frac{\partial F_1}{\partial\theta}-\frac{1}{r}\frac{\partial^2 F_1}{\partial r\partial\theta}\right)\right]$$

$$=\frac{1}{r^2}\frac{\partial F_1}{\partial\theta}+\frac{1}{r}\frac{\partial^2 F_1}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{1}{r}\frac{\partial^2 F_3}{\partial\theta\partial z}$$

$$+\left[-\frac{1}{r}\frac{\partial^2 F_3}{\partial z\partial\theta}+\frac{\partial^2 F_2}{\partial z^2}-\frac{F_2}{r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r^2}\frac{\partial F_1}{\partial\theta}-\frac{1}{r}\frac{\partial^2 F_1}{\partial r\partial\theta}\right]$$

$$=\left(\frac{1}{r}\frac{\partial F_2}{\partial r}+\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{\partial^2 F_2}{\partial z^2}\right)-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial\theta}$$

$$=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F_2}{\partial\theta^2}+\frac{\partial^2 F_2}{\partial z^2}\right)-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial\theta}$$

$$\boxed{b=\nabla^2 F_2-\frac{F_2}{r^2}+\frac{2}{r^2}\frac{\partial F_1}{\partial\theta}}$$

$$c=\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]\right]$$

$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1F_1)-\frac{\partial}{\partial x_1}(h_3F_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3F_3)-\frac{\partial}{\partial x_3}(h_2F_2)\right]\right)\right]$$

$$=\frac{\partial}{\partial z}\left[\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_1)+\frac{\partial}{\partial\theta}(F_2)+\frac{\partial}{\partial z}(rF_3)\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\left[\frac{\partial}{\partial z}(F_1)-\frac{\partial}{\partial r}(F_3)\right]\right)-\frac{\partial}{\partial\theta}\left(\frac{1}{r}\left[\frac{\partial}{\partial\theta}(F_3)-\frac{\partial}{\partial z}(rF_2)\right]\right)\right]$$

$$=\frac{\partial}{\partial z}\left[\frac{1}{r}\left[F_1+r\frac{\partial F_1}{\partial r}+\frac{\partial F_2}{\partial\theta}+r\frac{\partial F_3}{\partial z}\right]\right]$$

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial F_1}{\partial z}-r\frac{\partial F_3}{\partial r}\right)-\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial F_3}{\partial\theta}-\frac{\partial F_2}{\partial z}\right)\right]$$

$$=\frac{\partial}{\partial z}\left[\frac{F_1}{r}+\frac{\partial F_1}{\partial r}+\frac{1}{r}\frac{\partial F_2}{\partial\theta}+\frac{\partial F_3}{\partial z}\right]$$

$$-\frac{1}{r}\left[\left(\frac{\partial F_1}{\partial z}+r\frac{\partial^2 F_1}{\partial r\partial z}-\frac{\partial F_3}{\partial r}-r\frac{\partial^2 F_3}{\partial r^2}\right)-\left(\frac{1}{r}\frac{\partial^2 F_3}{\partial\theta^2}-\frac{\partial^2 F_2}{\partial\theta zz}\right)\right]$$

$$= \frac{1}{r}\frac{\partial F_1}{\partial z} + \frac{\partial^2 F_1}{\partial z \partial r} + \frac{1}{r}\frac{\partial^2 F_2}{\partial z \partial \theta} + \frac{\partial^2 F_3}{\partial z^2}$$

$$+\left[-\frac{1}{r}\frac{\partial F_1}{\partial z} - \frac{\partial^2 F_1}{\partial r \partial z} + \frac{1}{r}\frac{\partial F_3}{\partial r} + \frac{\partial^2 F_3}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F_3}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 F_2}{\partial \theta \partial z}\right]$$

$$= \frac{1}{r}\frac{\partial F_3}{\partial r} + \frac{\partial^2 F_3}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F_3}{\partial \theta^2} + \frac{\partial^2 F_3}{\partial z^2}$$

$$\boxed{c = \nabla^2 F_3}$$

Derivatives of the unit vectors in orthogonal curvilinear coordinate systems

The last topic to be discussed concerning curvilinear coordinates is the procedure to obtain the derivatives of the unit vectors, i.e. $\frac{\partial}{\partial x_j}\hat{e}_i = \hat{e}_{ij}$

$$e_{ij} = \nabla \mathbf{V}, \quad \varepsilon_{ij} = \frac{1}{2}\left(e_{ij} + e_{ij}^T\right) = \frac{1}{2}(\nabla\mathbf{V} + \mathbf{V}\nabla), \quad \omega_{ij} = \frac{1}{2}\left(e_{ij} - e_{ij}^T\right) = \frac{1}{2}(\nabla\mathbf{V} + \mathbf{V}\nabla)$$

To simplify the rotation, we define:

$$\mathbf{R}_{x_i} = \mathbf{r}_i, \qquad \mathbf{R}_{x_i} = h_i\hat{\mathbf{e}}_i = \mathbf{r}_i; \quad \frac{\partial}{\partial x_j}\mathbf{R}_{x_i} = \mathbf{r}_{ij} \text{ and } \frac{\partial}{\partial x_j}h_i = h_{ij}$$

Note that $\mathbf{r}_{ij}$ is symmetric, i.e. $\mathbf{r}_{ij} = \mathbf{r}_{ji}$

$$\mathbf{r}_1 = h_1\hat{\mathbf{e}}_1, \; \mathbf{r}_2 = h_2\hat{\mathbf{e}}_2, \; \mathbf{r}_3 = h_3\hat{\mathbf{e}}_3$$

$$\mathbf{r}_{11} = a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2 + c\hat{\mathbf{e}}_3 = h_{11}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{11}, \quad \mathbf{r}_{12} = h_{12}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{12}, \quad \mathbf{r}_{13} = h_{13}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{13}$$

E.g. Derivation of $\boxed{\hat{\mathbf{e}}_{11} = -\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3}$

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = h_1^2, \; \mathbf{r}_1 \cdot \mathbf{r}_{11} = h_1 h_{11}, \; \mathbf{r}_1 \cdot \mathbf{r}_{12} = h_1 h_{12}, \; \mathbf{r}_1 \cdot \mathbf{r}_{13} = h_1 h_{13}, \; \mathbf{r}_1 \cdot \mathbf{r}_2 = 0$$

$$\rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_2)}{\partial x_1} = 0, \rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_{21} = 0, \rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_2 = -\mathbf{r}_1 \cdot \mathbf{r}_{12}, \rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_2 = -h_1 h_{12}$$

$$\mathbf{r}_1 \cdot \mathbf{r}_3 = 0, \rightarrow \frac{\partial(\mathbf{r}_1 \cdot \mathbf{r}_3)}{\partial x_1} = 0, \rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_3 + \mathbf{r}_1 \cdot \mathbf{r}_{31} = 0, \rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_3 = -\mathbf{r}_1 \cdot \mathbf{r}_{13}, \rightarrow \mathbf{r}_{11} \cdot \mathbf{r}_3 = -h_1 h_{13}$$

$$\mathbf{r}_{11} = h_{11}\hat{\mathbf{e}}_1 - \frac{h_1 h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_1 h_{13}}{h_3}\hat{\mathbf{e}}_3 = h_{11}\hat{\mathbf{e}}_1 + h_1\hat{\mathbf{e}}_{11} \rightarrow \hat{\mathbf{e}}_{11} = -\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2 - \frac{h_{13}}{h_3}\hat{\mathbf{e}}_3$$

Just few examples:

Curvilinearcoordinate(q_1, q_2, q_3)	**Transformation from Cartesian (x,y,z)**	**Scale factors**
Spherical polar coordinates $(r,\theta,\phi)\in[0,\infty)\times[0,\pi]\times[0,2\pi)$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$h_1 = 1$ $h_2 = r$ $h_3 = r\sin\theta$
Cylindrical polar coordinates $(r,\phi,z)\in[0,\infty)\times[0,\pi]\times[\infty,\infty)$	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$h_1 = h_3 = 1$ $h_2 = r$
Parabolic cylindrical coordinates $(u,\upsilon,z)\in[-\infty,\infty)\times[0,\infty]\times[-\infty,\infty)$	$x=\frac{1}{2}\left(u^2-\upsilon^2\right)$ $y = uv$ $z = z$	$h_1 = h_2 = \sqrt{u^2+\upsilon^2}$ $h_3 = 1$

In Fluid Dynamics (paper-2); we'll have following results. So here it's just given for that purpose.

Incompressible N-S equations in orthogonal curvilinear coordinate systems

Continuity equation $\nabla\,.\,\mathbf{V} = 0$

Since $\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3F_1)+\frac{\partial}{\partial x_2}(h_3h_1F_2)+\frac{\partial}{\partial x_3}(h_1h_2F_3)\right]$

and $\mathbf{V}=v_1\hat{\mathbf{e}}_1+v_2\hat{\mathbf{e}}_2+v_3\hat{\mathbf{e}}_3$

$$\nabla\cdot\mathbf{V}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]=0$$

Momentum equation $\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{V}$, (where p piezometric pressure)

Since $\mathbf{V}=v_1\hat{\mathbf{e}}_1+v_2\hat{\mathbf{e}}_2+v_3\hat{\mathbf{e}}_3$, we can expand the momentum equation term by term

Local derivative $\frac{\partial\mathbf{V}}{\partial t}=\frac{\partial v_1}{\partial t}\hat{\mathbf{e}}_1+\frac{\partial v_2}{\partial t}\hat{\mathbf{e}}_2+\frac{\partial v_3}{\partial t}\hat{\mathbf{e}}_3$

Convective derivative (**V**. ∇) **V**

Since $\mathbf{V}=v_1\hat{\mathbf{e}}_1+v_2\hat{\mathbf{e}}_2+v_3\hat{\mathbf{e}}_3$ and $\mathbf{V}\cdot\nabla=\frac{v_1}{h_1}\frac{\partial}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial}{\partial x_3}$

$$(\mathbf{V}\cdot\nabla)\mathbf{V}=(\mathbf{V}\cdot\nabla)\left(v_1\hat{\mathbf{e}}_1\right)+(\mathbf{V}\cdot\nabla)\left(v_2\hat{\mathbf{e}}_2\right)+(\mathbf{V}\cdot\nabla)\left(v_3\hat{\mathbf{e}}_3\right)$$

$$(\mathbf{V}\cdot\nabla)\left(v_1\hat{\mathbf{e}}_1\right) == \frac{v_1}{h_1}\frac{\partial\left(v_1\hat{\mathbf{e}}_1\right)}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial\left(v_1\hat{\mathbf{e}}_1\right)}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial\left(v_1\hat{\mathbf{e}}_1\right)}{\partial x_3}$$

$$=\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}\hat{\mathbf{e}}_1+\frac{v_1v_1}{h_1}\frac{\partial\hat{\mathbf{e}}_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}\hat{\mathbf{e}}_1+\frac{v_2v_1}{h_2}\frac{\partial\hat{\mathbf{e}}_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\hat{\mathbf{e}}_1+\frac{v_3v_1}{h_3}\frac{\partial\hat{\mathbf{e}}_1}{\partial x_3}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1+\left(\frac{v_1v_1}{h_1}\hat{\mathbf{e}}_{11}+\frac{v_2v_1}{h_2}\hat{\mathbf{e}}_{12}+\frac{v_3v_1}{h_3}\hat{\mathbf{e}}_{13}\right)$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1+\frac{v_1v_1}{h_1}\left(-\frac{h_{12}}{h_2}\hat{\mathbf{e}}_2-\frac{h_{13}}{h_3}\hat{\mathbf{e}}_3\right)+\frac{v_2v_1}{h_2}\left(\frac{h_{21}}{h_1}\hat{\mathbf{e}}_2\right)+\frac{v_3v_1}{h_3}\left(\frac{h_{31}}{h_1}\hat{\mathbf{e}}_3\right)$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}\right)\hat{\mathbf{e}}_1+\left(\frac{v_2v_1h_{21}}{h_1h_2}-\frac{v_1v_1h_{12}}{h_1h_2}\right)\hat{\mathbf{e}}_2+\left(\frac{v_3v_1h_{31}}{h_3h_1}-\frac{v_1v_1h_{13}}{h_3h_1}\right)\hat{\mathbf{e}}_3$$

$$(\mathbf{V}\cdot\nabla)\left(v_2\hat{\mathbf{e}}_2\right)=\frac{v_1}{h_1}\frac{\partial\left(v_2\hat{\mathbf{e}}_2\right)}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial\left(v_2\hat{\mathbf{e}}_2\right)}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial\left(v_2\hat{\mathbf{e}}_2\right)}{\partial x_3}$$

$$=\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}\hat{\mathbf{e}}_2+\frac{v_1v_2}{h_1}\frac{\partial\hat{\mathbf{e}}_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}\hat{\mathbf{e}}_2+\frac{v_2v_2}{h_2}\frac{\partial\hat{\mathbf{e}}_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\hat{\mathbf{e}}_2+\frac{v_3v_2}{h_3}\frac{\partial\hat{\mathbf{e}}_2}{\partial x_3}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2+\frac{v_1v_2}{h_1}\hat{\mathbf{e}}_{21}+\frac{v_2v_2}{h_2}\hat{\mathbf{e}}_{22}+\frac{v_3v_2}{h_3}\hat{\mathbf{e}}_{23}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2+\frac{v_1v_2}{h_1}\left(\frac{h_{12}}{h_2}\hat{\mathbf{e}}_1\right)+\frac{v_2v_2}{h_2}\left(-\frac{h_{21}}{h_1}\hat{\mathbf{e}}_1-\frac{h_{23}}{h_3}\hat{\mathbf{e}}_3\right)+\frac{v_3v_2}{h_3}\left(\frac{h_{32}}{h_2}\hat{\mathbf{e}}_3\right)$$

$$=\left(\frac{v_1v_2h_{12}}{h_1h_2}-\frac{v_2v_2h_{21}}{h_2h_1}\right)\hat{\mathbf{e}}_1+\left(\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}\right)\hat{\mathbf{e}}_2+\left(\frac{v_3v_2h_{32}}{h_2h_3}-\frac{v_2v_2h_{23}}{h_2h_3}\right)\hat{\mathbf{e}}_3$$

$$(\mathbf{V}\cdot\nabla)\left(v_3\hat{\mathbf{e}}_3\right) = \frac{v_1}{h_1}\frac{\partial\left(v_3\hat{\mathbf{e}}_3\right)}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial\left(v_3\hat{\mathbf{e}}_3\right)}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial\left(v_3\hat{\mathbf{e}}_3\right)}{\partial x_3}$$

$$=\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}\hat{\mathbf{e}}_3+\frac{v_1v_3}{h_1}\frac{\partial\hat{e}_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}\hat{e}_3+\frac{v_2v_3}{h_2}\frac{\partial\hat{e}_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\hat{\mathbf{e}}_3+\frac{v_3v_3}{h_3}\frac{\partial\hat{e}_3}{\partial x_3}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3+\frac{v_1v_3}{h_1}\hat{\mathbf{e}}_{31}+\frac{v_2v_3}{h_2}\hat{\mathbf{e}}_{32}+\frac{v_3v_3}{h_3}\hat{\mathbf{e}}_{33}$$

$$=\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3+\frac{v_1v_3}{h_1}\left(\frac{h_{13}}{h_3}\hat{\mathbf{e}}_1\right)+\frac{v_2v_3}{h_2}\left(\frac{h_{23}}{h_3}\hat{\mathbf{e}}_2\right)+\frac{v_3v_3}{h_3}\left(-\frac{h_{31}}{h_1}\hat{\mathbf{e}}_1-\frac{h_{32}}{h_2}\hat{\mathbf{e}}_2\right)$$

$$=\left(\frac{v_1v_3h_{13}}{h_1h_3}-\frac{v_3v_3h_{31}}{h_3h_1}\right)\hat{\mathbf{e}}_1+\left(\frac{v_2v_3h_{23}}{h_2h_3}-\frac{v_3v_3h_{32}}{h_3h_2}\right)\hat{\mathbf{e}}_2+\left(\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}\right)\hat{\mathbf{e}}_3$$

Pressure gradient $\nabla p = \frac{1}{h_1}\frac{\partial p}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial p}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial p}{\partial x_3}\hat{\mathbf{e}}_3$

Viscous term $\nabla^2\mathbf{V}= \frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_1$

$$-\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)\right]\hat{\mathbf{e}}_1$$

$$+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_2$$

$$-\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)\right]\hat{\mathbf{e}}_2$$

$$+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]\hat{\mathbf{e}}_3$$

$$-\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)\right]\hat{\mathbf{e}}_3$$

Combine terms in $\hat{e}_1$ direction to get momentum equation in $\hat{e}_1$ direction

$$\frac{\partial v_1}{\partial t}+\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_1}{\partial x_3}+\frac{v_1v_2h_{12}}{h_1h_2}-\frac{v_2v_2h_{21}}{h_2h_1}+\frac{v_1v_3h_{13}}{h_1h_3}-\frac{v_3v_3h_{31}}{h_3h_1}$$

$$= -\frac{1}{\rho}\frac{1}{h_1}\frac{\partial p}{\partial x_1}+v\frac{1}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]$$

$$-v\frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)-\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)\right]$$

Combine terms in $\hat{e}_2$ direction to get momentum equation in $\hat{e}_2$ direction

$$\frac{\partial v_2}{\partial t}+\frac{v_2v_1h_{21}}{h_1h_2}-\frac{v_1v_1h_{12}}{h_1h_2}+\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_2}{\partial x_3}+\frac{v_2v_3h_{23}}{h_2h_3}-\frac{v_3v_3h_{32}}{h_3h_2}$$

$$= -\frac{1}{\rho}\frac{1}{h_2}\frac{\partial p}{\partial x_2}+v\frac{1}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]$$

$$-v\frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)-\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2v_2)-\frac{\partial}{\partial x_2}(h_1v_1)\right]\right)\right]$$

Combine terms in $\hat{e}_3$ direction to get momentum equation in $\hat{e}_3$ direction

$$\frac{\partial v_3}{\partial t}+\frac{v_3v_1h_{31}}{h_3h_1}-\frac{v_1v_1h_{13}}{h_3h_1}+\frac{v_3v_2h_{32}}{h_2h_3}-\frac{v_2v_2h_{23}}{h_2h_3}+\frac{v_1}{h_1}\frac{\partial v_3}{\partial x_1}+\frac{v_2}{h_2}\frac{\partial v_3}{\partial x_2}+\frac{v_3}{h_3}\frac{\partial v_3}{\partial x_3}$$

$$= -\frac{1}{\rho}\frac{1}{h_3}\frac{\partial p}{\partial x_3} + v\frac{1}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1)+\frac{\partial}{\partial x_2}(h_3h_1v_2)+\frac{\partial}{\partial x_3}(h_1h_2v_3)\right]\right]$$

$$-v\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1v_1)-\frac{\partial}{\partial x_1}(h_3v_3)\right]\right)-\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3v_3)-\frac{\partial}{\partial x_3}(h_2v_2)\right]\right)\right]$$

PREVIOUS YEARS QUESTIONS

Q1. Derive expression of ∇f in terms of spherical coordinates.

Prove that $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$ for any two vector point functions $f(r,\theta,\phi)$ and $g(r,\theta,\phi)$. Construct one example in three dimensions to verify this identity. **[8a 2020 IFoS]**

Hint: Refer the article for this already discussed above in theory part. Take help from the cylindrical coordinates example below that article.

Q2. Derive $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$ in spherical coordinates. **[8c 2019 IFoS]**

Hint: Refer the article for this already discussed above in theory part. Take help from the cylindrical coordinates example below that article.

Q3. For what values of the constants a,b and c the vector $\bar{V} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$ is irrotational. Find the divergence in cylindrical coordinates of this vector with these values. **[5d UPSC CSE 2017]**

Hint: curl=0; get a,b,c. already solved in line integral example#3.

For other part- Refer the article for this already discussed above in theory part. Take help from the cylindrical coordinates example.