
Mindset Makers for UPSC & IIT
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WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFoS
01 Conceptual Development
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MINDSET MAKERS
Mindset Makers
Strengthening Brains for IAS & IFoS
Fluid Mechanics
Mathematics Optional
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UPSC Mathematics Optional

BRAIN STORMING: Fluid Dynamics UPSC CSE & IFoS

$P(x, y, z) : \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

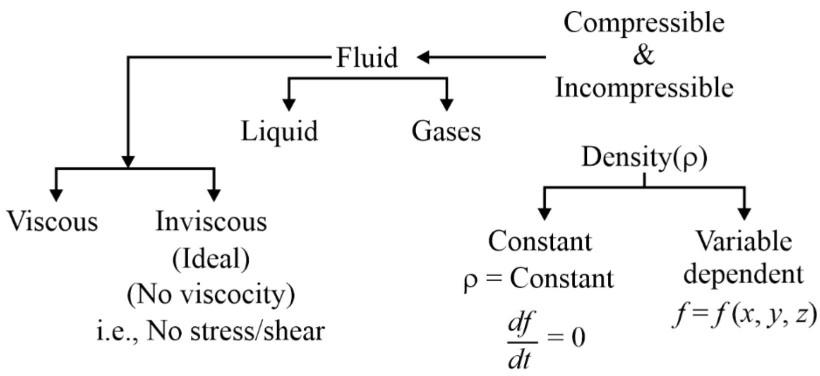
$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{i} + \frac{dz}{dt}\hat{k}$

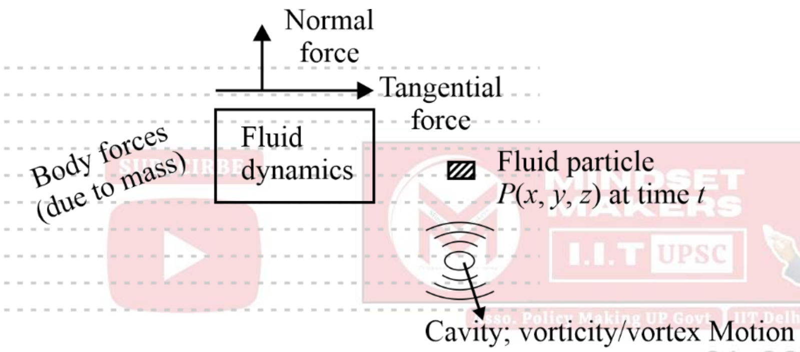
vel. Vector; $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

u; vel. component of $\vec{q}$ in x-axis

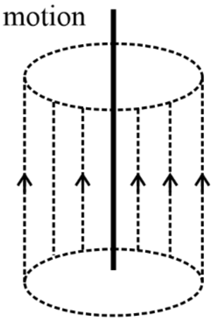
v; vel. Component of $\vec{q}$ in y-axis

w; vel. Component of $\vec{q}$ in z-axis

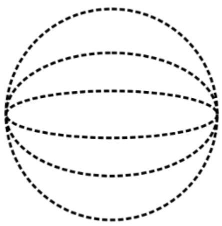




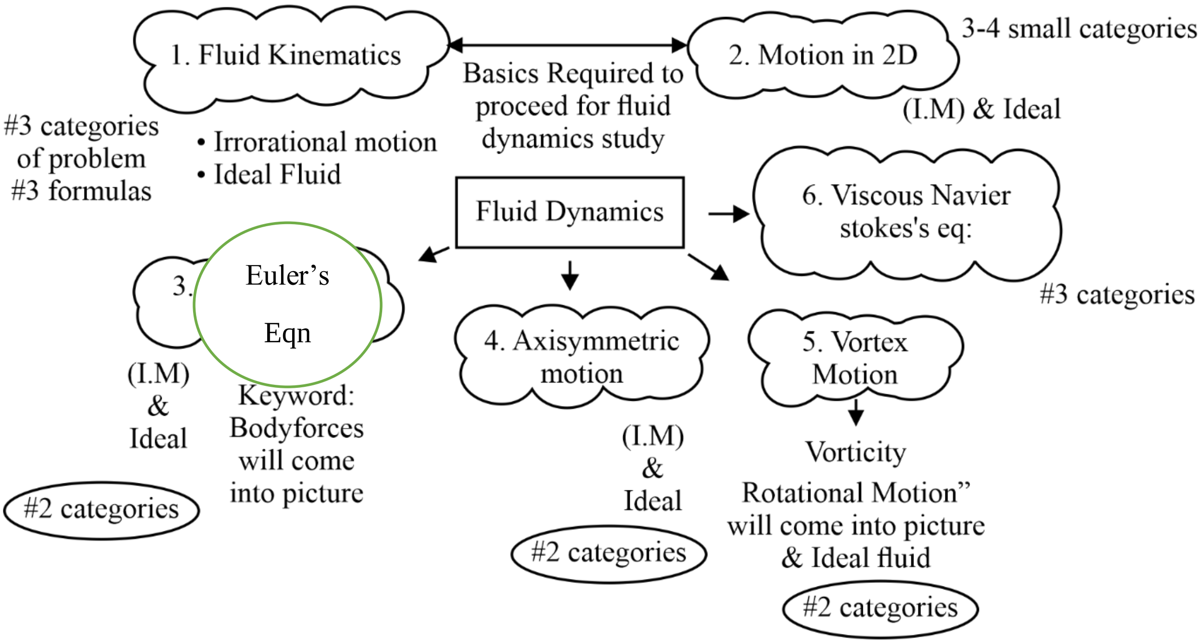
Axisymmetric
motion



Cylindrical
motion



Spherical motion



Basics form calculus required for fluid dynamics

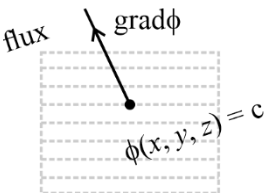
• Del operator ($\vec{\nabla}$) = $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

e.g. Applying $\vec{\nabla}$ on some scalar function $\phi(x, y, z)$ means:-

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

• Gradient of a scalar function $\phi(x, y, z)$;-



Mathematically

$$\boxed{\text{grad } \phi = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}}$$

• grad ϕ gives the direction; in which the change in ϕ; occurs most rapidly (greatest rate of increase)

(–grad ϕ) : gives the direction; in which ϕ decreases most rapidly.

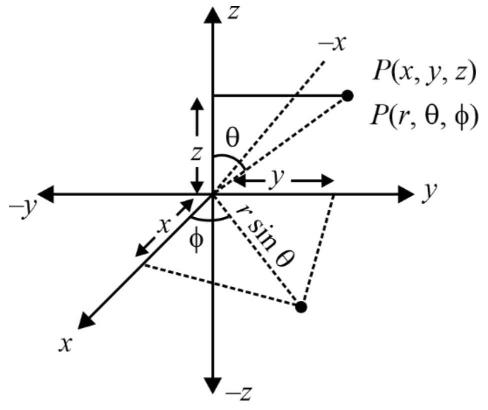
- **grad ϕ in spherical coordinates (r, θ, ϕ)**

$$\boxed{\vec{q} = q_r\hat{i} + q_\theta\hat{j} + q_\phi\hat{k}}$$

Will be required in fluid

So, let's say $f = f(r, \theta, \phi)$;

then



$z = r\cos\theta$

$x = (r\sin\theta)\cos\phi$

$y = (r\sin\theta)\sin\phi$

$$\boxed{\text{grad } f = \frac{\partial f}{\partial r}e_r + \frac{1}{r}\frac{\partial f}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}e_\phi}$$

where

$e_r = \sin\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + \cos\theta\,\hat{k}$

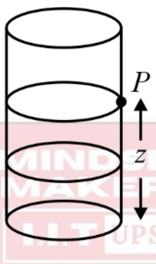
$e_\theta = \cos\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) - \sin\theta\,\hat{k}$

$e_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

- **Gradient in cylindrical coordinates (r, θ, z)**

Let $f = f(r, \theta, z)$

$$\boxed{grad\ f = \frac{\partial f}{\partial r}e_r + \frac{1}{r}\frac{\partial f}{\partial\theta}e_\theta + \frac{\partial f}{\partial z}e_z}$$



Where $e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$e_z = \hat{k}$

$x^2 + y^2 + r^2$

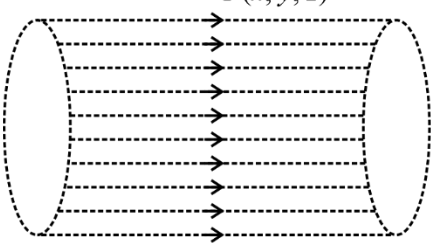
$(r\sin\theta, r\sin\theta, z)$

$\Uparrow$

cylindrical

- **Divergence of a vector field function :-**

$$q_r\hat{i} + q_\theta\hat{j} + q_\phi\hat{k} = \vec{q} = u\hat{i} + v\hat{j} + w\hat{k} = u(x, y, z)\,\hat{i} + v(x, y, z)\,\hat{j} + w(x, y, z)\,\hat{k}$$



Div. : loss in the fluid per unit volume per unit time

Let $\vec{F} = F_1(x, y, z)\,\hat{i} + F_2(x, y, z)\,\hat{j} + F_3(x, y, z)\,\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x}\left(\hat{i}\cdot\hat{i}\right) + \frac{\partial F_2}{\partial y}\left(\hat{j}\cdot\hat{j}\right) + \frac{\partial F_3}{\partial z}\left(\hat{k}\cdot\hat{k}\right)$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

$\vec{\nabla}\cdot\vec{F}$ is known as divergence of $\vec{F}$

Notice !! grad is of a scalar function but grad itself is a vector.

div. is of a vector function but div. itself is scalar

• **In spherical coordinates:-**

$$\vec{\nabla}\cdot\vec{q} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 q_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta q_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(q_\phi\right)$$

• **In cylindrical coordinates:-**

$$\vec{\nabla}\cdot\vec{q} = \frac{1}{r}\frac{\partial}{\partial r}\left(rq_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(q_\theta\right) + \frac{\partial}{\partial z}\left(q_z\right)$$

• **Curl of a vector field function:-**

As we know that $\vec{w} = \frac{1}{2}\text{curl } \vec{v}$; where $\vec{v}$ is linear velocity

$\vec{w}$ is angular velocity

i.e. curl is associated with the "Rotational" property in motion

$\vec{F} = F_1(x, y, z)\ \hat{i} + F_2(x, y, z)\ \hat{j} + F_3\ (x, y, z)\ \hat{k}$

$$\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

Note:- We need mainly in cartesian form only **(**UPCS CSE / IFoS**)**

• $\vec{\nabla}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$

• Spherical coordinates

$$\vec{\nabla}\times\vec{q} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ q_r & rq_\theta & r\sin\theta q_\phi \end{vmatrix}$$

• Cylindrical coordinates

$$\vec{\nabla}\times\vec{q}=\frac{1}{r}\begin{vmatrix}\vec{e}_r & r\vec{e}_\theta & \vec{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z}\\ q_r & q_\theta & q_z\end{vmatrix}$$

• Laplacean operator (∇^2); $\vec{\nabla}\cdot\vec{\nabla}$

Let's consider a scalar $\phi(x, y, z)$, then

$$\nabla^2\phi=\left(\vec{\nabla}\cdot\vec{\nabla}\right)\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi$$

• Cartesian form $\boxed{\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2 d}{\partial z^2}}$

• Spherical co-ordinate system:

$$\boxed{\nabla^2\psi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}}$$

• Cylindrical co-ordinate system

$$\boxed{\nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{d\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}}$$

• Some important observations :-

(i) div (curl $\vec{F}$) = 0 i.e., $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right)=0$

(ii) curl (grad ϕ) = $\vec{0}$ i.e., $\vec{\nabla}\times\left(\vec{\nabla}\phi\right)=\vec{0}$

(iii) A vector field $\vec{F}$ is said to be "solenoidal" if divergence of $\vec{F}$ is always (during motion) is zero

i.e., $\vec{\nabla}\cdot\vec{F}=0$

i.e., there is no loss in the fluid per unit time per unit volume,

(iv) A vector field $\vec{F}$ = $\vec{\nabla}\times\vec{F}=\vec{0}$

i.e., there is no rotation in $\vec{F}$ during the motion

Interpretation $\leftarrow$ $\left\{\begin{array}{l}\text{then there exists a scalar}\\ \quad\text{function } \phi \; st\\ \quad\quad \vec{F}=\vec{\nabla}\phi\\ \text{or } \vec{F}=-\vec{\nabla}\phi\begin{pmatrix}\text{we will take}\\ \text{in fluid dynamics}\end{pmatrix}\end{array}\right.$

$\downarrow$

If F irrorational; then there must exists a function ϕ such that $\vec{F}$ is of the form grad ϕ.

If, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$; then $\vec{F} = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

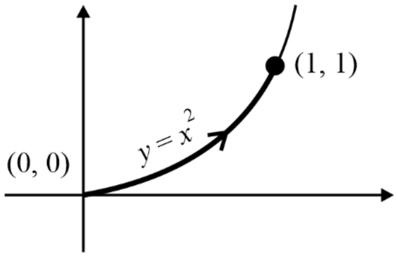
i.e.; $F_1 = \frac{\partial\phi}{\partial x}, F_2 = \frac{\partial\theta}{\partial y}, F_3 = \frac{\partial\phi}{\partial z}$

• **Vector Integration**

• Line integral (1D)

$\int_c \vec{F} \cdot d\vec{r}$; where c is the curve over which the line integral $\int \vec{F} \cdot d\vec{r}$ is being calculated

e.g.



C is a curve: parabola $y = x^2$ form (0, 0) to (1, 1)

$\therefore \int_c \vec{F} \cdot d\vec{r} = \int_c \left(F_1\hat{i} + F_2\hat{j}\right) \cdot \left(dx\hat{i} + dy\hat{j}\right)$

$\because \vec{F} = F_1\hat{i} + F_2\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

C: $y = x^2$, from (0, 0) to (1, 1)

: $dy = 2xdx$

$\int_c \vec{F} \cdot d\vec{r} = \int_{x=0}^{1} F_1\left(x, x^2\right)dx + F_2\left(x, x^2\right) \cdot 2xdx$

Or

$\int_c \vec{F} \cdot d\vec{r} = \int_{y=0}^{1} F_1\left(\sqrt{y}, y\right)\frac{1}{2\sqrt{y}}dy + F_2\left(\sqrt{y}, y\right)dy$

e.g. Line integral in 3D

Let $\vec{F} = F_1(x, y, z)\,\hat{i} + F_2(x, y, z)\,\hat{j} + F_3(x, y, z)\,\hat{k}$

& *path* is $x = \phi(t), y = \psi(t), z = g(t)$;

Where t is a parameter running; $t = a$ to b.

then the line integral

$\int_c \vec{F} \cdot \overline{d\vec{r}} = \int_c \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

$= \int_{t=a}^{b} F_1(t)\phi'(t)dt + F_2(t)\psi'(t)dt + F_3(t)g'(t)dt$

Surface Integral:-

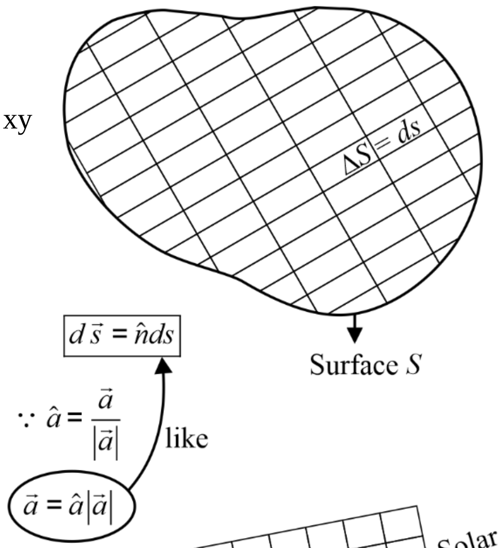
$$\boxed{d\vec{s} = ds\cos\alpha\hat{i} + ds\cos\beta\hat{j} + ds\cos\gamma\hat{k}}$$

Where ds cos α, ds cos β, ds cos γ

Are orthogonal projection of $\overrightarrow{ds}$ on the yz plane, xz plane & xy plane respectively:

So, ds cos α = *dy dz*, ds cos β = *dz dx*

ds cos γ = *dx dy*



Projection of above plate on some plane.

$$\boxed{dS = dydz\hat{i} + dzdx\hat{j} + dxdy\hat{k}}$$

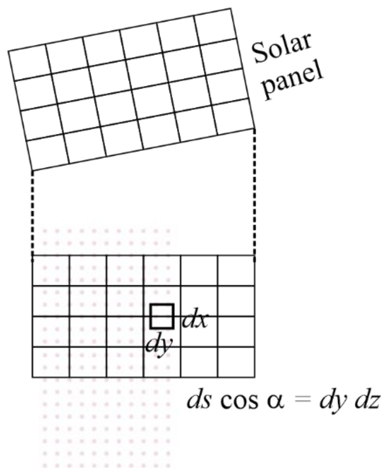
↓

$$\hat{n}dS = dydz\hat{i} + dzdx\hat{j} + dxdy\hat{k}$$

$$\Rightarrow \hat{i}\cdot\hat{n}ds = dydz + 0 + 0 \Rightarrow \boxed{ds = \frac{dydz}{|\hat{i}\cdot\hat{n}|}}$$

$$\hat{j}\cdot\hat{n}ds = 0 + dzdx + 0 \Rightarrow \boxed{ds = \frac{dzdx}{|\hat{j}\cdot\hat{n}|}}$$

$$\hat{k}\cdot\hat{n}ds = dxdy \Rightarrow \boxed{ds = \frac{dxdy}{|\hat{k}\cdot\hat{n}|}}$$



$$\int_s \vec{F}\cdot d\vec{s} = \int_s \vec{F}\cdot\hat{n}ds$$

$$\iint_{x\,y} \vec{F}\cdot\hat{n}\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|} \quad \text{or,} \quad \iint_{z\,x} \vec{F}\cdot\hat{n}\frac{dz\,dx}{|\hat{j},\hat{n}|} \quad \text{or,} \quad \iint \vec{F}\cdot\hat{n}\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

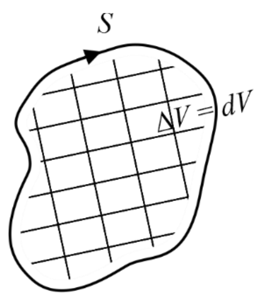
• **Volume integral**

$$\int_v \vec{F}dV = \iiint_{x\,y\,z}\left\{F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\right\}dx\,dy\,dz$$

Or If *F* is a scalar f :

$$\int_v Fdv = \iiint_{x\,y\,z} F(x,y,z)dx\,dy\,dz$$

V is the volume enclosed by the surface *S*

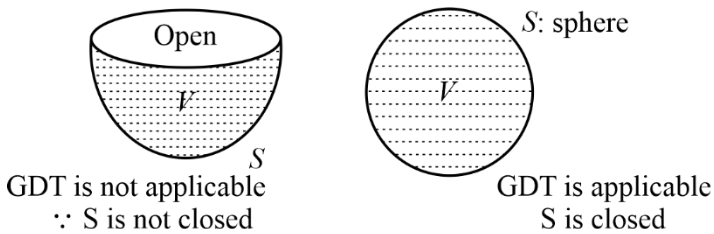


• some important theorems: defining relation between line, surface, volume integrals : -

(i) **Gauss' Divergence theorem(GDT):-**

Gives relation between volume & surface integral

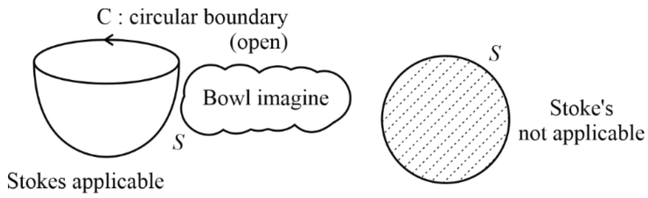
$\int_s \vec{F}\cdot\hat{n}ds = \int_v \text{div}\,\vec{F}\,dV$; when V is the volume (region) enclosed by a "closed" surface



(ii) **Stoke's Theorem:**

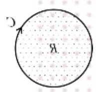
Gives relation between line integral & surface integral

$\int_c \vec{F}\cdot d\vec{r} = \int_s \text{curl}\,\vec{F}\cdot\hat{n}ds$; where c is a closed boundary enclosing surface S



(iii) **Green's theorem in the plane (xy-plane)**

Let R be the region enclosed by closed curve c; then



$$\int_c Pdx + Qdy = \int_x\int_y \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Fluid Kinematics

Brainstorming

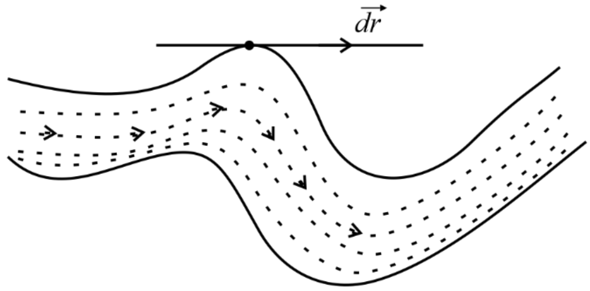
$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$; velocity vector $\vec{q}$

TYPE-(I): PROBLEMS : "STREAMLINES"

These are imaginary lines which are along the motion (in the direction of velocity)

Mathematically,

$\vec{q} \times d\vec{r} = 0$ ($\because$ $\vec{q}$ is parallel to $\overrightarrow{dr}$)



$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$$

$\Rightarrow (vdz - wdy)\ \hat{i}\ + (wdx - udz) + (udy\text{-}\ vdx)$

$= 0\,\hat{i}\ + 0\,\hat{j}\ + 0\,\hat{k}$

$\Rightarrow\ vdz - wdy = 0\ ,\ wdx - udz = 0,\ udy - vdx = 0$

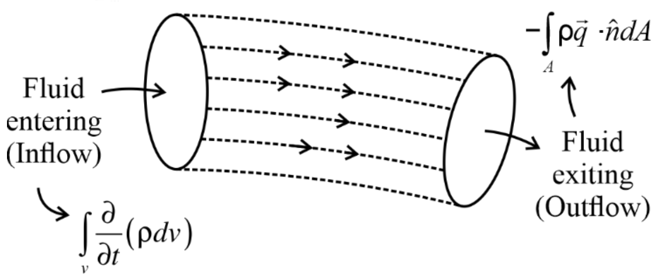
Exam point

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}}$$

Gives equations of streamlines.

Type II: Equation of continuity (conservation of mass)

Flux:- "*Rate of flow of mass*"



(i) $\frac{d}{dt}(m) = \frac{d}{dt}(\text{volume} \times \text{density})$

$= \frac{d}{dt}(A \times x \times \rho)$

$= \rho \cdot A\frac{dx}{dt}$; As motion is along one axis i.e. x-axis

$= \rho\ .\ A.\ u$, if ρ is constant

(ii) In general; flux is written as

$\rho\vec{q}\cdot d\vec{A} = \rho\vec{q}\cdot\hat{n}dA$

where $\hat{n}$ is unit outward normal vector to the surface area A

Speed = $\frac{\text{distance}}{\text{time}}$

for per unit time speed = distance length

Mathematically; for equation of continuity (or conservation of mass)

We have

$$\frac{d}{dt}\int_v \rho dV = -\int_A \rho \cdot \vec{q} \cdot \hat{n} dA$$

$$\Rightarrow \frac{d}{dt}\int_v \rho dV = -\int_A \operatorname{div}(\rho\vec{q}) dV \text{ ; applying GDT}$$

$$\Rightarrow \frac{d}{dt}\int_v \rho dV = -\int_V \vec{\nabla}(\rho\vec{q}) dV$$

$$\Rightarrow \int_V \left\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q})\right\} dV = 0$$

$$\Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q}) = 0} \quad \ldots(1) \text{ called eq. of continuity}$$

Supporting stuff from calculus:-

$$\vec{\nabla}\cdot(\rho\vec{q}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\rho u\hat{i} + \rho v\hat{j} + \rho w\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

$$= \left(\rho\frac{\partial u}{\partial x} + \frac{\partial\rho}{\partial x}u\right) + \left(\rho\cdot\frac{\partial v}{\partial y} + \frac{\partial\rho}{\partial y} + v\right) + \left(\rho\frac{\partial w}{\partial z} + \frac{\partial\rho}{\partial z}w\right)$$

$$= \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \left(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right)$$

$$= \rho\left\{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(u\hat{i} + v\hat{j} + w\hat{k}\right)\right\} + \left\{\left(u\hat{i} + v\hat{j} + w\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\rho\right\}$$

$$\boxed{\vec{\nabla}\cdot(\rho\vec{q}) = \rho\left(\vec{\nabla}\cdot\vec{q}\right) + \vec{q}\cdot\left(\vec{\nabla}\rho\right)} \quad \ldots(2)$$

Point to be noted:

$f = f(x, y, z, t)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt$$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial f}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial f}{\partial z}\left(\frac{dz}{dt}\right) + \frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{df}{dt} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}$$

$$\boxed{\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}}$$

$$\frac{d}{dt} = \left\{\left(u\hat{i} + v\hat{j} + w\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right\} + \frac{\partial}{\partial t}$$

** $\boxed{\frac{d}{dt} = \left(\vec{q}\cdot\vec{\nabla}\right) + \frac{\partial}{\partial t}}$ **Exam point**…(3)

$\because\ \rho = \rho\ (x, y, z, t)$

E.g. $\frac{d}{dt}(\rho) = \left(\vec{q}\cdot\vec{\nabla}\right)\rho + \frac{\partial\rho}{\partial t}$

$\Rightarrow\ \boxed{\frac{d}{dt} = \rho\left(\vec{q}\cdot\vec{\nabla}\right) + \frac{\partial\rho}{\partial t}}$

$\Rightarrow \frac{\partial\rho}{\partial t} = \frac{d\rho}{dt} - \left(\vec{q}\cdot\vec{\nabla}\right)\rho\ \ \ldots(4)$

Using (4) & (2) in (1)

$$\left(\rho\left(\vec{\nabla}\cdot\vec{q}\right) + \left(\vec{q}\cdot\vec{\nabla}\right)\rho\right) + \left(\frac{d\rho}{dt} - \left(\vec{q}\cdot\vec{\nabla}\right)\rho\right) = 0$$

$$\rho\left(\vec{\nabla}\cdot\vec{q}\right) + \frac{d\rho}{dt} = 0$$

$\boxed{\left(\vec{\nabla}\cdot\vec{q}\right) + \frac{1}{\rho}\frac{d\rho}{dt} = 0}$; Also an expression for equation of continuity

$\boxed{\left(\vec{\nabla}\cdot\vec{q}\right) + \frac{d}{dt}(\log\rho) = 0}$

Note:- If fluid is incompressible i.e., ρ = constant $\Rightarrow \frac{d\rho}{dt} = 0$

Then equation of continuity is $(\vec{\nabla}\cdot\vec{q}) = 0$

Exampoint:- For an incompressible fluid ; (steady flow)

Motion is possible only when; eq. of continuity holds i.e., $\vec{\nabla}\cdot\vec{q}$ = 0

Category 3:- Irrotational motion; finding velocity/scalar potential ϕ

Curl $\vec{q}$ = 0

$\vec{\nabla}\times\vec{q} = 0$

i.e., there exists a scalar function ϕ s.t

$\boxed{\vec{q} = -\vec{\nabla}\phi}$ here ϕ is called ***velocity potential***

$\Downarrow$

$$u\hat{i} + v\hat{j} + w\hat{k} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{\partial\phi}{\partial z}\hat{k}$$

On comparing we get

$$\frac{\partial\phi}{\partial x} = -u\ \ \ldots(1),\ \frac{\partial\phi}{\partial y} = -v\ \ldots(2),\ \frac{\partial\phi}{\partial z} = -w\ \ \ldots(3)$$

(3) is giving differential eq. and by solving these, we get required ϕ

Exampoints:-

1. $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{q} \cdot \vec{\nabla})$

2. flux = $-\int_A f_q \cdot d\vec{A} = -\int_A f_q \cdot \hat{n} dA$

3. type I

Streamlines are given by: $\frac{dx}{u} = \frac{dv}{v} = \frac{dz}{w}$

4. type II

For possible fluid motion (Incompressible fluid/ steady flow) eq. of continuity $\vec{\nabla} \cdot \vec{q} = 0$; must hold.

5. type III

Finding velocity potential :

$\vec{\nabla} \times \vec{q} = 0 \therefore \vec{q} = -\vec{\nabla}\phi$

$\Rightarrow u\hat{i} + v\hat{j} + w\hat{k} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{\partial\phi}{\partial z}\hat{k}$

$\therefore \boxed{\frac{\partial\phi}{\partial x} = -u, \frac{\partial\phi}{\partial y} = -v, \frac{\partial\phi}{\partial z} = -w}$

EXAMPLES TO SUBSTANTIATE

Example 1. Find the eq. of streamlines for the flow

$\vec{q} = -\hat{i}(3y^2) - \hat{j}(6x)$ at the point (1, 1)

Solutions: We know that, for streamlines

$\vec{q} \times d\vec{r} = \vec{0}$

$\therefore$ eq- of streamlines are given by,

$\frac{dx}{u} = \frac{dy}{v}$; where $\vec{q} = u\hat{i} + v\hat{j}$

Given $u = -3y^2$, $v = -6x$

$\because$ eq. of streamlines are given by

$\frac{dx}{-3y^2} = \frac{dy}{-6x}$

$6xdx = 3y^2 dy$

On integrating

$3x^2 = y^3 + c$; c is integration constant

At (1, 1) we have,

$3 - 1 = c \quad \Rightarrow c = 2$

$\therefore$ Required streamlines are given by

$3x^2 + y^3 = 2$

Example 2. The velocity components of a two dimensional flow fluid for an incompressible fluid are given by $u = e^x \cosh y$ & $v = -e^{-x} \sinh y$. Determine the eq. of streamline for flow.

Solutions: we know, eq. of streamlines is given by

$$\frac{dx}{u} = \frac{dv}{v} \ \{\because \ \vec{q} \times \overrightarrow{dr} = 0\}; \text{ where } \vec{q} = u\hat{i} + v\hat{j}$$

Given, $u = e^x \cos hy$, $v = -e^{-x} \sin hy$

$$\therefore \ \frac{dx}{e^x \cosh y} = \frac{dy}{-e^{-x} \sinh y}$$

$$e^{-2x} dx = -\frac{\cosh y}{\sinh y} dy$$

On integrating,

$$\frac{e^{-2x}}{-2} = -\log(\sin hy) + \log c, \text{ where c is integration constant.}$$

$$e^{-2x} = 2\log(\sin hy) - 2\log c$$

$$\boxed{e^{-2x} = \log\left(\frac{\sinh^2 y}{c^2}\right)}$$

Example 3: Show that $cy = e^{\frac{-1}{x}}$ are surfaces which are orthogonal to streamlines for an incompressible homogenous fluid at the point (x, y, z) with the velocity distribution given by

$$u = \frac{-c^2 y}{r^2}, v = \frac{c^2 x^2}{r^2}, \text{ w} = 0 \text{, where}$$

r denotes the distance of (x, y, z) from z – axis

Step (1):- Finding streamlines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dx}{w}$$

$$\frac{dx}{\frac{-c^2 y}{r^2}} = \frac{dy}{\frac{c^2 x^2}{r^2}} = \frac{dx}{0}$$

$$\Rightarrow \ \frac{dx}{-y} = \frac{dy}{x^2}$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{-x^2}{y} \quad \ldots(1) \text{ is the diff. eq. of streamline}$$

Now, replacing $\frac{dy}{dx}$ by $\frac{-1}{\left(\frac{dy}{dx}\right)}$ in (1)

$$\frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-x^2}{y}$$

$$\frac{dx}{dy} = \frac{x^2}{y}; \ \frac{dx}{x^2} = \frac{dy}{y} \Rightarrow \frac{-1}{x} = \log y + \log c \ \Rightarrow \frac{-1}{x} = \log(yc)$$

$$\Rightarrow \boxed{e^{\frac{-1}{x}} = cy}$$

Revising from ODE; Orthogonal trajectories

Curves which behaves according to some predefined condition/rule

Let $f_1(x, y) = c_1$ …(1) is

Some given family of curve; then orthogonal trajectories to (1); is the family of curves which cuts every member of (1) at an angle of 90°

Geometrically speaking:-

• Given family f_1: $y = mx$; family of straight lines (∵ m is a parameter/arbitrary constant)

$y = x, y = -x, y = \frac{3}{2}x, \ldots.$

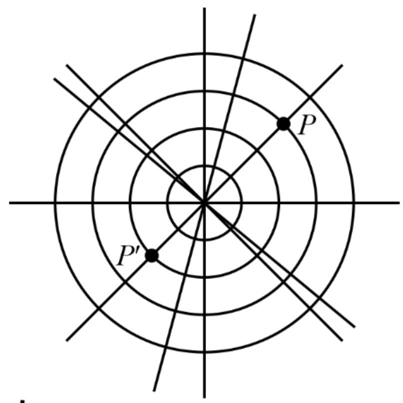
f_2; Family of concentric circles is an orthogonal trajectories of family of straight lines

At any point of intersection, we can crosscheck;

$$\left(\frac{dy}{dx}\right)_{f_1} \times \left(\frac{dy}{dx}\right)_{f_2} = -1$$

$\Rightarrow m_1 \times m_2 = -1$

$\Rightarrow f_1$ & f_2 cuts orthogonally



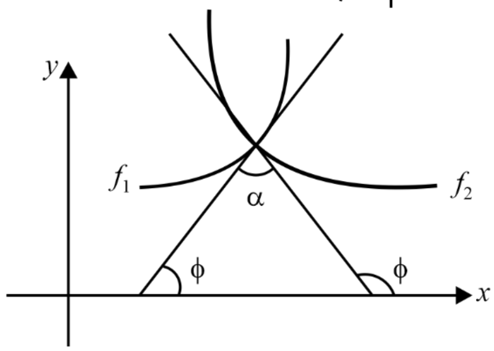
• Determining/ finding any oblique trajectories for some given family of curves:-

∵ $\alpha + \phi = \psi$; in the figure external angle is ψ

$\alpha = \psi - \phi$

$\tan \alpha = \tan(\psi - \phi)$

$$\tan \alpha = \frac{\tan\psi - \tan\phi}{1 + \tan\psi \tan\phi}$$



e.g. To get orthogonal trajectory $\alpha = 90°$

$$\therefore \tan 90° = \frac{\tan\psi - \tan\phi}{1 + \tan\psi \tan\phi}$$

$$\frac{1}{0} = \frac{\tan\psi - \tan\phi}{1 + \tan\psi \tan\phi}$$

$\tan\phi \tan\psi = -1$

$$\left(\frac{dy}{dx}\right)_{f_1} \times \left(\frac{dx}{dy}\right)_{f_2} = -1$$

$$\left(\frac{dy}{dx}\right)_{f_2} = \frac{-1}{\left(\frac{dy}{dx}\right)_{f_1}} \quad \ldots(1)$$

1. indicates that If we replace $\frac{dy}{dx}$ by $\frac{-1}{\left(\frac{dy}{dx}\right)}$ in the differential eq. of given family then we get diff. eq. for required family.

Example   Let $y = mx$ …(1) is given family

$$\frac{dy}{dx} = m$$

$\therefore$ diff eq. (1) is, $y = \frac{dy}{dx} \cdot x \Rightarrow \frac{dy}{dx} = \frac{y}{x}$ …(2) is the diff. eq. of given family.

Now, replace $\frac{dy}{dx}$ by $\frac{-1}{\left(\frac{dy}{dx}\right)}$ in (2), we get

$$\frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{y}{x}$$

$$\frac{dx}{dy} = \frac{-y}{x}$$

$$xdx - y\,dy$$

$$\boxed{\frac{x^2}{2} + \frac{y^2}{2} = c} \Rightarrow \text{family of concentric circles}$$

Example 4 : Determine the streamlines and path lines of the particle when the components of the velocity field are given by

$$u = \frac{x}{1+t},\ v = \frac{y}{2+t},\ w = \frac{z}{3+t}$$

streamlines are given by $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$$\frac{dx}{\frac{x}{(1+t)}} = \frac{dy}{\frac{y}{(2+t)}} = \frac{dz}{\frac{z}{(3+t}}$$

$$(1+t)\frac{dx}{x} = (2+t)\frac{dy}{y} = (3+t)\frac{dz}{z} \quad \ldots(1)$$

Taking first two fractious,

$$(1 + t)\,\frac{dx}{x} = (2 + t)\,\frac{dy}{y}$$

$$\left(\frac{1}{x}dx - \frac{2}{y}dy\right) = t\left(\frac{dy}{y} - \frac{1}{x}dx\right)$$

On integrating

$$\log_e x - 2\log_e y = t\{\log_e y - \log_e x\} + \log c_1$$

$$\log_e\left(\frac{x}{y^2}\right) = \log_e\left(\frac{y}{x}\right)^t + \log c_1$$

$$\frac{x}{y^2} = c_1\left(\frac{y}{x}\right)^t \ \ldots(2)$$

Path lines are given by:

$$\frac{dx}{dt} = u \Rightarrow \ x \ldots(1)$$

$$\frac{dy}{dt} = v \Rightarrow y \ldots(2)$$

$$\frac{dz}{dt} = w \Rightarrow z \ldots(3)$$

(1), (2), (3); gives req. path lines

Taking last two fractious of eq. (1).

$$(2 + t)\,\frac{dy}{y} = (3 + t)\,\frac{dz}{z}$$

$$\left(\frac{2}{y}dy - \frac{3}{2}dz\right) = t\left(\frac{dz}{z} - \frac{dy}{y}\right)$$

On integrating,

$2 \log_e y - 3 \log_e z = t\,(\log_e z - \log_e y) + \log c_2$

$$\log_e\left(\frac{y^2}{z^3}\right) = \log_e\left(\frac{z}{y}\right)^t + \log c_2$$

$$\frac{y^2}{z^3} = \left(\frac{z}{y}\right)^t c_2 \ \ldots(2)$$

Eq. (1) & (2) are required streamlines:

Path lines:-

$$\frac{dx}{dt} = \mathrm{u} \Rightarrow \frac{dx}{dt} = \frac{x}{1+t}$$

$$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}$$

$\log x = \log (1 + t) + \log c$

$x = (1 + t)\, c_3 \ldots(A)$

Similarly , $y = c_4\,(2 + t)$ …(B)

$z = c_5\,(3 + t)$ …(C)

$\therefore$ (A), (B), (C) are required path lines.

Example 5: Show that the velocity potential $\phi = \frac{a}{2}(x^2 + y^2 - 2z^2)$ satisfies the laplace eq. Also, determine streamlines.

Solutions:

$\because\ \phi = \frac{a}{2}(x^2 + y^2 - 2z^2)$

$$\frac{\partial\phi}{\partial x} = ax,\ \frac{\partial\phi}{\partial y} = ay,\ \frac{\partial\phi}{\partial z} = -2az$$

$$\frac{\partial^2\phi}{\partial x} = a,\ \frac{\partial^2\phi}{\partial y^2} = a,\ \frac{\partial^2\phi}{\partial z^2} = -2a$$

$$\therefore\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = a + a - 2a = 0$$

$\Rightarrow \phi$ satisfies $\nabla^2\phi = 0 \Rightarrow \phi$ satisfies Laplace's eq

$\because\ \phi$ is given $\Rightarrow$ velocity potential exists $\Rightarrow\ \vec{q} = -\vec{\nabla}\phi$

$$\Rightarrow u\hat{i} + v\hat{j} + w\hat{k} = \frac{-\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{a\phi}{\partial z}\hat{k}$$

$\Rightarrow u = -ax,\ v = -ay,\ w = 2az$

$\therefore$ streamlines are given by,

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \quad \ldots(1)$$

Taking first two fractions of (1),

$$\frac{dx}{x} = \frac{dy}{y}$$

$\log x = \log y + \log e_1 \Rightarrow x = y\,c_1$ …(A)

Taking last two fraction of (1)

$$\frac{dy}{-ay} = \frac{dz}{2az}$$

$$\frac{dy}{y} = \frac{-dz}{2z}$$

$\log y = \frac{-1}{2}\log z = \log c_2$

$\log y = \log \dfrac{c_2}{\sqrt{z}}$

$y = c_2 \; z^{\frac{-1}{z}}$ …(B)

$\therefore$ Eq. (A) & (B) gives the required streamlines.

Example 6 : $\vec{q} = \left(\dfrac{3x^2}{r^5}, \dfrac{3yz}{r^5}, \dfrac{kz^2 - r^2}{r^5}\right)$. For an incompressible fluid. Find the value of k for which it constitutes a possible fluid motion. Also find the scalar potential ϕ.

Exampoint:-

$r^2 = x^2 + y^2 + z^2$ …(1)

$2r\dfrac{\partial}{\partial x} = 2x \Rightarrow \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

We always have these in back of the mind; whatever needed we use accordingly

we know that for an incompressible find, eq. of motion is possible where

$\vec{\nabla} \cdot \vec{q} = 0$; i.e., $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ …(1)

$\because$ given $\vec{q} = \left(\dfrac{3xz}{r^5}, \dfrac{3yz}{r^5}, \dfrac{kz^2 - r^2}{r^2}\right)$

$\Rightarrow u = \dfrac{3xz}{r^5}$, $v = \dfrac{3yz}{r^5}$, $w = \dfrac{kz^2 - r^2}{r^5}$

$\therefore \dfrac{\partial u}{\partial x} = 3z\dfrac{\partial}{\partial x}\left(\dfrac{x}{r^5}\right)$

$= 3z\left\{x \cdot \dfrac{-5}{r^c} \times \dfrac{\partial r}{\partial x} + \dfrac{1}{r^5} \times 1\right\} = 3z\left\{x \cdot \dfrac{-5}{r^6} \times \dfrac{x}{r} + \dfrac{1}{r^5}\right\} = 3z\left\{\dfrac{-5x^2}{x^7} + \dfrac{1}{x^5}\right\}$ …(A)

Similarly, $\dfrac{\partial v}{\partial y} = 3y\dfrac{\partial}{\partial y}\left(\dfrac{y}{r^5}\right)$

$\dfrac{\partial v}{\partial y} = 3z\left\{\dfrac{-5y^2}{r^7} + \dfrac{1}{r^5}\right\}$ …(B)

$\dfrac{\partial w}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{Kz^2 - r^2}{r^5}\right) = \left[\left(Kz^2 - r^2\right) \cdot \dfrac{-5}{r^6} \times \dfrac{z}{r} + \dfrac{1}{r^5} \times \left\{K.2z - 2r \cdot \dfrac{dr}{dz}\right\}\right]$

$= \dfrac{-5\left(Kz^2 - r^2\right)}{r^7} + \dfrac{1}{r^5}\left\{2Kz - 2r \times \dfrac{z}{r}\right\}$

$$\frac{\partial w}{\partial z} = \frac{-5z\left(Kz^2 - r^2\right)}{r^7} + \frac{2z}{r^5}(K-1)$$

$$\frac{\partial w}{\partial z} = K\left\{\frac{-5z^3}{r^7} + \frac{2z}{r^5}\right\} + \frac{3z}{r^5} \quad \ldots(C)$$

Using (A), (B) & (C) in (1),

$$3z\left\{\frac{-5x^2}{r^7} + \frac{1}{r^5}\right\} + 3z\left\{\frac{-5y^2}{r^7} + \frac{1}{r^5}\right\} + K\left\{\frac{-5z^3}{r^7} + \frac{2z}{r^5}\right\} + \frac{3z}{r^5} = 0$$

$$\frac{1}{r^5}\{3z + 3z + 2Kz + 3z\} = \frac{5z}{r^7}\{3x^2 + 3y^2 + Kz^2\}$$

$$9 + 2K = \frac{5z}{r^2}\{3x^2 + 3y^2 + Kz^2\}$$

Gives possible choice for $K = 3$,

LHS = 9 + 6 = 15

$$\text{RHS} = \frac{5}{r^2}\left(3r^2\right) = 15$$

$\therefore$ Req. value for $K = 3$.

Now, for finding velocity potential ϕ taking

$$\vec{q} = -\nabla\phi$$

$$\Rightarrow \; u\hat{i} + v\hat{j} + w\hat{k} = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{\partial\phi}{\partial z}\hat{k}$$

$$u = \frac{-\partial\phi}{\partial x}, \; v = \frac{-\partial\phi}{\partial y}, \; w = \frac{-\partial\phi}{\partial z}$$

$$\Rightarrow \frac{3xz}{r^5} = \frac{-\partial\phi}{\partial x} \ldots(A), \; \frac{3yz}{r^5} = \frac{-\partial\phi}{\partial y} \ldots(B), \; \frac{3z^2 - r^2}{r^5} = \frac{-\partial\phi}{\partial z} \ldots(C)$$

From (A), (B) & (C),

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -\left\{\frac{3xzdx + 3yzdy + \left(3z^2 - r^2\right)dz}{r^5}\right\}$$

$$d\phi = \frac{-3z(xdx + ydy + zdz) + r^2 dz}{r^5}$$

$$= \frac{-3zrdr + r^2 dz}{r^5} = \frac{r^3 dz - z3r^2 dr}{\left(r^3\right)^2}$$

$$d\phi = d\left(\frac{z}{r^3}\right)$$

On integrating ; $\boxed{\phi = \frac{z}{r^3}}$

Example 7 : Consider a two dimensional incompressible steady flow field with velocity component in spherical coordinates (r, θ, ϕ) are given by $v_r = e_r\left(1 - \frac{3}{2}\frac{r_0}{r} + \frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta$

$v_\theta = 0$, $v_\phi = -C_1\left(1 - \frac{3r_0}{4r} - \frac{1}{4}\frac{r_0^3}{r^3}\right)$

where $r \geq r_0 > 0$ and where C_1 & r_0 are arbitrary constants.

Is the eq. of continuity satisfied?

Solution: $\because$ fluid is incompressible $\therefore$ $\frac{d\rho}{dt} = 0$

$\therefore$ for possible fluid motion, eq. of continuity is $\vec{\nabla}\cdot\vec{q} = 0$, where $\vec{q} = v_r\hat{i} + v_\theta\hat{j} + v_\phi\hat{k}$

$\because$ In spherical coordinates,

$\rho\left(\vec{\nabla}\cdot\vec{q}\right) = \rho$ (Divergence of $\vec{q}$)

$$= \frac{\rho}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{\rho}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{\rho}{r\sin\theta}\frac{\partial}{\partial\phi}\left(V_\phi\right)$$

$$= \frac{\rho c_1}{r^2}\frac{\partial}{\partial r}\left\{r^2 - \frac{3}{2}r_0 r + \frac{1}{2}\frac{r_0^3}{r}\right\}\cos\theta - \frac{\rho c_1}{r\sin\theta}\left\{1 - \frac{3r_0}{4r} - \frac{1}{4}\frac{r_0^3}{r^3}\right\}\frac{\partial}{\partial\theta} + 0$$

$$= = \frac{\rho c_1}{r^2}\left\{2r - \frac{3}{2}r_0 - \frac{1}{2}\frac{r_0^3}{r^2}\right\}\cos\theta - \frac{\rho c_1}{r\sin\theta}\left\{1 - \frac{3r_0}{4r} - \frac{1}{4}\frac{r_0^3}{r^3}\right\} \times 2\sin\theta\cos\theta$$

$$= \frac{\rho c_1}{r_1}\left\{2 - \frac{3}{2}\frac{r_0}{r} - \frac{1}{2}\frac{r_0^3}{r^3}\right\}\cos\theta \; - \frac{\rho c_1}{r}\left\{2 - \frac{3r_0}{2r} + \frac{1}{2}\frac{r_0^3}{r_3}\right\}\cos\theta$$

$\Rightarrow \rho(\vec{\nabla}\cdot\vec{q}) = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{q} = 0$; holds eq. of continuity

Q.1. What is the irrotational velocity field associated with the velocity potential $\phi = 3x^2 - 3x + 3y^2 + 16t^2 + 12zt$. Does the flow field satisfy eq. of continuity?

Solution

$\because$ There exists velocity potential $\phi = 3x^2 - 3x + 3y^2 + 16t^2 + 12zt$

$\therefore$ $\vec{q} = -\vec{\nabla}\phi \Rightarrow$ fluid $\vec{q}$ i.e. $\vec{q} = u\hat{i} + v\hat{j} + wk$ then

Eq. of continuity $\vec{\nabla}\cdot\vec{q} = 0$

$\vec{\nabla}\cdot(-\vec{\nabla}\phi) = 0$

$-\nabla^2\phi = 0$

$$-\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0$$

$-(6 + 6 + 0) = 0$

$-12 = 0$: Not possible

i.e., $\vec{\nabla} \cdot \vec{q} = 0$ not satisfied.

$\therefore$ Eq. of continuity not satisfied.

Q.2. In a fluid flow, the velocity vector is given by, $\vec{v} = 2x\hat{i} + 3y\hat{j} - 5z\hat{k}$. Determine the eq. of streamline passing through a point (4, 8, 1)

$\therefore \; \vec{v} = 2x\hat{i} + 3y\hat{j} - 5z\hat{k} = u_1\hat{i} + v_1\hat{j} + w_1\hat{k}$

$\therefore \; u_1 = 2x, v_1 = 3y, w_1 = -5z$

Now, eq. of streamline are given by,

$$\frac{dx}{u_1} = \frac{dy}{v_1} = \frac{dz}{w_1}$$

$$\frac{dx}{2x} = \frac{dy}{3y} = \frac{dz}{-5z} \quad \ldots(A)$$

On taking 1st two fraction of (A)

$$\frac{1}{2} \log x + \frac{1}{3} \log y + \log c_1$$

$$x^{\frac{1}{2}} = y^{\frac{1}{3}} c_1 \quad \ldots(1)$$

On taking last two fractious of (A)

$$\frac{1}{3} \log y = \frac{-1}{5} \log z + \log c_2$$

$$y^{\frac{1}{3}} = \frac{c_2}{z^{\frac{1}{5}}} \quad \ldots(2)$$

$\therefore$ passing through (4, 8, 1);

$$\therefore \; 4^{\frac{1}{2}} = 8^{\frac{1}{3}} c_1 \;\Rightarrow\; c_1 = 1$$

$$\text{Also, } 8^{\frac{1}{3}} = \frac{c_2}{1^{\frac{1}{5}}} \;\Rightarrow\; c_2 = 2$$

$\therefore$ Req. eq. of streamline are given by

$$x^{\frac{1}{2}} = y^{\frac{1}{3}} \quad \ldots(3)$$

$$\& \; y^{\frac{1}{3}} = \frac{2}{z^{\frac{1}{5}}} \quad \ldots(4)$$

Q.3. for an incompressible fluid flow, the component of velocity (u, v, w) are given by $u = x^2 + 2y^2 + 3z^2$, $v = x^2 y - y^2 z + zx$. Determine the third component w so, that they satisfy the eq. of continuity also find the z-component of acceleration.

Solutions

$\because$ for incompressible fluid; eq. of continuity is $\vec{\nabla}\cdot\vec{q} = 0$

i.e., $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

$2x + x^2 - 2yz + \frac{\partial w}{\partial z} = 0$

$\frac{\partial w}{\partial z} = 2yz - x^2 - 2x$

On integrating w.r.t z

$w = 2y\frac{z^2}{2} - x^2z - 2xz + f(x,y)$ {$\because$ x, y, z are independent variable}

$\therefore$ $w = yz^2 - x^2z - 2xz + f(x, y)$

where $f(x, y)$ is an integrate and

$\because$ z-component of velocity = w

$\therefore$ z-component of acc. = $\frac{dw}{dt}$

$\because$ $\frac{dw}{dt}=\frac{\partial w}{dt}+\left(\vec{q}\cdot\vec{\nabla}\right)w$

$a_z = \frac{dw}{dt} = 0 + \left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$

$= u \times (-2xz - 2z) + v(z^2) + w(2yz - x^2 - 2x)$

$= (x^2 + 2y^2 + 3z^2)(-2xz - 2z) + (x^2y - y^2z + zx)(z^2) + (yz^2 + x^2z - 2zx)(2yz - x^2 - 2x)$

Chapter 2: Motion in 2D

$$\vec{q} = u\hat{i} + v\hat{j}$$

Streamlines:- $\frac{dx}{u} = \frac{dy}{v} \Rightarrow vdx - udy = 0$

Exampoint

Along streamline ; $\psi(x, y) = \text{constant}$

Exact differential eq:-

Let's consider a differential eq, $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ …(1)

The differential eq. (1) is said to be exact; if there exists a fun./ curve $u(x, y) = c$; c is arbitrary constant s.t

The total differentiation of (2) gives (1) directly (without any manipulation/substitution)

e.g.:- The diff eq. ;

$x\,dx + y\,dy = 0$ is exact

Because $\exists\ x^2 + y^2 = c$ s.t

$d(x^2 + y^2) = d(c)$

$\Rightarrow 2xdx + 2ydy = 0$ …(1)

• The necessary & sufficient condition for the diff. eq. $M(x, y)\,dx + N(x, y) = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Discussing Stream function $\therefore\ vdx - udy = 0$ ...(1)

Now, if $\frac{\partial}{\partial y}(v) = \frac{\partial}{\partial x}(-u)$

then the differential eq. (1) is exact differential eq.

$\downarrow\ \therefore$ by def. of exactness,

$\exists\ \psi(x, y) = c$ …(2) ; such that

the total differentiation of (2) gives (1)

Total diff. of (2) : $d\psi = d(c)$

$$d\psi = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad \ldots(3)$$

$\therefore$ (3) & (1) must be same

$\therefore$ we get $\boxed{v = \frac{\partial \psi}{\partial x}}^*$, $\boxed{-u = \frac{\partial \psi}{\partial y}}^*$ **Exam points**

Hence ψ $(x, y) = c$; ψ is called the "stream function"

Type I problem

Finding the stream f^n- if $\vec{q} = u\hat{i} + v\hat{j}$ is given

$$\frac{\partial \psi}{\partial x} = v \;\ldots(1)$$

$$\frac{\partial \psi}{\partial y} = -u \;\ldots(2)$$

v & u are given, so use & get ψ

Observation (Imp for Exam)

• Let if "irrotational" motion

↓

Then exists velocity potential ϕ

s.t $\vec{q} = -\vec{\nabla}\phi$

$$\Rightarrow u\hat{i} + v\hat{j} = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y}$$

$$\Rightarrow u = \frac{-\partial \phi}{\partial x}, v = \frac{-\partial \phi}{\partial y} \;\ldots(1)$$

• for any fluid motion (be it irrotational or not there exists stream function ψ s.t

$$u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \;\ldots(2)$$

Exampoints:-

For an irrotational fluid motion, we have

• $u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}$

$\ldots(3)$

• $v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$ → Relation between ϕ & ψ

$$\boxed{\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}\end{aligned}} \;\ldots(4)$$

From complex Analysis:-

A function $w = f(z) = u(x, y) + iv(x, y)$; analytic f, then Cauchy–Riemann eq. are satisfied

(C-R eq.)

Here $z = x + iy$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Clearly ϕ & ψ satisfy C-R eq. (see (4))

So, we can define a function

$\boxed{w = f(z) = \phi + i\psi}$; w is an analytic (differentiable) function

Here w is called the **"complex potential"** for the given fluid motion.

Exam point:- (Summary for Type I problems)

Let's say (given) $\vec{q} = u\hat{i} + v\hat{j} = u(x, y)\hat{i} + v(x, y)\hat{j}$

$\therefore$ we can find :

- Eq. for streamlines $\frac{dx}{u} = \frac{dy}{v}$
- stream function ψ $(x, y) = c$:

  By using $u = \frac{-\partial\psi}{\partial y}$, $v = \frac{\partial\psi}{\partial x}$

- Now, we can find ϕ (for it irrotational motion)

By $\vec{q} = -\vec{\nabla}\phi$ i.e., $u = \frac{-\partial\phi}{\partial x}$, $v = \frac{-\partial\phi}{\partial y}$

- Magnitude of velocity i.e. speed = $\left|\frac{dw}{dz}\right| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2} = \sqrt{u^2 + v^2} = |\vec{q}|$

$$\therefore \frac{dw}{dz} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$$

- Stagnation point:-

Speed = 0

$\left|\frac{dw}{dz}\right| = 0$

- Velocity components in terms of ψ in polar coordinates

$$q_r = \frac{1}{r}\frac{\partial\psi}{\partial\phi},\; q_\theta = \frac{\partial\psi}{\partial r}$$

- Stationary points: where velocity is zero.
- stream function is also known as current f.

Example 1: If $\phi = A(x^2 - y^2)$ represents a possible flow phenomenon, determine the stream function.

Solution

$\because \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$, $\quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$

$\Rightarrow 2Ax = \frac{\partial\psi}{\partial y}$, $\quad -2Ay = -\frac{\partial\psi}{\partial x}$

Integrating w.r.t. y, Integrating w.r.t. x

$\Rightarrow \psi(x, y) = 2Axy + f(x)$, $\quad \psi(x, y) = 2Axy + g(y)$

Where $f(x)$ is integration constant where $g(y)$ is integration constant

Clearly, we can choose $f(x) = g(y) = 0$; then

Getting $\psi(x, y) = 2Axy$ in both cases. This is the required stream fun.

Example 2: Determine the stream function $\psi(x, y, t)$ for the given velocity field $u = Ut$, $v = x$.

$\because u = -\frac{\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x}$

$\therefore \frac{\partial\psi}{\partial y} = -Ut \quad , \frac{\partial\psi}{\partial x} = x$

Integrating w.r.t. y On integrating w.r.t. x,

$\psi = -Uty + f(x, t)$ …(1) $\quad \psi = \frac{x^2}{2} + g(y, t)$ ...(2)

$f(x, t)$ Integration constant $g(y, t)$ Integration constant

Now, if we choose integration constants appropriately

For (1) & (2), to be same,

$f(x,t) = \frac{x^2}{2}$, $g(y, t) = -Uty$

$\therefore$ Required stream function is

$$\psi - Uty = \frac{x^2}{2}$$

Example 3: The velocity potential function for a two dimensional flow is $\phi = x(2y - 1)$. At a point (4, 5) determine the speed & the value of stream function.

Solutions: we know,

$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$, $\quad \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x}$

$2y - 1 = \frac{\partial\psi}{\partial y}$, $\quad 2x = \frac{-\partial\psi}{\partial x}$

On integrating w.r.t. y, On integrating w.r.t. x,

$y^2 - y + f(x) = \psi(x, y)$…(1) $\quad -x^2 + g(y) = \psi(x, y)$ …(2)

Now, if we choose appropriately $f(x) = -x^2$ & $g(y) = y^2 - y$

We get,

$\psi(x, y) = y^2 - y - x^2$

$\therefore\ \psi(4, 5) = 25 - 5 - 16 = 4$

$\therefore$ Req. sped :-

$$\left|\frac{dw}{dz}\right| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}$$

$$\left|\frac{dw}{dz}\right| = \sqrt{(2y-1)^2 + (2x)^2}$$

At (4, 5)

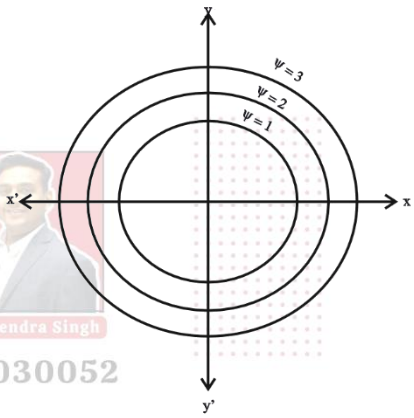
$$\left|\frac{dw}{dz}\right| = \sqrt{9^2 + 8^2} = \sqrt{165}$$

Ex4. If $\phi = A\left(x^2 - y^2\right)$ *represents a possible flow phenomenon, determine the stream function.*

Sol. Here $\qquad \phi = A\left(x^2 - y^2\right) \ldots.(1)$

$\partial\psi/\partial y = \partial\phi/\partial x = 2Ax$, using (1)

Integrating it w.r.t. 'y', $\qquad \psi = 2Axy + f(x), \ldots(2)$



pattern of streamlines for $\psi = x^2 + y^2$

Where $f(x)$ is an arbitrary function of x. (2) gives the required stream function.

Ex. 5. The streamlines are represented by (a) $\psi = x^2 - y^2$ and (b) $\psi = x^2 + y^2$ Then

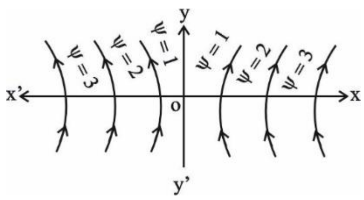
(i) determine the velocity and its direction at (2, 2)

(ii) (ii) sketch the streamlines and show the direction of flow in each case.

Part (i) Given that

Now, $\qquad u = \partial\psi/\partial y = -2y \qquad$ and $\qquad v = -\partial\psi/\partial x = -2x.$

At (2, 2) $\qquad u = -4 \qquad$ and $\qquad v = -4.$



The resultant velocity = $\left(u^2 + v^2\right)^{1/2}$ $\qquad = (16 + 16)^{1/2} \qquad = 4\sqrt{2}$ units.

And its direction has a slope = v/u = 1 showing that the velocity vector is inclined at 45° to x-axis.

The required stream line are given by ψ = c, where c is a constant, i.e. $x^2 - y^2$ = c, which represents a family of hyperbolas. In figure, we have sketched the steam lines for various values of ψ . The direction of arrowhead shows the direction of flow in each case.

Part (ii) Given that $\qquad \psi = x^2 + y^2$

Now, $\qquad \psi = x^2 + y^2 \qquad\qquad v = -\partial\psi/\partial x = -2x$

At (2, 2) $\qquad u = 4 \qquad$ and $\qquad v = -4$

$\therefore$ The resultant velocity

$= \left(u^2 + v^2\right)^{1/2}$ = (16 16)$^{1/2}$ = 4√2 units.

And its direction has a slope = v/u = –1, showing that the velocity vector is inclined at 135° to x-axis. The required stream lines are given by ψ = c, where c is a constant, i.e. $x^2 + y^2$ = c, which represents a family of circles. In figure, we have sketched the stream lines for various values of ψ . The direction of arrowhead shows the direction of flow in each case.

Next, m2 = the slope of tangent to $\psi = -\dfrac{\partial\psi/\partial x}{\partial\psi/\partial y} = \dfrac{-2y}{-2x} = -\dfrac{y}{x}$, by (3)

∴ m^2 = slope of tangent to stream lines ψ = c_2 at (2, 2) = – (2/2) = –1

Here m1m2 = –1 showing that the streamlines and the potential lines intersect orthogonal.

Ex.6. Determine the steam function $\psi(x, y, t)$ for the given velocity field $u = Ut$, $v = x$.

Sol. We know that $u = -(\partial\psi/\partial y)$ and $v = \partial\psi/\partial x.$

∴ $\partial\psi/\partial y = -Ut$ …(1)

$\partial\psi/\partial x = x.$ …(2)

Integrating (1), $\psi(x, y, t) = -Uty + f(x, t),$ …(3)

Where $f(x, t)$ is an arbitrary function of x and t.

From (3), $\partial\psi/\partial x = \partial f/\partial x$ (4)

Then (2) and (4) ⇒ $\partial f/\partial x = x.$ (5)

Integrating (5), $f(x, t) = x^2/2 + F(t),$ (6)

where F(t) is an arbitrary function of t.

Form (3) and (6), $\psi(x, y, t) = -Uty + x^2/2 + f(t).$

Ex. 7. To show that the curves of constant velocity potential and constant stream functions cut orthogonally at their points of intersection.

OR

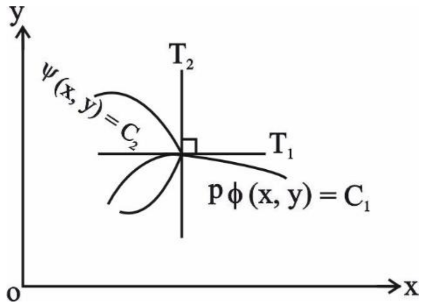
To shows that the family of curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2, c_1, c_2$ being constants, cut orthogonally at their point of intersection.

Proof. Let the curves of constant velocity potential and constant stream function be given by

$\phi(x, y) = c_1$ and $\psi(x, y) = c_2$

where c_1 and c_2 are arbitrary constants.

Let m_1 and m_2 be gradients of tangents PT_1 and PT_2 at point of intersection P of (1) and (2)



Then, we have

$$m_1 = \frac{\partial\phi/\partial x}{\partial\phi/\partial y} \quad \text{and} \quad m_2 = \frac{\partial\psi/\partial x}{\partial\psi/\partial y} \quad \ldots(3)$$

We know that ϕ and ψ satisfy the Cauchy-Riemann equations,

$$\partial\phi/\partial x = \partial\psi/\partial y \text{ and } \partial\phi/\partial y = -\partial\psi/\partial x. \quad \ldots(4)$$

Now, from (3), $$m_1 m_2 = \frac{(\partial\phi/\partial x)(\partial\psi/\partial x)}{(\partial\phi/\partial y)(\partial\psi/\partial y)} = \frac{(\partial\psi/\partial y)(\partial\psi/\partial x)}{-(\partial\psi/\partial x)(\partial\psi/\partial y)}, \text{ by (4)}$$

Hence $m_1 m_2 = -1$, showing that the curves (1) and (2) cut each other orthogonally.

Ex.8. Find the lines of flow in the two dimensional fluid motion given by

ϕ = xy = –(n/2) × (x + iy)^2 e^{2int}

Prove or verify that the paths of the particles of the fluid may be obtained by eliminating t from the equations.

$$r\cos(nt+\theta) - x_0 = r\sin(nt+\theta) - y_0 = nt(x_0 - y_0.)$$

Sol. Given $$\phi + i\psi = -(n/2)\times(x+iy)^2 e^{2int}.$$

Let $x = r\cos\theta$ and $y = r\sin\theta$ Then $x + iy = r(\cos\theta + i\sin\theta)$

So (1) becomes $$\phi + i\psi = -(n/2)\times\left(re^{i\theta}\right)^2 e^{2int} = -(n/2)\times r^2 e^{2i(\theta+nt)}$$

Equating the real and imaginary parts on both sides of (2), we get

$$b = -(n/2)\times r^2\cos 2(\theta+nt) \quad \text{and} \quad \psi = -(n/2)\times r^2 \sin 2\theta(\theta+nt)$$

The lines of flow are given by ψ = constant,

$$-(n/2)\times r^2\sin 2(\theta+nt) = \text{constant} \quad \text{or} \quad r^2\sin 2(\theta+nt) = \text{constant}.$$

We now proceed to find the path of the particles, we have

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = nr\cos 2(\theta+nt) = nr\cos 2\lambda, \text{ by (2)} \quad \ldots(3)$$

And $$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -nr\sin 2(\theta+nt) = -nr\sin 2\lambda, \text{by (2)} \quad \ldots(4)$$

Where $$nt + \theta = \lambda$$

Now $$(3) \Rightarrow nr\cos 2\lambda = \frac{dr}{dt} = \frac{dr}{d\lambda}\frac{d\lambda}{dt} = \frac{dr}{d\lambda}\left(\frac{d\theta}{dt} + n\right) \text{ by (5)}$$

or $$nr\cos 2\lambda = \frac{dr}{d\lambda}(-n\sin 2\lambda + n), \text{ using (4)}$$

or $$(2/r)dr - [2\cos 2\lambda(1-\sin 2\lambda)]d\lambda = 0$$

Integrating, $2\log r + \log(1-\sin 2\lambda) = lag\, C$ or $r^2(1-\sin 2\lambda) = C$

or $r^2(\sin^2\lambda + \cos^2\lambda - 2\sin\lambda\cos\lambda) = C$ or $[r(\cos\lambda - \sin\lambda)]^2 = C$

or $r(\cos\lambda - \sin\lambda)C'$, where $C'(=\sqrt{C})$ is are arbitrary constant. …(6)

Initially, let $\lambda = \theta_0$ and $r = r_0$ when $t = 0$. Then (6) gives

$\therefore$ (6) becomes $r\cos\lambda - r\sin\lambda = x_0 - y_0$ ..(7)

or $r\cos(\theta + nt) - x_0 = r\sin(\theta - nt) - y_0$, using (5) …(8)

Now, from (5), $d\lambda/dt = n + (d\theta/dt)$ or $d\lambda/dt = -r - n\sin 2\lambda$, using (4)

or $\dfrac{d\lambda}{1-\sin 2\lambda} = ndt$ or $\dfrac{d\lambda}{(\cos\lambda - \sin\lambda)^2} = n\,dt$

$\therefore$ $\dfrac{d\lambda}{(\cos\lambda - \sin\lambda)^2} = n\,dt$ or $-\int \dfrac{du}{u^2} = nt + D$

$(putting\, 1 - \tan\lambda = u \quad so\, that \quad -\sec^2\lambda\, d\lambda = du)$

or $\dfrac{1}{u} = nt + D$ or $\dfrac{1}{1-\tan\lambda} = nt + D$

or $\dfrac{\cos\lambda}{(\cos\lambda - \sin\lambda)} = nt + D$

As before, initially $\lambda = \theta_0$ *and* $t = 0$. Hence (9) gives

$D = \dfrac{\cos\theta_0}{\cos\theta_0 - \sin\theta_0} = \dfrac{r_0\cos\theta_0}{r_0\cos\theta_0 - r_0\sin\theta_0} = \dfrac{x_0}{x_0 - y_0}$, as before

Then, (9) becomes $\dfrac{r\cos\lambda}{r\cos\lambda - r\sin\lambda} = nt + \dfrac{x_0}{x_0 + y_0}$

or $\dfrac{r\cos(\theta + nt)}{x_0 - y_0} = nt + \dfrac{x_0}{x_0 + y_0}$ or $r\cos(\theta + nt) = nt(x_0 - y_0) + x_0$

or $r\cos(nt + \theta) - x_0 = nt(x_0 - y_0)$ …(10)

$\therefore$ Then, from (8) and (10), we have

$r\cos(nt + \theta) - x_0 = r\sin(nt + \theta) - y_0 = nt(x_0 - y_0)$

Ex. 9. A single source is placed in an infinite perfectly elastic fluid, which is also a perfect conductor of heat. Show that if the motion be steady, the velocity v at a distance r from the source satisfies the equation $\left(v - \dfrac{k}{v}\right)\dfrac{\partial v}{\partial r} = \dfrac{2k}{r}$ and hence that $r = \dfrac{1}{\sqrt{v}} e^{v^2/4k}$.

Sol. Since we have an infinite perfectly elastic fluid, there would be hardly any change in temperature, and hence Boyle's law would be obeyed and so $p = k\rho$ …(1)

Since the motion is symmetrical about the source, the equation of continuity may be written as

$\rho r^2 v = \text{constant}$, …(2)

Where v is the velocity at a distance r and ρ is the density of fluid. The pressure equation takes the form

$$\int \frac{dp}{\rho} + \frac{v^2}{2} = \text{constant} \qquad \text{or} \qquad k\int \frac{d\rho}{\rho} + \frac{v^2}{2} = \text{constant, by (1)} \qquad \ldots(3)$$

Differentiating (2) and (3) w. r. t. 'r', we have

$$vr^2 \frac{\partial \rho}{\partial r} + \rho\left[r^2 \frac{\partial v}{\partial r} + 2rv\right] = 0 \qquad \ldots(4)$$

and $$\frac{k}{\rho}\frac{\partial \rho}{\partial r} + v\frac{\partial v}{\partial r} = 0 \qquad \text{i.e.,} \qquad \frac{\partial \rho}{\partial r} = -\frac{v\rho}{k}\frac{\partial v}{\partial r} \qquad \ldots(5)$$

Substituting the value of $\partial\rho/\partial r$ given by (5) in (4), we get

$$vr^2\left(-\frac{v\rho}{k}\frac{\partial v}{\partial r}\right) + \rho\left(r^2\frac{\partial v}{\partial r} + 2rv\right) = 0$$

$$\frac{r^2}{k}\frac{\partial v}{\partial r}\left(k - v^2\right) = -2rv \qquad \text{or} \qquad \left(v - \frac{k}{v}\right)\frac{\partial v}{\partial r} = \frac{2k}{r}, \qquad \ldots(6)$$

Which proves the first part of the problem.

Integrating (6), $(v^2/2) - k \log v = 2k \log C$, C being an arbitrary constant.

or $$(1/2)\times \log v + \log r - \log C = v^2/4k \qquad \text{or} \qquad r\sqrt{v} = Ce^{v^2/4k}$$

or $$r = \left(1/\sqrt{v}\right)e^{v^2/4k} \quad \text{taking } C = 1$$

Ex. 10. *Prove that the radius of curvature R at any point of a streamline ψ = constant given by*

$$R = \frac{\left(u^2+v^2\right)^{3/2}}{u^2\left(\partial v/\partial x\right) - 2uv\left(\partial u/\partial x\right) + v^2\left(\partial v/\partial y\right)}$$ *where u, v are respectively the velocity components of a fluid motion along OX and OY.*

Sol. From Differential Calculus, we know that the radius of curvature R at a point (x, y) streamline ψ (x, y) = constant is given by

$$R = \frac{\left[1 + \left(dy/dx\right)^2\right]^{3/2}}{\left(d^2y/dx^2\right)} \qquad \ldots(1)$$

Given streamline is $$\psi(x, y) = 0 \qquad \ldots(2)$$

Also, we have $$u = -\partial\psi/\partial y \qquad \text{and} \qquad v = -\partial\psi/\partial x \qquad \ldots(3)$$

Differentiating (2) w. r. r. x, $$(\partial\psi/\partial x) + (\partial\psi/\partial y)(\partial y/\partial x) = 0$$

or $$v - u\left(dy/dx\right) = 0 \qquad \text{or} \qquad dy/dx = v/u \qquad \ldots(4)$$

Differentiating (4) w. r. t. x $$\frac{d^2y}{dx^2} = \frac{\partial}{\partial x}\left(\frac{v}{u}\right) + \frac{\partial}{\partial y}\left(\frac{v}{u}\right)\frac{dy}{dx}$$

or $$\frac{d^2y}{dx^2} = \frac{u\left(\partial v/\partial x\right) - v\left(\partial u/\partial x\right)}{u^2} + \frac{u\left(\partial v/\partial y\right) - v\left(\partial u/\partial y\right)}{u^2}, \frac{v}{u} \quad \text{using (4)}$$

or $$\frac{d^2y}{dx^2}=\frac{u\left[u(\partial v/\partial x)-v(\partial u/\partial x)\right]+u\left[u(\partial v/\partial y)-v(\partial u/\partial y)\right]}{u^3}$$

or $$\frac{d^2y}{dx^2}=\frac{u^2(\partial v/\partial x)-2uv(\partial u/\partial x)+v^2(\partial u/\partial y)}{u^3} \qquad \ldots(5)$$

$$\left[\because \frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)=\frac{\partial u}{\partial x}, by\,(3)\right]$$

Putting the values of dy/dx and d^2y/dx^2 from (4) and (5) in (1), we get

$$R=\frac{\left(1+v^2/u^2\right)^{3/2}}{\left|\left\{u^2(\partial v/\partial x)-2uv(\partial u/\partial x)-v^2(\partial u/\partial y)/u^3\right\}\right|}=\frac{\left(u^2/v^2\right)^{3/2}}{\left|u^2(\partial v/\partial x)-2uv(\partial u/\partial x)-v^2(\partial u/\partial y)/u^3\right|}$$

Ex. 11. *Show that $u = 2cxy$, $v = c(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.*

Sol. Given $\quad u = 2cxy \qquad v = c(a^2 + x^2 - y^2) \qquad \ldots(1)$

Equation of continuity in xy-plane is given by

$$\partial u/\partial x+\partial v/\partial y=0 \qquad \ldots(2)$$

From (1), $\partial u/\partial x=2cy$ and $\partial v/\partial y=-2cy$, putting these values in (2) we get $0 = 0$, showing (2) is satisfied by u, v given by (1). Hence u and v constitute a possible fluid motion.

Let ψ be the required stream function. Then, we have

$$u=-(\partial\psi/\partial y) \quad \text{or} \quad \partial\psi/\partial y=-2cxy \qquad \ldots(3)$$

and $$v=\partial\psi/\partial x \quad \text{or} \quad \partial\psi/\partial x=c\left(a^2+x^2-y^2\right) \qquad \ldots(4)$$

or $$x+iy=C\left(\cos\phi\cos i\psi-\sin\phi\sin i\psi\right)$$

or $$x+iy=C\cos\phi\cosh\psi-iC\sin\phi\sinh\psi$$

Equating real and imaginary parts, (2) gives

$$x=C\cos\phi\cosh\psi \quad \text{and} \quad y=-C\sin\phi\sinh\psi$$

so that $$\cos\phi=\frac{x}{C\cosh\psi} \quad \text{and} \quad \sin\phi=\frac{y}{C\sinh\psi}$$

Squaring and adding these, we obtain

$$\frac{x^2}{C^2\cosh^2\psi}+\frac{y^2}{C^2\sinh^2\psi}=1$$

Which give the streamlines in two-dimensions.

Again, given that the streamlines are confocal ellipses

$$x^2/\left(a^2+\lambda\right)+y^2/\left(b^2+\lambda\right)=1$$

Since (3) and (4) must be identical, we have

$$C^2\cosh^2\psi=a^2+\lambda \quad \text{and} \quad C^2\sinh^2\psi=b^2+\lambda$$

$$\therefore \quad C\left(\cosh\psi+\sinh\psi\right)=\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda} \quad \text{or} \quad Ce^{\psi}=\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}$$

$$\left[\because \cosh\psi = \left(e^{\psi} + e^{-\psi}\right)/2\right] \quad \text{and}$$

$$\sinh\psi = \left(e^{\psi} + e^{-\psi}\right)/2$$

Or $$\psi = \log\left(\sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}\right) - \log C$$

If ϕ, ψ are velocity potential and stream function, so also will be $A\phi$ and $A\psi$ where a constant. Hence (5) may be-written as

$$\psi = A\log\left(\sqrt{a^1+\lambda} + \sqrt{b^2+\lambda}\right) + B$$

From (1), $$\frac{dz}{dw} = -C\sin w = -C\sqrt{1-\cos^2 w} = -C\left(1 - z^2/C^2\right)^{1/2}$$

$$= \sqrt{C^2 - z^2} = -\sqrt{(C+z)(C-z)} = -\sqrt{r_1 r_2}$$

Where r1 and r2 are the focal distances (radii) of any point. P(z) from the foci S (C, 0) S' (– C, 0) of the ellipses.

Thus $$p = |dw/dz| = 1/\sqrt{r_1 r_2}$$

Ex. 12. *A velocity field is given by q = – xi + (y + t)j. Find the stream function and the streamlines for this field at t = 2.*

Sol. We have $$-\partial\psi/\partial y = u = -x$$

and $$-\partial\psi/\partial x = v = y + t$$

Integrating (1) and (2), we get $$\psi = xy + f_1(x,t)$$

and $$\psi = xy + tx + f_2(x,t)$$

Note that f_2 must be a function of t alone, otherwise (4) will not be satisfied, $f_1 = tx + f_2$. Thus

$$\psi = xy + tx + f_2(t)$$

The function f_2 cannot be obtained from the given data. However since we deal only differences in ψ values at a given t or with the derivatives $\partial\psi/\partial x$ and $\partial\psi/\partial y$, the determine of f_2 is not necessary. At t = 2, (5) becomes

$$\psi = xy + 2x + f_2(2)$$

The stream lines (ψ = constant) are given by $x(y+2)$ = constant,

Which are rectangular hyperbolas.

Ex. 13. *A two–dimensional flow field is given by $\psi = xy$ (a) Show that the flow is irritation (b) Find the velocity potential. (c) Verify that ψ and ϕ satisfy le Laplace equation . (d) find streamlines and potential lines.*

Sol. (*a*) The velocity components are given by $$u = -\partial\psi/\partial y = -x, \quad v = -\partial\psi/\partial y = -y,$$ so that

$q = ui + vj$ or $q = -ix + yj$

and $\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = 0.$

Hence the flow is irrotational.

(b) We have $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}, \qquad \dfrac{\partial \phi}{\partial y} = \dfrac{\partial \psi}{\partial x}$

$$\phi = \int (\partial\psi/\partial y) dx + f_1(y) = x^2/2 + f_1(y)$$

and $\phi = -\int (\partial\psi/\partial y) dx + f_2(y) = y^2/2 + f_2(x)$

(1) and (2) show that

$f_1(y) = -y^2/c + \text{constant}$ and $f_2(x) = x^2/2 + \text{constant},$

so that $\phi = (x^2 - y^2/2) + \text{constant}$

$(c) \nabla^2\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 + 0 = 0$ and $(c) \nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 1 - 1 = 0$

Hence ψ and ϕ satisfy the Laplace equation.

(*d*) The streamlines ($\psi = $constant) and the potential lines ($\phi = $constant) are given by $xy = C_1$ and $x^2 - y^2 = C_2$, respectively, where C_1 and C_2 are constants.

Ex.14. Show that $u = 2cxy$, $v = c\,(a^2 + x^2 - y^2)$ are the velocity component of a possible fluid motion. Determine the stream function.

$\because$ For possible fluid motion; eq of continuity holds

$\vec{\nabla} \cdot \vec{q} = 0$

$\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

$\Rightarrow 2cy - 2cy = 0$; holds,

$\therefore$ Yes, given components constitute a possible fluid motion.

$\because\ v = \dfrac{\partial \psi}{\partial x},\ -u = \dfrac{\partial \psi}{\partial y}$

$ce(a^2 + x^2 - y^2) = \dfrac{\partial \psi}{\partial x}, \quad -2xcy = \dfrac{\partial \psi}{\partial y}$

$\psi = ca^2\, c + c\dfrac{x^3}{3} - cy^2x + f(y) \qquad \psi = -xcy^2 + f(x)$

On choosing appropriately,

$f(y) = 0$, $f(x) = ca^2x + \dfrac{cx^3}{3}$ then

$\psi - ca^2x + \dfrac{cx^3}{3} - cy^2x$

Ex.15. Show that the velocity potential $\phi = \dfrac{1}{2}\log\{(x+a)^2+y^2\} - \dfrac{1}{2}\log\{(x-a)^2+y^2\}$ gives a possible motion.

Determine the streamline & show also that curves of equal speed are given as ovals of cassin $rr' =$ constant.

Solutions: For possible fluid motion; Eq. of continuity holds

i.e., $\vec{\nabla}\cdot\vec{q} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\because\ u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y}$$

$$\therefore\ \frac{\partial u}{\partial x} = -\frac{\partial^2 d}{\partial x},\ \frac{\partial v}{\partial y} = -\frac{\partial^2 \phi}{\partial y^2}$$

$\therefore$ eq. of continuity holds if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Now,

$$\frac{\partial \phi}{\partial x^2} = \frac{1}{2\left\{(x+a)^2+y^2\right\}} \times 2(x+a) - \frac{1}{2\left\{(x-a)^2+y^2\right\}} \times 2(x-a)$$

$$\frac{\partial^2 \phi}{\partial x} = \frac{\left\{(x+a)^2+y^2\right\}\times 1-(x+a)\times 2(x+a)}{\left\{(x+a)^2+y^2\right\}^2} - \left[\frac{\left\{(x+a)^2+y^2\right\}\times 1-(x-a)\times 2(x-a)}{\left\{(x-a)^2+y^2\right\}^2}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{y^2-(x+a)^2}{\left\{(x+a)^2+y^2\right\}^2} - \left\{\frac{y^2-(x-a)^2}{\left\{(x-a)^2+y^2\right\}^2}\right\} \quad \ldots(A)$$

$$\frac{\partial \phi}{\partial y} = \frac{2y}{2\left\{(x+a)^2+y^2\right\}} - \frac{2y}{2\left\{(x-a)^2+y^2\right\}}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\left\{(x+a)^2+y^2\right\}-y\{2y\}}{\left\{(x+a)^2+y^2\right\}^2} - \frac{\left\{(x-a)^2+y^2\right\}-y\{2y\}}{\left\{(x-a)^2+y^2\right\}^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{(x+a)^2-y^2}{\left\{(x+a)^2+y^2\right\}^2} - \left\{\frac{(x-a)^2-y^2}{\left\{(x-a)^2+y^2\right\}^2}\right\} \quad \ldots(B)$$

$\therefore$ from (A) & (B)

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0$$; holds eq. of continuity

Now, determining ψ : -

$$\frac{\partial\phi}{\partial x^2} + \frac{\partial\psi}{\partial y}, \ \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x}$$

$$\Rightarrow \ \frac{\partial\psi}{\partial y} = \frac{(x+a)}{(x+a)^2-y^2} - \frac{(x+a)}{(x+a)^2+y^2}$$

On integrating w.r.t. y.

$\psi = \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) + f(x)$; $f(x)$ is an integrate constant

$$\therefore \ \frac{\partial\psi}{\partial x} = \frac{y}{(x-a)^2+y^2} - \frac{y}{(x-a)^2+y^2} + f'(x)$$

But comparing above expression with $\frac{-\partial\phi}{\partial y}$ we get

$f'(x) = 0$

$\Rightarrow f(x) = \text{constant (say } c)$

$\therefore$ stream function

$\psi = \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) + c$; c is some constant

$$\therefore \ \psi = \tan^{-1}\left\{\frac{\frac{v}{x+a}-\frac{y}{x-a}}{1+\left(\frac{v}{x+a}\right)\left(\frac{v}{x-a}\right)}\right\}$$

$$\psi = \tan^{-1}\left(\frac{-2ay}{x^2-a^2+y^2}\right);$$

streamlines are given by,

$\psi(x, y) = \text{constant}$

$\Rightarrow \ \tan^{-1}\left(\frac{-2ay}{x^2-a^2+y^2}\right) = \text{constant}$; gives streamlines

Extra Observation:-

Now, if constant = 0 : then streamlines are given by

$$-\frac{2ay}{x^2-a^2+y^2} = 0$$

$\Rightarrow \ -2ay = 0$

$\Rightarrow y = 0$; streamline

Now,

$w = \phi + i\psi$

$$w = \frac{1}{2}\log\left\{(x+a)^2 + y^2\right\} - \frac{1}{2}\log\left\{(x-a)^2 + y^2\right\} + i\cdot\left\{\tan^{-1}\frac{v}{x+a} - \tan^{-1}\left(\frac{y}{x-a}\right)\right\}$$

$= \log[(x + a) + iy] - \log[(x - a) + iy]$

$= \log[(x + iy) + a\} - \log[(x + iy) - a]$

$w = \log(z + a) - \log(z - a)$

$w = \log\left(\frac{z+a}{z-a}\right)$

$\because \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \text{ speed} = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z+a} - \frac{1}{z-a}\right| = \frac{2a}{|z+a|\cdot|z-a|} = \frac{2a}{rr'}$$

For constant speed

$\frac{2a}{rr'}$ = constant

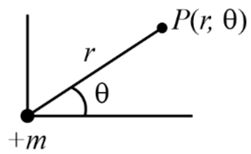
$\Rightarrow rr'$ = constant

Exam point:-

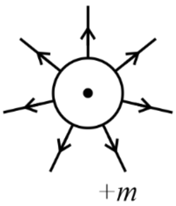
$|z - z_0| = r$: represents eq. of a circle with centre at $z = z_0$ & radius r

Sources and Sinks

Source :- A "source of mass m or strength m" is ; a flow across the circular boundary as $2\pi m$



We want to study the motion of fluid.

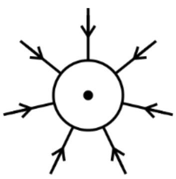


Source

Sink:-

A sink is of strength $- m$

Flow $= - 2\pi m$



Flow :- $- 2\pi m$

Sink of strength - m

Exampoint:-

$q_r = \frac{-1}{r}\frac{\partial\psi}{\partial\theta}$, $q_r = \frac{-\partial\phi}{\partial r}$

We have already study above velocity component q_r in polar form in terms of shi and phi

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Exampoint:- For a source of strength $+m$

$2\pi r q_r = 2\pi m$

$$\Rightarrow 2\pi r\left(\frac{-1}{r}\frac{\partial\psi}{\partial\theta}\right) = 2\pi m$$

$$\Rightarrow -\frac{\partial\psi}{\partial\theta} = -m \;\ldots(1) \qquad \Rightarrow \psi = -m\theta$$

For circular & radius 'r' flow in termes of q_r is $2\pi r q_r$

Similarly

$$2\pi r\left(\frac{-\partial\phi}{2r}\right) = 2\pi m$$

$$\Rightarrow -r\frac{\partial\phi}{\partial r} = m \;\ldots(2)$$

$$\Rightarrow \boxed{\phi = -m\log r}$$

$\therefore$ The complex potential due to a source of strength m is

$W = \phi + i\psi$

$W = -m\log r + i\,(-m\theta)$

$W = -\,m[\log^r + \log e^{i\theta}]$

$W = -\,m\log(re^{i\theta})$

$z = x + iy$

$z = r.e^{i\theta}$

$r^2 = x^2 + y^2$

$$\boxed{W = -m\log z}$$

Similarly, for a sink of strength $-m$.

$$\boxed{W = m\log(z)}$$

Exampoints: -

1. $W = -\,m\log(z)$: For source of strength m.
2. $W = m\log(z)$: for sink of strength $-\,m$.
3. Let source is at the point $z = z_0$; then

• $W = -m\log(z - z_0)$

• $W = m\log(z - z_0)$ {for sink}

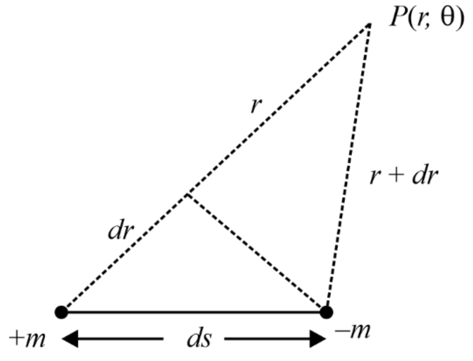
4. Let there are n sources at points $z_1, z_2, \ldots, z_n$ of strength m.

$W = -m\log(z - z_1) - m\log(z - z_2) - m\log(z - z_3) + \ldots + (-m\log(z - z_n))$ {for source}

$W = m\log(z - z_1) + m\log(z - z_2) + \ldots + m\log(z - z_n)$ {for sink}

Doublet

Source & sink of same strength at a small distance *ds*



Now, $\phi = m \log (r + dr) + m \log r$

$= -m \log \left(\dfrac{r+dr}{r}\right)$

$\phi = -m \log \left(1+\dfrac{dr}{r}\right)$

$= -m \left\{\dfrac{dr}{r} - \dfrac{(dr)^2}{2r^2} + \ldots\right\}$ [Apply $\log (1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^2}{3}$]

$= -m\dfrac{dr}{r}$; neglecting higher order terms

$\phi = \dfrac{-mds\cos\theta}{r}$

$\phi = \dfrac{-\mu}{r} \cos \theta$; Taking $mds = \mu$

$\because$ ψ is the complex conjugate of ϕ ($\because$ $w = \phi + i\psi$ is analytic)

$\therefore$ $\psi = \dfrac{+\mu}{r} \sin \theta$

$\therefore$ $w = \phi + i\psi$

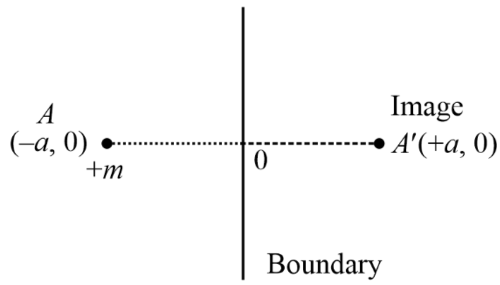
$= \dfrac{-\mu}{r}\cos \theta + \dfrac{\mu}{r} i \sin \theta = \dfrac{-\mu}{r} (\cos \theta - i \sin \theta) = \dfrac{-\mu}{r} e^{-i\theta} = \dfrac{-\mu}{re^{i\theta}}$

Exampoint

$\boxed{w = \dfrac{-\mu}{z}}$ & $\boxed{\mu = mds}$

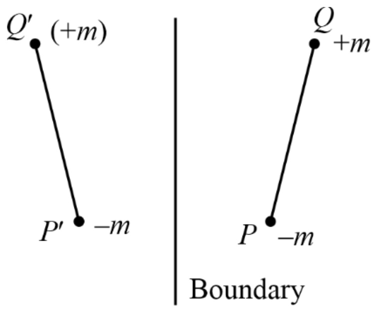
Article 3:- To determine the complex potential due to sources, sinks doublets in the presence of rigid boundaries:-

(i) Image of a source with respect to a line:-



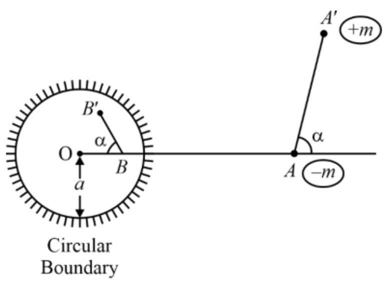
The image of a source of strength *m* with respect to a line in 2-D; is an equal source equidistant from the line to the opposite of source.

Image of doublet with respect to a line



The image of doublet (PQ) w.r.t. a line is the doublet ($P'\,Q'$)

Note :- **Image system w.r.t. a circular boundary:-**



Let's determine the image of **doublet AA'**
(with its axis, making an angle α)
in the presence of circular boundary:
is a doublet (again) BB'

B' is inverse point of A'
B is inverse point of A] $OA.OB = OA'.OB' = a^2$

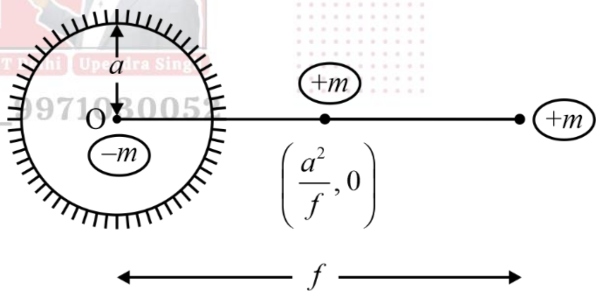
Explanation:-

• Image of source +m at A' ; consist of a source $+m$ at B' and a sink at O;

∵ source $+m$ & sink $-m$ cancel each other at O.

Exampoint

Image of a source of strength 'm' consisting of two things sink at the origin & source at a distance of $\dfrac{a^2}{f}$ from the origin.



$$\boxed{W = -m\log(z-f) - m\log\left(z - \frac{a^2}{f}\right) + m\log z}$$

Some important results regarding conformal transformations

(i) **In a conformal transformation:**

→ source is transformed into equal source

→ a sink is transformed into equal sink

→ a doublet is transformed into equal a doublet

(ii) The complex potential $W = \phi + i\psi$ is invariant under the conformal transformation

↓

To solve questions; to study the motion; we'll try to transform given system into a simpler system (through conformal transformation)

(iii) Let $\xi = f(z)$ be the conformal transformation then
the total kinetic energy of fluid in z-plane (per unit depth)
= total K.E. of fluid (per unit depth) in ξ-plane

(iv) Under a conformal transformation, a streamline in z-plane is transformed into a streamline in ξ-plane

(v) Important Point (for question solving)

While using conformal transformation $\xi = z^n$;

** n is found by; dividing $\frac{\pi}{2}$ by the half the angle between two rigid boundaries

Example 1: What are arrangements of sources & sinks which will give rise to the function $W = \log\left(z - \frac{a^2}{z}\right)$.

Also, find streamlines.

$$\because\ W = \log\left(z - \frac{a^2}{z}\right)$$

$$W = \log\left(\frac{z^2 - a^2}{z}\right)$$

$W = \log(z^2 - a^2) - \log z$

$W = \log(z + a) + \log(z - a) - \log z$

$W = 1.\log(z - (-a)) + 1.\log(z - a) + (-1).\log(z - 0)$ ...(1)

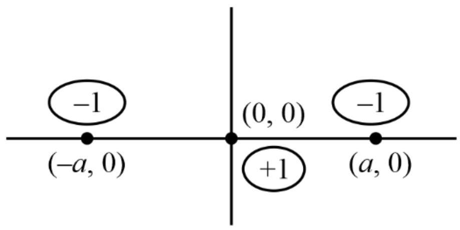
$W = m_1 \log(z - z_1) + m_2 \log(z - z_2) - m_3 \log(z - z_3)$

$\downarrow$	$\downarrow$	$\downarrow$
Sink of strength m_1 at $z = z_1$	Sink of strength m_2 at $z = z_2$	Sink of strength m_3 at $z = z_3$

$\therefore$ (1) is combination of

- a source of unit strength at origin
- Two sinks of unit strength at $z = a$, $z = -a$.

Finding streamlines : ψ = constants



$\because\ W = \log(z + a) + \log(z - a) - \log z$

$W = \log(x + iy + a) + \log(x + iy - a) - \log(x + iy)$

$$\phi + i\psi = \left(\left(\frac{1}{2}\log\left((x+a)^2 + y^2\right) + i\tan^{-1}\left(\frac{y}{x+a}\right)\right) + \right.$$

$$\left(\frac{1}{2}\log\left((x-a)^2 + y^2\right) + i\tan^{-1}\left(\frac{y}{x+a}\right)\right) - \left(\frac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\therefore\ \psi = \tan^{-1}\left(\frac{y}{x+a}\right) + \tan^{1}\left(\frac{y}{x-a}\right) - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left\{\frac{\frac{y}{x+a}+\frac{y}{x-a}}{1-\frac{y}{(x+a)}\frac{y}{(x-a)}}\right\} - \tan^{-1}\left(\frac{y}{x}\right)$$

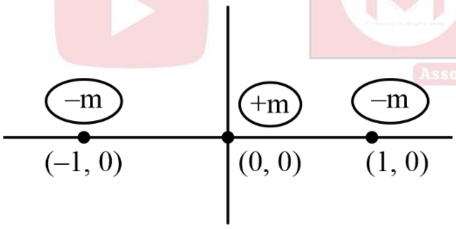
$$\psi = \tan^{-1}\left\{\frac{\frac{y}{(x+a)}+\frac{y}{x-a}-\frac{y}{x}}{1+\left\{1-\frac{y}{(x+a)}\frac{y}{x-a}\right\}\left(\frac{y}{x}\right)}\right\}$$

$\therefore$ Required streamlines are given by,

ψ = constant = c

$$\frac{\frac{y}{x+a}+\frac{y}{x-a}-\frac{y}{x}}{1+\left(1-\frac{y^2}{(x^2-a^2)}\right)\frac{y}{x}} = \tan c$$

Example 2: There is a source of strength m at (0, 0) and equal sinks at (1, 0) & (–1, 0). Discuss about 2-D motion,



$$\therefore\ w = m\log(z-(-1)) + m\log(z-1) - m\log(z-0)$$

$$W = m[\log(x+iy+1) + \log(x+iy-1) - \log(x+iy)]$$

$$\phi + i\psi = m\left[\frac{1}{2}\log\left((x+1)^2+y^2\right) + \tan^{-1}\left(\frac{y}{x+1}\right) + \frac{1}{2}\left((x-1)^2+y^2\right) + i\tan^{-1}\frac{y}{x^{-1}} - \frac{1}{2}\log\left(x^2+y^2\right) - i\tan^{-1}\frac{y}{x}\right]$$

$$\therefore\ \psi = \mathrm{m}\left[\tan^{-1}\frac{y}{x+1} + \tan^{-1}\frac{y}{x-1} - \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\psi = \mathrm{m}\left[\tan^{-1}\left\{\frac{\frac{y}{x+1}+\frac{y}{x-1}}{1-\frac{y^2}{x^2-1}}\right\} - \tan^{-1}\frac{y}{x}\right]$$

$$\psi = \text{m}\left[\tan^{-1}\left\{\frac{2yx}{x^2 - y^2 - 1}\right\} - \tan^{-1}\frac{y}{x}\right]$$

$$\psi = \text{m}\ \tan^{-1}\left\{\frac{\dfrac{2yx}{x^2 - y^2 - 1} - \dfrac{y}{x}}{1 + \dfrac{2xy^2}{x\left(x^2 - y^2 - 1\right)}}\right\}$$

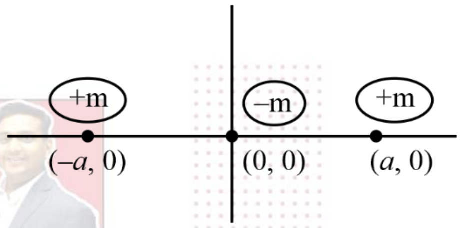
for streamlines

ψ = constant

$$\therefore\ \tan^{-1}\left\{\frac{2x^2y - x^2y + y^2 + y}{x^3 - xy^2 - x + 2xyz}\right\} = c$$

$$\Rightarrow \frac{x^2y + y + y^3}{x^3 - x + xy^2} = c$$

Ex.3. Find the stream fn. of 2-D motion due to two equal sources & an equal sink situated midway between them.



$W = -m\log(z + a) - m\log(z - a) + m\log z$

$$W = -m\left[\frac{1}{2}\log\left((x+a)^2 + y^2\right) + i\tan^{-1}\frac{y}{x+a} + \frac{1}{2}\log\left((x-a)^2 + y^2\right) + i\tan^{-1}\frac{x}{x-a} - \frac{1}{2}\log\left(x^2 + y^2\right) - i\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$\because\ W = \phi + i\psi$;

$$\text{So, } \psi = -m\left[\tan^{-1}\left(\frac{y}{x-a}\right) + \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x}\right)\right]$$

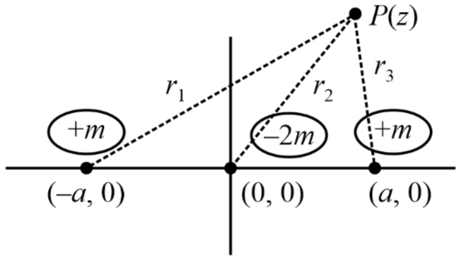
Ex.4. Two source each of strength m are placed at the point $(-a, 0)$ & $(a, 0)$ & a sink of strength 2m at th origin. Show that streamlines are the curves.

$(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a parameter

Show also that the fluid speed at any point is $\dfrac{2ma^2}{r_1 \cdot r_2 \cdot r_3}$; where r_1, r_2, r_3 are the distance of the point from the sources & sinks ?

Complex potential due to sources & sinks at an arbitrary points P;

$W = -m\log(z+a) - m\log(z-a) + 2m\log z \;\ldots(A)$



$$\psi = -m\tan^{-1}\left(\frac{y}{x+a}\right) - m\tan^{-1}\left(\frac{y}{x-a}\right) + 2m\ \tan^{-1}\left(\frac{y}{x}\right)$$

for streamlines

$$\therefore\quad \tan^{-1}\left(\frac{y}{x+a}\right) + \tan^{-1}\left(\frac{y}{x-a}\right) - 2\ \tan^{-1}\left(\frac{y}{x}\right) = \text{c}$$

$$\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1} + \tan^{-1}\left(\frac{y}{x-a}\right) - \tan^{-1}\left(\frac{y}{x}\right) = \text{e}$$

$$\tan^{-1}\left\{\frac{\frac{y}{x+a}-\frac{y}{x}}{1+\frac{y^2}{x(x+a)}}\right\} + \tan^{-1}\left\{\frac{\frac{y}{x-a}-\frac{y}{x}}{1+\frac{y^2}{x^2-ax}}\right\} = c$$

$$\tan^{-1}\left\{\frac{-ay}{x^2+y^2+ax}\right\} + \tan^{-1}\left\{\frac{ay}{x^2+y^2-ax}\right\} = \text{c}$$

$$\tan^{-1}\left\{\frac{\frac{ay}{x^2+y^2-ax}-\frac{ay}{x^2+y^2+ax}}{1+\frac{a^2y^2}{(x^2+y^2-ax)(x^2+y^2+ax)}}\right\} = c$$

$$\tan^{-1}\left\{\frac{ay\{x^2+y^2+ax-x^2-y^2+ax\}}{(x^2+y^2+ax)(x^2+y^2+ax)+a^2y^2}\right\} = c$$

$$\frac{2a^2xy}{(x^2+y^2)^2-a^2x^2+a^2y^2} = \tan c$$

$$\frac{(x^2+y^2)^2+a^2(y^2-x^2)}{a^2xy} = 2\cot c = \lambda \text{ (say)}$$

$(x^2+y^2)^2 + a^2(y^2-x^2) = \lambda\, a^2 + xy$

$(x^2+y^2)^2 = a^2\{x^2-y^2+\lambda xy\}$ pd

For speed,

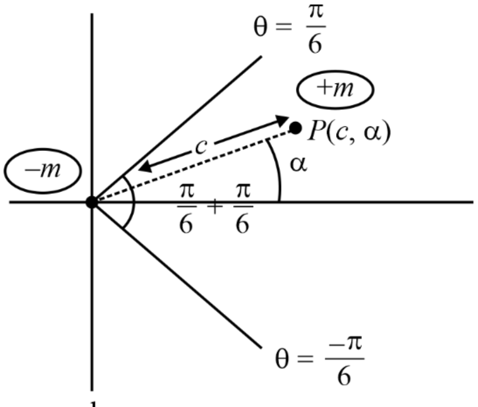
$$\left|\frac{dw}{dz}\right| = \left|\frac{-m}{z+a} - \frac{m}{z-a} + \frac{2m}{z}\right|$$

$$= \frac{2a^2m}{|z||z-a||z+a|}$$

$$\left|\frac{dw}{dz}\right| = \frac{2ma^2}{r_1 \cdot r_2 \cdot r_3}$$

Ex.5. Between the fixed boundaries $\theta = \frac{\pi}{c}$ & $\theta = \frac{-\pi}{6}$, there is a 2-D motion due to a source at the point ($r = c$, $\theta = \alpha$) and a sink at the origin absolving water at the same rate as source produces. Find the stream function and show that one of streamlines is a part of curve $r^3 \sin 3\alpha = c^3 \sin 3\theta$

Step (1) : configuration through figure



z-plane

Step (ii):-

$\boxed{\xi = z^3}$ $\qquad \because$ n = $\dfrac{\frac{\pi}{2}}{\frac{\pi}{6}} = 3$

$$R \cdot e^{i\theta_2} = \left(r \cdot e^{i\theta_1}\right)^3$$

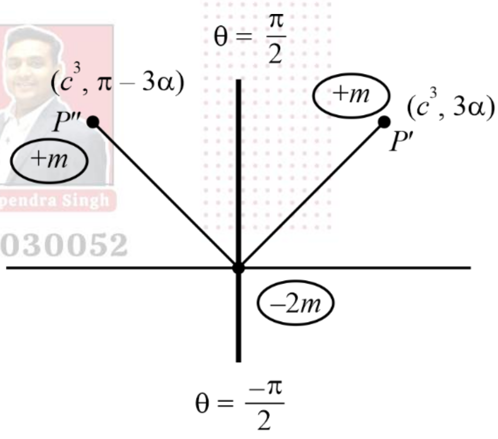
$$\Rightarrow \; Re^{i\theta_2} = r^3 e^{i3\theta_1}$$

$\Rightarrow R = r^3$ & $\theta_2 = 3\theta_1$

Now, boundaries of θ_2:

$-3 \times \frac{\pi}{6}$ to $3 \times \frac{\pi}{6}$

i.e., θ_2: $\frac{-\pi}{2}$ to $\frac{\pi}{2}$



(ξ-plane)

$\therefore$ Image of source P' is P'' ; image of sink at origin is sink at origin itself.

Step (iii): ultimately, complex potential (in ξ plane)

$W = -m \log(\xi - z_1) - m \log(\xi - z_2) + 2m \log(\xi - z_3)$

$W = -m \log(\xi - c^3 e^{i3\alpha}) - m \log(\xi - r.e^{i3(\pi - \alpha)}) + 2\, m \log(\xi)$

$\therefore W = -m \log(\xi - c^3 e^{i3\alpha}) - m \log(\xi + re^{-i3\alpha}) + 2m\log(\xi)$

$\because e^{i3\pi} = cos3\pi + i\, sin\, 3\pi$

$$= -\, m\log\left[\left(z^3 - c^3 e^{i3\alpha}\right)\cdot\left(z^3 + c^3 e^{-i3\alpha}\right)\right] + 2m\log z^3$$

$$W = -m\log\left\{\frac{z^6 - c^6 - 2ic^3 z^3 \sin 3\alpha}{z^6}\right\}$$

$W = -m\log\{1 - c^6 z^{-6} - 2ic^3 z^{-3} \sin 3\alpha\}$

$\phi + i\psi = W = -m\log(1 - c^6 r^{-6} e^{-6i\theta} - 2ic^3 r^{-3}.e^{-3i\theta} \sin 3\alpha)$

On comparing imaginary part.

$$\psi = -m \tan^{-1}\left\{\frac{c^6 r^{-6} \sin 6\theta - 2c^3 r^{-3} \sin 3\alpha \cos 3\theta}{1 - c^6 r^{-6} \cos 6\theta - 2c^3 r^{-3} \sin 3\alpha \sin 3\theta}\right\} \text{ is the req. streamline}$$

the streamlines are given by ψ = constant

So, corresponding to $\psi = 0$, we get streamlines as

$c^6 r^{-6} \sin 6\theta - 2c^3 r^{-3} \sin 3\alpha \cos 3\theta = 0$

$2c^3 \sin 3\theta \cos 3\theta = 2r^3 \sin 3\alpha \cos 3\theta$

$c^3 \sin 3\theta = r^3 \sin 3\alpha$

Ex.6. Between the fixed boundaries $\theta = \frac{\pi}{4}$ & $\theta = \frac{-\pi}{4}$ there is a 3-D motion due to a source of strength m at the point ($r = a$, $\theta = 0$) and an equal sink at ($r = b$, $\theta = 0$). Show that the stream function is

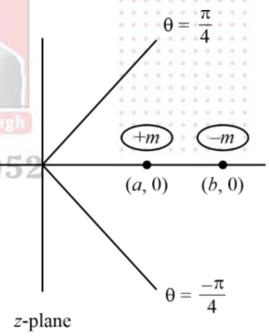
$$-m \tan^{-1} \frac{r^4\left(a^4 - b^4\right)\sin 4\theta}{r^8 - r^4\left(a^4 + b^4\right)\cos 4\theta + a^4 b^4}$$

and show that velocity at (r, θ) is (speed)

$$\frac{4m\left(a^4 - b^4\right)r^3}{\left(r^8 - 2a^4 r^4 \cos 4\theta + a^8\right)^{1/2}\left(r^8 - 2b^4 r^4 \cos 4\theta + b^8\right)^{1/2}}$$

The conformal transformation $\xi = z^n$

where $n = \dfrac{\frac{\pi}{2}}{\frac{\pi}{4}} = 2$



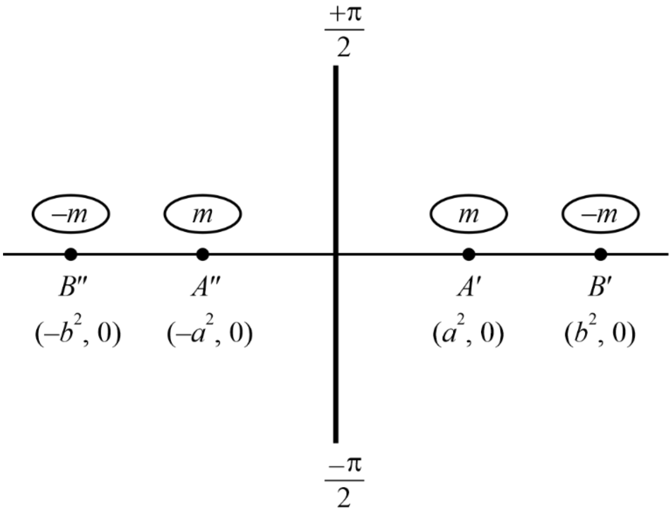
we take $\xi = z^2$

$$Re^{i\theta_2} = \left(re^{i\theta_1}\right)^2$$

$\therefore\ R = r^2$ & $\theta_2 = 2\theta_1$

Now, the boundaries of θ_2 : $2 \times \frac{-\pi}{4}$ to $2 \times \frac{\pi}{4}$

i.e. $\frac{-\pi}{2}$ to $\frac{\pi}{2}$



ξ-plane

Now, complex potential of ξ-plane,

$W = -m\log(\xi - a^2) + m\log(\xi - b^2) - m\log(\xi + a^2) + m\log(\xi + b^2)$

$W = -m\log(\xi^2 - a^4) + m\log(\xi^2 - b^4)$

$W = -m(z^4 - a^4) + m\log(z^4 - b^4)$ …(A)

$W = -m\log(r^4 e^{4i\theta} - a^4) + m\log(r^4 e^{i4\theta} - b^4)$

$\phi + i\psi = -m\log(r^4 \cos 4\theta - a^4 + ir^4 \sin 4\theta) + m\log(r^4 \cos 4\theta - b^4 + ir^4 \sin 4)$

$\log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

Comparing the imaginary part,

$$\psi = -m\left[\tan^{-1}\left\{\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - a^4}\right\} - \tan^{-1}\left\{\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - b^4}\right\}\right]$$

$$\psi = -m\left[\tan^{-1}\left\{\frac{\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - a^4} - \frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - b^4}}{1 + \frac{r^8 \sin^2 4\theta}{(r^4 \cos 4\theta - a^4)(r^4 \cos 4\theta - b^4)}}\right\}\right]$$

$$\psi = -m\left[\tan^{-1}\left\{\frac{r^4(a^4 - b^4 \sin 4\theta)}{r^8 \cos^2 4\theta - r^4 \cos 4\theta(a^4 + b^4) + a^4 b^4 + r^8 \sin^2 4\theta}\right\}\right]$$

$$\psi = -m\tan^{-1}\left\{\frac{r^4(a^4 - b^4)\sin\theta}{r^8 - r^4(a^4 + b^4)\cos 4\theta + a^4 b^4}\right\}$$

From (A)

$$\frac{dw}{dz} = -\mathrm{m} \times \frac{4z^3}{z^4 - a^4} + \mathrm{m} \times \frac{4z^3}{z^4 - b^4}$$

$$\frac{dw}{dz} = -4z^3 m\left[\frac{z^4 - b^4 - z^4 + a^4}{(z^4 - a^4)(z^4 - b^4)}\right]$$

$$\frac{dw}{dz} = \frac{-4mr^3(\cos 3\theta + i\sin 3\theta)(a^4 - b^4)}{(r^4(\cos 4\theta + i\sin 4\theta) - a^4)(r^4(\cos 4\theta + i\sin 4\theta) - b^4)}$$

$$\frac{dw}{dz} = \frac{-4mr^3(a^4 - b^4)(\cos 3\theta + i\sin 3\theta)}{(r^4 \cos 4\theta - a^4 i\sin 4\theta)(r^4 \cos 4\theta - b^4 + ir^4 \sin 4\theta)}$$

$$\left|\frac{dw}{dz}\right| = \frac{\sqrt{\left(-4mr^3\left(a^4-b^4\right)\cos 3\theta\right)^2+\left(-4mr^3\left(a^4-b^4\sin 3\theta\right)^2\right)}}{\sqrt{\left(r^4\cos 4\theta-a^4\right)^2+\left(r^4\sin 4\theta\right)^2}\sqrt{\left(r^4\cos 4\theta-b^4\right)^2+\left(r^4\sin 4\theta\right)^2}}$$

$$\left|\frac{dw}{dz}\right| = \frac{4mr^3\left(a^4-b^4\right)}{\left(r^8-2a^4r^4\cos 4\theta+a^8\right)^{1/2}\left(r^8-2b^4r^4\cos 4\theta+b^8\right)^{1/2}}$$

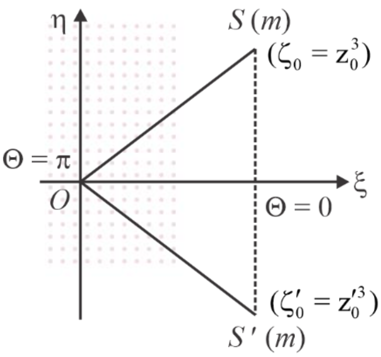
Level-2 of Preparation (Mentor's words: Interested students can go for this exercise once they're done with above concepts and examples)

Q.1. Use the method of image to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \pi/3$, the solution is

$\phi + i\psi = -m \log \left\{\left(z^3 - z_0^3\right)\left(z^3 - z_0'^3\right)\right\}$ where $z_0 = x_0 + iy_0$ and $z_0' = x_0 - iy_0$.

Sol. Consider the following transformation from z-plane (xy-plane) to ζ-plane ($\xi\eta$-plane) :

$\zeta = z^3$ where $z = re^{i\theta}$ $\Rightarrow$ $R = r^3$ and $\Theta = 3\theta$.



Hence the boundaries $\theta = 0$ and $\theta = \pi/3$ in z-plane transform to $\Theta = 0$ and $\Theta = \pi$ i.e., real axis in ζ-plane. The point z_0 in z-plane transforms to point ζ_0 in z-plane such that $\zeta_0 = z_0^3$. Hence the image system with respect to real axis in ζ-plane consists of

(i) a source m at $\zeta_0 = z_0^3$ (ii) a source m at $\zeta_0' = z_0'^3$

Hence, $w = -m \log(\zeta - \zeta_0) - m \log(\zeta - \zeta_0')$

or $w = -m \log(z_3 - z_0^3) - m \log(z^3 - z_0'^3)$

or $\phi + i\psi = -m \log\{(z^3 - z_0^3)(z3 - z_0'^3)\}$.

Q.2. If fluid fills the region of space on the positive side of the x-axis, which is a boundary and if there be a source m at the point (0, a) and an equal sink at (0, b) and if pressure on the negative side be the same as the pressure at infinity, show that the pressure on the boundary is $\pi\rho m^2(a-b)2/2ab(a+b)$, where ρ is the density of the fluid.

Sol. Here the image system with respect to x-axis in z-plane consists of

(i) a source m at $(0, a)$ i.e., at $z = ai$

(ii) a sink – m at $(0, b)$ i.e., at $z = bi$

(iii) a source m at $(0, -a)$ i.e., at $z = -ai$

(iv) a sink –m at $(0, -b)$ i.e., at $z = -b$

Clearly this image system does away with boundary $y = 0$ (i.e., x-axis).

Thus, the complete potential of this entire system is given by

$\therefore$ $w = -m \log(z - ai) + m \log(z - bi) - m \log(z + ai) + \log(z + bi)$

or $w = -m \log(z^2 + a^2) + m \log(z^2 + b^2)$

$\therefore$ velocity $= \left|\dfrac{dw}{dz}\right| = \left|-\dfrac{2zm}{z^2+a^2}+\dfrac{2zm}{z^2+b^2}\right|$

The velocity q at a point on the boundary (i.e., $y = 0$) is given by (setting $z = x + iy$)

$$q = \left|-\frac{2zm}{x^2+a^2}+\frac{2zm}{z^2+b^2}\right| = \frac{2xm\left(a^2-b^2\right)}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$$

Let p_0 be the pressure at infinity. Then by Bernoulli's theorem, the pressure p at any point given by

$$\frac{1}{2}q^2+\frac{p}{\rho} = \frac{1}{2}\times 0^2+\frac{p^2}{\rho} \qquad \text{or} \qquad \frac{p_0-p}{\rho} = \frac{1}{2}q^2.$$

$\therefore$ The resultant pressure on the boundary

$$= \int_0^{\infty}\left(p_0-p\right)dx = \frac{1}{2}\rho\int_0^{\infty}q^2dx = 2\rho\, m^2\int_0^{\infty}\frac{x^2\left(a^2-b^2\right)^2}{\left(x^2+a^2\right)^2\left(x^2+b^2\right)^2}dx, \text{ by (1) and (2)}$$

$$= 2\rho m^2\int_0^{\infty}\left[-\frac{a^2+b^2}{a^2-b^2}\left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)-\frac{a^2}{\left(x^2+a^2\right)^2}-\frac{b^2}{\left(x^2+b^2\right)^2}\right]dx$$

$$= 2\rho m^2\left\{\frac{a^2+b^2}{b^2-a^2}\left(\frac{\pi}{2a}-\frac{\pi}{2b}\right)-\frac{\pi}{4a}-\frac{\pi}{4b}\right\}, \text{ on simplification}$$

$$= \frac{\pi\rho m^2}{2ab}\left[\frac{2\left(a^2+b^2\right)-\left(a+b\right)^2}{\left(a+b\right)}\right] = \frac{\pi\rho m^2\left(a-b\right)^2}{2ab\left(a+b\right)}.$$

Q.3. Parallel line sources (perpendicular to *xy*-plane) of equal strength m are parallel to the points $z = nia$ where $n = \ldots, -2, -1, 0, 1, 2, \ldots$. Prove that the complex potential is $w = -m \log \sinh(\pi z/a)$. Hence, show that the complex potential for two dimensional doublets (lines doublets), with their axes parallel to the *x*-axis, of strength μ at the same points is given by $w = \mu \coth(\pi/a)$.

Sol. The complex potential due to sources of strength m situated at the points $z = 0, ia, -ia, 2ia, -2ia, \ldots$ is given by

$w = -m\log(z-0) - m\log(z-ia) - m\log(z+ia) - m\log(z-2ia) - m\log(z+2ia) - \ldots$

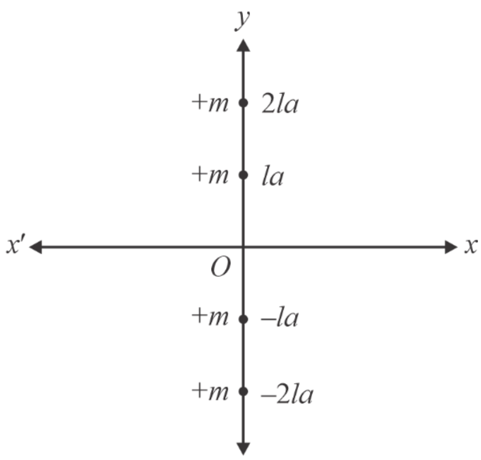
$= -m\log z - m\{(z-ia)(z+ia)\} - m\log\{z-2ia)(z+2i)\} - ..,$

$= -m\log z - m\log(z^2+a^2) - m\log(z^2+2^2a^2) - \ldots$

$= -m\log[z(z^2+a^2)(z^2+2^2a^2)(z^2+3^2a^2)\ldots]$

$$= -m\log\left\{\frac{\pi}{a}z\left(1+\frac{z^2}{a^2}\right)\left(1+\frac{z^2}{2^2a^2}\right)\left(1+\frac{z^2}{3^2a^2}\right)\cdots\right\}.$$

$$-m\log\left[\left(\frac{a}{\pi}\right)a^2\left(2^2a^2\right)\left(3^2a^2\right)\ldots\right]$$



$w = -m\sinh(\pi z/a) + \text{constant}.$

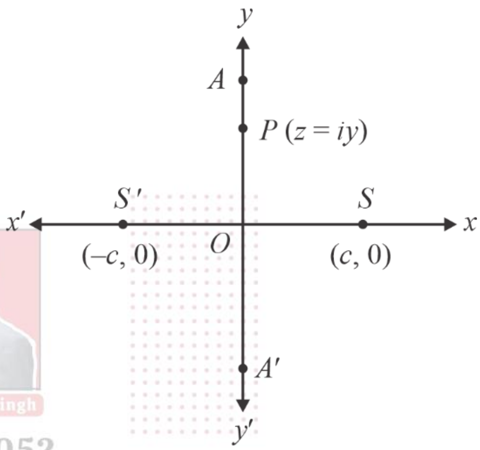
The complex potential w_1 for the doublets at the same point is

$$w_1 = -\frac{\partial w}{\partial z} = \frac{m\pi}{a}\coth\left(\frac{\pi z}{a}\right) = \mu\coth\left(\frac{\pi z}{a}\right), \text{ where } \quad \mu = \frac{m\pi}{a}.$$

Q.4. In the case of the motion of liquid in a part of a plane bounded by a straight line due to a source in the plane, prove that if $m\,\rho$ is the mass of fluid (of density ρ) generated at the source per unit of time the pressure on the length $2l$ of the boundary immediately opposite to the source is less than that on an equal length at a great distance by

$$\frac{1}{\rho}\frac{m^2\rho}{\pi^2}\left[\frac{1}{c}\tan^{-1}\frac{l}{c} - \frac{l}{l^2+c^2}\right],$$ where c is the distance of source to the boundary.

Sol. Let y-axis be the bounding line and let the given source of strength (μ, say) be situated at S where $OS = c$. Now, by the definition of strength μ of the source, we have $2\pi\mu\rho = m\rho$ so that $\mu = m/2\pi$. Now, the image system consists of



(i) a source of strength $m/2\pi$ at $S(c,0)$

(ii) a source of strength $m/2\pi$ at $S'\ (-c,0)$

Here S' is image of S such that $OS = OS' = c$.

The complex potential w is given by

$w = -(m/2\pi)\log(z-c) - (m/2\pi)\log(z+c) = -(m/2\pi)\log(z^2-c^2)$

The velocity is given by

$$\left|\frac{dw}{dz}\right| = \left|-\frac{m}{2\pi}\cdot\frac{2z}{z^2-c^2}\right| = \frac{m}{\pi}\left|\frac{z}{z^2-c^2}\right|.$$

Hence velocity q at any point P (where $z = iy$) is given by

$$q = \frac{m}{\pi}\left|\frac{iy}{-y^2-c^2}\right| = \frac{my}{\pi\left(y^2+c^2\right)}. \qquad \ldots(1)$$

Bernoulli's equation for steady motion is given by

$p/\rho + q^2/2 = \text{constant} = c$, say. ...(2)

Let p_0 be the pressure on y-axis at great distance from O so that $p = p_0$ and $q = 0$ when $y = \infty$.

Then (2) reduces to $p_0/\rho = c$ and hence (2) becomes

$$\frac{p}{\rho} + \frac{q^2}{2} = \frac{p_0}{\rho} \quad \text{or} \quad \frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

Writing $w = \phi + i\psi$ and equating real parts, we get

$\phi = -(m/2) \times [\log \{(r\cos\theta - f)^2 + (r\sin\theta)^2\} + \log \{(r\cos\theta - a^2/f)^2 + (r\sin\theta)^2\}]$

$$-\frac{m}{2}\left[\log\left(r^2 + f^2 - 2frc\text{os}\theta\right) + \log\left(r^2 + \frac{a^4}{f^2} - \frac{2ra^2}{f}\cos\theta\right)\right]$$

$$\therefore \frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2(r - f\cos\theta)}{r^2 + f^2 - 2fr\cos\theta} + \frac{2\{r - (a^2/f)\cos\theta\}}{r^2 + a^4/f^2 - 2r(a^2/f)\cos\theta}\right]$$

Hence normal velocity at any point Q on the circle

$$= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = m\left[\frac{a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \frac{(a/f)(f - a\cos\theta)}{(a^2/f^2)(f^2 + a^2 - 2af\cos\theta)}\right] =$$

$$m\left[\frac{a - f\cos\theta + f^2/a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \right] = \frac{m}{a}.$$

Now, if we place a source of strength $-m$ at O, the normal velocity due to it at Q will be $-(m/a)$ and hence the normal velocity of the system will reduce to zero.

Hence the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

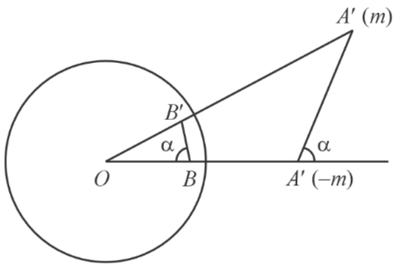
Image of a doublet with regard to a circle.

Let us determine the image of a doublet AA' with its axis making an angle α with OA, outside the circle, there being a sink $-m$ at A and a source m at A'. Join OA and OA'. Let B and B' be the inverse points of A and A' with regard to the circle with O as centre.

Then

$$OA \cdot OB = OA' \cdot OB' = a^2, \qquad \ldots(1)$$

where a is the radius of the circle.



Now the image of source m at A' consists of a source m at B' and and a sink $-m$ at O. Similarly, the image of sink $-m$ at A consists of a sink at B and a source m at O. Compounding these, we see that source m and sink $-m$ at O cancel each other and hence the image of the given doublet AA' is another doublet BB'.

Let the strength of the given doublet AA' be μ.

Then $$\mu = \lim_{A \to A'}(m \cdot AA'). \qquad \ldots(2)$$

From (1) $OA/OA' = OB'/OB$, …(3)
showing that triangles OAA' and $OB'B$ are similar. From these similar triangles, we have

$$\frac{BB'}{AA'} = \frac{OB'}{OA} = \frac{OB'}{OA} \cdot \frac{OA'}{OA'} = \frac{a^2}{OA \cdot OA'} \quad \ldots(4)$$

$\therefore$ μ'= strength of doublet $B'B$ = $\lim_{B' \to B}(m \cdot B'B) = \lim_{A \to A'} \frac{a^2}{OA \cdot OA'} \cdot (m \cdot AA')$, by (4)

$= \mu a^2/f^2$, using (2) and taking $OA = OA' = f$

Thus the image of a two-dimensional doublet at A with regard to a circle is another doublet an the inverse point B, the axes of the doublets making supplementary angles with the radius OBA.

To determine image system for a source outside a circle (or a circular cylinder) of radius *a* with help of the circle theorem.

Let OA = f. Suppose there is a source of strength m at A when $z = f$, outside the circle of radius a whose centre is at O. When the source is alone in the fluid complex potential at a point $P(z)$ is given by

$f(z) = -m\log(z - f)$ *Then* $\bar{f}(z) = -m\log(z - f)$

$\therefore$ $\bar{f}(a^2/z) = -m\log(a^2/z - f)$

When the circle of section $|z| = a$ is introduced, then the complex potential in the region $|z| \geq a$ is given by $w = f(z) + \bar{f}(a^2/z) = -m\log(z - f) - m\log(a^2/z - f)$

$$= -m\log(z - f) - m\log\left(\frac{a^2 - zf}{z}\right)$$

$= -m\log(z - f) - m\log(a^2 - zf) + m\log z$

$= -m\log(z - f) - m\log[(-f)(z - a^2/f] + m\log z$

$= -m\log(z - f) - m\log(z - a^2/f) + m\log z - m\log(-f))$

$\therefore$ $w = -\log(z - f) - m\log(z - a^2/f) + m\log z + \text{constant}$, …(1)

the constant (real or complex, $-m\log(-f)$) being immaterial from the view point of analysing the flow. (1) shows that w is the complex potential of

(i) a source m at " $A, z = f$

(ii) a source m at B, $z = a^2/f$

(iii) a sink $-m$ at the origin

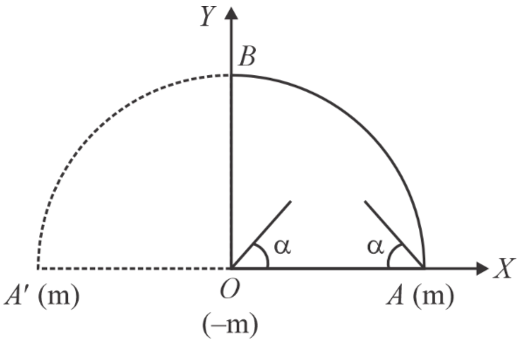
Since $OA \cdot OB = a^2$, A and B are the inverse points with respect to the circle $|z| = a$ and so B is inside the circle.

Thus the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

Q.7.(i) In the region bounded by a fixed quadrantal arc and its radii, deduce the motion

due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle α with the radius is $r^2 \sin(\alpha + \theta) = a^2 \sin(\alpha - \theta)$.

(ii) In a region bounded by a fixed quadrant arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle π/6 with radius is $r^2 \sin(\pi/6 + \theta) = a^2 \sin(\pi/6 - \theta)$, where a is radius of the quadrant.



Sol. (i). Let AOB be the circular quadrant of radius a with OA and OB as bounding radii. Consider a source of strength m at A and a sink of strength $-m$ at O. Then the image system consists of (i) a source m at $A(a, 0)$

(ii) a source m at $A'(-a, 0)$

(iii) a sink $-m$ at $O(0, 0)$.

Hence the complex potential w for the motion of the fluid at any point $P(z + x + iy + re^{i\theta})$ is given by

$$w = -\text{m}\log(z-a) - \text{m}\log(z+a) + \text{m}\log z = -\text{m}\log\frac{z^2-a^2}{z} = -\text{m}\log(z - a^2 z^{-1})$$

$$w = -m\log(z-a) - m\log(z+a) + m\log z = -m\log\frac{z^2-a^2}{z} = -m\log(z - a^2 z^{-1})$$

or $\quad w = -m\log(re^{i\theta} - a^2 r^{-1} e^{-i\theta})$, as $z = re^{i\theta}$

$w = -m\log[r(\cos\theta + i\sin\theta) - a^2 r^{-1}(\cos\theta - i\sin\theta)]$

$\phi + i\psi = -m\log[(r - a^2/r)\cos\theta + i(r + a^2/r)\sin\theta]$

Equating imaginary parts, we obtain

$$\psi = -\text{m}\tan^{-1}\frac{(r + a^2/r)\sin\theta}{(r - a^2/r)\cos\theta} = -m\tan^{-1}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right\}$$

The streamline leaving the end A and O at an angle α is given by

$$\psi = -\text{m}(\pi - \alpha) \qquad \text{i.e.,} \qquad -\text{m}\tan^{-1}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right\} = -\text{m}(\pi - \text{a})$$

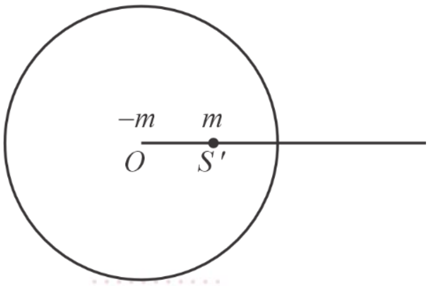
or $$\frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta} = \tan(\pi-\alpha) = -\tan\alpha = -\frac{\sin\alpha}{\cos\alpha}$$

or $(r^2 + a^2)\sin\theta\cos\alpha = -(r^2 - a^2)\cos\theta\sin\alpha$ or $r^2\sin(\alpha+\theta) = a^2\sin(\pi-\alpha)$

(ii) Proceed as above by taking $\alpha = \pi/6$.

Q.8. In the case of the two-dimensional fluid motion produced by a source of placed at a point S outside a rigid circular disc of radius a whose centre is O, show velocity of slip of the fluid in contact with the disc is greatest at the points where joining S to the ends of the diameter at right angles to OS meet the circle, prove that its at these points is $(2m \times OS)/(OS^2 - a^2)$

Sol. Let S' be the inverse point of S with respect to the circular disc, with O as its



Let $OS = c$. Then $OS \times OS' = a^2$ so that $OS' = a^2/c$.

The equivalent image system consists of

(i) a source of strength m at S $(c, 0)$,

(ii) a source of strength m at S' $(a^2/c, 0)$,

(iii) a sink of strength $-m$ at $O(0, 0)$.

Let OS be taken as x-axis. Then the complex potential for the motion of the fluid point $z(= x + iy = re^{i\theta})$ is given by

$$w = -m\log(z-c) - m\log(z-a^2/c) + m\log z$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z-a^2/c} + \frac{m}{z}$$

Let $q\ (= |dw/dz|)$ be the velocity at any point z. Then

$$q = m\left|\frac{1}{z-c} + \frac{1}{z-a^2/c} - \frac{1}{z}\right| = m\left|\frac{(z-a)(z+a)}{z(z-c)(z-a^2/c)}\right|$$

Hence the velocity at any point $z = ae^{i\theta}$ on the boundary of the circular disc is give

$$q = m\left|\frac{(ae^{i\theta}-a)(ae^{i\theta}+a)}{ae^{i\theta}(ae^{i\theta}-c)(ae^{i\theta}-a^2/c)}\right| = m\left|\frac{c(e^{i\theta}-1)(e^{i\theta}+1)}{e^{i\theta}(ae^{i\theta}-c)(ce^{i\theta}-a)}\right|$$

$$q = mc\left|\frac{(1-e^{-i\theta})(1+e^{i\theta})}{(ae^{i\theta}-c)(ce^{i\theta}-a)}\right| = \frac{2mc\sin\theta}{a^2+c^2-2ac\cos\theta}$$

For maximum q, $dq/d\theta = 0$. Hence (1) gives

$$2mc\,\frac{\left(a^2+c^2-2ac\cos\theta\right)\cos\theta-\sin\theta\left(2ac\sin\theta\right)}{\left(a^2+c^2-2ac\cos\theta\right)^2}=0$$

or $\quad (a^2+c^2)\cos\theta-2ac=0 \qquad$ *or* $\qquad \cos\theta=(2ac)/(a^2+c^2)$

Since $\theta = 0$ gives the minimum velocity [q becomes zero at $\theta = 0$ by (1)], the value given by (2) must correspond to the maximum value of velocity q. Moreover (2) gives the angles which the diameter through the point where the line joining S to the end of the

From (2), $\sin\theta = \sqrt{1-\cos^2\theta} = (c^2-a^2)/(c^2+a^2)$ …(3)

Using (1), (2) and (3), the maximum value of q is given by

$$q=\frac{2mc\cdot\left(\dfrac{c^2-a^2}{c^2+a^2}\right)}{a^2+c^2-\dfrac{4a^2c^2}{a^2+c^2}}=\frac{2mc\left(c^2-a^2\right)}{\left(a^2+c^2\right)^2-4a^2c^2}\quad\text{or}\quad q=\frac{2mc}{c^2-a^2}=\frac{2m\cdot OS}{OS^2-a^2}$$

Since the boundary of the circular disc is a streamline, the velocity on the boundary is the velocity of the slip.

Q.9. A source S and a sink T of equal strengths m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distances from O on opposite sides of it and on the same diameter AOB, show that the velocity of the liquid at any point P is

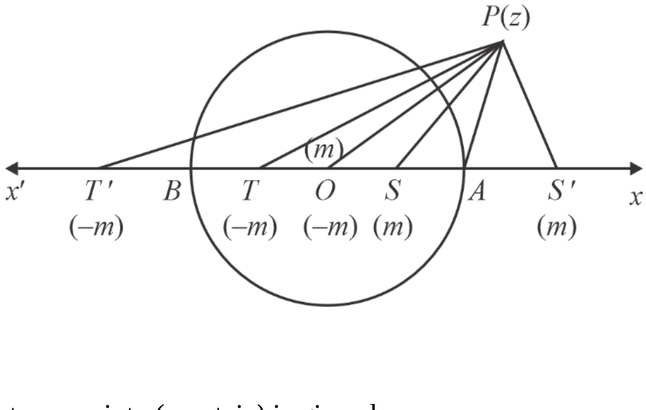
$$2m\,\frac{OS^2+OA^2}{OS}\,\frac{PA\cdot PB}{PS\cdot PS'\cdot PT\cdot PT'},$$

where S' and T' are the inverses of S and T with respect to the circle.

Sol. Let $OS = OT = c$. Then, we have $OA = a$, $OS\cdot OS' = a^2$ and $OT\cdot OT' = a^2$ so that

$(OS' = a^2/c$ and $OT' = a^2/c$ …(1)

Now the image system of source m at S consists of a source m at S' and a sink $-m$ at O. Again the image system of sink $-m$ at T consists of a sink $-m$ at T' and a source m at O. Compounding these, we find that source m and sink $-m$ at O cancel each other. Hence the equivalent image system finally consists of



(i) a source of strength m at $S(c, 0)$

(ii) a source of strength m at S' $(a^2/c, 0)$

(iii) a sink of strength $-m$ at $T(-c,0)$

(iv) a sink of strength $-m$ at T' $(-a^2/c,0)$

Taking OS as the x-axis, the complex potential at any point $z(= x + iy)$ is given by

$$w = -m \log(z-c) - m\log\left(z - \frac{a^2}{c}\right) + m\log(z+c) + m\log\left(z + \frac{a^2}{c}\right)$$

$$\frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z-a^2/c} + \frac{m}{z+c} + \frac{m}{z+a^2/c}$$

The velocity q (=$|dw/dz|$) at any point is given by

$$q = m\left|-\frac{2c}{z^2-c^2} - \frac{(2a^2/c)}{z^2-(a^4/c^2)}\right| = 2m\left|\frac{c(z^2-a^2)+(a^2/c)(z^2-a^2)}{(z^2-c^2)(z^2-a^4/c^2)}\right|$$

$$= 2m\,\frac{c^2+a^2}{c}\left|\frac{z^2-a^2}{(z^2-c^2)\left(z^2-\frac{a^4}{c^2}\right)}\right| = 2m\frac{c^2+a^2}{c}\,\frac{|z-a||z+a|}{|z-c||z+c|\left|z-\frac{a^2}{c}\right|\left|z+\frac{a^2}{c}\right|}$$

$$= 2m\,\frac{OS^2+OA^2}{OS}\cdot\frac{PA\cdot PB}{PS\cdot PS'\cdot PT\cdot PT'}$$

Q.10. In the part of an infinite plane bounded by a circular quadrant AB and the of the radii OA, OB, there is a two-dimensional motion due to the production of the and its absorption at B, at the uniform rate m. Find the velocity potential of the motion that the fluid which issues from A in the direction making an angle μ with OA follows whose polar equation is

$$r = a\sqrt{\sin 2\theta}\left[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}\right]^{1/2}$$

the positive sign being taken for all square roots.

Sol. The image system of source $m/2\pi$ at A with respect to the circular boundary of a source $m/2\pi$ at A (since A is the inverse point of itself) and a sink $-m/2\pi$ at O, the of the circle. Next, the image of system of the above mentioned image system with respect line OA and OB consists of

(i) a source of strength $m/2\pi + m/2\pi$ i.e. m/π *at* $A(a, 0)$

(ii) a source of strength $m/2\pi + m/2\pi$ i.e. m/π *at* A' $(-a,0)$

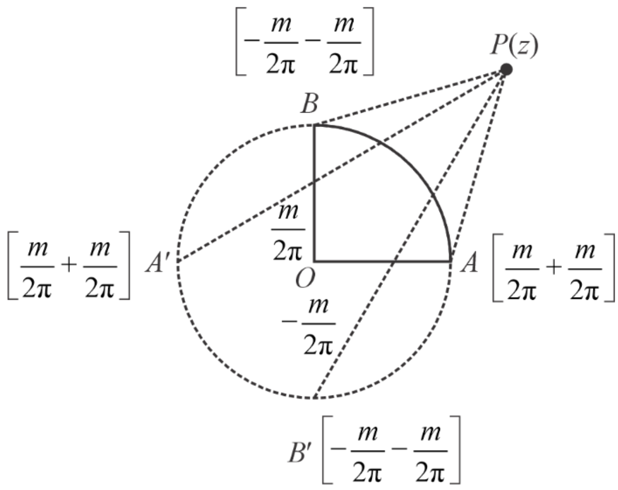
(iii) a sink of strength $-\dfrac{m}{2\pi}$ at $O(0, 0)$

Again there is a sink of strength $-m/2\pi$ at B. The image system of this sink with

to the circular boundary consists of a sink $-m/2\pi$ at B (since B is the inverse point of its

a source $m/2\pi$ and O. Again the image of the system of the above mentioned image system

respect to lines OA and OB as before consists of



(i) a sink of strength $-(m/2\pi) - (m/2\pi)$ *i.e.* $-(m/\pi)$ *at* $B(0, a)$

(ii) a sink of strength $-(m/2\pi) - (m/2\pi)$ *i.e.* $-(m/\pi)$ *at* $B'(0, -a)$

(iii) a source of strength $m/2\pi$ *at* O (0,0)

Compounding these we find that source $m/2\pi$ and sink $-m/2\pi$ at O cancel each

Taking OA as the x-axis, the complex potential at any point $P(z = x + iy = re^{i\theta})$ is given by

$$w = -\frac{m}{\pi}\log(z-a) - \frac{m}{\pi}\log(z+a) + \frac{m}{\pi}\log(z-ai) + \frac{m}{\pi}\log(z+ai)$$

$$\therefore \phi + i\psi = -\frac{m}{\pi}\log(z^2-a^2) + \frac{m}{\pi}\log(z^2+a^2)$$

Equating real parts, (1) gives

$$\phi = -\frac{m}{\pi}\log\left|z^2-a^2\right| + \frac{m}{\pi}\log\left|z^2-i^2a^2\right| = -\frac{m}{\pi}\{|z-a|\cdot|z+a|\} + \frac{m}{\log}\{|z-ia|z+\text{ia}|\}$$

$$\text{or } \phi = -\frac{m}{\pi}\log(AP\cdot A'P) + \frac{m}{\pi}\log(BP\cdot B'P) = \frac{m}{\pi}\log\frac{BP\cdot B'P}{AP\cdot A'P}$$

Putting $s = e^{i\theta}$ in (1) and equating imaginary parts, we get

$$\psi = -\frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} + \frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}$$

$$= -\frac{m}{\pi}\tan^{-1}\frac{\dfrac{r^2\sin 2\theta}{r^2\cos^2 2\theta - a^2} - \dfrac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}}{1+\dfrac{r^4\sin^2 2\theta}{r^4\cos^2 2\theta - a^4}} = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4-a^4}$$

The required streamline that leaves A at an inclination μ is given by $\psi = -(m/\pi)\mu$, i.e.,

$$-\frac{m}{\pi}\mu = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4-a^4} \quad \text{or} \quad r^4 - 2a^2r^2\sin 2\theta\cot\mu - a^4 = 0$$

$$r^2 = \left[2a^2\sin 2\theta\cot\mu + \sqrt{\left(4a^4\sin^2 2\theta\cot^2\mu + 4a^4\right)}\right]/2$$

wherein negative sign has been omitted because r^2 is non-negative quantity. Thus, we have

$$\mathrm{r} = a\sqrt{\sin 2\theta}\left[\cot\mu + \sqrt{\left(\cot^2\mu + \mathrm{cosec}^2 2\theta\right)}\right]^{1/2}.$$

Q.11. Prove that in the two-dimensional liquid motion due to any number of sources at points on a circle, the circle is a streamline provided that there is no boundary and that the algebraic sum of the strengths of sources is zero. Show that the same is true if the region of flow is bounded by a circle which cuts orthogonally the circle in question.

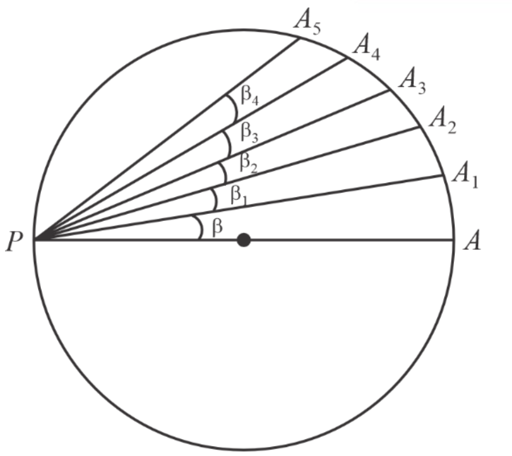
Sol. Let $A_1, A_2, A_3, \ldots$ be the positions of the sources of strengths $m_1, m_2, m_3, \ldots$ respectively. Let P be any point on the circle and let the diameter through P be taken as the initial line.

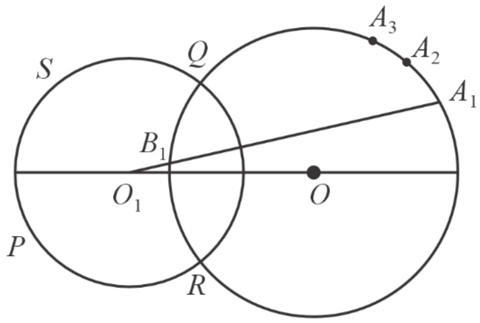
Let $\angle A_1PA = \delta, \angle A_2PA_1 = \beta_1, \angle A_3PA_2 = \beta_2$ and so on. Then the stream function ψ of the system is given by

$\psi = -m_1\delta - m_2(\delta + \beta_1) - m_3(\delta + \beta_1 + \beta_2) - \ldots$

$= -\delta(m_1 + m_2 + m_3 + \ldots) - [m_2\beta_1 + m_3(\beta_1 + \beta_2) + \ldots] = -\delta(m_1 + m_2 + m_3 + \ldots) - \text{constant},$

since $\beta_1, \beta_2, \beta_3, \ldots$ do not depend on the position of P. If we take $m_1 + m_2 + m_3 + \ldots = 0$, then $\psi =$ constant is a streamline i.e. the circle is a streamline.





Second Part. Let O_1 be the centre of a circle which cuts the above circle (with centre O) orthogonally. The image of m_1 at A is m_1 at B_1, the inverse point of A and a sink $-m_1$ at O_1. If the barriers are omitted, we see that the system reduces to a source 2 $(m_1 + m_2 + \ldots)$ on the

of the given circle and a sink $-(m_1 + m_2 + \ldots)$ at O_1. Since $m_1 + m_2 + \ldots = 0$, the result

Q.12. A line source is in the presence of an infinite plane on which is placed circular cylindrical boss, the direction of the source is parallel to the axis of the boss, is at a distance c from the plane and the axis of the boss, whose radius is a. Show

radius to the point on the boss at which the velocity is a maximum makes an angle

radius to the source, where $\theta = \cos^{-1}\dfrac{a^2+c^2}{\sqrt{\{2(a^4+c^4)\}}}$

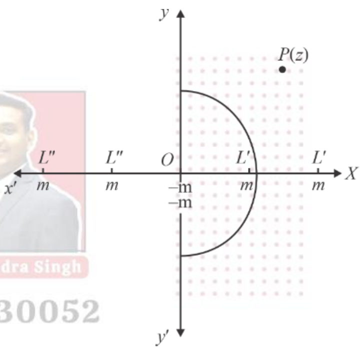
OR If the axis of y and the circle $x^2 + y^2 = a^2$ are fixed boundaries and there is a dimensional source at the point $(c, 0)$ where $c > a$, show that the radius drawn from,

the point on the circle, where the velocity is a maximum, makes with the axis of x an

$$\cos^{-1}\frac{a^2+c^2}{\sqrt{\{2(a^4+c^4)\}}}$$

When $c = 2a$, show that the required angle is $\cos^{-1}\left(5/\sqrt{34}\right)$.

Sol. Let there be a source of strength m at $L(c,0)$. Let L' be the inverse point of L with respect to the circular boundary so that $OL \times OL' = a^2$ i.e. $OL' = a^2/c$. The image of source m at L in the circular boundary (cylindrical boundary) is a source m at L' and a sink $-m$ at O.



For the above system the equivalent image system with respect to the y-axis (i.e. the line $x = 0$) consists of

(i) a source m at $L(c, 0)$ and $L''(-c, 0)$

(ii) a source m at $L'(a^2/c, 0)$ *and* $L''(-a^2/c, 0)$

(iii) a sink $-m - m$ i.e. $-2m$ at $O(0,0)$

Thus, if $P(z = x + iy = re^{i\theta})$ is any point in the fluid, the complex potential at P due above system is given by

$$w = -m\log(z-c) - m\log(z+c) - m\log(z - a^2/c) - m\log(z + a^2/c) + 2m\log z$$

or $\qquad w = 2m\log z - m\log(z^2 - c^2) - m\log(z^2 - a^4/c^2)$

$$\therefore \frac{dw}{dz} = \frac{2m}{z} - \frac{2mz}{z^2-c^2} - \frac{2mz}{z^2-a^4/c^2} \quad \text{or} \quad \frac{dw}{dz} = -\frac{2m\left(z^4-a^4\right)}{z\left(z^2-c^2\right)\left(z^2-a^4/c^2\right)}$$

The velocity $q(=|dw/dz|)$ at any point $P(z = ae)^{i\theta}$ on the circular boundary is given

$$q = \frac{2m\left|a^4e^{4i\theta}-1\right|}{\left|ae^{i\theta}\left(a^2e^{2i\theta}-c^2\right)\left(a^2e^{2i\theta}-a^4/c^2\right)\right|} \quad \text{or} \quad q = \frac{4mac^2\sin 2\theta}{a^4+c^4-2a^2c^2\cos 2\theta}$$

or $\quad (4mac^2/q) = (a^4 + c^4 - 2a^2 c^2 \cos 2\theta)/\sin 2\theta$

Let $f = 4max^{2i}/q$. When q is maximum, then f will be minimum. From (1), wet have

$f = (a^4 + c^4)\text{cosec}2\theta - 2a^2 c^2 \cot 2\theta$

$df/d\theta = -2(a^4 + c^4)\text{cosec } 2\theta\cot 2\theta + 4a^2 c^2 \text{cosec}^2 2\theta \quad \ldots(3)$

$d^2 f/d\theta^2 = 4(a^4 + c^4)\text{cosec } 2\theta(\text{cosec}^2 2\theta + \cot^2 2\theta) - 8a^2 c^2 \text{cosec}^2 2\theta\cot 2\theta$

$= 4\text{cosec } 2\theta[(a^2 \text{cosec } 2\theta - c^2 \cot 2\theta)2 + a^4 \cot^2 2\theta + c^4 \text{cosec}^2 2\theta]$

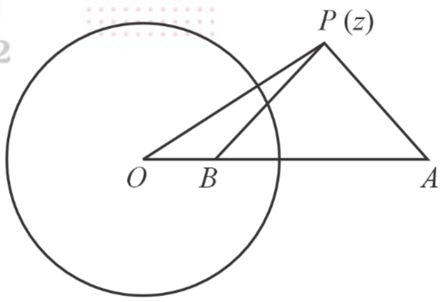
Since $\theta \le \pi/2$, clearly $d^2 f / d\theta^2$ is positive and hence f will be minimum and consequently will be maximum. From (3), setting $df/d\theta=0$, we get

$$(a^4 + c^4)\text{cosec } 2\theta\cot 2\theta = 4a^2 c^2 \text{cosec}^2 2\theta \qquad \text{or} \qquad \cos 2\theta = 2a^2 c^2/(a^4 + c^4)$$

$\therefore\ 2\cos^2\theta - 1 = 2a^2 c^2/(a^4 + c^4)$, *or* $\cos^2\theta = (a^2 + c^2)^2 / 2(a^4 + c^4)$

$$\cos\theta = \frac{\left(a^2 + c^2\right)}{\sqrt{\left\{2\left(a^4 + c^4\right)\right\}}}.$$

Q.13. A source of fluid situated in space of two dimensions, is of such strength that $2\pi\rho\mu$ presents the mass of fluid of density ρ emitted pet unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is $2\pi\rho\mu^2 a^2/r(r^2 - a^2)$, where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure?



Sol. Since the mass of fluid emitted is $2\pi\rho\mu$ per unit of time, by definition the strength of the given source is μ. Let this source be situated at A such that $OA = r$ and let B be the inverse point of A. Then, $OA \cdot OB = a^2$ so that $OB = a^2/r$. Here the equivalent image system consists of (taking OA as x-axis)

(i) a source of strength μ at $A(r, 0)$

(ii) a source of strength μ at $B(a^2/r, 0)$

(iii) a sink of strength μ at $O(0,0)$

Hence the complex potential at any point $P(z = x + iy)$ is given by

$w = -\mu \log(z - r) - \mu \log(z - a^2/r) + \mu\log z$

$$\frac{dw}{dz} = -\frac{u}{z-r} - \frac{\mu}{z - a^2/r} + \frac{\mu}{z} \quad \ldots(1)$$

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius'

Theorem(*just remember) , we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \quad \ldots(2)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}] \quad \ldots(3)$$

wherein the indicated sum of the residues is calculated at poles of $(dw/dz)^2$ lying within the circular boundary. Using (3), (2) reduces to

$$X - iY = -\pi\rho \times [\text{sum of the residues}] \qquad \ldots(4)$$

We proceed to find the residues of $(dw/dz)^2$. From (1), we have

$$\left(\frac{dw}{dz}\right)^2 = \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} - \frac{2}{z(z-a^2/r)} + \frac{2}{(z-r)(z-a^2/r)}\right]$$

$$= \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} + \frac{2}{rz} - \frac{2}{(a^2/r)(z-a^2/r)}\right.$$
$$\left. + \frac{2}{(a^2/r)z} + \frac{2}{(r-a^2/r)(z-r)} + \frac{2}{(a^2/r-r)(z-a^2/r)}\right]$$

$$\left(\frac{dw}{dz}\right)^2 = \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} - \frac{2}{z(z-a^2/r)} + \frac{2}{(z-r)(z-a^2/r)}\right].$$

From (5), we find that the poles inside the circular contour C are $z = 0$ and $z =$

$\therefore$ The required sum of the residues (from complex analysis)

= the sum of the coefficients of z^{-1} and $(z - a^2/r)^{-1}$ in R.H.S. of (5)

$$= \frac{2\mu^2}{r} + \frac{2\mu^2}{a^2/r} - \frac{2\mu^2}{a^2/r} + \frac{2\mu^2}{a^2/r - r} = \frac{2\mu^2 a^2}{r(a^2 - r^2)}$$

Using (6) in (4) and then equating real and imaginary parts, we have

$X = 2\pi\rho\mu^2 a^2/r\,(r^2 - a^2)$ and $Y = 0$.

Thus the disc is attracted towards the source along OA. Hence the disc will be urgent along *OA*.

Q.14. Within a circular boundary of radius a there is a two-dimensional liquid

to source producing liquid at the rate m, at a distance f from the centre, and an equal
centre. Find the velocity potential and show that the resultant pressure on the

$\rho m^2 f^3/2a^2(a^2 - f^2)$, where ρ is the density. Deduce as a limit velocity potential doublet at the centre.

Sol. Since the rate of production of liquid is m, by definition the strength of the given

is $m/2\pi$. Let this sources be situated at B such that $OB = f$

inverse point of B. Then $\quad OA \cdot OB = a^2$ so that $OA = a^2/f$.

Taking OA as x-axis, the equivalent image system consists of
(i) a source of strength $m/2\pi$ at $B(f,0)$
(ii) a source of length $m/2\pi$ at A (a^2

(iii) a sink of strength $-m/2\pi$ at $O(0,0)$

Hence the complex potential w at any point $P(z = x + iy)$ is

$w = -(m/2\pi)\log(z - f) - (m/2\pi)\log(z - a^2/f) + (m/2\pi)\log z$

$$\frac{dw}{dz} \; -\frac{m}{2\pi}\left[\frac{1}{z-f} + \frac{1}{z-a^2/f} - \frac{1}{z}\right]\left(\frac{dw}{dz}\right)^2 =$$

$$\frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2} + \frac{1}{(z-a^2/f)^2} + \frac{1}{z^2} + \frac{2}{(z-f)(z-a^2/f)} - \frac{2}{z(z-a^2/f)}\right.$$

$$= \frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2} + \frac{1}{(z-a^2/f)^2} + \frac{1}{z^2} + \frac{2}{(f-a^2/f)(z-f)} + \frac{2}{\left(\frac{a^2}{f} - f\right)(z-}\right.$$

$$. + \frac{2}{za^2/f} - \frac{2}{(a^2/f)(z-a^2/f)} - \frac{2}{f(z-f)} + \frac{2}{fz}\right].$$

If the pressure thrusts on the given circular disc are represented by (X, Y), then theorem, we have
the fluid.
Second part. By Bernoulli's equation. $p + (pq^2)/2 =$ constant. So it follow that p is
when q is maximum. Hence as explained in solution of Ex. 6 at a point $P(\alpha \cos\theta$,
where θ is given by $\cos\theta = (a^2 + f^2)/[2(a^4 + f^4)]^{1/2}$, the pressure is least.

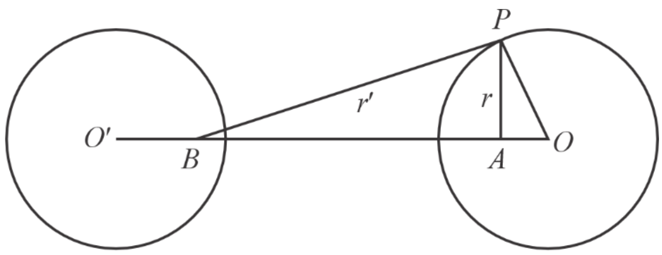
Q.15. Prove that for liquid circulating irrotationally in part of the fluid between

intersecting circles the curves of constant velocity are Cassini's Ovals.

Sol. Let O and O' be the centres of the two non-intersecting circles. Let $A(a, 0)$ and B $(-a, 0)$ be the inverse points with respect to both the circles. Let P be any point on one of the given circles such that $PA = r$ and $PB = r'$

Since A and B are inverse points of the circle with centre O, so by definition, we have

$OA.\ OB = OP^2$



Now, from similar triangle *OPA* and *OPB*, we have

$PA/PB = OP/OB$ = constant $\Rightarrow r/r'$ = constant.

Hence the equations of the two circles may taken as $r/r' = c_1$ and $r/r' = c_2$, where c_1 and c_2 are constants. Since these circles are two streamlines, it follows that the stream function ψ is of the form $f(r/r')$ and it being a harmonic, we take $\psi = k \log (r/r')$ because $\log r$ is the only function of *r* which is plane harmonic. Here *k* is a constant.

Now, if θ is the conjugate harmonic of *r*, $\phi + i\psi$ or $\psi - i\phi$ must be an analytic function of *z*, so that

$$\phi = -k(\theta - \theta')$$

$$w = \psi - i\phi = k\log(r/r') + ik(\theta - \theta') = k[\log r - \log r' + i\theta - i\theta']$$

$$= k[(\log r + i\theta) - (\log r' + i\theta')] = k[\log (re^{i\theta}) - \log (r' e^{i\theta'})]$$

or $\quad w = k[\log (z - a) - \log (z + a)]$, as $re^{i\theta} = z - a$ and $r'e^{i\theta'} = z + a$

$$q = \left|\frac{dw}{dz}\right| = \left|k\left[\frac{1}{z-a} - \frac{1}{z+a}\right]\right| = \frac{2ak}{|z-a+|z+a|} = \frac{2ak}{rr'}$$

Hence the curves of equal velocity are given by q = constant or $(2ak)/rr'$ = constant rr' = constant, which are Cassini's ovals.

PREVIOUS YEARS QUESTIONS

CHAPTER 1. FLUID KINEMATICS

Q1. If the velocity of an incompressible fluid at the point (x, y, z) is given by $(-Ay, Ax, 0)$, then prove that the surfaces intersecting the stream lines orthogonally exist and are the planes through *z*-axis, although the velocity potential does not exist. Discuss the nature of the fluid flow. **[6c IFoS 2022]**

Q2. The velocity components of an incompressible fluid in spherical polar coordinates (r, θ, ψ) are $(2Mr^{-3}\cos\theta, Mr^{-2}\sin\theta, 0)$, where M is a constant. Show that the velocity is of the potential kind. Find the velocity potential and the equations of the streamlines. **[5e UPSC CSE 2022]**

Q3. Verify whether the motion given by $\vec{q} = (3x\hat{i} - 2y\hat{j})xy^2$ is a possible fluid motion. If so, is it of the potential kind? Accordingly find out the streamlines and the velocity potential or the angular velocity if the fluid was replaced by a rigid solid. **[6c IFoS 2021]**

Q4. Show that $\vec{q} = \dfrac{\lambda(-y\hat{i} + x\hat{j})}{x^2 + y^2}$, ($\lambda$ =constant) is a possible incompressible fluid motion. Determine the streamlines. Is the kind of the motion potential? If yes, then find the velocity potential.**[7c UPSC CSE 2021]**

Q5. A velocity potential in a two-dimensional fluid flow is given by $\phi(x,y) = xy + x^2 - y^2$. Find the stream function for this flow. **[7c UPSC CSE 2020]**

Q6. In a fluid flow, the velocity vector is given by $\vec{V} = 2x\vec{i} + 3y\vec{j} - 5z\vec{k}$. Determine the equation of the streamline passing through a point $A = (4,9,1)$. **[6c 2020 IFoS]**

Q7. Consider the flow field given by $\psi = a(x^2 - y^2)$, 'a' being a constant. Show that the flow is irrigational. Determine the velocity potential for this flow and show that the streamlines and equivelocity potential curves are orthogonal. **[5d 2019 IFoS]**

Q8. Consider that the region $0 \le z \le h$ between the planes $z = 0$ and $z = h$ is filled with viscous incompressible fluid. The plane $z = 0$ is held at rest and the plane $z = h$ moves with constant velocity $V\hat{j}$. When conditions are steady, assuming there is no slip between the fluid and either boundary, and neglecting body forces, show that the velocity profile between the plates is parabolic. Find the tangential stress at any point $P(x,y,z)$ of the fluid and determine the drag per unit area on both the planes. **[8a 2019 IFoS]**

Q9. For an incompressible fluid flow, two components of velocity (u,v,w) are given by $u = x^2 + 2y^2 + 3z^2$, $v = x^2y - y^2z + zx$. Determine the third component w so that they satisfy the equation of continuity. Also, find the z-component of acceleration. **[(5c) UPSC CSE 2018]**

Q10. For a two-dimensional potential flow, the velocity potential is given by $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$.Determine the velocity components along the directions x and y. Also, determine the stream function ψ and check whether ϕ represents a possible case of flow or not.**[8b UPSC CSE 2018]**

Q11. If the velocity of an incompressible fluid at the point (x,y,z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5}\right), r^2 = x^2 + y^2 + z^2$$

then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. Further, determine the streamlines. **[8c UPSC CSE 2017]**

Q12. A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d. If V and v be the corresponding velocities of the stream and if the motion is assumed to be steady and diverging from the vertex of the cone, then prove that

$$\frac{u}{V} = \frac{D^2}{d^2} e^{(u^2 - V^2)/2K}$$

where K is the pressure divided by the density and is constant. **[7c UPSC CSE 2017]**

Q13. Find the streamlines and pathlines of the two dimensional velocity field:

$u = \frac{x}{1+t}, v = y, w = 0$. **[8b 2017 IFoS]**

Q14. In a steady fluid flow, the velocity components are $u = 2kx, v = 2ky$ and $w = -4kz$. Find the equation of a streamline passing through $(1,0,1)$. **[(6c) 2015 IFoS]**

Q15. Suppose $\vec{v} = (x-4y)\hat{i} + (4x-y)\hat{j}$ represents a velocity field of an incompressible and irrotational flow. Find the stream function of the flow. **[(8b) 2015 IFoS]**

Q16. Given the velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$, determine the streamline.

[(7c) UPSC CSE 2014]

Q16. Find the condition that $f(x,y,\lambda)=0$ should be a possible system of streamlines for steady irrotational motion in two dimensions, where λ is a variable parameter.

[5e 2014 IFoS]

Q17. Prove that

$\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$ is a possible form for the bounding surface of a liquid and find the velocity components.**[8c 2014 IFoS]**

Q18. Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are

$$u, v, w = \mu\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

where μ and φ are functions of x, y, z, t. **[5d UPSC CSE 2013]**

Q19. Find the values of a and b in the 2-D velocity field $\vec{v} = (3y^2 - ax^2)\hat{i} + bxy\hat{j}$ so that the flow becomes incompressible and irrational. Find the stream function of the flow. **[7a 2013 IFoS]**

Q20. Show that $\phi = xf(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed.

[8b UPSC CSE 2012]

Q21. Show that

$$u = \frac{A(x^2-y^2)}{(x^2+y^2)^2},\ v = \frac{2Axy}{(x^2+y^2)^2}, w = 0$$

are components of a possible velocity vector for invisoid incompressible fluid flow. Determine the pressure associated with this velocity field. **[7a 2012 IFoS]**

Q22. Is $\vec{q} = \dfrac{k^2\left(x\hat{j} - y\hat{i}\right)}{x^2 + y^2}$ a possible velocity vector of an incompressible fluid motion? If so, find the stream function and velocity potential of the motion. **[8c 2011 IFoS]**

Q23. A two-dimensional flow field is given by $\psi = xy$. Show that -

(i) the flow is irrotational;

(ii) ψ and ϕ satisfy Laplace equation

Symbols ψ and ϕ convey the usual meaning. **[5e 2010 IFoS]**

Q24. Show that $\phi = (x-t)(y-t)$ represents the velocity potential of an incompressible two-dimensional fluid. Further show that the streamlines at time *t* are the curves

$(x-t)^2 - (y-t)^2 =$ constant. **[7b 2010 IFoS]**

CHAPTER 2. MOTION IN 2D- SOURCES & SINK

Q1. Two sources of strength $\dfrac{m}{2}$ are placed at the point $(\pm a, 0)$. Show that at any point on the circle $x^2 + y^2 = a^2$, the velocity is parallel to the *y*-axis and is inversely proportional to *y*.

[8c UPSC CSE 2020]

Q2. In a two-dimensional fluid flow, the velocity components are given by $u = x - ay$ and $v = -ax - y$, where a is constant. Show that the velocity potential exists for this flow and determine the appropriate velocity potential. Also, determine the corresponding stream function that would represent the flow. **[7b 2020 IFoS]**

Q3. Two sources, each of strength *m*, are placed at the point $(-a, 0)$, $(a, 0)$ and a sink of strength $2m$ at origin. Show that the stream lines are the curves $\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2 + \lambda xy\right)$, where λ is a variable parameter.

Show also that the fluid speed at any point is $\left(2ma^2\right)/r_1 r_2 r_3$, where r_1, r_2 and r_3 are the distances of the points from the sources and the sink, respectively. **[8c UPSC CSE 2019]**

Q4. In the case of two-dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is u in a fixed direction, where u is a variable. Show that the maximum value of the velocity at any point of the fluid is $2u$. Prove that the force necessary to hold the disc is $2m\dot{u}$, where *m* is the mass of the liquid displaced by the disc. **[7d 2018 IFoS]**

Q5. Two sources, each of strength m, are placed at the points $(-a,0),(a,0)$ and a sink of strength $2m$ at the origin. Show that the streamlines are the curves $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2+\lambda xy\right)$, where λ is a variable parameter.

Show also that the fluid speed at any point is $\left(2ma^2\right)/r_1r_2r_3$, where r_1, r_2, r_3 are the distances of the point from the sources and the sink. **[8d 2018 IFoS]**

Q6. A simple source of strength m is fixed at the origin O in a uniform stream of incompressible fluid moving with velocity $U\vec{i}$ .Show that the velocity potential ϕ at any point P of the stream is $\frac{m}{r} - Ur\cos\theta$, where $OP = r$ and θ is the angle which $\overrightarrow{OP}$ makes with the direction $\vec{i}$. Find the differential equation of the streamlines and show that they lie on the surfaces $Ur^2\sin^2\theta - 2m\cos\theta =$constant. **[6b UPSC CSE 2016]**

Q7. Consider a uniform flow U_0 in the positive x-direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points.

[5d UPSC CSE 2015]

Q8. If fluid fills the region of space on the positive side of the x-axis, which is a rigid boundary and if there be a source m at the point $(0,\text{a})$ and an equal sink at $(0,\text{b})$ and if the pressure on the negative side be the same as the pressure at infinity, show that the resultant pressure on the boundary is $\frac{\pi\rho m^2(a-b)^2}{\{2ab(a+b)\}}$ where ρ is the density of the fluid. **[8b UPSC CSE 2013]**

Q9. With usual notations, show that ϕ and ψ for a uniform flow past a stationary cylinder are given by $\phi = U\cos\theta\left(r+\frac{a^2}{r}\right)$, $\psi = U\sin\theta\left(r-\frac{a^2}{r}\right)$. **[5e 2011 IFoS]**

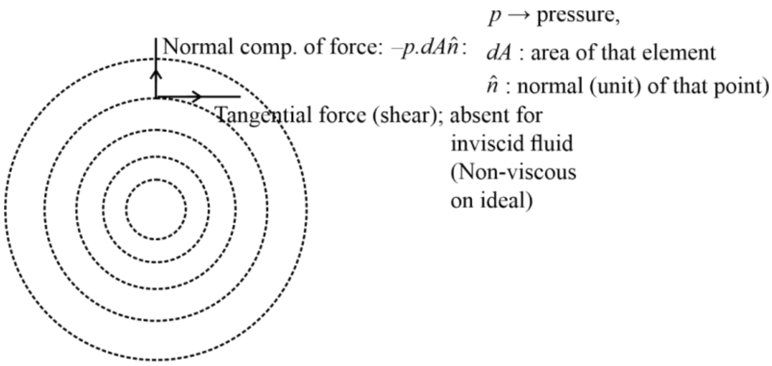
Note: The beauty of systematic learning is- You'll find solutions of almost every PYQ in above examples or questions attached with detailed answers. So to avoid repetition in this book, we have not put those solutions again as answers to PYQs.

3. Euler's Equation of motion: - (For ideal fluids)

• Body force, inertia, gravity, etc.

**(forces per unit mass) $\vec{B}$

$\therefore$ Total Body forces = $\int_v \vec{B}\rho dV$



• Rate of change of momentum $\left(\vec{M}\right)$

$$\left(\vec{M}\right) = \int_v \rho \cdot \vec{q} dv$$

$$\therefore \frac{d\vec{M}}{dt} = \frac{d}{dt}\int_v \rho \cdot \vec{q} dv \quad \ldots\text{(i)}$$

$\because$ mass × vel ; $m \,.\, v$; $\rho \cdot dv \cdot \vec{q}$

• Net force:-

$$\vec{F}_{net} = \int_A -pdA\hat{n} + \int_V \vec{B}\rho dV$$

By Newton's second law of motion:-

$$\frac{d\vec{M}}{dt} = \vec{F}_{net}$$

$$\Rightarrow \frac{d}{dt}\int_V \rho\vec{q}dV = -\int_A pdA\hat{n} + \int_V \vec{B}\rho dV$$

$$\int(\rho dV)\frac{d\vec{q}}{dt} + \int_v \vec{q}\cdot\frac{d}{dt}(\rho dv) = -\int_v\left(\vec{\nabla}\vec{p}\right)dV + \int_v \vec{B}\rho dV \quad \left\{\begin{matrix}\text{Using gauss}\\ \text{Divergence theorem}\end{matrix}\right\}$$

$$\int_v(\rho dV)\frac{d\vec{q}}{dt} + 0 = +\int_v \vec{B}\rho dv - \int_v\left(\vec{\nabla}\vec{P}\right)dV$$

$$\left\{\begin{matrix}\because \rho dV = \text{mass \& mass is fixed.}\\ \therefore \frac{d}{dt}(\rho dv) = 0\end{matrix}\right\}$$

$$\rho\cdot\frac{d\vec{q}}{dt} = \vec{B}\cdot\rho - \vec{\nabla}\vec{p}$$

$\boxed{\frac{d\vec{q}}{dt} = \vec{B} - \frac{1}{\rho}\vec{\nabla}\cdot\vec{p}}$ called euler's eq. of motion

Interpretations:-

$$\frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}\cdot\vec{\nabla}\right)\vec{q}$$

$\because \; \dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \left(\vec{q}\cdot\vec{\nabla}\right)$

Now using it in Euler's eq. of motion, we get

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}\cdot\vec{\nabla}\right)\vec{q} = \vec{B} - \frac{1}{\rho}\vec{\nabla}p$$

$$\frac{\partial}{\partial t}\left(u\hat{i} + v\hat{j} + w\hat{k}\right) + \left(u\hat{i} + v\hat{j} + w\hat{k}\right).\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(u\hat{i} + v\hat{j} + w\hat{k}\right)$$

$$= B_x\hat{i} + B_y\hat{j} + B_z\hat{k} - \frac{1}{\rho}\vec{\nabla}p$$

Exampoint 2

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = B_x - \frac{1}{\rho}\frac{\partial p}{\partial x}; \text{coeff. of } \hat{i}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = B_y - \frac{1}{\rho}\frac{\partial p}{\partial y}; \text{coeff. of } \hat{j}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = B_z - \frac{1}{\rho}\frac{\partial p}{\partial z}; \text{coeff. of } \hat{k}$$

Example just to do some mental prep about how to use above exam point!

Given, steady motion i.e., $\dfrac{\partial}{\partial t}(\;) = 0$

Incompressible inviscuid: possible motion $\vec{\nabla}\cdot\vec{q} = 0$

Component of velocity: $u = f_x$, $v = f_y$, $w = 0$

Q. Derive an expression for pressure p: if given p(0, 0, 0) = p_0

Recall; Euler's Eq. of motion,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = B_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = B_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = B_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Step II:-

$$0 + f_x\frac{\partial u}{\partial x} + f_y\frac{\partial u}{\partial y} + 0 = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \ldots(1)$$

$$0 + f_x\frac{\partial v}{\partial x} + f_y\frac{\partial v}{\partial y} + 0 = 0 - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \ldots(2)$$

$$0 + f_x \times 0 + f_4 \times 0 + 0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad ..(3)$$

Step II:

$\because$ p is a f-of x, y, z

$$\therefore\ dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

Now,

Using (1), (2), (3) $\rightarrow$ $dp = \_\ dx + \ldots(4)$

Now, on integrating eq. (4) we get expression for $P(x, y, z)$ including integration constant
To find this

↓

Use given initial condition
i.e. $P(0, 0, 0) = \_\_\_\_\_$

Ex.1 A steady inviscid incompressible fluid flow has velocity field u = fx, v = - fy, w = 0, where if is a constant Derive an expression for the pressure field p (x,y, z) if the pressure p (0, 0, 0) = p_0 and F = - gz.

Sol. Given $u = fx$, $v = -fy$, w = 0, f being a constant ...(1)

Also, given that $p = p_0$, when $x = 0$, $y = 0$, $z = 0$ ...(2)

Again, $F = -giz \Rightarrow x = 0$, $y = 0$ and $z = -gz$ ...(3)

Equations of motion for steady motion $(\partial / \partial t) = 0$ of an incompressible fluid flow are given by

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = X - (1/\rho)\times(\partial p/\partial x) \ldots (4)$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = Y - (1/\rho)\times(\partial p/\partial y) \ldots (5)$$

$$u(\partial s/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = Z - (1/\rho)\times(\partial p/\partial z) \ldots (6)$$

Using (1) and (3), (4), (5) and (6) reduce to

$$f^2 x = -(1/\rho)\times(\partial p/\partial x), \quad -f^2 y = -(1/\rho)\times(\partial p/\partial y), \quad 0 = -gz - (1/\rho)\times(\partial p/\partial z) \ldots (7)$$

Now, $dp = (\partial p/\partial x)dx + (\partial p/\partial y)dy + (\partial p/\partial z)dz$

$$dp = -(f^2 \rho x)dx + (f^2 \rho y)dy - (\rho g z)dz, \text{ using (7)}$$

Integrating, $p = -(f^2 \rho x)/2 + (f^2 \rho y^2)/2 - (\rho g z^2)/2 + C$, C being a constant... (8)

Putting $x = y = z = 0$ and $p = p_0$ (see condition (2)), in (8) we get C = p_0

Thus, the required expression for the pressure field is given by

$$p(x, y, z) = p_0 - \rho(f^2 x^2 - f^2 y^2 + g z^2)/2$$

Ex. 2. For a steady motion of inviscid incompressible fluid of uniform density under conservative forces, show that the vorticity w and velocity q satisfies.

$$(q.\nabla)w = (w.\nabla)q.$$

Sol. Vector equation of motion for inviscid incompressible fluid is

$$\partial q / \partial t + \nabla(q^2/2) - q \times curl\, q = F - (1/\rho)\nabla p \quad \ldots(1)$$

Since the motion is steady, $\partial q/\partial t = 0$ …(2)

Since ρ is uniform, $(1/\rho)\nabla p = \nabla(p/\rho)$ … (3)

Since F s conservative, $F = -\nabla\Omega$, where Ω s some scalar function. …(4)

Again, by definition, vorticity vector = w = curl q.

Using (2), (3), (4) and (5) in (1), we obtain

$$\nabla(q^2/2) - q \times w = -\nabla\Omega - \nabla(p/\rho) \quad \text{or} \quad q \times w = \nabla(q^2/2 + \Omega + p/\rho)$$

Taking the curl of both sides of the above equation and using the vector identity curl gral $\phi = 0$, we have

curl (q × w) = 0 or $(\nabla.w)q - (q.\nabla)w + (w.\nabla)q - (\nabla.q)w = 0$

or $-(q.\nabla)w + (w.\nabla)q = 0$ or $(q.\nabla)w = (w.\nabla)q.$

Where we have used the following two results

$\nabla.w = \nabla.\nabla \times q = 0$ and $\nabla.q = 0$ (continuity equation)

Ex. 3. Show that f the velocity field

$$u(x,y) = \frac{B(x^2 - y^2)}{(x^2 + y^2)^2}, \quad v(x,y) = \frac{2Bxy}{(x^2 + y^2)^2}, \quad w(x,y) = 0$$

Satisfies the equations of motion for inviscid incompressible flow. Then determine the pressure associated with this velocity field, B being a constant.

Sol. The equations of motion for steady inviscid incompressible flow are given by

$$u(\partial u / \partial x) + u(\partial u / \partial y) + w(\partial u / \partial z) = -(1/\rho)(\partial p / \partial x), \quad \ldots(1)$$

$$u(\partial v / \partial x) + v(\partial v / \partial y) + w(\partial v / \partial z) = -(1/\rho)(\partial p / \partial y), \quad \ldots(2)$$

and

$$u(\partial w / \partial x) + v(\partial w / \partial y) + w(\partial w / \partial z) = -(1/\rho)(\partial p / \partial z), \quad \ldots(3)$$

From the given values of u, v and w, we have

$$\frac{\partial u}{\partial x} = B\frac{2x(x^2+y^2)^2 - 4x(x^2-y^2)(x^2+y^2)}{(x^2+y^2)^4} = \frac{2Bx(3y^2-x^2)}{(x^2+y^2)^3},$$

$$\frac{\partial u}{\partial y} = B\frac{-2y(x^2+y^2)^2 - 4y(x^2-y^2)(x^2+y^2)}{(x^2+y^2)^4} = \frac{2By(3x^2-y^2)}{(x^2+y^2)^3}, \qquad \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial x} = 2B\frac{y(x^2+y^2)^2 - 4x^2y(x^2+y^2)}{(x^2+y^2)^4} = \frac{2By(y^2-3x^2)}{(x^2+y^2)^3},$$

$$\frac{\partial v}{\partial y} = 2B\frac{x(x^2+y^2)^2 - 4xy^2(x^2+y^2)}{(x^2+y^2)^4} = \frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3}, \qquad \frac{\partial v}{\partial z} = 0,$$

$$\partial w/\partial x = 0, \qquad \partial w/\partial y = 0 \qquad \text{and} \qquad \partial w/\partial z = 0$$

Substituting the given values of u, v and w and also using the above relations, (1), (2) and (3) reduce to

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2Bx(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2By(3x^2-y^2)}{(x^2+y^2)^3} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2By(y^2-3x^2)}{(x^2+y^2)^3} + \frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3} = -\frac{1}{0}\frac{\partial p}{\partial y},$$

And $\qquad 0 = -(1/\rho)\,(\partial p/\partial z)$

Simplifying the above equations, we have

$$\frac{2B^2x}{(x^2+y^2)^5}[(x^2-y^2)(3y^2-x^2) - 2y^2(3x^2-y^2)] = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{2B^2y}{(x^2+y^2)^5}[(x^2-y^2)(y^2-3x^2) + 2x^2(x^2-3y^2)] = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

And $\qquad 0 = \partial p/\partial z$

Again simplifying the above equations, we have

Or $\qquad \frac{2B^2x}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4) = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ i.e, $\frac{2B^2x\rho}{(x^2+y^2)^3} = \frac{\partial p}{\partial x}$ …(1)

$\frac{2B^2y}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4) = -\frac{1}{\rho}\frac{\partial p}{\partial z}$ i.e, $\frac{2B^2y\rho}{(x^2+y^2)^3} = \frac{\partial p}{\partial y}$ ...(2)

And $\qquad 0 = \partial p/dz,$ ...(3)

Relation (3) shows that the pressure p is independent of z, i.e, p = p (x, y), Hence, we have

$$dp = (\partial p/\partial x)dx + (\partial p/\partial y)dy$$

$$dp = \frac{2B^2x\rho}{(x^2+y^2)^3}dx + \frac{2B^2y\rho}{(x^2+y^2)^3}dy = B^2\rho(x^2+y^2)^{-3}(2xdx+2ydy)$$

$$dp = B^2\rho(x^2+y^2)^{-3}d(x^2+y^2).$$

Integrating, $p = C - (1/2)\times B^2\rho(x^2+y^2)^{-2} = C - \{B^2\rho/2(x^2+y^2)^2\}$

where C is a constant of integration. It gives the required pressure distribution.

Ex.4. The particle velocity for a fluid motion referred to rectangular axes is given by the components

u = A cos $(\pi x/2a)\cos(\pi z/2a)$. $\qquad$ v = 0, $\qquad w = A\sin(\pi x/2a)\sin(\pi x/2a)$, where A is a constant. Show that this is a passible motion of an incompressible fluid under no body forces n an infinite fixed rigid tube, $-a \le x \le a, 0 \le z \le 2a.$ Also find the pressure associated with this velocity field.

Sol. Given $u = A\cos(\pi x/2a)\cos(\pi z/2a), v = 0, \quad w = A\sin(\pi x/2a)\sin(\pi z/2a) \quad \ldots(1)$

From (1), $\partial u/\partial x = -(A\pi/2a) sin(\pi x/2a)\cos(\pi z/2a), \quad \partial v/\partial y = 0$

And $\partial w/\partial z = (A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a) \quad \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0, \ldots(2)$

Showing that the given velocity components represent a physically possible flow.

The equations of motion for steady inviscid incompressible flow under no body force are

$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = -(1/\rho)(\partial p/\partial x), \quad \ldots(3)$

$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = -(1/\rho)(\partial p/\partial y) \ldots(4)$

And $u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = -(1/\rho)(\partial p/\partial z) \quad \ldots(5)$

From (1) $\partial u/\partial y = 0; \quad \partial u/\partial z = -(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)$

$\partial v/\partial x = \partial v/\partial z = 0, \quad \partial w/\partial x = (A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a) \quad \ldots(6)$

And $1\,\partial w/\partial y = 0$

Using (1), (2) and (6), the equations of motion (3), (4) and (5) become

$$-A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}.\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a} - A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}.\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad 0 = -(1/\rho)(\partial p/\partial y)$$

$$A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}.\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a} + A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}.\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Simplifying the above equations, we have

$\partial p/\partial x = (\pi\rho A^2/2a)\cos(\pi x/2a)\sin(\pi x/2a). \quad \ldots(7)$

$\partial p/\partial y = 0 \quad \ldots(8)$

And $\partial p/\partial z = -(\pi\rho A^2/2a)\cos(\pi z/2a)\sin(\pi z/2a). \ldots(9)$

Equation (8) shows that the pressure p is independent of y so that $p = p(x, z)$. Then

$$dp = (\partial p/dx)dx + (\partial p/\partial z)dz$$

$dp = (\pi\rho A^2/2a)[\cos(\pi x/2a)\sin(\pi x/2a)dx - \quad \cos(\pi z/2a)sn(\pi z/2a)dz]$, Using (7) and (9)

Integrating, $\quad p = (\pi\rho A^2/2a)[(a/\pi)\sin^2(\pi x/2a) - (a/\pi)sin^2(\pi z/2a)] + C$

$p = (\rho A^2/2)[sin^2(\pi x/2a) - sin^2(\pi x/2a)] + C$, C being a constant of integration...(10)

(10) gives the required pressure associated with the velocity field by (1).

Ex.5. Prove that if $\lambda = (\partial u/\partial t) - v(\partial v/\partial x - \partial u/\partial y) + w(\partial u/\partial t - \partial w/\partial x)$ and μ, ν are two similar expressions, then $\lambda\, dx + \mu\, dy + \nu\, dz$ is a perfect differential, if the external forces are conservative and the density is constant.

Sol. Let (X, Y, Z) be the components of external forces. Since the external forces are conservative, there exists force potential $V(x, y, z)$ such that

$X = -\partial V/\partial x$, $\quad Y = -\partial V/\partial y$ and $\quad Z = -\partial V/\partial z$. ... (1)

Euler's dynamical equations of motion are

$$Du/Dt = X - (1/\rho)(\partial p/\partial x), \quad ...(2)$$

$$Dv/Dt = Y - (1/\rho)(\partial p/\partial y), \quad ... (3)$$

And $$Dw/Dt = Z - (1/\rho)(\partial p/\partial z), \quad ... (4)$$

Where $p(x, y, z)$ is the pressure at, any point (x, y, z).

Using (1), (2), (3) and (4) can be rewritten as

$$Du/Dt = -\partial V/\partial x - (1/\rho)(\partial p/\partial x), \quad ... (5)$$

$$Dv/Dt = -\partial V/\partial y - (1/\rho)(\partial p/\partial y) \quad ... (6)$$

And $$Dw/Dt = -\partial V/\partial z - (1/\rho)(\partial p/\partial z) \quad ...(7)$$

Multiplying (5), (6) and (7) by *dx, dy dy* dz and then adding, we have

$$\frac{Du}{Dt}dx + \frac{dv}{Dt}dy + \frac{dw}{Dt}dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

$$\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz = -dV - \frac{1}{\rho}dp. \quad ...(8)$$

Re- writing the given value of λ we have

$$\lambda = \frac{\partial u}{\partial t} - v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - w\frac{\partial w}{\partial x}$$

$$= \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right)$$

$$= \frac{Du}{Dt} - \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2 + w^2\right) = \frac{Du}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial x} \quad ...(9)$$

$$\left[\because \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \text{ and } q^2 = u^2 + v^2 + w^2\right]$$

Similarly, $\mu = \frac{Dv}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial y}$ and $\nu = \frac{Dw}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial z}$ ...(10)

$\therefore$ Using (9) and (10) we have,

$$\lambda\, dx + \mu\, dy + v\, dz = \frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz - \frac{1}{2}\left[\frac{\partial q^2}{\partial x}dx + \frac{\partial q^2}{\partial y}dy + \frac{\partial q^2}{\partial z}dz\right]$$

$$= -dV - (1/\rho)dp - (1/2)\times dq^2 = -d[V + (p/\rho) + (1/2)\times q^2]$$

Which is a perfect differential which is what we wished to prove.

spherical polar coordinates

Ex. 6. For an inviscid, incompressible, steady flow with negligible body forces, velocity components in spherical polar coordinates are given by

$u_r = V(1 - R^3/r^3)\cos\theta,\ u_\theta = -V(1 + R^3/2r^3)\sin\theta, \qquad u_\phi = 0$

Show that it is a possible solution of momentum equations (i.e. equations of motion). R and V are constants.

Sol. Here equations of motion in spherical polar coordinates are

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \quad \ldots(1)$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r + u_\theta}{r} - \frac{u_\theta^2\cot\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \quad \ldots(2)$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\phi u_\theta\cot\theta}{r} = F_\phi - \frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial \phi} \ldots(3)$$

For steady flow ($\partial/\partial t = 0$) with negligible body forces ($F_r = F_\theta = F_\phi = 0$), the above equations reduces

to $$u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \ldots(4)$$

$$u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \ldots(5)$$

$$0 = \frac{1}{\rho r\sin\phi}\frac{\partial p}{\partial \phi} \quad \ldots(6)$$

Equation (6) shows that p is function of r and θonly.

Give : $u_r = V\left(1 - \frac{R^3}{r^3}\right)\cos\theta, \quad u_\theta = -V\left(1 - \frac{R^3}{2r^3}\right)\sin\theta, \qquad \ldots(7)$

From (7), $\frac{\partial u_r}{\partial r} = \frac{3VR^3}{r^4}\cos\theta; \quad \frac{\partial u_r}{\partial \theta} = -V\left(1 - \frac{R^3}{r^3}\right)\sin\theta \qquad \ldots(8)$

Using (7) and (8), (4) reduces to

$$V\left(1 - \frac{R^3}{r^3}\right)\cos\theta . \frac{3VR^3}{r^4}\cos\theta - \frac{V}{r}\left(1 + \frac{R^3}{2r^3}\right)\sin\theta . \left[-V\left(1 - \frac{R^3}{r^3}\right)\sin\theta\right] - \frac{1}{r}\left[V^2\left(1 + \frac{R^3}{2r^3}\right)\sin^2\theta\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{3V^2R^3}{r^4}\left(1 - \frac{R^3}{r^3}\right)\cos^2\theta - \frac{3V^2R^3}{2r^r}\left(1 + \frac{R^3}{2r^3}\right)\sin^2\theta = -\frac{1}{\rho}\frac{\partial \rho}{\partial r} \ldots(9)$$

From (7), $\frac{\partial u_\theta}{\partial r}=\frac{3VR^3\sin\theta}{2r^4}$ and $\frac{\partial u_\theta}{\partial \theta}=-V\left(1+\frac{R^3}{2r^3}\right)\cos\theta$…(10)

Using (7) and (10), (5) reduces to

$$V\left(1-\frac{R^3}{r^3}\right)\cos\theta.\frac{3VR^3}{2r^4}\sin\theta+\frac{1}{r}\left[-V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right]\left[-V\left(1+\frac{R^3}{2r^3}\right)\cos\theta\right]$$

$$+\frac{1}{r}\left[V\left(1-\frac{R^3}{r^3}\right)\cos\theta\right]\left[-V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right]=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \ldots(11)$$

or $$\frac{3V^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta+\frac{3V^2R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial\theta}$$

Differentiating (9) with respect to θ, we get

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r}=\frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right).2\cos\theta(-\sin\theta)-\frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^4}\right)\times 2\sin\theta\cos\theta$$

or $$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r}=\left(-\frac{9V^2R^3}{r^4}+\frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \ldots(12)$$

Next, differentiating (11) with respect to *r*, we get

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta}=\frac{3V^2R^3}{2}\left(-\frac{3}{r^4}+\frac{6R^3}{r^7}\right)\sin\theta\cos\theta+\frac{3V^2R^3}{2}\left(-\frac{3}{r^4}-\frac{6R^3}{2r^7}\right)\times\sin\theta\cos\theta$$

or $$-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta}=\left(-\frac{9V^2R^3}{r^4}+\frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \ldots(13)$$

Since (12) and (13) are identical, the equations of motion (i.e., momentum equations) are satisfied.

Ex.7. The velocity components $u_r(r,\theta)=-V\left(1-\frac{a^2}{r^2}\right)\cos\theta$, $u_\theta(r,\theta)=u\left(1-\frac{a^2}{r^2}\right)\sin\theta$ satisfy the equations of motion for a two-dimensional inviscid incompressible flow. Find the pressure associated with this velocity field. U and a are constants.

Sol. The equations of motion for inviscid incompressible fluid in cylindrical polar coordinates are given

by $$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u^2}{r}=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(1)$$

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r+u_\theta}{r}=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \ldots(2)$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \ldots(3)$$

For steady ($\partial/\partial t = 0$) and two dimensional flow ($\partial/\partial z = 0$, $\underline{u}_z = 0$) with negligible body forces ($F_r = F_\theta = F_z = 0$), the above equations (1) to (3) reduces to

$$u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}=\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(4)$$

$$u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}=\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \ldots(5)$$

And $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

Which implies that p is function of r and θonly.

Given $$u_r = -U\left(1-\frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = U\left(1+\frac{a^2}{r^2}\right)\sin\theta \qquad \ldots(6)$$

Using (6), (4) reduces to

$$\left[-U\left(1-\frac{a^2}{r^2}\right)\cos\theta\right]\left(-\frac{2a^2U}{r^3}\right)\cos\theta+\frac{1}{r}U\left(1+\frac{a^2}{r^2}\right)\sin\left[-U\left(1-\frac{a^2}{r^2}\right)\times(-\sin\theta)\right]$$

$$-\frac{1}{r}.U^2\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{2U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta+\frac{U^2}{r}\sin^2\theta\left[\left(1-\frac{a^4}{r^4}\right)-\left(1+\frac{a^2}{r^2}\right)^2\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{2U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta-\frac{2U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta = -\frac{1\partial r}{\rho\partial r} \quad \ldots (7)$$

Again using (6), (5) reduces to

$$-U\left(1-\frac{a^2}{r^2}\right)\cos\theta\times U\left(-\frac{2a^2}{r^3}\right)\sin\theta+\frac{U}{r}\left(1+\frac{a^2}{r^2}\right)\sin\theta\times U\left(1+\frac{a^2}{r^2}\right)\cos\theta-\frac{1}{r}U\left(1-\frac{a^2}{r^2}\right)\cos\theta\times U$$
$$\left(1+\frac{a^2}{r^2}\right)\sin\theta = -\frac{1}{rp}\frac{\partial\rho}{\partial\theta}$$

$$\frac{2a^2U^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta = -\frac{1}{r\rho}\frac{\partial p}{\partial\theta}$$

$$\frac{2a^2U^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2a^2U^2}{r^2}\left(1+\frac{a^3}{r^2}\right)\sin\theta\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial\theta}$$

$$\frac{4U^2a^2}{r^2}\sin\theta\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial\theta} \quad \ldots (8)$$

Differentiating (7) with respect to θ, we have

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\partial r} = -\frac{4U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta-\frac{4U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta$$

Or $$\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\partial r} = \frac{8U^2a^2}{r^3}\sin\theta\cos\theta \qquad \ldots(9)$$

Differencing (8) with respect to r, we have

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \frac{8U^2a^2}{r^3}\sin\theta\cos\theta \;\ldots\ldots\; (10)$$

Since (9) and (10) are identical, it follows that the given velocity components satisfy the equations of motion.

Since p is function of r and θ, we have

$$dp = (\partial p/\partial r)dr + (\partial p/\partial\theta)d\theta$$

Substituting the value of $\partial p/\partial r$ and $\partial p/\partial\theta$ given by (7) and (8) respectively in the above equation, we obtain

$$\therefore\ dp = 2\rho U^2a^2\left\{\left(\frac{1}{r^3}+\frac{a^2}{r^5}\right)\sin^2\theta - \left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta\right\}dr - \frac{4\rho U^2a^2}{r^2}\sin\theta\cos\theta\,d\theta \;\ldots.\; (11)$$

Let dp=Mdr +Ndθ. Then, by comparison, we have

$$M = 2\rho U^2a^2\{(1/r^3 + a^2/r^2)\sin^2\theta - (1/r^3 - a^2/r^5)\cos^2\theta\}$$

And $N = -(4\rho U^2a^2/r^2)\times\sin\theta\cos\theta$

$$\frac{\partial M}{\partial\theta} = \frac{\partial}{\partial\theta}\left[2\rho U^2a^2\left\{\left(\frac{1}{r^3}+\frac{a^2}{r^5}\right)\sin^2\theta - \left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta\right\}\right]$$

$$= 2\rho U^2a^2\{(1/r^3 + a^2/r^5)\}\times 2\sin\theta\cos\theta + (1/r^3 - a^2/r^5)\times 2\sin\theta\cos\theta\}$$

$$= (8/r^3)\times\rho U^2a^2\sin\theta\cos\theta$$

And $$\frac{\partial N}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{4\rho U^2a^2}{r^2}\sin\theta\cos\theta\right) = \frac{8\rho U^2a^2\sin\theta\cos\theta}{r^3}$$

Thus, $\partial M/\partial\theta = \partial N/\partial r.$

Hence (11) must be exact and so its solution by the usual rule of an exact equation is

$$p = 2\rho U^2a^2$$

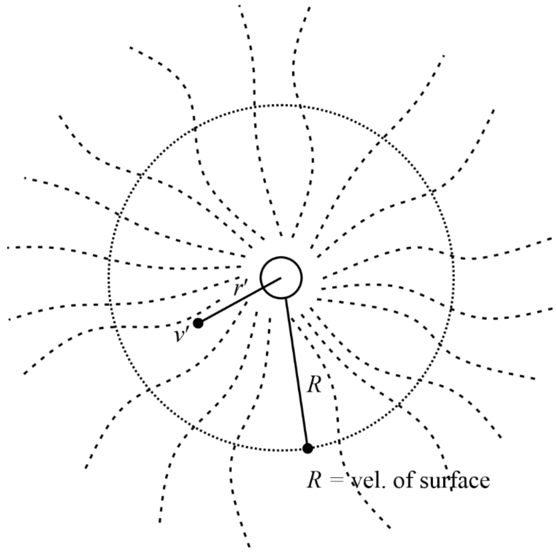
$$\left[\left(-\frac{1}{2r^2}-\frac{a^2}{4r^4}\right)\sin^2\theta - \left(-\frac{1}{2r^2}+\frac{a^2}{4r^4}\right)\cos^2\theta\right] + c$$

$p = 2\rho U^2a^2\left(\frac{\cos 2\theta}{2r^2} - \frac{a^2}{4r^4}\right) + C$, C being arbitrary constant

To decode type II problems

Ex.8. A sphere of radius R whose center is at rest. Vibrates radically in an infinite incompressible fluids of density ρ. If the pressure at infinity is π. Show that the pressure at the surface of the sphere at time *t* is

$$\pi+\frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2}+\left(\frac{dR}{dt}\right)^2\right\}$$



Here the motion will take place in a manner such that : each particle of the fluid moves towards centre of sphere.

∴ The free surface will be spherical.

** The velocity v' will be radial only

(i.e., v' is function of r' (radius) & time 't' only)

Eq. of continuity:-

$\boxed{r'^2v'=F(t)=R^2V}$ …(1); here V is the velocity at R distance from the centre or on surface of sphere(in figure, it's V=vel. on surface)

** Remember: if v' is function of r' & t only there (sphere) eq. of continuity reduces into

$$r'^2v'=F(t)$$

Step II:- If velocity is function of r & t only, then Euler's eq of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=B(r)-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{……(2)}$$

Exampoint:- Now, using (1) in (2), we need to proceed

This is interesting & need to remember

∵ from (1), we have,

$$r'^2 v' = F(t) = R^2V$$

$$\frac{\partial v'}{\partial t}=\frac{1}{r'^2}\frac{\partial}{\partial t}F(t)$$

$$\Rightarrow \quad \frac{\partial v'}{\partial t}=\frac{F'(t)}{r'^2}$$

Now, using in (2), we have,

$$\frac{F'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=B(r')-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On Integrating w.r.t r'

$$\frac{F'(t)}{r'^2}+\frac{1}{2}v'^2=\int B(r').dr'-\frac{p}{\rho}+A; \qquad \text{………(3)}$$

where A is integration constant.

Answer (above example):-

Eq" of continuity gives

$r'^2 v' = F(t) = R^2v$ ………..(1)

$$\Rightarrow \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots\ldots\ldots\ldots(2)$$

Euler's equation of motion,

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = B(r') - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$\because$ $B(r') = 0$: (No body force given)

$$\therefore \frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\Rightarrow \frac{F'(t)}{r'2} + \frac{1}{2}\frac{\partial}{\partial r'}\left(v'^2\right) = (-1/\rho)\frac{\partial}{dr'}(p)$$

On integrating w. r. t. r'

$$\frac{-F'(t)}{r'2} + \frac{1}{2}v'^2 = \frac{p}{\rho} + A; \qquad \text{where A is integration constant.}$$

$\because$ given when $r' \to \infty$, $v' = o$; $p = \Pi$

$\therefore$ from (3);

$$\frac{-F(t)}{\infty} + \frac{1}{2}\times 0 = \frac{-\Pi}{\rho} + A$$

$$\Rightarrow \quad A = \frac{\Pi}{\rho}$$

$\therefore$ eq. – (3) glues,

$$\frac{-F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad \ldots\ldots\ldots\ (4)$$

For Target

- But p = P and v′ = V when r′ = R

$\therefore$ We have from (4)

$$p\rho\left[\frac{-f'(t)}{R} + \frac{v^2}{2}\right] = \Pi - p$$

$$p = \Pi + \rho\left[\frac{\{f'(t)\}_{r=R}}{R} - \frac{1}{2}V^2\right]$$

$$p = \Pi + \frac{1}{2}\rho\left[\frac{2}{R}\{f'(t)\}_{r'=R} - V^2\right]\ldots\ldots(5)$$

$\because \quad V = \frac{dR}{dt}$, Now, $\because$ from (1), we have

$$\frac{d}{dt}(F(t)) = \frac{d}{dt}\left(R^2 v\right) = \frac{d}{dt}\left(R^2\frac{dR}{dt}\right) = \frac{d}{dt}\left(\frac{R}{2}\frac{d}{dt}(R^2)\right)$$

$$= \frac{R}{2}\left(\frac{d^2}{dt^2}(R^2)\right) + \left(\frac{1}{2}\frac{d}{dt}(R^2)\right) + \left(\frac{1}{2}\frac{d}{dt}(R^2)\frac{dR}{dt}\right)$$

$$F'(t) = \frac{d}{dt}(F(t)) = \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

Using this $F'(t)$ & V in (5) we get desired result.

$$\therefore \quad p = \prod + \frac{1}{2} f\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

Ex.9. An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure $\prod$ and contains spherical cavity of radius 'a' filled with a gas at pressure $m\prod$. Prove that if inertia of gas be neglected and boyle's law be supposed to hold through the ensuing motion, the radius of the sphere will oscillate between a and na, where n is determined by the eq. $1 + 3\, m \log n - n^3 = 0$

Eq. of continuity

$r'^2 v' = F(t) = R^2 v$ ………..(1

$$\Rightarrow \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad …………(2)$$

Euler's eq. of motion.

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r} = B(r') - \frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

Using 2

$$\frac{f'(t)}{r'^2} + \frac{1}{2}\frac{\partial}{\partial r'}\left(v'^2\right) = 0 - \frac{1}{\rho}\frac{\partial p'}{\partial r'} \quad \{\because\ B(r')=0\}$$

On instigating w.r.t.r'

$$\frac{-f'(t)}{r'} + \frac{1}{2}v'^2 = \frac{-p'}{\rho} + A$$, when A to integrator constant.

$$\because \text{ at } r' \to \infty, v' = 0 \ \& p = \prod \quad \therefore A = \frac{\prod}{\rho}$$

$$\therefore \quad \frac{-F(t)}{r'} + \frac{1}{2}\left(v'^2\right) = \frac{\prod - p'}{\rho} \qquad ……………… (3)$$

$\because$ Gas inside cavity follows Boyl's law

$$\therefore \left(\frac{4}{3}\pi a^3\right) \times m\prod = \left(\frac{4}{3}\pi R^3\right) \times P$$

$$P = \frac{a^3 m \prod}{R^3}$$

Now, using P for further,

- When r'=R, v' = V, p' = P = $\frac{a^3 m \prod}{R^3}$

$\therefore$ from (3), we have

$$\frac{-F'(t)}{R} + \frac{1}{2}V^2 = \frac{1}{\rho}\left(\prod - \frac{a^3 m \prod}{R^3}\right) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$\because$ From (1);

F'(t) = $2R\frac{dR}{dt}.V + R^2\frac{dV}{dt} = 2RV^2 + R^2\frac{dV}{dR}.\frac{dR}{dt}$

$$\therefore \; F'(t) = 2RV^2 + R^2V\left(\frac{dV}{dR}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Using (5) in (4)

$$-\left\{\frac{2RV^2 + R^2V\left(\frac{dV}{dR}\right)}{R}\right\} + \frac{1}{2}V^2 = \frac{1}{\rho}\left(\prod - \frac{a^3 m \prod}{R^3}\right)$$

$$\Rightarrow RV\frac{dV}{dR} + \frac{3}{2}V^2 = -\frac{\prod}{\rho} + \frac{a^3 m \prod}{R^3 \rho}$$

- Multiplying by $2R^2dR$, We get

$2R^3VdV + 3R^2V^2dR = \left(\frac{-2\prod R^2}{\rho} + \frac{2a^3 m \prod}{\rho R}\right)dR$

$$d(R^3V^2) = \left(\frac{-2\prod R^2}{\rho} + \frac{2a^3 m \prod}{\rho R}\right)dR$$

On integrating; $R^3V^2 = \frac{-2R^3}{3\rho} + \frac{2a^3 m \prod}{\rho}\log R + B \quad \ldots\ldots (6)$

Where B s integration onstant.

- Initially when R = a, V = 0

$\therefore$ (6) gives $B = \frac{2\prod a^3}{3\rho} - \frac{2\pi a^3 m \prod}{\rho}\log a$

i.e., we have,

$$R^3V^2 = \frac{2\prod}{3\rho}(a^3 - R^3) + \frac{2a^3 m \prod}{\rho}\log\left(\frac{R}{a}\right) \; \ldots\ldots(7)$$

$\because$ Radius of sphere oscillates b/w a & na

i.e, we have V = 0 at R = a & R = na

$\therefore$ Putting R = na, V = 0 in (7) we get

$$0 = \frac{2\prod}{3\rho}\left\{a^3 - n^3a^3 + 3ma^3\log\left(\frac{na}{a}\right)\right\}$$

$\Rightarrow \quad 1 + 3m \log n - n^3 = 0 \quad$ as $a \neq 0$

Ex.10. A mass of gravitating fluid is at rest under its own attraction only, the free surface being a sphere of radius b & the inner surface a rigid concentric shell of radius a. show that if the shell suddenly disappear, the initial pressure at any point of the fluid at a distance r from the centre so $\frac{2}{3}\pi r\delta^2(b-a)(r-a)\left(\frac{a+b}{r}+1\right)$

Attraction at a distance r'; $B(r) = \dfrac{\frac{4}{3}\pi r\rho(r'^3 - r^3)}{r'^2}$

- $r'^2 v' = F(t) = \Rightarrow \dfrac{\partial v'}{\partial t} = \dfrac{F'(t)}{r'^2}$
- $\dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r'} = B(r) - \dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

$$\therefore \frac{F'(t)}{r'^2} + \frac{1}{2}\frac{\partial}{\partial r'}\left(v'^2\right) = \frac{-\frac{4}{5}\pi r\rho(r'^3 - r^3)}{r'^2}$$

Negative sign is attached due to the nature of motion.

When shell is present. When shall disappears

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{4}{3}\pi r\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

$$\frac{-F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{-4}{3}\pi r\rho\left(\frac{r'^2}{2} + \frac{r^3}{r'}\right) - \frac{p'}{\rho} + A \quad \ldots\ldots\ldots (2)$$

- Initially when $t = 0$, $v' = 0$, $r = a$, $p' = p$ (Let)

$\therefore$ From (2) we get

$$\frac{-F'(o)}{r'} = \frac{-4}{3}\pi r\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{p}{\rho} + A \quad \ldots\ldots\ldots (3)$$

- But, $p = 0$; when $r' = a$ & $r' = b$

$\therefore$ eq. (3) gives.

$$\frac{-F'(o)}{a} = \frac{-4}{3}\pi r\rho\left(\frac{a^2}{2} + a^2\right) + A \quad \ldots\ldots\ldots\ldots (4)$$

$$\frac{-F'(o)}{b} = \frac{-4}{3}\pi r\rho\left(\frac{b^2}{2} + \frac{a^2}{b}\right) + A \quad \ldots\ldots\ldots\ldots (5)$$

Subtracting (5) from (4) ; we get,

$$F'(o)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{4}{3}\pi r\rho\left(\frac{b^2 - a^2}{2} + a^2\left(\frac{a}{b} - 1\right)\right)$$

$$\Rightarrow \quad F'(o) = -\frac{2}{3}\pi r\rho ab(a+b) + \frac{4}{3}\pi r\rho a^3 \quad \ldots. (6)$$

- Multiplying (4) by a, (5) by a there subtracting we get

$$0=\frac{4}{3}\pi r\rho\left(\frac{b^3}{2}-\frac{a^3}{2}\right)+A(a-b)$$

$\Rightarrow \quad A=\frac{2}{3}\pi r\rho(a^2+b^2+ab) \qquad \ldots\ldots\ldots\ldots (7)$

- Now using F'(o) & A from (6) & (7) in (3), we get

$$\frac{-1}{r'}\left\{\frac{-2}{3}\pi r\rho ab(a+b)+\frac{4}{3}\pi r\rho a^3\right\}=\frac{-4}{3}\pi r\rho\left(\frac{r'^2}{2}+\frac{a^3}{r'}\right)-\frac{p}{\rho}+\frac{2}{3}\pi r\rho(a^2+b^2+ab)$$

$$\frac{p}{\rho}=\frac{2}{3}\pi r\rho\left[a^2+b^2+ab-2\left(\frac{r'^2}{2}+\frac{a^3}{r'}\right)-\frac{ab(a+b)}{r'}+\frac{2a^3}{r'}\right] \; P=\frac{2}{3}\pi r\rho^2\left[a^2+b^2+ab-r'^2-\frac{ab(a+b)}{r'}\right]$$

Ex.11. Liquid contained b/w two parallel planes the free surface is a circular cylinder of radius a whose axis is perpendicular to planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if π is the pressure at the outer surface, the initial pressure at any point on the liquid at the distance r from center is $\pi\frac{\log r-\log b}{\log a-\log b}$

- Eq. of continuity *r'v' = F (t) = R'V'*

Explanation.

Here the motion will take place in such a manner: each particle (element) of the liquid moves towards the axis of cylinder; |z| = b

∴ The free surface would he cylindrical thus the velocity v' will he radial and v' will he function of r' (the radial distance from the centre of cylinder |Z| = b which is taken as origin and time t only.)

Let p' he the pressure at distance r'

- ∴ eq. of continuity: r' v' = F(t) ....(1)
- Euler's equation of motion

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=B(r')-\frac{1}{\rho}\frac{\partial p'}{\partial r'} \qquad \ldots\ldots\ldots (2)$$

Now, using (1) in (2)

$$\frac{F'(t)}{r'}+\frac{1}{2}\frac{\partial}{\partial r'}\left(v'^2\right)=0-\frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

From (1)

$$v'=\frac{F'(t)}{r'}$$

$$\frac{\partial v'}{\partial t}=\frac{p'(t)}{r'}$$

On integrating w.r.t r'

$F'(t)\log r' + \frac{1}{2}v'^2 = -\frac{1}{\rho}p' + A$; Where A is integration constant. ……(3)

- Initially, where $t = 0,\ v' = 0,\ p = P$

From (3), we have

$F'(o) = \log r' = \frac{-P}{\rho} + A$ …….(4)

Again, $p = \pi$, when $r' = a$ & $p = o$ when $r' = b$

$\therefore$ eq. (4) gives,

$F'(o)\log a = -\frac{\pi}{\rho} + A$ & $F'(o)\log b = A$

Sowing above eq.s we get

$F'(o)$ & A

$A = -\log b\frac{\pi}{\rho\log\left(\frac{a}{b}\right)}$, $F'(o) = -\frac{\pi}{\rho\log(a/b)}$

We have, from (4),

$$\frac{p}{\rho} = \frac{\pi}{\rho\log(a/b)}\log r' - \frac{\pi\log b}{\rho\log(a/b)}$$

$$\Rightarrow \quad P = \pi\frac{\log r' - \log b}{\log(a/b)}$$

$$\Rightarrow P = \pi\frac{\log r' - \log b}{\log a - \log b}$$

Q.1. A mass of liquid of density ρ whose external surface is a long circular cylinder of radius a which is subject to a constant pressure Π, surrounds a coaxial long circular cylinder of radius b. The internal cylinder is suddenly destroyed; show that if v is the velocity at the internal surface, when the radius is r, then

$$v^2 = \frac{2\Pi\left(b^2 - r^2\right)}{\rho r^2\log\left\{\left(r^2 + a^2 - b^2\right)/r^2\right\}}$$

Sol. So let's start doing the mental exercise!

When the inner cylinder is suddenly destroyed, the motion of the liquid will take place along the radii of the normal sections of the cylinder.

Hence the velocity will be function of r' (the radial distance from the centre of the cylinder $|z| = a$ which is taken as origin) and time t only. Let p be the pressure at a distance r'.

Then the equation of continuity is $r'v' = F(t)$…(1)

From (1), $\partial v'/\partial t = F'(t)/r'$ …(2)

The equation of motion is $\dfrac{\partial v'}{\partial t}+v'\dfrac{\partial v'}{\partial r'}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$ As there is no body for**ce.**

So $$\frac{F'(t)}{r}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r'},\ \text{using (2)}$$

Integrating; $F'(t)\log r' + \dfrac{1}{2}v'^2 = -\dfrac{p}{\rho} + C$, being an arbitrary constant …(3)

Let r and R be the radii of the internal and external surfaces of the cylinder and let v and V be the velocities there at any time t. Hence, we have

When $r' = r$, $v' = v$ $p = 0$ ...(4)

and when $r' = R$, $v' = V$, $p = \Pi$ …(5)

Using (4) and (5), (2) reduces to $F'(t)\log r + v^2/2 = C$ ….(6)

and $F'(t)\log R + V^2/2 = -\Pi/\rho + C$ …(7)

Subtracting (7) from (6), we have; $F'(t)(\log r - \log R) + (v^2 - V^2)/2 = \Pi/\rho$ …(8)

From (1), $rv = RV = F(t)$ …(9)

But $v = dr/dt$ and $V = dR/dt$. So (9) becomes $2rdr = 2RdR = 2F(t)dt$ …(10)

Also $R^2 - r^2 = a^2 - b^2$ …(11)

From (9), $F'(t) = \dfrac{d}{dt}(rv)=\dfrac{d}{dr}(rv).\dfrac{dr}{dt}=v\dfrac{d}{dr}(rv)$, as $v=\dfrac{dr}{dt}$ …(12)

Putting the values of $F'(t)$ and V given by (12) and (9) respectively in (8) yields

$$v=\frac{d}{dt}(rv).\log\frac{r}{R}+\frac{1}{2}\left(v^2-\frac{r^2v^2}{R^2}\right)=\frac{\Pi}{\rho}$$

$$rv\frac{d}{dr}(rv).\log\frac{r}{R}+\frac{1}{2}rv^2\left(1-\frac{r^2}{R^2}\right)=\frac{\Pi r}{\rho}$$

$$\frac{1}{2}\frac{d}{dr}\left\{(rv)^2\right\}.\log\frac{r}{R}+\frac{1}{2}r^2v^2\left(\frac{1}{r}-\frac{r}{R^2}\right)=\frac{\Pi r}{\rho}$$

$$\frac{d}{dr}\left(\frac{1}{2}r^2v^2\log\frac{r}{R}\right)=\frac{\Pi r}{\rho}, \qquad \text{…(13)}$$

where we have used (10) *i.e.* $RdR = rdr$.

Integrating (13), $\dfrac{1}{2}r^2v^2\log\dfrac{r}{R}=\dfrac{\Pi r^2}{2\rho}+C'$, ; C' being an arbitrary constant …(14)

But $v = 0$ when $r = b$. So $C' = -\pi b^2/2\rho$.

$\therefore$ From (14), $$r^2v^2 \log\frac{r}{R} = \frac{\Pi}{\rho}(r^2 - b^2)$$

$$r^2v^2 \log\left(\frac{r}{R}\right)^2 = \frac{2\Pi}{\rho}(r^2 - b^2)$$

$$v^2 = \frac{2\Pi\left(r^2 - b^2\right)}{\rho r^2 \log\left(r^2 / R^2\right)} = \frac{2\Pi\left(r^2 - b^2\right)}{\rho r^2 \log\left(R^2 / r^2\right)^{-1}} = -\frac{2\Pi\left(r^2 - b^2\right)}{\rho r^2 \log\left(R^2 / r^2\right)} = \frac{2\Pi\left(b^2 - r^2\right)}{\rho r^2 \log\left(R^2 / r^2\right)}$$

Thus, $$v^2 = \frac{2\Pi\left(b^2 - r^2\right)}{\rho r^2 \log\left\{\left(r^2 + a^2 - b^2\right)/ r^2\right\}}, \text{ using (11)}$$

Q.2. A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure on which at an infinite distance is Π and is such that the work done by this pressure on a unit area through a unit of length is one-half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity; prove that the time filling up the cavity will be $\pi a(\rho/\pi)^{1/2} \{2 - (3/2)^{3/2}\}$, a being the initial radius of the cavity, and P the density of the fluid.

Sol. At any time *t*, let *v*' be the velocity at a distance *r*' and *p* be the pressure there. Let *r* be the radius of the cavity at that time and *v* be the velocity there.

Equation of continuity is $r'^2v' = F(t) = r^2v$ …(1)

From (1), $\partial v'/\partial t = F'(t)/r'^2$ …(2)

The equation of motion is $\dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r} = \dfrac{\mu}{r'^2} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$; **body force is given as inversely proportional to square of distance r'.**

So $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{P}{\rho} + C,$$ *C* being an arbitrary constant

But *v*' = 0 and *p* = Π when *r*' = ∞. So C = Π/ρ. Hence the above equation beomes

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{\Pi - P}{\rho} \qquad \text{…(3)}$$

Also *v*' = *v* and *p* = 0 when *r*' = *r*. So from (3), we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{\Pi}{\rho} \qquad \text{…(4)}$$

From (1), $F'(t) = \frac{d}{dt}\left(r^2 v\right) = 2r\frac{dr}{dt}.v + r^2\frac{dv}{dt} = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\frac{dr}{dt}$

$= 2rv^2 + r^2 v\frac{dv}{dr}$, as $v = \frac{dr}{dt}$

Using the above value of F'(t), (4) gives

$$-\frac{1}{r}\left\{2rv^2 + r^2 v\frac{dv}{dr}\right\} + \frac{1}{2}v^2 = \frac{\mu}{r} + \frac{\Pi}{\rho} \quad \text{or} \quad 2rvdv + 3v^2 dr = -2\left(\frac{\mu}{r} + \frac{\Pi}{\rho}\right)dr$$

or $$2r^3 vdv + 3v^2 r^2\, dr = -2r^2\left(\frac{\mu}{r} + \frac{\Pi}{\rho}\right)dr \quad \text{or} \quad d(r^3v^2) = -2\left(\mu r + \frac{\Pi}{\rho}r^2\right)dr$$

Integrating, $r^3v^2 = -\left(\mu r^2 + \frac{2\Pi}{3\rho}r^3\right) + C'$, C' being an arbitrary constant …(5)

Initially, when $r = a$, $v = 0$. So $C' = \mu a^2 + (2\Pi/3\rho)a^3$.

∴ From (5), $$r^3v^2 = \mu(a^2 - r^2) + \frac{2\Pi}{3\rho}(a^3 - r^3)\text{….(6)}$$

Since the work done by Π is half the work done by the attractive force, we have

$$\Pi \times 1 \times 1 = \frac{1}{2}\int_\infty^a\left(-\frac{\mu}{r^2}\right)\rho dr \quad \text{so that} \quad \mu = \frac{2\Pi a}{\rho}.$$

Putting this value of μin (6), we get

$$r^3v^3 = \frac{2\Pi a}{\rho}(a^2 - \rho^2) + \frac{2\Pi}{3\rho}(a^3 - r^3)$$

or $$r^3v^2 = \frac{2\Pi}{3\rho}\left\{3a\left(a^2 - r^2\right) + a^3 - r^3\right\} \quad \text{or} \quad v^2 = \frac{2\Pi}{3\rho}\frac{\left\{3a\left(a^2 - r^2\right) + a^3 - r^3\right\}}{r^3}$$

or $$\frac{dr}{dt} = -\left(\frac{2\Pi}{3\rho}\right)^{1/2}\frac{\left\{3a\left(a^2 - r^2\right) + a^3 - r^3\right\}^{1/2}}{r^{3/2}} \quad \text{…(7)}$$

wherein negative sign is taken as r decreases when t increases.

Let T be the time of filling the cavity. Then we have, $r = a$ when $t = 0$ and $r = 0$ when $t = T$. Hence (7) gives on integration.

$$\int_0^T dt = -\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_a^0\frac{r^{3/2}dr}{\left\{3a\left(a^2 - r^2\right) + a^3 - r^3\right\}^{1/2}}$$

$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^a \frac{r^{3/2}dr}{(r+2a)\sqrt{(a-r)}} \quad \ldots(8)$$

Put $r = a\sin^2\theta$ so that $dr = 2a\sin\theta\cos\theta$. Then (8) reduces to

$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{a^{3/2}\sin^3\theta.2a\sin\theta\cos\theta d\theta}{a\left(2+\sin^2\theta\right).a^{1/2}\cos\theta} = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{\sin^4\theta d\theta}{2+\sin^2\theta}$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \left(\sin^2\theta - 2 + \frac{4}{2+\sin^2\theta}\right)d\theta$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[\frac{\pi}{4} - \pi + 4\int_0^{\pi/2} \frac{d\theta}{2+\sin^2\theta}\right] = 2a\left(\frac{3\rho}{\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + 4\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{2\sec^2\theta + \tan^2\theta}\right]$$

$$= 2a\left[-\frac{3\pi}{4} + 4\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{2+3\tan^2\theta}\right] = 2a\left(\frac{3\rho}{\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + \frac{4}{3}\int_0^{\infty} \frac{dt}{(2/3)+t^2}\right]$$

[Putting $\tan\theta = t$ and $\sec^2\theta d\theta = dt$]

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left\{-\frac{3\Pi}{4} \times \frac{4}{3}.\sqrt{\frac{3}{2}}\left[\tan^{-1}\left(t\sqrt{\frac{3}{2}}\right)\right]_0^{\infty}\right\} = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + \frac{4}{3}\left(\frac{3}{2}\right)^{1/2} . \frac{1}{2}\pi\right]$$

$$= a\pi\left(\frac{\rho}{\Pi}\right)^{1/2} \left\{-\frac{3}{2} \times \left(\frac{3}{x}\right)^{1/2} + \frac{4}{3} \times \frac{3}{2}\right\} = \pi a\left(\frac{\rho}{\Pi}\right)^{1/2} \left[2 - \left(\frac{3}{2}\right)^{3/2}\right].$$

Q.3. A spherical hollow of radius a initially exists in an infinite fluid, subject to constant pressure at infinity. Show that the pressure at distance r' from the centre when the radius of the cavity is r is to the pressure at infinity as $3r^2r'^4 + (a^3 - 4r^3)r'^3(a^3 - r^3)r^3 : 3r^2r'^4$.

Sol. Let v' be the velocity at a distance r' at any time t and p be the pressure there.

Let v be the velocity of the inner surface of radius r. Then the equation of continuity is

$r'^2v' = F(t) = r^2v$ …(1)

From (1), $\partial v'/\partial t = F'(t)/r'^2$…(2)

The equation of motion is $\dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \quad \text{using (2)}$$

Integrating, $-\dfrac{F'(t)}{r'} + \dfrac{1}{2}v'^2 = -\dfrac{p}{\rho} + C$, C being an arbitrary constant …(3)

Let Π be the pressure at infinity. Thus $v' = 0$ and $p = \Pi$ when $r' = \infty$. So (3) gives $C = \Pi/\rho$.

Then (3) reduces to $-\dfrac{F'(t)}{r'}+\dfrac{1}{2}v'^2=\dfrac{\Pi p}{\rho}$ …(4)

But $p = 0$ and $v' = v$ when $r' = r$. Then (4) gives $-\dfrac{F'(t)}{r'}+\dfrac{1}{2}v'^2=\dfrac{\Pi}{\rho}$ …(5)

From (1), $F'(t) = \dfrac{d}{dt}(r^2v) = 2r\dfrac{dr}{dt}v + r^2\dfrac{dv}{dt} = 2rv\dfrac{dr}{dt} + r^2\dfrac{dv}{dr}\dfrac{dr}{dt}$

$$2rv^2 + r^2v\frac{dv}{dr} \qquad \left[\because v=\frac{dr}{dt}\right]$$

Using the above value of F'(t), (5) gives

$$-\frac{1}{r}\left\{2rv^2+r^2v\frac{dv}{dr}\right\}+\frac{1}{2}v^2=\frac{\Pi}{\rho} \quad \text{or} \quad -rv\frac{dv}{dr}-\frac{3}{2}v^2=\frac{\Pi}{\rho} \qquad \ldots(6)$$

Multiplying both sides by $(-2r^2\,dr)$, (6) gives

$$2r^3v\,dv + 3r^2v^2dr = -\frac{2\Pi}{\rho}r^2dr \qquad \text{or} \qquad d(r^3v^2) = -\frac{2\Pi}{\rho}r^2dr$$

Integrating, $r^3v^2 = -\dfrac{2\Pi r^3}{3\rho} + C'$, C' being an arbitrary constant. …(7)

But when $r = a$, $v = 0$. Hence $C' = (2\Pi a^3)/(3\rho)$

∴ From (7), $r^3v^2 = \dfrac{2\Pi}{3\rho}(a^3 - r^3)$…(8)

Putting the value of v from (8) in (5), we get

$$F'(t) = r\left(\frac{1}{2}v^2-\frac{\Pi}{\rho}\right)=r\left[\frac{\Pi}{3\rho}\frac{a^3-r^3}{r^3}-\frac{\Pi}{\rho}\right]$$

or $F'(t) = \dfrac{\Pi}{3\rho}\dfrac{a^3-4r^3}{r^2}$ …(9)

From (1), $v' = (r^2v)/r'^2$ …(10)

Using (9) and (10), (4) reduces to

$$\frac{\Pi-p}{\rho}=\frac{1}{r'}.\frac{\Pi}{3\rho}.\frac{a^3-4r^3}{r^2}+\frac{1}{2}\frac{v^2r^4}{r'^4}=\frac{\Pi}{3\rho}.\frac{a^3-4r^3}{r^3r'}+\frac{\Pi}{3\rho}.\frac{r\left(a^3-r^3\right)}{r'^4}, \text{ using (8)}$$

$$\therefore \qquad \frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{\Pi}{3\rho}.\frac{a^3-4r^3}{r^2r'}-\frac{\Pi}{3\rho}.\frac{r\left(a^3-r^3\right)}{r'^4}$$

$$\frac{p}{\Pi}=\frac{3r^2r'^4+\left(a^3-4r^3\right)r'^3-\left(a^3-r^3\right)r^3}{3r^2r'^4}$$

which gives the required ratio of two pressures under consideration

Q.4. A solid sphere of radius a is surrounded by a mass of liquid whose volume is $(4\pi c^3)/3$ and its centre is a centre of attractive force varying directly as the square of the distance. If the solid sphere be suddenly annihilated, show that the velocity of the inner surface, when its radius is x, is given by

$$x^3x^3[(x^3 + c^3)^{1/3}\, x] = \left(\frac{2\Pi}{3\rho}+\frac{2\mu c^3}{9}\right)(a^3 - x^3)\,(a^3 - x^3)^{1/3},$$

where ρ is the density, Π the external pressure, μ the absolute force and $x = dx/dt$.

Sol. Let v' be the velocity at a distance r' at any time t and p be the pressure there. Let r and R be the radii and v and V the velocities of the inner and outer surfaces at time t.

Then the equation of continuity is $r'^2 v' = F(t) = r^2 v = R^2 V$ …(1)

From (1), $\partial v'/\partial t = F'(t)/r'^2$ …(2)

The equation of motion is $\frac{\partial v'}{dt} + v'\frac{\partial v'}{\partial r'} = -\mu r'^2 - \frac{1}{\rho}\frac{\partial p}{\partial r}$, where here $\mu r'^2$ is the attractive force

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\mu r'^2 - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating, $-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{\mu r'^3}{3} - \frac{p}{\rho}$ + C, C being an arbitrary constant …(3)

Now, when $r' = r$, $v' = v$ and $p = 0$

and when $r' = R$, $v' = V$ and $p = \Pi$

$\therefore$ (3) yields

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{\mu r^3}{3} + C \quad …(4)$$

and

$$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = -\frac{\mu R^3}{3} - \frac{\Pi}{\rho} + C \quad …(5)$$

Subtracting (4) from (5), we have

$$F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{1}{2}(v^2 - V^2) = \frac{\mu}{3}(r^3 - R^3) - \frac{\Pi}{\rho}$$

But $(4/3) \times \pi R^3 - (4/3) \times \pi r^3 = (4/3) \times \pi c^3$ so that $r^3 - R^3 = -c^3$.

$\therefore$

$$F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{1}{2}(v^2 - V^2) = \frac{\mu c^3}{3} - \frac{\Pi}{\rho} \quad …(6)$$

From (1), $F'(t) = \frac{d}{dt}(r^2 v) = \frac{d}{dr}(r^2 v).\frac{dr}{dt}$ or $F'(t) = v\frac{d}{dr}(r^2 v)$ …(7)

Again from (1), we get $V = (r^2 v)/R^2$ …(8)

Using (7) and (8), (6) gives

$$v\frac{d}{dr}(r^2 v).\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{1}{2}\left(v^2 - \frac{r^4 v^2}{R^4}\right) = \frac{\mu\rho c^3 + 3\Pi}{3\rho}$$

Multiplying both sides by r^2, we get

$$2r^2 v\frac{d}{dr}(r^2 v).\left(\frac{1}{r} - \frac{1}{R}\right) - v^2 r^4\left(\frac{1}{r^2} - \frac{r^2}{R^4}\right) = -\frac{\mu\rho c^3 + 3\Pi}{3\rho}r^2$$

$$\frac{d}{dr}(vr^2)^2.\left(\frac{1}{r} - \frac{1}{R}\right) - (vr^2)^2\left(\frac{1}{r^2} - \frac{r^2}{R^4}\right) = -\frac{\mu\rho c^3 + 3\Pi}{3\rho}r^2 \quad …(9)$$

From (1), $r^2 v = R^2 V$ *or* $r^2\frac{dr}{dt} = R^2\frac{dR}{dt}$ i.e, $r^2 dr = R^2 dR$ ..(10)

Integrating (9) and using (10), we have

$$r^4v^2\left(\frac{1}{r}-\frac{1}{R}\right)=-\frac{2\left(\mu c^3\rho=3\prod\right)}{3\rho}\int r^2dr+C'=-\frac{2\left(\mu c^3\rho+3\prod\right)}{9\rho}r^3+C'$$

When $r = a$, $v = 0$ so that $C'=\frac{2\left(\mu c^3\rho+3\prod\right)}{9\rho}a^3$

$$\therefore \quad r^4v^2\left(\frac{1}{r}-\frac{1}{R}\right)=\frac{2\left(\mu c^3\rho+3\prod\right)}{9\rho}(a^3-r^3)$$

i.e. $$r^4v^2\left[\frac{1}{r}-\frac{1}{\left(r^3+c^3\right)^{1/3}}\right]=\left(\frac{2\mu c^3}{9}+\frac{2\prod}{3\rho}\right)(a^3-r^3)$$

Now, for the inner surface, $r = x$, $v = x$. Hence, the above relation reduces to

$$x^2x^3[(x^3+c^3)^{1/3}-x]=\left(\frac{2\mu c^3}{9}+\frac{2\prod}{3\rho}\right)(a^3-x^3)\,(x^3-c^3)^{1/3}.$$

Q.5. A sphere is at rest in an infinite mass of homogenous liquid of density ρ, the pressure at infinity being P. If the radius R of the sphere varies in such a way that R = a + b cos nt, where b > a, show that pressure at the surface of the sphere at any time is

$$P+\frac{bn^2\rho}{4}(b-4a\cos nt-5b\cos 2nt).$$

Sol. Let v' be the velocity at a distance r' at any time t and p' be the pressure there. Again, let v be the velocity on the surface of sphere of radius R, where $R = a + b\cos nt$ …(1)

Then the equation of continuity is $r'^2v' = F(t) = R^2v$…(2)

From (2), $\partial v'/\partial t = F'(t)/r'^2$ …(3)

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p'}{\partial r'} \quad \text{or} \quad \frac{F'(t)}{r'}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial p'}{\partial r'},$$

using (3)

Integrating, $\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\frac{p'}{\rho}+C$, C being an arbitrary constant

Given: when $r' = \infty$, $v = 0$, $p' = P$. So C = P/ρ. So the above equation gives

$$\therefore \quad -\frac{F'(t)}{r}+\frac{1}{2}v'^2=\frac{P-p'}{\rho} \quad \text{…(4)}$$

Let $p' = p$ when $r' = R$. Also, $v' = v$ when $r' = R$ Then, (4) yields.

$$\therefore -\frac{F'(t)}{R}+\frac{1}{2}v^2=\frac{P-p}{\rho} \quad \text{or} \quad p = P+\rho\left[\frac{F'(t)}{R}-\frac{1}{2}v^2\right] \quad \text{…(5)}$$

From (2); $F'(t)=\frac{d}{dt}(vR^2)=2R\frac{dR}{dt}.v+R^2\frac{dv}{dt}=2R\left(\frac{dR}{dt}\right)^2+R^2\frac{d^2R}{dt^2}$ $\left[\because v=\frac{dR}{dt}\right]$

Using the above value of $F'(t)$ and noting $v = dR/dt$, we have

$$\frac{F'(t)}{R}-\frac{1}{2}v^2=2\left(\frac{dR}{dt}\right)^2+R\frac{d^2R}{dt^2}-\frac{1}{2}\left(\frac{dR}{dt}\right)^2=\frac{3}{2}\left(\frac{dR}{dt}\right)^2+R\frac{d^2R}{dt^2}$$

$= (3/2) \times (-bn \sin nt)^2 + (a + b \cos nt)(-bn^2 \cos nt)$, using

$= (bn^2/2) \times (3b \sin^2 nt - 2b \cos^2 nt - 2a \cos nt)$

$= (bn^2/4) \times [3b(1 - \cos 2nt) - 2b(1 + \cos 2nt) - 4a \cos nt]$

$= (bn^2/4) \times (b - 4a \cos nt - 5b \cos 2nt)$

Hence (5) reduces to $\quad p = P + \dfrac{bn^2 \rho}{4}(b - 4a \cos nt - 5b \cos 2nt)$.

Q.6. A sphere whose radius at time t is b + a cos nt, is surrounded by liquid extending to infinity under no forces. Prove that the pressure at distance r from the centre is less than the pressure Π at infinity by.

$$\rho \frac{n^2 a}{r}(b + a\cos nt)\left\{a(1 - 3\sin^2 nt) + b\cos nt + \frac{a^3 \sin^2 nt}{2r^3}(b + a\cos nt)^3\right\}$$

Prove also that least pressure at the surface of the sphere during the motion is $\Pi - n^2\rho a(a + b)$

Sol. Let v' be the velocity of the fluid at a distance *r'* from the origin at any time *t* and *p* be the pressure there.

Let $r' = b + a \cos nt$ and let *r* be the radius of any concentric sphere and *v* be the velocity there.

Then the equation continuity is $\quad r'^2 v = F(t) - r^2 v. \ldots(1)$

From (1), $\quad \partial v / \partial t = F(t)/r^2 \quad \ldots(2)$

The equation of motion s

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{or} \quad \frac{F'(t)}{r^2} + \frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right) = \frac{1}{\rho}\frac{\partial p}{\partial r}, \text{ using (2)}$$

Integrating it with respect to r, we have

$$\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C, \text{ C being an arbitrary constant} \quad \ldots(3)$$

When $r = \infty, v = 0, p = \Pi$, so (3) given C = Π/ρ. Hence (3) reduces to

$$\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi - p}{\rho} \ldots(4)$$

Now, $\quad r' = b + a\cos nt \quad \Rightarrow \quad v' = dr'/dt = -an\sin nt.$

Then, (1) $\quad \Rightarrow F(t) = r^2 v' = (b + a\cos nt)^2(-an\sin nt)$

$$F(t) = -an(b + a\cos nt)^2 \sin nt. \ \ldots(5)$$

Differentiating (5) with respect to 't', we have

$$F'(t) = 2a^2n^2(b + a\cos nt)\sin^2 nt - an^2(b + a\cos nt)^2 \cos nt$$

$$F'(t) = an^2(b + a\cos nt)[2a\sin^2 nt - (b + a\cos nt)\cos nt] \quad \ldots(6)$$

Now, (4) $\Rightarrow \Pi - \rho = -(\rho / r) F'(i) + (1/2) \times \rho v^2 \ldots(7)$

$$\Pi - \rho = -(\rho / r) F'(t) + (\rho / 2)\{F(t)/r^2\}^2, \text{ using (1)}$$

Using (5) and (6) the above equation becomes.

$$\Pi - \rho = -(\rho / r) \times an^2 (b + a\cos nt)[2a\sin^2 nt - (b + a\cos\ nt)\cos nt] + (\rho / 2r^4) \times a^2 n^2 (b + a\cos nt)^4 sin^2 nt$$

$$= (\rho a n^2 / r) \times (b + a\cos nt)\{a(1 - 3\sin^2 nt) + b\cos\ nt\ \ + (a/2r^3) \times \sin^2 nt (b + a\cos nt)^2\}$$

Second part :

At surface *r = r' = b + a* cos *nt, v = v' = dr' / dt* = - an sin t.

Also, using (6), (4) reduces to

$$\frac{\Pi - p}{\rho} = \frac{F'(t)}{r'} + \frac{1}{2} v'^2 = -\frac{1}{b + a\cos nt} an^2 (b + a\cos nt)$$

$$[2a\sin^2 nt - (b + a\cos nt)\cos nt] + (1/2) \times a^2 n^2 \sin^2 nt$$

$$= n^2 a[a(1 - 3\sin^2 nt) + b\cos nt + (1/2) \times a\, sin^2 nt] \qquad \ldots(8)$$

For the maximum or minimum of p, we must have $\frac{d}{dt}\left(\frac{\Pi - p}{\rho}\right) = 0$

i.e., $n^2 a[-6an\sin nt\cos nt - bn\sin nt + na\sin nt\ \ \cos nt] = 0$

Giving *sin nt = 0* or *cos nt = - (b/5a)* i.e. *nt = 0* or *nt = cos*$^{-1}$ *(-b/5a)*.

Now, $\frac{d^2}{dt^2}\left(\frac{\Pi - P}{\rho}\right) = \frac{d}{dt}[n^2 a\{-3an\sin 2nt - bn\, sn\, nt + (1/2) \times an\sin 2nt\}$

$$= n^2 a[-6an^2\cos 2nt - bn^2\cos nt + an^2\cos 2nt] = n^2 a[-6an^2 - bn^2 + an^2], when\ nt = 0$$

$\therefore \frac{d^2}{dt^2}\left(\frac{\Pi - P}{\rho}\right)$ is negative when nt = 0 $\Rightarrow \frac{d^2 P}{dt^2}$ is positive when nt = 0

Putting *nt = 0* in (8), the least pressure *p* is given by $(\Pi - p)/\rho = n^2 a(a + b)$

and hence, the required least pressure $= p = \Pi - \rho n^2 a(a + b)$

Similar question as above. A sphere of radius a is alone in an unbounded liquid which is at rest at a great distance from the sphere and is subject to no external force. The sphere s forced to vibrate radially keeping its spherical shape, the radius r at any time being given by r = a + b cos nt. Show that if Π is the pressure in the liquid at a great distance from the sphere, the least pressure (assumed positive) at the surface of the sphere during the motion is $\Pi - n^2 \rho b(a + b)$.

Q.7. A volume (4/3) $\times \pi c^3$ or gravitating liquid of density ρ s initially in the form a spherical shell of infinitely great radius. If the liquid shell contract under the influence of it's own attraction, there being no external or internal pressure, show that when the radius of the inner spherical surface s r, its velocity will be given by. $V^2 = (4\pi / \rho R / 15r^3)(2R^4 + 2R^3 r + 2R^3 r^2 - 3Rr^3 - 3r^4)$.

Where γ s the constant of gravitation, and $R^3 = r^3 + c^3$.

We now apply Newton's second law for impulsive motion to the fluid enclosed by the parallelopiped, namely,
Total impulse applied along *x*-axis = Change of momentum along *x*-axis

$$\therefore \quad -\delta x \delta y \delta z \frac{\partial \bar{\omega}}{\partial x} + \rho \delta x \delta y \delta z \, I_x = \rho \delta x \delta y \delta z \left(u_2 - u_1\right)$$

or $$\rho\left(u_2 - u_1\right) = \rho I_x - \left(\partial \bar{\omega} / \partial x\right) \quad ...(6)$$

Similarly $$\rho\left(v_2 - v_1\right) = \rho \, I_y - \left(\partial \bar{\omega} / \partial y\right) \quad ...(7)$$

and $$\rho\left(w_2 - w_1\right) = \rho \, I_z - \left(\partial \bar{\omega} / \partial z\right). \quad ...(8)$$

Equations (6), (7) and (8) are the required equations of motion of an incompressible fluid under impulsive forces.

Q.8. A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being ∏. The sphere is suddenly annihilated. Show that the pressure at a distance *r* from the centre immediately falls to ∏ (1 – *a*/*r*.)
Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius *a*/2, the impulsive pressure sustained by the surface of this sphere is $(y\prod\rho^2/6)^{1/2}$.

Sol. Let v' be the velocity at a distance *r*' from the centre of the sphere at any time *t* and *p* the pressure there. Then the equation of continuity is $r'^2 v' = F(t)$ ...(1)

From (1), $\partial v'/\partial t = F'(t)/r'^2$ ...(2)

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1 \partial p}{\rho \partial r'}$$

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2} v'^2\right) = -\frac{1}{\rho} \frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $-\dfrac{F'(t)}{r'} + \dfrac{1}{2} v'^2 = -\dfrac{p}{r'} + C$, C being an arbitrary constant.

When $r' = \infty$, then $p = \prod$ and $v' = 0$ so that $C = \prod/\rho$.

$$\therefore \quad -\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{\Pi - p}{\rho} \quad ...(3)$$

When the sphere is suddenly annihilated, we have

$t = 0$, $r' = a$, $v' = 0$ and $p = 0$

$$\therefore \text{ From (3), } -\frac{F'(0)}{a} = \frac{\Pi}{\rho} \quad \text{so that} \quad F'(0) = -\frac{a\Pi}{\rho}$$

Hence immediately after the annihilation of the sphere (with $t = 0$, $\tau = 0$), (3) reduces to

$$\frac{a\Pi}{\rho r'} + 0 = \frac{\Pi - p}{\rho} \qquad \text{or} \qquad p = \Pi\left(1 - \frac{a}{r'}\right) \qquad \text{...(4)}$$

Thus at the time of annihilation when $r' = r$, the pressure is given by

$$p = \Pi(1 - a/r'). \qquad \text{...(5)}$$

Second Part. If $\bar{\omega}$ be the impulsive pressure at distance r', then we have

$$d\bar{\omega} = -\rho v' \, dr' \qquad \text{...(6)}$$

Let r be the radius of the inner surface and v the velocity there. Then by the equation of continuity, we have

$$F(t) = r^2 v = r'^2 v' \text{ so that } v' = (r^2 v)/r'^2 \qquad \text{...(7)}$$

$\therefore$ (6) gives $\quad d\omega' = \rho v \, (r^2/r'^2) dr'$

Integrating with respect to r', we get $\bar{\omega} = \rho v\left(r^2 / r'\right) + C'$ ...(8)

When $\quad r' = \infty, \quad \bar{\omega} = 0 \quad$ so that $\; C' = 0$.

Hence, (8) reduces to $\quad \bar{\omega} = \rho v\left(r^2 / r'\right)$, ...(9)

which gives the impulsive pressure $\bar{\omega}$ at a distance r'. Since $r = a/2$, (9) reduces to

$$\bar{\omega} = \frac{1}{4}\rho v a^2 \cdot \frac{1}{r'} \qquad \text{...(10)}$$

We now determine velocity v at the inner surface of the sphere. Setting $r' = r$, $v' = v$ and $p = 0$ in (3), we get

$$\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \qquad \text{...(11)}$$

From (7), $\quad F'(t) = \frac{d}{dt}\left(r^2 v\right) = 2r\frac{dr}{dt}v + r^2\frac{dv}{dt} = 2r\frac{dr}{dt}v + r^2\frac{dv}{dt}\frac{dr}{dr}$

Thus, $\quad F'(t) = 2rv^2 + r^2 v\frac{dv}{dr}$, as $v = \frac{dr}{dt}$

$\therefore$ (11) gives $\quad -\frac{1}{r}\left(2rv^2 + r^2 v\frac{dv}{dr}\right) + \frac{1}{2}v^2 = \frac{\Pi}{\rho}$

Multiplying both sides of the above equation by ($-2r^2\,dr$), we get

$$2r^3 v\,dv + 3r^2 v^2 dr = -\frac{2\Pi r^2}{\rho}dr \qquad \text{or} \qquad d\left(r^3 v^2\right) = -\frac{2\Pi r^2}{\rho}dr$$

Integrating, $\quad r^3 v^2 = -\frac{2\Pi r^3}{3\rho} + C''$, C'' being an arbitrary constant

When $\quad r = a, \quad v = 0$ so that $C'' = -\frac{2\Pi a^3}{3\rho}$.

$$\therefore \qquad r^3 v^2 = \frac{2\Pi}{3\rho}\left(a^3 - r^3\right)$$

The velocity v on the surface of the sphere of radius $a/2$ (which would be the inner surface on which the liquid impinges) is given by (12) by replacing r by $a/2$

$\therefore \qquad v^2 = \dfrac{2\Pi}{3\rho} \times \dfrac{a^3 - a^3/8}{a^3/8} = \dfrac{14}{3} \times \dfrac{\Pi}{\rho}$

Putting this value of v in (10), the impulsive pressure at a distance r' is given by

$$\bar{\omega} = \frac{\rho}{r}\left(\frac{14}{3} \times \frac{\Pi}{\rho}\right)^{1/2} \frac{a^2}{r'} \qquad \text{...(13)}$$

Hence the desired impulsive pressure on the surface of the sphere of radius $a/2$ is given by setting $r' = a/2$ in (13).

$$\therefore \qquad \bar{\omega} = \frac{\rho}{r}\left(\frac{14}{3} \times \frac{\Pi}{\rho}\right)^{1/2} \times \frac{a^2}{(a/2)} = \left(\frac{7\Pi\rho a^2}{6}\right)^{1/2}$$

Q.9. A portion of homogeneous fluid is contained between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance. The inner spherical surface is suddenly annihilated and when the radii of the inner and outer surfaces of the fluid are r and R the fluid impinges on a solid ball concentric with these surfaces, prove that the impulsive pressure at any point of the ball for different values of R and r varies as

$$\{(a^2 - r^2 - A^2 + R^2)(1/r - 1/R)\}^{1/2}$$

Sol. Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is $r'^2 v' = F(t)$ ...(1)

From (1), $\qquad \partial v'/\partial t = F'(t)/r'^2 \qquad$ ...(2)

Taking μ/r'^2 as the force towards the centre of the sphere, the equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{dr'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'} \quad \text{or} \quad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $\qquad -\dfrac{F'(t)}{r'} + \dfrac{1}{2}v'^2 = \dfrac{\mu}{r'} - \dfrac{p}{\rho} + C$, C being an arbitrary constant ...(3)

Let r and R be the internal and external radii of the fluid at any time t and v and V be the velocities there. Thus, we have

When $\qquad r' = R, \quad v' = V, \quad p = 0$ and also when $r' = v, \qquad v' = v, \quad p = 0$

$\therefore$ (3) yields $\qquad -\dfrac{F'(t)}{R} + \dfrac{1}{2}V^2 = C + \dfrac{\mu}{R}$ ...(4)

and $\qquad -\dfrac{F'(t)}{r} + \dfrac{1}{2}v^2 = C + \dfrac{\mu}{r}$ ...(5)

Subtracting (4) from (5), we have

$$-F'(t)\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{1}{2}\left(v^2 - V^2\right) = \mu\left(\frac{1}{r} - \frac{1}{R}\right) \qquad \text{...(6)}$$

From the equation of continuity (1), we have

$$r^2 v = R^2 V = F(t) \qquad \text{...(7)}$$

From (7), $\qquad r^2 \dfrac{dr}{dt} = R^2 \dfrac{dR}{dt} = F(t)$

$r^2\, dr = R^2\, dR = F(t)\, dt \quad$ ...(8)

Using (7), (6) reduces to

$$-F(t)\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{1}{2}\{F(t)\}^2\left[\frac{1}{r^4} - \frac{1}{R^4}\right] = \mu\left[\frac{1}{r} - \frac{1}{R}\right]$$

Multiplying both sides by $2F(t)\,dt$, we get

$$-F(t)\,F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{1}{2}\{F(t)\}^2\left[\frac{2F(t)}{r^4}-\frac{2F(t)}{R^4}\right]dt=\mu\left[\frac{2F(t)}{r}-\frac{2F(t)}{R}\right]dt$$

or $$-2F(t)\,F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{1}{2}\{F(t)\}^2\left[\frac{2dr}{r^2}-\frac{2dR}{R^2}\right]=\mu\left(2rdr-2RdR\right),\ \text{using (8)}$$

Integrating, $$-\{F(t)\}^2\left[\frac{1}{r}-\frac{1}{R}\right]=\mu\left(r^2-R^2\right)+C',\text{being arbitrary constant...(9)}$$

Since velocity is zero when $r = a$ and $R = A$, if follows that $F(t) = 0$. Then (9) reduces to

$$0 = \mu\,(a^2 - A^2) + C' \qquad i.e., \qquad C' = -\mu(a^2 - A^2)$$

$\therefore$ (9) becomes $$-\{F(t)\}^2\left[\frac{1}{r}=\frac{1}{R}\right]=\mu\left(r^2-R^2-a^2+A^2\right)...(10)$$

it $\bar{\omega}$ be the impulsive pressure at a distance r', then we have

$$d\,\bar{\omega}=-\rho v'\,dr'=-\rho\frac{F(t)}{r'^2}dr',\ \text{using (1)}$$

Integrating, $$\bar{\omega}=\frac{\rho F(t)}{r'^2}+C'',\ C''\text{ being an arbitrary constant}$$

But when, $r' = R$, $\bar{\omega}=0$ so that $C' = [\rho\, F(t)]/R$. So the above equation gives

$$\therefore \quad \bar{\omega}=\rho F(t)\left(1/r'-1/R\right)$$

Hence the impulsive pressure at any point of the ball where $r' = r$ is given by

$$\bar{\omega}=\rho F(t)\left(1/r-1/R\right) \quad ...(11)$$

From (10), $$F(t)=\left\{\frac{\mu\left(a^2-r^2-A^2+R^2\right)}{\left(1/r-1/R\right)}\right\}^{1/2} \quad ...(12)$$

Using (12), (11) reduces to $\bar{\omega}=\rho\sqrt{\mu}\left\{\left(a^2-r^2-A^2+R^2\right)\left(1/r-1/R\right)\right\}^{1/2}$,

showing that the required impulsive pressure varies as $\left\{\left(a^2-r^2-A^2+R^2\right)\left(1/r-1/R\right)\right\}^{1/2}$

Exam Point. Many problems solved so far in this chapter may also be solved by using the **energy equation.** This principle is used to shorten the solution.

In what follows, we will give two methods to solve many problems.

The energy equation is stared as follows: *The rate of increase of energy in the system is equal to the rate at which work is done on the system.*

Note: "the volume integral form of Bernoulli's equation".

$$\frac{d}{dt}(T+W)=R-\frac{dI}{dt}=\int_s pq.nds+\int_v p\nabla.q\,dv$$

Energy equation for for incompressible fluids.

Since $I = 0$ for incompressible fluids, so above equation reduces to

$\frac{d}{dt}(T+W)=R.$

Q.10. An infinite mass of fluid is acted on by a force $\mu/r^{\frac{3}{2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere r = c in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{\frac{1}{2}} c^{\frac{5}{4}}$.

Sol. Method I At any time *t*, let *v*' be the velocity at distance *r*' from the centre. Again, let *r* be the radius of the cavity and *v* its velocity. Then the equation of continuity yields

$$r'^2 v' = r^2 v \qquad \ldots(1)$$

When the radius of the cavity is r, then

Kinetic energy = $\int_r^\infty \frac{1}{2}\left(4\pi r'^2 \rho dr'\right).v'^2$ [$\because$ Kinetic energy = 1/ 2 × mass × (velocity)^2]

$= 2\pi\rho r^4 v^2 \int_r^\infty \frac{dr'}{r'^2}$ using (1)

$= 2\pi\rho r^3 v^2$

The initial kinetic energy is zero.

Let *V* be the work function (or fore potential) due to external forces. Then, we have

$-\frac{\partial V}{\partial r'} = \frac{\mu}{r'^{3/2}}$ so that $V = \frac{2\mu}{r'^{1/2}}$

$\therefore$ the work done = $\int_r^c V\,dm$ being the elementary mass

$= \int_r^c \left(\frac{2\mu}{r'^{1/2}}\right).4\pi r'^2\, dr'\rho = 8\pi\mu\rho \int_r^c r'^{3/2}\, dr' = \frac{16}{5}\pi\rho\mu\left(c^{5/2} - r^{5/2}\right)$

We now use energy equation, namely, Increase in kinetic energy = work done

This $\Rightarrow$ $2\pi\rho r^3 v^3 - 0 = (16/5)\times \pi\rho\mu\left(c^{5/2} - r^{5/2}\right)$

$$v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2} \frac{\left(c^{5/2} - r^{5/2}\right)^{1/2}}{r^{3/2}} \qquad \ldots(2)$$

Wherein negative sign is taken because *r* decreases as *t* increases.

$$T = \int_V \frac{1}{2}\rho q^2 dV, \qquad W = \int_V \rho\Omega dV, \qquad I = \int_V \rho E\, dV, \qquad \ldots(6)$$

Where E is the intrinsic energy per unit mass,

Since $\nabla.(pq) = p\nabla.q + q.\nabla p$, we have $q.\nabla p = \nabla.(pq) - p\nabla.q$

$\therefore$ R.H.S. of (4) $= -\int_V \nabla.(pq)\,dV + \int_V p\nabla.q dV = \int_S pq.n\, dS + \int_S p\nabla.q dV, \qquad \ldots(7)$

[By Gauss divergence theorem]

When **n** is unit inward normal and *dS* is the element of the fluid surface *S*. We now prove that

$$\int_V p\nabla\cdot q dV = -\frac{dI}{dt}$$

Now, E is defined as the work done by the unit mass of the fluid against external pressure p (assuming that there exists a relation between pressure and density) from its actual state to some standard state in which p_0 and ρ_0 are the values of pressure and density respectively.

$\therefore \quad E = \int_V^{V_0} p dV, \quad$ where $V\rho = 1 \qquad$ *i.e.*, $\quad V = 1/\rho$

or
$$E = \int_\rho^{\rho_0} p d\left(\frac{1}{\rho}\right) = -\int_\rho^{\rho_0} \frac{p}{\rho^2} d\rho = \int_{\rho_0}^{\rho} \frac{p}{\rho^2} dp \qquad \dots(9)$$

From (9), $\quad \dfrac{dE}{d\rho} = \dfrac{p}{\rho^2} \quad$ and so $\quad \dfrac{dE}{dt} = \dfrac{dE}{d\rho}\dfrac{d\rho}{dt} = \dfrac{p}{\rho^2}\dfrac{d\rho}{dt}$

Multiplying both side by ρdV and then integrating over a volume *V*, we have

$$\int_V \frac{dE}{dt} \rho dV = \int_V \frac{p}{\rho}\frac{d\rho}{dt} dV \qquad \dots(10)$$

But
$$\frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV + E\frac{d}{dt}(\rho dV)$$

$$\frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV, \text{ using (4)} \qquad \dots(11)$$

Also from the equation of continuity $\quad \dfrac{d\rho}{dt} = -\rho\nabla\cdot q \qquad \dots(12)$

Using (11) and (12) , (10) reduces to

$$\frac{d}{dt}\int_V E\rho dV = -\int_V p\nabla.q dV \quad \text{or} \quad \frac{dI}{dt} = -\int_V p\nabla q dV, \text{ by (6)}$$

Which proves (8).

Again the rate of work done by the fluid pressure on an element δS of S is $p\delta S$ **n. q.**

Hence the rate at which work is being done by the fluid pressure is

$$\int_V pq.n\, dS = R, \text{ (say)} \qquad \dots(13)$$

Using (8) and (13), (7) reduces to

R.H.S. of (4) = $R - dI/dt$ $\qquad \dots(14)$

Hence using (6) and (14), (4) reduces to $\quad \dfrac{d}{dt}(T + W + I) = R \qquad \dots(15)$

PREVIOUS YEARS QUESTIONS

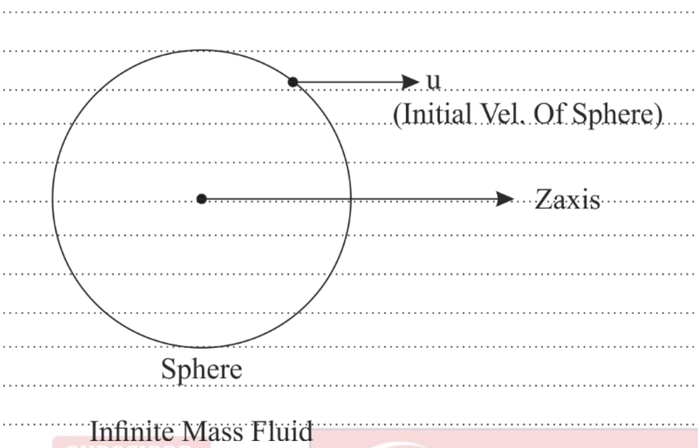
CHAPTER 3. EULER'S EQUATION OF MOTION

Q1. A sphere of radius R, whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, so that the pressure at the surface of the sphere at time *t* is

$\Pi + \dfrac{1}{2}\rho\left\{\dfrac{d^2R^2}{dt^2} + \left(\dfrac{dR}{dt}\right)^2\right\}$. **[8b UPSC CSE 2019]**

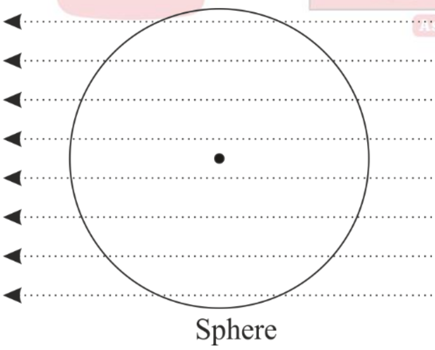
Q2. Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if ρ be the density and v be the velocity at a distance x from a fixed point at time t, then $\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left\{\rho\left(v^2 + k\right)\right\}$. **[5d UPSC CSE 2018]**

4. IRROTATIONAL MOTION IN 3 D: MOTION OF SPHERE & CYLINDER



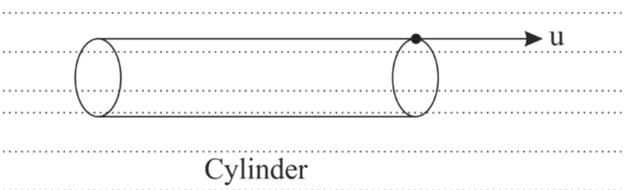
CASE – (1)

mass of fluid is moving but sphere is NOT moving along z – axis

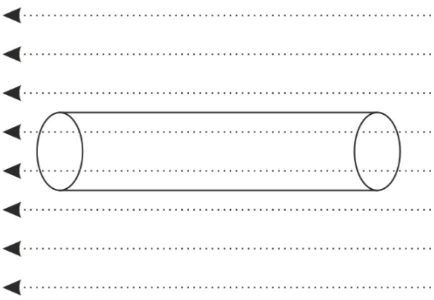


Case (2):

Fluid is NOT moving but Cylinder is moving along (-z axis), now we're interested in studying the motion/ position.



Case (3)



Case (4); we may have:

→ Two concentric spheres; motion

→ Two concentric cylinders ; motion

Source Basic ideas:

- $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$
- $\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{1}{r^2 sn^2\theta}\frac{\partial^2\phi}{\partial w^2}=0$

in spherical coon (r, q, w)

$\because$ Rotation is along z – axis i.e., symmetric about z – axis, so we can ignore the last term containing derivative w.r.t w.

$\therefore$ We have,

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}=0 \qquad \ldots\ldots\ldots(1)$$

- Now to solve PDE (1); We use variable separable method

  $\because$ ϕ is depending on r & θ so; we start by assuming ϕ as funtion of r multiplied by some funtion of θ.

i.e., $\phi(r,\theta)=f(r).g(\theta)$

Now, getting $\frac{\partial\phi}{\partial r},\frac{\partial\phi}{\partial\theta},\frac{\partial^2\phi}{\partial r^2}$, ………… & using in (1), we try to get sol. = $\phi(r,\theta)$

Note: For diff. eq. (1) we suppose

$\boxed{\phi=f(r)\cos\theta}$ as by the variable seperable method.

$\phi=f(r)\cos\theta$

$\therefore\ \frac{\partial\phi}{\partial r}=f'(r)\cos\theta,\frac{\partial^2\phi}{\partial r^2}=f''(r)\cos\theta$

$\frac{\partial\phi}{\partial\theta}=-f(r)\sin\theta,\frac{\partial^2\phi}{\partial\theta^2}=-f(r)\cos\theta$

Now, using these in (1) we get a diff. eq. which can be easily solved.

$$f''(r)\cos\theta+\frac{2}{r}f'(r)\cos\theta+\frac{1}{r^2}(-f(r)\cos\theta)+\frac{\cot\theta}{r^2}(-f(r)\sin\theta)=c$$

$$\Rightarrow \cos\theta\left\{f''(r)+\frac{2}{r}f'(r)-\frac{2}{r^2}f(r)\right\}=0$$

$$\Rightarrow \frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}-\frac{2}{r^2}f(r)=0$$

$$r^2\frac{d^2f}{dr^2}+2r\frac{df}{dr}-2f=0 \qquad \ldots\ldots\ldots\ldots (2)$$

Now solving the diff. eq.

$$r^2\frac{d^2f}{dr^2}+2r\frac{df}{dr}-2f=0 \qquad \ldots\ldots\ldots\ldots (3)$$

$\therefore$ (3) is turned into,

(D(D-1) + 2D - 2) f = 0 where D=d/dz

Auxiliary eq.

$M^2 + m - 2 = 0$

$M^2 + 2m - m - 2 = 0$

$(m + 2)(m - 1) = 0$

$M = 1, -2$

$\therefore$ $C.F = e^z + e^{-2z}$

$C.F = c_1 e^{\log_e r} + C_2 e^{-2\log_e r}$

$C.F = c_1 r + \frac{c_2}{r^2} = Ar + \frac{B}{r^2}$

$\therefore$ Sol. of (3) is $\boxed{f(r) = Ar + \frac{B}{r^2}}$

Where A & B also arbitrary constants.

Exam pout

i.e., Ultimately; ϕ is given as

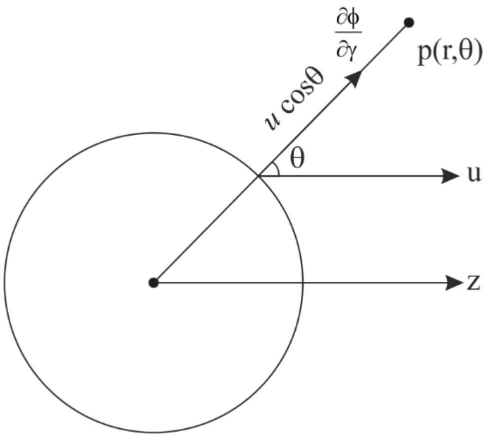
$\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$: where A & B are arbitrary constants, Now by using given initial condition, we try to find A & B; To get finally the in quid velocity potential ϕ.

Article 1:

Motion of a sphere in a infinite mass of fluid which is at rest at infinity.

Step -1

To study this motion; we need to follow following three constraints:



(i) ϕ satisfies Laplace eq. i.e, $\nabla^2\phi = 0$

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} = 0 \quad \ldots\ldots\ldots\ldots(1)$$

(ii) $\frac{-\partial\phi}{\partial r} = u\cos\theta \ldots.(2)$ at $r = a$ i.e., at surface of sphere

"Normal component of velocity" = vel. At that point i.e. r = a

(iii) $\frac{\partial\phi}{\partial r} = 0$ ………. (3)at $r = \infty$: at infinity

$\because$ it is at rest $\therefore$ v = 0; $\frac{\partial\phi}{\partial r} = 0$

Step -2

$\because$ For motion of sphere: $\nabla^2\phi = 0$ & motion is symmetric about z – axis.

$\therefore$ We have by (1) as $\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$; ………… (4)

where A & B are arbitrary constant

$$\therefore \frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{r^3}\right)\cos\theta = -u\cos\theta \, at \, r = a$$

$$\Rightarrow \left(A - \frac{2B}{a^3}\right)\cos\theta = -u\cos\theta \quad \&$$

$$\frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{r^3}\right)\cos\theta = 0 \; at \; r = \infty$$

$\Rightarrow$ (A-0) $\cos\theta$ = 0

$\Rightarrow$ A = 0

$$\therefore B = \frac{1}{2}ua^3.$$

i.e., we have the velocity potential as

$$\boxed{\phi = \frac{1}{r^2}\left(\frac{1}{2}ua^3\right)\cos\theta = \frac{1}{2}\frac{ua^3}{r^2}\cos\theta} \quad \text{..........} (5)$$

Now, we can get lines (streamlines) of flow.

Remember streamlines if ϕ is f' of r, θ. Then streamlines are given by

$$\frac{dr}{-\partial\phi/\partial r} = \frac{rd\theta}{-\partial\phi/r\partial\theta}$$

$\because$ from (5),

$$\frac{\partial\phi}{\partial r} = \frac{-1}{r^3}ua^3\cos\theta, \frac{\partial\phi}{\partial\theta} = \frac{-1}{2}\frac{ua^3}{r^2}\sin\theta$$

So, we have, lines of flow as,

$$\frac{dr}{\frac{-1}{r^3}ua^3\cos\theta} = \frac{rd\theta}{\frac{-1}{r}\left(\frac{1}{2r^2}ua^3\sin\theta\right)}$$

$$\Rightarrow \quad \frac{dr}{\cos\theta} = \frac{rd\theta}{\frac{1}{2}\sin\theta}$$

$$\Rightarrow \quad \frac{dr}{r} = 2\frac{\cos\theta}{\sin\theta}d\theta$$

$\Rightarrow$ log r =2log (sin θ) + log c

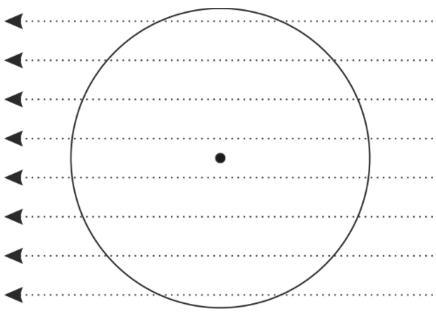
$\Rightarrow$ $\boxed{r = c\sin^2\theta}$ formula deriving is also important.

Exampoint

$\boxed{r = c\sin^2\theta}$: lines of flow/ Streamlines.

Case (ii): studying the motion : when fluid is flowing & sphere is at rest:

Zaxis



Let fluid is flowing with velocity u along –ve Z axis. So here; we manage the potential by adding one factor. $ur\cos\theta$ n ϕ (eq. (5))

So, in this case, $\boxed{\phi = \frac{1}{2r^2}ua^3\cos\theta + ur\cos\theta}$

Now, let's try to find streamlines (lines of flow)

$$\frac{dr}{\frac{\partial\phi}{\partial r}} = \frac{r d\theta}{\frac{\partial\phi}{r\partial\theta}}$$

$$\frac{dr}{4\left(1-\frac{a^3}{r^3}\right)\cos} = \frac{r d\theta}{\frac{-1}{r}\left(\frac{4a^3}{2r^2}\sin\theta + 4r\sin\theta\right)}$$

$$\frac{dr}{4\left(1-\frac{a^3}{r^3}\right)\cos\theta} = \frac{r d\theta}{-4\left(1+\frac{a^3}{2r^3}\right)sin\theta}$$

$$2\cot\theta d\theta = \frac{2r^3+a^3}{r^3-a^3}\frac{dr}{r}$$

$$2\cot\theta d\theta = \left(\frac{3r^2}{r^3-a^3} - \frac{1}{r}\right)dr$$

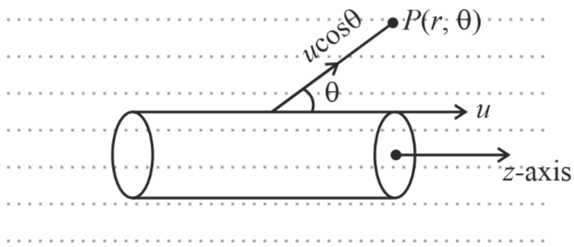
On integrating

$$-2\log\sin\theta = \log\left(r^3-a^3\right) - \log r - \log c$$

$$\sin^2\theta = \frac{cr}{r^3-a^3}$$

$$\boxed{r^2\sin^2\theta\left(1-\frac{a^3}{r^3}\right) = c}$$

Motion of cylinder in an infinite incompressible fluid :-



(i) $\nabla^2\psi = 0$

(ii) $-\frac{\partial\psi}{\partial r} = u\cos\theta : r = a$

(iii) $-\frac{\partial\psi}{\partial r} = 0 ; r = \infty$

Note: - by ψ : we can get ϕ easily (conjugate to ψ)

$$\therefore \boxed{\psi(r,\theta) = \left(Ar + \frac{B}{r}\right)\cos\theta}$$

Now the interpretation will be same as we discussed for sphere.

Note:- For two concentric cylinders:-

$$\psi(r, \theta) = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta$$

Ex.1. Show that when a sphere of radius a moves with initial velocity U through a perfect incompressible infinite fluid, the acceleration of a particle of the fluid at (r,0) is

$$3U^2\left(\frac{a^3}{r^4}-\frac{a^6}{r^7}\right)$$

Step-1

Note:- If we superimpose a negative velocity – U to both sphere and liquid; then the sphere will come at rest.

So fluid will be in flow.

$\therefore$ by case (ii), we have

$$\phi = u\left(r+\frac{a^3}{2r^2}\right)\cos\theta \qquad \text{........ (1)}$$

Now velocity components(from dynamics)

$$\because \quad \dot r = \frac{-\partial\phi}{\partial r} = -U\left(1-\frac{a^3}{r^3}\right)\cos\theta \quad \text{............ (2)}$$

$$r\dot\theta = \frac{-1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1+\frac{a^3}{2r^3}\right)\sin\theta \quad \text{........... (3)}$$

Also,

$$\ddot r = U\left(1-\frac{a^3}{r^3}\right)\sin\theta.\dot\theta - U\frac{3a^3}{r^4}\dot r\cos\theta$$

$$\ddot r = U\left(1-\frac{a^3}{r^3}\right)\frac{u}{r}\left(1+\frac{a^3}{2r^3}\right)\sin^2\theta + U^2\frac{3a^3}{r^4}\left(1-\frac{a^3}{r^3}\right)\cos^2\theta \quad \{\text{Using (2) \& (3)}\}$$

Step – 2

At the point (r,0), the velocity is only along the direction of r; as $\theta=0$

$\therefore$ For acceleration,

Req. Acc = $\ddot r \; at \; \theta = 0$

$$= 3U^2\frac{a^3}{r^4}\left(1-\frac{a^3}{r^3}\right)$$

$$= 3U^2\left(\frac{a^3}{r^4}-\frac{a^6}{r^7}\right).$$

Ex.2. An infinite ocean of an incompressible perfect liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is distributed by the sphere and the pressure in the liquid at a great distance from the obstacle is π. Show that the thrust on that half of the sphere on which the liquid impinges is $\pi a^2\left(\pi-\frac{\rho v^2}{16}\right)$

Mental Exercise:

From the first sentence:- liquid is in motion & sphere is at rest

$\Rightarrow \quad \phi = u\left(r + \frac{a^3}{2r^2}\right)\cos\theta$ .... (1)

Ans. $\because \quad \phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$ .... (1)

$$\frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$\therefore \left(\frac{\partial\phi}{\partial r}\right)_{r=a} = U\left(1 - \frac{a^3}{a^3}\right)\cos\theta = 0$$

$$\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=a} = \left[\frac{U}{r}\left(r + \frac{a^3}{2r^2}\right)(-sin\theta)\right] = \frac{-3}{2}U\sin\theta$$

$$\because \quad q^2 = \left\{\left(\frac{-\partial\phi}{\partial r}\right)^2 + \left(\frac{-1}{r}.\frac{\partial\phi}{\partial\theta}\right)^2\right\} = 0 + \frac{9}{4}U^2\sin^2\theta.$$

$q^2 = \frac{9}{4}U^2\sin^2\theta.$ .... (2)

- In steady motion in, the pressure at any point by Bernoulli's eq. is given by.

  $p/\rho + \frac{1}{2}q^2 = c$ .... (3)

  But $p = \pi, q = U$ at infinity

$$\therefore \quad c = \frac{\pi}{\rho} + \frac{1}{2}u^2$$

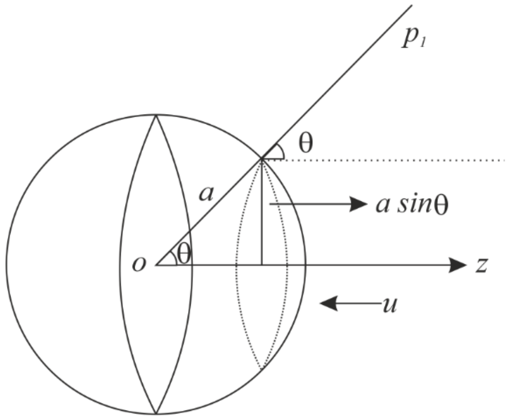
$\therefore$ From (3)

$p = \pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2$ .... (4)

- Now, using (2) in (4) the pressure (p_1) at any arbitrary point on surface of the sphere r = a s given by

  $p_1 = \pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho q^2 \sin^2\theta$

- Hence, the required thrust on that half of sphere our which the liquid impinges.



- $$= \int_0^{\pi/2} (p_1 \cos\theta) 2\pi a \sin\theta . a d\theta$$

$$= 2\pi a^2 \int_0^{\pi/2} \left(\pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta \right) \sin\theta \cos\theta d\theta$$

$$= 2\pi a^2 \left[\pi \times \frac{\pi\pi}{2\sqrt{2}} + \frac{1}{2}\rho U^2 \times \frac{\pi\pi}{2\sqrt{2}} - \frac{9}{8}\rho U^2 \frac{\sqrt{2}\sqrt{1}}{2\sqrt{3}} \right]$$

$$= \pi a^2 \left[\pi + \frac{1}{2}\rho U^2 - \frac{9}{16}\rho U^2 \right]$$

$$= \pi a^2 \left[\pi - \frac{\rho U^2}{16} \right]$$

$$= \pi a^2 \left[\pi - \frac{\rho U^2}{16} \right]$$

Article #2: Concentric sphere (Problem of initiate motion.)

Explanation:-

- A sphere of radius ‘a’ is surrounded by a concentric sphere of radius b; the space b/w being filled with fluid at rest.
- The inner sphere is given a velocity u and outer sphere a velocity v in the same direction.

  Now, to determine the initial motion of fluid.

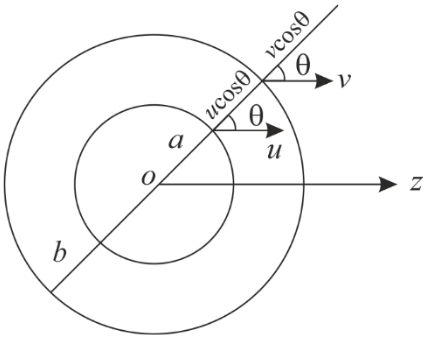
Step (1)

(i) $\nabla^2\phi = 0$ ……… (1)

(ii) $\frac{-\partial\phi}{\partial r} = u\cos\theta$ at $r = a$ ……….. (2)

(iii) $\frac{-\partial\phi}{\partial r} = v\cos\theta$ at r = b ………… (3)

Step (2):



$\because$ From (1), we have $\phi = \left(Ar + \dfrac{B}{r^2}\right)\cos\theta$ ….. (4)

Now, using (ii) and (iii) we get from (4)

$$A = \frac{4a^3 - vb^3}{b^3 - a^3} \;\&\; B = \frac{(u-v)a^3b^3}{2(b^3 - a^3)}$$

$$\therefore \quad \boxed{\phi = \left(\frac{ua^3 - vb^3}{b^3 - a^3}\right) r\cos\theta + \frac{(u-v)a^3b^3}{2(b^3 - a^3)}\frac{\cos\theta}{r^2}}$$

Ex. 3. Prove that for liquid contained between the two instantaneously concentric spheres, when the outer (radius a) is moving parallel to the x-axis with a velocity u and the inner (*radius* b) is moving parallel to the axis of y with velocity v, the velocity potential is

$$-\frac{1}{a^3 - b^3}\left\{a^3ux\left(1 + \frac{b^3}{2r^3}\right) - b^3vy\left(1 + \frac{a^3}{2r^3}\right)\right\}$$

and find the kinetic energy.

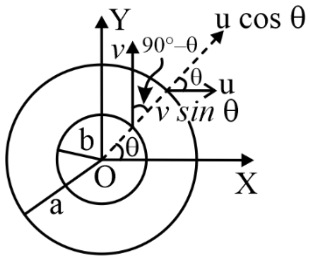
Sol. Here boundary conditions are

$-\,\partial\phi/\partial r = u \cos\theta,$ when $r = a$ ...(1)

and $-\,\partial\phi/\partial r = v \sin\theta,$ when $r = b.$ ...(2)

Moreover ϕ must satisfy the Laplace's equation

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} = 0. \qquad ...(3)$$



The above considerations suggest that ϕ must involve terms containing sin θ and *cos* θ.

So we assume that

$$\phi = (Ar + B/r^2)\cos\theta + (Cr + D/r^2)\sin\theta \qquad ...(4)$$

$$\therefore \quad -\frac{\partial\phi}{\partial r} = \left(-A + \frac{2B}{r^3}\right)\cos\theta + \left(-C + \frac{2D}{r^3}\right)\sin\theta \;.....(5)$$

Using (1) and (2), (5) gives

$$(-A + 2B/a^3)\cos\theta + (-C + 2D/a^3)\sin\theta = u\cos\theta$$

and $$(-A + 2B/b^3)\cos\theta + (-C + 2D/b^3)\sin\theta = v\sin\theta$$

Comparing the coefficients of $\cos\theta$ and $\sin\theta$, (6) and (7) give

$$-A + 2B/a^3 = u, \qquad -C + 2D/a^3 = 0$$

$$-A + 2B/b^3 = 0, \qquad -C + 2D/b^3 = v$$

Solving (8) and (9), we get

$$A = -\frac{ua^3}{a^3 - b^3}, \quad B = -\frac{ua^3b^3}{2\left(a^3 - b^3\right)}, \quad C = \frac{ub^3}{a^3 - b^3}, \quad D = \frac{ua^3b^3}{2\left(a^3 - b^3\right)}.$$

$$\phi = -\frac{ua^3}{a^3 - b^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta + \frac{vb^3}{a^3 - b^3}\left(r + \frac{a^3}{2r^2}\right)\sin\theta$$

$$= -\frac{1}{a^3 - b^3}\left[a^3u\left(1 + \frac{b^3}{2r^3}\right)r\cos\theta - b^3v\left(1 + \frac{a^3}{2r^3}\right)r\sin\theta\right]$$

$$= -\frac{1}{a^3 - b^3}\left[a^3u\left(1 + \frac{b^3}{2r^3}\right)x - b^3v\left(1 + \frac{a^3}{2r^3}\right)y\right] \quad \text{as } x = r\cos\theta, \text{ and } y = r\sin\theta$$

To determine *K.E.* The kinetic energy of the liquid is given by

$$T = -\frac{1}{2}\rho\int\int\phi\frac{\partial\phi}{\partial n}dS = -\frac{1}{2}\rho\int\int\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a}dS - \frac{1}{2}\rho\int\int\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=b}dS$$

Also, $-\partial\phi/\partial n$ denotes the outwards normal velocity

$$= \frac{1}{2}\rho\frac{1}{a^3 - b^3}\iint_{r=a}\left[a^3ux\left(1 + \frac{b^3}{2a^3}\right) - b^3vy\left(1 + \frac{1}{2}\right)\right](u\cos\theta)\,dS$$

$$-\frac{1}{2}\rho\frac{1}{a^3 - b^3}\iint_{r-b}\left[a^3ux\left(1 + \frac{1}{2}\right) - b^3vy\left(1 + \frac{a^3}{2b^3}\right)\right](v\sin\theta)\,dS$$

$$\left[Using\ (10)\ and \left(-\frac{\partial\phi}{\partial r}\right)_{r=a} = -u\cos\theta, \left(-\frac{\partial\phi}{\partial r}\right)_{r=b} = v\sin\theta\right]$$

$$= \frac{1}{4}\rho\frac{u^3\left(2a^3 + b^3\right)}{\left(b^3 - a^3\right)a}\iint_{r=a}x^2dS - \frac{3}{4}\rho\frac{uvb^3}{\left(a^3 - b^3\right)}\iint_{r=a}xy\,dS$$

$$-\frac{3}{4}\rho\frac{uva^3}{\left(b^3 - a^3\right)b}\iint_{r=b}xy\,dS + \frac{1}{4}\rho\frac{v^2\left(a^3 + 2b^3\right)}{\left(a^3 - b^3\right)b}\iint_{r=b}y^2\,dS$$

[Since, when $r = a$, $a\cos\theta = x$ and $a\sin\theta = y$ and when $r = b$, $b\cos\theta = x$ and $b\sin\theta = y$

$$= \frac{1}{4}\rho\frac{u^2\left(2a^3 + b^3\right)}{\left(a^3 - b^3\right)a}\cdot\frac{4}{3}\pi a^4 - 0 - + \frac{1}{4}\rho\frac{v^2\left(a^3 + 2b^3\right)}{\left(a^3 - b^3\right)b}\cdot\frac{4}{3}\pi b^4$$

$$\left[\because \iint_{r=a}x^2dS = M.I.\ \text{of the hollow sphere of radius } a \text{ about a diameter}\right.$$

$$= \frac{1}{2}\cdot\frac{2Ma^2}{3} = \frac{Ma^2}{3} = \frac{4\pi a^2\cdot a^2}{3} = \frac{4\pi a^4}{3}.$$

Similarly, $\iint_{r=b}y^2dS = \frac{4\pi b^4}{3}$

Also, $\iint_{r=a} xy\,dS = 0$ and $\iint_{r=b} xy\,dS = 0$ (being product of inertia)]

$$T = \frac{1}{3}\frac{\pi\rho}{a^3 - b^3}\left[2\left(u^2a^6 + v^2b^6\right) + a^3b^3\left(u^2 + v^2\right)\right]$$

Ex.4(i). A hollow spherical shell of inner radius a contains a concentric solid uniform sphere of radius b and density σ and the space between the two is filled with liquid of density ρ. If the shell is suddenly made to move with speed u, prove that a velocity v is imparted to the inner sphere, where

$$v = \frac{3ua^3}{2(\sigma/\rho)\left(a^3 - b^3\right) + a^3 2b^3}.$$

(ii) A spherical shell of internal radius a contains a concentric sphere of radius λa and density σ, the intervening space being filled with lived of density ρ and the whole system is at rest. If a velocity u is communicated to the shell prove that the initial velocity v communicated to the shell is given by

$$v = \frac{3u}{2(\sigma\rho)\left(1 - \lambda^3\right) + 1 + 2\lambda^3}$$

Sol. (i) The velocity potential φ must satisfy Laplace's equation $\nabla^2 \phi = 0$ and it must satisfy the following boundary conditions

$$-\partial\phi/\partial r = u\cos\theta, \quad \text{when} \quad r = a \qquad \text{...(1)}$$

and $$-\partial\phi/\partial r = v\cos\theta, \quad \text{when} \quad r = b \qquad \text{...(2)}$$

Accordingly, we assume that $\phi = (Ar + B/r^2)\cos\theta$ ...(3)

∴ $-\partial\phi/\partial r = -(A - 2B/r^3)\cos\theta$

Using (1) and (2), (4) gives

$$(-A + 2B/a^3)\cos\theta = u\cos\theta \quad \text{and} \quad (-A + 2B/b^3)\cos\theta = v\cos\theta.$$

These give $A = \dfrac{b^3v - a^3u}{a^3 = b^3}$ and $B = \dfrac{a^3b^3(v-u)}{2\left(a^3 - b^3\right)}$

$$\therefore \quad \phi = \frac{1}{a^3 - b^3}\left[\left(b^3v - a^3u\right)r + \frac{a^3b^3(v-u)}{2r^2}\right]\cos\theta \qquad \text{...(5)}$$

The impulsive pressure at any point of the solid sphere $r = b$ is given by

$$\bar{\omega} = (\rho\phi)_{r=b} = \frac{\rho b}{a^3 - b^3}\left[b^3v - a^3u + \frac{a^3}{2}(v-u)\right]\cos\theta$$

∴ Resultant impulsive pressure on the inner sphere

$$= \int_0^x \bar{\omega}\cos\theta \cdot b\,d\theta \cdot 2\pi b\sin\theta = \frac{2\rho b^3\pi}{a^3 - b^3}\left[b^3v - a^3u + \frac{a^3}{2}(v-u)\right]\int_0^\pi \cos^2\theta\sin\theta\,d\theta$$

$$= -\frac{4\rho b^3\pi}{3\left(a^3 - b^3\right)}\left[b^3v - a^3u + \frac{1}{2}a^3(v-u)\right].$$

Since the solid sphere of density σ and radius b moves with velocity v, the equation of motion gives

$$\frac{4}{3}\pi b^3\sigma v = -\frac{4\rho b^3\pi}{3\left(a^3 - b^3\right)}\left[b^3v - a^3u + \frac{1}{2}a^3(v-u)\right] \quad \text{or}$$

$$v = \frac{3ua^3}{3(\sigma/\rho)\left(a^3 - b^3\right) + a^3 2b^3}.$$

Part (ii). Here $b = a\lambda$.

Ex. 5. Liquid of density ρ fills the space between a solid sphere of radius a and density σ and a fixed concentric spherical envelope of radius b. Prove that the work done by an impulse which starts the solid sphere with velocity U is

$$\frac{1}{3}\pi a^3 U^2\left(2\sigma+\frac{2a^3+b^3}{b^3-a^3}\rho\right).$$

Sol. The total impulse I is given by

$$I = MU + \iint \bar{\omega} \cos\theta\, dS$$

But $$\iint \bar{\omega} \cos\theta\, dS = \frac{2}{3}\pi\rho U a^3 \frac{2a^3+b^3}{b^3-a^3},$$

and $$M = \text{mass of inner solid sphere} = (4.3) \times \pi a^3 \sigma$$

$$I = \frac{2\pi a^3}{3}\left(2\sigma+\frac{2a^3+b^3}{b^3-a^3}\rho\right)$$

Hence the work done by impulse $I = I \times$ (mean of the initial and final velocities)

$$= I \times \frac{0+U}{2} = \frac{1}{2}UI = \frac{\pi a^3 U^2}{3}\left(2\sigma+\frac{2a^3+b^3}{b^3-a^3}\rho\right)$$

Ex.6. The space between two concentric spherical shells of radii a and b ($a > b$) is with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V, in the same direction. Prove that the resultant impulsive pressure on the inner shell is

$$\frac{2\pi\rho b^3}{3\left(a^3-b^3\right)}\left[3a^3U-\left(a^3+2b^3\right)V\right]$$

Further show that the $K.E.$ of the liquid is

$$\frac{\pi\rho}{3\left(a^3-b^3\right)}\left[a^3b^3\left(V-U\right)^2+2\left(b^3V-a^3U\right)^2\right]$$

Sol. The velocity potential is given by

$$\phi = \frac{Ua^3-Vb^3}{b^3-a^3} r\cos\theta + \frac{\left(U-V\right)a^3b^3}{2\left(b^3-a^3\right)}\frac{\cos\theta}{r^2} = \frac{1}{a^3-b^3}\left[r\left(b^3V-a^3U\right)+\frac{a^3b^3\left(V-U\right)}{2r^2}\right]\cos\theta$$

The impulsive pressure at a point on the sphere $r = b$ is given by

$$\bar{\omega} = \left(\rho\phi\right)_{r=b} = \frac{\rho\cos\theta}{a^3-b^3}\left[b\left(b^3V-a^3U\right)+\frac{1}{2}a^3b\left(V-U\right)\right] \qquad \text{...(2)}$$

The resultant impulsive pressure on the inner shell ($r = b$)

$$= \int_0^x \bar{\omega}\cos\theta \cdot b\theta \cdot 2\pi b\sin\theta = \frac{2\pi b^3}{a^3-b^3}\left[b^3V-a^3U+\frac{1}{2}a^3\left(V-U\right)\right]\int_0^\pi \cos^2\theta\sin\theta\, d\theta, \text{by (2)}$$

$$= \frac{\pi b^3}{a^3-b^3}\left[2\left(b^3V-a^3U\right)+a^3\left(V-U\right)\right]\cdot\left(-\frac{2}{3}\right) = \frac{2\pi\rho b^3}{3\left(a^3-b^3\right)}\left[3a^3U-\left(a^3+2b^3\right)V\right]$$

From (1), $$\frac{\partial\phi}{\partial r} = \frac{1}{a^3-b^3}\left[b^3V-a^3U-\frac{a^3b^3\left(V-U\right)}{r^3}\right]\cos\theta$$

and $\qquad \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = \dfrac{1}{a^3-b^3}\left[b^3V - a^3U + \dfrac{a^3b^3(V-U)}{2r^3}\right](-sin\,\theta)$

$\therefore \qquad q^2 = (-\partial\phi/\partial r)^2 + (-\partial\phi/r\partial\theta)^2$

$$= \frac{1}{\left(a^3b^3\right)^2}\left[\left\{b^3V - a^3U - \frac{a^3b^3(V-U)}{r^3}\right\}^2 cos^2\theta + \left\{b^3V - a^3U + \frac{a^3b^3(V-U)}{2r^2}\right\}^2 sin^2\theta\right]$$

$$= \frac{1}{\left(a^3-b^3\right)^2}\left[\left(b^3V - a^3U\right)^2 + \frac{a^6b^6(V-U)^2}{r^6}\right]\left(cos^2\theta + \frac{1}{4} + sin^2\theta\right)$$

$$-\frac{2a^3b^3\left(b^3V - a^3U\right)(V-U)}{r^3}cos^2\theta + \frac{a^3b^3\left(b^3V - a^3U\right)(V-U)}{r^3}sin^2\theta\Bigg]$$

$$= \frac{1}{\left(a^3-b^3\right)^2}\left[\left(b^3V - a^3U\right)^2 + \frac{a^6b^6(V-U)^2}{4r^6}\left(1+3cos^2\theta\right) + \frac{a^3b^3\left(b^3V - a^2U\right)(V-U)}{r^3}\left(1-3cos^2\theta\right)\right]$$

$$K.E. = \int_0^x\int_b^a\left(\frac{1}{2}\rho q^2\right)2\pi r\,sin\,\theta\cdot r d\theta\,dr = \frac{\pi\rho}{\left(a^3-b^3\right)^2}\int_b^a\left[2\left(b^3V - a^3U\right)^2 + \frac{a^6b^6}{r^6}(V-U)^2\right]r^3dr$$

on putting value of q^2 and integrating w.r.t. θ]

$$= \frac{\pi\rho}{3\left(a^3-b^3\right)}\left[a^3b^3(V-U)^2 + 2\left(b^3V - a^3U\right)^2\right].$$

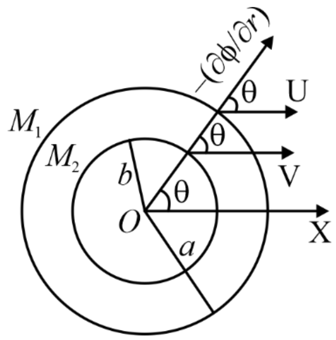
Ex.7. Incompressible fluid of density ρ is contained between two rigid concentric spherical the outer one of mass M_1 and radius *a*, the inner one of mas M_2 and radius *b*. A blow *P* is given to the outer surface. Prove that the initial velocities of the two containing (*U* of the outer and *V* for the inner) are given by the equations

$$\left\{M_1 + \frac{2\pi\rho a^3\left(2a^3+b^3\right)}{3\left(a^3-b^3\right)}\right\}U - \frac{2\pi\rho a^3b^3}{a^3-b^3}V - P,$$

$$\left\{M_2 + \frac{2\pi\rho b^3\left(2a^3+b^3\right)}{3\left(a^3-b^3\right)}\right\}V = \frac{2\pi\rho a^3b^3}{a^3-b^3}U.$$

Sol. $\quad \phi = \dfrac{1}{a^3-b^3}\left[\left(Vb^3 - Ua^3\right)r + \dfrac{(V-U)a^3b^3}{2r^2}\right]cos\,\theta \qquad ...(1)$

The normal blow *P* in the outer surface imparts velocity *U* to the outer and *V* to the inner spherical surface. Let $\bar{\omega}_1$, $\bar{\omega}_2$ be the impulsive pressures on an element *dS* of the boundary surface *r* = *a* and *r* = *b* respectively. Then



$$M_1 U = P - \iint \bar{\omega}_1 \cos\theta\, dS \quad \text{on} \quad r = a \quad ...(2)$$

and $$M_2 V = -\iint \bar{\omega}_2 \cos\theta\, dS \quad \text{on} \quad r = b \quad ...(3)$$

On $r = a$, from (1), $$\bar{\omega}_1 = (\rho\phi)_{r=a} = \frac{1}{a^3 - b^3}\left[\left(Vb^3 - Ua^3\right)a + \frac{1}{2}ab^3(V - U)\right]\cos\theta$$

$\therefore$ $$(2) \Rightarrow M_1 U = P - \int_0^\pi \bar{\omega}\cos\theta \cdot a\,d\theta \cdot 2\pi a \sin\theta$$

$$= P - \frac{2\pi a^2}{a^3 - b^3}\left[\left(Vb^3 - Ua^3\right)a + \frac{1}{2}ab^3(V - U)\right]\int_0^\pi \cos^2\theta \sin\theta\, d\theta$$

$$= P - \frac{\pi a^3}{a^3 - b^3}\left[3Vb^3 - U\left(2a^3 + b^3\right)\right] \times \left(-\frac{2}{3}\right)$$

Thus, $$\left\{M_1 + \frac{2\pi\rho a^3\left(2a^3 + bh3\right)}{3\left(a^3 - b^3\right)}\right\}U - \frac{2\pi\rho a^3 b^3}{a^3 - b^3}V = P.$$

Again on $r = b$ $$\bar{\omega}_2 = (\rho\phi)_{r=b} = \frac{1}{a^3 - b^3}\left[\left(Vb^3 - Ua^3\right)b + \frac{(V - U)a^3 b}{2}\right]\cos\theta$$

$\therefore$ (3) $\Rightarrow$ $$M_2 V = -\int_0^\pi \bar{\omega}_2 \cos\theta\, b\,d\theta \cdot 2\pi b \sin\theta$$

$$= -\frac{2\pi b^2 \rho}{a^3 - b^3}\left[\left(Vb^3 - Ua^3\right)b + \frac{(V - U)a^3 b}{2}\right]\int_0^\pi \cos^2\theta \sin\theta\, d\theta$$

$$= -\frac{2\pi b^3 \rho}{a^3 - b^3}\left[Vb^3 - Ua^3 + \frac{1}{2}a^3(V - U)\right] \cdot \left(-\frac{2}{3}\right)$$

Thus, $$\left\{M_2 + \frac{2\pi\rho b^3\left(2b^3 + a^3\right)}{3\left(a^3 - b^3\right)}\right\}V = \frac{2\pi\rho b^3 a^3}{a^3 - b^3}U.$$

PREVIOUS YEARS QUESTIONS

CHAPTER 4. AXISYMMETRIC MOTION

Q1. The space between two concentric spherical shells of radii $a, b(a < b)$ is filled with a liquid of density ρ. If the shells are set in motion, the inner one with velocity U in the *x*-

direction and the outer one with velocity V in the y-direction, then show that the initial motion of the liquid is given by velocity potential

$$\phi = \frac{\left\{ a^3 U \left(1+\frac{1}{2}b^3 r^{-3}\right)x - b^3 V \left(1+\frac{1}{2}a^3 r^{-3}\right)y \right\}}{\left(b^3 - a^3\right)}$$

where $r^2 = x^2 + y^2 + z^2$, the coordinates being rectangular. Evaluate the velocity at any point of the liquid. **[7b UPSC CSE 2016]**

Q2. A sphere is at rest in an infinite mass of homogenous liquid of density ρ, the pressure at infinity being P. If the radius R of the sphere varies in such a way that $R = a + b\cos nt$, where $b < a$, then find the pressure at the surface of the sphere at any time. **[8c 2016 IFoS]**

Q3. In an axisymmetric motion, show that stream function exists due to equation of continuity. Express the velocity components in terms of the stream function. Find the equation satisfied by the stream function if the flow is irrotational. **[8c UPSC CSE 2015]**

Q4. A rigid sphere of radius a is placed in a stream of fluid whose velocity in the undisturbed state is V. Determine the velocity of the fluid at any point of the disturbed stream.

[5e UPSC CSE 2012]

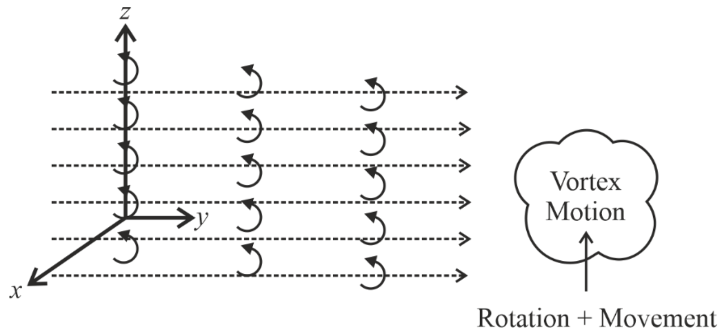
Note: The beauty of systematic learning is- You'll find solutions of almost every PYQ in above examples or questions attached with detailed answers. So to avoid repetition in this book, we have not put those solutions again as answers to PYQs.

THEME: VORTEX MOTION

$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

<u>Mental Exercise</u>

- Now let's start today's story:
- Rotational Motion



Vorticity $\Omega = \text{curl } \vec{q}$

As we have idea from calculus and dynamics that curl of a vector is associated with rotation property.

<u>Till now we have studied:</u>

Irrotational motion of fluid

→ $\text{curl } \vec{q} = 0$

$\exists$ some function $\phi = c$ (vel. Potential)

s.t $\vec{q} = -\text{grad } \phi$

↓

Motion in 2D

↓

Stream function

Complex potential

$\Rightarrow w = \phi + i\psi$

Fluid Kinematics + Motion in 2D

→ Sources & sinks

→ Doublet

→ Image system

Now we're going to study about Rotational & Irrotational Motion of fluids

- Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\therefore \text{curl } \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} - \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

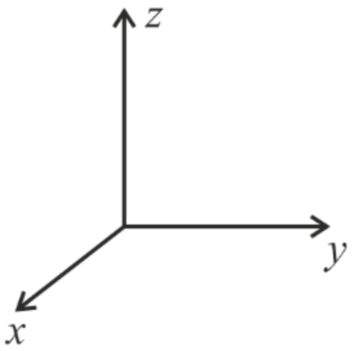
$$\vec{\Omega} = \Omega_x \hat{i} + \Omega_y \hat{i} + \Omega_z \hat{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

On comparing, we get

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Here Ω_x, Ω_y, Ω_z are called **"vorticity components".**

Vortex lines: $\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$



• **Motion in 2D:-**

If vorticity is along z-axies → $\Omega_x = 0, \Omega_y = 0, \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}$

Keywords:-

• Inside vortex

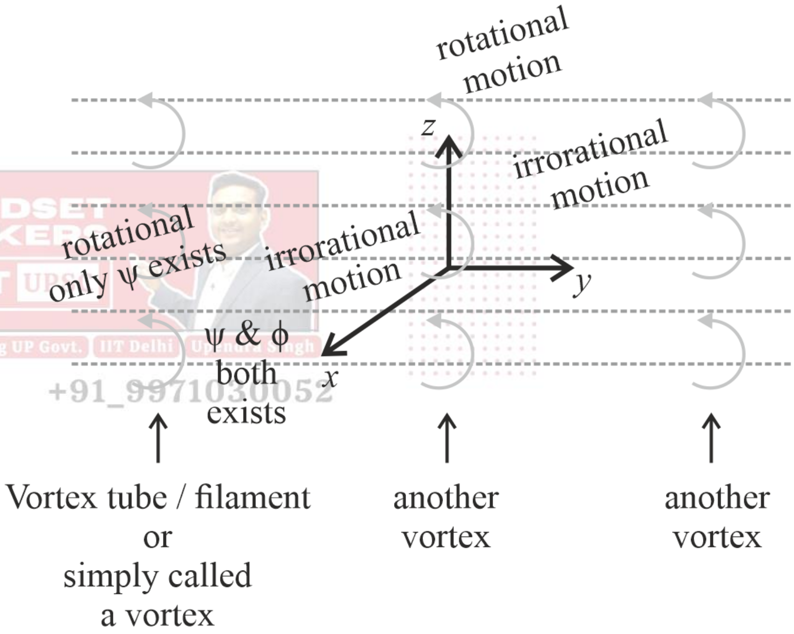
• Outside vortex

"Vortex has strength"

Here, Vortex is along z-axis & Motion is in *xy* plane

∴ $\Omega_x = 0$ and $\Omega_y = 0$, only Ω_z will be in picture

∵ we'll have ϕ & ψ in this motion



Let's study complex potential.

$w = \phi + i\psi$

(i) Inside vortex:- For above motion

∵ we have $\Omega_x = 0$, $\Omega_y = 0$, $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ …(1)

We know that : (Motion in 2D);

$$\frac{dx}{u} = \frac{dv}{v}$$

⇒ $vdx - udy = 0$ …(2)

For perfect (exact) diff. eq.; $\frac{\partial v}{\partial y} = \frac{-\partial u}{\partial x}$

i.e., $\exists\ \psi(x, y) = c$ …(3) s.t that the

total differentiation of (3) gives (2),

i.e., $d\psi = 0$ gives (2),

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{dy} dy = v\, dx - u\, dy$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial v}{\partial x}, \frac{\partial^2 \psi}{\partial y^2} = \frac{-\partial u}{\partial y} \quad \ldots(4)$$

Using (4) in (1),

Exampoint $\boxed{\Omega_z = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}}$

Remember :-

$$\Omega_z = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} : \textit{inside vortex}$$

$\Omega_z = 0$; *Outside vortex*

(ii) Outside the vortex:-

$\because$ Motion is irrotational,

$\downarrow$

There exists velocity potential ϕ

So, now we can try to establish some result including ϕ & ψ.

- Let $P(r, \theta)$ be an aubituary point.

$$\rightarrow \frac{\partial \psi}{\partial r} = -\frac{1}{r} \frac{\partial \phi}{\partial r}$$

Also, $\because$ Outside the vortex $\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

$$\because \nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

Note:- There is symmetry about origin, ψ must be independent of θ

We have

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + 0 = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{r} \right) = 0$$

On integrating ; $r \frac{\partial \psi}{dr} = c$; c is integration constant …(6)

$\psi = c \log r$ …(7)

$$\frac{\partial \psi}{\partial r} = \frac{c}{r} \Rightarrow \frac{-1}{r} \frac{\partial \phi}{\partial \theta} = \frac{c}{r}$$

On integrating; $\boxed{\phi = -c\theta}$ …(8)

Now, summarizing above discussion; we have

The complex potential $w = \phi + i\,\psi\ = -c\theta + ic\log r$ …(9)

Let K be the 'circulation' in the circuit embracing the vortex (strength of vortex)

Remember

$$\boxed{K = \int_{\theta=0}^{2\pi}\left(\frac{-1}{r}\frac{\partial\theta}{\partial\theta}\right)r\,d\theta} = c\int_{\theta=0}^{2\pi} d\theta = 2\pi c \Rightarrow c = \frac{K}{2\pi} \quad \ldots(10)$$

Using (1) in (9), we have

$$\therefore\ \phi = -\frac{K}{2\pi}\theta,\ \psi = \frac{K}{2\pi}\log r$$

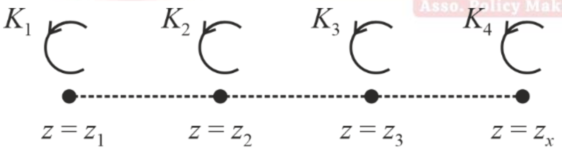
$\boxed{W = i\dfrac{K}{2\pi}\log z}$ As; $z = re^{i\theta}$; $\log z = \log(re^{i\theta})$ $= \log r + i\,\theta \therefore\ i\log z = i(\log r + i\theta)$

Note:-

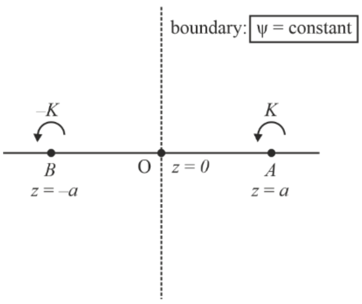
i. If vortex is not at origin but a some point $z = z_0$; then $\boxed{w = \dfrac{iK}{2\pi}\log(z - z_0)}$

ii. If there are several rectilinear vortices, then

$$\boxed{w = \frac{iK_1}{2\pi}\log(z - z_1) + \frac{iK_2}{2\pi}\log(z - z_2) + \ldots + \frac{iK_n}{2\pi}\log(z - z_n)}$$



Boundary: - (i) Plane boundary:



$$W = \frac{iK}{2\pi}\log(z - a) + \frac{i(-K)}{2\pi}\log(z + a)$$

$$\boxed{W = \frac{iK}{2\pi}\log\frac{(z - a)}{(z + a)}}$$

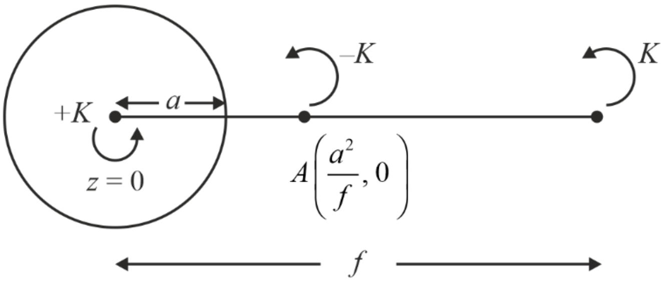
$\because\ z = x + iy$

$z = r(\cos\theta + i\sin\theta)$

As $z = re^{i\theta}$

Image of vertex at A (strength K) is vortex B (strength – K)

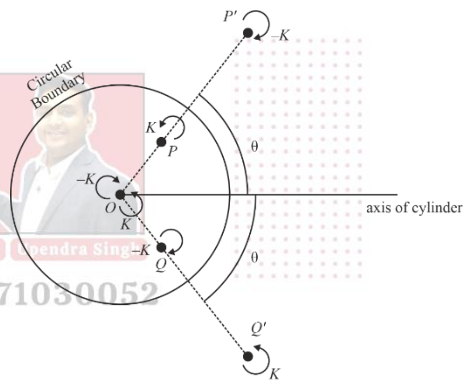
(ii) for circular boundary:-



∴

$$w = \frac{iK}{2\pi}\log(z-f) + \frac{i(-K)}{2\pi}\log\left(z-\frac{a^2}{f}\right) + \frac{i(+K)}{2\pi}\log(z-0)$$

Ex.1. When a pair of equal and opposite rectilinear vertices are situated in a long circular cylinder at equal distance from its axis, show that the path of each vortex is given by the eq. $(r^2 \sin^2\theta - b^2)(r^2 - a^2)^2 = 4a^2b^2r^2\sin^2\theta$, θ being measured through the centre perpendicular to the joint of the vertices.



Note:- At origin, vortex K & $-K$

∴ cancelled

• Let K be the strength of the vortex at $P(r, \theta)$ & $-K$ be for vortex at $Q(r_1 - \theta)$

• Let P'& Q' be the inverse points of P & Q respectively with regard to the circle of radius a & centroid at origin i.e., $|z| = a$

$$\therefore\ OP' = \frac{a^2}{r} = OQ'$$

• Then the image of vortex K at P is a vortex $-K$ at P' and image of $-K$ at Q is a vortex at Q'

∴ the complex potential for the whole system is

$$w = \frac{iK}{2\pi}\left[\log\left(z - r\cdot e^{i\theta}\right) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log\left(z - r\cdot e^{-i\theta}\right) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

Attention! motion in any vortex will be due to presence of other (remaining) vertices

The complex potential or to read the motion of the vortex P i.e., we will take

$$w = \frac{iK}{2\pi}\left[-\log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log\left(z - r\cdot e^{-i\theta}\right) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$w = \frac{-iK}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log\left(re^{i\theta} - re^{-i\theta}\right) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$= \frac{-iK}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + i\theta + \log(2ir\sin\theta) - \log\left\{\left(r - \frac{a^2}{r}\right)\cos\theta + i\left(r^2 + \frac{a^2}{2}\right)\sin\theta\right\}\right]$$

$$\because \ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Also using $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

Comparing imaginary parts on both sides, we get

$$\psi = \frac{-K}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{\left(r - \frac{a^2}{r}\right)^2\cos^2\theta + \left(r + \frac{a^2}{r}\right)^2\sin^2\theta\right\}\right]$$

$$\psi = \frac{-K}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta\right\}\right]$$

$$\psi = \frac{-K}{4\pi}\log\left[\frac{\left(r - \frac{a^2}{r}\right)^2 \times (2r\sin\theta)^2}{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta}\right]$$

so, the required streamline are given by ψ = constant

i.e., $\dfrac{(r^2 - a^2)^2 \times r^2\sin^2\theta}{r^4 + a^4 - 2a^2r^2\cos 2\theta} = b^2$ (Let's say constant = b^2)

$b^2 (r^4 + a^4 - 2a^2r^2 \cos {}^2\theta) = (r^2 - a^2)^2 r^2 \sin^2 \theta$

$b^2\{(r^2 - a^2)^2 + {}^2a^2 r^2 (1 - \cos 2\theta)\} = r^2 (r^2 - a^2)^2 \sin^2 \theta$

$2a^2 b^2 r^2 (1 - \cos^2 \theta) = (r^2 - a^2)^2\{r^2\sin^2\theta - b^2\}$

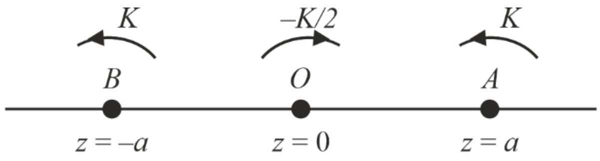
$4 a^2 b^2 r^2 \sin^2 \theta = (r^2 - a^2)^2 \{r^2 \sin^2 \theta - b^2\}$.

Ex.2. Two point vortices each of strength K are situated at $(\pm a, 0)$ and a point vortex of strength $\frac{-K}{2}$ is situated at the origin. Show that the fluid motion is stationary and find eq. of streamlines. Show that the streamline which passes through the stagnation points meets the x-axis at $(\pm b, 0)$ where, $3\sqrt{3}\ (b^2 - a^2)^2 = 16a^3 b$.

Step-1 Complex potential of fluid motion is

$$w = \frac{iK}{2\pi}\log(z-a) + \frac{iK}{2\pi}\log(z+a) + \frac{i\left(-\frac{K}{2}\right)}{2\pi}\log z$$

$w = \frac{iK}{2}\left[\log\left(z^2 - a^2\right) - \frac{1}{z}\log z\right]$ …(1)



Step-2

Now, the complex potential for the vortex *A*:

$$w' = w - \frac{iK}{2\pi}\log(z-a)$$

$$w' = \frac{iK}{2\pi}\left[\log(z+a) - \frac{1}{2}\log z\right] \quad \ldots(2)$$

∵ The velocity (u_A, v_A) of the vortex *K* at *A* is solely produced by the other vortices

$$\therefore\ u_A - iv_A = \left(\frac{-dw'}{dz}\right)_{z=a} = -\frac{iK}{2\pi}\left[\frac{1}{(z+a)} - \frac{1}{2z}\right]_{z=a}$$

$$= \frac{-iK}{2\pi}\left[\frac{1}{2a} - \frac{1}{2a}\right] = 0$$

$\Rightarrow u_A - iv_A = 0 \Rightarrow u_A = 0,\ v_A = 0$ (∵ velocity is zero)

$\Rightarrow$ *A* is a stationary point.

Now, the complex potential for the vortex O:

$$w_0 = w - \frac{i\left(-\frac{K}{2}\right)}{2\pi}\log z$$

$$w_0 = \frac{iK}{2\pi}\left[\log(z+a) + \log(z-a)\right] \quad \ldots(3)$$

∵ The velocity (u_0, v_0) of the vortex $\frac{-K}{2}$ at 0 is solely produced by the other vertices.

$$\therefore\ u_0 - iv_0 = \left(\frac{-dw_0}{dz}\right)_{z=0} = \frac{-iK}{2\pi}\left[\frac{1}{z+a} + \frac{1}{z-a}\right]_{z=0} = 0$$

∴ O is a stationary point,

Similarly, we can show *B* is also stationary points.

Therefore, The fluid water is stationary.

• finding streamlines :-

$$w = \phi + i\psi = \frac{iK}{2\pi}\left[\log\left(z^2 - a^2\right) - \frac{1}{2}\log z\right]$$

$$= \frac{iK}{2\pi}\left[\log\left\{(x+iy)^2 - a^2\right\} - \frac{1}{2}\log(x-iy)\right]$$

$$= \frac{iK}{2\pi}\left[\frac{1}{2}\log\left\{\left(x^2-y^2-a^2\right)^2+(2xy)^2+(2xy)^2\right\} + i\tan^{-1}\left(\frac{2yx}{x^2-y^2-a^2}\right) - \frac{1}{4}\log\left(x^2+y^2\right)-\frac{1}{2}\tan^{-1}\frac{y}{x}\right]$$

$$\therefore\ \psi = \frac{K}{2\pi}\left[\frac{1}{2}\log\left\{\left(x^2-y^2-a^2\right)^2+4x^2y^2\right\}-\frac{1}{4}\log\left(x^2+y^2\right)\right]$$

$$\psi = \frac{K}{4\pi}\log\left\{\frac{\left(x^2-y^2-a^2\right)^2+4x^2y^2}{\left(x^2+y^2\right)^{1/2}}\right\}$$

Now, streamlines are given as,

$$\psi = \text{constant} = \frac{K}{4\pi}\log c.\ \text{(say)}$$

$$\therefore\ \frac{K}{4\pi}\log\left\{\frac{\left(x^2-y^2-a^2\right)^2+4x^2y^2}{\left(x^2+y^2\right)^{1/2}}\right\} = \frac{K}{4\pi}\log c$$

$(x^2 - y^2 - a^2)^2 + 4x^2 y^2 = c(x^2 + y^2)^{1/2}$

$(x^2 - y^2)^2 + a^4 - 2a^2(x^2 - y^2) + 4x^2y^2 = c(x^2 + y^2)^{1/2}$

$(x^2 + y^2)^2 + a^4 - 2a^2(x^2 - y^2) = c(x^2 + y^2)^{1/2}$ …(A); c is aub constant

For stagnation point:-

$$\frac{dw}{dz} = 0$$

$$\Rightarrow \frac{iK}{2\pi}\left[\frac{2z}{z^2-a^2}-\frac{1}{2z}\right] = 0 \ \{\text{from}(1)\}$$

$4z^2 - z^2 + a^2 = 0$

$3z^2 + a^2 = 0$

$$z^2 = \frac{-a^2}{3}$$

$$z = \frac{\pm ia}{\sqrt{3}} \ \text{ i.e., } x = 0,\ y = \pm\frac{ia}{\sqrt{3}}$$

$$\therefore \text{ stagnation points are :- } \left(0, \frac{a}{\sqrt{3}}\right) \ \& \ \left(0, \frac{-a}{\sqrt{3}}\right)$$

As the above point passes through (A); this gives

$$\left(\frac{a^2}{3}\right)^2 + a^4 + 2^2 \times \frac{a^2}{3} = \frac{ca}{\sqrt{3}}$$

$$\Rightarrow c = \frac{\sqrt{3}}{a}\left[\frac{a^4+9a^4+6a^4}{9}\right]$$

$$c = \frac{16\sqrt{3}a^3}{9}$$

$\therefore$ eq. (A) becomes,

$$(x^2 + y^2)^2 + a^4 - 2a^2 (x^2 - y^2) = \frac{16\sqrt{3}a^3}{9}(x^2 + y^2)^{1/2} \quad \ldots(B)$$

As, the streamline also passes through ($\pm b$, 0);

$\therefore$ from (B),

$$b^4 + a^4 - 2a^2b^2 = \frac{16\sqrt{3}a^3}{9} \times b$$

$$(b^2 - a^2)^2 = \frac{16a^3b}{3\sqrt{3}}$$

$$3\sqrt{3}\,(b^2 - a^2)^2 = 16a^3b.$$

Ex.3. Prove that the necessary and sufficient condition that the vortex lines may been right angles to the stream lines are

$u, v, w, = \mu\left(\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z\right)$, where μ, ϕ are functions of x, y, z, t.

Find the necessary and sufficient condition that vortex lines may be at right angles streamlines.

Sol. Streamlines are given by $dx/u = dy/v = dz/w$

and vortex lines are given by $dx/\Omega_x = dy/\Omega_y = dz/\Omega_z$

(1) and (2) will be right angles, if $u\Omega_x + v\Omega_y + w\Omega_z = 0$

But $\Omega_x + \partial w/\partial y - \partial v/\partial z$, $\quad \Omega_y + \partial u/\partial z - \partial w/\partial x \quad \Omega_z + \partial v/\partial x - \partial u/\partial y$

Using (4), (3) may be re-written as

$$x\left(\partial w/\partial y - \partial v/\partial z\right) + v\left(\partial u + \partial z - \partial w/\partial x\right) + w\left(\partial v/\partial x - \partial u/\partial y\right) = 0$$

Which is the necessary and sufficient condition in order that $udx + vdy + wdy$ may be a differential. So we may write

$$udx + vdy + wdy = \mu d\phi = \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$\therefore \quad u = \mu\left(\partial\phi/\partial x\right), \quad v = \mu\left(\partial\phi/\partial y\right)$ and $\quad w = \mu\left(\partial\phi/\partial z\right)$

Ex. 4. It $udx + vdy + wdz = d\theta + \lambda d_\chi$, where θ, λ, χ are functions of x, y, z, t, prove that vortex lines at any time are the lines of inter – section of the surface λ = constant and χ = constant

Sol. Given $udx + vdy + wdz = d\theta + \lambda d\chi$

$$\therefore\ udx + vdy + wdz = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz + \frac{\partial\theta}{\partial t}dt + \lambda\left(\frac{\partial\chi}{\partial x}dx + \frac{\partial\chi}{\partial y}dy + \frac{\partial\chi}{\partial z}dz + \frac{\partial\chi}{\partial t}dt\right)$$

$$\Rightarrow \quad \left.\begin{aligned} u &= \frac{\partial\theta}{dx} + \lambda\frac{\partial\chi}{\partial x}, & v &= \frac{\partial\theta}{dy} + \lambda\frac{\partial\chi}{\partial y} \\ w &= \frac{\partial\theta}{dz} + \lambda\frac{\partial\chi}{\partial z}, \text{ and} & 0 &= \frac{\partial\theta}{dt} + \lambda\frac{\partial\chi}{\partial t} \end{aligned}\right\} \quad \ldots..(i)$$

Hence the components of spin $\Omega_x, \Omega_y, \Omega_z$ are given by

$$2\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial z} + \lambda \frac{\partial \chi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \theta}{\partial y} + \lambda \frac{\partial \chi}{\partial y}\right), \text{ using (1)}$$

$$= \frac{\partial^2 \theta}{\partial y \partial z} + \frac{\partial \lambda}{\partial y}\frac{\partial \chi}{\partial z} + \lambda \frac{\partial^2 \chi}{\partial y \partial z} - \frac{\partial^2 \theta}{\partial z \partial y} - \lambda \frac{\partial^2 \chi}{\partial z \partial y} - \frac{\partial \lambda}{\partial z}\frac{\partial \chi}{\partial y} = \frac{\partial \lambda}{\partial y}\frac{\partial \chi}{\partial z} - \frac{\partial \lambda}{\partial z}\frac{\partial \chi}{\partial y}$$

or
$$2\Omega_x = \begin{vmatrix} \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\chi/\partial y & \partial\chi/\partial z \end{vmatrix}$$

Similarly, $2\Omega_y = \begin{vmatrix} \partial\lambda/\partial z & \partial\lambda/\partial x \\ \partial\chi/\partial z & \partial\chi/\partial x \end{vmatrix}$ and $2\Omega_z = \begin{vmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y \\ \partial\chi/\partial x & \partial\chi/\partial y \end{vmatrix}$

$$\therefore \quad 2\left(\Omega_x \frac{\partial \lambda}{\partial x} + \Omega_y \frac{\partial \lambda}{\partial y} + \Omega_z \frac{\partial \lambda}{\partial z}\right) = \begin{vmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\chi/\partial x & \partial\chi/\partial y & \partial\chi/\partial x \end{vmatrix} = 0$$

$$\therefore \quad \Omega_x(\partial\lambda/\partial x) + \Omega_y(\partial\lambda/\partial y) + \Omega_z(\partial\lambda/\partial z) = 0. \qquad \text{.....(2)}$$

Similarly, we have $\Omega_x(\partial\chi/\partial x) + \Omega_y(\partial\chi/\partial y) + \Omega_z(\partial\chi/\partial z) = 0$ .....(3)

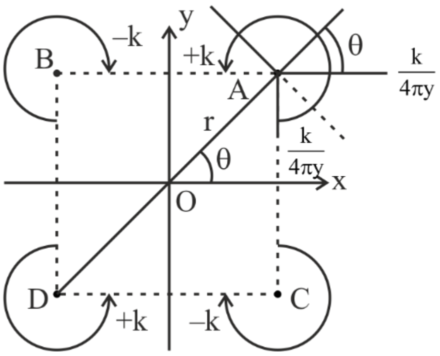
Equation (2) and (3) show that the vortex lines at any time are the lines of intersection of the surface λ = constant and χ = constant.

Image of vortex in a quadrant.

The image system of vortex of strength *k*, at the point *A*(*x, y*) in *xy*-plane with respect to quadrant *XOY* consists of (*i*) a vortex of strength –*k* at $B(-x, y)$

(*ii*) a vortex of strength –*k* at $C(x, -y)$

(*iii*) a vortex of strength *k* at $D(-x, -y)$



The velocity at *A* is only on accounts of its images and hence its components are as indicated in the figure. Thus the radial and transverse components of velocity at *A* are given by

$$\frac{dr}{dt} = \frac{k\cos\theta}{4\pi y} - \frac{k\sin\theta}{4\pi x} = \frac{k\cos\theta}{4\pi r\sin\theta} - \frac{k\sin\theta}{4\pi r\cos\theta} = \frac{k\left(\cos^2\theta - \sin^2\theta\right)}{2\pi r\sin 2\theta} = \frac{k\cos 2\theta}{2\pi r\sin 2\theta} \qquad \text{.....(1)}$$

$$r\frac{d\theta}{dt} = \frac{k}{4\pi r} - \frac{k\sin\theta}{4\pi y} - \frac{k\cos\theta}{4\pi x} = \frac{k}{4\pi r} - \frac{k\sin\theta}{4\pi r\cos\theta} = -\frac{k\cos\theta}{4\pi r\cos\theta} = -\frac{k}{4\pi r} \qquad \text{.....(2)}$$

On dividing (1) by (2), $\frac{1}{r}\frac{dr}{d\theta} = -2\frac{\cos 2\theta}{\sin 2\theta}$ or $\frac{1}{r}dr = -\frac{\cos 2\theta}{\sin 2\theta}d\theta$

Integrating it, $\log r = -\log \sin 2\theta + \log c$ *i.e.,* $r \sin 2\theta = c$,

Which is Cote's spiral. Transforming into cartesian, it becomes (using $x = r\cos\theta$, $y = r\sin\theta$)

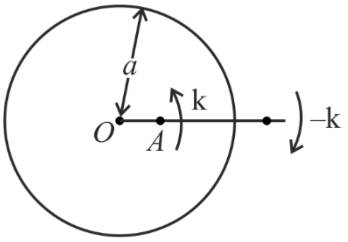
$2r\sin\theta\cos\theta = c$ or $4r^4\cos^2\theta\sin^2\theta = c^2r^2$ or $4(r\cos\theta)^2(r\sin\theta)^2 = c^2r^2$

i.e,. $4x^2y^2 = c^2(x^2+y^2)$ *or* $1/x^2 + 1/y^2 = 4/c^2$

Vortex inside an infinite circular cylinder

Let the vortex of strength *k* be situated at *A*(*OA* = *f*) inside the circular cylinder of radius *a* with axis parallel to the axis of the cylinder.

Let a vortex of strength –*k* be placed at *B*, where *B* is the inverse point of *A* with respect to the circular section of the cylinder so that



$OB.OA = a^2$ or $OB.f = a^2$

$\Rightarrow$ $OB = a^2/f$

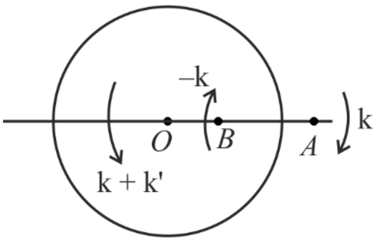
The circle is one of the co-axial system having *A* and *B* as limiting points and so it is a streamline.

The velocity of $A = \dfrac{k}{2\pi.AB} = \dfrac{k}{2\pi(OB-OA)} = \dfrac{k}{2\pi(a^2/f - f)} = \dfrac{kf}{2\pi(a^2-f^2)}$

Which is perpendicular to *OA*. *B* also has the above mentioned velocity so that *OAB* will not remain a straight line at the next instant. But if *A* describes a circle about *O* with the above velocity, then at every instant the circle will be a streamline, the positions of *B*, of course, changing from instant to instant.

Vortex outside a circular cylinder.

Let he vortex of strength *k* be situated at *A* (*OA* = *f*) outside the circular cylinder of radius *a* with axis parallel to the axis of the cylinder. Let a vortex of strength –*k* be placed *B*, where *B* is the inverse point of *A* with respect to the circular section of the cylinder so that



$$OB.OA = a^2 \Rightarrow OB.f = a^2 \Rightarrow OB = a^2/f.$$

Then the circle will be an instantaneous streamline due to this vortex pair and *A* will describe a circle with velocity

$$= \frac{k}{2\pi.AB} = \frac{k}{2\pi.(OB-OA)} = \frac{k}{2\pi(f - a^2/f)} = \frac{kf}{2\pi(f^2-a^2)}$$

But the introduction of a vortex of strength -*k* at *B* gives a circulation on -*k* about the cylinder and let the circulation about the cylinder be k'. The circulation –*k* about the cylinder due to the vortex *B* can be annulled by putting a vortex *k* at *O* and therefore to get the final circulation *k'* about the cylinder, we must put an additional vortex *k'* at *O*.

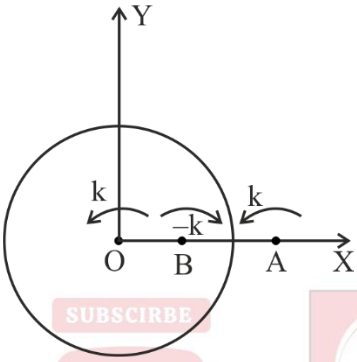
Thus we have a vortex *k* at *A*, –*k* at *B*, *k* + *k'* at *O*. Hence the velocity of *A* due to the above system.

$$=\frac{k+k'}{2\pi.OA}-\frac{k}{2\pi.AB}=\frac{k+k'}{2\pi f}-\frac{k}{2\pi\left(AB-OB\right)}=\frac{k+k'}{2\pi f}-\frac{k}{2\pi\left(f-a^2/f\right)}=\frac{k+k'}{2\pi f}-\frac{kf}{2\pi\left(f^2-a^2\right)}$$

and *A* describes a circle with this velocity

Image of a vortex outside a circular cylinder.

To show that the image system of a vortex *k* outside the circular cylinder consists of a vortex of strength –*k* at the inverse point and vortex of strength *k* at the centre.



Let us determine the image of a vortex filament of strength *k* placed at *A*($z = c > a$) with respect to a circular cylinder $|z| = a$ with *O* as centre. Let *B* be the inverse point of *A* with respect to $|z| = a$ so that

$$OA\times OB=a^2 \text{ and so } OB=a^2/c$$

In absence of $|z| = a$, the complex potential at any point due to vortex at *A* is given by

$$\left(ik/2\pi\right)\times\log\left(z-c\right).$$

When the circular cylinder $|z| = a$ is inserted in the fluid, the modified complex potential by Milne-Thomson's circle theorem is given by

$$w=\frac{ik}{2\pi}\log\left(z-c\right)-\frac{ik}{2\pi}\log\left(\frac{a^2}{z}-c\right)=\frac{ik}{2\pi}\log\left(z-c\right)-\frac{ik}{2\pi}\log\left[-\frac{c}{z}\left(z-\frac{a^2}{c}\right)\right]$$

$$=\left(ik/2\pi\right)\left[\log\left(z-c\right)-\log\left(z-a^2/c\right)+\log z-\log\left(-c\right)\right]$$

On adding the constant term $\left(ik/2\pi\right)\log\left(-c\right)$ to the above value, the complex potential takes the form

$$w=\frac{ik}{2\pi}\log\left(z-c\right)-\frac{ik}{2\pi}\log\left(z-\frac{a^2}{c}\right)+\frac{ik}{2\pi}\log z\,. \qquad .....(1)$$

Putting $w=\phi+i\psi,\qquad z=ae^{i\theta}$ for any point on $|z| = a$ and equating imaginary parts, (1) gives $\psi = 0$. Thus there would be no flow across the boundary $|z| = a$. Hence motion would remain unchanged if the cylindrical boundary $|z| = a$ where made a rigid barrier. From (1) the required image system follows.

Note 1. Complex potential *w*' induced at *A*, by a vortex –*k* at *B* and a vortex *k* at *O* is

$$w' = w - (ik/2\pi)\log(z-c) = -(ik/2\pi)\log\left(z - a^2/c\right) + (ik/2\pi)\log z$$

$\therefore$ $$-\frac{dw'}{dz} = -\frac{ik}{2\pi} \times \frac{1}{z - a^2/c} + \frac{ik}{2\pi} \times \frac{1}{z} = -\frac{ik}{2\pi}\left[\frac{c}{cz - a^2} - \frac{1}{z}\right]$$

$\therefore$ $$\left|-\frac{dw'}{dz}\right|_{z=c} = \frac{k}{2\pi}\left|\frac{c}{cz-a^2} - \frac{1}{z}\right|_{z=c} = \frac{ik}{2\pi c} \times \frac{a^2}{c^2 - a^2}$$

Which given velocity of the vortex *A* with which it moves round the cylinder.

Note 2. Since the term *ik* log *z* denotes the circulation round the cylinder, the result of the above image system may be restated as under.

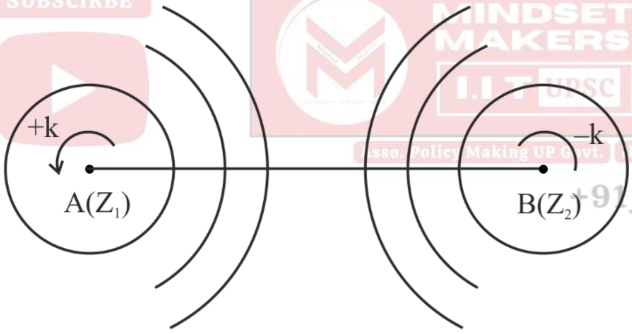
The image system of a vortex *k* outside the circular cylinder consists of a vortex of strength –*k* a the inverse point and a circulation of strength *k* round the cylinder.

Note 3. Proceeding as above we can also show that the image system of a vortex –*k* outside the circular cylinder consists of a vortex of strength *k* at the inverse point and a vortex of strength –*k* at the centre.

Image of a vortex inside a circular cylinder.

To show that the image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point.

Let there be a vortex pair consisting of two vortices of strength *k* at $A(z = z_1)$ and –*k* at *B* ($z = z_2$). Then the complex potential at any point is given by



$$w = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - z_2)$$

or $$\phi + i\psi = \frac{ik}{2\pi}\log\left(r_1 e^{i\theta_1}\right) - \frac{ik}{2\pi}\log\left(r_2 e^{i\theta_2}\right)$$

$\therefore$ $$\psi = \frac{ik}{2\pi}\log\frac{r_1}{r_2}$$

Where $r_1|z - z_1|$, $r_2|z - z_2|$.

Hence the streamlines are given by ψ = const. i.e., $r_1 / r_2 = c$, which represents a family of co-axial circles with *A* and *B* as limiting points.

Moreover the motion is unsteady and hence streamlines go on changing and following the vortices which move through the liquid. However, if a particular circle of the family of coaxial circle be replaced by a similar rigid boundary and held fixed, then it follows that the image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point

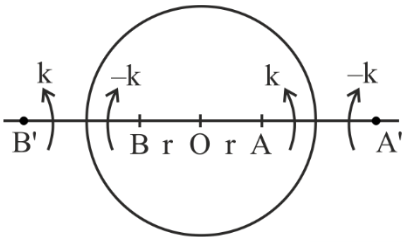
Note. Let O be the centre of the cylinder. Let $OA = c$. Then, if B is the inverse point of $A, OB = a^2/c$, where a is the radius of the circular cylinder. The vortex at A will move round the circular cylinder with velocity q given by

$$q = \frac{k}{2\pi AB} = \frac{k}{2\pi(OB-c)} = \frac{k}{2\pi\left(a^2/c-c\right)} = \frac{kc}{2\pi\left(a^2-c^2\right)}$$

Let ω be the angular velocity of vortex at A. Then

$$\omega = \frac{q}{OA} = \frac{q}{c} = \frac{k}{2\pi\left(a^2-c^2\right)}$$

Q.1. A vortex pair is situated within a cylinder Show that it will remain at rest if the distance of either from the centre is given by $\left(\sqrt{5}-2\right)^{1/2} a,$ where a is the radius of the cylinder.



Sol. Let vortices of strengths k and $-k$ situated at A and B respectively within the circular cylinder form the given vortex pair. Let $OA = r = OB$ and let A', B' be the inverse points of A and B respectively with regard to the cylinder so that $OA' = a^2/r = OB'$.

The image system consists of a vortex of strength $-k$ and A' and vortex of strength k at B'. The vortex will remain at rest if its velocity due to other three vortices is zero, that is

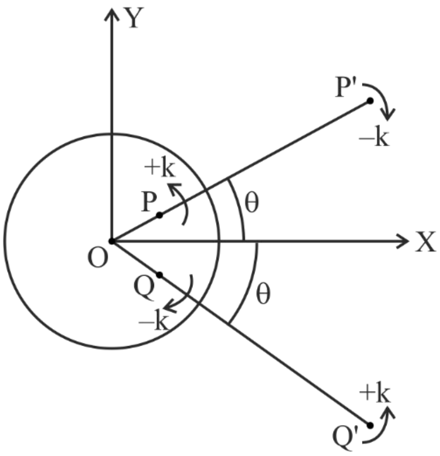
$$\frac{k}{2\pi}\left[\frac{1}{AA'}-\frac{1}{BA'}+\frac{1}{B'A}\right]=0 \quad \text{or} \quad \frac{1}{\left(a^2/r\right)-r}-\frac{1}{2r}+\frac{1}{\left(a^2/r\right)+r}=0$$

or $r\left\{1/\left(a^2-r^2\right)+1/\left(a^2+r^2\right)\right\}-(1/2r)=0$ or $4r^2a^2-\left(a^4-r^4\right)=0$

or $r^4+4a^2r^2-a^4=0$ or $\left(r^2/a^2\right)^2+4\left(r^2/a^2\right)-1=0$

or $r^2/a^2=\left\{-4+(16+4)^{1/2}\right\}/2\sqrt{5}-2$ or $r=\left(\sqrt{5}-2\right)^{1/2} a$

Q.2. When a pair of equal and opposite rectilinear vortices are situation a long circular cylinder at equal distance from its axis, show that the path of each vortex is given by the equation $\left(r^2\sin^2\theta-b^2\right)\left(r^2-a^2\right)^2=4a^2b^2r^2\sin^2\theta, \theta$ being measured from the line through the centre perpendicular to the joint of the vortices.



Sol. Let k be the strength of the vortex at $P(r,\theta)$ and $-k$ at $Q(r,\theta)$. Let P' and Q' be the inverse points of P and Q respectively with regard to the circular cylinder $|z| = a$ so that $OP' = a^2/r = OQ'$. Then the image of vortex k at P is a vortex $-k$ at P' and the image of vortex -k at Q is a vortex k at Q'.

Hence the complex potential of the system of four vortices is given by

$$w = \frac{ik}{2\pi}\left[\log\left(z - re^{i\theta}\right) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log\left(z - re^{-i\theta}\right) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

or
$$w = (ik/2\pi)\log\left(z - re^{i\theta}\right) + w',$$

Since the motion of vortex P is solely due to other vortices, the complex potential of the vortex at P is given by value of w' at $z = re^{i\theta}$.

$$\therefore \quad [w']_{z=re^{i\theta}} = \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log\left(z - re^{-i\theta}\right) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]_{z=re^{i\theta}}$$

$$\therefore \quad \phi + i\psi = -\frac{ik}{2\pi}\left[-\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log\left(re^{i\theta} - re^{-i\theta}\right) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]_{z=re^{i\theta}}$$

or
$$\phi + i\psi = -\frac{ik}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + i\theta + \log(2ir\sin\theta) - \log\left[\left(r - \frac{a^2}{r}\right)\cos\theta + i\left(r + \frac{a^2}{r}\right)\sin\theta\right]\right]$$

$$\therefore \quad \psi = -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{\left(r - \frac{a^2}{r}\right)^2\cos^2\theta + \left(r + \frac{a^2}{r}\right)^2\sin^2\theta\right\}\right]$$

[Using the formula: $\log(x+iy) = (1/2)\times\log\left(x^2+y^2\right) + i\tan^{-1}(y/x)$]

$$= -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{\left(r^2 + \frac{a^4}{r^2} - 2r.\frac{a^2}{r}\cos 2\theta\right)\right\}\right]$$

Thus,
$$\psi = -\frac{k}{4\pi}\log\frac{\left(r - a^2/r\right)^2 \times (2r\sin\theta)^2}{r^2 + a^4/r^2 - 2a^2\cos 2\theta}$$

So, the required streamlines are given by ψ = const., i.e., $\dfrac{\left(r^2-a^2\right)^2 r^2 \sin^2\theta}{r^4+a^4-2a^2r^2\cos 2\theta}=b^2$, say

i.e., $$b^2\left(r^4+a^4-2a^2r^2\cos 2\theta\right)=r^2\left(r^2-a^2\right)^2\sin^2\theta$$

i.e., $$b^2\left\{\left(r^2-a^2\right)^2+2a^2r^2\left(1-\cos 2\theta\right)\right\}=r^2\left(r^2-a^2\right)^2\sin^2\theta$$

$\therefore$ $r\sin 2\theta = 4/C$ =constant = A, say .....(5)

Again, $\tan\theta = y_0,/x_0$, as $x_0 = r\cos\theta$, $y_0 = r\sin\theta$

Differentiating both sides of $\tan\theta = y_0/x_0$ w.r.t. 't', we get

$$\sec^2\theta\dot{\theta}=\left(x_0\dot{y}_0-y_0\dot{x}_0\right)/x_0^2 \qquad \text{or}$$

$$x_0^2\sec^2\theta\dot{\theta}=x_0\dot{y}_0-y_0\dot{x}_0$$

or $$r^2\dot{\theta}=x_0\dot{y}_0-y_0\dot{x}_0=x_0v-y_0u,\ \text{by (3)} \qquad \left[\because\ x_0=r\cos\theta\right]$$

Thus, $$r^2\dot{\theta}=-\frac{k}{4\pi}\frac{y_0^2}{x_0^2+y_0^2}-\frac{k}{4\pi\pi}\frac{x_0^2}{x_0^2+y_0^2} \qquad \text{using (2)}$$

$\therefore$ $r^2\dfrac{d\theta}{dt}=-\dfrac{k}{4\pi}$ or $\dfrac{A^2}{\sin^2 2\theta}\dfrac{d\theta}{dt}=-\dfrac{k}{4\pi}$, using (5)

or $$dt=-\left(4\pi A^2/k\right)\cos ec^2 2\theta d\theta$$

Integrating, $t=\left(2\pi A^2/k\right)\cot 2\theta$, so that t is proportional to cot2θ

Vortex rows.

When a body moves slowly through a liquid rows of vortices are often generated in its wake. When these vortices are stable, then they can be photographed. In the next two articles we wish to consider infinite system of parallel rectilinear vortices in two dimensional flow.

Infinite number of parallel vortices of the same strength in one row.

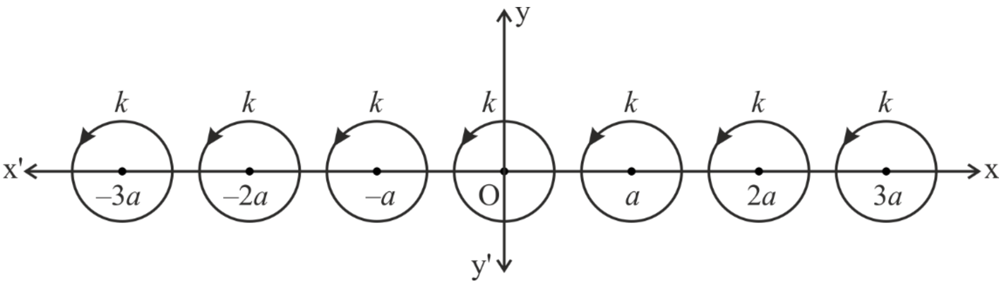
To show that the motion due to a set of line vortices of strength k at point $z=\pm na\left(n=0,1,2,3,....\right)$ is given by the relation $w=\left(ik/2\pi\right)\log\sin\left(\pi z/a\right)$

Proof. Let there be (2n + 1) vortices of strength k each situated at the points (0, 0) $\left(\pm a,0\right),\left(\pm 2a,0\right),\left(\pm 3a,0\right),....\left(\pm na,0\right)$. The complex potential of these (2n + 1) vortices at any point z is given by

$$w_{2n+1}=\left(ik/2\pi\right)\left[\log z+\log\left(z-a\right)+\log\left(z+a\right)+\log\left(z-2a\right)+\log\left(z+2a\right)+.....+\log\left(z-na\right)+\log\left(z+na\right)\right]$$

$$=\left(ik/2\pi\right)\log\left[z\left(z^2-a^2\right)\left(z^2-a^2\right)\left(z^2-2^2a^2\right)\left(z^2-3^2a^2\right)....\left(z^2-n^2a^2\right)\right]$$

$$=\frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)....\left(1-\frac{z^2}{n^2a^2}\right)+\frac{ik}{2\pi}\log\left[\left(-1\right)^n\frac{a}{\pi}.a^2.2^2a^2......n^2a^2\right]\right] \quad .....(1)$$



The second terms on R.H.S. of (l) being constant, it may be neglected for the purpose of complex potential. Hence the complex potential given by (l) may be also written as

$$w_{2l l+1} = \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right).....\left(1-\frac{z^2}{n^2a^2}\right)\right]$$

Making $n \to \infty$ in (2), then complex potential w of the entire system of vortices at point $t = \pm na\,(n = 0,1,2,3,....,)$ is given by

$$w = \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\left(1-\frac{z^2}{3^2a^2}\right)....\right] \qquad .....(3)$$

But $$\sin\theta = \theta\left(1-\theta^2/\pi^2\right)\left(1-\theta^2/2^2\pi^2\right)\left(1-\theta^2/3^2\pi^2\right)..... \qquad .....(4)$$

Putting $\theta = \pi z/a$ i.e., $z/a = \theta/\pi$ in (4), we get

$$\sin(\pi z/a) = (\pi z/a)\left(1-z^2/a^2\right)\left(1-z^2/2^2a^2\right)..... \qquad .....(5)$$

Using (5), (3) becomes $$w = (ik/2\pi)\log\sin(\pi z/a) \qquad .....(6)$$

Let u and v be the velocity components at any point of the fluid not occupied by any vortex filament. Then, we have

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cos\frac{\pi z}{a}, \text{ using (6)}$$

$$= -\frac{ik}{2a}\cot\frac{\pi(x+iy)}{a} = -\frac{ik}{2a}\frac{\cos\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}{\sin\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}$$

$$= -\frac{ik}{2a}\frac{\sin(2\pi x/a)-\sin(2\pi iy/a)}{\cos(2\pi iy/a)-\cos(2\pi x/a)} = -\frac{ik}{2a}\frac{\sin(2\pi x/a)-i\sinh(2\pi y/a)}{2a\cosh(2\pi y/a)-\cos(2\pi x/a)}$$

Equating real and imaginary parts, we have

$$u = -\frac{k}{2a}\frac{\sinh(2\pi y/a)}{\cosh(2\pi y/a)-\cos(2\pi x/a)} \qquad .....(7)$$

$$v = -\frac{k}{2a}\frac{\sinh(2\pi x/a)}{\cosh(2\pi y/a)-\cos(2\pi x/a)}$$

Since the motion of the vortex at the origin in due to other vortices only, the velocity q_0 of vertex at the origin is given by

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z\right]\right\}_{z=0} = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z}\right]_{z=0}$$

$$= -\frac{ik}{2\pi}\lim_{z\to 0}\left[\frac{\pi\cos(\pi z/a)}{a\sin(\pi z/a)} - \frac{1}{2}\right] \qquad \text{Form: } [\infty - \infty]$$

$$= -\frac{ik}{2\pi a}\lim_{z\to 0}\frac{\pi z\cos(\pi z/a) - a\sin(\pi z/a)}{z\sin(\pi z/a)} \qquad \text{Form: } \left[\frac{0}{0}\right]$$

[On evaluating the above indeterminate form with help of *L*' Hospital's rule]

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are so at rest, Thus we find that the vortex row induces no velocity on itself.

$$\phi + i\psi = \frac{ik}{2\pi}\log\sin\left\{\frac{\pi}{a}(x+iy)\right\}$$

$$\phi - i\psi = -\frac{ik}{2\pi}\log\sin\left\{\frac{\pi}{a}(x-iy)\right\}$$

Subtracting (10) from (9), $$2i\psi = \frac{ik}{2\pi}\left[\log\sin\left\{\frac{\pi}{a}(x+iy)\right\} + \log\sin\left\{\frac{\pi}{a}(x-iy)\right\}\right]$$

or $$\psi = \frac{k}{4\pi}\log\left[\sin\left\{\frac{\pi}{a}(x+iy)\right\}\sin\left\{\frac{\pi}{a}(x-iy)\right\}\right] = \frac{k}{4\pi}\log\left[\frac{1}{2}\left(\cos\frac{2\pi iy}{a} - \cos\frac{2\pi x}{a}\right)\right]$$

or $$\psi = \frac{k}{4\pi}\log\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right),$$

on omitting the irrelevant constant. The required streamlines are given by ψ = const.

i.e., $$\cosh(2\pi y/a) - \cos(2\pi x/a) = \text{const.}$$

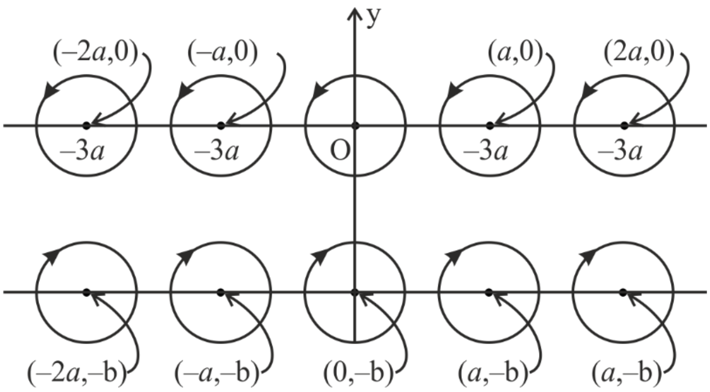
When y is very large, the second therm on L.H.S. of (12) may be omitted. Then the streamlines are given by

$\cosh(2\pi y/a)$ = const, so that y = const.,

Showing that at a great distance from the row of vortices the streamlines are parallel to the

Two infinite rows of parallel rectilinear vortices.

Let there be two infinite rows of vortices one above the other at a distance b, the upper one having vortices each of strength k and lower one each of strength $-k$, one vertex of the upper row being exactly above each of the lower row. Taking the upper row as x-axis and y-axis passing through the centre of one of the vortices of strength k each are at the points $(0,0),(\pm a,0),(\pm 2a,0)$ …. And those of strength $-k$ each are at the points $(0,-b),(\pm a,-b),(\pm 2a,-b)$



The complex potential of the entire system is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib)$$

Let u and v be the velocity components at any point of the fluid not occupied by any filament. Then

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cot\frac{\pi z}{a} + \frac{ik}{2\pi}\cot\frac{\pi}{a}(z+ib)$$

The velocity of the vortex at the origin is given by

$$u_0 - iv_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib)\right]\right\}_{z=0}$$

$$\Rightarrow\ u_0 - iv_0 = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z} - \frac{\pi}{a}\coth\frac{\pi}{a}(z+ib)\right]_{z=0} = -\frac{ik}{2a}\cot\frac{i\pi b}{a} = \frac{k}{2a}\coth\frac{\pi b}{a}$$

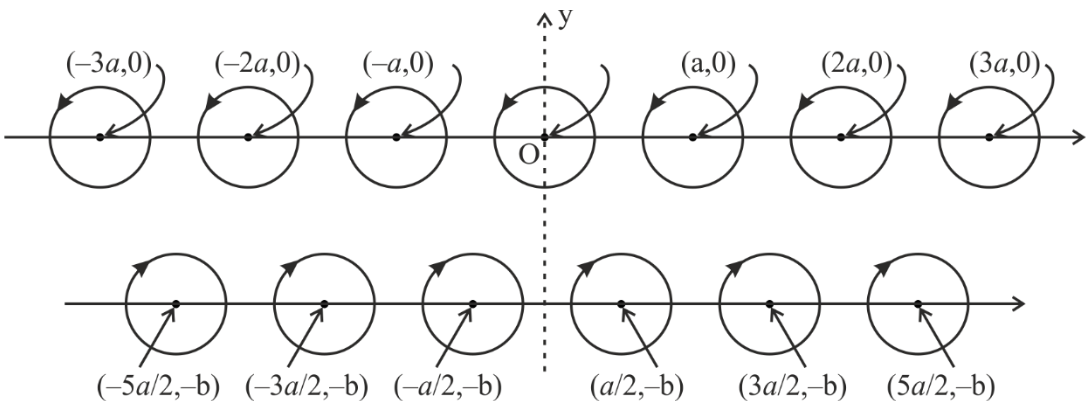
$\because \lim_{z\to 0}\{(\pi/a)\cot(\pi z/a) - (1/z)\} = 0$. Prove yourself as in Art.

So that $u_0 = (k/2a)\coth(\pi b/a)$, and $v_0 = 0$

Showing that the vortex system moves parallel to itself with velocity $(k/2a)$ coth $(\pi b/a)$.

Karman Vortex Street.

Let there be two parallel rows of vortices of equal but opposite strength placed in such a way that each vortex in row is opposite to the point midway between two vortices of the other row. $(\pm 2a, 0)$, ……
and the vortices of strength $-k$ each be situated at the points $(\pm a/2, -b), (\pm 3a/2, -b)$



If u and v be the velocity components at any point of the fluid not occupied by any vortex filament. Then

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cot\frac{\pi z}{a} + \frac{ik}{2a}\cot\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right) \qquad \ldots..(2)$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity of vortex at the origin is given by

$$= -\frac{ik}{2a}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z} - \frac{\pi}{a}\cot\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right)\right]_{z=0} = -\frac{ik}{2a}\cot\left(\frac{\pi}{2} + \frac{i\pi b}{a}\right) = -\frac{ik}{2a}\tan\frac{i\pi b}{2}$$

$[\because \lim_{z\to 0}\{(\pi/a)\cot(\pi z/a) - (1/z)\} = 0.$

Thus, $u_0 - iv_0 = (k/2a)\tanh(\pi b/a)$

So that $u_0 = (k/2a)\tanh(\pi b/a)$ and $v_0 = 0$,

Showing that the entire system would move parallel to itself with a uniform velocity $(k/2a)\tanh(\pi b/a)$.

Note. A Karman vortex street is often realized when a flat plate moves broadside through a liquid.

PREVIOUS YEARS QUESTIONS

CHAPTER 5. VORTEX MOTION

Q1.Verify that $w = ik\log\left(\frac{z-ia}{z+ia}\right)$ is the complex potential of a steady fluid flow about a circular cylinder, the plane $y = 0$ being a rigid boundary. Further show that the fluid exerts a downward force of magnitude $\left(\frac{\pi\rho k^2}{2a}\right)$ per unit length on the cylinder, where ρ is the fluid density. **[7b IFoS 2022]**

Q2. Two point vortices each of strength k are situated at $(\pm a, 0)$ and a point vortex of strength $-\frac{k}{2}$ is situated at the origin. Show that the fluid motion is stationary and also find the equations of streamlines. If the streamlines, which pass through the stagnation points, meet the x-axis at $(\pm b, 0)$, then show that $3\sqrt{3}\left(b^2 - a^2\right)^2 = 16a^3 b$. **[7c UPSC CSE 2022]**

Q3. Discuss the flow given by the complex potential

$w = \log_e\left(z - \frac{a^2}{z}\right)$.Draw sketches of the streamlines and explain the flow directions along the streamlines. **[7b IFoS 2021]**

Q4. What arrangements of sources and sinks can have the velocity potential $w = \log_e\left(z - \frac{a^2}{z}\right)$? Draw the corresponding sketch of the streamlines and prove that two of them subdivide into the circle $r = a$ and the axis of y. **[5e UPSC CSE 2021]**

Q5. The velocity vector in the flow field is given by

$\vec{q} = (az - by)\hat{i} + (bx - cz)\hat{j} + (cy - ax)\hat{k}$;where a, b, c are non-zero constants. Determine the equations of vortex lines.**[8c 2017 IFoS]**

Q6. Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r} \right]$ possess vorticity, where $\vec{q}(u, v, m)$ is the velocity in the Cartesian frame, $\vec{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$? What is the circulation in the circle $x^2 + y^2 = 9, z = 0$? **[5b UPSC CSE 2016]**

Q7. Prove that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving in the absence of an external force satisfies the differential equation

$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \nabla)\vec{q} + \nu \nabla^2 \vec{\Omega}$ where $\vec{q}$ is the velocity vector with $\vec{\Omega} = \nabla \times \vec{q}$. **[5d 2014 IFoS]**

Q8. If *n* rectilinear vortices of the same strength K are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^3}{(n-1)K}$. Find the velocity at any point of the liquid. **[8c UPSC CSE 2013]**

Q9. Prove that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving in the absence of an external force satisfies the differential equation

$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \nabla)\vec{q} + \nu \nabla^2 \vec{\Omega}$. **[5d 2012 IFoS]**

Q10. An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength K but they are alternately of opposite signs. Find the Complex function that determines the velocity potential and the stream function. **[8b UPSC CSE 2011]**

Q11. In an incompressible fluid the vorticity at every point is constant in magnitude and direction; show that the components of velocity u, v, w are solutions of Laplace's equation.

[5f UPSC CSE 2010]

Q12. When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation $(r^2 \sin^2\theta - b^2)(r^2 - a^2) = 4a^2 b^2 r^2 \sin^2\theta$, θ being measured from the line through centre perpendicular to the joint of the vortices.**[8b UPSC CSE 2010]**

Q13. Show that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving under no external forces satisfies the differential equation

$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \nabla)\vec{q} + \nu \nabla^2 \vec{\Omega}$ where ν is the kinematic viscosity. **[8c 2010 IFoS]**

Note: The beauty of systematic learning is- You’ll find solutions of almost every PYQ in above examples or questions attached with detailed answers. So to avoid repetition in this book, we have not put those solutions again as answers to PYQs.

THEME- VISCOSITY: NAVIER STOKE'S EQUATIONS

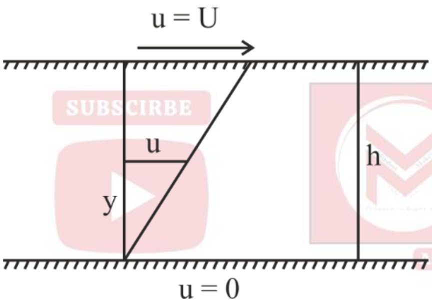
INTRODUCTION: So far we have been concerned with perfect (ideal) fluids (frictionless and incompressible). In the motion of such perfect fluid, two contacting layers of the fluid experience no tangential forces (shearing stress) but act on each other with normal forces (pressure) only or in other sense we can define that a perfect fluid exerts no internal resistance to a change in shape.

In this chapter we shall consider the cases of actual (real) fluids. In real fluids the inner layers of the fluid transmit tangential as well as normal stresses. Viscosity of the fluid is that property of actual fluids which exerts such resistance.

Because of the absence of tangential forces, a difference in relative tangential velocities exists on the boundary between a perfect fluid and a solid wall i.e. there is a slip, on the other hand, in actual fluids the existence of inter-molecular attraction causes the fluid to adhere to a solid wall and it gives rise to shearing stress.

The difference between a perfect and a real fluid is the existence of shearing stress and the condition of no slip.

Measurement of Viscosity.



Consider the motion of a fluid between two very long parallel plates, at a distance *h* apart.

Let the lower plate be at rest and the upper plate is moving with a constant velocity *U* parallel to itself. The pressure being constant throughout the flued.

We see that the fluid adheres to both the walls, so that its velocity at the lower plate is zero and that at the upper plate is equal to the velocity *U*.

Again, the velocity distribution in the fluid between the plates is linear, is

Liner, so that the fluid velocity is proportional to the distance y from the lower plate (there being no slip on the walls).

Then $$u = U\frac{y}{h}$$

Since the tangential force to the upper plate be in equilibrium with the frictional forces in the fluid.

Also the experiments shows that this force is proportional to the velocity U of the upper plate and inversely proportional to the distance h. Let τ denotes the frictional force per unit area or $\tau \propto \frac{U}{h}$

{In general $\frac{U}{h}$ can be replaced by the velocity gradient $\frac{du}{dy}$.

or $\tau \propto \frac{du}{dy}$

or $\tau = \mu \frac{du}{dy}$ .....(i)

where μ is a constant of proportionality depending on the pressure and temperature.

Note: For gases μ is independent of the pressure at ordinary temperature. The relation (i) is known as Newton's equation of viscosity. B transformation. We have

$$\mu = \frac{\tau}{du/dy}$$

which is known as the coefficient of viscosity or Absolute viscosity or Dynamic viscosity.

A fluid for which the constant of proportionality (i.e. viscosity) does not change with rate of deformation is said to be a Newtonian fluid..

$$\mu = \frac{\text{shearing stress}}{\text{velocity gradient}}$$

{Shearing stress $\Rightarrow$ Force/unit area and velocity gradient = velocitylength

or μ(force per unit area/rate of shear)

In all fluid motions in which frictional and inertial forces interact, we consider the ratio of the viscosity to the density such as

$$v = \frac{\mu}{\rho}$$

Which is known as kinemetic Viscosity.

Strain Analysis.

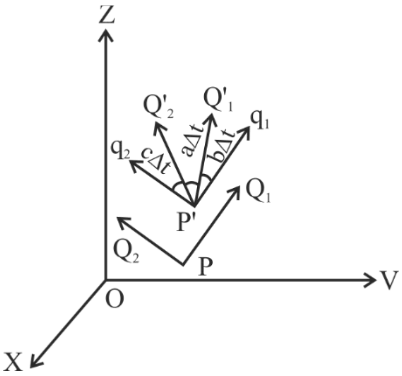
When the various elements of a system undergo relative displacements under the action of impressed forces, it is said to be strained.

(1) Normal Strain is defined as the ratio of the change in length to the, riginal length of a straight line element.
(2) Shearing Strain is defined as the change in angle between two linear elements from the unstrained state to the strained state.

Since the motion of the fluid is completely determined when the velocity vector **q** is given as a function of position and time; q = q(xyzt).

∃ kinematic relations between the components of the rate of strain and this function.

Let the velocity of an infinitesimal element at P(x y z) at any time be (u v w).



Let PQ_1 and PQ_2 be two perpendicular lines through P having infinitesimal length δS_1 and δS_2,

direction cosines are $(l_1\ m_1\ n_1)$ and $(l_2\ m_2\ n_2)$ respectively.

The coordinates and velocities at Q_1 and Q_2 be at the same time t are.

$(x+\delta x_1, y+\delta y_1, z+\delta z_1; u+\delta u_1, v+\delta v_1, w+\delta w_1)$

and $(x+\delta x_2, y+\delta y_2, z+\delta z_2; u+\delta u_2, v+\delta v_2, w+\delta w_2)$ respectively.

From Analytical geometry 3D; Evidently $\delta S^2 = \sum_{xyz} \delta x^2$

or $\qquad \delta x = l\delta S, \delta y = m\delta S, \delta z = n\delta S \qquad$ …..(i)

Since PQ_1 and PQ_2 are perpendicular

Then $\sum_{xyz} \delta x_1 \delta x_2 = 0 \qquad$ …..(ii)

The relative velocity $(\delta u, \delta v, \delta w)$ of Q relative to the point P can be written as

$$\delta u = \sum_{xyz} u_x \delta x, \delta v = \sum_{xyz} v_x \delta x, \delta w = \sum_{xyz} w_x \delta x$$

Now assuming the following symbols,

$$\left.\begin{aligned} & e_{xx} = 2u_x, e_{yy} = 2v_{v,} e_{zz} = 2w_z \\ \text{and} \quad & e_{yz} = e_{zy} = w_y + v_z \\ & e_{zx} = e_{xz} = u_z + w_x \\ & e_{xy} = e_{yx} = v_x + u_y \end{aligned}\right] \quad \text{.....(iii)}$$

also $\xi = w_y - y_{z,} \eta = u_z - w_x, \xi = v_x - u_y$

Form the above assumed relation (iii), we have

$$\left.\begin{aligned} & u_x = \frac{1}{2} e_{xx}, && u_y = \frac{1}{2}(e_{xy} - \zeta), && u_z = \frac{1}{2}(e_{xz} + \eta) \\ & v_x = \frac{1}{2}(e_{xy} + \zeta), && v_y = \frac{1}{2} eyy. && v_z = \frac{1}{2}(e_{yz} - \xi) \\ & w_x = \frac{1}{2}(e_{xz} - \eta), && w_y = \frac{1}{2}(e_{yz} + \xi), && w_x = \frac{1}{2} e_{zz} \end{aligned}\right\} \quad \text{.....(iv)}$$

Where ξ, η, ζ are the **components of vorticity about the coordinate axes *OX, OY, OZ.***

Now the velocity of Q in terms of these symbols is

$$\left.\begin{aligned} & u_Q = u_P = \delta u \\ & = up + \frac{1}{2}\left(e_{xx}\delta x + e_{xy}\delta y + e_{xz}\delta z + \frac{1}{2}(\eta\delta z - \zeta\delta y)\right) \\ & v_Q = v_P + \delta v \\ & = v_P + \frac{1}{2}\left(e_{yz}\delta_x + e_{yy}\delta y + e_{yz}\delta z + \frac{1}{2}(\zeta\delta x - \xi\delta z)\right) \\ \text{and} \quad & w_Q = w_P + \delta w \\ & = w_P + \frac{1}{2}\left(e_{xx}\delta x + e_{xy}\delta y + e_{xz}\delta z + \frac{1}{2}(\xi\delta y - \eta\delta x)\right) \end{aligned}\right] \quad \text{.....(v)}$$

The velocity at Q consists of three parts:

(a) Velocity of translation (u_p) which is the same as that of P.

(b) Rate of deformation (Rate of component of strain) as

$$\frac{1}{2}\left(e_{xx}\delta x + e_{xy}\delta y + e_{xz}\delta z\right), \frac{1}{2}\left(e_{yx}\delta_x + e_{yy}\delta_y + e_{yz}\delta z\right)$$

$$\frac{1}{2}\left(e_{xx}\delta x + e_{xy}\delta y + e_{xz}\delta z\right)$$

(c) Velocity produced by rigid body due to rotation of angular velocity $\left(\frac{1}{2}\xi, \frac{1}{2}\eta, \frac{1}{2}\zeta\right)$ about straight lines parallel to the axes of reference through P.

Velocity of Q relative to the point P is

$$\left.\begin{aligned}\delta u &= \frac{1}{2}\delta S\left\{le_{xx} + me_{xy} + ne_{xz} + (n\eta - m\zeta)\right\}\\ \delta v &= \frac{1}{2}\delta S\left\{le_{yx} + me_{yy} + ne_{yz} + (l\zeta - n\xi)\right\}\\ \delta w &= \frac{1}{2}\delta S\left\{le_{zx} + me_{zv} + ne_{zz} + (m\xi - l\eta)\right\}\end{aligned}\right\} \qquad \text{.....(vi)}$$

Rate of elongation

Consider P', Q_1', Q_2' be the position of P, Q_1, Q_2 respectively at time $t + \Delta t.$ Evidently, the coordinates of P' are

$$(x + u\Delta t, y + v\Delta t, z + w\Delta t)$$

and that of $Q'_1\left\{x + \delta x + (u + \delta u)\Delta t, y + \delta y + (v + \delta v)\Delta t, z + \delta z + (w + \delta w)\Delta t\right\}$

then $(P'Q_1')^2 = (\delta x + \delta u\Delta t)^2 + (\delta y + \delta v\Delta t)^2 + (\delta z + \delta w\Delta t)^2$

or $P'Q_1' = \left\{(\delta x + \delta u\Delta t)^2 + (\delta y + \delta v\Delta t)^2 + (\delta z + \delta w\Delta t)^2\right\}^{1/2}$

or $P'Q_1' = \left\{(\delta S)^2 + 2\Delta t\left(l\delta S\delta u + m\delta S\delta v + n\delta S\delta w\right)\right\}^{1/2}$

Using the relation (i) and (vi), we have

$$P'Q_1' = \left[1 + \frac{1}{2}\Delta t\left\{l^2 e_{xx} + m^2 e_{yy} + n^2 e_{xx} + 2lme_{xy} + 2mne_{yz} + 2nle_{xx}\right\} + 0(\Delta t)^2\right. \qquad \text{.....(i)}$$

Rate of elongation

$$= \frac{P'Q_1' - PQ_1}{PQ_1}.\frac{1}{\Delta t}$$

$$= \frac{1}{2}\left[l^2 e_{xx} + m^2 e_{yy} + n^2 e_{xx} + 2lme_{xy} + 2mne_{vz} + 2nl\,e_{xx}\right] \qquad \text{.....(ii)}$$

Which gives the relative rate of elongation of PQ_1.

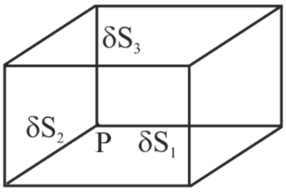
Consider PQ_1, parallel to the X-axis then the direction cosines becomes (1, 0, 0), hence from (ii),

$\frac{1}{2}e_{xx}$ represents the relative rate of elongation in the direction of X-axis.

Similarly $\frac{1}{2}e_{xx}$ represents the relative rate of elongation in the direction of Y-axis.

And $\frac{1}{2}e_{xx}$ represents he relative rate of elongation in the direction of Z-axis.

Ex.1. Consider a rectangular parallelopiped with edges PQ_1, PQ_2 and PQ_3, parallel to the axis of reference of lengths δS_1, δS_2 and δS_3 respectively, then the relative rate of increase of its volume is given by



$$= \underset{\Delta t\to 0}{Lt} \frac{\delta S_1\left(1+\frac{1}{2}e_{xx}\Delta t\right).\delta S_2\left(1+\frac{1}{2}e_{yy}\Delta t\right).\delta S_3\left(1+\frac{1}{2}e_{xz}\Delta t-\delta S_1\delta S_2\delta S_4\right)}{\delta S_1\delta S_2\delta S_3\Delta t}$$

$$= \underset{\Delta t\to 0}{Lt}\left\{\left(1+\frac{1}{2}e_{xx}\Delta t\right)\left(1+\frac{1}{2}e_{yy}\Delta t\right)\left(1+\frac{1}{2}e_{zz}\Delta t\right)-1\right\}$$

$$=\frac{1}{2}\left\{e_{xx}+e_{yy}+e_{zz}\right\}$$ {neglecting the term of higher orders of Δ t.

$$=u_\infty+v_y+w_e$$

$$=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

= div. q (where q is the velocity vector)

The relative rate of increase in the volume is called dilation generally is denoted by Δ.

Thus $\Delta=\frac{1}{2}\left(e_{xx}+e_{yy}+e_{zz}\right)$

If the rate of increase vanishes then it is known as equation of continuity.

Rate of Shear.

e_{xy} represents the rate of the decreases of the angle between the line which were originally parallel to the axis of X and Y respectively i.e., $e_{xy} \Rightarrow$ the rate of shear in the XY-plane.

Similarly we can say that e_{yz} and e_{zx} as the rates of shear in YZ-plane and in ZX-plane.

Rate of strain tensor

The rate of strain matrix

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}$$

A symmetric tensor, i.e.,

$e_{xy} = e_{yz}, e_{yz} = e_{zy}$ and $e_{zx} = e_{xz}$

Rate of strain components.

Let (u v w) be the components of the velocity parallel to the coordinates axes at the point (x, y, z) at time t. The components of the relative velocity at an infinitely near point $(x+\delta x, y+\delta y, z+\delta z)$ are:

$$\delta u = \frac{1}{2}\left(e_{xx}\delta x + e_{xy}\delta y + e_{xx}\delta z\right) + \frac{1}{2}\left(\eta\delta z - \zeta\delta y\right)$$

$$\delta v = \frac{1}{2}\left(e_{yx}\delta x + e_{yy}\delta y + e_{vz}\delta z\right) + \frac{1}{2}\left(\zeta\delta x - \xi\delta z\right)$$

and $$\delta w = \frac{1}{2}\left(e_{zx}\delta x + e_{xy}\delta y + e_{zz}\delta z\right) + \frac{1}{2}\left(\xi\delta y - \eta\delta x\right)$$

Where $e_{xx} = 2\frac{\partial u}{\partial x}, e_{yy} = 2\frac{\partial v}{\partial y}, e_{zz} = 2\frac{\partial w}{\partial z}$

$$e_{yx} = e_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$e_{xx} = e_{xx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$e_{xy} = e_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

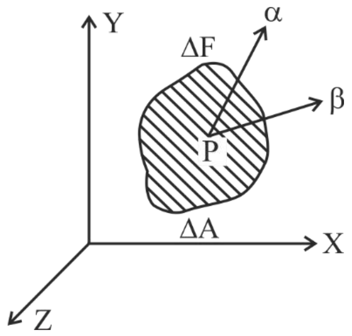
and $$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

ξ, η, ζ are the components of the vorticity vector $\vec{\omega}$ and $\vec{\omega} = \text{Curl}\,\vec{V}$

Where V (u, v, w) is the velocity vector.

The quantities e_{xx}, e_{yy} etc. are called the rate of strain components and $\frac{1}{2}e_{xx}$ represents the rate of extension of a line element in the direction of the X-axis. e_{yx} is the rate of change of the angle between two lines along the axis of X and axis of Y.

Stress Analysis.



Consider a point $P(x\ y\ z)$ in the fluid medium and take infinitesimal area ΔA surrounding the point P. The fluid on each side of the area exerts of force ΔF on it.

Then the stress S of the fluid at P on the area A is define as

$$S = \underset{\Delta A \to 0}{Lt.} \frac{\Delta F}{\Delta A}; \text{ (This is finite and non-zero)}$$

In other words the forces per unit area which two neighbouring elements of volume with a common surface exerts on each other are called stresses.

For a fluid at rest, stress is normal to the surface and is in the nature of a pressure. When fluids are in motion, ∃ also shearing stress in addition to normal stress.

The stress components can be represented by $P_{\alpha\beta}$, where α denotes the direction of the normal to the area and β is the direction in which the stress component is taken.

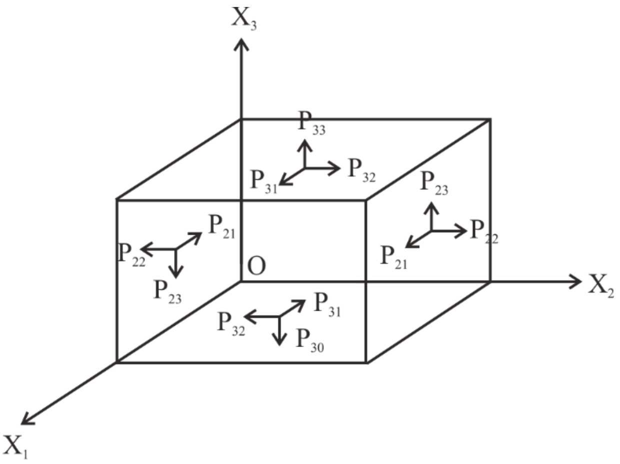
Considering the right handed system of coordinate axes, we define the stress matrix

$$P = P_{ij}$$

Where P_{ij} is the component of stress acting on a are ΔA perpendicular to the axis x_1 taken in the direction parallel to x_1 axis.

Stress Tensor.

Now we shall prove that P is a symmetric tensor. We know by D' Alembert's principle that the reversed effective forces and the impressed forces acting on a dynamical system at any instant are in equilibrium, and the fact that the force on an infinitesimal area in any direction can be taken as the product of the area and the stress acting at its centre in that direction.



P is symmetric, thus stress matrix is diagonally symmetric and contains only six unknowns.

The three sets of stress components are given by

$$\begin{matrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{matrix}$$

The diagonal elements p_{xx}, p_{yy}, p_{zz} of this array are called normal of direct stresses. The remaining six elements are known shearing stress. For an inviscid fluid

$$p_{xx} = p_{vv} = p_{zz} = -p$$

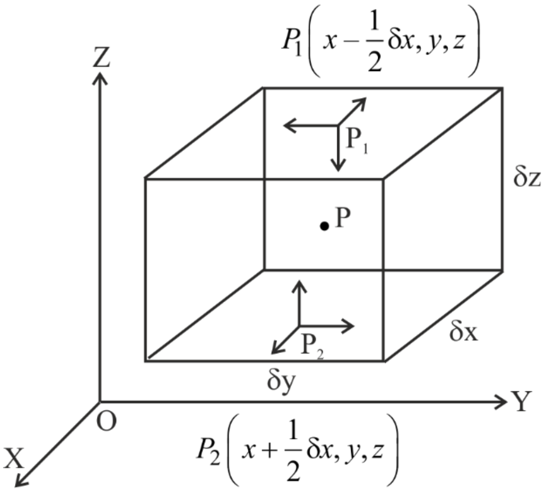
and $$p_{xy} = p_{xz} = \square = 0$$

The matrix $\begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yx} \\ p_{zx} & p_{xy} & p_{zz} \end{bmatrix}$ is called a **stress matrix**

The quantities p_{ij} where $I, j = x, y, z$, are called the stress tensor which is a second order tensor.

Translation motion of fluid element.

Consider the motion of a small rectangular parallelopiped of viscous fluid, having $P(x\,y\,z)$ as centre and its edges of lengths $\delta x, \delta y, \delta z$ parallel to the fixed rectangular axes.



Mass of fluid element

$=\rho\delta x\,\delta y\,\delta z$; (which will remain constant)

Suppose the element move along with the fluid. The components of the forces parallel to the co-ordinate axes OX, OY, OZ on the surface of area $\delta y\,\delta z$ through the point $P(x, y, z)$ are

$$\left(p_{xx}\delta y\,\delta z, p_{xy}\delta y\,\delta z, p_{xz}\delta y\,\delta z\right)$$

At the point $P_2\left(x+\frac{1}{2}\delta x, y, z\right)$, the corresponding force components across the parallel plane of area $\delta y\,\delta z$ are (**i** is the unit normal measured outwards from the fluid).

$$\left[\left\{p_{xx}+\frac{1}{2}\delta_x\left(\frac{\partial p_{xx}}{\partial x}\right)\right\}\delta y\,\delta z, \left\{p_{xy}+\frac{1}{2}\delta x\left(\frac{\partial p_{xy}}{\partial x}\right)\right\}\delta y\,\delta z, \left\{p_{xz}+\frac{1}{2}\delta x\left(\frac{\partial p_{xz}}{\partial x}\right)\right\}\delta y\,\delta z\right]$$

Similarly for the parallel plane through $P_1\left(x-\frac{1}{2}\delta x, y, z\right)$ the corresponding components are,

$$\left[-\left\{p_{xx}-\frac{1}{2}\delta_x\left(\frac{\partial p_{xx}}{\partial x}\right)\right\}\delta y\,\delta z, -\left\{p_{xy}-\frac{1}{2}\delta x\left(\frac{\partial p_{xy}}{\partial x}\right)\right\}\delta y\,\delta z, -\left\{p_{xz}-\frac{1}{2}\delta x\frac{\partial p_{xz}}{\partial x}\right\}\delta y\,\delta z\right]$$

(Since –**i** is the unit normal drawn outwards from the fluid element).

The force on the parallel planes through P_1 and P_2 are equivalent to a single force at P having components

$$\left\{\frac{\partial p_{xx}}{\partial x}, \frac{\partial p_{xy}}{\partial x}, \frac{\partial p_{xz}}{\partial x}\right\}\delta x\,\delta y\,\delta z$$

Together with the couple whose moments are

$$-p_{xz}\delta x\delta y\delta z \qquad \text{about } OY.$$

and $$+p_{xy}\delta x\delta y\delta z \qquad \text{about } OZ.$$

Similarly the pair of faces perp. To Y axis give a force at P having components

$$\left[\frac{\partial p_{yx}}{\partial y},\frac{\partial p_{yy}}{\partial y},\frac{\partial p_{yz}}{\partial y}\right]\delta x\,\delta y\,\delta z$$

Together with couples of moments

$$-p_{yx}\,\delta x\,\delta y\,\delta z \text{ about } OZ.$$

$$+p_{yz}\,\delta x\,\delta y\,\delta z \text{ about } OX.$$

And the pair of faces perp. to the Z-axis give a force at P having components

$$\left[\frac{\partial p_{xx}}{\partial z},\frac{\partial p_{xy}}{\partial z},\frac{\partial p_{zz}}{\partial z}\right]\delta x\,\delta y\,\delta z$$

Together with couples of moments.

$$-p_{xy}\,\delta x\,\delta y\,\delta z \text{ about } OX.$$

$$+p_{xx}\,\delta x\,\delta y\,\delta z \text{ about } OY.$$

Thus the surface forces on all six faces of the cuboid reduce to a singles force at P having components.

$$\left[\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial z}+\frac{\partial p_{xx}}{\partial z}\right),\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial z}+\frac{\partial p_{xy}}{\partial z}\right),\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\right]\delta x\,\delta y\,\delta z$$

Together with a vector couple having cartesian components

$$\left\{\left(p_{yz}-p_{xy}\right),\left(p_{xx}-p_{xz}\right),\left(p_{xy}-p_{yx}\right)\right\}\delta x\,\delta y\,\delta z$$

Consider the external body forces are $(X\ Y\ Z)$ per unit mass at the point P. Then the total body force on the element has components.

$$\left(X\ Y\ Z\right)\rho\delta x\,\delta y\,\delta z\ .$$

The total force component acting on fluid element P along the i-direction

$$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z+\rho X\,\delta x\delta y\,\delta z$$

Let q(u v w) be the velocity at the point P at any time t, then the equation of motion along the i direction

$$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z+\rho X\,\delta x\,\delta y\,\delta z$$

$$=\left(\rho\delta x\,\delta y\,\delta z\right)\frac{du}{dt}$$

or $$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)+\rho X=\rho\frac{du}{dy}$$

Since $$u=u\left(x\,y\,z\,t\right)$$

and $$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

Exam Point: Above discussion is just to have an idea about how these equations are coming out . Ultimately we need to remember below final equations for exam.

Thus we have the equations of motion in the direction of i, j and k

or $$\frac{du}{dt}=X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)$$

or $$\frac{dv}{dt}=Y+\frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}\right)$$

or $$\frac{dv}{dt}=Z+\frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)$$

Can be represented in tensor form

$$\frac{\partial p_x}{\partial t}+u_j u_{i.j}=X_i\frac{1}{\rho}P_{ij.j}$$

$$\left(ij=1,2,3\right)$$

Where $x_1 \Rightarrow$ the co-ordinate

$u_1 \Rightarrow$ the velocity components.

$X_1 \Rightarrow$ the external body force components.

Newtonian fluids.

The fluids in which the stress components are linear functions of rate of strain components are called Newtonian fluids.

Navier – Stokes equations of Motion of a Viscous fluid.

We know that the equation of translation motion of fluid element is

$$\frac{du}{dt} = X + \frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right). \qquad \text{.....(i)}$$

We know that

$$p_{xx} = -p + 2\mu\frac{\partial u}{\partial x} + \lambda\Delta\,,\ \ p_{yz} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\ \ p_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Substituting the above values in (i), we have

$$\frac{du}{dt} = X + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x} + \lambda\Delta\right) + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\}\right]$$

or $$\frac{du}{dt} = X + \frac{1}{\rho}\left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu + \frac{\partial u}{\partial x} + \lambda\Delta\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

or $$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + v\Delta^2 u + \left(v + \frac{\lambda}{\rho}\right)\frac{\partial\Delta}{\partial x} \qquad \text{.....(ii)}$$

Since $\lambda = -\frac{2}{3}\mu$ for compressible fluid and $\Delta = 0$ for an incompressible fluid, then (ii) reduces to

$$\frac{du}{dt} = X - \frac{i}{\rho}\frac{\partial p}{\partial x} + v\Delta^2 u + \frac{1}{3}v\frac{\partial\Delta}{\partial x}$$

Thus the equations of motion along the co-ordinate axes are given by

as $$v = \frac{\mu}{\rho}$$

or $$v + \frac{\lambda}{\rho} = \frac{\mu}{\rho} + \frac{\lambda}{\rho}$$

$$= \frac{\mu}{\rho} - \frac{2}{3}\frac{\mu}{\rho}$$

$$= \frac{1}{3}v$$

Similarly

$$\left.\begin{aligned}\frac{du}{dt} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} + v\Delta^2 u + \frac{1}{3}v\frac{\partial \Delta}{\partial x}\\ \frac{dv}{dt} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + v\Delta^2 v + \frac{1}{3}v\frac{\partial \Delta}{\partial y}\\ \text{and}\quad \frac{dw}{dt} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + v\Delta^2 w + \frac{1}{3}v\frac{\partial \Delta}{\partial x}\end{aligned}\right\} \quad \text{.....(iii)}$$

Known as Navier-Stoke's equation of motion.

These equations can be written in tensor form as

$$\frac{du_1}{dt} = X_1 - \frac{1}{\rho}p_{ij} + vu_{ij} + \frac{1}{3}v\nabla_{ij}$$

The relation (iii), can also be represented in vectorial form

$$\frac{dq}{dt} = F - \nabla\int\frac{dp}{\rho} + v\nabla^2 q + \frac{2}{3}v\nabla(\nabla.q) \quad \text{.....(iv)}$$

Where $q = (x\,y\,z)$ and $F = (X, Y, Z)$

$$\left\{\text{Since } \frac{dq}{dt} = \frac{\partial q}{\partial t} + (q.\nabla)q = \frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times(\nabla\times q)\right.$$

$$\left\{\text{and } \nabla\times(\nabla\times q) = \nabla(\nabla.q) - \nabla^2 q\right.$$

Thus the equation (iv) reduces to,

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times(\nabla\times q) = F - \nabla\int\frac{dp}{\rho} + v\left[\nabla(\nabla.q) - \nabla\times(\nabla\times q)\right] + \frac{1}{3}v\nabla(\nabla.q)$$

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times(\nabla\times q) = F - \nabla\int\frac{dp}{\rho} + \frac{4}{3}v\nabla(\nabla.q) - v\nabla\times(\nabla\times q) \quad \text{.....(v)}$$

Which is another form of Navier-Stoke's equation of motion.

For incompressible flow, the relations (iv) and (v) reduce to

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p + v\nabla^2 q$$

$$= F - \frac{1}{\rho}\nabla p - v\nabla\times(\nabla\times q). \quad \text{.....(vi)}$$

Boundary Conditions:

The equation (vi) represents that for an incompressible flow the equation of motion differs from Euler's equation of motion in inviscid flow by the form $-\nu\nabla\times(\nabla\times q)$. This term arises due to Viscosity which increases the order of differential equation and therefore and additional boundary condition is needed. This is satisfied by the condition that there must be no slip between a viscous fluid and its boundary. So at fixed boundary q = 0. It follows that the normal and tangential velocity components both must vanish.

Equations for vorticity and circulation.

We know that the Navier-stoke's equation of motion is

$$\frac{\partial q}{\partial t}+\nabla\left(\frac{1}{2}q^2\right)-q\times(\nabla\times q)=F-\frac{1}{\rho}\nabla p+\nu\nabla^2 q$$

Let the external forces are conservative and density is a function of pressure only.

Then $\quad \vec{\zeta}=\nabla\times q$

or $$\frac{\partial q}{\partial t}-q\times\vec{\zeta}=-\nabla\left[\Omega+\int\frac{dp}{\rho}+\frac{1}{2}q^2\right]+\nu\nabla^2 q$$

Taking curl of both the sides we have

$$\text{curl}\frac{\partial q}{\partial t}-\text{curl}\left(q\times\vec{\zeta}\right)=\nu\ \text{curl}\left(\nabla^2 q\right)$$

or $$\frac{\partial\vec{\zeta}}{\partial t}+(q.\nabla)\vec{\zeta}-\left(\vec{\zeta}.\nabla\right)q=\nu\nabla^2\vec{\zeta}$$

$$\begin{cases}\text{as div. }\zeta\\ =\text{div. cural q}\\ =0\end{cases}$$

or $$\frac{d\vec{\zeta}}{dt}=\left(\vec{\zeta}.\nabla\right)q+\nu\overrightarrow{\nabla^2\zeta}$$

Which is known the **equation to vorticity.**

Let Γ be the circulation round a closed circuit,

then $\quad \Gamma=\int_c udx+vdy+wdz$

or $$\frac{D\Gamma}{Dt}=\frac{D}{Dt}\int_c udx+vdy+wdz$$

$$\frac{D\Gamma}{Dt}=\int_c\left(\frac{Du}{Dt}dx+\frac{Dv}{Dt}dt+\frac{Dw}{Dt}dz\right)+\int_c(udu+vdv+wdw)$$

{The second vanishes as circuit being closed.

or $$\frac{D\Gamma}{Dt} = \int_c \left[-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+V\right)dx - \frac{\partial}{\partial y}\left(\frac{p}{\rho}+V\right)dy - \frac{\partial}{\partial z}\left(\frac{p}{\rho}+V\right)dz + v\left(\nabla^2 udx + \nabla^2 vdy + \nabla^2 wdz\right)\right]$$

$$= -\int_c d\left(\frac{p}{\rho}+V\right) + v\int_c \nabla^2\left(udx+vdy+wdz\right)$$

$$= v\nabla^2 \int_c udx+vdy+wdz$$

$$\left\{\text{as } \Gamma = \int_c udx+vdy+wdz\right.$$

= $v\nabla^2\Gamma$ (other integral vanishes for a closed circuit)

Equations of motion in cylindrical polar coordinates.

We know that the Navier-stoke's equation is

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times(\nabla\times q)$$

$$= F - \frac{1}{\rho}\nabla p + v\nabla\times(\nabla\times q) \quad \text{.....(i)}$$

Let (r, θ, z) be the coordinates of a point, then it reduces to

$$\frac{d}{dt}(q_r) - \frac{qe^2}{r} = -\frac{\partial\Omega}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$+v\left(\nabla^2 q_r - \frac{2}{r^2}\frac{\partial p_e}{\partial\theta} - \frac{\partial r}{r^2}\right)$$

$$\frac{d}{dt}(q_\theta) + \frac{q_r q_\theta}{r} = -\frac{\partial\Omega}{r\partial\theta} - \frac{1}{\rho}\frac{\partial\Omega}{r\partial\theta} + v\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r^2}\right)$$

and $$\frac{d}{dt}(q_z) = -\frac{\partial\Omega}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + v\nabla^2 q_z$$

where $$\frac{d}{dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + q_\theta\frac{\partial}{r\partial\theta} + q_z\frac{\partial}{\partial z}$$

and $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

(iii) Spherical Polar Coordinates.

$$\frac{dq_r}{dt} - \frac{1}{r}\left(q_e^2 + q_\phi{}^2\right) = -\frac{\partial\Omega}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + v\left(\nabla^2 q_r - \frac{2q_r}{r^2} - 2\frac{\cot\theta}{r^2}q_\theta - \frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial q_\phi}{\partial\phi}\right)$$

Note: Navier Stoke's equation can also be written as

$$\frac{\partial V}{\partial t} + grad.\left(\frac{1}{2}V^2\right) - 2V\times\omega = -F - \frac{1}{\rho}\,\text{grad.}\,p + v\nabla^2 V$$

Now curl $V = 2(\xi, \eta, \zeta)$

$$= 2\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \nabla^2 u = -\nabla^2 u$$

Thus the equation reduces to, if F = –grad. V

or $$\frac{\partial V}{\partial t} - 2V\times\omega = -\text{grad.}\left(\frac{p}{\rho} + V + \frac{1}{2}V^2\right) - 2v\,\text{curl}\,\vec{\omega}$$

and $$\frac{\partial q_\theta}{\partial t} - \frac{q_\phi{}^2\cot\theta}{r} + \frac{q_r q_\theta}{r} = -\frac{1}{r}\frac{\partial\Omega}{\partial\theta} - \frac{1}{\rho}\frac{\partial p}{r\partial\theta} + v\left(\nabla^2 q_\theta - \frac{q_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi}\right)$$

$$\frac{\partial q_\phi}{\partial t} + \frac{q_r q_\theta}{r} + \frac{q_\theta q_\phi\cot\theta}{r} = -\frac{\partial\Omega}{r\sin\theta\partial\phi} - \frac{1}{\rho}\frac{\partial p}{r\sin\theta\partial\phi} + v\left(\nabla^2 q_\phi - \frac{q_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\phi}\right)$$

where $$\frac{d}{dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + q_\theta.\frac{\partial}{r\partial\theta} + q_\phi\frac{\partial}{r\sin\theta\partial\phi}$$

and $$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

(iii) Orthogonal curvilinear coordinates.

We know

$$dS^2 = (h_1 d\lambda_1)^2 + (h_2 d\lambda_2)^2 + (h_3 d\lambda_3)^2$$

then $$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial\lambda_1}l_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial\lambda_2}m_1 + \frac{1}{h_3} + \frac{\partial\phi}{\partial\lambda_3}.n_1$$

$$\nabla \times F = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 l_1 & h_2 m_1 & h_3 n_1 \\ \frac{\partial}{\partial \lambda_1} & \frac{\partial}{\partial \lambda_2} & \frac{\partial}{\partial \lambda_2} \\ h_1 F_1 & h_2 F_2 & h_2 F_3 \end{vmatrix}$$

Energy Dissipation due to Viscosity.

Consider a particle of viscous fluid of fixed mass $\rho \delta v$ moving at any time *t* with velocity *q*.

The kinetic energy is = $\frac{d}{dt}\left\{\frac{1}{2}(\rho \delta v) q^2\right\} = \rho \delta v q . \frac{dq}{dt}$

Thus the total rate of gain kinetic energy of the entire fluid of volume *V* is

$$= \int_V \rho q . \left(\frac{dq}{dt}\right) dv$$

$$= \rho \int_V q . \left(\frac{dq}{dt}\right) dv \qquad \text{for an incompressible fluid.}$$

We know that the Navier-stoke's equations for a viscous fluid is

$$\frac{dq}{dt} = F - \frac{1}{\rho} \nabla p - v \nabla \times (\nabla \times q) \qquad \text{.....(i)}$$

Multiply both the side of (i) scalarly be $\rho q dv$ and integrating over the volume *V* of the fluid.

$$\int_V \rho q . \frac{dq}{dt} dv = \int q . F \rho dv - \int \nabla . (\rho q) dv - v . \rho \int q . \{\nabla \times (\nabla \times q)\} dv$$

Thus the rate of energy dissipation (*E*) due to viscosity is

$$E = \mu \int_V q . \{\nabla \times (\nabla \times q)\} dv \qquad \left\{ \text{as } v = \frac{\mu}{\rho} \right.$$

We know that

$$\left\{\nabla . \left\{q \times \text{curl}\, q\right) = (\text{curl } q)^2 - q . \{\nabla \times (\nabla \times q)\}\right\}$$

or $$E = \mu \int_V (\nabla \times q)^2 dv - \mu \int_V \nabla . (q \times \text{curl } q) dv$$

$$E = \mu \int_V (\nabla \times q)^2 dv - \mu \int_S n . (q \times \text{curl } q) ds$$

{Changing from volume integral to surface integral

(where *S* is the total surface enclosing the volume *V*).

When the boundary S is at rest, and there is not slips between fluid and boundary.

Then $\quad q = 0$ on the surface S

thus $$E = \mu\int_v (\nabla \times q)^2 dv = \mu\int_v (\text{curl } q)^2 dv = \mu\int_v \zeta^2 dv$$

Ex. 1. Prove that

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial(x, y)}$$

Where ψ is a stream function for a two-dimensional motion of a viscous liquid.

We know that the Navier stoke's equation for compressible viscous fluid.

$$\frac{\partial q}{\partial t} + (q.\nabla)q = -\frac{1}{\rho}\nabla p \times \nu\nabla^2 q$$

(since external body forces are absent)

or $$\frac{\partial q}{\partial t} - q \times (\nabla \times q) = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}q^2\right) + \nu\nabla^2 q \qquad \text{.....(i)}$$

Taking curl of the relation (i) both the sides, we have

$$\frac{\partial\vec{\zeta}}{\partial t} - \text{curl}\left(q \times \vec{\zeta}\right) = \nu \text{ curl } \nabla^2 q$$

$\{$as $\vec{\zeta} = \nabla \times q$

or $$\frac{\partial\vec{\zeta}}{\partial t} + (q.\nabla)\vec{\zeta} - \left(\vec{\zeta}.\nabla\right)q = \nu\nabla^2\vec{\zeta} \qquad \text{.....(i)}$$

Since there is a two dimensional motion of a viscous fluid then

$q = (u, v, 0)$

and $\quad \vec{\zeta} = (0, 0, \zeta)$

Now (ii) can be written as

$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\vec{\zeta} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\vec{\zeta} \qquad \text{.....(iii)}$$

The stream function ψ exist, then

$$u = -\frac{\partial\psi}{\partial y} \text{ and } v = \frac{\partial\psi}{\partial x}$$

or $\quad \zeta = \nabla^2\psi$

Substituting the value of ζ in (iii), we have

$$\left(v\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = v\frac{\overrightarrow{\partial\zeta}}{\partial y} + u\frac{\overrightarrow{\partial\zeta}}{\partial x}$$

or $$\left(v\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\psi}{\partial x}.\frac{\partial}{\partial y}\nabla^2\psi - \frac{\partial\psi}{\partial y}.\frac{\partial}{\partial x}\nabla^2\psi$$

or $$\left(v\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial}{\partial x}(\psi).\frac{\partial}{\partial y}\left(\nabla^2\psi\right) - \frac{\partial}{\partial y}(\psi)\frac{\partial}{\partial x}\left(\nabla^2\psi\right)$$

or $$\left(v\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\phi = \frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial(x\,y)}$$

Ex. 2. Prove that, in the slow steady motion of a viscous liquid in two dimensions

$v\nabla^2\psi = \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y}$ where (X, Y) is the impressed force per unit area

We know that the Navier-stoke's equation of motion is

$$\frac{\partial q}{\partial t} + (q.\nabla)q = F - \frac{1}{\rho}\nabla p + v\nabla^2 q \qquad .....(i)$$

Here $\frac{\partial q}{\partial t} = 0$, motion being steady. Also the inertia term $(q.\nabla)$ q is negligible, that of slow motion. Since the motion of the liquid is in two dimensions, so

F = (*X Y*) ⇒ Impressed force or external body force

q = (*u, v*) ⇒ Components of the velocity.

The equation (1) reduces to

$$F - \frac{1}{\rho}\nabla p + v\nabla^2 q = 0 \qquad .....(ii)$$

Taking curl of the above relation, we have

Curl F + $v\nabla^2$ curl q = 0

or $\quad$ Curl F + $\nabla v^2 \overrightarrow{\zeta} = 0$ $\qquad$ .....(iii)

Thus $\vec{\zeta} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

Since $\exists$ a stream function ψ, therefore, we have

$$u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}$$

$$\vec{\zeta} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\vec{\zeta} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$$

From (iii), we have

$$\text{Curl } F = -v\nabla^4 \psi$$

or $$v\nabla^4 \psi = -\text{Curl } F$$

or $$v\nabla^4 \psi = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}$$

Ex.3. Prove that for a liquid filling up a vessel in the form of surface of revolution which is rotating about its axis (Z-axis) with angular velocity ω, the rate of dissipation of energy has on addition term

$$2\mu\omega \int\int (lDu + mDv)\, dS$$

Where $D = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right), (l, m, n)$ are the direction cosines of the inward normal.

Since the liquid rotates about Z-axis with an angular velocity ω.

Here $u = -\omega y, v = \omega x, w = 0$.

Consider the additional terms is

$$= 4\mu \int\int\int \left(\frac{\partial v}{\partial x} . \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} . \frac{\partial v}{\partial y} \right) dx\, dy\, dz \qquad \text{....(i)}$$

$$= 4\mu \int\int\int \left\{ \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x} \right) \right\} dx\, dy\, dz$$

$$= -4\mu \int\int \left\{ lv\frac{\partial}{\partial x} - mv\frac{\partial u}{\partial x} \right\} dS$$

$$= -4\mu \int\int v\left(l\frac{\partial u}{\partial y} - m\frac{\partial u}{\partial x}\right)dS$$

$$= -4\mu \int\int v\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)dS$$

$$= -4\mu w \int\int x\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)dS \qquad \text{.....(ii)}$$

{as $v = \omega x$

{from the equation of continuity

(i) can also be represented, as follows

$$= 4\mu \int\int\int \left\{\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)\right\} dx\,dy\,dz$$

$$= -4\mu \int\int \left(mu\frac{\partial v}{\partial x} - lu\frac{\partial v}{\partial y}\right)dS$$

$$= -4\mu \int\int u\left(m\frac{\partial v}{\partial x} - l\frac{\partial v}{\partial y}\right)dS$$

$$= -4\mu \int\int u\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial y}\right)dS$$

{as $u = -\omega y$

$$= 4\mu \int\int y\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x}\right)dS \qquad \text{....(iii)}$$

Taking the mean of (ii) and (iii), we get

$$= 2\mu\omega \int\int \left\{y\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x}\right) - x\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)\right\}dS$$

$$= 2\mu\omega \int\int \left\{l\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)u + m\left(m\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)v\right\}dS$$

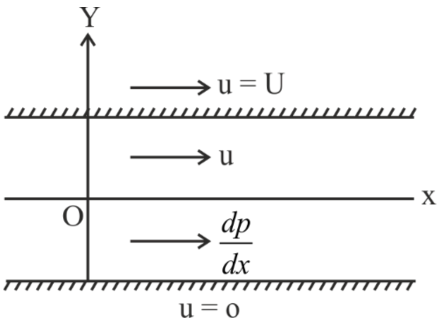
$$= 2\mu\omega \int\int (l.Du + m.Dv)\,dS$$

Laminar flow between parallel plate.

By laminar flow we mean that the fluid moves in layers parallel to the plates.

Consider the two-dimensional laminar flow of an incompressible fluid of constant viscosity between parallel straight plates.

In order to maintain such a motion, the pressure difference in the direction of axis of X, i.e., along the plates must be balanced by the shearing stress.



A flow is called parallel if only one velocity component is different from zero i.e., all fluid particles move in one direction.

Here for parallel flow, we have

$u = u(x, y, t)$ and $v = w = 0$ everywhere.

Also, $p = p(x, y, t)$

The equation of continuity is: $\dfrac{\partial u}{\partial x} = 0$ {as $v = 0 = w$

$\rightarrow$ that the velocity component u is independent from x.

or $u = u(y, t)$

The equation of motion is given by

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad \text{.....(i)}$$

Also $p = p(x, t)$

We see that $\dfrac{dp}{dx}$ must be a constant or a function of t, Since p is not a function of y and u is not a function of x.

Integrating (i) with regard to y for steady flow, we have

$$-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}=0$$

or $$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\frac{\partial p}{\partial x} \quad \text{.....(ii)}$$

or $$\frac{\partial u}{\partial y}=\frac{1}{\mu}\frac{\partial p}{\partial x}y+B$$

or $$u=\frac{1}{\mu}.\frac{\partial p}{\partial x}.\frac{y^2}{2}+By+A$$

or $$u=\frac{1}{2\mu}.\frac{dp}{dx}y^2+By+A$$

where A and B are arbitrary constants to be determined by the boundary conditions.

Case I. Plane Couette flow.

Here we shall determine the solution of equation (ii) between two parallel plates when the upper plate is moving in its own plane with a velocity U and the lower plate is stationary i.e., at rest.

Here $\frac{dp}{dx}=0$; one wall is at rest and other is in uniform motion.

The boundary conditions are: $y=-\frac{d}{2}, u=0$ and $y=+\frac{d}{2}, u=U$

From relation (iii), we have

$$0=-\frac{Ad}{2}+B \quad \text{and} \quad U=+\frac{Ad}{2}+B$$

Solving these two, we get

$$A=\frac{U}{d} \text{ and } B=\frac{U}{2}$$

Substituting the values of the constants A and B in (iii), we have

$$u=\frac{U}{d}y+\frac{U}{2} \quad \text{or} \quad u=\frac{U}{2}\left(1+\frac{2y}{d}\right)$$

Such a flow is known a plane couette flow or shear flow, when the upper plate is moving with velocity U. The velocity distribution is linear.

Case. II. Plane Poiseuille flow.

In this case the pressure gradient $\frac{dp}{dx}$ is not equal to zero but both the plates are at rest

i.e., $\frac{dp}{dx}$ = Constant. The boundary conditions are : $y = -\frac{d}{2}, u = 0$ and $y = +\frac{d}{2}, u = 0$

From (iii), we have; $0 = \frac{1}{2\mu}\frac{dp}{dx}.\frac{d^2}{4} + A\frac{d}{2} + B$

and $$0 = \frac{1}{2\mu}\frac{dp}{dx}.\frac{d^2}{4} - A\frac{d}{2} + B$$

which gives $A = 0$ and $B = -\frac{1}{8\mu}\frac{dp}{dx}.d^2$

Substituting the values of the constants A and B in (iii),

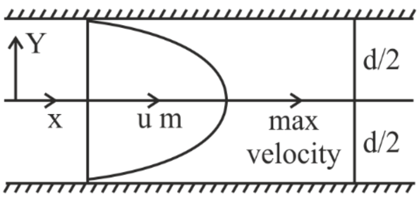
$$u = \frac{1}{2\mu}\frac{dp}{dx}y^3 - \frac{1}{8\mu}\frac{dp}{dx}d^2$$

or $$u = -\frac{1}{8\mu}\frac{dp}{dx}d^2\left(1 - \frac{4y^2}{d^2}\right)$$ or $$u = u_m\left(1 - \frac{4y^2}{d^2}\right)$$

where $$u_m = -\frac{1}{8\mu}\frac{dp}{dx}d^2$$

is the maximum velocity in the flow occurring at $y = 0$.

The velocity distribution is parabolic in the interval between the two plates.



(Parallel flow with parabolic velocity distribution)

Case III. Generalised plane Couette flow.

In this case the pressure gradient $\frac{dp}{dx}$ is constant and one plate is at rest, the other plate is in motion. The boundary conditions are given by

$$y = -\frac{d}{2}, u = 0 \quad \text{and} \quad y = +\frac{d}{2}, u = U$$

Then from (iii), we have; $0 = \frac{1}{2\mu}\frac{dp}{dx}.\frac{d^2}{4} - A\frac{d}{2} + B$

$$U = \frac{1}{2\mu}\frac{dp}{dx}.\frac{d^2}{4} + A\frac{d}{2} + B$$

Which gives $A = \frac{U}{d}$ and $B = A\frac{d}{2} - \frac{1}{2\mu}\frac{dp}{dx}.\frac{d^2}{4}$

or $B = \frac{U}{2} - \frac{d^2}{8\mu}\frac{dp}{dx}$

Substituting the values of the constants *A* and *B* in (iii), we have

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + \frac{U}{d}y + \frac{U}{2} - \frac{d^2}{8\mu}\frac{dp}{dx} \qquad \text{…..(iv)}$$

Total flux across a plane perpendicular to *X* is

$$\int_{-d/2}^{d/2} Udy \int_{-d/2}^{d/2}\left\{\frac{1}{2\mu}\frac{dp}{dx}y^2 + \frac{U}{d}y + \frac{U}{2} - \frac{d^2}{8\mu}\frac{dp}{dx}\right\}.dy$$

$$= \left[\frac{1}{2\mu}\frac{dp}{dx}.\frac{y^3}{3} + \frac{U}{d}.\frac{y^2}{2} + \frac{U}{2}y - \frac{d^2}{8\mu}\frac{dp}{dx}y\right]_{-6/2}^{6/2}$$

$$= \frac{1}{2\mu}\frac{dp}{dx}.\frac{d^3}{12} + \frac{Ud}{2} - \frac{d^2}{8\mu}\frac{dp}{dx}d$$

$$\int_{-d/2}^{d/2} Udy = \frac{Ud}{2} - \frac{d^3}{12\mu}.\frac{dp}{dx} \qquad \text{….(v)}$$

Differentiating (iv) with regard to *y*, we have

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + \frac{U}{d}$$

At $y = \pm\frac{d}{2}$

$$\frac{du}{dy} = \frac{U}{d} \pm \frac{d}{2\mu}\frac{dp}{dx}$$

Thus drag on the boundaries

$$= \left(\mu\frac{du}{dy}\right)y = \pm\frac{d}{2} = \left(\frac{\mu U}{d} \pm \frac{d}{2}\frac{dp}{dx}\right) \text{ per unit area.}$$

Laminar flow between concentric rotation cylinders. Couette flow.

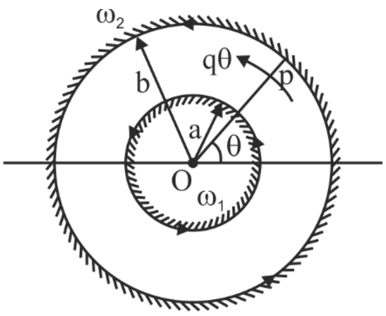
Consider the two-dimensional steady flow of an incompressible fluid between two concentric rotating cylinders.

Let a and b be the radii of the inner and outer cylinder respectively, and ω_1 and ω_2 be their angular velocities.

Here the components of velocity in cylindrical coordinates are given by

$$u = 0, v = v(r), w = 0 \text{ and } p = p(r) \qquad \text{.....(i)}$$

Substituting these values in equation of motion, we have



$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{.....(ii)}$$

and $$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = 0 \qquad \text{.....(iii)}$$

and $$\frac{\partial p}{\partial z} = 0 \qquad \text{......(iv)}$$

Let any point P the angular velocity be ω then $v = \omega r$

or $$\frac{dv}{dr} = \omega + r\frac{d\omega}{dr}$$

and $$\frac{d^2 y}{dr^2} = \frac{d\omega}{dr} + \frac{d\omega}{dr} + r\frac{d^2\omega}{dr^2} = r\frac{d^2\omega}{dr^2} + 2\frac{dw}{dr}$$

From (iii), we have

$$r\frac{d^2\omega}{dr^2} + 2\frac{d\omega}{dr} + \frac{d\omega}{dr} + \frac{\omega}{r} - \frac{\omega}{r} = 0$$

or $$r\frac{d^2\omega}{dr^2} + 3\frac{d\omega}{dr} = 0 \text{ or } \frac{d^2\omega / dr^2}{d\omega / dr} = -\frac{3}{r}$$

By integrating, we have; $\log\left(\frac{d\omega}{dr}\right) = -3\log r + \log A$

or $\quad \frac{d\omega}{dr} = \frac{A}{r^3}$ or $\quad \omega = B - \frac{A}{2r^2} \quad$ .....(v)

The boundary conditions are

I $\quad r = a, \omega = \omega_1$

II $\quad r = b, \omega = \omega_2$

or $\quad \omega_1 = B - \frac{A}{2a^2}$ and $\quad \omega_2 = B - \frac{A}{2b^2}$

Solving the above equations, we have

$$A = \frac{2a^2b^2(\omega_1 - \omega_2)}{a^2 - b^2} \text{ and } B = \frac{\omega_2 b^2 - \omega_1 a^2}{b^2 - a^2}$$

Substituting the values of the constants A and B in (v), we have

$$\omega = \frac{\omega_2 b^2 - \omega_1 a^2}{b^2 - a^2} - \frac{a^2b^2(\omega_2 - \omega_1)}{b^2 - a^2} \cdot \frac{1}{r^2}$$

If the inner cylinder is at rest, then $\omega_1 = 0$.

So $\frac{P}{4\mu}$ Or $\quad \omega = \frac{\omega_2 b^2}{r^2} \cdot \frac{r^2 - a^2}{b^2 - a^2}$

There will be the tangential stress Pre only in the fluid

i.e, $\quad Pre = \mu\left(\frac{dv}{dr} - \frac{v}{r}\right)$

$$Pre = \mu\left(\omega + r\frac{d\omega}{dr} - \omega\right)$$

$$Pre = \mu r\frac{d\omega}{dr}$$

Its moment about the axis is given by

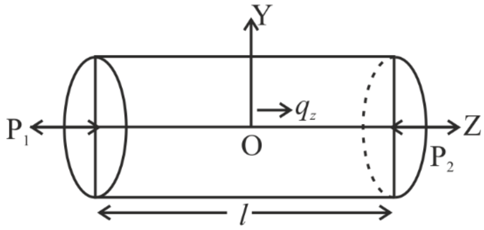
$$= 2\pi r(Pre).r = 2\pi r^2 .pr\frac{d\omega}{dr} = 2\pi\mu r^3\frac{d\omega}{dr} = 2\pi\mu r^3 . \frac{A}{r^3} = 4\pi\mu, \frac{a^2b^2}{a^2 - b^2}(\omega_1 - \omega_2)$$

$$= 4\pi\mu, \frac{a^2b^2}{b^2 - a^2}\omega_2 \qquad \{\omega_1 = 0 \text{ as the inner cylinder is at rest.}$$

Hagen-Poiseuille flow in a circular pipe.

Here we shall consider the steady laminar flow through a long straight pipe of circular cross section. We know that the shearing force on the surface of any cylindrical shape of fluid must be balanced by the difference of pressure between the ends.

Let Z-axis be chosen along the axis of the pipe.



Consider q_z be the component of velocity parallel to the axis of pipe when is a function of r only.

The velocity component in the tangential and radial directions are zero.

Equation of continuity in cylindrical coordinates

$\frac{\partial q_z}{\partial z} = 0$ …(i) $\Rightarrow$ that q_z is independent of z or a function of r only.

Also the equations of motion in cylindrical coordinates are given by

$$\mu\left(\frac{d^2 q_z}{dr^2} + \frac{1}{r}\frac{dq_s}{dr}\right) = \frac{dp}{dz} \qquad \text{…(ii)}$$

and
$$\frac{\partial p}{\partial r} = 0; \frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \qquad \text{…(iii)}$$

Since the velocity q_z is a function of r only and the pressure p is independent of r, therefore the pressure gradient $\frac{dp}{dz}$ must be a constant and let it be equal to $\frac{p_2 - p_1}{l}$, from relation (ii), we have

$$\frac{d^2 q_z}{dr^2} + \frac{1}{r}\frac{dq_z}{dr} = \frac{p_2 - p_1}{\mu l}$$

Or
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right) = -\frac{p_1 - p_2}{\mu l}$$

Or
$$r\frac{dq_z}{dr} = \frac{p_1 - p_2}{2\mu l}r^2 + A$$

Or
$$\frac{dq_z}{dr} = \frac{p_1 - p_2}{2\mu l}r^2 + \frac{A}{r}$$

Or
$$q_z = -\frac{p_1 - p_2}{4\mu l}r^2 + A\log r + B \qquad \text{…(iv)}$$

The velocity is finite at $r = 0$, so A must be zero. The boundary condition is

$$r = a,\ q_z = 0.$$

Then (iv) reduces to

$$0 = -\frac{p_1 - p_2}{4\mu l}a^2 + B \qquad \text{Or} \qquad B = \frac{p_1 - p_1}{4\mu l}a^2$$

Then $$q_z = \frac{p_1 - p_z}{4\mu l}\left(a^2 - r^2\right)$$

Since the maximum velocity occurs on the axis, then

$\left(q_z\right)_{max} = \dfrac{p_1 - p_2}{4\mu l}a^2$ (as $r = 0$ on the axis of the pipe)

The volume V_0 of the fluid flowing through the pipe per unit time is

$$V_0 = \frac{1}{2}\left(q_z\right)_{max}.\pi a^2 \quad \text{Or} \quad V_0 = \frac{1}{2}\left(\frac{p_1 - p_2}{4\mu l}a^2\right).\pi a^2$$

Or $$V_0 = \frac{\pi a^4}{8\mu}.\frac{p_1 - p_z}{l}$$

This relation was obtained experimentally by Hagen and afterwards independently by Poiseuille. With the help of this relation, the coefficient of viscosity of the fluid can be determined.

Again *total flux across any section*

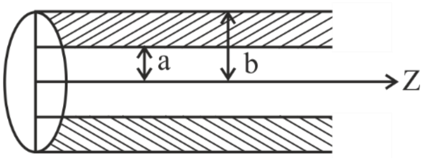
$$= \int_0^a q_z\, 2\pi r dr = 2\pi\frac{p_1 - p_2}{4\mu l}\int_0^a\left(a^2 - r^2\right).r\, dr = \frac{p_1 - p_2}{8\mu l}.\pi a^4$$

and the *drag* on the cylinder is

$$= 2\pi a l\mu\left\{\frac{p_1 - p_2}{4\mu l}(-2r)\right\}_{r=a} = \pi a^2\left(p_1 - p_2\right)$$

Steady flow between co-axial circular pipes.

Let the flow take place between two co-axial cylinders of radii a and b ($b > a$). Consider the inner boundary have a velocity V while the outer is at rest.



The boundary conditions are $r = a$, $q_z = V$ and $r = b$, $q_z = 0$

Then as we have from previous discussion, we have

$$V = \frac{p_1 - p_2}{4\mu l}a^2 + A\log a + B$$

and $$0=\frac{p_1-p_2}{4\mu l}b^2+A\ \ lob\ b+B$$

Substituting the values of the constants A and B in relation (iv) § 9.91.

$$q_z = V\,\frac{log\left(\frac{r}{b}\right)}{log\left(\frac{a}{b}\right)}-\frac{p_1-p_2}{4\mu l}$$

$$\left\{r^2-\frac{b^2\,log\left(\frac{r}{a}\right)-a^2\,log\left(\frac{r}{h}\right)}{log\left(\frac{b}{a}\right)}\right\}$$

The flux relative to the fixed boundary is given by

$$\int_a^b q_z.2\pi rdr$$

$$=\ \pi V\left\{\frac{\frac{1}{2}\left(b^2-a^2\right)}{log\left(\frac{b}{a}\right)}-a^2\right\}+\frac{\pi}{8\mu}.\frac{p_1-p_2}{l}\left\{b^4-a^4-\frac{\left(b^2-a^2\right)^2}{log\left(\frac{b}{a}\right)}\right\}$$

Steady flow in tubes of cross-section other than circular.

Consider the axis of z along the axis of the tube. Let the component of velocity w is a function of x and y but not of z, and that $u = 0 = v$.

The equation to continuity reduced to

$$\frac{\partial w}{\partial z}=0\,, \qquad \ldots\text{(i)}$$

$\Rightarrow$ that w is independent of z i.e., a function of x and y only. There are no external forces and the inertia terms vanish in steady motion, then the equations of motion reduce to,

$$\frac{\partial p}{\partial x}=0 \quad \text{and} \quad \frac{\partial p}{\partial y}=0 \qquad \ldots\text{(ii)}$$

and $$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{\partial p}{\partial z} \qquad \ldots\text{(iii)}$$

Since w is independent of z, p is independent of x and y then in steady flow along a tube the pressure gradient $\frac{\partial p}{\partial z}$ must be a constant, let it be equal to $(-P)$.

then $$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{P}{\mu} \qquad \text{...(iv)}$$

with a boundary condition w = 0 on the surface of the tube.

Consider $w = \lambda - \frac{P}{4\mu}(x^2 + y^2)$, then λ has to satisfy the equation

$$\frac{\partial^2 \lambda}{\partial x^2}+\frac{\partial^2 \lambda}{\partial y^2}=0$$

with the boundary condition $\lambda = \frac{P}{4\mu}(x^2 + y^2)$ on the surface of the tube.

Thus to solve the problem for a particular boundary we consider

$$w = \lambda - \left\{as \frac{1}{y}\frac{\partial \omega}{\partial r}=\frac{1}{r}.\frac{d\omega}{dr}(x^2 + y^2) + B \right. \qquad \text{...(v)}$$

where B is an arbitrary constant, λ is a suitable solution of the two dimensional Laplace's equation. The constant B can be determined by applying the condition w = 0 on the surface of the tube.

(a) Elliptic section.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

Consider $w = A(x^2 - y^2) + B - \frac{P}{4\mu}(x^2 + y^2)$ ...(i)

Since on the surface of the elliptic section

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(ii)}$$

On the boundary $w = 0$

then $$\left(\frac{P}{4\mu}-A\right)x^2+\left(\frac{P}{4\mu}+A\right)y^2=B \qquad \text{...(iii)}$$

This required that, from (ii) and (iii), we have

$$a^2\left(\frac{P}{4\mu}-A\right)=b^2\left(\frac{P}{4\mu}+A\right)=B$$

Or $$A=\frac{P}{4\mu}.\frac{a^2-b^2}{a^2+b^2} \quad \text{and} \quad B=\frac{P}{2\mu}.\frac{a^2-b^2}{a^2+b^2}$$

Substituting the values of A and B in (i), we have

$$w=\frac{P}{4\mu}.\frac{a^2-b^2}{a^2+b^2}\left(x^2-y^2\right)+\frac{P}{2\mu}\frac{a^2b^2}{a^2+b^2}-\frac{P}{4\mu}\left(x^2+y^2\right)$$

Or $$w=\frac{P}{2\mu}.\frac{a^2-b^2}{a^2+b^2}\left\{1+\frac{1}{2}\frac{a^2-b^2}{a^2b^2}\left(x^2-y^2\right)\right.$$

$$\left.-\frac{1}{2}\frac{a^2+b^2}{a^2b^2}\left(x^2+b^2\right)\right\}$$

Or $$w=\frac{P}{2\mu}.\frac{a^2b^2}{a^2+b^2}\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}$$

Flux of the fluid over the area of the ellipse is given by

$$=\iint w\,dx\,dy=\frac{P}{2\mu}.\frac{a^2b^2}{a^2+b^2}\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$$

$$=\frac{P}{2\mu}.\frac{a^2b^2}{a^2+b^2}\left\{\iint dx\,dy-\frac{1}{a^2}\iint x^2dx\,dy-\frac{1}{b^2}\iint y^2\,dx\,dy\right\}$$

$$=\frac{P}{2\mu}\frac{a^2b^2}{a^2+b^2}\left\{\pi ab-\frac{1}{a^2}.\pi ab\frac{a^2}{4}-\frac{1}{b^2}.\pi ab\frac{b^2}{4}\right\}$$

$$=\frac{P}{2\mu}\frac{a^2b^2}{a^2+b^2}.\frac{1}{2}\pi ab=\frac{\pi P}{4\pi}.\frac{a^3b^3}{a^2+b^2}$$

(b) Equilateral triangle.

Consider

$$w=A\,(x^3+3xy^2)+B-\frac{P}{4\mu}\left(x^2+y^2\right) \qquad \ldots\text{(i)}$$

Since $w = 0$ at all points of the boundary, then from (i), we have

$$A(x^2-3xy^2)+B-\frac{P}{4\mu}(x^2+y^2)=0 \qquad \ldots\text{(ii)}$$

If $x = a$ be a part of the boundary, then

$$A\,(a^3-3ay^2)+B-\frac{P}{4\mu}(a^2+y^2)=0$$

Or $$Aa^3+B-\frac{Pa^2}{4\mu}=0 \text{ and } -3aA-\frac{P}{4\mu}=0$$

Thus $$A=-\frac{P}{12a\mu} \text{ and } B=\frac{Pa^2}{3\mu}$$

Substituting the values of A and B in (ii), we have

$$-\frac{P}{12a\mu}\left(x^3-3xy^2\right)+\frac{Pa^2}{3\mu}-\frac{P}{4\mu}\left(x^2+y^2\right)=0$$

Or $$x^3-3xy^2+3ax^2+3ay^2-4a^2=0$$

Or $(x-a)(x+2a-\sqrt{3}y)(x+2a+\sqrt{3}y)=0.$

Therefore the boundary consists of

$x=a,\ x+2a-\sqrt{3}y=0$ and $x+2a+\sqrt{3}y=0$

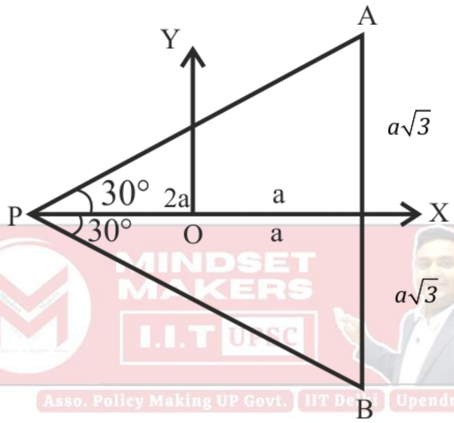
or $x=a,\ y=\frac{1}{\sqrt{3}}x+\frac{2}{\sqrt{3}}a$ and $y=-\frac{1}{\sqrt{3}}x+\frac{2a}{\sqrt{3}}$ …(iii)

Which forms an equilateral triangle

So $w=-\frac{P}{12a\mu}\left(x^3-3xy^2+3ay^3-4a^3\right)$

Flux of the fluid over the cross-section is

$$\iint w\,dx\,dy$$



$$=-\frac{P}{12a\mu}\iint\left(x^3-3xy^2+3ax^2+3ay^2-4a^3\right)*\,dxdy$$

*(i) $\iint x^3dx\,dy=\int_{-2a}^{a}x^3(y)\Big|_{-\frac{x+2a}{\sqrt{3}}}^{\frac{x+2a}{\sqrt{3}}}.dx \qquad =\frac{2}{\sqrt{3}}\int_{-2a}^{a}x^3(x+2a)dx=-\frac{9a^5}{5\sqrt{3}}$

(ii) $3\iint xy^2dx\,dy=\int x\left(y^3\right)\Big|_{-\frac{x+2a}{\sqrt{3}}}^{\frac{x+2a}{\sqrt{3}}}.dx \qquad =\frac{2}{3\sqrt{3}}\int_{-2a}^{a}x(x+2a)^3dx=\frac{2la^5}{5\sqrt{3}}$

(iii) $3a\iint\left(x^2+y^2\right)dx\,dy$ = 3a (sum of the moments of Inertia)= 3a ($\frac{1}{3}$. 3a. a$\sqrt{3}$)

$$\left\{\frac{3a^2}{4}+\frac{3a^2}{4}+a^2+\frac{a^2}{4}+\frac{a^2}{4}\right\}=9\sqrt{3}\,a^5$$

(iv) $4a^3 \iint dx\, dy = 4a^3 .3a.a\sqrt{3} = 12\sqrt{3}\, a^5$ $= -\dfrac{P}{12a\mu}\left\{-\dfrac{9a^5}{5\sqrt{3}} - \dfrac{27a^5}{5\sqrt{3}} + \dfrac{27a^5}{\sqrt{3}} - \dfrac{36a^5}{\sqrt{3}}\right\}$

$$= \frac{27}{20\sqrt{3}} . \frac{Pa^4}{\mu}$$

Average flow = $\dfrac{Flux}{Area} = \dfrac{\dfrac{27}{10\sqrt{3}}}{\dfrac{1}{2}.3a.2a\sqrt{3}} = \dfrac{3}{20}\dfrac{Pa^2}{\mu}$

Steady motion due to a slowly rotating sphere.

Consider the component of velocity are

$$u = -\omega y,\ v = \omega x \text{ and } \omega = 0$$

where ω is the angular velocity and is a function of r ($r^2 = x^2 + y^2 + z^2$) only.

The equations of motion are; (neglecting the squares of velocities)

Or $0 = \dfrac{\partial p}{\partial x} + \mu \nabla^2 u$ …(i)

Or $0 = 0 = \dfrac{\partial p}{\partial x} + \mu \nabla^2 v$ …(ii)

Or $0 = \dfrac{\partial p}{\partial z}$ …(iii)

Since $\dfrac{\partial u}{\partial x} = -y\dfrac{\partial u}{\partial x} or \dfrac{\partial^3 u}{\partial x^2} = -y\dfrac{\partial^2 u}{\partial x^2}$

Or $\dfrac{\partial u}{\partial y} = -y\dfrac{\partial \omega}{\partial y} - \omega or \dfrac{\partial^2 u}{\partial x^2} = -y\dfrac{\partial^2 \omega}{\partial y^2} - 2\dfrac{\partial \omega}{\partial y}$

and $\dfrac{\partial^2 u}{\partial z^2} = -y\dfrac{\partial^2 \omega}{\partial z^2}$

Thus $\nabla^2 u = -y\left\{\nabla^2 \omega + \dfrac{2}{y}, \dfrac{\partial \omega}{\partial y}\right\}$

Or $\nabla^2 u = -y\left\{\dfrac{d^2\omega}{dr^2} + \dfrac{2}{r}.\dfrac{d\omega}{dr} + \dfrac{2}{r}\dfrac{d\omega}{dr}\right\}$; $\left\{as \dfrac{1}{y}\dfrac{\partial \omega}{\partial r} = \dfrac{1}{r}.\dfrac{d\omega}{dr}\right.$

Or $0 = -\dfrac{\partial p}{\partial x} - \mu y\left(\dfrac{d^2\omega}{dr^2} + \dfrac{4}{4}\dfrac{d\omega}{dr}\right)$

Now the equation (i), (ii) and (iii) reduce to

$$0 = \frac{\partial p}{\partial z};\ 0 = -\frac{\partial p}{\partial y} + \mu x\left(\frac{d^2\omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr}\right);\ 0 = \frac{\partial p}{\partial z}$$

These are satisfied by p = constant.

the $$\frac{d^2\omega}{dr^3}+\frac{4}{r}\frac{d\omega}{dr}=0$$

By integrating, we have

$r^6\frac{d\omega}{dr}=A$ Or $\frac{d\omega}{dr}=\frac{A}{r^4}$ Or $\omega=\frac{B}{r^3}+C$ ...(iv); (where B and C are arbitrary constant).

Let the motion is produced by a solid sphere of radius a rotating with angular velocity Ω and the liquid extends to infinity, we have

$$C = 0,\ B = a^3\,\Omega$$

So $$\omega=\frac{a^3}{r^3}\Omega$$

If there is an outer fixed concentric sphere of radius b, then the boundary conditions are

I $\quad r = a,\ \omega = \Omega$

II $\quad r = b,\ \omega = 0$

From (*iv*), we have

$$\Omega=\frac{B}{a^3}+C \text{ And } 0=\frac{B}{b^2}+C$$

or $$B=\frac{a^3b^3}{b^3-a^3}\Omega \text{ or } C=-\frac{a^3}{b^3-a},\Omega b$$

Substituting the values of B and C in (*iv*), we have

$$\omega=\frac{a^3b^3}{b^3-{}^3a}\Omega\frac{1}{r^3}-\frac{a^3}{b^3-a^3}\Omega \text{ i.e. } \omega=\frac{a^3\Omega}{r^8}\cdot\frac{b^3-r^3}{b^3-a^3}$$

Ex. 4. One surface (nearly plane) is fixed and another near surface (plane) rotates with angular velocity ω about an axis perpendicular to its plane and there is a film of viscous fluid between them. Prove that the pressure p in the film satisfies the equation

$$h^3\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}\right)+\frac{\partial h^3}{\partial r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial h^3}{\partial\theta}\frac{\partial p}{\partial\theta}=6\mu\omega\frac{\partial h}{\partial\theta},$$

where (r, θ) are polar coordinates in the plane of the film, the origin being in the axis of rotation, and h is the thickness of the film.

Consider any point (x, y) on the upper surface

then $\quad U = -\,\omega y,\ V = \omega x$

The total flux across a plane perpendicular to X-axis is

$$\int_0^h udz=\frac{1}{2}hU-\frac{h^3}{12\mu}\frac{\partial p}{\partial x} \qquad \{\text{Ref. equation } (v) \text{ Case } III$$

$$=-\frac{1}{2}h\omega y-\frac{h^3}{12rl}\frac{\partial p}{\partial x} \qquad ...(ii)$$

Similarly the total flux across a plane perpendicular to Y-axis,

$$\int_0^h vdz=\frac{1}{2}h\omega x-\frac{h^3}{12\mu}\frac{\partial p}{\partial y} \qquad ...(ii)$$

Now from the equation of continuity, we have

$$\frac{\partial}{\partial x}\left\{-\frac{1}{2}h\omega y-\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\frac{1}{2}h\omega x-\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right\}=0$$

or
$$\frac{h^3}{12\mu}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\frac{1}{12_{\mu}}\left(\frac{\partial h^3}{\partial x}\cdot\frac{\partial p}{\partial x}+\frac{\partial h^3}{\partial y}\cdot\frac{\partial p}{\partial y}\right)$$

$$=\frac{1}{2}\omega\left(x\frac{\partial h}{\partial y}-y\frac{\partial h}{\partial x}\right) \qquad ...(iii)$$

Since (r, θ) are the polar coordinates in the plane of the film,

$$\left.\begin{aligned} &\text{then} \quad \frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\\ &\text{and} \quad \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\\ &\text{also} \quad \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\end{aligned}\right\} \qquad (iv)$$

Substituting the results of (*iv*) in (*iii*), we have

$$\frac{h^3}{12\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)p+\frac{1}{12\mu}\left\{\left(\cos\theta\frac{\partial h^3}{\partial r}-\frac{\sin\theta}{r}\frac{\partial h^3}{\partial\theta}\right)\right.$$

$$\left.\left(\cos\theta\frac{\partial p}{\partial r}-\frac{\sin\theta}{r}\frac{\partial p}{\partial\theta}\right)+\left(\sin\theta\frac{\partial h^3}{\partial r}+\frac{\cos\theta}{r}\frac{\partial h^3}{\partial\theta}\right)\left(\sin\theta\frac{\partial p}{\partial r}+\frac{\cos\theta}{r}\frac{\partial p}{\partial\theta}\right)\right\}$$

$$=\frac{1}{2}\omega\left\{r\cos\theta\left(\sin\theta\frac{\partial h}{\partial r}+\frac{\cos\theta}{r}\frac{\partial h}{\partial\theta}\right)-r\sin\theta\left(\cos\theta\frac{\partial h}{\partial r}-\frac{\sin\theta}{r}\frac{\partial h}{\partial\theta}\right)\right\}$$

or
$$h^3\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)p+\frac{\partial h^3}{\partial r}\cdot\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial h^3}{\partial\theta}\cdot\frac{\partial p}{\partial\theta}=6\mu\omega\frac{\partial h}{\partial\theta}\cdot$$

Viscosity

Ex. 5. A liquid occupying the space between two co-axial circular cylinders is acted upon by a force $\frac{c}{r}$ per unit mass, where r is the distance from the axis, the lines of force being circles round the axis. Prove that in the steady motion the velocity at any point is given by the

$$\frac{C}{2v}\left\{\frac{b^2}{r^2}\frac{r^2-a^2}{b^x-a^2}\log\left(\frac{b}{a}\right)-r\log\frac{r}{a}\right\}$$

where v is the coefficient of kinematric viscosity.

Consider the axis of the cylinder be the z-axis. Here

$$q_r = 0 = q_z$$

and q_θ is independent of θ and z *i.e.* it is a function of r only.

So $\qquad q_\theta = r\omega \qquad$ {$q_z = 0$ considering the cylinders to be sufficiently long.

where ω is the angular velocity of the liquid at the point (r, θ, z).

thus the equation of motion for viscous fluid reduces to

$$v\left(\nabla^2 q_\theta - \frac{q_\theta}{r^2}\right) + \frac{C}{r} = 0.$$

or
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\omega r) - \frac{\omega}{r} = -\frac{C}{vr}$$

or
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\omega r) - \frac{\omega}{r} = -\frac{C}{vr}$$

or
$$r\frac{d^2\omega}{dr^2} + 3\frac{d\omega}{dr} = -\frac{C}{vr}$$

Multiplying both the sides with r^2 and integrating, we have

$$r^2\frac{d^2\omega}{dr^2} + 3r^2\frac{d\omega}{dr} = -\frac{C}{v}r$$

or
$$r^3\frac{d\omega}{dr} + \frac{Cr^2}{2v} = A$$

or
$$\frac{d\omega}{dr} = -\frac{C}{2vr} + \frac{A}{r^2}$$

or
$$\omega = -\frac{C}{2v}\log lr - \frac{A}{2r^2} + B \qquad ...(i)$$

where A and B are arbitrary constant.

The boundary conditions are,

I. $\omega = 0, \quad r = a$

II. $\omega = 0, \quad r = b.$

Now the relation (*i*) reduces to with the help of condition *I* and *II*,

$$0 = -\frac{C}{2v}\log a - \frac{A}{2a^2} + B \qquad ...(ii)$$

or
$$0 = -\frac{C}{2v}\log b - \frac{A}{2b^2} + B. \qquad ...(iii)$$

By subtracting, we have

$$0 = \frac{C}{2v}(\log b - \log a) - \frac{A}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

or
$$A = \frac{Ca^2b^2}{v(b^2 - a^2)}\log\left(\frac{b}{a}\right). \qquad ...(iv)$$

Form (*i*) and (*ii*), by subtracting, we have

$$\omega = -\frac{C}{2v}(\log r - \log a) - \frac{A}{2}\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$

or
$$\omega = -\frac{C}{2v}(\log r - \log a) - \frac{Ca^2b^2}{2v(b^2 - a^2)}\cdot\frac{a^2 - r^2}{r^2a^2}\log\left(\frac{b}{a}\right) \quad \{\text{from } (iv)$$

or
$$\omega = -\frac{C}{2v}\log\left(\frac{r}{a}\right) + \frac{C}{2v}\cdot\frac{r^2 - a^2}{b^2 - a^2}\cdot\frac{b^2}{r^2}\log\left(\frac{b}{a}\right).$$

Thus $\quad q_\theta = rw$

$$= -\frac{Cr}{2v}\log\left(\frac{r}{a}\right) + \frac{C}{2v}\cdot\frac{r^2-a^2}{b^2-a^2}\cdot\frac{b^2}{r}\log\left(\frac{b}{a}\right)$$

$$= \frac{C}{2v}\left\{\frac{r^2-a^2}{b^2-a^2}\cdot\frac{b^2}{r}\log\left(\frac{b}{a}\right) - \log\left(\frac{r}{a}\right)\right\}. \quad \text{Proved.}$$

Ex. 6. Incompressible viscous liquid is moving steadily under pressure between planes $y = 0$, $y = h$. The plane $y = 0$ has a constant velocity U in the direction of the axis x, and the plane $y = h$ is fixed. The planes are porous, and the liquid is sucked in uniformly over one and ejected uniformly over the other. Show that a possible solution is given by

$$u = \frac{\left(Ue^{h/a} + Ah\right) - \left(U + Ah\right)}{e^{h/a} - 1} + Ay, v = \frac{v}{a}$$

where v is the kinematic coefficient of viscosity. Determine the meaning of the constants A and a.

Since the planes $y = 0$ and $y = h$ are taken infinitely large, the velocity components (u, v) at any point (x, y) will be independent from x. Thus the equation of continuity reduces to

$$\frac{\partial v}{\partial y} = 0 \qquad \ldots(i)$$

or $\quad v = \text{constant} = \dfrac{v}{a}.$

The equations of motion are

$$v\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2}. \qquad \ldots(ii)$$

and $\quad 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}. \qquad \ldots(iii)$

The boundary conditions are,

I. $\quad y = 0, u = U$

II. $\quad y = h, u = 0.$

Substituting the given value of u and the value of $v = \dfrac{v}{a}$ in the equations of motion (ii), we have

$$\frac{v}{a}\left\{-\frac{1}{a}\frac{\left(U + Ah\right)}{e^{h/a} - 1} + A\right\} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$+v\left\{-\frac{1}{a^2}\frac{\left(U + Ah\right)e^{v/a}}{e^{h/a} - 1}\right\} \qquad \ldots(iv)$$

i.e. $\dfrac{Av}{a} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ i.e. $\dfrac{\partial p}{\partial x} = -\dfrac{A}{a}v\rho$ i.e. $\dfrac{\partial p}{\partial x} = -\dfrac{A\mu}{a}$

$$p = -\frac{A\mu}{a}x + B$$

which also satisfies the condition (iii), $\dfrac{\partial p}{\partial y} = 0$

Hence the given velocity components satisfy the equation of motion, and forms the possible solutions.

Consider the mass of liquid sucked per unit area per unit time at $y = 0$ be m, then $m = \rho v$

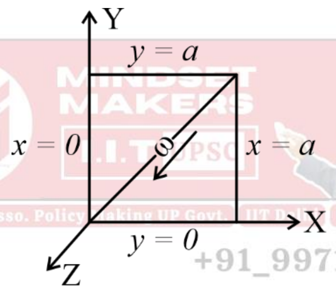
$$m = \rho \frac{v}{a}\frac{\mu}{a} \qquad \{\text{as } \mu = \rho v] \quad \text{or } a = \frac{\mu}{m} \cdot$$

Substituting the value of a in (iv), we have

$$A = -\frac{1}{\rho} \cdot \frac{a}{v}\frac{\partial p}{\partial x} \text{ i.e. } A = -\frac{\mu}{m\rho v}\frac{\partial p}{\partial x} \text{ i.e. } A = -\frac{1}{m}\frac{\partial p}{\partial x} \cdot$$

Ex. 7. Viscous liquid flowing steadily under pressure through an infinitely long rectangular tube whose axis is parallel to the axis of z. The sides $x = 0$ and $x = a$ are smooth and the sides $y = 0$, $y = a$ do not permit of slipping of liquid in contact with them. The pressure gradient Maintaining the motion is suddenly annulled. Show that the total flux across any section is $\frac{Qa^2}{10v}$ where Q is the flux per unit time across a section in the initial steady motion, given that $\sum_{0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \cdot$

Since the rectangular tube is infinitely long and the sides $x = 0$ and $x = a$ are smooth. The velocity component w of an element at $(x\ y\ z)$ parallel to Z-axis is a function of y only, the other two components u and v are zero.



The equations of motion reduce to

$$\mu \frac{\partial^2 w}{\partial y^2} = \frac{\partial p}{\partial z} \qquad \text{...}(i)$$

and

$$\frac{\partial p}{\partial x} = 0 = \frac{\partial p}{\partial y} \cdot \qquad \text{...}(ii)$$

Integrating (i), we have

$$\mu \frac{\partial w}{\partial y} = \frac{\partial p}{\partial z} y + A$$

or

$$\mu w = \frac{1}{2}\frac{\partial p}{\partial z} y^2 + Ay + B \qquad \text{...}(iii)$$

The boundary conditions are

I. $w = 0$, $y = 0$ and *II.* $w = 0$, $y = a$

which gives from (iii),

$$B = 0 \text{ and } A = -\frac{1}{2} a \frac{\partial p}{\partial z}$$

or $\mu w = \frac{1}{2}\left(y^2 - ay\right)\frac{\partial p}{\partial z}\cdot$ ...(*iv*)

Thus flux $Q = \int_0^a w(ady)$

or $Q = \frac{a}{2\mu}\frac{\partial p}{\partial z}\int_0^a \left(y^2 - ay\right)dy\ ;\ Q = \frac{a}{2\mu}\frac{\partial p}{\partial z}\left(\frac{a^3}{3} - \frac{a^2}{2}\right);\ Q = -\frac{a^4}{12\mu}\frac{\partial p}{\partial z}$

or $\frac{\partial p}{\partial z} = -\frac{12\mu Q}{a^4}$ ...(*v*)

So $\mu w = -\frac{1}{2}\left(y^2 - ay\right)\cdot\frac{12\mu Q}{a^4}$ Or $w = \frac{6Q}{a^4}y(a - y)$....(*vi*)

When the pressure gradient is suddenly annihilated, the equation of motion becomes

$$\frac{\partial w}{\partial t} = -v\frac{\partial^2 w}{\partial y^2}\cdot \qquad \{\text{Here } u = 0 = v \qquad ...(vii)$$

Consider $w = f(y)e^{-vk^2 t}$ be the solution of (*vii*), then

$$\frac{\partial^2 f(y)}{\partial y^2} = -k^2 f(y)$$

which shows that $f(y)$ is of the from cos *ky* or *sin ky*

or $w = \sum Ak\, e^{-vk^2 t}\begin{cases}\cos ky\\ \sin ky\end{cases}$ ...(*viii*)

At $t = 0$, we have; $w = \frac{6Q}{a^4}y(a - y)$.

Expressing $y(a - y)$ in the form of Fourier Series $0 < y < a$,

we have $y(a - y) = \frac{8a^2}{\pi^3}\sum_{\pi-1}^{\infty}\frac{\sin(2n+1)\frac{\pi y}{a}}{(2n+1)^2}$ ...(*ix*)

Consider $k = (2\pi + 1)\frac{\pi}{a}\cdot$

Thus *w* for any time *t* is given by; $w = \frac{6Q}{a^4}\cdot\frac{8a^2}{\pi^2}\sum_{n-1}^{\infty}\frac{\sin(2n+1)\frac{\pi y}{a}}{(2n+1)^2}\cdot e^{-vt\left(\frac{2n+1}{a}\pi\right)^2}$

Thus the total flux is

$$= \int_{t=0}^{\infty}\int_{v=0}^{a}(a\,dy)w.dt = a\frac{48Q}{\pi^3 a^2}\sum_{n-1}^{\infty}\frac{1}{(2n+1)^3}\int_0^{\infty} e^{-vt\left(\frac{2n+1}{a}\pi\right)^2_{dt}}\int_0^a \sin\left\{(2n+1)\frac{\pi y}{a}\right\}dy$$

$$= \frac{48Q}{\pi^3 a}\cdot\frac{2a^2}{v\pi^2}\cdot\frac{a}{\pi}\sum\frac{1}{(2n+1)^3}\left\{\frac{1}{v\left(\frac{2n+1}{a}\right)^2}\right\}\left\{\frac{2}{(2n+1)\frac{\pi}{a}}\right\}$$

$$=\frac{48Q}{\pi^3 a}\cdot\frac{2a^2}{v\pi^2}\cdot\frac{a}{\pi}\sum\frac{1}{(2n+1)^6}=\frac{96Qa^2}{v\pi^6}\sum\frac{1}{(2n+1)^6}=\frac{96Qa^2}{v\pi^6}\cdot\frac{\pi^6}{960}=\frac{Qa^2}{10v}.$$

PREVIOUS YEARS QUESTIONS

CHAPTER 6. NAVIER STOKES EQUATION

Q1. Find Navier-Stokes equation for a steady laminar flow of a viscous incompressible fluid between two infinite parallel plates. **[8c UPSC CSE 2014]**

Q2. For a steady Poiseuille flow through a tube of uniform circular cross-section, show that

$w(R)=\frac{1}{4}\left(\frac{p}{\mu}\right)\left(a^2-R^2\right)$. **[7a UPSC CSE 2011]**

Note: The beauty of systematic learning is- You'll find solutions of almost every PYQ in above examples or questions attached with detailed answers. So to avoid repetition in this book, we have not put those solutions again as answers to PYQs.