Brain mapping UPSC CSE & IFoS Syllabus

Paper I: Dynamics & Statics

Paper II: Rigid dynamics & Fluid dynamics

Dynamics

- **Kinematics in 2-D**

  Linear: Velocity & Acceleration

  Angular: Velocity & Acceleration

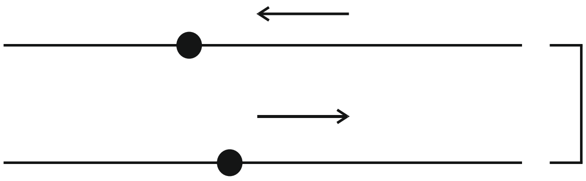
  Radial & Transverse: velocity & Acceleration

  Tangential & Normal: velocity & Acceleration

  Definition; *decode the formula & Remember.*

- **Rectilinear Motion**

  (i) Particle is moving along the line (Rectilinear motion)/ one-directional motion

  (ii) 

  (i) Simple Harmonic Motion (SHM)

  Required to remember: Newton's II law of motion,

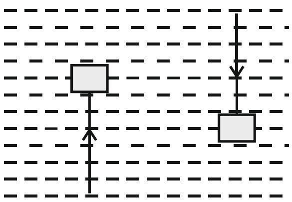
  $$F = ma = m\frac{d^2x}{dt^2}$$

  Keywords: velocity : $\frac{dx}{dt}$, Acceleration : $\frac{d^2x}{dt^2}$

  e.g. If $\frac{d^2x}{dt^2} \alpha (-x)$; We get a differential equation

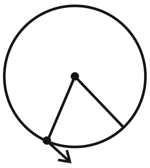
  Solve this diff eq^n &interpret about required result.

  (iii)  (ii) Hooke's Law (Elasticity comes into picture).

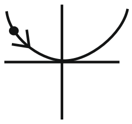
  (iv)  (Resisting liquid)

  (iii) Motion in a resisting medium

- **Constraint Motion**



cercular motion:-

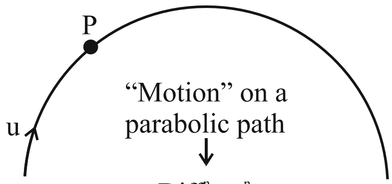


cycloid just
an example

constrained Motion

Normal	Velocity	
&	&	$\rightarrow$ diff. eq $\rightarrow$ solve & get the result
Tangential	acceleration	

- **Projectiles**



"Motion" on a parabolic path

$\downarrow$

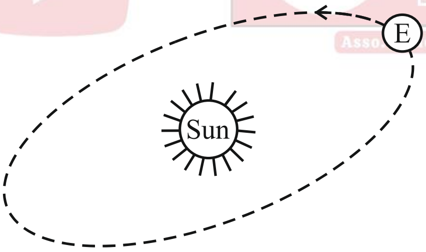
Diff^n eq^n

$\downarrow$

Sol^n

- Range
- Maximum weight

- **(i) Central Orbit** **(ii) Inverse square law.**



Kepler's law

Diff. eq^n $\rightarrow$ Sol^n $\rightarrow$ Interpret

e.g. $F = G\dfrac{m_1 m_2}{r^2}$

$F \times \dfrac{1}{r^2}$

Like $\dfrac{d^2x}{dt^2}$

$\downarrow$ Motivation

Diff^n eq^n

$\downarrow$

Solve

$\downarrow$

Interpret

- **Work, Energy, Impulse**

Def^n & formula

Interpreting given eq^n

$\downarrow$

Reaching to diff. eq^n

$\downarrow$

Sol^n

Noticed! We're interested in studying the **motion of a particle** in this topic dynamics (all chapters)

Rigid Dynamics

Motion of "system of particles"

$\downarrow$

a rigid body e.g:- bodies like ball, cylinder, rectangular lamina, triangle, circular disk etc.

Flow of Questions:

Newton's second law of motion

$\downarrow$

Diff. eq^n

$\downarrow$

Sol

$\downarrow$

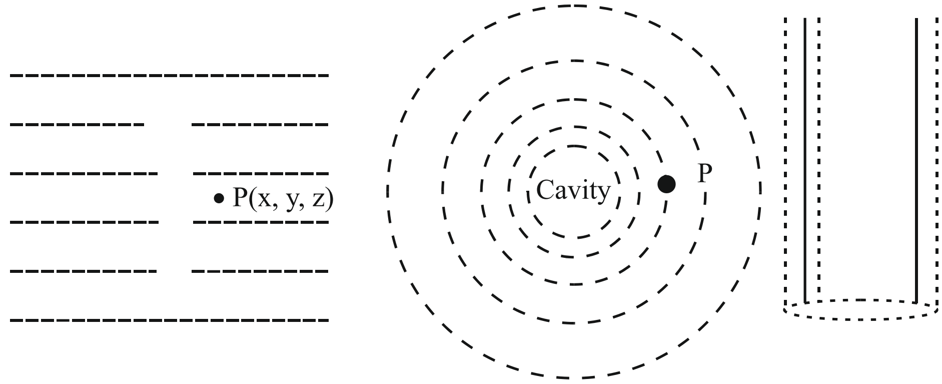
Interpretation

1. Moment of Inertia:-
2. D'Alembert's Principle
3. Motion about a fined axis
4. Motion in 2D.
5. Lagrangian & Hamiltonian

(fixed formulae $\rightarrow$ standard examples $\rightarrow$ Remember the procedure)

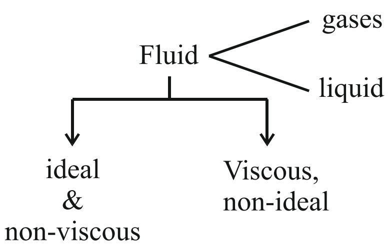
Fluid Dynamics (P-II)

Velocity vector $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$



Spherical motion Cylindrical motion

fluid particle at the point P, we study the **motion of the fluid particle** at P(x, y, z)



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1. **Kinematics in 2D:-** (100% one question)

Type I Q:- eq^n of continuity & Possible fluid motion

Formula & solve

$\vec{\Delta} \cdot \vec{q} = 0$ solve

Type II Q:- Streamlines

formula & solve: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

solve using vector calculus.

Type III Q:- Existence of scalar potential then

$\vec{q} = \vec{\Delta}\phi$

$$\Rightarrow u\hat{i} + v\hat{j} + w\hat{k} = \frac{-\partial\phi}{\partial x}dx = \frac{\partial\phi}{\partial y}dy - \frac{\partial\phi}{\partial z}dz$$

compare & find $\phi(x, y, z)$; called scalar potential

2. **Euler's equation of motion:**

Two categories of (force) eq^n:-

↓

Solve (P.D.E eq).

Interpret

3. **Motion in 2D:- Sources & sinks.**

Chapter 1 & chapter 3 : should be studied together

Omplex potential $\omega = \phi + i\psi$

From chapter 1: 3-Types of Q^n based on formulae (Direct)-

4. Axisymmetric motion:-

Spherical Cylindrical

↓

Formula → Solving

Till chapter 4: ideal (non-viscous) fluid (No vorticity)

5. **Vortex Motion**

Vorticity comes into picture

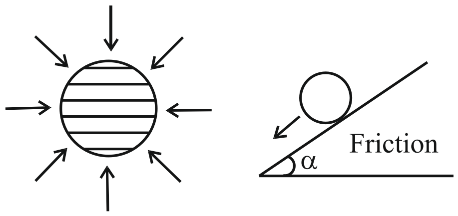
Two categories of Q^n

Diff eq^n → solving → Interpret

6. **Navier stoke's eqn:-**

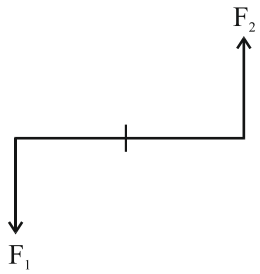
Viscous fluid (3 Types)

Statics (P–I)

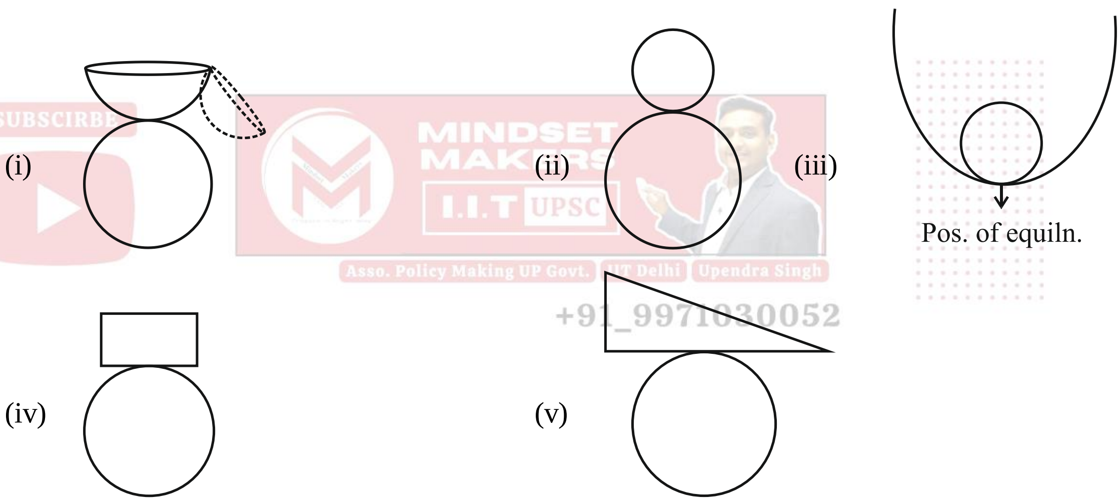


1. **Equilibrium & coplanar forces** [Many Q^n asked (Basic logic formula) based Q^n]
2. **Friction**
3. **Forces in 3D**

(Moment, couple)



4. **Stable & unstable equilibrium**



Different cases based on bodies

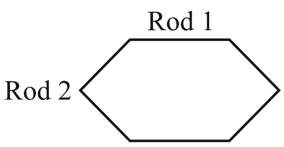
++: Very standard (famous) repeated categories of Q^n (4–5) : Just solve twice & go:

Giving a jerk (tilting a bity) to hemisphere

If it comes back in its original position (stable) | did not come back

5. **Principle of virtual Work:-**

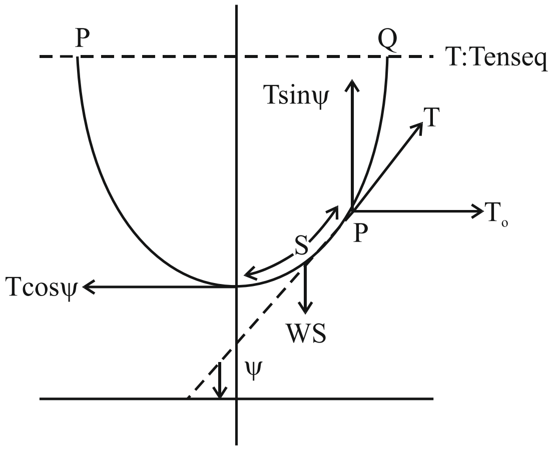
Categories of Q^n



6. **Common Category**

Flexible string (tied at two points P & Q) hanging under gravity

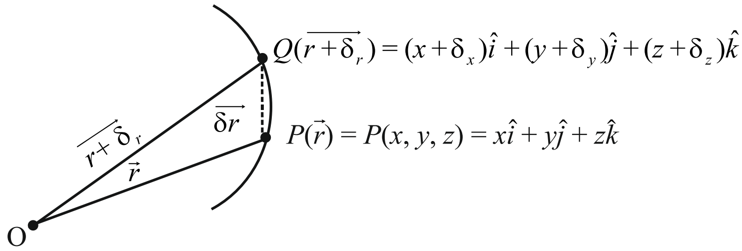
Fixed kind of problem (can be guessed)



W is; weight per unit length of chain/string

(Kinematics)

Velocity & Acceleration



Let a particle moves from P to Q in time interval δt i.e., at the time t; particle is at the position P & at the time t + δt, particle is at the position Q

$\therefore$ displacement from P to Q is $\delta\vec{r}$

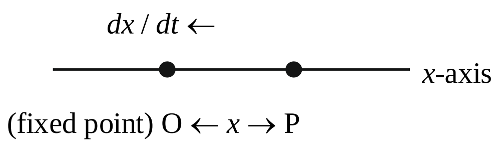
- $\dfrac{\delta\vec{r}}{\delta t}$ is called the average velocity & $\lim\limits_{\delta t\to 0}\dfrac{\delta\vec{r}}{\delta t} = \dfrac{d\vec{r}}{dt}$: velocity
- $\vec{v} = \dfrac{d\vec{r}}{dt}$: velocity
- $v = |\vec{v}| = \left|\dfrac{d\vec{r}}{dt}\right|$: speed
- $\dfrac{\delta\vec{v}}{\delta t}$ is called average acceleration R

  $\lim\limits_{\delta t\to 0}\dfrac{\delta\vec{v}}{\delta t} = \dfrac{d^2\vec{r}}{dt^2}$: acceleration.

  Note: (1) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

  $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

  Here, $\dfrac{dx}{dt}$ is called component/resolved part of $\vec{v}$ along x-axis. Similarly others are along y and z axis respectively.
- (It is taken to be positive or negative according as the direction of $\vec{v}$ in the direction of increasing or decreasing of x



$$\frac{\overleftarrow{dx}}{dt} = -ve$$

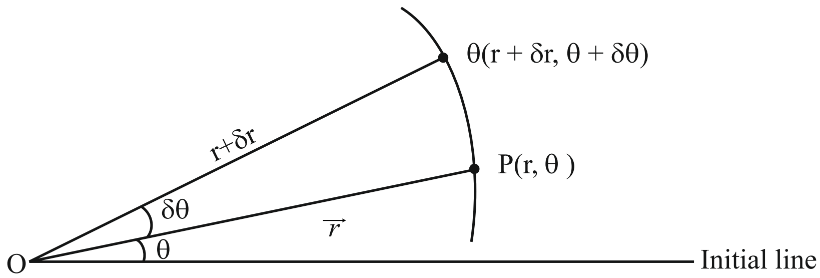
Similarly, we can discuss for $\dfrac{dy}{dt}$ & $\dfrac{dz}{dt}$ respectively

Note: (2) Similar discussion works for acceleration components.

$$\vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

$\frac{d^2x}{dt^2}$: +ve or –ve : According as it is in the direction of x is increasing or x decreasing ;i.e. force is in the direction towards or opposite to fixed point.

Angular Velocity, Angular Acceleration



The average rate of change of angle of OP about the point (; is called the angular velocity of point P (Particle at P);

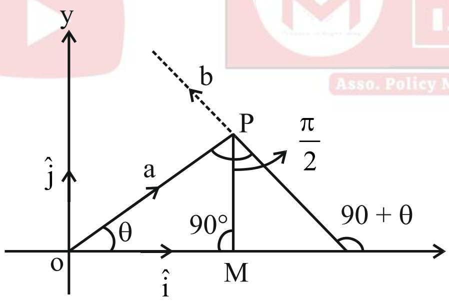
It is given by $\vec{w} = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta}$; Angular velocity

Angular acceleration = $\frac{d^2\theta}{dt^2} = \ddot{\theta}$ radian/s^2

- **Rate of change of unit vector in a plane:-**

Let a : unit vector

b : unit vector : normal to plane of a



Let $\vec{a} = \overrightarrow{OM} + \overrightarrow{MP}$

$\vec{a} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$\frac{d\vec{a}}{dt} = -\sin\theta \cdot \frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}$

$\frac{d\vec{a}}{dt} = \frac{d\theta}{dt}\left[\cos\left(\frac{\pi}{2}+\theta\right)\hat{i} + \sin\left(\frac{\pi}{2}+\theta\right)\hat{j}\right]$

$\frac{d\vec{a}}{dt} = \frac{d\theta}{dt}\vec{b}$

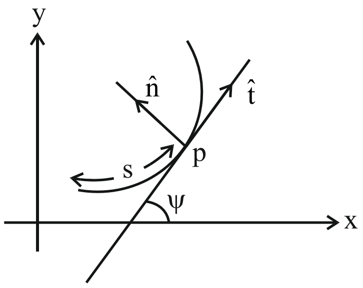
Where $\vec{b} = \cos\left(\frac{\pi}{2}+\theta\right)\hat{i} + \sin\left(\frac{\pi}{2}+\theta\right)\hat{j}$ is a unit vector

Inclined at an angle $\left(\frac{\pi}{2}+\theta\right)$ with OX (i.e., x-axis)

Particular case

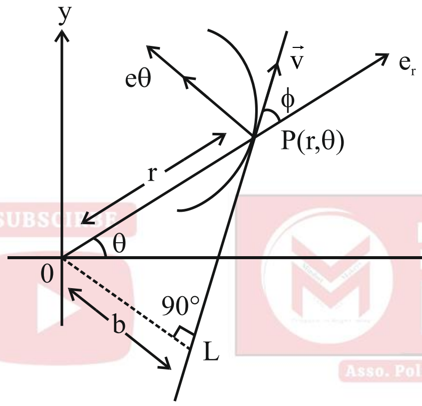
Exam point

$$\boxed{\frac{d\hat{n}}{dt} = -\frac{d\psi}{dt}\hat{t} = -\psi \cdot \hat{t}}$$



here $\hat{t}$ is in the direction of s increasing (along tangent at p) & $\hat{n}$: normal $\hat{n}$ is in the direction of ψ increasing.

- Relation between Angular & linear velocities



v_r : component of $\vec{v}$ along radius vector $\vec{r}$

v_θ : component of $\vec{v}$ along v

$v = |\vec{v}|$

$\angle POX = \theta$

$$\vec{w} = \frac{d\theta}{dt}$$

e_r, e_θ : be unit vectors along & perpendicular to OP respectively.

$\boldsymbol{r} = |\vec{r}|\ e_r = re_r$ and $\frac{d}{dt}(e_r) = \frac{d\theta}{dt} e_\theta$……(2)

The linear velocity of $\vec{v}$ at the point P along OP is:

Now, $\because\ \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r \cdot e_r)$

$$= \frac{dr}{dt}e_r + r \cdot \frac{d}{dt}(e_r)$$

$$\vec{v} = \frac{dr}{dt}e_r + r\frac{d\theta}{dt}e_\theta$$

We know that,

Now the component of a vector $\vec{a}$ in the direction of unit vetor $\vec{b}$ is given by $\vec{a}\cdot\vec{b}$. If v_θ is the component of the velocity $\vec{v}$ in the direction perpendicular to OP, then

$$v_\theta = v_r e_\theta = \left(\frac{dr}{dt}e_r + r\frac{d\theta}{dt}e_\theta\right)e_\theta$$

$$= r\frac{d\theta}{dt} = r.w \;\; [\because e_r \cdot e_\theta = 0]$$

So, $w = \dfrac{v_\theta}{r}$

Exam point

$$w = \frac{\text{component of velocity } v \text{ at P perpendicular to OP}}{r}$$

$\because$ The angle between $\vec{v}$ & e_θ is $\dfrac{\pi}{2}-\phi$, then

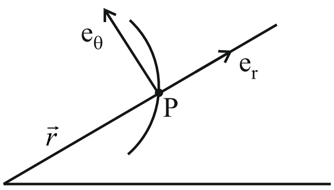
$$v_\theta = \vec{v}\cdot e_\theta = v\cdot 1.\cos\left(\frac{\pi}{2}-\phi\right) = v\sin\phi$$

$$w = \frac{v\sin\phi}{r} = \frac{v\vec{r}\sin\phi}{r^2}$$

Exam point

$$\boxed{w = \frac{d\theta}{dt} = \frac{vp}{r^2} \quad \because p = r\sin\phi}$$

- **Radial (along radius vector) & transverse (Normal to radial vector) velocities & accelerations**



$$\left.\begin{aligned}&\because \frac{de_r}{dt} = \frac{d\theta}{dt}e_\theta\\ &\text{and } \frac{d}{dt}e_\theta = \frac{d\theta}{dt}e_r\end{aligned}\right\}\ldots\ldots.(1)$$

$\because$ $\vec{v}$: the velocity of particle at P:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r.e_r) = \frac{dr}{dt}e_r + r\frac{de_r}{dt} = \frac{dr}{dt}e_r + r\frac{d\theta}{dt}e_\theta \ldots\ldots(2)$$

$$\because e_r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

here, the component $\frac{dr}{dt}$ is called the radial velocity & the component $r\frac{d\theta}{dt}$ is called the transverse velocity of particle at P.

Exam point

Radial vel. = $\frac{dr}{dt}$ (+ve : in the direction of r increasing, –ve : in the direction of r decreasing)

Transverse vel. = $r\frac{d\theta}{dt}$ (+ve : in the direction of ; θ increasing & –ve : θ decreasing)

Basic idea

If $\vec{r} = x\hat{i} + yj$ (x-compo, y compo,) (y axis is 1r to x-axis)

- If $\vec{a}$ is acceleration at point P, then

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\left\{\frac{dr}{dt}e_r + r\frac{d\theta}{dt}e_\theta\right\}$$

$$\vec{a} = \left\{\frac{d^2r}{dt^2}e_r + \frac{dr}{dt}\cdot\frac{d\theta}{dt}e_\theta\right\} + \left\{\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right\} - \left\{r\cdot\frac{d\theta}{dt}\cdot\frac{d\theta}{dt}e_r\right\}$$

$$\vec{a} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}e_r + \left\{2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right\}e_\theta \ \{(\text{form }(1))\}$$

Exam point

Radial acceleration at P = $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r(\dot{\theta})^2$

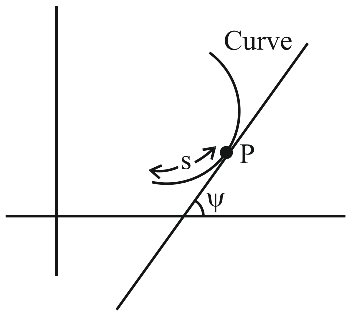
Transverse acceleration at P = $2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

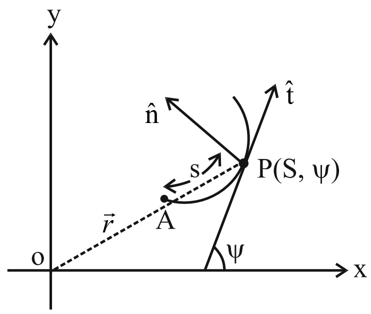
Tangential and Normal velocities & acceleration:

Intrinsic equations:- An equation involving s and ψ, is called intrinsic equation for given curve.

Here s : arc length

ψ: angle made by tangent at P, with the x-axis





$\hat{t}$: unit vector along tangent at P, in the direction of s increasing and $\hat{n}$: unit vector along the normal at P, in the direction of ψ increasing (i.e., in the direction of inwards down normal)

from vector calculus, we know that,

$\boxed{\frac{d\vec{r}}{ds} = \hat{t}}$; Remember that $\frac{d\vec{r}}{ds}$ denotes the unit tangent vector in the direction of s increasing.

Also, we know that

$\frac{d}{dt}(\hat{t}) = \frac{d\psi}{dt}(\hat{n})$; As we discussed in starting.

If $\vec{v}$ is velocity of particle at P, then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \frac{ds}{dt}\hat{t}$$

$$\therefore \ \vec{v} = \frac{ds}{dt}\hat{t} + 0 \cdot \hat{n}$$

Exam point

$\therefore$ Tangential velocity at P = $\frac{ds}{dt}$

Normal velocity at P = 0

$$\therefore \ v = |\vec{v}| = \sqrt{\left(\frac{ds}{dt}\right)^2 + 0^2} = \frac{ds}{dt}$$

For Tangential & normal acceleration

$$\because \ \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v\hat{t}\right)$$

$$= \frac{dv}{dt}\hat{t} + v\frac{d}{dt}(\hat{t})$$

$$= \frac{dv}{dt}\hat{t} + v\frac{d\psi}{dt}\hat{n}$$

$$= \frac{dv}{dt}\hat{t} + \frac{d\psi}{ds} \cdot \frac{ds}{dt}\hat{n}$$

$$= \frac{dv}{dt}\hat{t} + \left(v \cdot \frac{1}{\rho} \cdot v\right)\hat{n} \quad \left\{\because \rho = \frac{ds}{d\psi} \ \text{ Radius of curvature at P differential calculus}\right\}$$

$$\therefore \ \vec{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

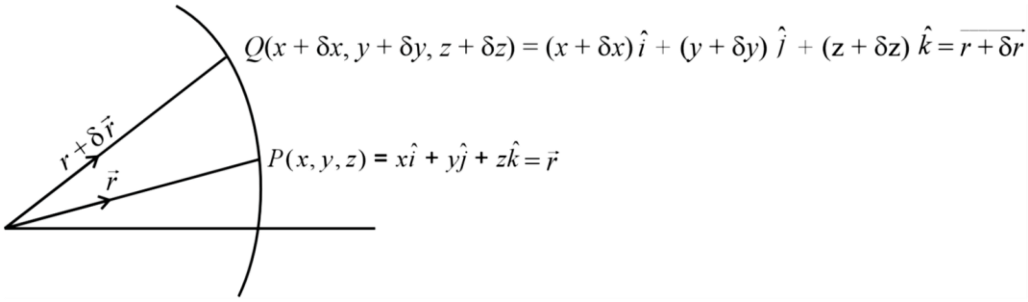
Exam point

Tangential acceleration = $\dfrac{dv}{dt}$

Normal acceleration = $\dfrac{v^2}{\rho}$

- Mindset required to recall (Information from previous discussion) for further chapters.

(1) **Velocity & Acceleration**



$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

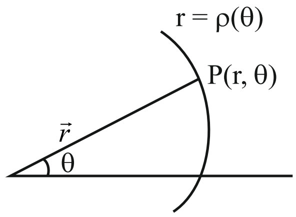
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

$\dfrac{d^2x}{dt^2}$ is component of $\vec{a}$ along x-axis

+ve or –ve according as $\vec{a}$ is in direction of increasing x or decreasing x.

$$\therefore\ |\vec{v}| = v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$a = |\vec{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$$



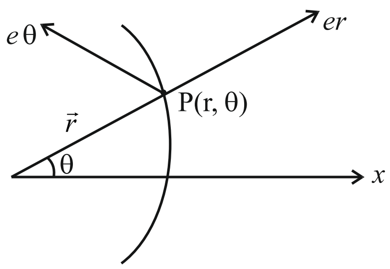
- Angular velocity $\vec{w} = \dfrac{d\theta}{dt} = \dot{\theta}$

- Angular Acceleration = $\dfrac{d^2\theta}{dt^2} = \ddot{\theta}$

Relation between linear & Angular velocities

$w = \dfrac{vp}{r^2}$; where $p = r\sin\phi$ (Pedal equation)

2. **Radial & Transverse**



(i) Radial velocity = $\dfrac{dr}{dt} = \dot{r}$

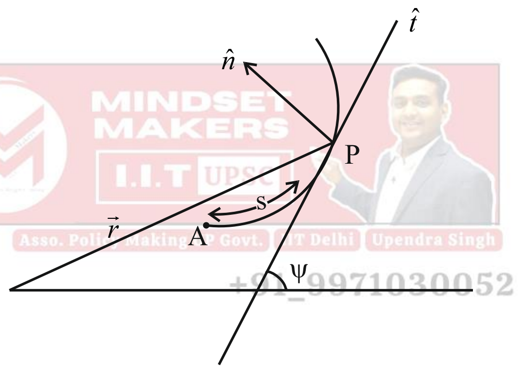
Transverse velocity = $r\dfrac{d\theta}{dt} = r\dot{\theta}$

$v = |\vec{v}| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$

(iii) Radial acceleration = $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = \ddot{r} - r\ddot{\theta}$

Transverse acceleration = $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2 \cdot \dfrac{d\theta}{dt}\right)$

3. Tangential & Normal:



$\rho = \dfrac{ds}{d\psi}$: Radius of curvature,

(1) Tangential velocity = $\dfrac{ds}{dt}$

Normal velocity = 0

(ii) Tangential acceleration = $\dfrac{d^2s}{dt^2}$

Normal acceleration = $\dfrac{v^2}{\rho}$

Resultant acceleration:

$$a = |\vec{a}| = \sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Ex-1: Prove that the acceleration of a point moving in a curve with uniform speed is $\rho\left(\frac{d\psi}{dt}\right)^2$.

Answer $\because\ v=\frac{ds}{dt}=\text{constant} \quad \therefore\ \frac{d^2s}{dt^2}=0$

$\therefore$ Acceleration = $\sqrt{\left(\frac{d^2s}{dt}\right)^2+\left(\frac{v^2}{\rho}\right)^2}=\sqrt{0+\left(\frac{v^2}{\rho}\right)^2}=\frac{v^2}{\rho}=\rho\cdot\left(\frac{v}{\rho}\right)^2$

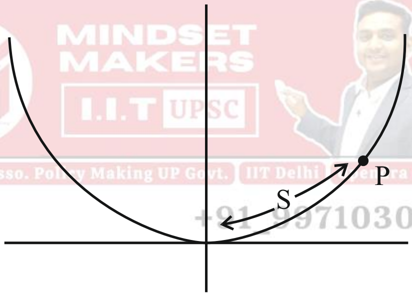
$$= \rho\cdot\left[\frac{ds/dt}{ds/d\psi}\right]^2 \qquad \left\{\begin{aligned}\because v&=\frac{ds}{dt}\\ \rho&=\frac{ds}{d\psi}\end{aligned}\right\}$$

$$=\rho\left(\frac{d\psi}{dt}\right)^2$$

Ex-2:- A point describes the cycloid s = 4a sinψ with uniform speed v. Find the acceleration at any point.

Answer The path of particle is s = 4a sin ψ…..(1)

Given, $\frac{ds}{dt}=v=\text{constant}$…..(2) $\therefore\ \frac{d^2s}{dt^2}=0$



From (1), we have $\frac{ds}{d\psi}$ = 4a cosψ = ρ …..(3)

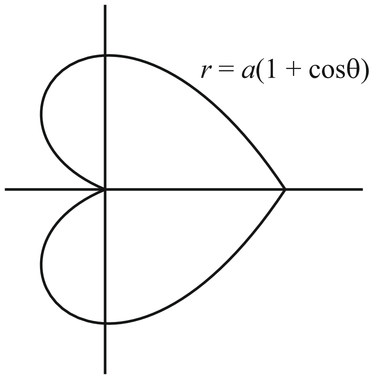
Now, the resultant acceleration is,

$$=\sqrt{\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2}=\sqrt{0+\left(\frac{v^2}{\rho}\right)^2}=\frac{v^2}{4a\cos\psi}=\frac{v^2}{4a\sqrt{1-\sin^2\psi}}=\frac{v^2}{4a\sqrt{1-\left(\frac{s}{4a}\right)^2}}$$

$$=\frac{v^2}{\sqrt{16a^2-s^2}}$$

Ex-3: A small head slides with constant speed v on a smooth wire in the shape of cardioid $r = a$ (1+ cosθ). Show that the angular velocity is $\frac{v}{2a}\left(\sec\frac{v}{2}\right)$ and that the radial component of acceleration is constant

Ans. $r = a(1+\cos\theta)$



$r = 2a\cos^2\theta/2$……(1)

$$\frac{dr}{d\theta} = -4a\cos\frac{\theta}{2}\sin\frac{\theta}{2}\times\frac{1}{2}$$

$$\frac{dr}{d\theta} = -2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\because \tan\phi = r\frac{d\theta}{dr} = \frac{2a\cos^2\theta/2}{-2a\sin\theta/2\cos\theta/2} = -\cot\frac{\theta}{2}$$

$$\tan\phi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\phi = \frac{\pi}{2}+\frac{\theta}{2}\;\ldots.(2)$$

Now, $p = r\sin\phi = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$ …..(3)

The angular velocity = $\frac{d\theta}{dt} = \frac{vp}{r^2} = \frac{v\cdot r\cos\theta/2}{r^2} = \frac{v\cos\theta/2}{2a\cos^2\theta/2} = \frac{v}{2a}\sec\frac{\theta}{2}$ ….(4)

Now, from (1), $\frac{dr}{dt} = -4a\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}\times\frac{1}{2}\frac{d\theta}{dt}$

$$= -2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\times\frac{v}{2a}\sec\left(\frac{\theta}{2}\right)$$

$$\frac{dr}{dt} = -v\sin\frac{\theta}{2}$$

$$\therefore \frac{d^2r}{dt^2} = -v\cos\frac{\theta}{2}\times\frac{1}{2}\cdot\frac{d\theta}{dt} = \frac{-v^2}{4a}$$

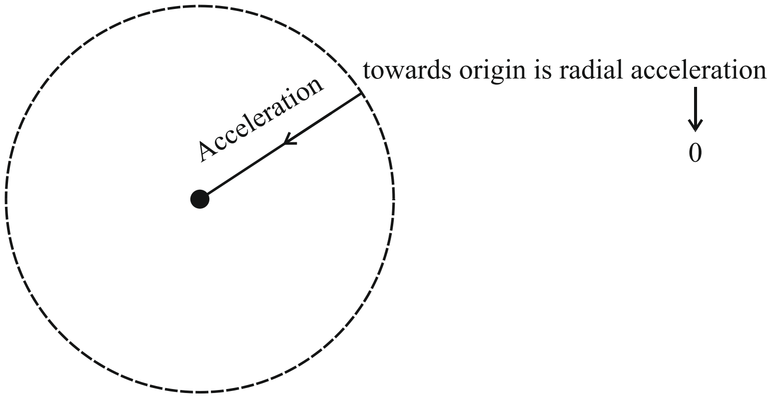
Now, Radial acceleration is,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{-v^2}{4a} - 2a\cos^2\frac{\theta}{2}\left(\frac{v}{2a}\sec\frac{\theta}{2}\right)^2 = -\frac{v^2}{4a} - \frac{v^2}{2a} = \frac{-3v^2}{4a}$$, which is constant

Ex-4: A particle moves along a circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Show that the transverse acceleration varies as the fifth power of θ.

Given Radial acceleration = 0

i.e., $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0$ …..(1)



Now, curve is given by

$r = 2a\cos\theta$

$$\frac{dr}{dt} = -2a\sin\theta\frac{d\theta}{dt}$$

$$\frac{d^2r}{dt^2} = -2a\sin\theta\cdot\frac{d^2\theta}{dt^2} - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$\therefore$ From (1),

$$-2a\sin\theta\cdot\frac{d^2\theta}{dt^2} - 2a\cos\theta\cdot\left(\frac{d\theta}{dt}\right)^2 - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} = -2\frac{\cos\theta}{\sin\theta}\cdot\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{d^2\theta/dt^2}{(d\theta/dt)} = -2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{dt}$$

Integrating w.r.t t, we get,

$\log\left(\frac{d\theta}{dt}\right) = -2\log(\sin\theta) + \log c$; c is arbitrary constant

Now, the transverse acceleration is, $= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

$$= \frac{1}{2a\cos\theta}\times\frac{d}{dt}(4a^2\cos^2\theta\cdot c\,\text{cosec}^2\theta)$$

$$\frac{4ca^2}{2a\cos\theta}\frac{d}{dt}\left(\cot^2\theta\right)$$

$$= \frac{2ac}{\cos\theta}2\cot\theta\left(-\text{cosec}^2\theta\cdot\frac{d\theta}{dt}\right)$$

$$= \frac{-4ac}{\cos\theta}\cot\theta\cdot\text{cosec}^2\theta\cdot c\,\text{cosec}^2\theta$$

$= -4ac^2\,\text{cosec}^5\theta$

Hence, the transverse acceleration α $\text{cosec}^5\theta$

Rectilinear Motion

"When a point (particle) moves along a straight line

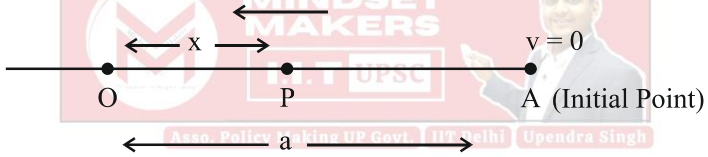
Horizontal | Vertical

Particle moves towards O from P (Rectilinear motion) at point P:-

- Velocity = $\frac{dx}{dx}$
- acceleration = $\frac{d^2x}{dt^2}$

S.H.M (Simple Harmonic Motion):-

When a particle (point) moves **towards a fixed point** O, such that the **acceleration** at some point P (which is at a distance *x* from the fined point O) **is proportional to** ***x***; then such motion is called simple harmonic motion



$\frac{d^2x}{dt^2} \times x$

$\Rightarrow \frac{d^2x}{dt^2} = -\mu x$, where μ is proportional constant

–ve sign attached (because motion is in the direction of decreasing *x*).

Now, the story begins; specifications/observations about S.H.M.

(1) ∵ The differential equation representing S.H.M is,

$$\frac{d^2x}{dt^2} = -\mu x \;.......(1)$$

We are interested in knowing what is velocity at the point P? What is x = ….? i.e displacement from fixed point O.

** Multiplying by $2\frac{dx}{dt}$ in (1);

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\mu x \cdot 2 \cdot \frac{dx}{dt}$$

On integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + c \ldots(2);\ c \text{ is integrating constant}$$

$v^2 = -\mu x^2 + c$

$\because$ At the initial point A,

$$\frac{dx}{dt} = v = 0\ ;\ x = a$$

Now, using this in (2)

$O^2 = -\mu a^2 + c \Rightarrow c = \mu a^2$

$\therefore$ We have, from (2),

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + \mu a^2$$

$$\frac{dx}{dt} = -\sqrt{\mu\left(a^2 - x^2\right)} \ \ldots..(3) \qquad \text{\{–ve sign (because motion is in the direction of decreasing x)}$$

Exampoint

$$v = -\sqrt{\mu\left(a^2 - x^2\right)}$$

Velocity of a particle at the point P.

Now, on integrating (3), we get

$$-\frac{dx}{\sqrt{\mu}\sqrt{a^2 - x^2}} = dt$$

$$\Rightarrow \frac{1}{\mu}\cos^{-1}\left(\frac{x}{a}\right) = t + c$$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}.t + \sqrt{\mu}.c \qquad \ldots..(4)$$

At the point A ; x = a, $t = 0$

$\therefore$ From (4), we have

$$0 = \sqrt{\mu}.0 + c \Rightarrow c = 0$$

$\therefore$ We have, $\quad \cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t$

$$x = a\cos\left(\sqrt{\mu}t\right) \qquad \ldots..(5)$$

(ii) At the point O:

$\because\ x = 0$

$$\therefore\ v = \frac{dx}{dt} = -\sqrt{\mu\left(a^2 - 0^2\right)} = -\sqrt{\mu}a$$

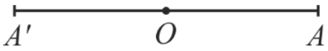
Time taken from A to O ; Let $t = t_1$

$\therefore$ From (5), $0 = a\cos\left(\sqrt{\mu}\,t_1\right)$

$\frac{\pi}{2} = \sqrt{\mu} t_1 \Rightarrow t_1 = \frac{\pi}{2\sqrt{\mu}}$

Time Period :– (T) is the time taken from A to O, O to A', A' to O, O to A

$T = 4t_1$



$$T = 4 \times \frac{\pi}{2\sqrt{\mu}}$$

$$T = \frac{2\pi}{\sqrt{\mu}}$$

Frequency (n) $= \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$

Suppose if the centre of force (O is not origin). Let if fixed point is $x = b$ then the equation of motion for SHM; written as $\frac{d^2x}{dt^2} = -\mu(x-b)$

(Now, proceed as previous)

When particle moves from A' to O; then $\frac{d^2x}{dt^2} = +\mu x$ (x is increasing direction now)

Phase & Epoch:-

$$\because \frac{d^2x}{dt^2} + \mu x = 0 \quad \text{.....(1)}$$

$$\therefore x = a\cos\left(\sqrt{\mu}t + \in\right) \quad \text{.....(A)}$$

The constant $\in$ is called the "starting phase" on the epoch of the motion and the quantity $\sqrt{\mu}.t + \in$ is called the agreement of the motion.

The phase at any time t of a SHM is the twine that has elapsed since the particle passed its extreme position in the positive direction.

$\because$ from (5), x is maximum when $\cos\left(\sqrt{\mu}t + \in\right)$ is maximum i.e., $\cos\left(\sqrt{\mu}t + \in\right) = 1$

Therefore, if t_1 to the time of reaching the internes position in the position direction then

$$\cos\left(\sqrt{\mu}t_1 + \in\right) = 1$$

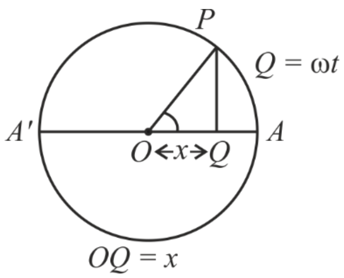
$$\sqrt{\mu}t_1 + \in = 0$$

$$t_1 = \frac{-\in}{\sqrt{\mu}}$$

$\therefore$ The phase at time (t) = $t - t_1$

$$= t + \frac{\in}{\sqrt{\mu}}$$

Geometrical representation of SHM



$OQ = x$

$\therefore\ x = a\cos(\omega t)$

As the particle comes sound the circumference, the foot Q of the perpendicular on the diameter *AA*' oscillates lecture *A* & *A'*

$$= \frac{dx}{dt} = -aw\sin\omega t \ \ \& \ \ \frac{d^2x}{dt^2} = -a\omega^2\cos\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\therefore\ \frac{d^2x}{dt^2}x - x \ \text{(SHM)} \ \& \ \mu = \omega^2$$

The periodic time

$$T = \frac{2\pi}{\omega}$$

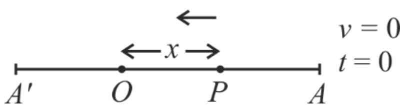
Thus, a particle decides a circle with constant angular velocity, the foot of the perpendicular from it on any dimeter executes a SHM

Periodic Motion

A motion is said to the periodic if after a certain interval of time; motion is repeated in the same manner.

Exampoint (About SHM (summary)

1. $\frac{d^2x}{dt^2} \propto -x$



2. $\frac{d^2x}{dt^2} = -\mu x$

3. $v = \frac{dx}{dt} = -\sqrt{\mu\left(a^2 - x^2\right)}$

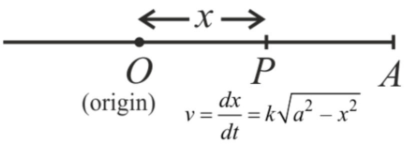
   v is maximum at point *O*; $x = 0$

   $\therefore\ v_{\max} = -\sqrt{\mu}.a$

4. $x = a\cos\left(\sqrt{\mu}t\right)$

5. Thene Period $T = \dfrac{2\pi}{\sqrt{\mu}}$

Ex-1. A particle moves in a strength line and its velocity at distance x from the sings. is $k\sqrt{a^2 - x^2}$, where a & k are constant from that the motion is SHM. Find the amplitude & time period.



Ans. Given $\dfrac{dx}{dt} = k\sqrt{a^2 - x^2}$

Fig.

$$\left(\frac{dx}{dt}\right)^2 = k^2\left(a^2 - x^2\right)$$

Diff. w.r.t. t,

$$\frac{dx}{dt} x \frac{d^2x}{dt^2} = -k^2 \times 2x.\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = -k^2 x \qquad \text{.....(1)}$$

$$\Rightarrow \frac{d^2x}{dt^2} \times x$$

Comparing (1) with $\dfrac{d^2x}{dt^2} = -\mu x$

We get $\mu = k^2$

$$\therefore \ T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{k}$$

For amplitude,

Putting $\dfrac{dx}{dt} = 0$

At A,

$$k\sqrt{a^2 - x^2} = 0$$

$$x = \pm a$$

$\Rightarrow x = a$ is the amplitude of motion.

Ex-2. Show that if the displacement of a particle in a straight hence impressed by the equ. $x = a\cos nt + b\sin nt$, it descales a SHM whose amplitude is $\dfrac{2\pi}{n}$

$$\because \ x = a\cos nt + b\sin nt \qquad \text{.....(1)}$$

$$\frac{dx}{dt} = -an\sin nt + bn\cos nt \qquad \text{.....(2)}$$

$$\frac{d^2x}{dt^2} = -an^2 \cos nt - bn^2 \sin nt$$

$$\frac{d^2x}{dt^2} = -n^2 \left(a \cos nt + b \sin t\right)$$

$$\frac{d^2x}{dt^2} = -n^2 x \Rightarrow \frac{d^2x}{dt^2} \propto x$$

$\therefore$ (SHM)

On comparing (3) with $\frac{d^2 3}{dt^2} = -\mu x$

For amplitude

Putting $\frac{dx}{dt} = 0$

$\therefore$ From (2)

$n(a\text{sint}) = bn\cos nt$

$$\tan nt = \frac{b}{a}$$

$$\therefore \ \cos nt = \frac{a}{\sqrt{a^2+b^2}}, \ \sin nt = \frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore x = a.\frac{a}{\sqrt{a^2+b^2}} + b.\frac{b}{\sqrt{a^2+b^2}} = \frac{a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$$

Ex-3. Show that the particle incenting SHM requires one sixth of its period to move from the position of maximum displacement to one in which the displacement is half of the amplitude.

Ans. As the particle incenting SHM, so,

$$\frac{d^2x}{dt^2} = -\mu x$$

$$\frac{2dx}{dt}\frac{d^2x}{dt^2} = \frac{2dx}{dt}(-\mu x)$$

On integrating

$$\left(\frac{dx}{dt}\right)^2 = \mu\left(a^2 - x^2\right)$$

$$\frac{dx}{dt} = -\sqrt{\mu\left(a^2-x^2\right)}$$

$$\Rightarrow \ -\frac{dx}{\sqrt{\mu}\sqrt{a^2-x^2}} = dt$$

$$= \frac{-1}{\sqrt{\mu}} \int_a^{a/2} \frac{dx}{\sqrt{a^2-x^2}} = \int_{t=0}^{t=t_1} dt$$

$$\Rightarrow t_1 = \frac{1}{\sqrt{\mu}} \int_{a/2}^{a} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\Rightarrow t_1 = \frac{1}{\sqrt{\mu}} \left[\sin \frac{x}{a} \right]_{a/2}^{a}$$

$$\Rightarrow t_1 = \frac{1}{\sqrt{\mu}} \left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right) \right]$$

$$\Rightarrow t_1 = \frac{1}{\mu} \left[\frac{\pi}{2} - \frac{\pi}{6} \right]$$

$$\Rightarrow t_1 = \frac{\pi}{3\sqrt{\mu}}$$

$\therefore$ We know, for SHM

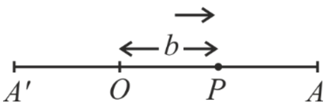
$$T = \frac{2\pi}{\sqrt{\mu}}$$

Now, $\dfrac{t_1}{T} = \dfrac{\pi / 3\sqrt{m}}{2\pi / \sqrt{\mu}} = \dfrac{1}{6}$

$$\Rightarrow t_1 = \frac{1}{6} T$$

Ex-4. A particle is performing a SHM of period T about a center O and it passes through a point P where $OP = b$, with velocity v in the direction OP, prove that the time elapses before it matures to the point P is, $\dfrac{T}{A} \tan^{-1}\left(\dfrac{vT}{2\pi b}\right)$

$\because$ The particle is in SHM.



$$T = \frac{2\pi}{\sqrt{\mu}}$$

Let a be the amplitude of the particle executing SHM.

We know,

$$\frac{dx}{dt} = -\sqrt{\mu\left(a^2 - x^2\right)} \qquad .....(1)$$

$$v^2 = \mu\left(a^2 - b^2\right) \qquad .....(2)$$

$\therefore$ Req. time = time taken from P to A & A to P

= 2 × time taken from A to P

Req. time = 2 × t_1 .....(3)

From (1), from A to P,

$$\frac{dx}{dt} = -\sqrt{\mu\left(a^2 - x^2\right)}$$

$$\int_{t=0}^{t=t_1} dt = \frac{1}{\sqrt{\mu}} \int_{x=a}^{b} -\frac{dx}{\sqrt{a^2 - x^2}}$$

$$t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\left(\frac{x}{a}\right)\right]_a^b$$

$$t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\left(\frac{b}{a}\right) - 0\right]$$

$$t_1 = \frac{\cos^{-1}\left(\frac{b}{a}\right)}{\sqrt{\mu}}$$

$$t_1 = \frac{1}{\mu} \times \tan^{-1}\left(\frac{\sqrt{a^2 - b^2}}{b}\right)$$

$$t_1 = \frac{T}{2\pi} \times \tan^{-1}\left(\frac{vT}{b.2\pi}\right)$$

$\therefore$ Req. Time = $2t_1 = \frac{T}{A}\tan^{-1}\left(\frac{vT}{2b\pi}\right)$

Ex-5. A particle is moving with SHM & while making are incursion from one position to rest to the other, its distance from middle point of its path at there consecutive seconds are observed to he x_1, x_2, x_3 prove that the time of a complete oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$

Ans. $\because$ It so SHM

$$x = a\cos\left(\sqrt{\mu} = t\right), \qquad T = \frac{2\pi}{\sqrt{\mu}}$$

$$x_1 = a\cos\left(\sqrt{\mu}t_1\right)$$

$$x_2 = a\cos\left(\sqrt{\mu}\left(t_1 + 1\right)\right)$$

$$x_3 = a\cos\left(\sqrt{\mu}\left(t_1 + 2\right)\right)$$

Now, $\dfrac{x_1 + x_3}{2x_2} = \dfrac{a\left(\cos\left(\sqrt{\mu}\left(t_1 + 2\right)\right) + a\cos\left(\sqrt{\mu}t_1\right)\right)}{2xa\cos\left(\sqrt{\mu}\left(t_1 + 1\right)\right)}$

$$= \frac{2\cos\left(\mu t_1 + \mu\right).\cos\mu}{2\cos\left(\mu\left(t + 1\right)\right)}$$

$$\frac{x_1 + x_3}{2x_2} = \frac{\cos\mu}{1}$$

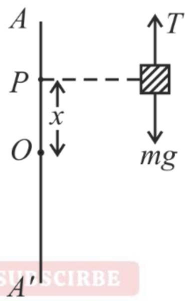
$$\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)=\mu$$

$$\frac{2\pi}{T}=\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$$

$$T=\frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$$

Next Category of Problem:

Q. A body is attached to one end of n inelastic sting, and the other end moves in a vertical line with SHM of amplitude a making n oscillations pure second. Slow that the string will not remain tight during the motion unless $n^2<\frac{g}{4\pi^2 a}$



T is Tension in sluing

mg = weight of particle

given n oscillations per second

$$\therefore\ T=\frac{2\pi}{\sqrt{\mu}}=\frac{1}{n}\Rightarrow\mu=4\pi^2n^2 \qquad \text{.....(1)}$$

Clearly, the impressed on effective force on particle will be $T-mg$

By, Newton's second law of motion ($F = ma$)

$$\text{at } P:\ F=\frac{md^2x}{dt^2}$$

$$\therefore\ \text{We have, } \frac{md^2x}{dt^2}=T-mg$$

$$\Rightarrow\ T=mg+m\frac{d^2x}{dt^2} \qquad \text{.....(2)}$$

For T to we least, we reed $\frac{d^2x}{dt^2}$ i.e., acceleration at point P as least

$\because$ For SHM, least acceleration at $P = -\mu a$

$\therefore$ Least tension $T = mg - \mu o\ \mu a$

We, reed T as to be position (otherwise stewing will not be tight enough to execute SHM)

$\therefore\ T > O \Rightarrow mg - m\mu a > 0$

$\Rightarrow mg - m4\pi^2 na > 0$

$\Rightarrow g - 4\pi^2 n^2 a > 0$

$\Rightarrow 4\pi^2 n^2 a > g$

$$\Rightarrow n^2 < \frac{g}{4\pi^2 a}$$

Q. A horizontal shelf in moved up & down with SHM of period $\frac{1}{2}$ second. What is the amplitude admissible in order that a weight placed one the shelf may not be jerked off.

We have

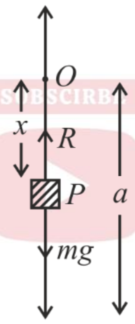
$$T = \frac{1}{2}\text{sec.} = \frac{2\pi}{\sqrt{\mu}}$$

Effective force on the body = $R - mg$

By, Newton's second law of motion at P

We have,

$$F = \frac{md^2x}{dt^2}$$



$\therefore$ We have

$$R - mg = \frac{md^2x}{dt^2}$$

$$R = mg + m \times \frac{d^2x}{dt^2} \qquad \ldots..(1)$$

Now, for no fcuk, we have $\frac{d^2x}{dt^2}$ should be least As, the shelf is in SHM,

$\therefore$ Least acceleration at $P = -\mu a$ $\qquad \ldots..(2)$

$\therefore$ Using (2) in (1),

$\therefore$ Least reaction (R) = $mg - m\mu a$

Now, for wo jeink,

R > O

$mg - \mu ma > 0$

$$g - \frac{4\pi^2}{T^2} a > 0$$

$$a < \frac{gT^2}{4\pi^2 a} \Rightarrow a < \frac{g}{16\pi^2}$$

Q. **A particle of mass *m* is attached to a light wire which is stretched between two point (fixed) with a Tension T. If *a*, *b* are the distances. of the particle from two ends, prove that the period of small transverse oscillation of mass *m* is** $\dfrac{2\pi}{\sqrt{\dfrac{T(a+b)}{mab}}}$

Step I: <u>Explanation :-</u>

Let a light wire be tightly stretched between two points A & B.

Let a particle of mass be attached at the point O s.t OA = a, OB = b.

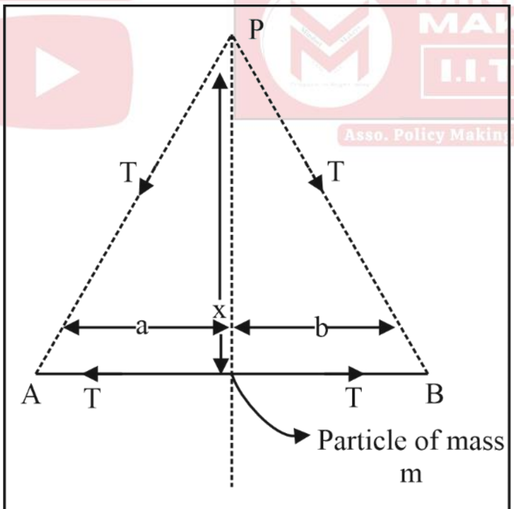
Let the particle is displaced slightly, perpendicular to AB. (i.e., in the tranverse direction) and let go.

Let P be the position of particle at any time t When OP = x

$\therefore$ The displacement is small, therefore the tension in the string at any displaced position can be taken as T which is the tension in the string in the angular position.

$\therefore$ The equation of the motion of particle is,

$$m\frac{d^2x}{dt^2} = -\left(T\cos\angle OPA + T\cos\angle OPB\right).$$



(Transverse perpendicular)

Step II :- Eqn. of motion is Ma = F

$$\frac{md^2x}{dt^2} = -T\cos(\angle OPA) + T\cos(\angle OPB)$$

Horizontal comp. of tension T ; T cos θ

$$m\frac{d^2x}{dt^2} = -T\left[\frac{OP}{AP} + \frac{OB}{PB}\right]$$

$$= -T\left[\frac{x}{\sqrt{a^2+x^2}} + \frac{x}{\sqrt{b^2+x^2}}\right]$$

$$= -T\left[x\left(a^2+x^2\right)^{-\frac{1}{2}} + x\left(b^2+x^2\right)^{-\frac{1}{2}}\right]$$

$$m\frac{d^2x}{dt^2} = -T\left[\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}} + \frac{x}{b}\left(1+\frac{x^2}{b^2}\right)^{-\frac{1}{2}}\right]$$

$$= -T\left[\frac{x}{a}\left(1-\frac{1}{2}\frac{x^2}{a^2}+....\right)+\frac{x}{b}\left(1-\frac{1}{2}\frac{x^2}{b^2}+....\right)\right]$$

$m\dfrac{d^2x}{dt^2} = -T\left[\dfrac{x}{a}+\dfrac{x}{b}\right]$; Neglecting higher powers of x as x is small

$$m\frac{d^2x}{dt^2} = -T\left(\frac{a+b}{mab}\right).x$$

$\therefore \dfrac{d^2x}{dt^2} \propto x$; SHM

$$\therefore \mu = \frac{T(a+b)}{mab}$$

$\therefore$ Time Period $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\sqrt{\dfrac{T(a+b)}{mab}}}$

Q. If in a SHM, u, v, w be the velocities at distances a, b, c from a fixed point on a straight, which is not the Centre of force show that the period T is given by the

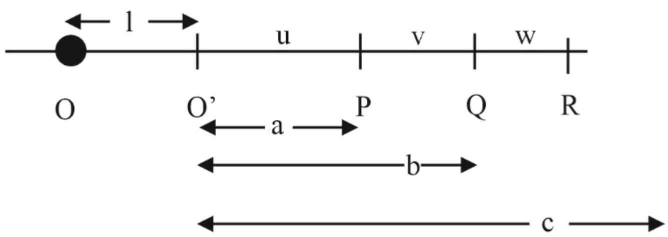
eq $= \dfrac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

Ans:- O: Centre of force, O' is fined point.

$\therefore$ From SHM

$v^2 = \mu\left(A^2 - x^2\right)$; where A is the amplitude ……………….(E)

$\therefore$ We have,



At P, $x = OP = l + a$, $v = u$ ..(1)

At Q, $x = OQ = l + b$, $v = v$ .. (2)

At R, $x = OR = l + c$ $v = w$ ..(3)

$\therefore$ Using (2) in (E)

$$u^2 = \mu\left\{A^2 - (l+a)^2\right\}$$

$$\frac{u^2}{\mu} = A^2 - l^2 - a^2 - 2al$$

$$\Rightarrow\left(\frac{u^2}{\mu}+a^2\right)+2a.l+\left(l^2-A^2\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\left(\frac{v^2}{\mu}+b^2\right)+2bl+\left(l^2-A^2\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\left(\frac{w^2}{\mu}+c^2\right)+2l.c+\left(l^2+a^2\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Eliminating $2l$ and l^2-A^2 Form (3), (4) & (5)

$$\begin{vmatrix} \frac{u^2}{\mu}+a^2 & a & 1 \\ \frac{v^2}{\mu}+b^2 & b & 1 \\ \frac{w^2}{\mu}+c^2 & c & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix}+\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}=\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$

$$\Rightarrow \quad \mu\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \mu(a-b)(b-c)(c-a)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$\therefore$ Time Period, $T=\frac{2\pi}{\sqrt{\mu}}\Rightarrow \mu=\frac{4\pi^2}{T^2}$ $\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$

Using (5) in (4), We get

$$\therefore \quad \frac{4\pi^2}{T^2}(a-b)(b-c)(c-a)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

Hooke's Law

The tension of an elastic string is proportional to the extension of length of string beyond its natural length.

Let AB is a wire of length l and it is attached (tied) at fixed point A, and a weight is attached at point B and let's say if on attaching weight, the length of that elastic wire is stretched by δl ; then

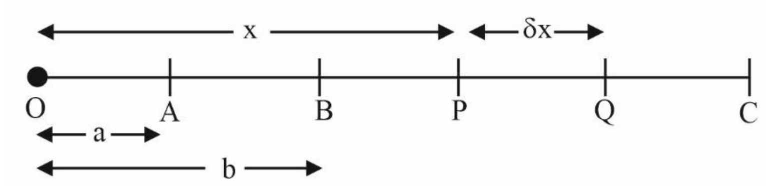
Tension in the string due to weight.

$$T \alpha \frac{\delta l}{l}$$

$\Rightarrow \quad T = \lambda . \frac{\delta l}{l}$ Where λ is elasticity constant or, modulus of elasticity.

Exam point :-

Theorem :- Prove that the wonk done against stretching a light elastic string, is equal to the product of its extension and the mean of its final & initial tensions.



Let OA = a; natural length of string, where end is fixed at O.

Let the string is stretched beyond its natural length,

Let B & C be the two positions of the string during its tension.

Let $OB = b, \quad OC = c$

$\therefore$ By Hook's Low

The tension at B; $\quad T_B = \lambda \frac{(b-a)}{a}$

The tension at C; $\quad T_C = \lambda \frac{(c-a)}{a}$ Where, λ is elasticity constant.

Now, as to find the work done; from B to C stretching, for this,

Let $OP = x$

Give a slight displacement of δx : P to Q

$$\therefore T_p = \lambda \frac{(x-a)}{a}$$

$\therefore$ Work done (on P to Q) $= T_p \delta x = \lambda \frac{(x-a)}{a} . \delta x$

$\therefore$ Required Work done $= \int_{x=b}^{c} \lambda \frac{(x-a)}{a} dx$

$$= \frac{\lambda}{2a} \left[(x-a)^2 \right]_b^c$$

$$= \frac{\lambda}{2a} \left[(c-a)^2 - (b-a)^2 \right]$$

$$= \frac{\lambda}{2a} \left[\{(c-a)-(b-a)\}\{(c-a)+(b-a)\} \right]$$

$$= \frac{\lambda}{2a}\left[(c-b)\{(c-a)+(b-a)\}\right]$$

$$= (c-b).\frac{1}{2}\left[\frac{\lambda}{a}(c-a)+\frac{\lambda}{a}(b-a)\right]$$

$$W_{B\ to\ C} = (c-b)\frac{1}{2}[T_c + T_B]$$

$W_{B\ to\ C}$ = BC × mean of the tension at B & C.

Exam Point 2:-

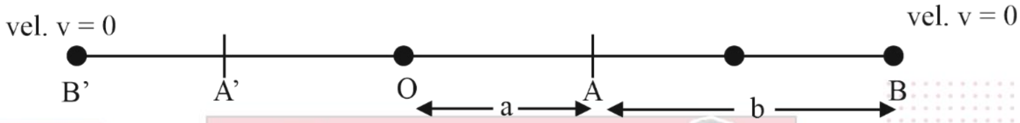
particle is attached to the end of a horizontal elastic string.

Let a particle of mass m is attached to the one end A of a horizontal elastic string OA whose other end is fixed to *a* point O on a smooth horizontal table.

Now, the particle is pulled to any distance in the direction of the string and then let go.

To Discuss Motion:-

$$vel. v = b\sqrt{\frac{\lambda}{am}}$$



Eq. of motion is,

$$m\frac{d^2x}{dt^2} = -\lambda.\frac{x}{a}$$

$$\frac{d^2x}{dt^2} = -\left(\frac{\lambda}{am}\right)x \qquad ....(1)$$

Which represents SHM & $\mu = \left(\frac{\lambda}{am}\right)$

Now, we can discuss velocity (v) formula, displacement (x) formula, Time Period (T) formula.

∴ We know, from SHM,

$$\left(\frac{dx}{dt}\right)^2 = \mu\left(A^2 - x^2\right);$$ here A is amplitude.

In this case,

$$\left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}\left(b^2 - x^2\right)$$

∴ at the point A, where x=0

$$\left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}b^2$$

i.e., velocity gained by the particle; at the point A;

$$v = b\sqrt{\frac{\lambda}{am}}$$

Similarly at A' ; $v = b\sqrt{\dfrac{\lambda}{am}}$

Now the particle will do the natural motion (SHM ceased at A & A') with velocity $b\sqrt{\dfrac{\lambda}{am}}$

Time taken by the particle for a complete round.

= Time in B to A (**SHM t_1** + Time in A to O 9 (**Natural t_2**)

+ Time in O to A'(**Natural t_2)** + Time in A' to B'(**SHM t_1)**

$\therefore$ Time period $T = (2t_1 + 2t_2)$ (Time taken in B to B' & B' to B)

$$\therefore \quad T = \frac{2\pi}{\sqrt{\dfrac{\alpha}{am}}} + \frac{4a}{b\sqrt{\dfrac{\lambda}{am}}} \rightarrow \frac{Distance}{Speed}$$

$$T = 2\pi\sqrt{\frac{am}{\lambda}} + \frac{4a}{b}\sqrt{\frac{am}{\lambda}}$$

Motion of a particle under the attraction of earth.

Newton's low of gravitation :-

1. when the particle moves (upwards / doenwords) outside the surface of earth, the acceleration varies as the square of distance of particle from the center of earth

Acceleration $\alpha \dfrac{1}{x^2}$

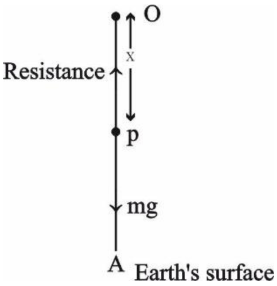
2. When particle moves inside earth then acceleration $\propto$ distance.

Motion in a resisting medium :-

resistance $\propto v^2$

resistance = kv^2

Scenario 1:-



fixed point

(particle of mass m is drooped from O)

Thinking process:

• We take an arbitrary point P; at a distance x; from the fixed point.

• Now, we check : the direction of motion.

If it is in the direction of x increasing

Then we take

$\frac{d^2x}{dt^2}$ as +ve

and if in opposite direction (i.e. in the direction of x decreasing)

Then we take

$\frac{d^2x}{dt^2}$ as – ve

Resolving forces; to use F = ma;

eq. of motion.

effective force = mg + (–resistance)

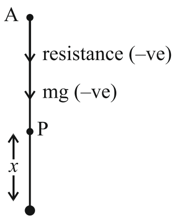
∵ the direction of resistance is in the direction of decreasing x.

$$\Rightarrow m\frac{d^2x}{dt^2} = \text{mg} + (\text{–resistance})$$

$$\Rightarrow m\frac{d^2x}{dt^2} = mg - m(kv^2)$$

$$\boxed{m\frac{d^2x}{dt^2} = mg - m(kv^2)}$$

Scenario-2



$\left.\begin{array}{l}\because \text{direction of mg} \\ \text{direction of resistance}\end{array}\right\}$ both are in the direction of x decreasing

O: fixed point (particle of mass m is projected upwards from point O)

Equation of motion at P,

$$\boxed{m\frac{d^2x}{dt^2} = -mg - m(kv^2)}$$

Terminal velocity:

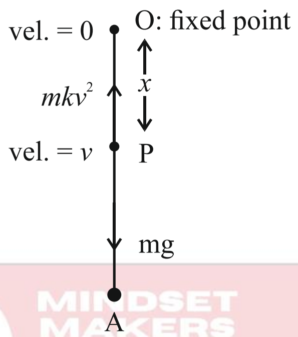
Let's say, if a particle is falling under gravity. After some time of falling; the velocity becomes constant (due to resistance of medium of travelling)

i.e., $\frac{dx}{dt}$ = constant$\Rightarrow$ $\frac{d^2x}{dt^2}$ = 0

So; when acceleration becomes zero; this constant velocity is called the terminal velocity.

Article 1:-

Discussing about motion of particle of mass m; falling from a fixed point.



Equation of motion is.

$$m\frac{d^2x}{dt^2} = mg - mkv^2$$

$$\frac{d^2x}{dt^2} = g\left(1-\frac{kv^2}{g}\right) \quad \text{….(1)}$$

If V is the terminal velocity, then

when v = V, $\frac{d^2x}{dt^2} = 0$

$\therefore$ equation (1) gives,

$$\frac{d^2x}{dt^2} = g\left(1-\frac{v^2}{V^2}\right) \Rightarrow \frac{d^2x}{dt^2} = \frac{g}{V^2}(V^2-v^2) \quad \text{…..(2)}$$

(i) To find relation of v & x.

$\because$ (2) can be written as,

$$v\frac{dv}{dx} = \frac{g}{V^2}(V^2-v^2) \qquad \left\{\because \frac{d^2x}{dt^2} = V\frac{dV}{dx}\right\}$$

$$\Rightarrow \frac{-2g}{V^2}dx = \frac{-2vdv}{V^2-v^2}$$

on integrating.

$$\frac{-2g}{V^2}\cdot x = \log(V^2 - v^2) + A;\ A \text{ is integral constant}$$

But initially at O, $x = 0$, $v = 0$

$\therefore$ 0= log V^2 + A $\Rightarrow$ A = –log V^2

So, we have,

$$-\frac{2gx}{V^2} = \log(V^2 - v^2) - \log V^2$$

$$\boxed{v^2 = V^2\left(1 - e^{-\frac{2gx}{V^2}}\right)}$$

(ii) *Relation between v and t:*

$$\because\ \frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2)\ ;\ \left\{\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right\}$$

$$\Rightarrow\ \frac{g}{V^2}dt = \frac{dv}{V^2 - v^2}$$

on integrating

$$\frac{g}{V^2}t = \frac{1}{2V}\log\frac{V+v}{V-v} + B;\ \text{B is integrating constant}$$

Initially at O, when $t = 0$, $v = 0$

$$0 = \frac{1}{2V}\log 1 + B \Rightarrow \text{B} = 0$$

$$\therefore\ \frac{gt}{V^2} = \frac{1}{2V}\log\left(\frac{V+v}{V-v}\right)$$

$$\Rightarrow\ t = \frac{V}{2g}\log\left(\frac{1+\frac{v}{V}}{1-\frac{v}{V}}\right)$$

$$\Rightarrow\ t = \frac{V}{g} + \tan h^{-1}\left(\frac{v}{V}\right)\ \left\{\text{Note: } \tan h^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right\}$$

$$\boxed{v = V + \tan h\left(\frac{gt}{V}\right)}$$

$$\sin h\theta = \frac{e^{\theta} - e^{-\theta}}{2},\quad \cos h\theta = \frac{e^{\theta} + e^{-\theta}}{2}\quad \because \tan h\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

(iii) *Relation between x & t*

Eliminating v from above two points

$$V^2 \tan h^2\left(\frac{gt}{V}\right) = V^2\left(1 - e^{\frac{-2gx}{V^2}}\right)$$

$$\Rightarrow\ e^{-\frac{2gx}{V^2}} = 1 - \tan h^2\left(\frac{gt}{V}\right)$$

$$\Rightarrow e^{-\frac{2gx}{V^2}} = \sec h^2\left(\frac{gt}{V}\right)$$

$$\Rightarrow e^{\frac{2gx}{V^2}} = \cos h^2\left(\frac{gt}{v}\right)$$

$$\Rightarrow \boxed{x = \frac{V^2}{g}\log\cos h\left(\frac{gt}{V}\right)}$$

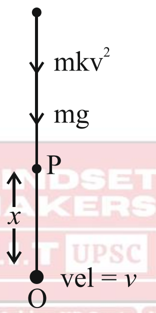
Article 2:-

Discussing the motion of particle projected vertically upwards.

Equation of motion at P is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

$$\frac{d^2x}{dt^2} = -(g + kv^2) \;\ldots..(1)$$



- ***Relation between v & x:***

  From (1), we have,

  $$v\cdot\frac{dv}{dx} = -(g + v^2)$$

  Let V be the terminal velocity; (1) gives

  $$0 = -g + kV^2 \Rightarrow K = \frac{g}{V^2}$$

  $\therefore$ We have equation of motion

  $$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right)$$

  $$\frac{d^2x}{dt^2} = -\frac{g}{V^2}\left(V^2 + v^2\right)\ldots.(2)$$

  $$\therefore \frac{vdv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$$

  $$\int -\frac{2g}{V^2}dx = \int\frac{2v}{V^2 + v^2}dv$$

  $$-\frac{2gx}{V^2} = \log(V^2 + v^2) + A$$

Exam point

In this scenario, Let if V is terminal velocity of particle during its downward motion i.e velocity when $\frac{d^2x}{dt^2}=0$ during downward motion

$\therefore\ 0 = mg - mkV^2$

But initially, at O, $x = 0$, $v = u$.

$A = -\log(V^2 + u^2)$

$$\therefore \; -\frac{2gx}{v^2} = \log(V^2 + v^2) - \log(V^2 + u^2)$$

$$\boxed{x = \frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2 + v^2}}$$

- ***Relation between v & t***

  equation (2) can be written as,

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2)$$

$$\int \frac{dv}{(V^2 + v^2)} = \int \frac{-g}{V^2} dt$$

$$\frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right) = -\frac{g}{V^2}t + B$$

But initially $t = 0$, $v = Vu$

$$B = \frac{1}{V}\tan^{-1}\left(\frac{u}{v}\right)$$

$$\therefore \; \frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right) = -\frac{g}{V^2}t + \frac{1}{V}\tan^{-1}\left(\frac{u}{V}\right)$$

$$\boxed{t = \frac{V}{g}\left[\tan^{-1}\left(\frac{u}{V}\right) - \tan^{-1}\left(\frac{v}{V}\right)\right]}$$

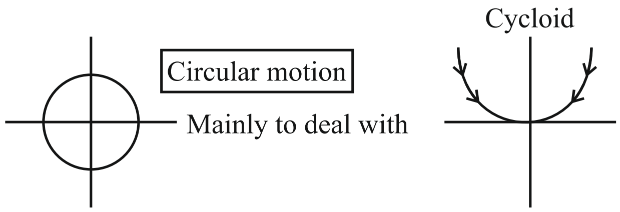
Exam point

- The above three relations need not be remembered. Only learn the procedure step by step while discussing the motion of particle in both scenarios.
- Here it must be take care that the sign of $\frac{dx}{dt}, \frac{d^2x}{dt^2}$ is taken to be what!

Constrained Motion

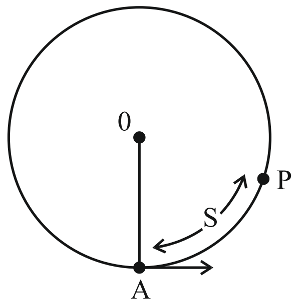
A motion is said to be constrained if it is under some particular condition or path of motion is some particular condition

Syllabus:



(i) 

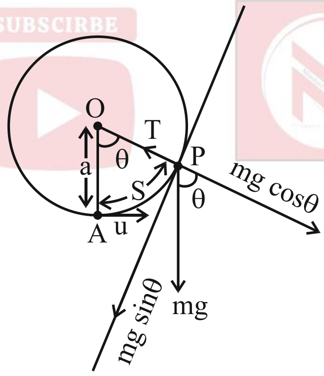
- OA is an inextensible string, where O be a fixed point,
- Let’s attach a particle of mass *m* at the point A.

(ii) 

Target: writing equation of motion at P.

Velocity = u; initial velocity

S: arc length AP.

(iii) 

We know that,

Tangential acceleration = $\dfrac{d^2s}{dt^2}$

Normal acceleration = $\dfrac{v^2}{\rho}$

here ρ is the radius of curvature,

∵ for circle; radius of curvature = radius of circle ∴ ρ = a,

Now, equation of motion at the point A;

- $m\dfrac{d^2s}{dt^2}$ = –mg sinθ…..(1)
- $m\dfrac{v^2}{a}$ = T – mg cos θ….(2)

Note: ∵ direction of mg sinθ is in direction of s decreasing. So, negative sign is attached

$\because$ $s = a\theta \Rightarrow \frac{ds}{dt} = a\frac{d\theta}{dt}$ and $\frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}$

Now, to solve differential equation (1) and (2);

We use, $s = a\theta \Rightarrow \frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}$ in (1) we get

$\therefore$ $m\left(a\frac{d^2\theta}{dt^2}\right) = -mg\sin\theta$

$a\frac{d^2\theta}{dt^2} = -g\sin\theta.$

Multiplying both side by $2a\frac{d\theta}{dt}$ and the integrating

$\left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$; where A is constant of integration

i.e., $v^2 = 2ag\cos\theta + A$ $\left[\because v = \frac{ds}{dt} = a\frac{d\theta}{dt}\right]$

Initially at point A;

$\theta = 0$, $v = u$.

$\therefore$ we get, $u^2 = 2ag + A \Rightarrow A = u^2 - 2ag$

So, we have

$\boxed{v^2 = 2ag\cos\theta + (u^2 - 2ag)}$

Exam point

$\boxed{v^2 = u^2 - 2ag + 2ag\cos\theta}$ ....(3)

Now, we discuss about tension in string; T

$\therefore$ from (2), we have,

$\frac{mv^2}{a} = T - mg\cos\theta$

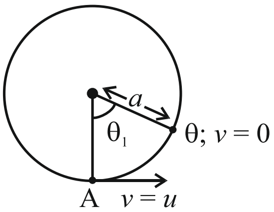
$\frac{m}{a}\left\{2ag\cos\theta + u^2 - 2ag\right\} = T - mg\cos\theta$

Exam point

$\boxed{T = \frac{m}{a}\left\{u^2 - 2ag + 3ag\cos\theta\right\}}$ ...(4)

Observations (For above circular motion)

(i) Let h, be the height from the lowest point A, at which $v = 0$ and let say at $\theta = \theta_1$; $v = 0$



So, from (2),

$0^2 = u^2 - 2ag + 2ag\cos\theta_1$

$$\cos\theta_1 = \frac{2ag - u^2}{2ag}$$

$\therefore\ h_1 = OA - a\cos\theta_1$

$$h_1 = a - a\left(\frac{2ag - u^2}{2ag}\right)$$

Exam point $\boxed{h_1 = \dfrac{u^2}{2g}}$ ....(5)

(ii) Let h_2 be the height; from lowest point A

where $T = 0$ and $\theta = \theta_2$

$\therefore$ From (4),

$$0 = \frac{m}{a}\{u^2 - 2ag + 3ag\cos\theta_2\}$$

$$\cos\theta_2 = \frac{2ag - u^2}{3ag}$$

$\therefore\ h_2 = OA - a\cos\theta_2$

$$h_2 = a - a\left\{\frac{2ag - u^2}{3ag}\right\}$$

Exam point $\boxed{h_2 = \dfrac{u^2 + ag}{3g}}$ ...(6)

Beautiful interpretation based on above observations:

(i) When the velocity v vanishes before the tension T vanishes:

This is possible if and only if $h_1 < h_2$

$\therefore$ from (5) and (6),

$$\frac{u^2}{2g} < \frac{u^2 + ag}{3g}$$

$u^2 \cdot 3g < u^2 \cdot 2g + 2ag^2$

$u^2 g < 2ag^2$

$u^2 < 2\,ag$

$$\boxed{u < \sqrt{2ag}}$$

If the initial velocity u is less than $\sqrt{2ag}$; then velocity will be zero before T becomes zero. In this case the particle will be oscillating about the point A. i.e., the particle will not cross above the horizontal diameter.

(ii) When v and T vanish simultaneously.

$\because v^2 = u^2 - 2ag + 2ag\cos\theta$; $h_1 = \dfrac{u^2}{2g}$

$T = \dfrac{m}{a}\{u^2 - 2ag + 3ag\cos\theta\}$; $h_2 = \dfrac{u^2 + ag}{3g}$

when $h_1 = h_2$,

$\dfrac{u^2}{2g} = \dfrac{u^2 + ag}{3g}$

$\Rightarrow u = \sqrt{2ag}$.

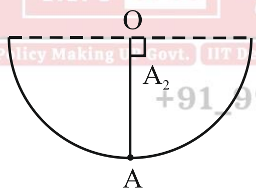
Now, for $v = 0$, $T = 0$,

$0 = u^2 - 2ag + 2ag\cos\theta_1$

$0 = u^2 - 2ag + 3ag\cos\theta_2$

using $u = \sqrt{2ag}$ in above equation we get

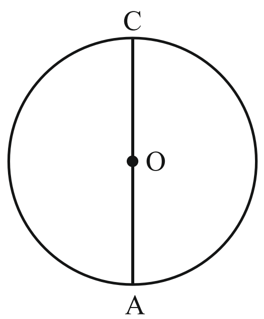
$\theta_1 = \theta_2 = \dfrac{\pi}{2}$



If we give initial velocity as $u = \sqrt{2ag}$; then the particle will oscillate about A; will go up to horizontal diameter and then return back.

(iii) Condition for describing the complete circle

To describe the complete circle, v and T should not vanished before the point C (or before $\theta = \pi$)



$\because$ At the highest point C : $\theta = \pi$

$\because v^2 = u^2 - 2ag + 2ag\cos\theta$

$\Rightarrow v^2 = u^2 - 2ag + 2ag \cos\pi$

$\Rightarrow v^2 = u^2 - 4ag \ldots(7)$

And $T = \frac{m}{a}\{u^2 - 2ag + 3ag\cos\theta\}$

$\Rightarrow T = \frac{m}{a}\{u^2 - 2ag + 3ag\cos\pi\}$

$T = \frac{m}{a}[u^2 - 5ag] \ldots.(8)$

Case (i): When $u > \sqrt{5ag}$ i.e., $u^2 > 5ag$.

So, $v > 0$, $T > 0$ i.e.

neither v nor T vanishes at C

So, particle will describe complete circle.

Case (iii): When $u = \sqrt{5ag}$

$\therefore\ v > 0, T = 0] \Rightarrow$ string becomes loose but particle has velocity; so, it will describe the complete circle.

Exam Point: If $u \geq \sqrt{5ag}$; then particle will describe the complete circle.

(iv) When T vanishes before v vanishes

Let at the height h_2, $\theta = \theta_2$; $T = 0$ but $v \neq 0$ i.e. $v > 0$ (particle has velocity)

Let h_1 be the height where $v = 0$

$\therefore$ here we need $h_1 > h_2$

$$\frac{u^2}{2g} > \frac{u^2 + ag}{3g}$$

$$\Rightarrow \boxed{u > \sqrt{2ag}}$$

In this case $\cos\theta_2$ is –ve

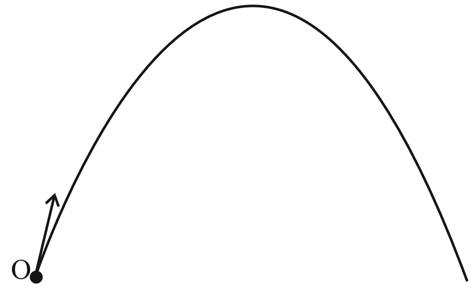
$\therefore\ \theta_2 > \frac{\pi}{2}$

But if we take, $\sqrt{2ag} < u < \sqrt{5ag}$; then the particle will start doing circular motion at $\theta = \theta_2$. But since it has +ve velocity at $v = v_2$, so it will keep doing the motion. Notice, here string becomes loose (slack).

So; particle will fall down at **parabolic path now**.

PROJETILES

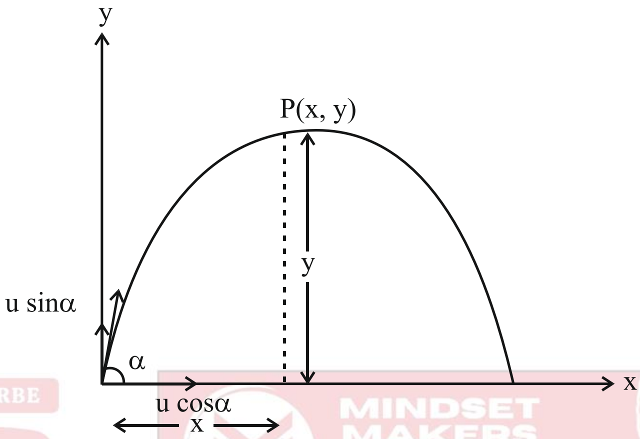
(i)



Point of projection

A particle (here called; the projectile) is projected with the initial velocity u from the initial point (point of projection) O; in the direction not vertically up ward; so this particle will follow a curved path.

(ii)



Notice: here we take:

(i) Air offers no resistance

(ii) The acceleration is only effective force which works downward (vertically) here, we assumed this as constant acceleration (vertically) as –g.

(iii) So, by above two points; we have,

- $\dfrac{d^2x}{dt} = 0 \ldots(1)$

($\because$ No resistance by air and so no force works in horizontal direction on path of motion)

- $\dfrac{d^2y}{dt^2} = -g \ldots(2)$

[horizontal comp. of acceleration = $\dfrac{d^2x}{dt^2}$]

[vertical comp. of acceleration = $\dfrac{d^2y}{dt^2}$]

(iv) Interpretation based on above differential equation :

- from (1); $\dfrac{dx}{dt}$ = constant

$\therefore$ Horizontal component of velocity at any point of the motion is constant and it is u cosα at O;

so we have $\frac{dx}{dt} = u \cos\alpha \Rightarrow x = u \cos\alpha \cdot t + A$

Initially at O, $x = 0$, $t = 0 \Rightarrow A = 0$

$\therefore$ Horizontal component of displacement is

Exam Point $\boxed{x = (u\cos\alpha)\cdot t}$ …(3)

from (2), we have; $\frac{d^2y}{dt^2} = -g$

$\Rightarrow \frac{dy}{dt} = -gt + B$; $\frac{dy}{dt}$ = vertical comp. of velocity

$\because$ at the point O; $\frac{dy}{dt} = u \sin\alpha$, $t = 0$.

So, we have,

$u \sin\alpha = 0 + B \Rightarrow B = u \sin\alpha$,

$\therefore$ we have,

$\frac{dy}{dt} = -gt + u \sin \alpha$

Exam Point $\boxed{\frac{dy}{dt} = u\sin\alpha - gt}$ …(4)

gives vertical comp. of velocity of the projectile at point $P(x, y)$ at time t.

$\Rightarrow y = (u \sin\alpha)\cdot t - g\frac{t^2}{2} + 0$

$\because$ At point 0, $t = 0$, $y = 0$, $c = 0$

Exam Point $\boxed{y = (u\sin\alpha), t - \frac{1}{2}gt^2}$ …(5)**;** Vertical comp. of displacement at point P.

Now, eliminate t; from equation (3) and (5)

$\because$ from (3),

$x = u \cos\alpha t$; $t = \frac{x}{u\cos\alpha}$.

$\therefore$ from (5),

$y = u\sin\alpha \cdot \frac{x}{u\cos\alpha} - \frac{1}{2}g \times \frac{x^2}{u^2\cos^2\alpha}$

Exam Point

$\boxed{y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2 x}}$ …(6)

This represents the path of motion (called trajectory of projectile)

$\because$ (6) is an equation of a parabola; So; projectile's trajectory is a parabola.

- **Point (i):** Let v be the resultant velocity at point P at time t, then

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(u\cos\alpha)^2 + (u\sin\alpha - gt)^2}$$

- **Point (ii):**

The direction of velocity here is along the tangent to the curve at that point:-

$\tan\theta = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{u\sin\alpha - gt}{u\cos\alpha}$; here θ is the angle which velocity vector makes with horizontal line (x-axis)

Discussing above this trajectory (parabola) geometrically.

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha} - x\tan\alpha = -y$$

$$x^2 - \frac{2u^2\cos^2\alpha}{g}\tan\alpha \cdot x = -\frac{2u^2\cos^2\alpha}{g}y$$

$$x^2 - 2\times\frac{u^2\sin\alpha\cos\alpha}{g}\times x + \left(\frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = \frac{-2u^2\cos^2\alpha}{g}y + \frac{u^4\sin^2\alpha\times\cos^2\alpha}{g^2}$$

$$\left(x - \frac{u^2\cos\alpha\sin\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$$

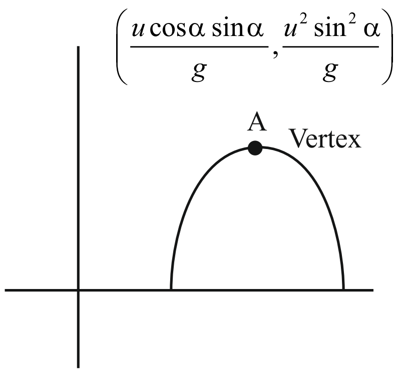
Now, if we shift the origin to the point

$\left(\frac{u^2\cos\alpha\sin\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$; then we get above equation of parabola as,

$$x^2 = -\frac{2u^2\cos^2\alpha}{g}y$$

- **Vertex of parabola is:-**

$$\left(\frac{u^2\cos\alpha\sin\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$$



- **Focus:**

Let S be the focus of trajectory S is a point on the axis of parabola

$\because$ x-coordinate of $S = \dfrac{u^2 \sin\alpha \cos\alpha}{g} = \dfrac{u^2 \sin 2\alpha}{2g}$

y-coordinate of s = the y-coordinate of $A - \dfrac{1}{4}$ latus rectum

$$= \frac{u^2 \sin^2\alpha}{2g} - \frac{1}{u}\frac{2u^2\cos^2\alpha}{g}$$

y coordinate of S = $\dfrac{u^2 \sin^2\alpha}{2g} - \dfrac{u^2\cos^2\alpha}{2g} = \dfrac{-u^2}{g}(\cos^2\alpha - \sin^2\alpha) = \dfrac{-u^2}{g}\cos 2\alpha$

Exam Point

The coordinate of focus of parabola is

$$\left(\frac{u^2}{2g}\sin 2\alpha, \frac{-u^2}{g}\cos 2\alpha\right)$$

Latus rectum = $\dfrac{2}{g}u^2\cos^2\alpha$; (Let A is vertex)

• Directrix:-

The directrix of a parabola is a line perpendicular to axis of parabola & so it is a horizontal line (here)

The height of the direction above O (i.e above point of projection)

= the height of the vertex A above O + $\dfrac{1}{4}$ latus rectum

$$= \frac{u^2\sin^2\alpha}{2g} + \frac{1}{4}\frac{2u^2\cos^2\alpha}{g} = \frac{u^2}{2g}(\cos^2\alpha + \sin^2\alpha) = \frac{u^2}{2g}$$

Exam point

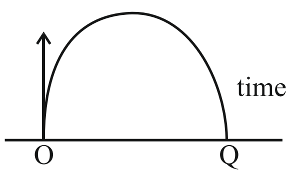
The directrix of parabola is $y = \dfrac{u^2}{2g}$; line

Not depending on α, so

Therefore, the trajectories of all projected in the same vertical plane from the same point with same velocity in different directions have the same direction.

• Time of Flight: The time taken by the projectile from O (the point of projection) to the point when the projectile is again touches the surface.

We know that



$S = ut - \dfrac{1}{2}gt^2$

for S to be zero, $u = u$ sinα, we have,

$u \sin\alpha \cdot t = \dfrac{1}{2}gt^2$; $\boxed{t = \dfrac{2u\sin\alpha}{g}}$

Horizontal Range

$\because x = u \cos\alpha \cdot t$

$R = (u\cos\alpha)\times\dfrac{2u\sin\alpha}{g}$; $\boxed{R = \dfrac{u^2\sin 2\alpha}{g}}$ $\therefore$ maximum horizontal range is at $\sin 2\alpha = 1 \Rightarrow \alpha = \dfrac{\pi}{4}$.

$\therefore$

$$\boxed{\text{Max horizontal range} = \frac{u^2}{g}}$$

i.e. For max. horizontal range, $\boxed{\alpha = \dfrac{\pi}{4}}$

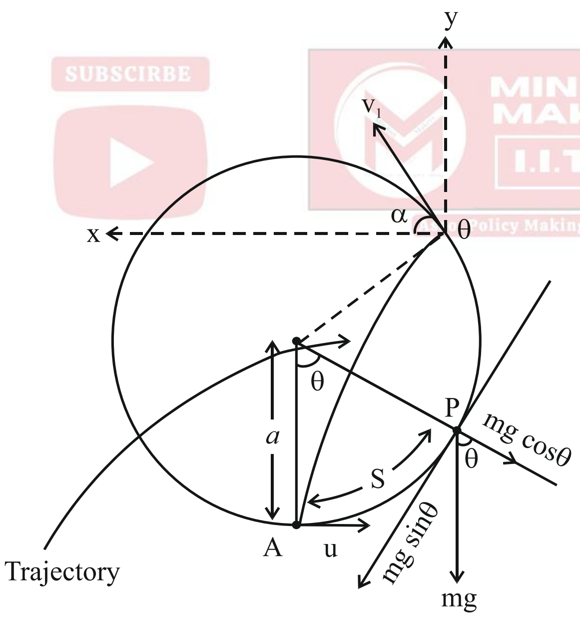
• **Greatest Height:**

$\because v^2 = u^2 - 2gH$

$\because$ at greatest height; $v = 0$

$\therefore$ $0 = (u \sin\alpha)^2 - 2g\text{H}$

$$\boxed{\text{H} = \frac{u^2\sin^2\alpha}{2g}}$$



T = 0, $v > 0$

at $\theta = \theta_1$: we can find cosθ, here so we can find v_1 here

$v^2 = u^2 - 2ag + 2ag\cos\theta$

$T = \dfrac{m}{a}[u^2 - 2ag + 3ag\cos\theta]$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$m\frac{v^2}{a} = T - mg\cos\theta$$

$$ma\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$2a\frac{d\theta}{dt}$ multiply

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2$$

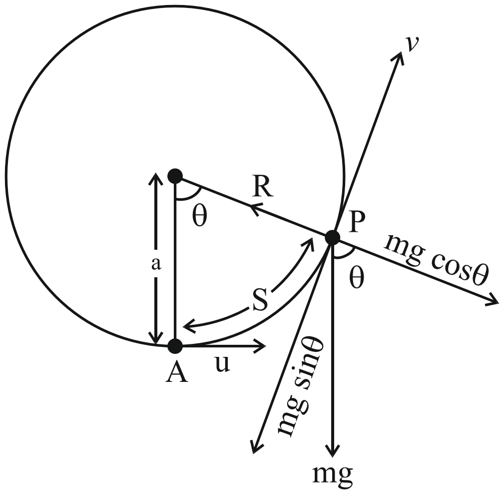
$\downarrow$ Interpretate

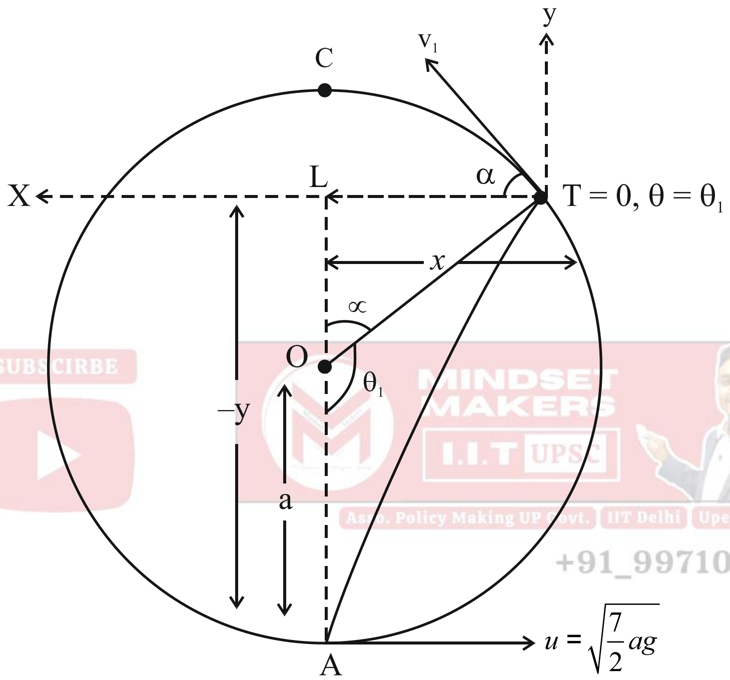
$$\boxed{\alpha = \pi - \theta_1}$$

$\cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1$

A particle inside and at the lowest point of a fixed smooth hollow sphere of radius a is projected horizontally with velocity $\sqrt{\frac{7}{2}ag}$. Show that it will leave the sphere at a height $\frac{3}{2}a$ above the lowest point and its subsequent path (trajectory) meets the sphere again at the point of projection,

Step (1): **Step (ii)**





Required:- AL $= \dfrac{3a}{2}$

For step (1):

Mathematics:

At point P,

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$\therefore\ v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + c_1$$

$$\therefore\ \text{At A, } \theta = 0,\ v = \sqrt{\frac{7}{2}ag}$$

$$\therefore\ \frac{7}{2}ag = 2ag \times 1 + c_1$$

$c_1 = \frac{3}{2}ag.$

$\therefore\ v^2 = \frac{3}{2}ag + 2ag\cos\theta \ldots.(1)$

At point P,

$m\frac{v^2}{a} = R - mg\cos\theta$

$R = \frac{m}{a}[v^2 + ag\cos\theta]$

$R = \frac{m}{a}[\frac{3}{2}ag + 2ag\cos\theta + ag\cos\theta]$

$R = \frac{m}{a}[\frac{3}{2}ag + 3ag\cos\theta]$

$R = 3mg\,(\frac{1}{2} + \cos\theta)\ldots(2)$

Now, $\because$ at the point Q; R = 0 and $\theta = \theta_1$

$\therefore$ From (2)

$0 = 3mg\left(\frac{1}{2} + \cos\theta_1\right)$

$\cos\theta_1 = -\frac{1}{2}\ldots.(3)$

$\therefore$ From (1), we have,

$v_1^2 = \frac{3}{2}ag + 2ag\left(-\frac{1}{2}\right)$

$v_1^2 = \frac{1}{2}ag\ldots(4)$

$\because$We know that the trajectory after Q is,

$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}\ldots.(A)$

$\because v_1$ is known (by (4))

Now, we need to find $\cos\alpha$

$\downarrow$for this

$\because\ \alpha = \pi - \theta_1$

$\cos\alpha = \cos(\pi - \theta_1)$

$\cos\alpha = -\cos\theta_1$

$\cos\alpha = \frac{1}{2}$ {using (3)}....(5)

Now, using (4) & (5) in (A); we get,

$y = x\cdot\sqrt{3} - \frac{1}{2}\frac{gx^2}{\frac{1}{2}ag \times \frac{1}{2^2}}$ …(6) $\{\because \cos\alpha = \frac{1}{2}, \therefore \tan\alpha \frac{\sqrt{3}}{1}\}$

$\because$ from the figure (ii); we have

$x = a\sin\alpha = a\cdot\frac{\sqrt{3}}{2}$ $\{\because \cos\alpha = \frac{1}{2}, \therefore \sin\alpha = \frac{\sqrt{3}}{2}\}$

Now, from (6),

$$y = a\frac{\sqrt{3}}{2} \times \sqrt{3} - \frac{1}{2} \times \frac{g \times \frac{3}{4}a^2}{\frac{1}{2}ag \times \frac{1}{4}}$$

$$y = \frac{3a}{2} - 3a$$

$y = \frac{-3a}{2}$ {$\because$Q is working as origin, $\therefore$ below : –ve y-axis}

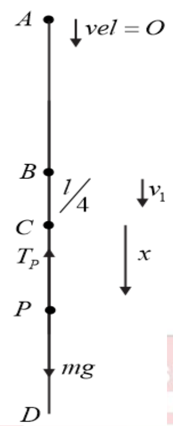
i.e., the trajectory or path meets the point A, because

$\text{AL} = \frac{3a}{2}$

Example 1:- A light elastic strings AB of length l is fixed at A and is such that if a weight w be attached to

B, the string will be stretched to a length $2l$. If a weight w be attached to B and let fall from the level of A prove that (i) the amplitude of the S.H.M. that ensues is $3l/4$; (ii) the distance through which it falls is $2l$; and (iii) the period of oscillation is $\sqrt{\left(\frac{l}{4g}\right)}\left(4\sqrt{2}+\pi+2\sin^{-1}\frac{1}{2}\right)$

Solution:- $AB=l$ is the natural length of an elastic string whose and end is fixed at A. Let λ be the modulus of elasticity of the string. If a weight w be attached to the other end of the string it extends the string to length $2l$ while hanging in equilibrium. Therefore $w=\lambda\frac{2l-l}{l}=\lambda$. (1)



Now in the actual problem a particle of weight w or mass $l(w/g)$ is attached of the free end of the string. Let C be the position of equilibrium of this weight w. Then considering the equilibrium of the weight at C, we have $\frac{1}{4}w=\lambda\frac{BC}{l}=w\frac{BC}{l}$ [$\because$ by (1), $\lambda=w$]

$\therefore$ $BC=\frac{1}{4}l$.

Now the weight $\frac{1}{4}w$ is dropped from A. It fall the distance $AB(=l)$ freely under gravity. If v_1 be the velocity gained by this weight at B, we have $\left(v_1=\sqrt{2gl}\right)$ in the downward direction. When this weight falls below B, the string begins to extend beyond its natural length and the tension to operate. The velocity of the weight continues increasing upto C, after which it starts decreasing. Suppose the weight to instantaneous rest at D, where $CD=a$.

During the motion of the weight below B, let P be its position after any time t ,where $CP=x$ [**Note** that we have taken C as origin and CP is the direction of x increasing]. If T_P be the tension in the string , AP, we have $T_P=w\frac{\frac{1}{4}l+x}{l}$ acting vertically upwards.

The equation of motion of this weight $w/4$ at P is

$$\frac{1}{4}\frac{w}{g}\frac{d^2x}{dt^2}=\frac{1}{4}w-w\frac{\frac{1}{4}1+x}{l}=\frac{1}{4}w-\frac{1}{4}w-w\frac{x}{l} \text{ or } \frac{1}{4}\frac{w}{g}\frac{d^2x}{dt^2}=-w\frac{x}{l} \text{ or } \frac{d^2x}{dt^2}=-\frac{4g}{l}x.$$
(2)

Which is the equation of a S.H.M. with centre at the origin C and amplitude $CD(=a)$. The equation (2) holds good so long as the string is stretched i.e., for the motion of the, weight from B to D. Multiplying (2) by $2(dx/dt)$ and integrating w.r.t, t, we get $\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}x^2+k$, where $\frac{1}{4}$ is a constant.

At $B,\ x=-\frac{1}{4}l$ and $dx/dt=\sqrt{(2gl)}$

$\therefore \qquad 2gl=-\frac{4g}{l}.\frac{1}{16}l^2+k$ or $k=\frac{9}{4}gl$.

Thus, we have $\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}x^2+\frac{9}{4}gl=\frac{4g}{l}\left(\frac{9}{16}l^2-x^2\right)$ (3)

The equation (3) gives velocity at any point between B and D. At $D,\ x=a, dx/dt=0$. Therefore (3) gives $0=\frac{4g}{l}\left(\frac{9}{16}l^2-a^2\right)$ or $a=\frac{3}{4}l$

Hence the amplitude a of the S.H.M. that ensure is $\frac{3}{4}l$. Also the total distance through which the weight falls $=AB+BC+CD=l+1l+\frac{3}{4}l=2l$.

Now let t_1 be the time taken by the weight to fall freely under gravity from A to B.

Then using the formula $v=u+ft$, we get $\sqrt{(2gl)}=0gt_1$ or $t_1=\sqrt{(2l/g)}$.

Again let t_2 be the time taken by the weight to fall from B to D while moving in S.H.M. From (3), on taking square root, we get $\frac{dx}{dt}=+\sqrt{\left(\frac{4g}{l}\right)}\sqrt{\left(\frac{9}{16}l^2-x^2\right)}$, where the $+ive$ sign has been taken because the weight is moving in the direction of x increasing. Separating the variable, we get $\sqrt{\left(\frac{l}{4g}\right)}\frac{dx}{\sqrt{\left(\frac{9l^2}{16}-x^2\right)}}=dt$. Integrating from B to D, we get

$$\int_0^2 dt=\sqrt{\left(\frac{l}{4g}\right)}\int_{-l/4}^{3l/4}\frac{dx}{\sqrt{\left(\frac{9}{16}l^2-x^2\right)}}$$

$$\therefore \qquad t_2=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}\frac{3}{4}\right]_{-l/4}^{3l/4}=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}1-\sin^{-1}\left(-\frac{1}{3}\right)\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\frac{1}{2}\pi+\sin^{-1}\frac{1}{3}\right]$$

Hence the total time taken to fall from A to $D=t_1+t_2$.

$$=\sqrt{\left(\frac{2l}{g}\right)}+\sqrt{\left(\frac{l}{4g}\right)}\left[\frac{1}{2}\pi+\sin^{-1}\frac{1}{3}\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\frac{\pi}{2}+\sin^{-1}\frac{1}{3}+2\sqrt{2}\right]$$

Now after instantaneous rest at D, the weight begins to move upwards. From D to B it moves in S.H.M. whose equation is (2). At B the string becomes stack and S.H.M. ceases. The velocity of the weight at B is $\sqrt{(2gl)}$ upwards. Above B the weight rises freely under gravity and comes to instantaneous rest at A. Thus it oscillates again and again between A and D.

The time period of one complete oscillation $=2$ time from A to $D=2(t_1+t_2)=\sqrt{\left(\frac{l}{4g}\right)}\left\{\pi+4\sqrt{2}+2\sin^{-1}\frac{1}{3}\right\}$.

Example 2:- A particle of mass m is attached to one end of an elastic string of natural length a and modulus of elasticity $2mg$, whose other end is fixed at O. The particle is let fall from A, when A is other end is fixed at O. Vertically above O and $OA=a$. Show that its velocity will be zero at B, where $OB=3a$. Calculate also the time from A to B.

Solution:-

Let $OC=a$, be the natural length of an elastic string suspended from the fixed point O.

The modulus of elasticity λ of the string is given to be equal to $2mg$, where m is the mass of the particle attached to the other end of the string.

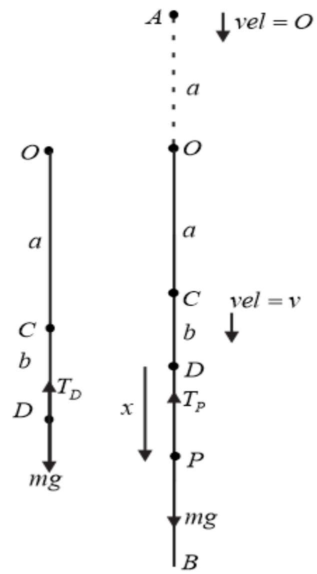
If D is the position of equilibrium of the particle such that $CD=b$, then at D then tension T_D in the string OD balances the weight of the particle.

$\therefore \quad mg=T_D=\lambda\frac{b}{a}=2mg.\frac{b}{a}$ or $b=a/2$.

The particle is let fall from A where $OA=a$. Then the motion from A to C will be freely under gravity.

If V is the velocity of the particle gained at the point C, then $V^2=0+2g.2a$ or $V=2\sqrt{(ag)}$ (1)

In the downward direction



As the particle moves below C, the string begins to extend beyond its natural length and the tension begins to operate.

The velocity of the particle continues increasing upto D after which it starts decreasing. Suppose that the particle comes to instantaneous rest at B.

During the motion below C, let P be the position of the particle at any time t, where $DP = x$. If T_P is the tension in the string OP, we have $T_P = \lambda \dfrac{b+a}{a}$,acting vertically upwards.

$\therefore$ The equation of motion of the particle at P is $m\dfrac{d^2x}{dt^2} = mg - T_P = mg - \lambda \dfrac{b+x}{a}$

$$= mg - 2mg\frac{\frac{1}{2}a + x}{a} = -\frac{2mg}{a}x \text{ or } \frac{d^2x}{dt^2} = -\frac{2g}{a}x \qquad (2)$$

Which represents a S.H.M with centre at D and holds good for the motion from C to B.

Multiplying both sides of (2) by $2(dx/dt)$ and then integrating, we have $\left(\dfrac{dx}{dt}\right)^2 = -\dfrac{2g}{a}x^2 + k$

where k is a constant.

But at C, $x = -DC = -b = -a/2$ and $(dx/dt)^2 = V^2 = 4ag$. $4ag = -\dfrac{2g}{a}.\dfrac{a^2}{4} + k$ or

$k = \dfrac{9}{2}ag$

$$\therefore \left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \frac{9}{2}ag \text{ or } \left(\frac{dx}{dt}\right)^2 = \frac{2g}{a}\left(\frac{9}{4}a^2 - x^2\right) \qquad (3)$$

If the particle comes to instantaneous rest at B where $DB = x_1$, (say) then at $B, x = x_1$ and $dx/dt = 0$. Therefore from (3), we have $0 = \dfrac{2g}{a}\left(\dfrac{9}{4}a^2 - x_1^2\right)$. Given $x_1 = \dfrac{3}{2}a$.

Now $OB = OC + CD + DB = a + \frac{1}{2}a + \frac{3}{2}a = 3a$, which proves the first part of the question.

To find the time from A to B

If t_1 is the time from A to C, then from $s = ut + \frac{1}{2}ft^2$

$$2a = 0 + \frac{1}{2}gt_1^2 \qquad \therefore\ t_1 = 2\sqrt{(a/g)} \qquad (4)$$

Now from (3), we have $\frac{dx}{dy} = \sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left(\frac{9}{4}a^2 - x^2\right)}$ the $+ive$ sign has been taken because the particle is moving in the direction of x increasing.

Or $dt = \sqrt{\left(\frac{a}{2g}\right)} \cdot \frac{dx}{\sqrt{\left(\frac{9}{4}a^2 - x^2\right)}}$

Integrating from C to B, the time t_2 from C to B is given by

$$t_2 = \sqrt{\left(\frac{a}{2g}\right)} \int_{x=-a/2}^{3a/2} \frac{dx}{\sqrt{\left(\frac{9}{4}a^2 - x^2\right)}}$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\sin^{-1}\left(\frac{x}{3a/2}\right)\right]_{-a/2}^{3a/2}$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\sin^{-1}1 - \sin^{-1}\left(-\frac{1}{3}\right)\right]$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\frac{\pi}{2} + \sin^{-1}\left(\frac{1}{3}\right)\right]$$

$\therefore$ The time from A to $B = t_1 + t_2$

$$= 2\sqrt{(a/g)} + \sqrt{(a/2g)}.\left[\pi/2 + \sin^{-1}(1/3)\right]$$

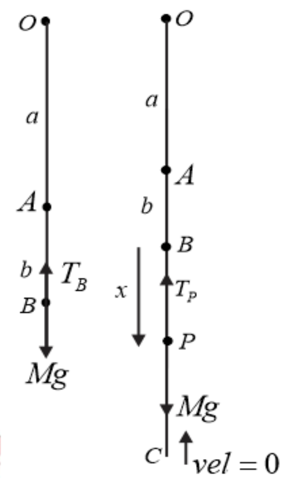
$$= \frac{1}{2}\sqrt{(a/2g)}\left[4\sqrt{2} + \pi + 2\sin^{-1}(1/3)\right]$$

Example 3:- Two bodies of masses M and M' are attached to the lower end of an elastic string whose upper end is fixed and hong at rest: M falls off; show that the distance of M from the upper end of the string at time t is $a + b + c\cos\left\{\sqrt{(g/b)}\ t\right\}$, where a is the entrenched length of the string, b and c the distance by which it would be stretched when supporting M and M' respectively.

Solution:- Let $OA = a$ be the natural length of and elastic string suspended from the fixed point O. If B is the position of equilibrium of the particle of mass M attached to the lower end of the string and $AB = b$, then $Mg = \lambda \frac{AB}{a} = \lambda \frac{b}{a}$ (1)

Similarly $M'g = \lambda \frac{c}{a}$ (2)

Adding (1) and (2), we have $(M + M')g = \lambda \frac{b+c}{a}$.



Thus the string will be stretched by the distance $b+c$ when supporting both the masses M and M' at the lower end.

Let OC be the stretched length of the string when both the masses M and M' are attached to its lower end.

Then $AC = b+c$ and so $BC = AC - AB = b+c-b = c$. Now when M' falls off at C, the mass M will begin to move towards B string with velocity zero at C.

Let P be the position of the particle of mass M at any time t, where $BP = x$.

If T_P be the tension in the string OP, then $T_P = \lambda \frac{b+x}{a}$,acting vertically upwards.

$\therefore$ The equation of motion of the particle of mass M at P is $M\frac{d^2x}{dt^2} = Mg - T_P = Mg - \lambda \frac{b+x}{a}$

$= Mg - \lambda \frac{b}{a} - \frac{\lambda x}{a}$

$= Mg - Mg - \frac{Mg}{b}x$, [$\because$ from (1), $Mg = \frac{\lambda b}{a}$]

$= -\frac{Mg}{b}x$.

$\therefore$ $\frac{d^2x}{dt^2} = -\frac{g}{b}x$ (3), which represents a S.H.M. with centre at B and amplitude BC grating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2 = -\frac{g}{b}x^2 + k$,where k is a constant.

But at the point C, $x = BC = c$ and $dx/dt = 0$

$\therefore \quad 0 = -(g/b)c^2 + k$ or $k = (g/b)c^2$

$\therefore \quad \left(\dfrac{dx}{dt}\right)^2 = \dfrac{g}{b}\left(c^2 - x^2\right)$ or $\dfrac{dx}{dt} = -\sqrt{\left(\dfrac{g}{b}\right)}\sqrt{\left(c^2 - x^2\right)}$, the $-ive$ sign has been taken since the particle is moving in the direction of x decreasing.

$\therefore \quad dt = -\sqrt{\left(\dfrac{b}{g}\right)}\dfrac{dx}{\sqrt{\left(c^2 - x^2\right)}}$, separating the variables.

Integrating $t = \sqrt{(b/g)}\cos^{-1}(x/c) + D$, where D is constant.

But at C, $t = 0$ and $x = c$;

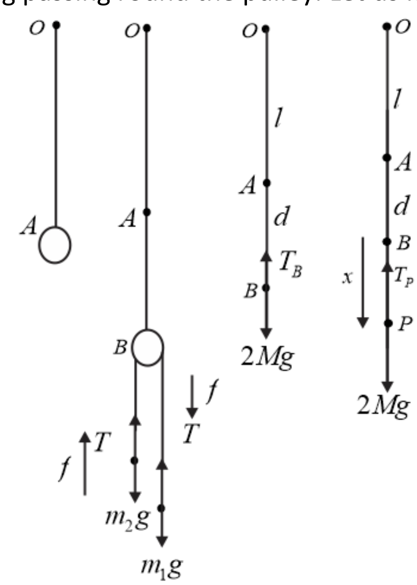
$\therefore \quad t = \sqrt{(b/g)}\cos^{-1}(x/c)$ or $x = BP = c\cos\left\{\sqrt{(g/b)}\ t\right\}$.

$\therefore$ The required distance of the particle of mass M at time t from the point O

$= OP = OA + AB + BP = a + b + c\cos\left\{\sqrt{(g/b)}\ t\right\}$

Example 4:- A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity g. If mass m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley, show that the pulley executes simple harmonic motion about a centre whose depth below the point suspension is $l\left\{1 + (2M/n)\right\}$, where M is the harmonic mean between m_1 and m_2.

Solution:- Let a smooth **light** pulley be suspended from a fixed point O by a spring OA of natural length l and modulus of elasticity $\lambda = ng$. Let B be the position of equilibrium of the pulley when masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley. Let T be the tension in the inextensible string passing round the pulley. Let us first find the value of T.



Let f be the common acceleration of the particles m_1, m_2 which hang at the ends of a light inextensible string passing round the pulley. If $m_1 > m_2$, then the equations of motion of m_1, m_2 are $m_1 g - T = m_1 f$ and $T - m_2 g = m_2 f$

Solving, we get $T = \frac{2m_1 m_2}{(m_1 + m_2)} g = Mg$, where $M = \frac{2m_1 m_2}{m_1 + m_2} =$ the harmonic mean between m_1 and m_2. Now the pressure on the pulley $= 2T = 2Mg$ and therefore the pulley, which itself is light, behaves like a particle of mass $2M$.

Now the problem reduces to the vertical motion of a mass $2M$ attached to the end A of the string OA whose end is fixed at O. If B is the equilibrium position of the mass $2M$ and $AB = d$, then the tension T_B in the spring OB is $\lambda(d/l)$, acting vertically upwards.

For equilibrium of the pulley of mass $2M$ at the point B, we have $2Mg = T_B = \lambda \frac{d}{l} = ng \frac{d}{l}$

Or $$d = \frac{2Ml}{n} \quad (1)$$

Now let the particle of mass $2M$ be slightly pulled down and then let go. If P is the position of this particle at time t such that $BP = x$, then the tension in the spring OP

$= T_P = \lambda \frac{d+x}{l} = ng . \frac{d+x}{l}$, acting vertically upwards.

$\therefore$ The equation of motion of the pulley is given by $2M . \frac{d^2x}{dt^2} = 2Mg - T_P$

$$= 2Mg - mg \frac{d+x}{l} = 2Mg - ng \frac{d}{l} - \frac{ng}{l} x = -\frac{ng}{l} x$$

$\therefore$ $$\frac{d^2x}{dt^2} = -\frac{ng}{2M\,l} x \quad \text{[by (1)]}$$

Which represent a simple harmonic motion about the centre B.

Hence the pulley executes simple harmonic motion with centre at the point B whose depth below the point of suspension O given by $OB = OA + AB = l + d$

$$= l + \frac{2M\,l}{n} = \left(1 + \frac{2M}{n}\right).$$

Example 4:- Show that the time of descent to the centre of force, varying inversely as the square of the distance from the centre through first half of its initial distance is to that thorough the last half as $(\pi + 2); (\pi - 2)$,

Solution:- Let the particle start from rest from the point A at a distance a from the centre of force O. If x is the distance of the particle from the centre of force at time t, then the equation of motion of the particle at time t is $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$.

Now proceeding as in, we find that the time t measured from the initial position $x = a$ to any point distant x from the centre O is given by the equation $t = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\right\}\left(1-\frac{x}{a}\right)}\right]$ (1)

[Give the complete proof for deducing this equation here]

Now let B be the middle of OA. Then at B, $x = a/2$.

Let t_1 be the time from A to B, $x = a/2$ and $t = t_1$, so putting $x = a/2$ and $t = t_1$ in (1), we get

$$t_1 = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\right] = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\frac{\pi}{4} + \frac{1}{2}\right]$$

Again let t_2 be the time from A to O, $x = 0$ and $t = t_2$. So putting $x = a$ and $t = t_2$ (1), we get

$$t_2 = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1} 0 + 0\right] = \sqrt{\left(\frac{a^3}{2\mu}\right)} \cdot \frac{\pi}{2}$$

Now if t_3 be the time from B to O, (i.e. the time to cover the has half of the initial displacement),

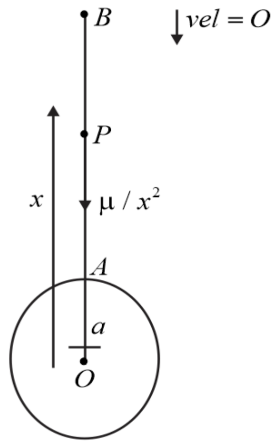
then $t_3 = t_2 - t_1 = \sqrt{\left(\frac{a^3}{2\mu}\right)} \cdot \left[\frac{\pi}{4} - \frac{1}{2}\right]$

We have $\frac{t_1}{t_2} = \frac{\frac{1}{4}\pi + \frac{1}{2}}{\frac{1}{4}\pi - \frac{1}{2}} = \frac{\pi + 2}{\pi - 2}$, which proves the required result.

Example:- If the earth's attraction vary inversely as the square of the distance from its centre and g be its magnitude at the surface, the time of falling from a height h above the surface to the surface is $\sqrt{\left(\frac{a+h}{2g}\right)}\left[\sqrt{\left(\frac{h}{a}\right)} + \frac{a+h}{g}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}\right]$, where a is the radius of the earth.

Solution:- Let O be the centre of the earth taken as origin. Let OB be the vertical line through O which meets the surfaces of the earth at A and let $AB = h$; $OA = a$ is the radius of the earth.

A particle falls from rest from B towards the surface of the earth. Let P be the position of the particle at any time t, where $OP = x$. [**Note** that O is the origin and OP is the direction of x increasing]. According to the Newton's law of gravitation the acceleration of the particle at P is μ / x^2 directed towards O i.e. in the direction of x decreasing. Hence the equation of motion of the particle at P is $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$ (1)



The equation (1) holds good for the motion of the particle from B to A. At A (i.e., on the surface of the earth $x=a$ and $d^2x/dt^2=-g$. Therefore $-g=-\mu/a^2$ or $\mu=a^2g$. Thus the equation (1) becomes $\frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}$.

Integrating, we get $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C$, at $B, x=OB=a+h, \frac{dx}{dt}=0$

$\therefore$ $\quad 0=\frac{2a^2g}{a+h}+C$ or $C=-\frac{2a^2g}{a+h}$. Thus, we have $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}-\frac{2a^2g}{a+h}=2a^2g\left(\frac{1}{x}-\frac{1}{a+h}\right)$.

For the sake of convenience let us put $a+h=b$. Then

$$\left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right)=\frac{2a^2g}{b}\left(\frac{b-x}{x}\right) \qquad (2)$$

The equation (2) gives velocity at any point from B to A.

From (2) on taking square root, we get $\frac{dx}{dt}=-a\sqrt{\left(\frac{2g}{b}\right)}\sqrt{\left(\frac{b-x}{x}\right)}$ where the negative sign has been taken because the particle is moving in the direction of x decreasing.

$$\therefore \quad dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\sqrt{\left(\frac{x}{b-x}\right)}dx \qquad (3)$$

Let t_1 be the time from B to A. Then integrating (3) from B to A, we get

$$\int_0^{t_1} dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a \sqrt{\left(\frac{x}{b-x}\right)}dx$$

$$\therefore \quad t_1-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a \sqrt{\left(\frac{x}{b-x}\right)}dx$$

Put $x=b\cos^2\theta$; so that $dx=-2b\cos\theta\sin\theta d\theta$

$$\therefore \quad t_1=\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_0^{\cos^{-1}(a/b)}\frac{\cos\theta}{\sin\theta}2b\cos\theta\sin\theta\, d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\ \frac{b}{a}\int_0^{\cos^{-1}\sqrt{(a/b)}} 2\cos^2\theta\ d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\ \frac{b}{a}\left[0+\frac{1}{2}\sin\theta\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\sin\theta\cos\theta\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\ \frac{b}{a}\left[\theta+\cos\theta\sqrt{\left(1-\cos^2\theta\right)}\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{a}{b}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\left[\frac{b}{a}\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{b}{a}\right)}+\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\cos^{-1}\sqrt{\left(\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a+h}{a}\right)}\sqrt{\left(1-\frac{a}{a+h}\right)}\right]$$

[replacing b by $a+h$]

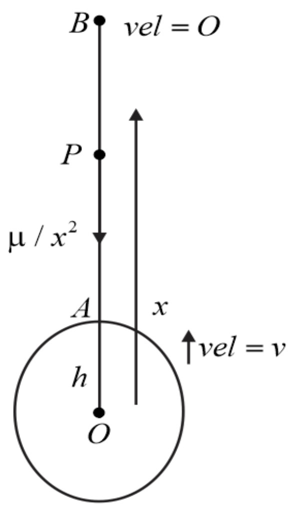
$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\left(1-\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a+h}{a}\right)}\sqrt{\left(\frac{h}{a+h}\right)}\right]$$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}+\sqrt{\left(\frac{h}{a}\right)}\right]$$

Example 5:- If h be the height due to the velocity v at the earth surface supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that $\frac{1}{h}-\frac{1}{H}=\frac{1}{r}$,where r is the radius of the earth.

Solution:- If h is the height of the particle due to the velocity v at the earth's surface, supposing its attraction constant (i.e., taking the acceleration due to gravity as constant and equal to g), then from the formula $v^2=u^2+2fs$, we have $0^2=v^2-2gh$.

$\therefore \qquad v^2=-2gh \qquad (1)$



When the variation of gravity is taken into account, let P be the position of the particle at any time t measured from the instant the particle is projected vertically upwards from the earth's surface with velocity v, and let $OP = x$

The acceleration of the particle at P is μ / x^2 direction towards O.

$\therefore$ The equation of motion of the particle at P is $\dfrac{d^2x}{dt^2} = -\dfrac{\mu}{x^2}$ (2)

[Here the $-ive$ sign is taken since the acceleration acts in the direction of x decreasing]

But at A i.e., on the surface of the earth.

$x = OA = r$ and $\dfrac{d^2x}{dt^2} = -g$.

From (2), we have $-g = -\mu / r^2$ or $\mu = gr^2$

Substituting in (2), we have $\dfrac{d^2x}{dt^2} = -\dfrac{gr^2}{x^2}$. (3)

Multiplying both sides of (3) by $2(dx/dt)$ and then integrating w.r.t. $'t'$, we have

$\left(\dfrac{dx}{dt}\right)^2 = \dfrac{2gr^2}{x} + A$, where A is a constant of integration.

But at the point A, $x = OA = r$ and $dx/dt = v$, which is the velocity of projection at A.

$\therefore$ $v^2 = \dfrac{2gr^2}{r} + A$ or $A = v^2 - 2gr$

$\therefore$ $\left(\dfrac{dx}{dt}\right)^2 = \dfrac{2gr^2}{x} + v^2 - 2gr$ (4)

Suppose the particle in this case rises upto the point B, where $AB = H$. Then at the point B, $x = OB = OA + AB = r + H$ and $dx/dt = 0$.

$\therefore$ From (4), we have $0 = \dfrac{2gr^2}{r+H} + v^2 - 2gr$

Or $v^2 = \dfrac{2gr^2}{r+H} + 2gr = \dfrac{2grH}{r+H}$ (5)

Equating the values of v^2 from (1) and (5), we have $2gh=\dfrac{2grH}{r+H}$ or $\dfrac{1}{h}=\dfrac{r+H}{rH}$

Or $\dfrac{1}{h}=\dfrac{1}{H}+\dfrac{1}{r}$ or $\dfrac{1}{h}-\dfrac{1}{H}=\dfrac{1}{r}$.

Example 6:- A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is $\dfrac{1}{3}\sqrt{\left(\dfrac{2a}{g}\right)}\left[\left(1+\dfrac{h}{a}\right)^{3/2}-1\right]$, where a is the radius of the earth.

Solution:- [Refer fig. of before example]

Let O be the centre of the earth and A the point of projection on the earth's surface.

If P is the position of the particle at any time t, such that $OP=x$, then the acceleration at $P=\mu/x^2$ directed towards O.

$\therefore$ The equation of motion of the particle at P is $\dfrac{d^2x}{dt^2}=-\dfrac{\mu}{x^2}$ (1)

But at the point A, on the surface of the earth $x=a$ and $d^2x/dt^2=-g$.

$\therefore$ $-g=-\mu/a^2$ or $\mu=a^2g$

$\therefore$ $\dfrac{d^2x}{dt^2}=-\dfrac{a^2g}{x^2}$

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't' ,we get $\left(\dfrac{dx}{dt}\right)^2=\dfrac{2a^2g}{x}+C$, where C is constant.

But when $x\to\infty$, $dx/dt\to 0$ $\quad\therefore\ C=0$

$\therefore$ $\left(\dfrac{dx}{dt}\right)^2=\dfrac{2a^2g}{x}$ or $\dfrac{dx}{dt}=\dfrac{a\sqrt{(2g)}}{\sqrt{x}}$ (2)

[Here $+ive$ sign is taken because the particle is moving in the direction of x increasing]

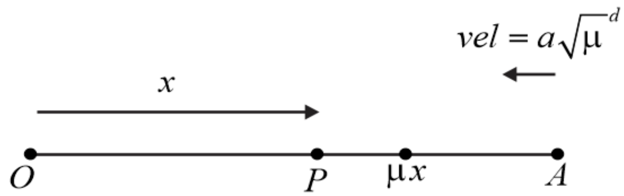
Separating the variables, we have $dt=\dfrac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx$

Integrating between the limits $x=a$ to $x=a+h$ the required time t to reach a height h is given by $t=\dfrac{1}{a\sqrt{(2g)}}\int_a^{a+b}\sqrt{(x)}\,dx=\dfrac{1}{a\sqrt{(2g)}}\left[\dfrac{2}{3}x^{3/2}\right]_a^{a+h}$

$$=\frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2}-a^{3/2}\right]=\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$$

Example 7:- If a particle is projected toward the centre of repulsion, varying as the distance from the centre, from a distance a from it with a velocity $a\sqrt{\mu}$; prove that the particle will approach the centre but will never reach it.

Solution:- Let the particle be projected from the point A with velocity $a\sqrt{\mu}$ towards the centre of repulsion O and and let $OA = a$.



If P is the position of the particle at time t such that $OP = x$, then the P, then acceleration on the particle is μx in the direction PA.

$\therefore$ The equation of motion of the particle is $\frac{d^2x}{dt^2} = \mu x$ [$+ive$ sign is taken because the acceleration is in the direction of x increasing]

Multiplying by $2(dx/dt)$ and integrating w.r.t. $'t'$, we have $(dx/dt)^2 = \mu x^2 + C$ where C is a constant.

But at $A, x = a$ c and $(dx/dt)^2 = a^2\mu$ $\therefore C = 0$

$\therefore$ $(dx/dt)^2 = \mu x^2$ or $dx/dt - = \sqrt{\mu x}$ (1)

[$-ive$ sign is taken because the particle is moving in the direction of x decreasing]

The equation (1) shows that the velocity of the particle will be zero when $x = 0$ and not before it and so the particle will approach the centre O.

From (1), we have $dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{x}$

Integrating between the limit $x = a$ to $x = 0$, the time t_1 from A to O is given by

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{dx}{x} = \frac{1}{\sqrt{\mu}}[\log x]_0^a = \frac{1}{\sqrt{\mu}}(\log a - \log 0)$$

$= \infty$. $[\because \ \log 0 = -\infty]$

Hence the particle with take an infinite time to reach the centre O or in order words it will never reach the centre O.

Example 8:- A particle moves in a straight line under a force to a point in it, varying as (distance)$^{-4/3}$. Show that the velocity in falling from rest at infinitely to distance a is equal to that acquired in falling from rest at a distance a to a distance $a/8$.

Solution:- If x is the distance of the particle from the fixed point at time t ,then the equation of motion of the particle is $\frac{d^2x}{dt^2} = -\mu x^{-4/3}$ (1)

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. t, we have $\left(\frac{dx}{dt}\right)^2 = \frac{6\mu}{x^{1/3}} + A$ (2)

If the particle falls from rest at infinitely i.e., $dx/dt = 0$ when $x = \infty$, we have from (2), $A = 0$

$\therefore \quad (dx/dt)^2 = 6\mu / x^{1/3}$

If v_1 is the velocity of the particle at $x = a$, then $v_1^2 = 6\mu / a^{1/3}$ (3)

Again if the particle falls from rest at a distance a, i.e., if $dx/dt = 0$ when $x = a$, we have from (2) $0 = \frac{6\mu}{a^{1/3}} + A$ or $A = -\frac{6\mu}{a^{1/3}}$

$\therefore \quad \left(\frac{dx}{dt}\right)^2 = 6\mu\left(\frac{1}{x^{1/3}} - \frac{1}{a^{1/4}}\right)$

If in this case v_2 is the velocity of the particle at $x = a/8$, then $v_2^2 = 6\mu\left[\left(\frac{8}{a}\right)^{1/8} - \frac{1}{a^{1/3}}\right] = 6\mu\left(\frac{2}{a^{1/3}} - \frac{1}{a^{1/3}}\right) = \frac{6\mu}{a^{1/3}}$ (4)

From (3) and (4), we observe that $v_1 = v_2$, which proves the required result.

Example 9:- A particle moves in a straight lien, its acceleration directed towards a fixed point O in the line and is alwaus equal to $\mu\left(a^3/x^2\right)^{1/3}$ when it is an a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity $a\sqrt{(6\mu)}$ after time $\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$.

Solution:- Take the centre of force O as origin. Suppose a particle starts from rest at A, where $OA = a$. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time t, where $OP = x$. The acceleration of the particle at P is $\mu a^{5/3} x^{-2/3}$ directed towards O.

Therefore the equation of motion of the particle is $\frac{d^2x}{dt^2} = -\mu a^{5/3} - x^{-2/3}$ (1)

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. $'t'$,we have $\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3} x^{1/3}}{1/3} + k = -6\mu a^{5/3} x^{1/3} + k$, where k is a constant.

At A, $x = a$ and $dx/dt = 0$, so that $-6\mu a^{5/3} a^{1/3} + k = 0$ or $k = 6\mu a^2$.

$\therefore \quad (dx/dt)^2 = -6\mu a^{5/3} x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right)$ (2)

Which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x = 0$ and $(dx/dt) = v_1^2$. So from (2), we have $v_1^2 = 6\mu a^{5/3}\left(a^{1/3} - 0\right) = 6\mu a^2$ or $v_1 = a\sqrt{(6\mu)}$

Now taking square root of (2), we get $dx/dt=-\sqrt{\left(6\mu a^{5/3}\right)}=\sqrt{\left(a^{1/3}-x^{1/3}\right)}$, where the $-ive$ sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get $dt=-\frac{1}{\sqrt{\left(6\mu a^{5/3}\right)}}\cdot\frac{dx}{\sqrt{\left(a^{1/3}-x^{1/3}\right)}}$. (3)

Let t_1 be the time from A to O. Then integrating (3) from A to O, we have

$$\int_0^{t_1} dt=-\frac{1}{\sqrt{\left(6\mu a^{5/3}\right)}}\int_a^0\frac{dx}{\sqrt{\left(a^{1/3}-x^{1/3}\right)}}$$

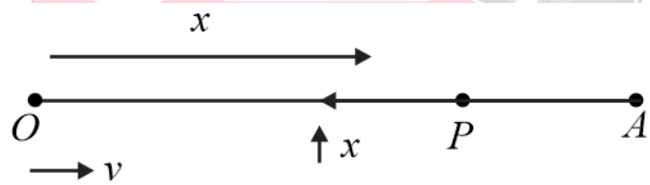
$$=\frac{1}{\sqrt{\left(6\mu a^{5/3}\right)}}\int_0^a\frac{dx}{\sqrt{\left(a^{1/3}-x^{1/3}\right)}}$$

Put $x=a\sin^6\theta$ so that $dx=6a\sin^5\theta\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a,\ \theta=\pi/2$.

$$\therefore \quad t_1=\frac{1}{\sqrt{\left(6\mu a^{5/3}\right)}}\int_0^{\pi/2}\frac{6a\sin^5\theta\cos\theta\,d\theta}{a^{1/6}\cos\theta}$$

$$=\sqrt{\left(\frac{6}{\mu}\right)}\int_0^{\pi/2}\sin^5\theta\,d\theta=\sqrt{\left(\frac{6}{\mu}\right)}\cdot\frac{4.2}{5.3.1}=\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}.$$

Que 1:- A particle starts with a given velocity v and moves under a retardation equal to k times the space described. Show that the distance traversed before if comes to rest is $v/\sqrt{k}$.



Solution:- Suppose the particle starts from O with velocity v and moves in the straight line OA. Let P be the position of the particle after any time t ,where $OP=x$. Then the retardation of the particle at P is $k\,x$ i.e., the acceleration of the particle at P is $k\,x$ and is directed towards O i.e., in the direction of x decreasing. Therefore the equation of motion of the particle at P is $d^2x/dt^2=-k\,x$ (1)

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. $'t'$, we have $(dx/dt)^2=-k\,x^2+C$, where C is constant.

At O, $x=0$ and $dx/dt=v$, so that $v^2=C$

$$\therefore \quad (dx/dt)^2=v^2-k\,x^2 \qquad (2)$$

Which gives the velocity of the particle at a distance x from O.

From (2), $dx/dt=0$ when $v^2=k\,x^2=0$ i.e., when $x=v/\sqrt{k}$.

Hence the distance traversed before the particle comes to rest is $v/\sqrt{k}$.

Que 2:- Assuming that at a distance x from a centre of force, the speed v of a particle, moving in a straight line is given by the equation $x = ae^{bv^2}$, where a and b are constant. Find the law and the nature of the force.

Solution:- Given $x = ae^{bv^2}$. Therefore $e^{bv^2} = x/a$

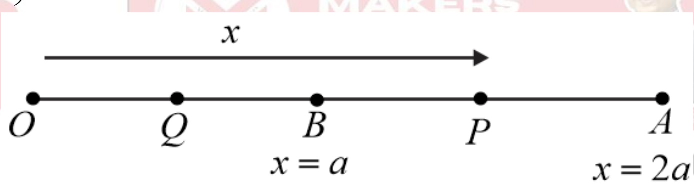
Or $bv^2 = \log(x/a) = \log x - \log a$ (1)

Differentiating both sides of (1) w.r.t. x, we get $2bv\dfrac{dv}{dx} = \dfrac{1}{x}$ or $v\dfrac{dv}{dx} = \dfrac{1}{2b}\dfrac{1}{x}$

$\therefore$ The equation of motion of the particle is $\dfrac{d^2x}{dt^2} = \dfrac{1}{2b}.\dfrac{1}{x}$ [Note that $v\dfrac{dv}{dx} = \dfrac{d^2x}{dt^2}$]

Hence the acceleration varies inversely as the distance of the particle from the centre of force. Also the force is repulsive or attractive according as b is positive or negative.

Que 3:- A particle of mass m moving in a straight line is acted upon by an attractive force which is expressed by the formula $m\mu a^2/x^2$ for values of $x \geq a$, and by the formula $m\mu x/a$ for $x \leq a$, where x is the distance from a fixed origin in the line. If the particle starts at a distance $2a$ from the origin, prove that it will reach the origin with velocity $(2\mu a)^{1/2}$. Prove further that, the time taken to reach the origin is $\left(1+\frac{3}{4}\pi\right)\sqrt{(a/\mu)}$.



Solution:- Let O be the origin and A the point from which the particle starts. We have $OA = 2a$ and $OB = a$, so that B is the middle point of OA.

Motion from A to B:- The particle starts from rest at A and it moves towards B. Let P be its position at any time t, where $OP = x$. According to the question the acceleration of P is $\mu a^2/x^2$ and is directed towards O i.e., in the direction of x decreasing.

Therefore, the equation of motion of P is $\dfrac{d^2x}{dt^2} = -\dfrac{\mu a^2}{x^2}$ (1)

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t 't', we have $\left(\dfrac{dx}{dt}\right)^2 = \dfrac{2\mu a^2}{x} + C$

When $x = 2a$, $dx/dt = 0$, so that $C = -2\mu a^2/2a$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2\mu a^2}{x} - \frac{2\mu a^2}{2a} = 2a^2\mu\left[\frac{1}{x} - \frac{1}{2a}\right] = a\mu\frac{2a-x}{x} \quad (2)$$

Which gives the velocity of the particle at any position between A and B. Suppose the particle reaches B with the velocity v_1. Then at B, $x = a$ and $(dx/dt)^2 = v_1^2$. So from (2), we get $v_1^2 = a\mu \dfrac{2a-a}{a} = a\mu$ or $v_1 = \sqrt{(a\mu)}$, its direction being towards the origin O.

Now taking square root of (2), we get $\dfrac{dx}{dt} = -\sqrt{(a\mu)}\sqrt{\left(\dfrac{2a-x}{x}\right)}$, where the $-ive$ sign has been takes because the particle is moving in the direction of x decreasing.

Separating the variables, we get $dt = -\dfrac{1}{\sqrt{(a\mu)}}\sqrt{\left(\dfrac{x}{2a-x}\right)}dx$ (3)

Let t_1 be the time from A to B. then at A, $x = 2a$ and $t = 0$, while at $B, x = a$ and $t = t_1$. So integrating (3) from A to B, we get $\int_0^{t_1} dt = -\dfrac{1}{\sqrt{(a\mu)}}\int_{2a}^{a}\sqrt{\left(\dfrac{x}{2a-x}\right)}dx$.

Put $x = 2a\cos^2\theta$, so that $dx = -4a\cos\theta\sin\theta\, d\theta$. When $x = 2a, \theta = 0$ and when $x = a,\ \theta = \pi/4$.

$$\therefore \quad t_1 = -\frac{1}{\sqrt{(a\mu)}}\int_0^{\pi/4} \frac{\cos\theta}{\sin\theta}(-4a\cos\theta\sin\theta)\,d\theta$$

$$= \sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4} 2\cos^2\theta\, d\theta = 2\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta$$

$$= 2\sqrt{\left(\frac{a}{\mu}\right)}\left[\theta + \frac{1}{2}\sin\theta\right]_0^{\pi/4} = 2\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{4}+\frac{1}{2}\right] = \sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right]$$

Motion from B to O:- Now the particle starts from B towards O with velocity $\sqrt{(a\mu)}$ gained b it during its motion from A to B. Let Q be its position after time t since it starts from B and let $OQ = x$. Now according to the question the acceleration of Q is $\mu x/a$ directed towards O.

Therefore the equation of motion of Q is $\dfrac{d^2x}{dt^2} = -\dfrac{\mu x}{a}$ (4)

Multiplying both sides of (4) by $2(dx/dt)$ and integrating w.r.t. t, we have $\left(\dfrac{dx}{dt}\right)^2 = -\dfrac{\mu}{a}x^2 + D$

At $B,\ x = a$ and $(dx/dt)^2 = v_1^2 = a\mu$, so that $a\mu = -a\mu + D$. Or $D = 2a\mu$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = -\frac{\mu}{a}x^2 + 2a\mu = \frac{\mu}{a}\left(2a^2 - x^2\right) \qquad (5)$$

Which gives the velocity of the particle at any position between B and O. Let v_2 be the velocity of the particle at O. Then putting $x=0$ and $(dx/dt)^2 = v_2^2$ in (5), we get $v_2^2 = \frac{\mu}{a}(2a^2 - 0) = 2a\mu$ or $v_2 = \sqrt{(2a\mu)}$.

Hence the particle reaches the origin with the velocity $\sqrt{(2a\mu)}$.

Now taking square root of (5), we get $\frac{dx}{dt} = -\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{(2a^2 - x^2)}$, where the $-ive$ sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have $dt = -\sqrt{\left(\frac{a}{\mu}\right)}\frac{dx}{\sqrt{(2a^2 - x^2)}}$ (6)

Let t_2 be the time from B to O Then at $B, t=0$ and $x=a$ while at O, $x=0$ and $t=t_2$. So integrating (6) from B to O, we get $\int_0^{t_2} dt = -\sqrt{\left(\frac{a}{\mu}\right)}\int_a^0 \frac{dx}{\sqrt{(2a^2 - x^2)}}$

i.e. $t_2 = \sqrt{\left(\frac{a}{\mu}\right)}\left[\cos^{-1}\frac{x}{a\sqrt{2}}\right]_a^0$

$= \sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}$

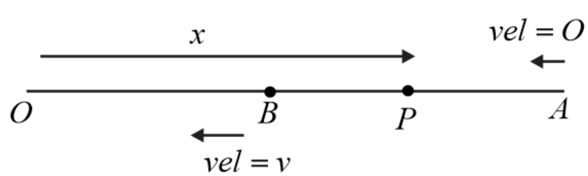
Hence the whole time taken to reach the origin $O = t_1 + t_2$

$= \sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2} + 1\right] + \sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4} = \sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{3\pi}{4} + 1\right]$

Que 4:- A particle moves along the axis of x starting from rest at $x=a$. For an interval t_1 from the beginning of the motion the acceleration is $-\mu x$ for a subsequent time t_2 the acceleration is μx, and at the end of this interval the particle is at the origin; prove that $\tan\left(\sqrt{\mu t_1}\right).\tanh\left(\sqrt{\mu t_2}\right) = 1$

Solution:- Let the particle moving along the axis of x start from rest at A such that $OA = a$.

Let $-\mu x$ be the acceleration for an interval t_1 from A to B and μx that for an interval t_2 from B to O, when $OB = b$.



For motion from A to B:- the equation of motion is $\frac{d^2x}{dt^2} = -\mu x$ (1)

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. t, we have $(dx/dt)^2 = -\mu x^2 + A$,where A is a constant.

But $x = a, dx/dt = 0$ $\therefore\ 0 = -\mu a^2 + A$ or $A = \mu a^2$

$\therefore$ $(dx/dt)^2 = \mu(a^2 - x^2)$ (2)

Or $dx/dt = -\sqrt{\mu}\sqrt{(a^2 - x^2)}$

[the $-ive$ sign is taken because the particle is moving in the direction of x decreasing]

Or $dt = \dfrac{1}{\sqrt{\mu}} \cdot \dfrac{dx}{\sqrt{(a^2 - x^2)}}$, [separating the variables]

Integrating between the limits $x = a$ to $x = b$, the time t_1 from A to B is given by

$$t_1 = \frac{1}{\sqrt{\mu}} \int_{x=a}^{b} \frac{dx}{\sqrt{(a^2 - x^2)}} = \frac{1}{\sqrt{\mu}} \left[\cos^{-1}\frac{x}{a}\right]_{x=a}^{b} = \frac{1}{\sqrt{\mu}} \cos^{-1}\frac{b}{a}$$

$\therefore$ $\cos(\sqrt{\mu t_1}) = b/a$ and $\sin(\sqrt{\mu t_1}) = \sqrt{\left[1 - \cos^2(\sqrt{\mu t_1})\right]}$

$$= \sqrt{\left(1 - \frac{b^2}{a^2}\right)} = \frac{\sqrt{(a^2 - b^2)}}{a}$$

Dividing, $\tan(\sqrt{\mu t_1}) = \dfrac{\sqrt{(a^2 - b^2)}}{b}$ (3)

If V is the velocity at B where $x = b$, then from (2)

$V^2 = \mu(a^2 - b^2)$ (4)

For motion from B to O:- the velocity at B is V and the particle moves towards O under the acceleration μx.

$\therefore$ the equation of motion is $\dfrac{d^2x}{dt^2} = \mu$ (5)

Integrating $(dx/dt)^2 = \mu x^2 + B$, where B is constant.

But at the point B, $x = b$ and $(dx/dt)^2 = V^2 = \mu(a^2 - b^2)$

$\therefore$ $\mu(a^2 - b^2) = \mu b^2 + B$ or $B = \mu(a^2 - 2b^2)$

$\therefore$ $\left(\dfrac{dx}{dt}\right)^2 = \mu\left[x^2 + (a^2 - 2b^2)\right]$ or $\dfrac{dx}{dt} = -\sqrt{\mu}\sqrt{\left[x^2 + (a^2 - 2b^2)\right]}$

Or $dt = -\dfrac{1}{\sqrt{\mu}} \dfrac{dx}{\sqrt{\left[x^2 + (a^2 + 2b^2)\right]}}$

Integrating between the limits $x = b$ to $x = 0$, the time t_2 form B to O is given by

$$t_2 = -\frac{1}{\sqrt{\mu}} \int_{x=b}^{0} \frac{dx}{\sqrt{\left[x^2 + (a^2 - 2b^2)\right]}}$$

$$= -\frac{1}{\sqrt{\mu}}\left[\sin^{-1}\frac{x}{\sqrt{(a^2-2b^2)}}\right]_b^0 = \frac{1}{\sqrt{\mu}}\sin^{-1}\frac{b}{\sqrt{(a^2-2b^2)}}$$

$\therefore$ $\sinh\left(\sqrt{\mu t_2}\right) = \frac{b}{\sqrt{(a^2-2b^2)}}$ so that $\cosh\left(\sqrt{\mu t_2}\right) = \sqrt{\left\{1+\sinh^2\left(\sqrt{\mu t_2}\right)\right\}}$

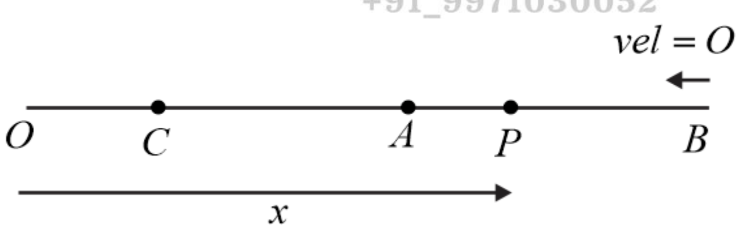
$$= \sqrt{\left(1+\frac{b^2}{a^2-2b^2}\right)} - \sqrt{\left(\frac{a^2-b^2}{a^2-2b^2}\right)}$$

Dividing, $\tanh\left(\sqrt{\mu t_2}\right) = \frac{b}{\sqrt{(a^2-b^2)}}$ (6)

Multiplying (3) and (6), we have $\tan\left(\sqrt{\mu t_1}\right).\ \tanh\left(\sqrt{\mu t_2}\right) = 1$.

Que 5 :- A particle starts from rest at a distance b from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu\left[1-\frac{a}{x}\right]$ towards the point when $x>a$ but $\mu\left[\frac{a^2}{x^2}-\frac{a}{x}\right]$ from the same point when $x<a$; prove that particle will oscillate through a space $\left[\frac{b^2-a^2}{b}\right]$.

Solution:- Let the particle start from rest at B, where $OB = b$, and move towards the centre of force. Let $OA = a$.



Motion from B to A:- i.e. when $x>a$.

Since the law of force, when $x>a$ is $\mu(1-a/x)$ towards O, therefore the equation of motion is $\frac{d^2x}{dt^2} = -\mu\left(1-\frac{a}{x}\right)$

Multiplying both sides by $2(dx/dt)$ and integrating w.r.t. t, we have $\left(\frac{dx}{dt}\right)^2 = -2\mu(x-a\log x)+C$, where C is a constant.

But at B, $x=0$ $B=b$ and $dx/dt=0$. $\therefore C = 2\mu(b-a\log b)$

$\therefore$ $\left(\frac{dx}{dt}\right)^2 = 2\mu(b-a\log b - x + a\log x)$ (1)

If V is the velocity at the point A where $x = OA = a$, then from (1), we have $V^2 = 2\mu(b - a - a\log b + a\log a)$.

Motion from A towards O i.e. when $x < a$.

The velocity of the particle at A is V and it moves towards O under the law of force $\mu\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$ at the distance x from the fixed point O.

$\therefore$ The equation of motion is $\frac{d^2x}{dt^2} = \mu\left[\frac{a^2}{x^2} - \frac{a}{x}\right]$

Multiplying both sides by $2(dx/dt)$ and integrating, we have $\left(\frac{dx}{dt}\right)^2 = 2\mu\left(-\frac{a^2}{x} - a\log x\right) + D$ where D is constant.

But at the point A,

$x = a$ and $(dx/dt)^2 = v^2 = 2\mu(b - a + a\log b + a\log a)$

$\therefore$ $D = 2\mu(b - a - a\log b + a\log a) + 2\mu(a + a\log a)$

$= 2\mu(b - a\log b + 2a\log a) = 2\mu\{b + a\log(a^2/b)\}$

$\therefore$ $\left(\frac{dx}{dt}\right)^2 = -2\mu\left(\frac{a^2}{x} + a\log x\right) + 2\mu\left\{b + a\log\left(\frac{a^2}{b}\right)\right\}$ (3)

If the particle comes to rest at the point C, where $x = c$, then putting $x = c$ and $dx/dt = 0$ in (3), we get $2\mu\left(\frac{a^2}{c} + a\log c\right) = 2\mu\left\{b + a\log\left(\frac{a^2}{b}\right)\right\}$

Or $\frac{a^2}{c} + a\log c = \frac{a^2}{(a^2/b)} + a\log\left(\frac{a^2}{b}\right)$

$\therefore$ $c = a^2/b$ i.e., $OC = a^2/b$

Since B and C are the position of instantaneous rest of the particle, therefore the particle oscillates through the space BC.

We have $BC = OB - OC = b - \frac{a^2}{c} = \frac{b^2 - a^2}{b}$ which proves the required result.

Motion in Resisting Medium

Example:- (1) A particle falls from rest under gravity through a distance x in a medium whose resistance varies as the square of the velocity. If v be the velocity actually acquired by it. v_0 the velocity it would have acquired, had there been no resisting medium and V the terminal velocity, show that $\frac{v^2}{v_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2.3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\ldots$

Solution:- If v is the velocity of the particle acquired in falling through a distance x in the given resisting medium then proceeding as in \$, it is given by $v^2=V^2\left(1-\varepsilon^{-2gx/V^2}\right)$ (1)

If v_0 is the velocity of the particle acquired in falling freely through a distance x, if there is no resisting medium, then $v_0^2=0+2gx=2gx$ (2)

Substituting $2gx=v_0^2$ in (1), we have $v^2=V^2\left(1-e^{-v_0^2/V^2}\right)$

$$=V^2\left[1-\left\{1-\frac{v_0^2/V^2}{1!}+\frac{v_0^4/V^4}{2!}-\frac{v_0^6/V^6}{3!}+\ldots.\right\}\right]$$

$$=V^2\left[\frac{v_0^2/V^2}{1!}-\frac{v_0^4/V^4}{2!}+\frac{v_0^6/V^6}{3!}-\ldots\ldots\right]$$

$$=v_0^2\left[1-\frac{1}{2!}.\frac{v_0^2}{V^2}+\frac{1}{3!}\frac{v_0^4}{V^4}-\frac{1}{4!}.\frac{v_0^6}{V^6}+\ldots\right] \text{ or } \frac{v^2}{v_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2.3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\ldots.$$

Example: - (2) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that greatest height attained by the particle is $\frac{V^2}{g}\left[\lambda-\log(1+\lambda)\right]$, where V is the terminal velocity of the particle and λV the initial velocity.

Solution:- Suppose a particle of mass m is projected vertically upwards from O with velocity λV in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP=x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are ,

(i) The force mkv due to the resistance acting vertically downwards i.e. against the direction of motion of the particle, and

(ii) The weight mg of the particle acting vertically downwards.

Since both these forces act in the direction of x decreasing, therefore the equation of motion of the particle of the particle at time t is

$$m\frac{d^2x}{dt^2}=-mg-mkv \text{ or } \frac{d^2x}{dt^2}=-g\left(1+\frac{k}{g}v\right) \quad (1)$$

Now V is given to the terminal velocity of the particle during its downward motion. Then V is the velocity of the particle when during the downward motion its acceleration is zero. If the particle falls vertically downwards, the resistance acts vertically upwards. Therefore the equation of motion of the particle in downward motion is $m\frac{d^2x}{dt^2} = mg - mkv$ (2)

Putting $v = V$ and $d^2x/dt^2 = 0$ in (2), we get $0 = mg - mkV$ or $k = g/V$

Substituting this value of k in (1), the equation of motion of the particle in the upward motion is

$$\frac{d^2x}{dt^2} = -g\left(1+\frac{v}{V}\right) \text{ or } v\frac{dv}{dx} = -\frac{g}{V}(V+v), \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

Or $dx = -\frac{V}{g}\frac{v\,dv}{V+v}$, separating the variables.

Or $dx = -\frac{V}{g}\left\{\frac{(v+V)-V}{v+V}\right\}dv = -\frac{V}{g}\left(1-\frac{V}{V+v}\right)dv$.

Integrating, we have $x = -\frac{V}{g}\{v - V\log(V+v)\} + A$, where A is a constant.

But initially when $x = 0$, $v = \lambda V$ (given)

$$\therefore \quad 0 = -\frac{V}{g}\{\lambda V - V\log(V+\lambda V)\} + A \text{ or } A = \frac{V}{g}\left[\lambda V - V\log\{V(1+\lambda)\}\right]$$

$$\therefore \quad x = \frac{V}{g}\left[\lambda V - v - V\log\frac{V(1+\lambda)}{(V+v)}\right]$$, giving the velocity of the particle at any position.

If h is the greatest height attained by the particle, we have $v = 0$ when $x = h$.

$$\therefore \quad h = \frac{V}{g}\left[\lambda V - V\log\frac{V(1+\lambda)}{V}\right] = \frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right].$$

Example:- (3) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Find the greatest height attained by the particle.

Solution:- Suppose a particle of mass m is projected vertically upwards from a point O with velocity u in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P.

Then proceeding as in Ex.3 the equation of motion of the particle at time t is $m\frac{d^2x}{dt^2} = -mg - mkv$ or

$$\frac{d^2x}{dt^2} = -(g+kv) \text{ or } v\frac{dv}{dx} = -(g+kv).$$

$$\therefore \quad dx = -\frac{v}{g+kv}dv = -\frac{1}{k}\frac{kv}{g+kv}dv$$

$$= -\frac{1}{k}\frac{(g+kv)-g}{g+kv}dv = -\frac{1}{k}\left[1-\frac{g}{g+kv}\right]dv.$$

Integrating, we get $x = -\frac{1}{k}\left[v - \frac{g}{k}\log(g+kv)\right] + A$, where A is a constant.

But initially when $x = 0$, we have $v = u$.

$\therefore$ $0 = -\frac{1}{k}\left[v - \frac{g}{k}\log(g+ku)\right] + A$ or $A = \frac{1}{k}\left[u - \frac{g}{k}\log(g+ku)\right]$.

$\therefore$ $x = -\frac{1}{k}\left[v - \frac{g}{k}\log(g+kv)\right] + \frac{1}{k}\left[u - \frac{g}{k}\log(g+ku)\right]$

$= \frac{1}{k}(u-v) - \frac{g}{k^2}\log\frac{g+ku}{g+kv}$, giving the velocity of the particle at any position.

If h is the greatest height attained by the particle, we have $v = 0$ when $x = h$.

$\therefore$ $h = \frac{u}{k} - \frac{g}{k^2}\log\left(\frac{g+ku}{g}\right) = \frac{u}{k} - \frac{g}{k^2}\log\left(1 + \frac{ku}{g}\right)$.

Example:- (4) A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. Show that the greatest height attained by the particle is $\frac{1}{2k}\log\frac{g+ku^2}{g}$.

Solution:- Let a particle of mass m be projected vertically upwards from a point O with velocity u. If v is the velocity of the particle time t at a distance x from the starting point O, then the resistance on the particle is mkv^2 in the downward direction i.e. in the direction of x decreasing. The weight mg of the particle also acts vertically downwards. So the equation of motion of the particle during its upward motion is $m\frac{d^2x}{dt^2} = -mg - mkv^2$ or $v\frac{dv}{dx} = -(g+kv^2)$, $\left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$

Or $\frac{2kv\,dv}{g+kv^2} = -2k\,dx$, separating the variables.

Integrating, $\log(g+kv^2) = -2kx + A$, where A is constant.

But initially $x = 0, v = u$; $\therefore$ $\log(g+kv^2) = -2kv + \log(g+ku^2)$

Or $2kx = \log(g+ku^2) - \log(g+kv^2)$ or $x = \frac{1}{2k}\log\frac{g+ku^2}{g+kv^2}$ (1)

Which gives the velocity of the particle at a distance x.

If h is the greatest height attained by the particle then at $x = h$, $v = 0$. Therefore from (1), we have $h = \frac{1}{2k}\log\frac{g+ku^2}{g}$.

Example:- (5) A particle is projected vertically upwards with a velocity V and the resistance of the air produces a retardation kv^2, where v is the velocity. Show that the velocity V' with which the particle will return to the point of projection is given by $\frac{1}{V'^2} = \frac{1}{V^2} + \frac{k}{g}$.

Solution:- Let a particle of mass m be projected vertically upwards wit ha velocity V.

If v is the velocity of the particle at time t, at distance x from the starting point the resistance there is mkv^2 in the downward direction (i.e. in the direction of x decreasing).
The weight mg of the particle also acts vertically downwards.

$\therefore$ The equation of motion of particle in the upward motion is $m\frac{d^2x}{dt^2} = -mg - mkv^2$ or

$v\frac{dv}{dx} = -\left(g + kv^2\right)$ or $\frac{2kv + dv}{g + kv^2} = -2k\,dx$.

Integrating $\log\left(g + kv^2\right) = -2kx + A$, where A is a constant.

Initially when $x = 0$, $v = V$; $\therefore$ $A = \log\left(g + kV^2\right)$

$\therefore$ $\log\left(g + kv^2\right) = -2kx + \log\left(g + kV^2\right)$ or $x = \frac{1}{2k}\log\frac{g + kV^2}{g + kv^2}$

If h is the maximum height attained by the particle, then $v = 0$ where $x = h$

$\therefore$ $h = \frac{1}{2k}\log\frac{g + kV^2}{g}$ (1)

Now from the highest point O' the particle falls vertically downwards.
Let y be the depth of the particle below the highest point O' after time t and v be the velocity there. Then the resistance at this point is mkv^2 acting in the vertically upwards direction.

$\therefore$ The equation of motion of the particle during its downward motion is $m\frac{d^2y}{dt^2} = mg - mkv^2$

or $v\frac{dv}{dy} = g - kv^2$ or $\frac{-2kv\,dv}{g - jv^2} = -2k\,dy$.

Integrating $\log\left(g - kv^2\right) = -2ky + B$, where B is a constant.

At the highest point O', $y = 0, v = 0$; $\therefore B = \log g$.

$\therefore$ $\log\left(g - kv^2\right) = -2ky + \log g$ or $y = \frac{1}{2k}\log\frac{g}{g - kv^2}$.

If the particle returns to the point of projection O with velocity V', then $v = V'$ when $y = h$
.

$\therefore$ $h = \frac{1}{2k}\log\frac{g}{g - kV'^2}$ (2)

From (1) and (2), equating the values of h, we have $\frac{1}{2k}\log\frac{g+kV^2}{g}=\frac{1}{2k}\log\frac{g}{g-kV'^2}$

Or $\frac{g+kV^2}{g}=\frac{g}{g-kV'^2}$ or $\left(g+kV^2\right)\left(g-kV'^2\right)=g^2$ or $-gkV'^2+gkV^2-k^2V^2V'^2=0$.

Dividing by $kgV^2V'^2$, we have $-\frac{1}{V^2}+\frac{1}{V'^2}-\frac{k}{g}=0$ or $\frac{1}{V'^2}=\frac{1}{V^2}+\frac{k}{g}$.

Que:- (1) A particle falls from rest in a medium in which the resistance is kv^2 per unit mass. Prove that the distance fallen in time t is $(1/k)\log\cos h\left\{t\sqrt{(gk)}\right\}$.

If the particle were ascending, show that at any instant its distance from the highest point of its path is $(1/k)\log\sec\left\{t\sqrt{(gk)}\right\}$, where t now denotes the time it will take to reach its highest point.

Solution:- When the particle is falling vertically downwards, let x be its distance from the starting point after time t. If v is its velocity at this point, then the resistance on the particle is mkv^2 in the vertically upwards direction. The weight mg of the particle acts vertically downwards.

$\therefore$ The equation of motion of the particle during the downward motion is $m\frac{d^2x}{dt^2}=mg-mkv^2$

or $\frac{d^2x}{dt^2}=g-kv^2$ or $v\frac{dv}{dx}=g-kv^2$ $\left[\because \frac{d^2x}{dt^2}=\frac{dv}{dt}\right]$

Or $\frac{dv}{g-kv^2}=dt$ or $\frac{dv}{k\left[(g/k)-v^2\right]}=dt$.

Integrating, we get $\frac{1}{k}\cdot\frac{1}{\sqrt{(g/k)}}=t+C_1\tan h^{-1}\frac{v}{\sqrt{(g/k)}}$.

But initially when $t=0,\ v=0$; $\therefore\ C_1=0$.

$\therefore$ $\frac{1}{\sqrt{(gk)}}\tan^{-1}\frac{v}{\sqrt{(g/k)}}=t$ or $\tanh^{-1}\frac{v}{\sqrt{(g/k)}}=t\sqrt{(g/k)}$ or $\frac{v}{\sqrt{(g/k)}}=\tan h\left\{t\sqrt{(gk)}\right\}$

Or $v=\sqrt{\left(\frac{g}{k}\right)}\cdot\frac{\sin h\left\{t\sqrt{(gk)}\right\}}{\cos h\left\{t\sqrt{(gk)}\right\}}$ or $\frac{dx}{dt}=\sqrt{\left(\frac{g}{k}\right)}\cdot\frac{1}{\sqrt{(gk)}}\cdot\frac{\sqrt{(gk)}\,.\sinh\left\{t\sqrt{(gk)}\right\}}{\cos h\left\{t\sqrt{(gk)}\right\}}$

Or $dx=(1/k)\,.\frac{\sqrt{(gk)}.\sin h\left\{t\sqrt{(gk)}\right\}}{\cos h\left\{t\sqrt{(gk)}\right\}}dt$.

Integrating, we get $x=(1/k).\log\cos h\left\{t\sqrt{(gk)}\right\}+C_2$

But initially when $t=0,\ x=0$.

$\therefore$ $0=(1/k).\log\cos h0+C_2=(1/k).\log1+C_2=0+C_2$.

$\therefore$ $C_2=0$.

$\therefore \quad x=(1/k)\log\cos h\left\{t\sqrt{(gk)}\right\}$, which proves the first part of the question.

Vertically Upwards Motion:- When the particle is ascending vertically upwards, let y be its distance from the starting point after time T. If v is its velocity at this point ,then the resistance is mkv^2 in the downward direction. The weight mg of the particle also acts vertically downwards.

$\therefore$ The equation of motion of the particle during the upwards motion is $m\dfrac{d^2v}{dT^2}=-mg-mkv^2$

or $\dfrac{d^2y}{dT^2}=-\left(g+kv^2\right)$ or $\dfrac{dv}{dT}=-\left(g+kv^2\right)$ $\qquad \left[\because \dfrac{d^2y}{dT^2}=\dfrac{dv}{dT}\right]$

Or $\dfrac{dv}{g+kv^2}=-dT$ or $\dfrac{dv}{k\left[(g/k)+v^2\right]}=-dT$.

Integrating, we get $\dfrac{1}{k}\cdot\dfrac{1}{\sqrt{(g/k)}}\tan^{-1}\dfrac{v}{\sqrt{(g/k)}}=-T+C_1$.

Let t_1 be the time from the point of projection to reach the highest point. Then $T=t_1,\ v=0$

$\therefore \quad 0=-t_1+C_1$ or $C_1=t_1$

$\therefore \quad \dfrac{1}{\sqrt{(g/k)}}\tan^{-1}\dfrac{v}{\sqrt{(g/k)}}=t_1-T$ or $\tan^{-1}\dfrac{v}{\sqrt{(g/k)}}=(t_1-T)\sqrt{(gk)}$

Or $\dfrac{v}{\sqrt{(g/k)}}=\tan\left\{(t_1-T)\sqrt{(gk)}\right\}$ or $v=\dfrac{dv}{dT}=\sqrt{\left(\dfrac{g}{k}\right)}.\tan\left\{(t_1-T)\sqrt{(gk)}\right\}$ (1)

If h is the greatest height attained by the particle and x be the depth below the highest point of the point distant y from the point of projection, then $x=h-y$ (2)

Also if t denotes the time from the distance y from the point of projection to reach the highest point, then $t=t_1-T$

(3)

From (2), we have $dx=-dy$ and from (3), we have $dt=-dT$.

$\therefore \quad \dfrac{dx}{dt}=\dfrac{dy}{dT}$.

$\therefore$ From (1), we have $\dfrac{dx}{dt}=\sqrt{\left(\dfrac{g}{k}\right)}.\tan\left\{t\sqrt{(gk)}\right\}$

Integrating, we get $x=\sqrt{\left(\dfrac{g}{k}\right)}.\dfrac{\log\sec\left\{t\sqrt{(gk)}\right\}}{\sqrt{(gk)}}+C_2$

$\left[\because \int\tan x\,dx=\log\sec x\right]$

$x=(1/k)\log\sec\left\{t\sqrt{(gk)}\right\}+C_2$

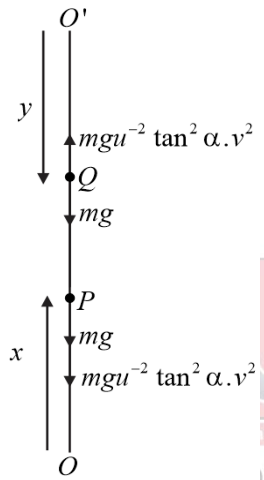
But from (2) and (3), it is obvious that $x=0$, when $t=0$.

$\therefore \quad 0=(1/k)\log\sec 0+C_2$ or $C_2=0$

$\therefore \quad x = (1/k)\log \sec\left\{t\sqrt{(gk)}\right\}$, which gives the required distance x of the particle from the highest point.

Que:- (2) A particle projected upwards with velocity u in a medium, the resistance of which is $gu^{-2}\tan^2\alpha$ times the square of the velocity, α being a constant. Show that the particle will return to the point of projection with velocity $u\cos\alpha$ after a time $ug^{-1}\cot\alpha\left[\alpha + \log\dfrac{\cos\alpha}{1-\sin\alpha}\right]$.

Solution:- Upward motion. Let a particle of mass m be projected vertically upwards from the point O with velocity u. If v is the velocity of the particle at time t at the point P such that $OP = x$,then resistance at P is $mgu^{-2}\tan^2\alpha.v^2$ acting in vertically downward direction. Since the weight mg of the particle also acts vertically downwards, therefore the equation of motion of the particle during its upwards motion is $m\dfrac{d^2x}{dt^2} = -mg - mgu^{-2}\tan^2\alpha.v^2$ or $\dfrac{d^2x}{dt^2} = -gu^{-2}\tan^2\alpha\left(u^2\cot^2\alpha + v^2\right)$.



From (1), we have $v\dfrac{dv}{dx} = -gu^{-2}\tan^2\alpha\left(u^2\cot^2\alpha + v^2\right)$ or

$$dx = -\frac{u^2\cot^2\alpha}{2g}.\frac{2v\,dv}{v^2 + u^2\cot^2\alpha}$$

Integrating, we have $x = -\dfrac{u^2\cot^2\alpha}{2g}\log\left(v^2 + u^2\cot^2\alpha\right) + A$,where A is a constant.

But at $O, x = 0$ and $v = u$.

$$\therefore \quad 0 = -\frac{u^2\cot^2\alpha}{2g}\log\left(u^2\operatorname{cosec}^2\alpha\right) + A \text{ or } A = \frac{u^2\cot^2\alpha}{2g}\log\left(u^2\operatorname{cosec}^2\alpha\right).$$

$$\therefore \quad x = -\frac{u^2\cot^2\alpha}{2g}\log\left(v^2 + u^2\cot^2\alpha\right) + \frac{u^2\cot^2\alpha}{2g}\log\left(u^2\operatorname{cosec}^2\alpha\right)$$

Or $x = \dfrac{u^2\cot^2\alpha}{2g}\log\dfrac{u^2\operatorname{cosec}^2\alpha}{v^2 + u^2\cot^2\alpha}$.

If the particle rises to the point O' such that $OO' = h$, then at $O', x = h\,v = 0$.

$$\therefore \quad h = \frac{u^2\cot^2\alpha}{2g}\log\frac{u^2\operatorname{cosec}^2\alpha}{u^2\cot^2\alpha} = \frac{u^2\cot^2\alpha}{2g}\log\sec^2\alpha \qquad (2)$$

To find the time from O to O', the equation (1) can be written as $\frac{dv}{dt}=-gu^{-2}\tan^2\alpha\left(u^2\cot^2\alpha+v^2\right)$ or $dt=-\frac{u^2\cot^2\alpha}{g}\frac{dv}{u^2\cot^2\alpha+v^2}$.

Let t_1 be the time from O to O'. Then from O to O', t varies from 0 to t_1 and v varies from u to 0. So integrating from O to O' we get $\int_0^{t_1} dt=-\frac{u^2\cot^2\alpha}{g.u\cot\alpha}\left[\tan^{-1}\frac{v}{u\cot\alpha}\right]_u^0$

Or $t_1=\frac{u}{g}\cot\alpha.\tan^{-1}\tan\alpha\frac{u\alpha}{g}\cot\alpha$ (3)

Downward Motion:- Now from the highest point O', the particle falls downwards. If y is its distance after time t from O' and if v is the velocity there, then the total resistance at this point is $mgu^{-2}\tan^2\alpha.v^2$, acting vertically upwards.

The weight mg of the particle acts vertically downwards.

$\therefore$ The equation of motion of the particle during its downwards motion is $m\frac{d^2y}{dt^2}=mg-mgu^{-2}\tan^2\alpha.v^2$ or $\frac{d^2y}{dt^2}=gu^{-2}\tan^2\alpha\left(u^2\cot^2\alpha-v^2\right)$ (4)

Or $v\frac{dv}{dy}=gu^{-2}\tan^2\alpha\left(u^2\cot^2\alpha-v^2\right)$ or $dy=-\frac{u^2\cot^2\alpha}{2g}.\frac{-2v\,dv}{u^2\cot^2\alpha-v^2}$

Integrating, we have $y=-\frac{u^2\cot^2\alpha}{2g}\log\left(u^2\cot^2\alpha-v^2\right)+B$

But at O', $y=0$ and $v=0$

$\therefore$ $0=-\frac{u^2\cot^2\alpha}{2g}\log\left(u^2\cot^2\alpha\right)+B$

Substituting, we get $y=\frac{u^2\cot^2\alpha}{2g}0\log\frac{u^2\cot^2\alpha}{u^2\cot^2\alpha-v^2}$

If v_1 is the velocity at the lowest point O, then at $o, y=h$, $v=v_1$.

$\therefore$ $h=\frac{u^2\cot^2\alpha}{2g}.\log\frac{u^2\cot^2\alpha}{u^2\cot^2\alpha-v_1^2}$ (5)

From (2) and (5), we have $\frac{u^2\cot^2\alpha}{2g}.\log\sec^2\alpha=\frac{u^2\cot^2\alpha}{2g}.\log\frac{u^2\cot^2\alpha}{u^2\cot^2\alpha-v_1^2}$ or

$\sec^2\alpha=\frac{u^2\cot^2\alpha}{u^2\cot^2\alpha-v_1^2}$ or $u^2\cot^2\alpha-v_1^2=u^2\cot^2\alpha.\cos^2\alpha$

or $v_1^2=u^2\cot^2\alpha.\left(1-\cos^2\alpha\right)=u^2\cot^2\alpha\sin^2\alpha=u^2\cos^2\alpha$ or $v_1=u\cos\alpha$ i.e. the particle returns to the point of projection with velocity $v_1=u\cos\alpha$. This proves the first part of the question. Again to find the time from O' to O, the equation (4) can be written as $\frac{dv}{dt}=gu^{-2}\tan^2\alpha\left(u^2\cot^2\alpha-v^2\right)$ or $dt=\frac{u^2}{g}\cot^2\alpha.\frac{dv}{u^2\cot^2\alpha-v^2}$.

Let t_2 be time from O' and O. Then from O' to O, t varies from 0 to t_2 and v varies from 0 to $u\cos\alpha$. Therefore integrating from O' to O, we have

$$\int_0^{t_2} dt = \frac{u^2}{g}\cot^2\alpha \, . \int_{v=0}^{v=u\cos\alpha} \frac{dv}{u^2\cot^2\alpha - v^2}.$$

$$\therefore \qquad t_2 = \frac{u^2\cot^2\alpha}{2gu\cot\alpha} . \left[\log\frac{u\cot\alpha + v}{u\cot\alpha - v}\right]_0^{u\cos\alpha}$$

$$= \frac{u}{2g}\cot\alpha . \left[\log\frac{u\cot\alpha + u\cos\alpha}{u\cot\alpha - u\cos\alpha} - \log 1\right]$$

$$= \frac{u}{2g}\cot\alpha \, .\log\frac{1+\sin\alpha}{1-\sin\alpha} = \frac{u}{2g}\cot\alpha . \log\frac{(1+\sin\alpha).(1-\sin\alpha)}{(1-\sin\alpha).(1-\sin\alpha)}$$

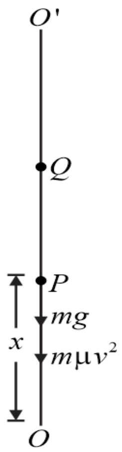
$$= \frac{u}{2g}\cot\alpha . \log\frac{(1-\sin^2\alpha)}{(1-\sin\alpha)^2} = \frac{u}{2g}\cot\alpha . \log\left(\frac{\cos\alpha}{1-\sin\alpha}\right)^2$$

$$= \frac{u}{g}\cot\alpha . \log\frac{\cos\alpha}{1-\sin\alpha}.$$

$\therefore$ The required time $= t_1 + t_2 = \dfrac{\mu}{g}\cot\alpha\left[\alpha + \log\dfrac{\cos\alpha}{1-\sin\alpha}\right].$

Que:- (3) A heavy particle is projected vertically upwards in a medium the resistance of which varies as the square of velocity. If it has a kinetic energy K in its upwards path at a given point, when it passes the same point on the way down, show that its loss of energy is $\dfrac{K^2}{K+K'}$, where K' is the limit to which the energy approaches in its downwards course.

Solution:- Let a particle of mass m be projected vertically upwards with a velocity u from the point O. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is $m\mu v^2$ acting vertically downwards. The weight mg of the particle also acts vertically downwards.



$\therefore$ The equation of motion of the particle during its upwards motion is $m\dfrac{d^2y}{dt^2} = -mg - m\,\mu v^2$

or $\dfrac{d^2x}{dt^2} = -g\left(1+\dfrac{\mu}{g}v^2\right)$ (1)

It H is the maximum height attained by the particle, then at the highest point O' the particle comes to rest and starts falling vertically downwards. If y is the distance fallen in time t from O' and v is the velocity of the particle at this point, then the resistance is $m\mu v^2$ acting vertically upwards.

$\therefore$ The equation of motion of the particle during its downwards motion is $m\dfrac{d^2y}{dt^2} = mg - m\mu v$

or $\dfrac{d^2y}{dt^2} = g - \mu v^2$ (2)

If V is the terminal velocity of the particle during its downward motion, then $d^2y/dt^2 = 0$ when $v = V$. Therefore $0 = g - \mu V^2$ or $\dfrac{\mu}{g} = \dfrac{1}{V^2}$ (3)

$\therefore$ From (2), the equation of motion of the particle in downward motion is $\dfrac{d^2y}{dt^2} = g\left(1-\dfrac{1}{V^2}v^2\right)$

or $v\dfrac{dv}{dy} = \dfrac{g}{V^2}\left(V^2 - v^2\right)$ or $\dfrac{-2v\,dv}{V^2 - v^2} = -\dfrac{2g}{V^2}dy$.

Integrating $\log\left(V^2 - v^2\right) = -\dfrac{2g}{V^2}y + A$, where A is a constant.

But at $O', y = 0$ and $v = 0$; $\therefore A = \log V^2$

$\therefore$ $\log\left(V^2 - v^2\right) = -\dfrac{2g}{V^2}y + \log V^2$ or $\dfrac{2gy}{V^2} = \log V^2 - \log\left(V^2 - v^2\right)$ or

$y = \dfrac{V^2}{2g}\log\left(\dfrac{V^2}{V^2 - v^2}\right)$ (4)

If v_1 is the velocity of the particle at the point Q at distance h from O', when falling downwards, then from (4) $h = \dfrac{V^2}{2g}\log\left(\dfrac{V^2}{V^2 - v_1^2}\right)$ (5)

Upward Motion:- When the particle is moving upwards from O, then from (1) with help of (3), the equation of motion of the particle is $\dfrac{d^2x}{dt^2} = -g\left(1+\dfrac{v^2}{V^2}\right)$ or $v\dfrac{dv}{dx} = -\dfrac{g}{V^2}\left(V^2 - v^2\right)$

or $\dfrac{2v\,dv}{V^2 + v^2} = -\dfrac{2g}{V^2}dx$,

Integrating, $\log\left(V^2 + v^2\right) = -\dfrac{2g}{V^2}x + B$, where B is a constant.

But at $O, x = 0, v = u$; $\therefore B = \log\left(V^2 + u^2\right)$ or $x = \dfrac{V^2}{2g}\log\left(\dfrac{V^2 + v^2}{V^2 + u^2}\right)$ (6)

If v_2 is the velocity of the particle at the point Q in its upward motion then at $Q, x = OQ = H - h, v = v_2$

$\therefore$ $$H - h = \left(\frac{V^2}{2g}\right)\log\left(\frac{V^2 + u^2}{V^2 + v_2^2}\right) \qquad (7)$$

Since H is the maximum height attained by the particle therefore putting $x = H$ and $v = 0$

in (6), we get $H = \frac{V^2}{2g}\log\left(\frac{V^2 + u^2}{V^2}\right)$ (8)

Substituting the values of h and H from (5) and (8) in (7), we get $\frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2} - \frac{V^2}{2g}\log\frac{V^2}{V^2 - v_1^2} = \frac{V^2}{2g}\log\frac{V^2 + u^2}{V^2 + v_2^2}$

or $\log\frac{\left(V^2 + u^2\right)}{V^2} - \log\left(\frac{V^2 + u^2}{V^2 + v_2^2}\right) = \log\left(\frac{V^2}{V^2 - v_1^2}\right)$

or $\log\left\{\left(\frac{V^2 + u^2}{V^2}\right) \cdot \left(\frac{V^2 + v_2^2}{V^2 + u^2}\right)\right\} = \log\frac{V^2}{V^2 - v_1^2}$

or $\frac{V^2 + v_2^2}{V^2} = \frac{V^2}{V^2 - v_1^2}$ or $\left(V^2 + v_2^2\right)\left(V^2 - v_1^2\right) = V^4$ or $\left(V^2 + v_2^2\right)V^2 - \left(V^2 - v_2^2\right)v_1^2 = V^4$ or

$$v_1^2 = \frac{v_2^2 V^2}{V^2 + v_2^2} \qquad (9)$$

Now the kinetic energy K of the particle at the point Q at depth h below O' during its upwards motion $= \frac{1}{2}mv_2^2$ and $K.E$ at Q during downward motion $\frac{1}{2}mv_1^2$.

Also the terminal $K.E. = \frac{1}{2}mV^2$.

The required loss of $K.E. = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

$= \frac{1}{2}m\left[v_2^2 - \frac{v_2^2 V^2}{V^2 + v_2^2}\right]$, substituting for v_1^2 from (9)

$= \frac{m}{2} \cdot \frac{v_2^4}{V^2 + v_2^2} = \frac{\left(\frac{1}{2}mv_2^2\right)^2}{\frac{1}{2}mV^2 + \frac{1}{2}mv_2^2} = \frac{K^2}{K' + K}$, where $K' = \frac{1}{2}mV^2 =$ limiting $K.E.$ in the medium.

Que:- (4) A particle is projected vertically upwards. Prove that if the resistance of air were constant and equal to $(1/n)$th of its weight, the time of ascent and descent would be as $(n-1)^{1/2} : (n+1)^{1/2}$.

Solution:- Suppose a particle of mass m is projected vertically upwards from O with velocity u. Let P be its position at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are,

(i) The weight mg of the particle acting vertically downwards and

(ii) The forces of resistance $(1/n)mg$ acting vertically upwards.

The equation of motion of the particle at time t is $m\frac{d^2x}{dt^2} = -mg - \frac{1}{n}mg$ or

$$\frac{d^2x}{dt^2} = -\left(\frac{n+1}{n}\right)g \qquad (1)$$

The equation (1) can be written as $\frac{dv}{dt} = -\left(\frac{n+1}{n}\right)g$

$$\therefore \quad dt = -\left(\frac{n}{n+1}\right)\frac{1}{g}dv. \qquad (2)$$

Let O' be the point of maximum height i.e. the velocity of the particle becomes zero at O'. Let $OO' = h$ and let t_1 be time of ascent from O to O'. Then from (2), we have

$$\int_0^{t_1} dt = -\left(\frac{n}{n+1}\right)\frac{1}{g}\int_u^0 dv = \left(\frac{n}{n+1}\right)\frac{1}{g}\int_0^u dv \text{ or } t_1 = \left(\frac{n}{n+1}\right)\frac{u}{g}. \qquad (3)$$

Again the equation (1) can also be written as $v\frac{dv}{dx} = -\left(\frac{n+1}{n}\right)g$.

$$\therefore \quad dx = -\left(\frac{n}{n+1}\right)\frac{1}{g}c\,dv.$$

Integrating from O to O', we get $\int_0^h dx = -\left(\frac{n}{n+1}\right)\frac{1}{g}\int_u^0 v\,dv$ or $h = \left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u^2}{2}$.

During the downwards motion from O' to O, the equation of motion of the particle I s

$$\frac{d^2x}{dt^2} = g - \frac{g}{n} = \left(\frac{n-1}{n}\right)g. \qquad (4)$$

The equation (4) can be written as $v\frac{dv}{ds} = \left(\frac{n-1}{n}\right)g$

$$\therefore \quad dx = \left(\frac{n}{n-1}\right)\frac{1}{g}\,vdv \qquad (5)$$

Suppose the particle reaches back O with velocity u_1. Then integrating (5) from O' to O we get $\int_0^h dx = \left(\frac{n}{n-1}\right)\frac{1}{g}\int_0^{u_1} v\,dv$ or $h = \left(\frac{n}{n-1}\right)\frac{1}{g}\cdot\frac{u_1^2}{2}$ or $\left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u^2}{2} = \left(\frac{n}{n+1}\right)\frac{1}{g}\cdot\frac{u_1^2}{2}$

Substituting for h or $u_1^2 = \left(\frac{n-1}{n+1}\right)u^2$ or $u_1 = \sqrt{\left(\frac{n-1}{n+1}\right)}u$.

Now the equation (4) can also be written as $\frac{dv}{dt} = \left(\frac{n-1}{n}\right)g$.

$$\therefore \qquad dt = \left(\frac{n}{n-1}\right)\frac{1}{g}\,dv \qquad (6)$$

Let t_2 be the time of descent from O' to O. Then integrating (6) from O' to O, we get

$$\int_0^{t_2} dt = \left(\frac{n}{n-1}\right)\frac{1}{g}\int_0^{u_1} dv \text{ or } t_2 = \left(\frac{n}{n-1}\right)\frac{1}{g}u_1 = \left(\frac{n}{n-1}\right)\frac{1}{g}.\sqrt{\left(\frac{n-1}{n+1}\right)}u.$$

$$= \frac{u}{g}.\frac{n}{\sqrt{(n-1)}\sqrt{(n+1)}}.$$

$$\therefore \qquad \frac{t_1}{t_2} = \left(\frac{n}{n+1}\right)\frac{u}{g}.\frac{g\sqrt{(n-1)}\sqrt{(n+1)}}{u\,n} = \frac{\sqrt{(n-1)}}{\sqrt{(n+1)}}.$$

Hence $t_1 : t_2 = (n-1)^{1/2} : (n+1)^{1/2}$.

Constrained Motion

Example 1:- A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension in the string has the values mW and nW respectively, when the particle is at the highest and lowest point in the path. Show that $n = m + 6$.

Solution:- Let M be the mass of the particle. Then $W = Mg$ i.e., $M = W / g$

Proceeding as in 2, in tension T in string in any position is given by $T = \frac{M}{a}\left(u^2 - 2ag + 3ag\cos\theta\right)$ [See equation (5) of 2 and deduce it here]

Or $T = \frac{W}{ag}\left(u^2 - 2ag + 3ag\cos\theta\right)$ (1)

Now mW is given to the tension in the string at the highest point and nW that at the lowest point. Therefore $T = mW$ when $\theta = \pi$ and $T = nW$ when $\theta = 0$. So from (1), we have

$mW = \frac{W}{ag}\left(u^2 - 2ag + 3ag\cos\pi\right)$ given $m = \frac{1}{ag}\left(u^2 - 5ag\right)$ (2)

And $nW = \frac{W}{ag}\left(u^2 - 2ag + 3ag\cos 0\right)$ given $n = \frac{1}{ag}\left(u^2 + ag\right)$ (3)

Subtracting (2) from (3), we have $n - m = 6$ or $n = m + 6$.

Example2:- A heavy particle hangs from a fixed point O, by a string of length a. It is projected horizontally with a velocity $v^2 = \left(2 + \sqrt{3}\right)ag$; show that the string becomes slack when it has described an angle $\cos^{-1}\left(-1/\sqrt{3}\right)$

Solution:- The equation of motion of the particle are $m\frac{d^2s}{dt^2} = -mg\sin\theta$ (1)

And $m\frac{v^2}{a} = T - mg\cos\theta$ (2)

Also $s = a\theta$ (3)

From (1) and (3), we have $a = \frac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying both sides by $2a\left(d\theta / dt\right)$ and then integrating w.r.t. t, we have

$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$, where A is the constant of integration.

But initially at A, $\theta = 0$ and $v^2 = \left(2 + \sqrt{3}\right)ag$.

$\therefore$ $\left(2 + \sqrt{3}\right)ag = 2ag\cos 0 + A$, given $A = \sqrt{3ag}$

$\therefore$ $v^2 = 2ag\cos\theta + \sqrt{3ag}$.

Substituting this value of v^2 in (2), we have $T = \frac{m}{a}\left[v^2 + ag\cos\theta\right]$

$\frac{m}{a}\left[3\sqrt{ag} + 3ag\cos\theta\right]$ (4)

The string becomes slack when $T=0$.

$\therefore$ from (4), we have $0=\frac{m}{a}\left[\sqrt{3ag}+3ag\cos\theta\right]$

$\cos\theta=-1/\sqrt{3}$ or $\theta=\cos^{-1}\left(-1/\sqrt{3}\right)$.

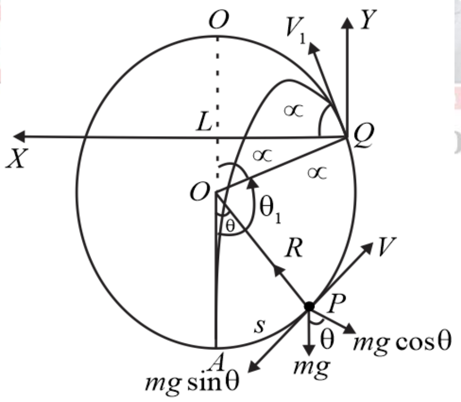
Example 3:- A particle inside and at the lowest point of a fixed smooth hollow sphere of radius is a projected horizontally with velocity $\sqrt{\left(\frac{7}{2}ag\right)}$. Show that it will leave the sphere at a height $\frac{3}{2}a$ above the lowest point and its subsequent path meets the sphere again at the point of projection.

Solution:- A particle is projected from the lowest point A to a sphere with velocity $u=\sqrt{\left(\frac{7}{2}ag\right)}$ to move along the inside of the sphere. Let P be the position of the particle at any time t where are $AP=s$ and $\angle AQP=\theta$. If v be the velocity of the particle at P, the equations of motion along the tangent and normal are $m\frac{d^2s}{dt^2}=-mg\sin\theta$ (1)

And $m\frac{v^2}{a}=R-mg\cos\theta$ (2)

$s=a\theta$ (3)

From (1) and (3), we have a $\frac{d^2\theta}{dt^2}=-g\sin\theta$.



Multiplying both sides by $2a\frac{d\theta}{dt}$ and the integrating, we have $v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A$

But at the point A, $\theta=0$ and $v=u=\sqrt{\left(\frac{7}{2}ag\right)}$.

$\therefore$ $A=\frac{7}{2}ag-2ag=\frac{3}{2}ag$.

$\therefore$ $v^2=\frac{3}{2}ag+2ag\cos\theta$ (4)

Now from (2) and (4), we have $R=\frac{m}{a}\left[v^2+ag\cos\theta\right]=\frac{m}{a}\left[\frac{3}{2}ag+2ag\cos\theta+ag\cos\theta\right]$

$$= 3mg\left(\frac{1}{2} + \cos\theta\right)$$

If the particle leaves the sphere at the point Q, where $\theta = 0$, then $0 = 3mg\left(\frac{1}{2} + \cos\theta_1\right)$ or

$\cos\theta_1 = -\frac{1}{2}$. If $\angle COQ = \propto$, then $\propto = \pi + \theta_1$

$\therefore \quad \cos\propto = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{1}{2}$

$\therefore \quad AL = AO + OL = a + a\cos\propto = a + \frac{a}{2} = \frac{3a}{2}$ i.e. the particle leaves the sphere at a height $\frac{3}{2}a$ above the lowest point.

If v_1 is the velocity of the particle at the point Q, then putting $v = v_1$ $R = 0$ and $\theta = \theta_1$ in (2), we get $v_1^2 = -ag\cos\theta_1 = -ag.\left(-\frac{1}{2}\right) = \frac{1}{2}ag$, the particle leaves the sphere at the point Q with velocity $v_1 = \sqrt{\left(\frac{1}{2}ag\right)}$ making an angle $\propto$ with the horizontal and subsequently describes a parabolic path.

The equation of the parabolic trajectory w.r.t QX and QY as co-ordinates axes is $y = x\tan\propto - \frac{1}{2}\frac{gx^2}{v_1^2\cos^2\propto}$ of $y = x.\sqrt{3} - \frac{gx^2}{2.\frac{1}{2}ag.\frac{1}{4}}$ $\left[\because \cos\propto = \frac{1}{2}\right.$ and so $\sin\propto = \sqrt{(1-\cos^2\propto)} = \sqrt{3/2}$. Thus $\tan\propto = a\sqrt{3}\,]$

Or $y = \sqrt{3x} - \frac{4x^2}{a}$ (6)

From the figure, for the point A, $x = QL = a\sin\propto = a\sqrt{3/2}$ and $y = -LA = -\frac{3}{2}a$.

If we put $x = a\sqrt{3/2}$ in the equation (6), we get $y = a\frac{\sqrt{3}}{2}.\sqrt{3} - \frac{4}{a}.\frac{3a^2}{4} = \frac{3a}{2} - 3a = -\frac{3}{2}a$

Thus the co-ordinates of the point A satisfy the equation (6). Hence the particle, after leaving the sphere at Q, describes a parabolic path which meets the sphere again at the point of projection A.

Example 4:- Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius a from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.

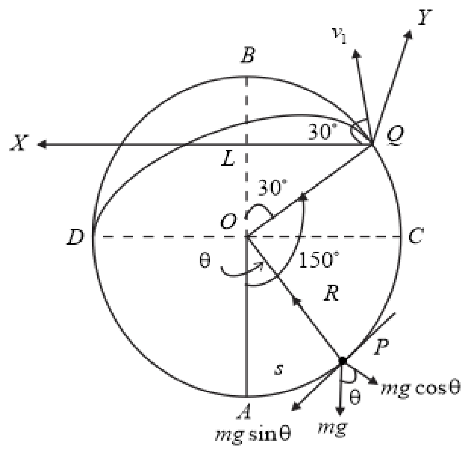
Solution:- Let a particle of mass m be projected with velocity u from the lowest point A of a smooth circular hoop of radius a along the interior of the hoop. If P is its position at any time t such that $\angle AOP = \theta$ and arc $AP = x$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{st^2} = -mg\sin\theta \qquad (1)$$

And $m\frac{v^2}{a} = R - mg\cos\theta$ (2)

Also $x = a\theta$ (3)

From (1) and (3), we have $a\frac{d^2\theta}{dt^2} = -g\sin\theta$



Multiplying both sides by $2a\frac{d\theta}{dt}$ and then integrating, we have $v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$

But the point A, $\theta = 0$, and $v = u$ $\therefore\ A = u^2 - 2ag$

$\therefore$ $v^2 = u^2 - 2ag + 2ag\cos\theta$ (4)

From (2) and (4), we have $-R = \frac{m}{a}\left(v^2 + ag\cos\theta\right)$

$= \frac{m}{a}\left(u^2 - 2ag + 3ag\cos\theta\right)$

If the particle leaves the circular hoop at the point Q where $\theta = 150^\circ$, then `

$0 = \frac{m}{a}\left(u^2 - 2ag + 3ag\cos 150^\circ\right)$ or $0 = u^2 - 2ag - \frac{3\sqrt{3}}{2}ag$.

$$u = \left[\frac{1}{2}ag\left(4 + 3\sqrt{3}\right)\right]^{1/2}$$

Hence the particle will leaves the circular hoop at an angular distance of 30° from the initial velocity of projection is $u = \left[\frac{1}{2}ag\left(4 + 3\sqrt{3}\right)\right]^{1/2}$.

Again $OL = OQ\cos 30^\circ = a\left(\sqrt{3/2}\right)$ and $QL = OQ\sin 30^\circ = a/2$.

If v_1 is the velocity of the particle at the point Q, then $v = v_1$ when $\theta = 150^\circ$. Therefore from (4), we have $v_1^2 = \frac{1}{2}ag\left(4 + 3\sqrt{3}\right) - 2ag + 2ag\cos 150^\circ = \frac{1}{2}ag\sqrt{3}$ so that $v_1 = \left(\frac{1}{2}ag\sqrt{3}\right)^{1/2}$

Thus the particle leaves the circular hoop at the point Q, with velocity $v_1 = \frac{1}{2}\left(\sqrt{3ag}\right)^{1/2}$ at an angle 30° to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinates axes is $y = x\tan 30^\circ - \dfrac{gx^2}{2v_1^2\cos^2 30^\circ} = \dfrac{x}{\sqrt{3}} \dfrac{gx^2}{2.\frac{1}{2}\sqrt{3ag}.\left(\sqrt{3/2}\right)^2}$ or $y = \dfrac{x}{\sqrt{3}} - \dfrac{4x^2}{3\sqrt{3}a}$ (5)

For the point D which is extremity of the horizontal diameter CD ,we have $x = QL + OD = \dfrac{1}{2}a + a = 3a/2, y = -LO = -a\sqrt{3/2}$.

Clearly the co-ordinates of the point D satisfying the equation (5). Hence the particle after leaving the circular hoop at Q, strikes the hoop again at an extremity of the horizontal diameter.

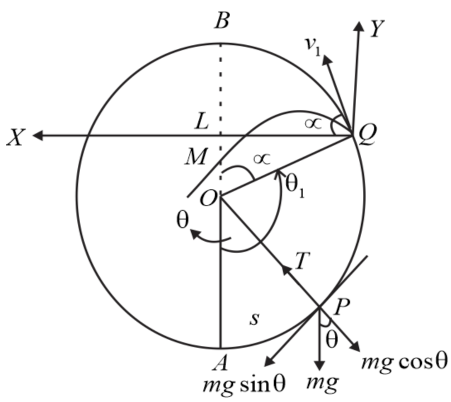
Example 5:- A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity $\sqrt{(2gh)}$. If $\dfrac{5a}{2} > h > a$, prove that the circular motion ceases when the particle has reached the height $\dfrac{1}{3}(a+2h)$. Prove also that the greatest height ever reached by the particle above the point of projection is $\dfrac{(4a-h)(a+2h)^2}{27a^2}$.

Solution:- Let a particle of mass m be attached to one end of a string of length a whose other end is fixed at O. The particle is projected horizontally with a velocity $u = \sqrt{(2gh)}$ from A. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$,then the equations of motion of the particle are $m\dfrac{d^2s}{dt^2} = -mg\sin\theta$ (1)

And $m\dfrac{v^2}{a} = T - mg\cos\theta$ (2)

Also $s = a\theta$ (3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$



Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have $v^2 = \left(a\dfrac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$.

But at the point A, $\theta = 0$,and $v = u - \sqrt{(2gh)}$.

$\therefore$ $A = 2gh - 2ag$

$\therefore \qquad v^2 = 2ag\cos\theta + 2gh - 2ag \qquad (4)$

From (2) and (4), we have $T = \frac{m}{a}\left(v^2 + ag\cos\theta\right) = \frac{m}{a}\left(3ag\cos\theta + 2gh - 2ag\right)$

If the particle leaves the circular path at Q where $\theta = \theta_1$, then $T = 0$ when $\theta = \theta_1$.

$\therefore \qquad 0 = \frac{m}{a}\left(3ag\cos\theta_1 + 2gh - 2ag\right)$ or $\cos\theta_1 = -\frac{2h-2a}{3a}$

Since $\frac{5}{2}a > h > a$ i.e. $5a > 2h > 2a$, therefore $\cos\theta_1$ is negative and its absolute value is < 1. So θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circular path at Q before arriving at the highest point.

Height of the point Q above A.

$= AL = AO + OL = a + a\cos(\pi - \theta) = a - a\cos\theta_1$

$= a + a.\frac{2h-2a}{3a} = \frac{1}{3}(a + 2h)$ i.e. the particle leaves the circular path when it has reached a height $\frac{1}{3}(a+2h)$ above the point of projection.

If v_1 is the velocity of the particle at the point Q, then from (4), we have $v_1^2 = 2ag\cos\theta_1 + 2gh - 2ag$

$= -2ag.\frac{(2h-2a)}{3a} + 2g(h-a)$

$= 2g(h-a)\left(1 - \frac{2}{3}\right) = \frac{2}{3}g(h-a)$.

If $\angle LOQ = x$, then $\propto = \pi - \theta_1$.

$\therefore \qquad \cos\propto \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{2(h-a)}{3a}$.

Thus the particle leaves the circular path at the point Q with velocity $v_1 = \sqrt{\left\{\frac{2}{3}g(h-a)\right\}}$ at an angle $\cos\propto = \cos^{-1}\left\{2(h-a)/3a\right\}$ to the horizontal and will subsequently describe a parabolic path.

Maximum height of the particle above the point Q

$= H = \frac{v_1^2\sin^2 x}{2g} = \frac{v_1^2}{2g}\left(1 - \cos^2\propto\right) = \frac{1}{3}(h-a)\left[1 - \frac{4}{9a^2}(h-a)^2\right]$

$= \frac{1}{27a^2}(h-a)\left[9a^2 - 4\left(h^2 - 2ah + a^2\right)\right]$

$= \frac{(h-a)}{27a^2}\left[5a^2 + 8ah - 4h^2\right] = \frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$

$\therefore$ Greatest height ever reached by the particle above the point of projection A.

$= AL + H = \frac{1}{3}(a+2h) + \frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$

$= \frac{1}{27a^2}(a+2h)\left[9a^2 + (h-a)(5a-2h)\right]$

$$= \frac{1}{27a^2}(a+2h)\left[4a^2+7ah-2h^2\right]$$
$$= \frac{1}{27a^2}(a+2h)(a+2h)(4a-h) = \frac{1}{27a^2}(4a-h)(a+2h)^2 .$$

Example 6:- A particle is free to move on a smooth vertical circular wire of radius a. If is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{(a/g)}.\log\left(\sqrt{5}+\sqrt{6}\right)$.

Solution:- Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.

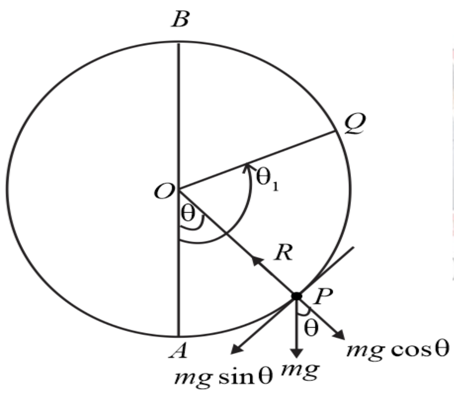
If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad (1)$$

And $$m\frac{v^2}{a} = R - mg\cos\theta \qquad (2)$$

Also $s = a\theta$ (3)

From (1) and (3) we have a $\frac{d^2\theta}{dt^2} = -g\sin\theta$



Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

But according to the question $v-0$ at the highest point B, where $\theta = \pi$.

$\therefore$ $0-2ag\cos\pi + A$ or $A = 2ag$

$$\therefore \quad v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag \qquad (4)$$

From (2) and (4), we have $R = \frac{m}{a}\left(v^2 + ag\cos\theta\right) = \frac{m}{a}\left(2a+3ag+\cos\theta\right)$ (5)

If the reaction $R=0$ at the point Q where $\theta = \theta_1$, then from (5), we have

$$0 = \frac{m}{a}\left(2ag + 3ag\cos\theta_1\right) \text{ or } \cos\theta_1 = -2/3 \qquad (6)$$

From (4), we have $\left(a\frac{d\theta}{dt}\right)^2 = 2ag(\cos\theta+1) = 2ag\,.2\cos^2\frac{1}{2}\theta = 2ag\cos^2\frac{1}{2}\theta$.

$\therefore$ $\frac{d\theta}{dt} = 2\sqrt{(g/a)} \cos\frac{1}{2}\theta$, the positive sign being taken before the radical sign because θ increases as t increases or $dt = \frac{1}{2}\sqrt{(a/g)} \sec\frac{1}{2}\theta \, d\theta$

Integrating from $\theta = 0$ to $\theta = \theta_1$, the required time t is given by $t = \frac{1}{2}\sqrt{(a/g)} \int_{\theta-0}^{\theta_1} \sec\frac{1}{2}\theta \, d\theta$

Or $t = \sqrt{(a/g)} \left[\log\left(\sec\frac{1}{2}\theta + \tan\frac{1}{2}\theta \right) \right]_0^{\theta_1}$

Or $t = \sqrt{(a/g)} \log\left(\sec\frac{1}{2}\theta_1 + \tan\frac{1}{2}\theta_1 \right)$ (7)

From (6), we have $2\cos^2\frac{1}{2}\theta_1 - 1 = -\frac{2}{3}$

Or $2\cos^2\frac{1}{2}\theta_1 = 1 - \frac{2}{3} = \frac{1}{3}$

Or $\cos^2\frac{1}{2}\theta_1 = \frac{1}{6}$ or $\sec^2\frac{1}{2}\theta_1 = 6$

$\therefore$ $\sec\frac{1}{2}\theta_1 = \sqrt{6}$ and $\tan\frac{1}{2}\theta_1 = \sqrt{\left(\sec^2\frac{1}{2}\theta_1 - 1 \right)} = \sqrt{(6-1)} = \sqrt{5}$

Substituting in (7), the required time is given by $t = \sqrt{(a/g)} \log\left(\sqrt{6} + \sqrt{5} \right)$.

Example 7:- A particle is placed on the outside of a smooth vertical circle. If the particle starts from a point whose angular distance is $\propto$ from the highest point of circle, show that it will fly off the curve when $\cos\theta = \frac{2}{3}\cos\propto$.

Solution:- A particle slides down on the outside of the arc of a smooth vertical circle of radius a, string from rest at a point B such that $AOB = \propto$. Let P be the position of the particle at any time t where arc $AP = s$ and $\angle POA = \theta$. The forces acting on the particle at P arc: (i) weight mg acting vertically downwards and (ii) the reaction R along the outwards drawn normal OP.

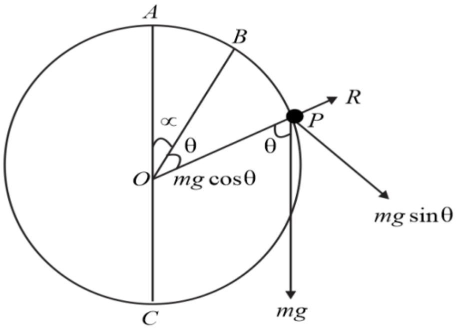
If v be the velocity of the particle at P, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2 s}{dt^2} = mg\sin\theta \quad (1)$$

And $$m\frac{v^2}{a} = mg\cos\theta - R \quad (2)$$

Also $s = a\theta$ (3)

From (1) and (3), we have $a\frac{d^2\theta}{dt^2} = g\sin\theta$



Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = -2ag\cos\theta + A$$

But initially at $B,\ \theta = \propto$ and $v = 0$ $\qquad\qquad \therefore\ A = 2ag\cos\propto$

$\therefore \qquad v^2 = 2ag\cos\propto - 2ag\cos\theta \qquad\qquad (4)$

From (2) and (4), we have $R = \frac{m}{a}\left(-v^2 + ag\cos\theta\right) = \frac{m}{a}\left(-2ag\cos\propto + 3ag\cos\theta\right)$

$= mg\left(-2\cos\propto + 3\cos\theta\right)$

At the point where the particle files off the circle, we have $R = 0$

$\therefore \qquad$ From (5), we have $0 = mg\left(-2\cos\propto + 3\cos\theta\right)$ or $\cos\theta = \frac{2}{3}\cos\propto$.

Example 8:- A particle is projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point of the outside of a fixed smooth sphere of radius a. Show that if will leave the sphere at the point whose vertical distance below the point of projection is $a/6$.

Solution:- Let a particle be projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point A on the outside of a fixed smooth sphere of radius a. If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2} - mg\sin\theta \qquad (1)$$

And $\qquad m\frac{v^2}{a} = mg\cos\theta - R \qquad (2)$

Here v is the velocity of the particle at P

Also $s = a\theta \qquad (3)$

From (1) and (3), we have $a\frac{d^2\theta}{dt^2} = g\sin\theta$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2\left(a\frac{d\theta}{dt}\right)^2 = -2ag\cos\theta + A$$

But initially at $A,\ \theta = 0$ and $v = \sqrt{(ag/2)}$

$\therefore \qquad ag/2 = -2ag + A$ or $A = \frac{1}{2}ag + 2ag = \frac{5}{2}ag$

$\therefore \qquad v^2 = \frac{5}{2}ag - 2ag\cos\theta \qquad (4)$

From (2) and (4), we have $R = \frac{m}{a}\left(ag\cos\theta - v^2\right) = \frac{m}{a}\left(3ag\cos\theta - \frac{5}{2}ag\right)$

Or $R = mg\left(3\cos\theta - \frac{5}{2}\right)$

If the particle leaves the sphere at the point Q where $\theta = \theta_1$ then putting $R = 0$ and $\theta = \theta_1$ in (5), we have $0 = mg\left(3\cos\theta_1 - \frac{5}{2}\right)$ or $\cos\theta_1 = 5/6$

Vertical depth of the point Q below the point of projection A.

$= AL = OA - OL = a - a\cos\theta_1 = a - \frac{5}{6}a = \frac{1}{6}a$.

Example 9:- A particle slides down a smooth cycloid whose axis is vertical and vertex downwards, starting from rest at the cusp. Find the velocity of the particle and the reaction on it at any point of the cycloid.

Solution:- Here particle starts at rest from the cusp A.

The equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad (1)$$

And $m\frac{v^2}{\rho} = R - mg\cos\psi \qquad (2)$

For the cycloid $s = 4a\sin\psi \qquad (3)$

From (1) and (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$.

Multiplying both sides by $2\frac{ds}{dt}$ and integrating, we have $v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$

But initially at the cusp. A, $s = 4a$ and $v = 0$

$\therefore \qquad A = \frac{g}{4a}.(4a)^2 = 4ag$

$\therefore \qquad v^2 = -\frac{g}{4a}s^2 + 4ag = -\frac{g}{4a}(4a\sin\psi)^2 + 4ag$

$= 4a\left(1 - \sin^2\psi\right)$

Or $v^2 = 4ag\cos^2\psi \qquad (4)$

Differentiating (3), $\rho = ds/d\psi = 4a\cos\psi$

Substituting for v^2 and ρ in (2), we have

$$R = m\frac{v^2}{\rho} + mg\cos\psi = m.\frac{4ag\cos^2\psi}{4a\cos\psi} + mg\cos\psi$$

Or $R = 2mg\cos\psi \qquad (5)$

The equations (4) and (5) give the velocity and the reaction at any point of the cycloid.

Example 10:- A particle oscillates cusp to cusp of a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P i.e. $v = V\cos\psi$.

Solution:- The velocity v of the particle at any point P of the cycloid is given by $v = 2\sqrt{(ag)}\cos\psi$.

[From equation (4)]

If V is the velocity of the particle at the vertex, where $\psi = 0$, then

$V = 2\sqrt{(ag)}\cos 0 = 2\sqrt{(ag)}$.

$\therefore$ $v = V\cos\psi =$ the resolved part of V along the tangent at ρ. Hence the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P.

Example 11:- A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and vertex downwards, prove that the magnitude of the acceleration is equal to g at every point of the path and the pressure when the particle arrives at the vertex is equal to twice the weight of the particle.

Solution:- Here the particle starts at rest from the cusp A.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad (1)$$

And $m\dfrac{v^2}{\rho} = R - mg\cos\psi$ (2)

For the cycloid, $s = 4a\sin\psi$ (3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2} = -\dfrac{g}{4a}s$

Multiplying both sides by $2(ds/dt)$ and integrating, we have $v^2 = \left(\dfrac{ds}{dt}\right)^2 = -\dfrac{g}{4a}s^2 + A$

But initially at the cusp A, $s = 4a$ and $v = 0$. $\therefore A = 4ag$.

$\therefore$ $v^2 = -\dfrac{g}{4a}s^2 + 4ag = -\dfrac{g}{4a}(4a\sin\psi)^2 + 4ag = 4ag\left(1 - \sin^2\psi\right)$

Or $v^2 = 4ag\cos^2\psi$ (4)

Differentiating (3), $\rho = ds/d\psi = 4a\cos\psi$

Now at the point P, tangent acceleration $= d^2s/dt^2 = -g\sin\psi$ [From (1)]

And normal acceleration $= \dfrac{v^2}{\rho} = \dfrac{4ag\cos^2\psi}{4a\cos\psi} = g\cos\psi$

$\therefore$ The resultant acceleration at any point P.

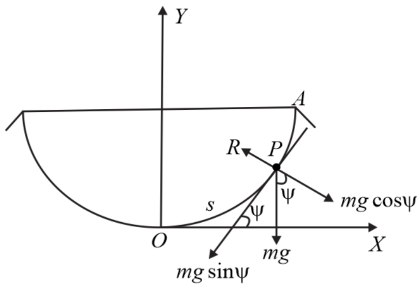
$= \sqrt{\ }$ [tangent. Acceleration)^2 +(normal acceleration)^2]

$= \sqrt{\left[(-g\sin\psi)^2 + (g\cos\psi)^2\right]} = g$

From (2) and (4), we have $R = m.\dfrac{4ag\cos^2\psi}{4a\cos\psi} + mg\cos\psi = 2mg\cos\psi$ (5)

At the vertex O, $\psi = 0$. Therefore putting $\psi = 0$ in (5), the pressure at the vertex $= 2mg =$ twice the weight of the particle.

Example 12:- Prove that for a particle, sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height.



Solution:- Let a particle of mass m slide down the arc of a cycloid starting at rest from the cusp A. If P is the position of the particle at any time t, then the equations of motion of the particle along the tangent and normal are $m\dfrac{d^2s}{dt^2}=-mg\sin\psi$ (1)

And $m\dfrac{v^2}{\rho}=R-mg\cos\psi$ (2)

For the cycloid, $s=4a\sin\psi$ (3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have $v^2=\left(\dfrac{ds}{dt}\right)^2=-\dfrac{g}{4a}s^2+A$

But initially at the cusp A, $s=4a$ and $v=0$ $\quad\therefore A=4ag$.

$$\therefore \quad v^2=4ag-\frac{g}{4a}s^2=4ag-\frac{g}{4a}(4a\sin\psi)^2=4ag\left(1-\sin^2\psi\right)$$

$$=4ag\cos^2\psi$$

Or $v=2\sqrt{(ag)}\cos\psi$, giving the velocity of the particle at the point P its direction being along the tangent at P. Let V be the vertical component of the velocity v at the point P.

Then $V=v\cos\psi\left(90^\circ-\psi\right)=v\sin\psi=2\sqrt{(ag)}\cos\psi.\sin\psi$

Or $V=\sqrt{(ag)}\sin 2\psi$, which is maximum when $\sin 2\psi=1$ i.e., $2\psi=\pi/2$ i.e., $\psi=\pi/4$.

When $\psi=\pi/4$, $\quad s=4a\sin(\pi/4)=2\sqrt{2a}$

Putting $s=2\sqrt{2a}$ in the relation $s^2=8a^2/8a=a$.

Thus at the point where the vertical velocity is maximum, we have $y=a$. The vertical depth fallen upto this point.

$=$(they-coordinates of A) $-a=2a-a=a=\dfrac{1}{2}(2a)$

$=$ half the vertical height of the cycloid.

Example 13:- A particle oscillates in a cycloid under gravity, the amplitude of the motion being b, and period being T. Show that its velocity at any time t measured from a position of rest is $\dfrac{2\pi b}{T}\sin\left(\dfrac{2\pi t}{T}\right)$

Solution:- The equation of motion of the particle are $m\dfrac{d^2s}{dt^2}=-mg\sin\psi$ (1)

And $m\frac{v^2}{\rho}=R-mg\cos\psi$ (2)

For the cycloid, $s=4a\sin\psi$ (3)

From (1) and (3), we have $\frac{d^2s}{dt^2}=-\frac{g}{4a}s$. (4)

Which represents a S.H.M.

$\therefore$ The time period T of the particle is given by $T-2\pi/\sqrt{(g/4a)}$

Or $T=4\pi\sqrt{(a/g)}$ (5)

Multiplying both sides of (4) by $2\frac{ds}{dt}$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A \quad (6)$$

But the amplitude of the motion is b. So the actual distance of a position of rest of the vertex O is b i.e., $v=0$ when $s=b$.

$\therefore$ from (6), we have $A=\frac{g}{4a}b^2$

Substituting this value of A in (6), we have $v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}\left(b^2-s^2\right)$ (7)

$\therefore$ $\frac{ds}{dt}=-\frac{1}{2}\sqrt{\left(\frac{g}{a}\right)}\sqrt{\left(b^2-s^2\right)}$

($-ive$ sign is taken because the particle is moving in the direction of s decreasing)

Or $dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{\left(b^2-s^2\right)}}$

Integrating $t=2\sqrt{(a/g)}\cos^{-1}(s/b)+B$

But $t=0$ when $s=b$. $\therefore B=0$

$\therefore$ $t=2\sqrt{(a/g)}\cos^{-1}(s/b)$

Or $s=b\cos\left\{\frac{t}{2}\sqrt{\left(\frac{g}{a}\right)}\right\}$

Substituting this value of s in (7), we have $v^2=\frac{g}{4a}\left[b^2-b^2\cos^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}\right]$

$=\frac{g}{4a}b^2\sin^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$

Or $v=\frac{b}{2}\sqrt{(g/a)}\sin\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$

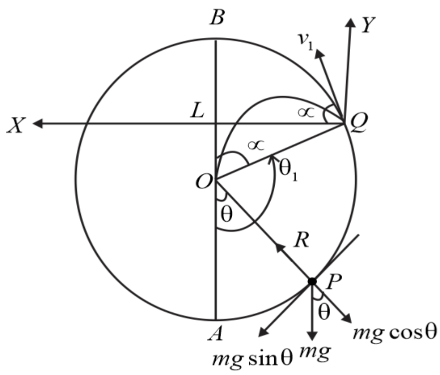
From (5), $\sqrt{(g/a)}=\frac{4\pi}{T}$

$\therefore$ The velocity of the particle at any time t measured from the position of rest is given by $v=\frac{b}{2}.\frac{4\pi}{T}\sin\left(\frac{t}{2}.\frac{4\pi}{T}\right)=\left(\frac{2\pi b}{T}\right)\sin\left(\frac{2\pi t}{T}\right)$.

Que 1:- A particle is projected along the inside of a smooth vertical circle of radius a from the lowest point. Show that the velocity of projection required in order that after leaving the circle, the particle may pass through the centre is $\sqrt{\left(\frac{1}{2}ag\right)}\ .\sqrt{(3+1)}$.

Solution:- Let the particle be projected from the lowest point A along the inside of a smooth vertical circle of radius a, with velocity u. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad (1)$$



And $m\frac{v^2}{a} = R - mg\cos\theta$ (2)

Also $s = a\theta$ (3)

From (1) and (3), we have $a\frac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying both sides by $2a\frac{d\theta}{dt}$ and integrating, we have $v^2 = \left(a\frac{d\theta}{dt}\right)^2 = ag\cos\theta + A$.

But at the lowest point A, $\theta = 0$ and $v = u$ $\quad \therefore A - u^2 - 2ag$

$\therefore \quad v^2 = 2ag\cos\theta + u^2 - 2ag$ (4)

From (2) and (4), we have $R = \frac{m}{a}\left(v^2 + ag\cos\theta\right) = \frac{m}{a}\left(u^2 - 2ag + 3ag\cos\theta\right)$ (5)

If the particle leaves the circle at Q ,where $\theta = \theta_1$, then from (5),

$0 = \frac{m}{a}\left(u^2 - 2ag + 3ag\cos\theta_1\right)$ or $\cos\theta_1 = -\left(\frac{u^2 - 2ag}{3ag}\right)$

If $\angle BOQ = \propto$, then $\propto = \pi - \theta_1$.

$\therefore \quad \cos\propto = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{u^2 - 2ag}{3ag}$

If v_1 is the velocity at Q, then putting $v = v_1$, $R = 0$ and $\theta = \theta_1$ in (2), we have $v_1^2 = -ag\cos\theta_1 = -ag\cos(\pi - \propto) = ag\cos\propto$

Thus the particle leaves the circle at Q with velocity $v_1 = \sqrt{(ag\cos\propto)}$ at angle $\propto = \cos^{-1}\left(\dfrac{u^2 - 2ag}{3ag}\right)$ to the horizontal and subsequently it describe a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinates axes is

$$y = x\tan\propto - \frac{gx^2}{2v_1^2\cos^2\propto} = x\tan\propto - \frac{2x^2}{2ag\cos^2\propto} \qquad (6)$$

$$\left[\because v_1^2 = ag\cos\propto\right]$$

The coordinates of the centre O w.r.t. QX and QY as coordinates axes are given by $x = QL = a\sin\propto$ and $y = -LO = -a\cos\propto$

If after leaving the circle at Q the particle passes through the centre $O(a\sin\propto - a\cos\propto)$, then the point O lies on the curve (6)

$\therefore \qquad -a\cos\propto = a\sin\propto.\tan\propto - \dfrac{ga^2\sin^2\propto}{2ag\cos^3\propto}$

Or $\dfrac{\sin^2\propto}{2\cos^2\propto} = \dfrac{\sin^2\propto}{\cos\propto} + \cos\propto = \dfrac{\sin^2\propto + \cos^2\propto}{\cos\propto} = \dfrac{1}{\cos\propto}$

Or $\sin^2\propto = 2\cos^2\propto$ or $1 - \cos^2\propto = 2\cos^2\propto$ or $3\cos^2\propto = 1$

Or $\cos^2\propto = 1/3$ or $\cos\propto = 1/\sqrt{3}$

$\therefore \qquad \dfrac{u^2 - 2ag}{3ag} = \dfrac{1}{\sqrt{3}} \qquad \left[\because \cos\propto = \dfrac{u^2 - 2ag}{3ag}\right]$

Or $u^2 - 2ag = \sqrt{3}ag$

Or $u^2 = \left(2+\sqrt{3}\right)ag = \left(\dfrac{4+2\sqrt{3}}{2}\right)ag = \dfrac{ag}{2}\left(1+\sqrt{3}\right)^2$

$\therefore \qquad u = \sqrt{\left(\dfrac{1}{2}ag\right)}.\sqrt{(3+1)}$

Thus the particle will pass through the centre if the velocity of projection at the lowest point is $\sqrt{\left(\dfrac{1}{2}ag\right)}\left(1+\sqrt{3}\right)^2$.

Que 2:- A particle tied to a string of length a is projected from its lowest point, so that after leaving the circular path it describes a free path passing through the lowest point. Prove that the velocity of projection is $\sqrt{\left(\dfrac{7}{2}ag\right)}$.

Solution:- Refer figure of before example

Take $R = T$ (i.e., the tension in the string)

Let a particle of mass m be attached to one end A of the string OA whose other end is fixed at O. Let the particle be projected from the lowest point A with velocity u. If the particle leaves the circular path at Q with velocity v_1 at an angle $\propto$ to the horizontal, then proceed as in before example to get $v_1 = \sqrt{(ag\cos\propto)}$ and $\cos\propto = \left(\dfrac{u^2 - 2ag}{3ag}\right)$

After Q the particle describes a parabolic path whose equation w.r.t. the horizontal and vertical line QX and QY as co-ordinate axes is

$$y = x\tan\propto - \frac{gx^2}{v_1^2\cos^2\propto} = x\tan\propto - \frac{gx^2}{2ag\cos^3\propto} \quad (1)$$

$$\left[\because\ v_1^2 = ag\cos\propto\right]$$

The co-ordinates of the lowest point A w.r.t. QX and QY at co-ordinates axes are given by

$x = QL = a\sin\propto$ and $y = -LA = -(LO + OA)$

$$= -(a\cos\propto + a) = -a(\cos\propto + 1)$$

If the particle passes through the lowest point $A\left[a\sin\propto, -a(\cos\propto+1)\right]$, then the point A lies on the curve (1).

$\therefore$ $$-a(\cos\propto + 1) = a\sin\propto\tan\propto - \frac{ga^2\sin^2\propto}{2ag\cos^3\propto}$$

Or $\dfrac{\sin^2\propto}{2\cos^2\propto} = \dfrac{\sin^2\propto}{\cos\propto} + \cos\propto + 1$

$= \dfrac{\sin^2\propto + \cos^2\propto + \cos\propto}{\cos\propto} = \dfrac{1+\cos\propto}{\cos\propto}$

Or $\sin^2\propto = 2\cos^2\propto(1+\cos\propto)$

Or $(1-\cos^2\propto) = 2\cos^2\propto(1+\cos\propto)$

Or $(1-\cos\propto)(1+\cos\propto) = 2\cos^2\propto(1+\cos\propto)$

Or $1-\cos\propto = 2\cos^2\propto$ $\quad [\because\ 1+\cos\propto]$

Or $2\cos^2\propto + \cos\propto - 1 = 0$ or $(2\cos\propto - 1)(\cos\propto + 1) = 0$

Or $2\cos\propto - 1 = 0$ $\quad [\because \cos\propto + 1 \neq 0]$

Or $\cos\propto = \dfrac{1}{2}$

Or $u^2 - \dfrac{2ag}{3ag} = \dfrac{1}{2}$ $\quad \left[\because \cos\propto = \dfrac{u^2 - 2ag}{3ag}\right]$

Or $u^2 = 2ag + \dfrac{3}{2}ag = \dfrac{7}{2}ag$ or $u = \sqrt{\left(\dfrac{7}{2}ag\right)}$

Que 3:- Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120°. If the ropes strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\dfrac{3mv^2}{2l}$ where m is the mass of the person and l the length of the rope.

Solution:- Let u be the velocity of a person of mass m at the lowest point. If v is the velocity of the person and T the tension in the rope of length l at a point P at an angular distance θ from the lowest point, then proceed as in to get.

$$v^2 = u^2 - 2\lg + 2\lg\cos\theta \quad (1)$$

And $$T = \frac{m}{l}\left(u^2 - 2\lg + 3\lg\cos\theta\right) \quad (2)$$

Now according to the question the ropes can support twice the person's weight at rest. Therefore the maximum tension the rope can bear is $2mg$. So for the greatest angle through which be such that $T = 2mg$ when $\theta = 0$.

Then from (2), we have $2mg = \frac{m}{l}\left(u^2 - 2\lg + 3\lg\cos 0\right)$ or $2gl = u^2 - 2\lg + 3\lg$ or $u^2 = \lg$.

Now from (1), we have $v^2 = \lg - 2\lg + 2\lg\cos\theta = 2\lg\cos\theta - \lg = \lg(2\cos\theta - 1)$

If $v = 0$ at $\theta = \theta_1$, then $0 = gl(2\cos\theta_1 - 1)$ or $\cos\theta_1 = \frac{1}{2}$. Therefore, $\theta_1 = 60^\circ$.

Thus the person can swing through an angle of 60° from the vertical on one side of the lowest point. Hence the person can oscillate through an angle of $60^\circ + 60^\circ = 120^\circ$.

Second part:- If the rope is strong enough and the person can swing through an angle of 180° i.e., through an angle 90° on one side of the lowest point, then $v = 0$ at $\theta = 90^\circ$.

$\therefore$ from (1), we have $0 = u^2 - 2\lg + 2\lg\cos 90^\circ$ or $u^2 = 2\lg$.

Thus if the person's velocity at the lowest point is $\sqrt{(2\lg)}$ then he can swing through an angle of 180°.

Then from (1) ,we have $v^2 = 2\lg - 2\lg + 2\lg\cos\theta$ or $\cos\theta = \frac{v^2}{2\lg}$.

Therefore from (2), the tension in the rope at an angular distance θ where the velocity is v, is given by $T = \frac{m}{l}\left[2\lg - 2\lg + 3\lg \cdot \frac{v^2}{2\lg}\right] = \frac{3mv^2}{2l}$.

Que 4:- A heavy bead slides on a smooth circular wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the highest point, prove that the radius through the bead in time t will turn through an angle $2\tan^{-1}\left[\sin h\left\{t\sqrt{(g/a)}\right\}\right]$ and that the bead will take an infinite time to reach the highest point.

Solution:- Refer figure of before example.

The equation of motion of the bead are $m\frac{d^2s}{dt^2} = -mg\sin\theta$ (1)

And $m\frac{v^2}{a} = R - mg\cos\theta$ (2)

Also $s = a\theta$ (3)

From (1) and (3), we have a $\frac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have $v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$

But according to the question at the highest point $v = 0$ i.e., when $\theta = \pi$, $v = 0$

$\therefore$ $0 = 2ag\cos\pi + A$ or $A = 2ag$

$\therefore \qquad v^2 = \left(a\dfrac{d\theta}{dt}\right)^2 = 2ag + 2ag\cos\theta = 2ag\left(1+\cos\theta\right)$

$= 2ag\ .\ 2\cos^2\dfrac{1}{2}\theta$ or $a\dfrac{d\theta}{dt} = 2\sqrt{(ag)}.\cos\dfrac{1}{2}\theta$

Or $dt = \dfrac{1}{2}\sqrt{(a/g)}\ .\ \sec\dfrac{1}{2}\theta\ d\theta$

Integrating, the time t from A to P is given by $t = \dfrac{1}{2}\sqrt{(a/g)}\ .\ \int_0^\theta \sec\dfrac{1}{2}\theta\ d\theta$

$$= \frac{1}{2}\sqrt{(a/g)}.2\left[\log\left(\tan\frac{1}{2}\theta + \sec\frac{1}{2}\theta\right)\right]_0^\theta$$

$$= \sqrt{(a/g)}\left[\log\left(\tan\frac{1}{2}\theta + \sec\frac{1}{2}\theta\right) - \log 1\right]$$

$$= \sqrt{(a/g)}.\left[\log\left\{\tan\frac{1}{2}\theta + \sqrt{\left(1+\tan^2\frac{1}{2}\theta\right)}\right\}\right]$$

$$= \sqrt{(a/g)}.\sinh^{-1}\left(\tan\frac{1}{2}\theta\right)$$

$$\left[\because \sinh^{-1}x = \log\left\{x + \sqrt{(1+x^2)}\right\}\right]$$

Or $t\sqrt{(g/a)} = \sinh^{-1}\left(\tan\dfrac{1}{2}\theta\right)$

Or $\tan\dfrac{1}{2}\theta \sinh\left\{t\sqrt{(g/a)}\right\}$

$\therefore \qquad \theta = 2\tan^{-1}\left[\sinh\left\{t\sqrt{(g/a)}\right\}\right]$

Again the time to reach the highest point B while starting from A.

$$= \frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\pi}\sec\frac{1}{2}\theta\ d\theta$$

$$= \frac{1}{2}\sqrt{(a/g)}.\ 2\left[\log\left(\tan\frac{1}{2}\theta + \sec\frac{1}{2}\theta\right)\right]_0^\pi$$

$$= \sqrt{(a/g)}.\left[\log\left(\tan\frac{1}{2}\pi + \sec\frac{1}{2}\pi\right) - \log\left(\tan 0 + \sec 0\right)\right]$$

$$= \sqrt{(a/g)}\ .\left[\log\infty - \log 1\right] = \infty$$

Therefore the bead takes an infinite time to reach the highest point.

Que 5:- A particle attached to a fixed $peg\,O$ by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string make an angle θ with the horizontal, the resultant acceleration is $g\sqrt{\left(1+3\sin^2\theta\right)}$.

Solution:- Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed $peg\,O$. Initially let the string be horizontal in the position OA such that $OA = l$. The particle starts from A and moves in a circle whose centre is O and radius is l. Let P be the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$.

The force acting on the particle at P are : (i) its weight mg acting vertically downwards and (ii) the tension T in the string along PO.

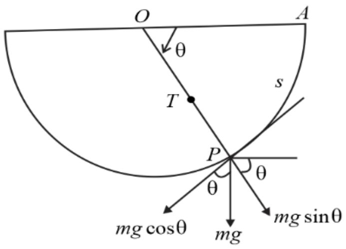
$\therefore$ The equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2} = mg \cos\theta \quad (1)$$

And $$m\frac{v^2}{l} = T - mg\sin\theta \quad (2)$$

Also $s = l\theta$ (3)

From (1) and (3), we have $l\frac{d^2\theta}{dt^2} = g\cos\theta$



Multiplying both sides by $2l(d\theta/dt)$ and integrating, we have $v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\sin\theta + A$.

But initially at the point A, $\theta = 0$, $v = 0$. $\therefore$ $A = 0$

$\therefore$ $v^2 = 2lg \sin\theta$ (4)

The resultant acceleration of the particle at P

$= \sqrt{\ }$ [(Tangential acceleration)^2+(Normal acceleration)^2]

$$= \sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{l}\right)^2\right]} \qquad \left[\because Normal\ accel = \frac{v^2}{p} = \frac{v^2}{l}\right]$$

$$= \sqrt{\left[(g\cos\theta)^2 + \left(\frac{2lg\sin\theta}{l}\right)^2\right]}$$

$$= g\sqrt{\left[1-\sin^2\theta + 4\sin^2\theta\right]} = g\sqrt{\left(1+3\sin^2\theta\right)}$$

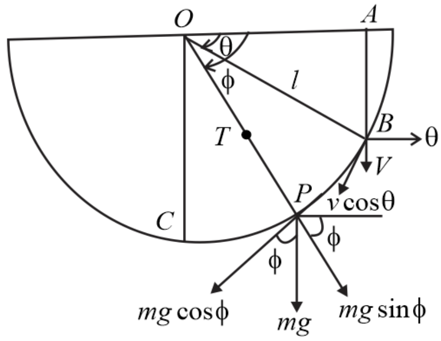
Que 6:- A particle attached to a fixed $peg\, O$ by a string of length l, is let fall from a point in the horizontal line through O of a distance $l\cos\theta$ from O; show that its velocity when it is vertically below O is $\sqrt{\left[2gl\left(1-\sin^2\theta\right)\right]}$.

Solution:- Let a particle of mass m be attached to a string of length l whose other end is attracted to a fixed $peg\, O$. Let the particle fall from a point A in the horizontal line through O such that $OA = l\cos\theta$. The particle will fall under gravity from A to B, where $OB = l$.

$\because$ $OA = l\cos\theta$ and $OB = l$, therefore $\angle AOB = 0$ and $AB = l\sin\theta$.

$\therefore$ The velocity of the particle at B.

$=V=\sqrt{(2g.\ AB)}=\sqrt{(2gl\sin\theta)}$, vertically downwards.



As the particle reaches B, there is jerk in the string and the impulsive tension in the string destroys the component of the velocity along OB and the component of the velocity along the tangent at B remains unaltered i.e., the particle moves in the circular path with centre O and radius l with the tangential velocity $V\cos\theta$ at B.

[Note:- In the figure write D at the end of the horizontal radius through O]

If P is the position of the particle at any time t such that $\angle DOP=\phi$ and arc $DP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\cos\phi \qquad (1)$$

And $\frac{mv^2}{l}=T-mg\sin\phi$ (2)

Also $s=l\phi$ (3)

From (1) and (3), we have $l\frac{d^2\phi}{dt^2}-g\cos\phi$

Multiplying both sides by $2l\left(d\phi/dt\right)$ and integrating, we have $v^2=\left(l\frac{d\phi}{dt}\right)^2=2\lg\sin\phi+A$

But at the point $B,\ \phi=\theta$ and $v=V\cos\theta$

$\therefore \quad A=V^2\cos^2\theta-2\lg\sin\theta=2gl\sin\theta.\cos^2\theta-2\lg\sin\theta$

$=-2\lg\sin\theta\left(1-\cos^2\theta\right)=-2\lg\sin^3\theta$

$\therefore \quad v^2=2\lg\sin\phi-2\lg\sin^3\theta$

When the particle is at C vertically below O, we have at C $\phi=\pi/2$. Therefore the velocity v at C is given by $v^2=2\lg\sin\frac{1}{2}\pi-2\lg\sin^3\theta=2\lg\left(1-\sin^3\theta\right)$

$\therefore$ The required velocity $v=\sqrt{\left[2\lg\left(1-\sin^3\theta\right)\right]}$.

Que 7:- A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Solution:- Let a particle start from rest from the cusp A of the cycloid. Proceeding as in the last example the velocity v of the particle at any point P at time t, is given by $v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}\left(16a^2-s^2\right)$

Or $\frac{ds}{dt}=-\frac{1}{2}(g/a)\sqrt{(16a^2-s^2)}$, the $-ive$ sign is taken because the particle is moving in the direction of s decreasing.

$\therefore$ $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(16a^2-s^2)}} \quad (1)$$

The vertical height of the cycloid is $2a$. At the point where the particle has fallen down the first half first half of the vertical height of the cycloid, we have $y=a$. Putting $y=a$ in the equation $s^2=8ay$, we get $s^2=8a^2$ or $s=2\sqrt{2a}$.

$\therefore$ Integrating (1) from $s=4a$ to $s=2\sqrt{2a}$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{2\sqrt{2a}}\frac{ds}{\sqrt{(16a^2-s^2)}}=2\sqrt{(a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2a}}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{2\sqrt{2a}}{4a}-\cos^{-1}\right]=2\sqrt{(a/g)}\left[\cos^{-1}\frac{1}{\sqrt{2}}-\cos^{-1}1\right]$$

$$=2\sqrt{(a/g)}\left[\frac{1}{4}\pi=0\right]=\frac{1}{2}\pi\sqrt{(a/g)}$$

Again integrating (1) from $s=2\sqrt{2a}$ to $s=0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by $t_2=-2\sqrt{(a/g)}\int_{s=2\sqrt{2a}}^{0}\sqrt{\frac{ds}{\sqrt{(16a^2-s^2)}}}$

$$=2\sqrt{(a/g)}.\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2a}}^{0}=2\sqrt{(a/g)}\left[\cos^{-1}0-\cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=2\sqrt{(a/g)}\left[\frac{1}{2}\pi-\frac{1}{4}\pi\right]=\frac{1}{2}\pi\sqrt{(a/g)}$$

Hence $t_1=t_2$ i.e. the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Que 8:- A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag)}/V\right]$.

Solution:- Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP=s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \quad (1)$$

And $$m\frac{v^2}{\rho}=R-mg\cos\psi \quad (2)$$

For the cycloid $s=4a\sin\psi$ (3)

From (1) and (3), we have $\frac{d^2s}{dt^2}=-\frac{g}{4a}s$

Multiplying both sides by $2(ds/dt)$ and integrating, we have $v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$

But initially at the cusp $A, s = 4a$ and $(ds/dt)^2 = V^2$.

$\therefore$ $V^2 = -(g/4a).16a^2 + A$ or $A = V^2 + 4ag$.

$$\therefore \quad v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2 = \left(\frac{g}{4a}\right)\left[\frac{4a}{g}\left(V^2 + 4ag\right) - s^2\right]$$

Or $\frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\sqrt{\left[\frac{4a}{g}\left(V^2 + 4ag\right) - s^2\right]}$

($-ive$ sign is taken because the particle is moving in the direction of s decreasing)

Or $dt = -2\sqrt{(a/g)} \cdot \frac{ds}{\sqrt{\left[(4a/g)\left(V^2 + 4ag\right) - s^2\right]}}$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{(a/g)} \int_{s=4a}^{0} \frac{ds}{\sqrt{\left[(4a/g)\left(V^2 + 4ag\right) - s^2\right]}}$$

$$= 2\sqrt{(a/g)} \int_{0}^{4a} \frac{ds}{\sqrt{\left[(4a/g)\left(V^2 + 4ag\right) - s^2\right]}}$$

$$= 2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)\left(V^2 + 4ag\right)}}\right]_0^{4a}$$

$$= 2\sqrt{(a/g)} \cdot \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{\left(V^2 + 4ag\right)}}\right\}$$

$$= 2\sqrt{(a/g)} \cdot \theta \qquad (4)$$

Where $\theta = \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{\left(V^2 + 4ag\right)}}\right\}$

We have $\sin\theta = \frac{2\sqrt{(ag)}}{\sqrt{\left(V^2 + 4ag\right)}}$.

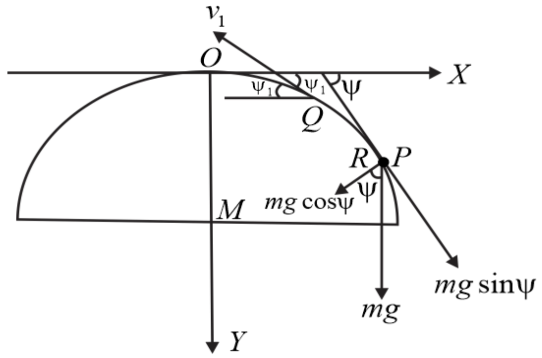
$$\therefore \quad \cos\theta = \sqrt{(1 - \sin\theta)} = \sqrt{\left[1 - \frac{4ag}{V^2 + 4ag}\right]} = \frac{V}{\sqrt{\left(V^2 + 4ag\right)}}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{V} = \frac{\sqrt{(4ag)}}{V}$$

Or $\theta = \tan^{-1}\left[\sqrt{(4ag)}/V\right]$

$\therefore$ From (4), the time of reaching the vertex is $= 2\sqrt{(a/g)}.\tan^{-1}\left[\sqrt{(4ag)}/V\right]$.

Que 9:- A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{(2gh)}$; prove that the latus rectum of the parabola described after leaving the arc is $\left(h^2/2a\right)$, where a is the radius of the generating circle.



Solution:- Let a particle of mass m be projected with velocity $\sqrt{(2gh)}$ from the cusp A up the concave side of the cycloid. If P is the position of the particle after any time t such that arc $OP = s$, the equations of motion along the tangent and normal are $m\left(d^2s/dt^2\right) = mg\sin\psi$ (1)

And $m\left(v^2/\rho\right) = R + mg\cos\psi$ (2)

[Note that here the reaction R of the curve acts along the inwards drawn normal and the tangential component of mg acts in the direction of s increasing]

For the cycloid $s = 4a\sin\psi$

From (1) and (3), we have $\dfrac{d^2s}{dt^2} = \dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and then integrating, we have $v^2 = (ds/dt)^2 = (g/4a)s^2 + A$.

Initially at A, $s = 4a$ and $v = \sqrt{(2gh)}$

$\therefore \quad A = 2gh - 4ag$.

$\therefore \quad v^2 = \dfrac{g}{4a}s^2 + 2gh - 4a = \dfrac{g}{4a}(4a\sin\psi)^2 + 2gh - 4ag$

$= 4ag\sin^2\psi + 2gh - 4ag = 2ag - 4ag\left(1-\sin^2\psi\right)$

$= 2gh - 4ag\cos^2\psi$ (4)

From (2) and (4), we have $R = \dfrac{m}{4a\cos\psi}\left(2gh - 4ag\cos^2\psi\right) - mg\cos\psi$

$[\because \ \rho = ds/d\psi = 4a\cos\psi]$

$= \dfrac{mg}{2a\cos\psi}\left(h - 2a\cos^2\psi\right) - mg\cos\psi$

$= \dfrac{mg}{2a\cos\psi}\left[h - 2a\cos^2\psi - 2\cos^2\psi\right]$

$= \dfrac{mg}{2a\cos\psi}\left[h - 4a\cos^2\psi\right]$ (5)

Suppose the particle leaves the cycloid at the point Q where $\psi=\psi_1$. Then putting $\psi=\psi_1$ and $R=0$ in (5), we have $h-4a\cos^2\psi_1=0$

Or $\quad \cos^2\psi_1 = h/4a$ (6)

If v_1 is the velocity at Q, then from (4), we have $v_1^2 = 2gh-4ag\cos^2\psi_1 = 2gh-4ag.(h/4a)=gh$

$\therefore$ The particle leaves the cycloid at the point Q with velocity $v_1=\sqrt{(gh)}$ inclined at an angle ψ_1 to the horizontal given by (6). Subsequently it describes a parabolic path.

The latus rectum of the parabolic path describes after Q

$=(2/g)$ (Square of the horizontal velocity at Q)

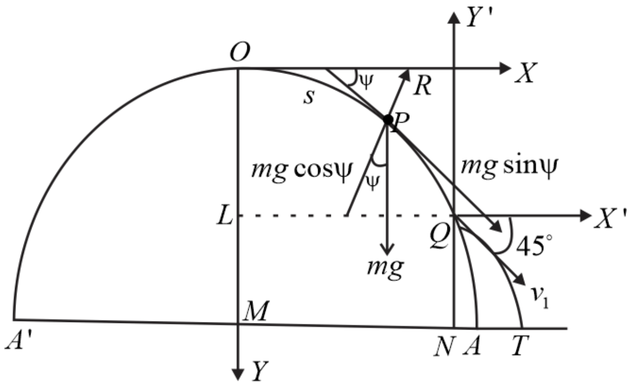
$=(2/g)(v_1^2\cos^2\psi_1)=(2/g)(gh)(h/4a)=h^2/2a$.

Que 10:- A particle is placed very near the vertex of a smooth cycloid whose axis is vertical and vertex upwards, and is allowed to rum down the curve. Prove that it will leave the curve when it has fallen through half the vertical height of the cycloid.

Also prove that the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Also show that it falls upon the base of the cycloid at a distance $\left(\frac{1}{2}\pi+\sqrt{3}\right)$ a from the centre of the base, a being the radius of the generating circle.

Solution:- Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at any time t such that arc $OP=s$. If the tangent at P makes an angle ψ with the horizontal, then the equations of motion of the particle along the tangent and normal at P are



$$m\frac{d^2s}{dt^2}=mg\ \sin\psi \quad (1)$$

And $m\dfrac{v^2}{\rho}=mg\cos\psi-R$ (2)

Also for the cycloid $s=4a\sin\psi$ (3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s$

Multiplying both sides by $2(ds/dt)$ and integrating, we have $v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}s^2 + A$.

Initially at O, $s = 0$ and $v = 0$ $\quad\therefore\quad$ $A = 0$

$$\therefore \quad v^2 = \frac{g}{4a}s^2 = \frac{g}{4a}(4a\sin\psi)^2 = 4ag\sin^2\psi \quad (4)$$

From (2) and (4), we have $R = mg\cos\psi - \frac{mv^2}{\rho} = mg\cos\psi - m\frac{4ag\sin^2\psi}{4a\cos\psi}$

$[\because \rho = ds/dt = 4a\cos\psi]$

$$= \frac{mg}{\cos\psi}\left(\cos^2\psi - \sin^2\psi\right)$$

If the particle leaves the cycloid at the point Q, then at Q, $R = 0$. From $R = 0$, we have

$$\frac{mg}{\cos\psi}\left(\cos^2\psi - \sin^2\psi\right) = 0$$

Or $\sin^2\psi = \cos^2\psi$ or $\tan^2\psi = 1$

Or $\tan\psi = 1$ or $\psi = 45^\circ$

Thus at Q, we have $\psi = 45^\circ$. Putting $\psi = \frac{1}{4}\pi$ in $s = 4a\sin\psi$, we have at Q, $s = 4a\sin\frac{1}{4}\pi = 4a.\left(1\sqrt{2}\right) = 2\sqrt{2a}$. Again putting $s = 2\sqrt{2a}$ in $s^2 = 8ay$, we have at $Q, y = s^2/8a = 8a^2/8a = a$.

Thus $OL = a$. Therefore $LM = OM - OL = 2a - a = a$. Hence the particle leaves the cycloid at the point Q, when it has fallen through half the vertical height of the cycloid.

Second Part:- If v_1 is the velocity of the particle at Q, then from (4) we have $v_1^2 = 4ag\sin^2 45^\circ = 2ag$.

Hence the particle leaves the cycloid at Q with velocity $v_1 = \sqrt{(2ag)}$ in a direction making an angle 45° downwards with the horizontal. After Q the particle will describe a parabolic path.

Latus rectum of the parabola describe after Q

$= \frac{2v_1^2\cos^2 45^\circ}{g} = \frac{2\,2ag.\frac{1}{2}}{g} = 2a$ i.e. the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Third Part:- The equation of the parabolic path described by the particle after leaving the cycloid at Q with respect to the horizontal and vertical line QX' and QY' as the coordinate axes is

$y = x\tan\left(-45^\circ\right) - \frac{gx^2}{2v_1^2\cos^2}\left(-45^\circ\right)$ [Note that here the angle of projection for the motion of the projectile is -45°]

Or $y = -x - \frac{gx^2}{2.2ag.\frac{1}{2}}$

Or $y=-x-\dfrac{x^2}{2a}$ (5)

Suppose after leaving the cycloid at Q the particle strikes the base of the cycloid at the point T Let (x_1, y_1) be the coordinates of T with respect to QX' and QY' as the coordinates axes. Then $x_1 = NT$ and $y_1 = -QN = -a$.

But the point $T(x_1, -a)$ lies on the curve (5).

$\therefore$ $$-a=-x_1-\frac{x_1^2}{2a}$$

Or $x_1^2+2ax_1-2a^2=0$.

$\therefore$ $$x_1=\frac{-2a\pm\sqrt{\{4a^2-4.1.(-2a^2)\}}}{2.1}$$

Neglecting the $-ive$ sign because x_1 cannot be negative, we have $x_1 = NT = -a + a\sqrt{3}$.

The parametric equations of the cycloid w.r.t. OX and OY as the coordinates axes are $x=a(\theta+\sin\theta)\, y=a(1-\cos\theta)$, where θ is the parameters and $\theta = 2\psi$.

$\therefore$ At the point Q, where $\psi=\frac{1}{4}\pi$, we have

$$x=LQ=a(2\psi+\sin 2\psi)=a\left[2.\frac{1}{4}\pi+\sin\left(2.\frac{1}{4}\pi\right)\right]=a\left(\frac{1}{2}\pi+1\right)$$

$\therefore$ The horizontal distance of the point T from the centre M of the base of the cycloid.

$= MT = MN + NT = LQ + NT$

$=a\left(\frac{1}{2}\pi+1\right)+\left(-a+a\sqrt{3}\right)=\left(\frac{1}{2}\pi+\sqrt{3}\right)a$.

Central Orbits:

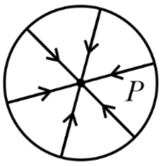
Mindset Required:- At least once, try to decode the whole article, to have clear understanding

• Resembling 3-4 eq. (Out of above step)

• Apply the eq. ; get the problem solved.

Central force:-

A force whose line of action always passes through a fixed point.

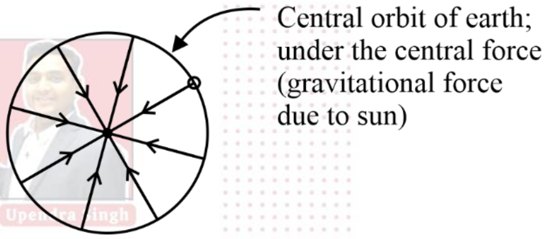


Here P is a central force

Central orbit of earth; under the central force

(gravitational force due to sun)

Practically:

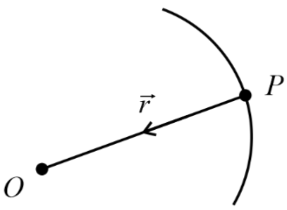


Notice!

Here, clearly, the force is radial only means; force is applied only along the radius vector.

Exam point

A central orbit is always a plain curve



fixed point

Clearly, the motion is along radial vector only.

So, the radial acceleration & the radial vector will be in same direction.

$$\therefore \quad \frac{d^2\vec{r}}{dt^2} \times \vec{r} = \vec{0}$$

$$\frac{d^2\vec{r}}{dt^2} \times \vec{r} + \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0 \left\{ \because \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0 \right\}$$

$$\frac{d}{dt}\left(\frac{d\vec{r}}{dt} \times \vec{r}\right) = 0$$

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} \times \vec{r} = a \text{ constant vector} = h \text{ (say)}$$

So we have,

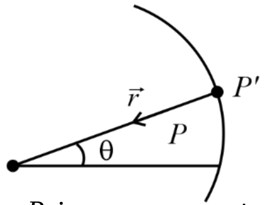
$$\vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \vec{r} \right) = \vec{r} \cdot \vec{h} \Rightarrow \vec{r} \cdot \vec{h} = 0 \quad ...(1)$$

$\therefore$ from (1), we can say $\vec{u}$ is always perpendicular to a constant vector; so the central orbit is a plane curve.

Differential eq. of a central Orbit

Radial acceleration:-

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \quad ...(1)$$



• negative sign will come into eq. because P is opp. to $\vec{r}$ .i.e., in the direction of decreasing 'r'

• Radial acceleration

• Transverse acc. will always be zero here.

Transverse acceleration:

$$\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \quad ...(2)$$

$$\Rightarrow \qquad \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0$$

$$\Rightarrow \qquad \boxed{r^2 \frac{d\theta}{dt} = h \text{ (say)} = \text{constant}} \quad ....(3)$$

Just a representation to get a differential equation easier to remember.

Let $\quad r = \dfrac{1}{u}$

From (3) we have

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$$

Also, $\dfrac{dv}{dt} = \dfrac{-1}{u^2}\dfrac{du}{dt} = \dfrac{-1}{u^2} \cdot \dfrac{du}{d\theta} \cdot \dfrac{d\theta}{dt} = \dfrac{-1}{u^2} \cdot \dfrac{du}{d\theta} u^2 \cdot h = -h\dfrac{du}{d\theta}$

Similarly

$$\frac{d^2 r}{dt^2} = -h\frac{d^2 u}{d\theta^2}\frac{d\theta}{dt} = -h^2 u^2 \frac{d^2 u}{d\theta^2}$$

Substituting these in (1), we have

$$-h^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}\left(u^2h\right)^2=-P$$

$\Rightarrow \quad h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3=P$

$\Rightarrow \quad \boxed{\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{P}{h^2u^2}}$ **Exampoint (1)**

Which is called the differential equation of central orbit.

- **In pedal form:-** In eq. involving *p* & *r*; where *p* is the perpendicular distance from O upon the tangent at point *P'*

Note:- from differential calculus, we have

$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2;$ But $u=\frac{1}{r}$

$\therefore \quad \frac{du}{d\theta}=\frac{-1}{r^2}\cdot\frac{dr}{d\theta}$ i.e., $\left(\frac{du}{d\theta}\right)^2=\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$\therefore \quad \frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$

Differentiating w.r.t θ,

$$\frac{-2}{p^3}\frac{dp}{d\theta}=2u\frac{du}{d\theta}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}=2\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)$$

$\frac{-1}{p^3}\frac{dp}{d\theta}=\frac{du}{d\theta}\frac{P}{h^2u^2}$ {using Exam point (1)}

$\frac{-1}{p^3}\frac{dp}{dr}\cdot\frac{dr}{d\theta}=\left(\frac{-1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{P}{h^2u^2}\right)$ $\left\{\because \frac{du}{d\theta}=\frac{-1}{r^2}\frac{dr}{d\theta}\right\}$

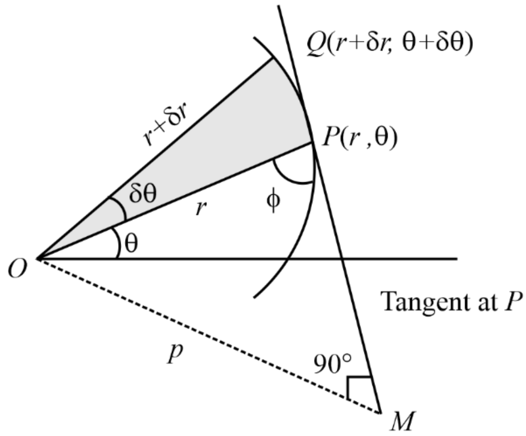
$\frac{1}{p^3}\frac{dp}{dr}=\frac{1}{r^2}\frac{P}{h^2u^2}$

$\frac{1}{p^3}\frac{dp}{dr}=\frac{P}{h^2};$ $\boxed{P=\frac{h^2}{p^3}\frac{dp}{dr}}$ **Exampoint**

Rate of description of sectoral area:-

The *Q* is very closer to *P*;

At last we'll take $Q \to P$



Sectoral area = area ΔPOQ

$$= \frac{1}{2} . OP . OQ \sin \angle POQ$$

$$= \frac{1}{2} r (r + \delta r) \sin \delta\theta$$

∴ Rate of description of sectoral area

$$= \lim_{\delta t \to 0} \frac{\text{sectoral area } POQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{1}{2} \text{r(r + δr)} (\sin \delta\theta)/\delta\theta \cdot \frac{\delta\theta}{\delta t}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}; \quad \lim_{\delta\theta \to 0} (\sin \delta\theta)/\delta\theta = 1$$

Now, for a central orbit,

$$r^2 \frac{d\theta}{dt} = h$$

∴ $r^2 \frac{d\theta}{ds} . \frac{ds}{dt} = h \Rightarrow r^2 \frac{d\theta}{ds} v = h$ ...(1) $\quad \left\{ \because \frac{ds}{dt} = v \right\}$

But from diff. calculus:

$r \frac{d\theta}{ds} = \sin \varphi$, where φ is the angle between the radius vector & the tangent,

∴ $r^2 \frac{d\theta}{ds} = r \sin \varphi = p \quad \{\because p = r \sin \varphi\}$

Putting $r^2 \frac{d\theta}{dS} = p$ in (1) we have

$\boxed{v = \frac{h}{p}}$ **Exam point** ∴ $v^2 = \frac{h^2}{p^2}$ ∴ $v \propto \frac{1}{p}$ **(Remember)**

But $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$

So, we have,

$\boxed{v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]}$; This eq. gives the linear velocity at any point P' of the path of a central orbit.

Summary:

If orbit (given)

e.g. elliptic

Hyperbolic

Parabolic

e.g. **Elliptic**

$\frac{l}{r} = 1 + e \cos\theta$

$\frac{1}{r} = \frac{1}{l} + \frac{e}{l}\cos\theta$ ...(1)

Now, we're interested in discussing

1. Law of force; $\propto \varphi(r)$!!

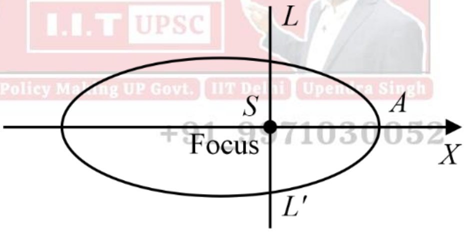
2. velocity at any point

3. time period

$$\boxed{\begin{array}{l} \frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \\ v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] \\ P = \frac{h^2}{p^3}\frac{dp}{dr} \\ v = \frac{h}{p} \end{array}}$$

Case (1):- Elliptic orbit:-

(focus as the centre of force)



$\frac{l}{r} = 1 + e \cos\theta$

$\Rightarrow \quad u = \frac{1}{l} + \frac{e}{l}\cos\theta$ ...(1)

$\frac{du}{d\theta} = \frac{-e}{l}\sin\theta$

$\frac{d^2u}{d\theta^2} = \frac{-e}{l}\cos\theta$ ...(2)

(i) Law of force:-

$\because$ diff. eq. of central orbit is

$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$

$\frac{-e}{l}\cos\theta + \frac{1}{l} + \frac{e}{l}\cos\theta = \frac{P}{h^2\left(\frac{1}{l}+\frac{e}{l}\cos\theta\right)^2}$;

where P is the central acc. assumed to be attracted

$P = h^2 u^2 \times \frac{1}{l}$

$$P = \frac{h^2}{l} \times \frac{1}{r^2}$$

$$P = \frac{\mu}{r^2} \Rightarrow \boxed{P \propto \frac{1}{r^2}}$$

i.e., acceleration values inversely as the square of distance of particle from the focus.

Also, the force is attractive because P is +ve.

(II) Velocity

$\because \quad v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$

$$= h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(\frac{-e}{l}\cos\theta\right)^2\right]$$

$$= \frac{h^2}{l}\left[\frac{1+e^2}{l} + 2\frac{l\cos\theta}{l}\right]$$

$$= \mu\left[\frac{1+e^2}{l} + 2\left(u - \frac{1}{l}\right)\right]$$

$$= \mu\left[2u - \frac{1-e^2}{l}\right]$$

$$v^2 = \mu\left[\frac{2}{r} - \frac{1-e^2}{l}\right]$$

If $2a$ and $2b$ are the length of major & minor axes of ellipse, then

$$l = \text{semi laltus rectum} = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

$\therefore \quad \frac{1-e^2}{l} = \frac{1}{a}$

$\therefore \quad \boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$

Which gives the velocity of the particle at any point on the path.

The above eq. shows that the magnitude of velocity at any point of the path depends only on the distance from the focus & that it is independent of the direction of motion

Also, $\boxed{v^2 < \frac{2\mu}{r}}$

III. Periodic Time :-

$\because \quad T\left(\frac{h}{2}\right)$ = The whole area of ellipse

$\downarrow$

Rate of description of sectoral area

$T\,\frac{h}{2} = \pi\, ab$ $\{\because$ area of ellipse $= \pi\, ab\}$

$T = \frac{2\pi ab}{\sqrt{\mu l}}$ $[\because h^2 = \mu l]$

$T = \frac{2\pi ab}{\sqrt{\mu \frac{b^2}{a}}}$; $\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}}$ $\therefore \quad T \propto a^{3/2}$

Case (ii) :- For hyperbolic orbit:-

(centre of force being the focus)

In case of hyperbola : e > 1

Also, $l = \frac{b^2}{a} = \frac{a^2\left(e^2-1\right)}{a} = a(e^2 - 1)$

Proceeding as previous case, we get

(i) $\boxed{P = \frac{\mu}{r^2}}$, where $h^2 = \mu l$

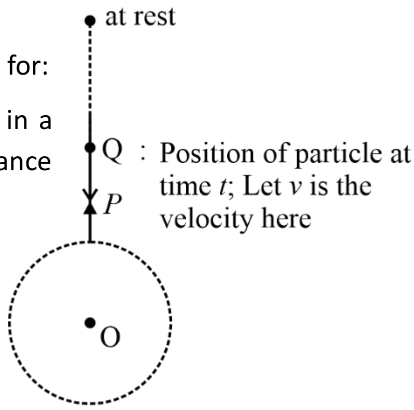
(ii) $\boxed{v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right]}$ $(\because e > 1)$

or $\boxed{v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]}$ Note that $\boxed{v^2 > \frac{2\mu}{r}}$

Case (iii):- $\because$ ***e* = 1 in this case,**

$$\boxed{\begin{aligned} &P = \frac{\mu}{r^2} \\ &\& \; v^2 = \frac{2\mu}{r} \end{aligned}}$$

Velocity from infinity



This phrase; "velocity from Infinity" (in connection with central orbit) is used for:

That the velocity a particle will acquire if it is moved from rest at infinity in a straight line to that point under the action of an attractive force in accordance with the law associated with the orbit.

At Q;

$$v\,\frac{dv}{dr} = -P$$

$$vdv = -Pdr$$

Let V be the velocity acquired in falling from rest at infinity to a point distance

'a' from the centre of force O

$$\int_0^v vdv = -\int_\infty^a Pdr$$

$$\frac{1}{2}V^2 = -\int_\infty^a Pdr \Rightarrow \boxed{V^2 = -2\int_\infty^a P\cdot dr}$$

Angular momentum or momentum of momentum

The expression $r^2\,\dfrac{d\theta}{dt}$ is called the angular momentum about the pole O of a particle of unit mass moving in a plane curve.

$\because \quad r^2\,\dfrac{d\theta}{dt} = h$ = constant

$\Rightarrow$ Angular momentum of a central orbit is conserved.

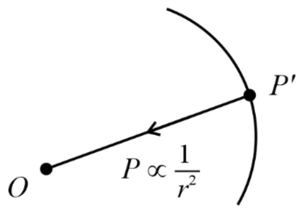
The Inverse square Law (Planetary motion)

**(Just a special case: central orbit)

Motivation:-

$$F = \frac{Gm_1m_2}{r^2}$$

$$F \propto \frac{1}{r^2}$$



Kepler's Law

Motion under inverse square law:-

The path of the particle which is moving so that its acceleration is always directed towards a fixed point & is equal to $\dfrac{\mu}{(\text{distance})^2}$, is a conic section and; we're interested in three cases which arise here clearly, path is a central orbit

The diff. eq. of path is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P = \frac{\mu}{r^2}$$

Gives, $\frac{-2h^2}{p^3}dp = \frac{-2\mu}{r^2}dr$

On integrating

$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + B$ ...(1) $\qquad \{\because v = \frac{h}{p}\}$

Note:- We refer; focus as pole

Need to remember

→ **Pedal eq. of Ellipse**

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$

→ **Parabola**

$p^2 = ar$

→ **Hyperbola**

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1$$

(1) Elliptical path

$\because \quad \frac{b^2}{p^2} = \frac{2a}{r} - 1$

$\therefore \quad \frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{-1}$

$\therefore \quad h = \frac{\mu b^2}{a}, B = \frac{-\mu}{a}$

Putting in (1), ; $\boxed{v^2 = \frac{2\mu}{r} - \frac{1}{a}\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$ **Exam point**

Obviously here $\boxed{v^2 < \frac{2\mu}{r}}$

(2) Parabolic path:

$\because \quad p^2 = ar$

$$\frac{h^2}{1} = \frac{2h}{\left(\frac{1}{a}\right)} = \frac{B}{0}$$

$\therefore$ (1) gives,

$$\boxed{v^2 = \frac{2\mu}{r}}$$ **Exampoint**

(3) Hyperbolic path

$\because$ $\frac{b^2}{p^2} = \frac{2a}{r} + 1$

$\therefore$ $\frac{h^2}{p^2} = \frac{\mu}{a} = \frac{B}{1}$

$\therefore$ $h^2 = \frac{\mu b^2}{a}$, $B = \frac{\mu}{a}$

$\therefore$ $$\boxed{v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)}$$ **Exampoint**

Obviously here $v^2 > \frac{2\mu}{r}$

Exampoint (1):- From above discussion, it is clear that

- If $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ or $v^2 < \frac{2\mu}{r}$, then path is ellipse
- If $v^2 = \frac{2\mu}{r}$ or $v^2 = \frac{2\mu}{r}$, then path is parabolic
- If $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$ or $v^2 > \frac{2\mu}{r}$, then path is hyperbolic

Exampoint (2):

Clearly, in all three cases:

The magnitude of velocity at any point is independent of the direction of velocity at that point.

Also, we found that

$h^2 = \mu\frac{b^2}{a} = \mu \,.\, l$: in case of elliptic path

$h^2 = 2\mu a = \mu \,.\, l$; in case of parabolic path

$h^2 = \mu\frac{b^2}{a} = \mu \,.\, l$: in case of hyperbolic path

Thus in all cases $h = \sqrt{\mu l}$, where l is the length of semi latus rectum

Exampoint (3): Kepler's law of motion

(i) Each planet describes an ellipse having the sun as one of its foci.

(ii) The radius vector drawn from sun on a planet sweeps out equal area in equal time.

(iii) the squares of periodic times of the various planets are proportional to the cubes of the semi-major axes of their orbits.

Deduction from kepler's Law,

$$T = \frac{\text{Area of ellipse (i.e. the area described)}}{\text{rate of description of sectoral area}}$$

$$T = \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{\sqrt{\mu\left(\frac{b^2}{a}\right)}} = \frac{2\theta a^{3/2}}{\sqrt{\mu}} \left\{\text{Using } h = \frac{\mu b^2}{a}\right\}$$

$$T^2 = \frac{4\pi a^3}{\mu}$$

$\therefore T^2 \propto a^3$

EXAMPLES TO SUBSTANTIATE

Example1:- Find the law of force towards the pole under which the curve $r^n \cos n\theta = a^n$ is described.

Solution:- The equation of the curve is $r^n \cos n\theta = a^n$.

Replacing r by $1/u$, we have $\frac{1}{n^u}\cos n\theta = a^n$ or $a^n u^n = \cos n\theta$ (1)

Taking logarithm of both sides of (1), we have $n\log a + n\log u = \log\cos n\theta$

Differentiating w.r.t 'θ', we have

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\cos n\theta}.(-n\sin n\theta) \text{ or } \frac{du}{d\theta} = -u\tan n\theta \quad (2)$$

Differentiating again w.r.t. 'θ' we have

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta - un\sec^2 n\theta = u\tan^2 n\theta - un\sec^2 n\theta$$

[Substituting for $du/d\theta$ from (2)]

The differential equation of the central orbits is $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$.

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\left(u + u\tan^2 n\theta - nu\sec^2 n\theta\right)$$

$$= h^2u^2\left(\sec^2 n\theta - n\sec^2 n\theta\right) = h^2u^3(1-n)\sec^2 n\theta$$

$$= h^2u^3(1-n).\left(\frac{1}{a^n n^u}\right)^2 = \frac{h^2}{a^{2n}}\frac{(1-n)}{u^{2n-3}} = \frac{h^2(1-n)}{a^{2n}}.r^{2n-3}$$

$\therefore \quad P \propto r^{2n-3}$ i.e. the force is proportional to the $(2n-3)^{th}$ power of the distance from the pole.

Example2:- A particle describes the curve $r^n = A\cos n\theta + B\sin n\theta$ under a force to the pole. Find the law of force.

Solution:- Here $r^n = A\cos n\theta + B\sin n\theta$.

Let $A = k\cos\alpha$ and $B = k\sin\alpha$, where k and α are constants.

Replacing r by $1/u$, we have $r^n = u^{-n} = k\cos(n\theta - \alpha)$ (1)

$\therefore$ $-n\log u = \log k + \log\cos(n\theta - \alpha)$.

Differentiating both sides w.r.t. 'θ', we have $\frac{-n}{u}\frac{du}{d\theta} = -n\tan(n\theta - \alpha)$ or

$$\frac{du}{d\theta} = u\tan(n\theta - \alpha)$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = \frac{du}{d\theta}.\tan(n\theta-\alpha) + un\sec^2(n\theta-\alpha)$$

$$= u\tan^2(n\theta-\alpha) + un\sec^2(n\theta-\alpha)$$

The differential equation of the path is $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$

$$\therefore \quad P = h^2u^2\left[u + u\tan^2(n\theta-\alpha) + un\sec^2(n\theta-\alpha)\right]$$

$$= h^2u^2\left[\sec^2(n\theta-\alpha) + n\sec^2(n\theta-\alpha)\right]$$

$$= (1+n)h^2u^3\sec^2(n\theta-\alpha)$$

$$= (1+n)h^2u^3(ku^n)^2 \qquad [\because \text{ from (1), } \sec(n\theta-\alpha) = ku^n]$$

$$= \frac{(1+n)h^2k^2}{r^{2n+3}}.$$

Thus $P \propto \frac{1}{r^{2n+3}}$ i.e. the force is inversely proportional to the $(2n+3)^{th}$ power of the distance from the pole.

Example3:- A particle describes the curve $r = 2a\cos\theta$ under the force P to the to the pole. Find the law of force.

A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.

Solution:- Let a be the radius of the circle. If we take pole on the circumference of the circle and the diameter through the pole as the initial line, the equation of the circle is $r = 2a\cos\theta$ or $1/u = 2a\cos\theta$ (1)

$\therefore$ $-\log u = \log(2a) + \log\cos\theta$.

Differentiating w.r.t. 'θ', we have $-\frac{1}{u}\frac{du}{d\theta} = -\tan\theta$ or $\frac{du}{d\theta} = u\tan\theta$ and

$$\frac{d^2u}{d\theta^2} = u.\sec^2\theta + \frac{du}{d\theta}\tan\theta$$

$$= u\sec^2\theta + u\tan\theta.\tan\theta = u\sec^2\theta + u\tan^2\theta.$$

The differential equations of the path is $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$.

$$\therefore \quad P = h^2u^2\left[u + u\sec^2\theta + u\tan^2\theta\right] = h^2u^2\left[\left(1+\tan^2\theta\right) + \sec^2\theta\right]$$

$= 2h^2u^3 \sec^2\theta$

$= 2h^2u^2(2au)^2$ [Substituting for $\sec\theta$ from (1)]

$= \dfrac{8a^2h^2}{r^5}$

$\therefore$ $P \propto 1/r^5$ i.e. the force varies inversely as the fifth power of the distance from the pole. Also the positive value of P indicates that the force is attractive.

Example4:- Find the law of force towards the pole under which the following curves are described.

(i) $r^2 = 2ap$, (ii) $p^2 = ar$ and (iii) $b^2/p^2 = (2a/r) - 1$,

Solution:- (i) The equation of the curve is $r^2 = 2ap$.

$\therefore$ $\dfrac{1}{p} = \dfrac{2a}{r^2}$ or $\dfrac{1}{p^2} = \dfrac{4a^2}{r^4}$

Differentiating w.r.t. $'r'$, we have $-\dfrac{2}{p^3}\dfrac{dp}{dr} = -\dfrac{16a^2}{r^5}$.

$\therefore$ $\dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{8a^2h^2}{r^5}$ (1)

Now from the pedal equation of a central orbit, we have

$P = \dfrac{h^3}{p^3}\dfrac{dp}{dr} = \dfrac{8a^2h^2}{r^5}$ [From (1)]

$\therefore$ $P \propto 1/r^5$ i.e. the force varies inversely as the fifth power of the distance from the pole.

(ii) The equation of the curve is $p^2 = ar$, which is the pedal equation of the parabola referred to the focus as the pole.

$\therefore$ $\dfrac{1}{p^2} = \dfrac{1}{a}\dfrac{1}{r}$.

Differentiating w.r.t $'r'$, we get $-\dfrac{2}{p^3}\dfrac{dp}{dr} = -\dfrac{1}{a}\dfrac{1}{r^2}$

$\therefore$ $\dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{h^2}{2a}\dfrac{1}{r^2}$. (1)

From the pedal equation of a central orbit, we have $P = \dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{h^2}{2a}\dfrac{1}{r^2}$ [From (1)]

$\therefore$ $P \propto 1/r^2$ i.e. the force varies inversely as the square of the distance from the pole.

(iii) The equation of the given central orbits is $\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$ (1)

(1) Is the pedal equation of an ellipse referred to the focus as pole.

Differentiating both sides of (1) w.r.t $'r'$, we get $-\dfrac{2b^2}{p^3}\dfrac{dp}{dr} = -\dfrac{2a}{r^2}$, or $\dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{a}{b^2}\dfrac{h^2}{r^2}$.

$\therefore$ $P = \dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{ah^2}{b^2}\dfrac{1}{r^2}$.

Thus $P \propto 1/r^2$ i.e. the acceleration varies inversely as the square of the distance from the focus of the ellipse.

Example5:- A particle describes the curve $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ under an attraction to the origin, prove that the attraction at a distance r is $h^2\left[2\left(a^2+b^2\right)r^2 - 3a^2b^2\right].r^{-7}$

Solution:- The equation of the given curve is $r^2 = a^2\cos^2\theta + b^2\sin\theta$ or $\frac{1}{u^2} = \frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta)$

Or $\frac{1}{u^2} = \frac{1}{2}\left(a^2+b^2\right) + \frac{1}{2}\left(a^2-b^2\right)\cos 2\theta$ (1)

Differentiating w.r.t. 'θ', we have $-\frac{2}{u^3}\frac{du}{d\theta} = -\left(a^2-b^2\right)\sin 2\theta$

Or $\frac{du}{d\theta} = \frac{1}{2}\left(a^2-b^2\right)u^3\sin 2\theta$

Differentiating again w.r.t. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{3}{2}\left(a^2-b^2\right)u^2.\frac{du}{d\theta}\sin 2\theta + \left(a^2-b^2\right)u^3\cos 2\theta$$

$$= \frac{3}{2}\left(a^2-b^2\right)u^2.\frac{1}{2}\left(a^2-b^2\right)u^3\sin 2\theta.\sin 2\theta + \left(a^2-b^2\right)u^3\cos 2\theta$$

$$= \frac{3}{4}u^5\left(a^2-b^2\right)^2\sin^2 2\theta + \left(a^2-b^2\right)u^3\cos 2\theta$$

$$= \frac{3}{4}u^5\left(a^2-b^2\right)^2\left(1-\cos^2 2\theta\right) + u^3\left(a^2-b^2\right)\cos 2\theta$$

$$= \frac{3}{4}u^5\left(a^2-b^2\right)^2 - \frac{3}{4}u^5.\left\{\left(a^2-b^2\right)\cos 2\theta\right\}^2 + u^3\left(a^2-b^2\right)\cos 2\theta$$

Now from (1), $\left(a^2-b^2\right)\cos 2\theta = \frac{2}{u^2} - \left(a^2+b^2\right)$.

$$\therefore \quad \frac{d^2u}{d\theta^2} = \frac{3}{4}u^5\left(a^2-b^2\right)2 - \frac{3}{4}u^5\left\{\frac{2}{u^2} - \left(a^2+b^2\right)\right\}^2 + u^3\left\{\frac{2}{u^2} - \left(a^2+b^2\right)\right\}$$

$$= \frac{3}{4}u^5\left(a^2-b^2\right)^2 - \frac{3}{4}u^5\left\{\frac{4}{u^4} - \frac{4}{u^2}\left(a^2+b^2\right) + \left(a^2+b^2\right)\right\} + 2u - \left(a^2+b^2\right)u^3$$

$$= \frac{3}{4}u^5\left(a^2-b^2\right)^2 - 3u + 3u^3\left(a^2+b^2\right) - \frac{3}{4}u^5\left(a^2+b^2\right)^2 + 2u - \left(a^2+b^2\right)u^3$$

$$= \frac{3}{4}u^5\left\{\left(a^2-b^2\right)^2 - \left(a^2+b^2\right)^2\right\} + 2u^3\left(a^2+b^2\right) - u$$

$$= 2\left(a^2+b^2\right)u^3 - 3a^2b^2u^5 - u$$

The differential equation of the central orbit is $\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$.

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\left[u + 2\left(a^2+b^2\right)u^3 - 3a^2b^2u^5 - u\right]$$

$$= h^2u^2\left[2\left(a^2+b^2\right)r^2 - 3a^2b^2\right] = h^2r^{-7}\left[2\left(a^2+b^2\right)r^2 - 3a^2b^2\right].$$

Example6:- Show that the only law for a central attraction for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to the distance is that of inverse cube.

Solution:- Let the central acceleration P be given by $P = f'(r)$ (1)

[Note the form we have assumed for P]

The equation of motion of the particle falling from infinity under the central acceleration given by (1) is $v\frac{dv}{dr} = -P = -f'(r)$ or $2vdv = -2f'(r)dr$.

Integrating $v^2 = -2\int f'(r)dr + A$, where A is constant of integration.

Or $v^2 = -2f(r) + A$. (2)

Thus the velocity v at a distance r acquired in falling from infinity is given by (2). Again let V be the velocity of the particle moving in a circle under the same central acceleration P at a distance r from the centre of the circle. For a circle with centre at the centre of force pole, we have the central radial attractive acceleration $P =$ the inward normal acceleration V^2 / P

$\therefore$ $P = V^2 / r$ [$\because$ for the circle $\rho = r$]

Or $V^2 = rP = r f'(r)$.

But according to the question $V = v$ or $V^2 = v^2$

$\therefore$ $r f'(r) = -2f(r) + A$ or $r^2 f'(r) + 2r f(r) = Ar$ or $\frac{d}{dr}\{r^2 f(r)\} = Ar$.

Integrating both sides w.r.t. $'r'$, we have $r^2 f'(r) = \frac{1}{2}Ar^2 + B$ m where B is a constant

Or $f(r) = \frac{A}{2} + \frac{B}{r^2}$.

Differentiating both sides w.r.t. $'r'$, we have $f'(r) = \frac{-2B}{r^3}$ so that $P = -\frac{2B}{r^3}$.

$[\because P = f'(r)]$

$\therefore$ $P \propto 1/r^3$ i.e. the law of force is that of inverse cube.

Example7:- In a central orbit described under a force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.

Solution:- If v is the velocity of the particle at any point at a distance r from the centre of force, then according to the question $v \propto \frac{1}{r}$ or $v = \frac{k}{r}$ (1)

Where k is a constant.

But in a central orbit $v = h / p$ (2)

Where p is the length of the perpendicular from the pole on the tangent at any point of the path.

From (1) and (2), we have $\frac{k}{r} = \frac{h}{p}$ or $p = \frac{h}{k}r$

Or $p = ar$, where $a = h/k = a$ constant.

This is the pedal equation of an equiangular spiral. Hence the path is an equiangular spiral.

Example9:- The velocity at any point of a central orbit is $(1/n)^{th}$ of what it would be for a circular orbit at the same distance. Show that the central force varies as $\dfrac{1}{r^{(2n^2+1)}}$ and that the equation of the orbit is $r^{n^2-1} = a^{n^2-1}.\cos(n^2-1)\theta$.

Solution:- Under the same central force P, let v and V be the velocities at a distance r from the centre of force in the central orbit and the circular orbit respectively. Then according to the question, we have $v = V/n$ or $v^2 = V^2/n^2$ (1)

But $V^2/r = P$ or $V^2 = Pr = P/u$ (2)

$\therefore$ From (1) and (2), we have $v^2 = \dfrac{P}{n^2u}$ or $h^2\left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right] = \dfrac{P}{n^2u}$ (3)

[$\because$ For a central orbit, $v^2 = h^2\left\{u^2 + (du/d\theta)^2\right\}$]

Differentiating both sides of (3), w.r.t. 'θ', we have

$$h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{d\theta} - \frac{P}{u^2}\frac{du}{d\theta}\right] = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du}\frac{du}{d\theta} - \frac{P}{u^2}\frac{du}{d\theta}\right].$$

$$\therefore \quad 2h^2\frac{du}{d\theta}.\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\frac{du}{d\theta}\left[\frac{1}{u}\frac{dP}{du} - \frac{P}{u^2}\right].$$

Dividing out by $du/d\theta$, we get $2h^2\left[u + \dfrac{d^2u}{d\theta}\right] = \dfrac{1}{n^2}\left[\dfrac{1}{u}\dfrac{dP}{du} - \dfrac{P}{u^2}\right]$

Or $2.\dfrac{P}{u^2} = \dfrac{1}{n^2}\left[\dfrac{1}{u}\dfrac{dP}{du} - \dfrac{P}{u^2}\right]$ $\left[\because \dfrac{P}{h^2u^2} = u + \dfrac{d^2u}{d\theta^2}\right]$

Or $2n^2.\dfrac{P}{u^2} = \left[\dfrac{1}{u}\dfrac{dP}{du} - \dfrac{P}{u^2}\right]$ or $(2n^2+1)\dfrac{P}{u^2} = \dfrac{1}{u}\dfrac{dP}{du}$

Or $\dfrac{dP}{P} = (2n^2+1).\dfrac{du}{u}$.

Integrating, $\log P = (2n^2+1)\log u + \log A$.

$$\therefore \quad P = Au^{2n^2+1} = \frac{A}{r^{2n^2+1}}$$

$\therefore$ $P \propto \dfrac{1}{r^{2n^2+1}}$, which proves the first result.

Substituting $P = Au^{2n^2+1}$ in (3), we have $h^2\left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right] = \dfrac{Au^{2n^2+1}}{n^2u} = \dfrac{A}{n^2}u^{2n^2}$.

Putting $u = \dfrac{1}{r}$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have $\dfrac{1}{r^2} + \left(-\dfrac{1}{r^2}\dfrac{dr}{d\theta}\right)^2 - \dfrac{A}{n^2h^2r^{2n^2}}$

Or $r^{2n^2} - 2 + r^{2n^2-4}\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{A}{n^2h^2}$

Or $r^{2n^2-4}\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{A}{n^2h^2} - r^{2n^2-2}$

Or $\left(r^{n^2-2}\right)^2\left(\frac{dr}{d\theta}\right)^2 = a^{2n^2-2} - r^{2n^2-2}$ setting $A/n^2h^2 = a^{2n^2-2}$ to get the required from of the answer.

$$\therefore \qquad \frac{dr}{d\theta} = \frac{\sqrt{\left\{a^{2n^2-2} - r^{2n^2-2}\right\}}}{r^{n^2-2}} \text{ or } \frac{r^{n^2-2}\,dr}{\sqrt{\left\{\left(a^{n^2-1}\right)^2 - \left(r^{n^2-1}\right)^2\right\}}} = d\theta.$$

Putting $r^{n^2-1} = z$ so that $\left(n^2-1\right)r^{n^2-2}\,dr = dz$, we have $\frac{dz}{\sqrt{\left\{\left(a^{n^2-1}\right)^2 - z^2\right\}}} = \left(n^2-1\right)d\,\theta$.

Integrating, $\sin^{-1}\left(\frac{z}{a^{n^2-1}}\right) = \left(n^2-1\right)\theta + B$

Or $\sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = \left(n^2-1\right)\theta + B$

Initially when $\theta = 0$, let $r = a$. Then $B = \sin^{-1}1 = \pi/2$

$$\therefore \qquad \sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = \left(n^2-1\right)\theta + \frac{1}{2}\pi \text{ or } \frac{r^{n^2-1}}{a^{n^2-1}} = \sin\left\{\left(n^2-1\right)\theta + \frac{1}{2}\pi\right\} = \cos\left(n^2-1\right)\theta$$

Or $r^{n^2-1} = a^{\left(n^2-1\right)}\cos\left(n^2-1\right)\theta$, which is the required equation of the orbit.

Example10:- A particle moves with a central acceleration μ / (distance)^2, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \frac{2\mu}{R}\right)^{1/2}}\right].$$

Solution:- If the particle describes a hyperbola under the central acceleration μ / (distance)^2, then the velocity v of the particle at a distance r from the centre of force is given by $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$. (1)

Where $2a$ is the transverse axis of the hyperbola?

Since the particle is projected with velocity V at a distance R, therefore from (1), we have

$$V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right) \text{ or } \frac{\mu}{a} = V^2 - \frac{2\mu}{R}. \qquad (2)$$

If α is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$, we have $h = Vp = VR\sin\alpha$ (3)

[$\because$ $p = r\sin\phi$ and initially $r = R, \phi = \alpha$]

Also $h = \sqrt{(\mu l)} = \sqrt{\left\{\mu.\left(b^2/a\right)\right\}} = \sqrt{(\mu a)}$ (4)

[$\because$ $b = a$ for a rectangular hyperbola]

From (3) and (4), we have $VR\sin\alpha = \sqrt{(\mu a)}$ or $\sin\alpha = \frac{\sqrt{(\mu a)}}{VR} = \frac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \frac{\mu}{VR\sqrt{(\mu/a)}}$

Substituting for μ/a from (2), we have $\sin\alpha=\mu/\left\{\left(VR\sqrt{\left(V^2-2\mu/R\right)}\right)\right\}$ or $\alpha=\sin^{-1}\left[\mu/\left\{VR\sqrt{\left(V^2-2\mu/R\right)}\right\}\right]$ which is the required angle of projection.

Example11:- A particle of unit mass describes an equiangular spiral of angle α, under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral; show that the force is $\mu r^{2\sec^2\alpha-3}$ and that the rate of description of sectorial area about the pole is $\frac{1}{2}\sqrt{\left(\mu\sin\alpha\cos\alpha\right)}.r^{\sec^2\alpha}$.

Solution:- Here the particle is moving under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral.

$\therefore$ The central radial acceleration $=\ddot{r}=-r\dot{\theta}^2=0$ (1)

If F is the force on the particle of unit mass, perpendicular to the line joining the particle to the pole, then $F=$ transverse acceleration.

i.e. $F=\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$. (2)

The equation of the equiangular spiral is $r=ae^{\theta\cot\alpha}$ (3)

Differentiating (3), w.r.t. 't', we have $\dot{r}=a\,e^{\theta\cot\alpha}\,\dot{\theta}\cot\alpha=r\,\dot{\theta}\cot\alpha$

or $\dot{\theta}=\frac{\dot{r}}{r}\tan\alpha$. (4)

$\therefore$ From (1) and (4), we have $\ddot{r}=r\left(\frac{\dot{r}}{r}\tan\alpha\right)^2$ or $\frac{\ddot{r}}{\dot{r}}=\frac{\dot{r}}{r}\tan^2\alpha$

Integrating, we have $\log\dot{r}=\left(\tan^2\alpha\right)\log r+\log A$, where A is a constant of integration or

$\log\dot{r}=\log\left(Ar^{\tan 2\alpha}\right)$ or $\dot{r}=Ar^{\tan^2\alpha}$. (5)

Substituting the value of $\dot{r}$ from (5) in (4), we have $\dot{\theta}=\frac{1}{t}\tan\alpha.Ar^{\tan^2\alpha}$

Or $\dot{\theta}=A\tan\alpha.\,r^{\tan^2\alpha-1}$ (6)

$\therefore$ From (2), we have $F=\frac{1}{r}\frac{d}{dt}\left(r^2A\tan\alpha.r^{\tan^2\alpha-1}\right)=\frac{A\tan\alpha}{r}\frac{d}{dt}\left(r^{\tan^2\alpha+1}\right)$

$=\frac{A\tan\alpha}{r}\frac{d}{dt}\left(r^{\sec^2\alpha}\right)=\frac{A\tan\alpha}{r}.\sec^2\alpha\,r^{\sec^2\alpha-1}\dot{r}$

$=A\tan\alpha\sec^2\alpha.\,r^{\sec^2\alpha-2}.Ar^{\tan^2\alpha}$ [Substituting from (5)]

$=A^2\tan\alpha\sec^2\alpha\,r^{\sec^2\alpha-2+\tan^2\alpha}$

$\mu r^{\sec^2\alpha-2+\sec^2\alpha=1}$, where $\mu=A^2\tan\alpha\sec^2\alpha$ (7)

Thus $F=\mu r^{2\sec^2\alpha-3}$, which proves the first part.

Second Part:- The rate of description of the sectorial area $=\frac{1}{2}r^2\dot{\theta}$

$$=\frac{1}{2}r^2 A\tan\alpha\ r^{\tan^2\alpha-1} \qquad \text{[Substituting from (6)]}$$

$$=\frac{1}{2}\tan\alpha\, r^{2+\tan^2\alpha-1}$$

$$=\frac{1}{2}\sqrt{\left(\mu\cot\alpha\cos^2\alpha\right)}\tan\alpha\, r^{\tan^2\alpha+1}$$

[Substituting $A=\sqrt{\left(\mu\cot\alpha\cos^2\alpha\right)}$, from (7)]

$$=\frac{1}{2}\sqrt{\left(\mu\cot\alpha\cos^2\alpha\tan^2\alpha\right)}r^{\sec^2\alpha}=\frac{1}{2}\sqrt{\left(\mu\sin\alpha\cos\alpha\right)}\,r^{\sec^2\alpha}.$$

Example12:- A particle subject to the central acceleration $\left(\mu/r^3\right)+f$ is projected from an apse at a distance $'a'$ with the velocity $\sqrt{\mu}/a$; prove that at any subsequent time $t, r=a-\frac{1}{2}ft^2$.

Solution:- Here the central acceleration $P=\frac{\mu}{r^3}+f=\mu u^3+f$, where $\frac{1}{r}=u$.

$\therefore$ The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{1}{u^2}\left(\mu u^3+f\right) \text{ or } h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu u+\frac{f}{u^2}$$

Multiplying both sides by $2\left(du/d\theta\right)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2-\frac{2f}{u}+A, \qquad (1)$$

Where A is a constant.

But initially when $r=a$ i.e. $u=1/a, du/d\theta=0$ (at an apse) and $v=\sqrt{\mu/a}$.

$\therefore$ From (1), we have $\frac{\mu}{a^2}=h^2\left(\frac{1}{a^2}\right)=\frac{\mu}{a^2}-2f\,a+A$.

$\therefore$ $h^2=\mu$ and $A=2\,fa$..

Substituting the values of h^2 and A in (1), we have $\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2-\frac{2f}{u}+2\,fa$

$$\text{Or } \mu\left(\frac{du}{d\theta}\right)^2=2\,fa-\frac{2f}{u} \qquad (2)$$

Now $u=1/r$, so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$. Therefore, from (2), we have

$$\mu\left(-\frac{1}{2}\frac{dr}{d\theta}\right)^2=2\,fa-2\,fr=2f\left(a-r\right) \text{ or } \left(\frac{dr}{d\theta}\right)^2=\frac{2f\,r^4}{\mu}\left(a-r\right)$$

$$\text{Or } \frac{dr}{d\theta}=-\sqrt{\left(2f/\mu\right)}.r^2\sqrt{\left(a-r\right)}$$

Also $h=r^2\frac{d\theta}{dt}=r^2\frac{d\theta}{dr}.\frac{dr}{dt}$.

$\therefore \qquad \sqrt{\mu} = r^2 . \sqrt{\left(\frac{\mu}{2f}\right)} . \frac{(-1)}{r^2\sqrt{(a-r)}} . \frac{dr}{dt}$

[Substituting for h and $dr/d\theta$]

Or $dt = \frac{-1}{\sqrt{(2f)}} . (a-r)^{-1/2} dr$

Integrating, $t = \frac{1}{\sqrt{(2f)}} . 2(a-r)^{1/2} + B$, where B is constant

But initially when $t = 0, r = a$; $\qquad \therefore B = 0$

$\therefore \qquad t = \sqrt{(2/f)} .(a-r)^{1/2}$

Or $t^2 = (2/f)(a-r)$

Or $a - r = \frac{1}{2} ft^2$. $\qquad \therefore r = a - \frac{1}{2} ft^2$.

Example13:- A particle moves under a repulsive force $m\mu$ / (distance)^3 and is projected from an apse at a distance a with a velocity V; show that the equation to the path is $r \cos p\theta = a$, and that the angle θ described in time t is $(1/p)\tan^{-1}(pVt/a)$, where $p^2 = (\mu + a^2V^2)/(a^2V^2)$.

Solution:- Since the particle moves under a repulsive force $\frac{m\mu}{(distane)^3} = \frac{m\mu}{r^3}$

$\therefore \qquad$ The central acceleration $P = -\frac{\mu}{r^3} = -\mu u^3$.

$\therefore \qquad$ The differential equation of the path is $h^2\left[u^2 + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{-\mu u^3}{u^2} = -\mu u$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + A \qquad (1)$$

Where A is a constant.

But initially at an apse, $r = a, u = 1/a,\ du/d\theta = 0$ and $v = V$

$\therefore \qquad$ From (1), we have $V^2 = h^2\left[\frac{1}{a^2}\right] = -\frac{\mu}{a^2} + A$.

$\therefore \qquad h^2 = a^2V^2$ and $A = V^2 + (\mu/a^2) \qquad (2)$

Substituting the values of h^2 and A in (1), we have $a^2V^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + V^2 + \frac{\mu}{a^2}$

Or $a^2V^2\left(\frac{du}{d\theta}\right)^2 = -(a^2V^2 + \mu)u^2 + \frac{(a^2V^2 + \mu)}{a^2}$

Or $a^2\left(\frac{du}{d\theta}\right)^2 = \frac{(a^2V^2 + \mu)}{a^2V^2}(1 - a^2u^2)$

Or $a^2\left(\frac{du}{d\theta}\right)^2 = p^2(1 - a^2u^2)$, where $p^2 = \frac{\mu + a^2V^2}{a^2V^2}$

Or $a\dfrac{du}{d\theta}=p\sqrt{\left(1-a^2u^2\right)}$ or $pd\,\theta=\dfrac{adu}{\sqrt{\left(1-a^2u^2\right)}}$

Integrating $p\theta+B=\sin^{-1}\left(au\right)$, where B is a constant.

But initially when $u=1/a$, let $\theta=0$. Then $B=\sin^{-1}1=\dfrac{1}{2}\pi$.

$\therefore \quad p\theta+\dfrac{1}{2}\pi=\sin^{-1}\left(au\right)$

Or $au=\sin\left(\dfrac{1}{2}\pi+p\,\theta\right)$

Or $a/r=\cos p\,\theta$

Or $r\cos p\theta=a$ (3)

Which is the equation of the path.

Second Part: - We have $h=r^2\dfrac{d\theta}{dt}$

[Note that in a central orbit for finding the time, we use this formula]

[Substituting for h from (2) and for r from (3)]

Or $aV=a^2\sec^2 p\theta\dfrac{d\theta}{dt}$,

Or $dt=\left(a/V\right)\sec^2 p\,\theta\, d\theta$.

Integrating $t+C=\dfrac{a}{pV}\tan p\theta$.

But initially $t=0$ and $\theta=0$. Therefore $C=0$.

$\therefore \quad t=\dfrac{a}{pV}\tan p\,\theta$ Or $\tan p\,\theta=pV\,t/a$

$\therefore \quad \theta=\left(1/p\right)\tan^{-1}\left(pV\,t/a\right)$, which gives the angle θ described in time t.

Example14:- A particle moves under a central force $m\lambda\left(3a^3u^4+8au^2\right)$. It is projected from an apse at a distance a from the centre of force with velocity $\sqrt{\left(10\lambda\right)}$. Show that second apsidal distance is half of the first and that the equations of the path is $2r=a\left[1+\sec h\left(\theta/\sqrt{5}\right)\right]$.

Solution:- Here the particle moves under the central force $m\,\lambda\left(3a^3u^4+8au^2\right)$. Therefore the central acceleration P is given by $P=\lambda\left(3a^3u^4+8au^2\right)$.

$\therefore$ The differential equation of the path is $h^2\left[u+\dfrac{d^2u}{d\theta^2}\right]=\dfrac{P}{u^2}=\dfrac{\lambda}{u^2}\left(3a^3u^4+8au^2\right)$

Or $h2\left[u+\dfrac{d^2u}{d\theta^2}\right]=\lambda\left(3a^3u^2+8a\right)$.

Multiplying both sides by $2\left(du/d\theta\right)$ and integrating, we have

$h^2\left[2.\dfrac{u^2}{2}+\left(\dfrac{du}{d\theta}\right)^2\right]=2\lambda.\left(a^3u^3+8au\right)+A$

Or $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \lambda\left(2a^3u^3 + 16au\right) + A$ (1)

Where A is a constant

But initially at an apse, $r = a, u = 1/a, du/d\theta = 0$ and $v = \sqrt{(10\lambda)}$.

$\therefore$ From (1), we have $10\lambda = h^2\left[\frac{1}{a^2}\right] = \lambda\left(2a^3.\frac{1}{a^3} + 16a.\frac{1}{a}\right) + A$

$\therefore$ $h^2 = 10a^2\lambda$ and $A = 10\lambda - 18\lambda = -8\lambda$

Substituting the values of h^2 and A in (1), we have

$$10a^2\lambda\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \lambda\left(2a^3u^3 + 16au\right) - 8\lambda$$

Or $10a^2\left(\frac{du}{d\theta}\right)^2 = 2a^3u^3 - 10a^2u^2 + 16au - 8$

Or $5a^2\left(\frac{du}{d\theta}\right)^2 = \left[a^3u^3 - 5a^2u^2 + 8au - 4\right]$

$= a^2u^2(au - 1) - 4au(au - 1) + 4(au - 1)$

$= (au - 1)\left(a^2u^2 - 4au + 4\right)$

$= (au - 1)(au - 2)^2$ (2)

To find the second apsidal distance:- At an apse, we have $du/d\theta = 0$.

$\therefore$ From (2), $0 = (au - 1)(au - 2)^2$

Or $u = 1/a$ and $2/a$ or $r = a$ and $a/2$.

But $r = a$ is the first apsidal distance. Therefore the second apsidal distance is $a/2$ which is half of the first.

To find the equation of the path:- From equation (2), we have

$$\sqrt{5}a\frac{du}{d\theta} = -(au - 2)\sqrt{(au - 1)}$$

$\therefore$ $$\frac{d\theta}{\sqrt{5}} = \frac{-a\,du}{(au - 2)\sqrt{au - 1}}$$

Substituting $au - 1 = z^2$, so that $adu = 2z\,dz$, we have $\frac{d\theta}{\sqrt{5}} = \frac{-2z\,dz}{\left(z^2 - 1\right)z}$

Or $\frac{d\theta}{2\sqrt{5}} = \frac{dz}{1 - z^2}$

Integrating, $\frac{\theta}{2\sqrt{5}} + B = \tan h^{-1} z$, where B is a constant

or $\frac{\theta}{2\sqrt{5}} + B = \tan h^{-1}\sqrt{(au - 1)}$ (3)

But initially when $u = 1/a$, $\theta = 0$

$\therefore$ From (3), $B = 0$.

Putting $B = 0$ in (3), we get $\frac{\theta}{2\sqrt{5}} = \tan h^{-1}\sqrt{(au - 1)}$

Or $\tan h\left(\frac{\theta}{2\sqrt{5}}\right)=\sqrt{(au-1)}$

Now $\cos h 2A=\frac{1+\tan h^2 A}{1-\tan h^2 A}$.

$\therefore \quad \cos h\left(\frac{\theta}{\sqrt{5}}\right)=\frac{1+\tan h^2\left(\theta/2\sqrt{5}\right)}{1-\tan h^2\left(\theta/2\sqrt{5}\right)}=\frac{1+(au-1)}{1-(au-1)}=\frac{au}{2-au}$

Or $2-au=au\sec h\left(\theta/\sqrt{5}\right)$

Or $2=au\left[1+\sec h\left(\theta\sqrt{5}\right)\right]=(a/r)\left[1+\sec h\left(\theta\sqrt{5}\right)\right]$

Or $2r=a\left[1+\sec h\left(\theta/\sqrt{5}\right)\right]$, which is the required equation of the path.

Example15:- A particle describes an orbit with a central acceleration $\mu u^3-\lambda u^5$ being projected from an apse at a distance a with velocity equal to that from infinity. Show that its path is $r=a\cos h(\theta/n)$, where $n^2+1=2\mu a^2/\lambda$.

Prove also that it will be at a distance r at the end of time $\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[a^2\log\left\{\frac{x+\sqrt{(r^2-a^2)}}{a}\right\}+r\sqrt{(r^2-a^2)}\right]$.

Solution:- Here, the central acceleration $P=\mu u^3-\lambda u^5=\frac{\mu}{r^3}-\frac{\lambda}{r^5}$.

Let V be the velocity from infinity at the distance a under the same acceleration. Then

$$V^2=-2\int_\infty^a P\,dr=-2\int_\infty^a\left(\frac{\mu}{r^3}-\frac{\lambda}{r^5}\right)dr$$

$$=-2\left[-\frac{\mu}{2r^2}+\frac{\lambda}{4r^4}\right]_\infty^a=\frac{\mu}{a^2}-\frac{\lambda}{2a^4}$$

$$=\frac{\lambda}{2a^4}\left(\frac{2\mu a^2}{\lambda}-1\right)=\frac{\lambda n^2}{2a^4} \qquad \left[\because\ n^2+1=\frac{2\mu a^2}{\lambda}\right]$$

$\therefore \quad V=\left(n/a^2\right)\sqrt{(\lambda/2)}$.

The differential equation of the path is $h^2\left\{u+\frac{d^2u}{d\theta^2}\right\}=\frac{P}{u^2}=\frac{\mu u^3-\lambda u^5}{u^2}=\mu u-\lambda u^3$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have $h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}=2\left(\frac{\mu u^2}{2}-\frac{\lambda u^4}{4}\right)+A$, where A is a constant.

Or $v^2=h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}=\mu u^2-\frac{\lambda u^4}{2}+A$. (1)

But initially when $r=a$ i.e. $u=1/a$, $du/d\theta=0$ (at an apse) $v=V=(n/a^2)\sqrt{(\lambda/2)}$.

Therefore from (1), we have $\frac{\lambda n^2}{2a^4}=h^2\left\{\frac{1}{a^2}\right\}=\frac{\mu}{a^2}-\frac{\lambda}{2a^4}+A$.

$\therefore \quad h^2=\frac{\lambda n^2}{2a^2}$ and $A=\frac{\lambda n^2}{2a^4}-\left(\frac{\mu}{a^2}-\frac{\lambda}{2a^4}\right)=\frac{\lambda}{2a^4}(n^2+1)-\frac{\mu}{a^2}$

$=\frac{\lambda}{2a^4}\cdot\left(\frac{2\mu a^2}{\lambda}\right)-\frac{\mu}{a^2}=0$. $\quad\left[\because n^2+1=\frac{2\mu a^2}{\lambda}\right]$

Substituting the values of h^2 and A in (1), we have $\frac{\lambda n^2}{2a^2}\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}=\mu u^2-\frac{\lambda u^4}{2}$

$=\frac{\lambda}{2a^2}(n^2+1)u^2-\frac{\lambda u^4}{2}$ $\quad\left[\because n^2+1=\frac{2\mu a^2}{\lambda}\right]$

Or $n^2u^2+n^2\left(\frac{du}{d\theta}\right)^2=(n^2+1)u^2-a^2u^4$

Or $n^2\left(\frac{du}{d\theta}\right)^2=u^2-a^2u^4$.

Putting $u=\frac{1}{r}$ so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$, we have $n^2\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}-\frac{a^2}{r^4}$ or

$n^2\left(\frac{dr}{d\theta}\right)^2=r^2-a^2$ or $\frac{dr}{d\theta}=\frac{\sqrt{(r^2-a^2)}}{n}$ (2)

Or $\frac{d\theta}{n}=\frac{dr}{\sqrt{(r^2-a^2)}}$

Integrating $\theta/n+B=\cosh^{-1}(r/a)$, where B is a constant.

But initially when $r=a$, $\theta=0$ (say). Then $B=\cosh^{-1}(1)=0$.

$\therefore \quad \theta/n=\cosh^{-1}(r/a)$ or $r=a\cosh(\theta/n)$, which is the required equation of the path.

Second Part:- We know that $h=r^2\frac{d\theta}{dt}$ or $h=r^2\frac{d\theta}{dr}\cdot\frac{dr}{dt}$.

Substituting for h and $dr/d\theta$, we have $\frac{n}{a}\sqrt{\left(\frac{\lambda}{2}\right)}=r^2\cdot\frac{n}{\sqrt{(r^2-a^2)}}\frac{dr}{dt}$.

Or $dt=a\sqrt{\left(\frac{2}{\lambda}\right)}\frac{r^2dr}{\sqrt{(r^2-a^2)}}$

Integrating, the time t from the distance a to the distance r is given by

$$t=a\sqrt{(2/\lambda)}\int_{r=a}^{r}\frac{r^2dr}{\sqrt{(r^2-a^2)}}=a\sqrt{(2/\lambda)}\int_a^r\frac{(r^2-a^2)+a^2}{\sqrt{(r^2-a^2)}}dr$$

$$= a\sqrt{(2/\lambda)}\ \int_a^r \left\{\sqrt{(r^2-a^2)} + \frac{a^2}{\sqrt{(r^2-a^2)}}\right\} dr$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2-a^2)} - \frac{a^2}{2}\log\left\{r+\sqrt{(r^2-a^2)}\right\} + a^2\log\left\{r^2+\sqrt{(r^2-a^2)}\right\}\right]_a^r$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2-a^2)} + \frac{a^2}{2}\log\left\{r+\sqrt{(r^2-a^2)}\right\}\right]_a^r$$

$$a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2-a^2)} + \frac{a^2}{2}\log\left\{r+\sqrt{(r^2-a^2)}\right\} - \frac{a^2}{2}\log a\right]$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2-a^2)} + \frac{a^2}{2}\log\left\{\frac{r+\sqrt{(r^2-a^2)}}{a}\right\}\right]$$

$$= \sqrt{(a^2/2\lambda)}\left[r\sqrt{(r^2-a^2)} + a^2\log\left\{\frac{r+\sqrt{(r^2-a^2)}}{a}\right\}\right].$$

Example16:- A particle is acted on by a central repulsive force which varies as the nth power of the distance. If the velocity at any point be equal to that would be acquired in falling from the centre to the point, show that the equation to the path is of the form $r^{(n+3)/2}\cos\frac{1}{2}(n+3)\theta =$ constant.

Solution:- Since the particle is acted on b y a central repulsive force which varies as the n[th] power of the distance, therefore the central acceleration $P = -\mu$ (distance)$^n = -\mu r^n = -\mu/u^n$.

While falling in a straight line from rest from the centre of force if v is the velocity of the particle at a distance x from the centre, then $v\frac{dv}{dx} = \mu x^n$ ro $v\,dv = \mu x^n\,dx$.

$\int_0^V v\,dv = \int_0^r \mu x^n\,dx$ or $\frac{1}{2}V^2 = \mu\left[\frac{x^{n+1}}{n+1}\right]_0^r = \frac{\mu}{n+1}r^{n+1}$

Or $V^2 = \{2\mu/(n+1)\}\ r^{n+1}$ (1)

The differential equation of the central orbit is $h^2\left[u^2+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = -\frac{\mu/u^n}{u^2} = -\mu u^{-n-2}$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have $v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{-2\mu u^{-n-1}}{(-n-1)} + A = \frac{2\mu}{(n+1)u^{n+1}} + A$, (2)

where A is constant and v is the velocity of the particle in the orbit at a distance r from the centre.

But according to the question, we have $v^2 = V^2$ i.e. $\frac{2\mu}{(n+1)u^{n+1}} + A = \frac{2\mu}{n+1}r^{n+1}$.

$\therefore \qquad A = 0 \qquad\qquad [\because u = 1/r]$

Substituting the value of A in (2), we have $h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{(n+1)u^{n+1}}$, or

$u^2+\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{(n+1)h^2u^{n+1}}=\frac{\lambda^2}{u^{n+1}}$, where $\lambda^2=\frac{2\mu}{(n+1)h^2}=$ constant or

$\left(\frac{du}{d\theta}\right)^2=\frac{\lambda^2}{u^{n+1}}-u^2=\frac{\lambda^2-u^{n+3}}{u^{n+1}}$ or $\frac{du}{d\theta}=-\frac{\sqrt{(\lambda^2-u^{n+3})}}{u^{(n+1)/2}}$ or $d\theta=\frac{-u^{(n+1)/2}du}{\sqrt{(\lambda^2-u^{n+3})}}$

Substituting $u^{(n+3)/2}=z$, so that $\frac{1}{2}(n+3)u^{(n+1)/2}\,du=dz$, we have

$d\theta=\frac{-2\,dz}{(n+3)\sqrt{(\lambda^2-z^2)}}$ or $\frac{1}{2}(n+3)d\theta=-\frac{dz}{\sqrt{(\lambda^2-z^2)}}$

Integrating $\frac{1}{2}(n+3)\theta+B=\cos^{-1}(z/\lambda)=\cos^{-1}\left\{u^{(n+3)/2}/\lambda\right\}$ (3)

Now choose λ such that when $u=1/a,\ \theta=0$, $(1/\lambda)(1/a)^{(n+3)/2}=1$.

Then from (3), $0+B=\cos^{-1}1=0$. Therefore $B=0$

Putting $B=0$ in (3), we have $\frac{1}{2}(n+3)\theta=\cos^{-1}\left\{u^{(n+3)/2}/\lambda\right\}$ or

$u^{(n+3)/2}=\lambda\cos\left\{\frac{1}{2}(n+3)\theta\right\}$

Or $r^{(n+3)/2}\cos\left\{\frac{1}{2}(n+3)\theta\right\}=1/\lambda=$ constant

This gives the required equation to the path.

Example17:- A particle subject to a force producing an acceleration $\mu(r+2a)/r^5$ towards the origin is projected from the point $(a,0)$ with a velocity equal to the velocity from infinity at an angle $\cot^{-1}2$ with the initial line; show that the equation to the path is $r=a(1+2\sin\theta)$.

Solution:- Here, the central acceleration $P=\frac{\mu(r+2a)}{r^5}=\mu\left(\frac{1}{r^4}+\frac{2a}{r^5}\right)=\mu\left(u^4+2au^5\right)$

Let V be the velocity of the particle acquired in falling from rest from infinity the same acceleration to the point of projection which is at a distance a from the centre. Then $V^2=-2\int_\infty^a P\,dr=-2\int_\infty^a\mu\left(\frac{1}{r^4}+\frac{2a}{r^5}\right)dr$

$=-2\mu\left[-\frac{1}{3r^3}-\frac{2a}{4r^4}\right]_\infty^a=2\mu\left[\frac{1}{3a^3}+\frac{1}{2a^3}\right]=\frac{5\mu}{3a^3}$ or $V=\sqrt{(5\mu/3a^3)}$

According to the question the velocity of projection of the particle is equal to V i.e. $\sqrt{(5\mu/3a^3)}$.

Now the differential equation of the path is $h^2\left[u^2+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left(u^4+2au^5\right)=\mu\left(u^2+2au^3\right)$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(\frac{2u^3}{3}+au^4\right)+A \qquad (1)$$

Where A is a constant.

Initially when $r=a$ i.e. $u=1/a, v=\sqrt{\left(5\mu/3a^3\right)}$.

Also initially $\phi=\cot^{-1}2$ or $\cot\phi=2$ or $\sin\phi=1/\sqrt{5}$.

But $p=r\sin\phi$. Therefore initially $p=a\left(1/\sqrt{5}\right)=a/\sqrt{5}$ or $1/p^2=5/a^2$.

But $1/p^2=u^2+\left(du/d\theta\right)^2$. Therefore initially, when $r=a$ we have $u^2+\left(du/d\theta\right)=5/a^2$.

Applying the above initial conditions in (1), we have $\frac{5\mu}{3a^3}=h^2\frac{5}{a^2}=\mu\left(\frac{2}{3a^3}+\frac{a}{a^4}\right)+A$

$\therefore \qquad h^2=\mu/3a, A=0$

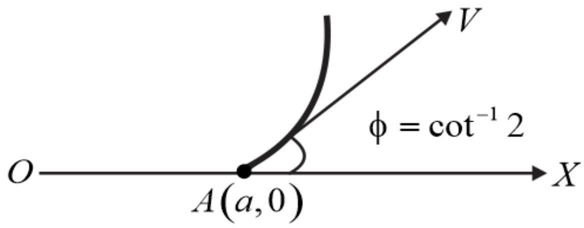
Substituting the values of h^2 and A in (1), we have $\frac{\mu}{3a}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(\frac{2}{3}u^3+au^4\right)$ or $\left(\frac{du}{d\theta}\right)^2=2au^3+3a^2u^4-u^2$,

Putting $u=\frac{1}{r}$, so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$, we have $\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{2a}{r^3}+\frac{3a^2}{r^4}-\frac{1}{r^2}$ or

$\left(dr/d\theta\right)^2=2ar+3a^2-r^2=3a^2-\left(r^2-2ar\right)=3a^2-\left(r-a\right)^2+a^2=4a^2-\left(r-a\right)^2$

Or $dr/d\theta=\sqrt{\left[\left(2a\right)^2-r\left(r-a\right)^2\right]}$

[Note that as the particle starts moving from A, r increase as θ increases. So we have taken $dr/d\theta$ with $+ive$ sign.]



Or $d\theta=\frac{dr}{\sqrt{\left[\left(2a\right)^2-\left(r-a\right)^2\right]}}$

Integrating $\theta+B=\sin^{-1}\left(\frac{r-a}{2a}\right)$

But initially when $r=a,\ \theta=0$. $\qquad \therefore\ B=\sin^{-1}0=0$

$\therefore \quad \theta = \sin^{-1}\left(\frac{r-a}{2a}\right)$ or $\sin\theta = \frac{r-a}{2a}$ or $r = a(1+2\sin\theta)$, which is the required equation of the path.

Example(18):- A particle moves in a curve under a central acceleration so that its velocities at any point is equal to that in a circle at the same distance and under the same attraction. Show that the law of force is that of inverse cube, and the path is an equiangular spiral.

Solution:- Let the central acceleration $P = u^2\phi'(u)$. [Note]

The differential equation of the central orbit is $h^2\left\{u+\frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \phi'(u)$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\} = 2\phi(u)+A \qquad (1)$$

But according to the question the velocity of the particle at any point in the orbit is equal to that in a circle at the same distance under the same acceleration.

$$\therefore \quad \frac{v^2}{r} = P \text{ or } v^2 = rP = ru^2\phi'(u) = u\phi'(u) \qquad (2)$$

Substituting the value of v^2 from (2) in (1), we have $u\phi' = 2\phi(u)+A$ or $\frac{\phi'(u)}{u^2} - \frac{2\phi(u)}{u^3} = \frac{A}{u^3}$ [dividing both sides by u^3] or $\frac{d}{du}\left[\frac{1}{u^2}\phi(u)\right] = \frac{A}{u^3}$.

Integrating $\frac{1}{u^2}\phi(u) = -\frac{A}{2u^2}+B$ or $\phi(u) = -\frac{A}{2}+Bu^2$

Differentiating w.r.t. 'u', we have $\phi'(u) = 2Bu$.

$$\therefore \quad P = u^2\phi'(u) = u^2.2Bu = 2Bu^3 = 2B/r^3 \qquad (3)$$

Or $P \propto 1/r^3$ i.e. the acceleration varies inversely as the cube of the distance from the centre.

To find the equation of the path. The differential equation of the path in pedal form is $\frac{h^2}{p^3}\frac{dp}{dr} = P$.

$$\therefore \quad \frac{h^2}{p^3}\frac{dp}{dr} = \frac{2B}{r^3} \qquad \left[\because \text{ from}(3), P = \frac{2B}{r^3}\right]$$

Or $-\frac{2h^2}{p^3}dp = -\frac{4B}{r^3}dr$. [Multiplying both sides by -2]

Integrating, $\frac{h^2}{p^2} = \frac{2B}{r^2}+C$.

If $p\to\infty$ when $r\to\infty$, we have $C=0$.

$\therefore \quad p = ar$, where a is a constant.

This is the pedal equation of an equiangular spiral and is the required path.

Example(19):- A particle moves in a plane under a central force which varies inversely as the square of the distance from the fixed point, find the orbit.

Solution:- We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is $\frac{h^2}{p^3}\frac{dp}{dr} = P$ (1)

Where P is the central acceleration assumed to be attractive.

Here $P = \mu / r^2$. Putting $P = \mu / r^2$ in (1), we get $\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$ or $\frac{h^2}{p^3}dp = \frac{\mu}{r^2}dr$ or

$-2\frac{h^2}{p^3}dp = -\frac{2\mu}{r^2}dr$

Integrating both sides, we get $v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C$ (2)

Let $v = v_0$ when $r = r_0$.

Then $v_0^2 = \frac{2\mu}{r_0} + C$ or $C = v_0^2 - \frac{2\mu}{r_0}$

Putting this value of C in (2), the pedal equation of the central orbit is

$\frac{h^2}{p^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0}$ (3)

Case I:- Let $v_0^2 = \frac{2\mu}{r_0}$. Then the equation (3) becomes $\frac{h^2}{p^2} = \frac{2\mu}{r}$ which is of the form $p^2 = ar$.

This is the pedal equation of a parabola referred to focus as pole.
Hence in this case the orbit is a parabola with centre of force at the focus.

Case II:- Let $v_0^2 < \frac{2\mu}{r_0}$. In this case the equation (3) reduces to the form $\frac{b^2}{p^2} = \frac{2a}{r} - 1$

This is the pedal equation of an ellipse referred to a focus as pole.
Hence in this case the orbit is an ellipse with centre of force at its focus.

Case III:- Let $v_0^2 > \frac{2\mu}{r_0}$. In this case the equation (3) reduces to the form $\frac{b^2}{p^2} = \frac{2a}{r} + 1$.

This is the pedal equation of a hyperbola referred to a focus as pole. It represents that branch of the hyperbola which is nearer to the focus taken as pole.
Hence we conclude the under inverse square law the central orbit is always a conic with centre of force at the focus.

Example(20):- If the central force varies inversely as the cube of the distance from a fixed point, find the orbit.

Solution:- We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is $\frac{h^2}{p^3}\frac{dp}{dr} = p$ (1)

Where P is the central acceleration assumed to be attractive.

Here $P = \mu / r^3$. Putting $P = \mu / r^3$ in (1), we get $\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^3}$ or $\frac{h^2}{p^3}dp = \frac{\mu}{r^3}dr$ or $-\frac{2h^2}{p^3}dp = -\frac{2\mu}{r^3}dr$.

Integrating both sides, we get $\frac{h^2}{p^2} = \frac{\mu}{r^2} + C$ (2)

Let $p \to \infty$ as $r \to \infty$. Then $0 = 0 + C$ or $C = 0$

Putting $C = 0$ in (2), the pedal equation of the orbit is $\frac{h^2}{p^2} = \frac{\mu}{r^2}$ or $p^2 = \frac{h^2}{\mu}r^2$ or $p = ar$ where a is some constant.

This is the pedal equation of an equiangular spiral. [Note that the pedal equation of the equiangular spiral $r = a e^{\theta \cot \alpha}$ is $p = r \sin \alpha$]

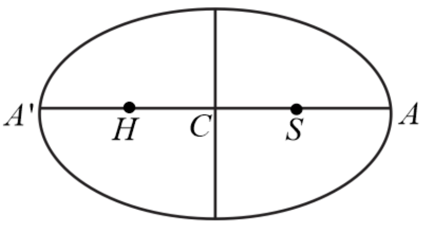
Hence under inverse cube law the central orbit is an equiangular spiral.

The Inverse Square Law

Example:- (1) If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that $(1-e)v_1 = (1+e)v_2$.

Solution:- The path of a planet is an ellipse with the sum at its focus. Therefore the velocity v of the planet at a distance r from the focus S (the sun) is given by $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ (1)

Let v_1 and v_2 be the velocities of a planet at the point A and A' which are nearest and farthest from the sun at S. Then at A, $r = SA = CA - CS = a - ae$, $v = v_1$ and at $A', r = sA' = CS + CA' = ae + a$, $v = v_2$.



Substituting these values in (1), we have $v_1^2 = \mu\left(\frac{2}{a-ae} - \frac{1}{a}\right) = \mu\left\{\frac{2-(1-e)}{a(1-e)}\right\} = \mu\frac{(1+e)}{a(1-e)}$

and $v_2^2 = \mu\left\{\frac{2}{ae+a} - \frac{1}{a}\right\} = \mu\left\{\frac{2-(1+e)}{a(1+e)}\right\} = \mu\frac{(1-e)}{(1+e)}$.

Dividing, we have $\frac{v_1^2}{v_2^2}=\frac{(1+e)^2}{(1-e)^2}$ or $\frac{v_1}{v_2}=\frac{1+e}{1-e}$ or $(1-e)v_1=(1+e)v_2$

Example:- (2) The greatest and least velocities of a certain planet in its orbit round the sum are 30km/sec. and 29.2 km/sec. respectively. Find the eccentricity of the orbit.

Solution:- Refer figure of Ex. 1 example:-

The path of the planet round the sum is an ellipse with the sun as the focus S. Therefore the velocity of the planet at a point distance r from S is given by $v^2=\mu(2/r-1/a)$. (1)

From(1), it is evident that the velocity of the planet is greatest or least according as r is least of greatest. Thus the velocity is greatest at A and least at A'. Therefore according to the question, $v=30\,km/\sec$. When $r=SA=CA-CS=a-ae=a(1-e)$ and $v=29.2\,km/\sec$. when $r=SA'=CA'+CS=a+ae=(1+e)$.

Putting these values in (1), we have $30^2=\mu\left[\frac{2}{a(1-e)}-\frac{1}{a}\right]=\frac{\mu(1+e)}{a(1-e)}$ and $(29.2)^2=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a}\right]=\frac{\mu(1-e)}{a(1+e)}$.

Dividing, we have $\left(\frac{1+e}{1-e}\right)^2=\left(\frac{30}{29.2}\right)^2$

Or $(1+e)/(1-e)=30/29.2$ or $(29.2)(1+e)=30(1-e)$ or $e(29.2+30)=30-29.2$ or $e=(0.8)/(59.2)=1/74$.

Example:- (3) A particle is projected from the earth's surface with velocity v. Show that if the diminution of gravity is taken into account, but the resistance of the air neglected, the path is an ellipse, of major axis $2ga^2/(2ga-v^2)$, where a is the earth's radius.

Solution:- If $2a_1$ is the major axis of the ellipse described by the particle, then its velocity V at a distance r from the centre of the earth is given by $V^2=\mu(2/r-1/a_1)$ (1)

But on the surface of the earth, we have
$r=$ the radius of the earth $=a$.
Also the particle has been projected with velocity v from the earth's surface. Therefore putting $r=a$ and $V=v$ in (1), we have $v^2=\mu\left(\frac{2}{a}-\frac{1}{a_1}\right)$

(2)

Now for a particle on the surface of the earth the acceleration $'g'$ due to gravity is given by $g=\mu/a^2$ so that $\mu=a^2g$,

Substituting the value of μ in (2), we have $v^2=ga^2(2/a-1/a_1)$

Or $2/a-1/a_1=v^2/ga^2$ or $1/a_1=2/a-v^2/ga^2=(2ga-v^2)/ga^2$ or $a_1ga^2/(2ga-v^2)$.

Hence the length of the major axis of the ellipse described $=2a_1=2ga^2/(2ag-v^2)$.

Example:- (4) A particle describes an ellipse under force μ /(distance)^2 towards the focus. If it was projected with velocity V from a point distant r from the centre of force, show that its periodic time is $\left(2\pi/\sqrt{\mu}\right).\left[2/r-V^2/\mu\right]^{-3/2}$.

Solution:- Let a be the length of the semi-major axis of the ellipse described by the particle. Then the velocity V at a point distant r from the centre of force is given b y $V^2=\mu\left[2/r-1/a\right]$

$\therefore \quad V^2/\mu=2/r-1/a$ or $1/a=2/r-V/\mu$ or $a=\left[2/r-V^2/\mu\right]^{-1}$.

$\therefore \quad$ The periodic time $=\dfrac{2\pi a^{3/2}}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\mu}}\left[\dfrac{2}{r}-\dfrac{V^2}{\mu}\right]^{-3/2}$

Example:- (5) Show that an unresisted particle falling to the earth's surface from a great distance would acquire a velocity $\sqrt{(2ga)}$, where a is the radius of the earth.

Solution:- When the particle is at a distance x from the centre of the earth, its acceleration due to the attraction of the earth is μ/x^2 and is directed towards the centre of the earth. On the surface of the earth $x=a$ and the acceleration due to gravity is g. Therefore $\mu/a^2=g$ or $\mu=a^2g$

Now a particle falls unresisted to the earth's surface from a great distance. The only force acting on the particle is the attraction of the earth. If v is velocity of the particle at a distance x from the centre of the earth, we have $v\left(dv/dx\right)=-\mu/x^2=-a^2g/x^2$

$\therefore \quad vdv=-\left(a^2g/x^2\right)dx$.

If V is velocity acquired by the particle on the surface of the earth, we have $\int_0^V v\,dv=-\int_\infty^a \dfrac{a^2g}{x^2}dx$ or $\left[\dfrac{1}{2}v^2\right]_0^V=a^2g\left[\dfrac{1}{x}\right]_\infty^a$ or $V^2/2=a^2g/a$ or $V^2=2ag$ or $V=\sqrt{(2ag)}$.

Example:- (6) If the velocity of the earth at any point of its orbit, assumed to be circular, were increased by about one-half, prove that it would describe a parabola about the sun as focus. Show also that , if a body was projected from the earth with a velocity exceeding 7 miles per second, if will not return to the earth and may even leave the solar system.

Solution:- Let a be the radius of the earth's orbit(supposed to be circular) with sun as centre. If v_1 is the velocity in a circular central orbit at a distance a, we have v_1^2/a (i.e. normal acceleration) $=\mu/a^2$ (i.e. the central acceleration),

$\therefore \quad v_1^2=\mu/a \qquad (1)$

If v_2 is the velocity in a parabolic path at a distance $r=a$ from the focus, then from 2 of this chapter, $v_2^2=2\mu/a \qquad (2)$

From (1) and (2), we have $v_2^2=2v_1^2$ or $v_2=\sqrt{2}v_1 \qquad (3)$

Or $v_2 = v_1 + \left(\sqrt{2} - 1\right)v_1 = v_1 + \frac{1}{2}v_1$ (approximately)

$[\because \sqrt{2} - 1 = \frac{1}{2}$ (approx.)]

Thus the velocity in a parabolic path is $\frac{3}{2}$ times the velocity at the same distance in the circular path. Hence if the velocity of the earth in circular path be increased by about one half of itself at the same distance, then it would describe a parabola about the sun as the focus.

Second Part:- Let V be the least velocity of projection from the surface of the earth so that the body will not return to the earth. Then for this velocity of projection the path of the body is a parabola with focus at the earth's same centre. Therefore if R be the radius of the earth, then from (2), this velocity V on the surface of the earth is given by $V = \sqrt{(2\mu / R)}$.

Also on the surface of the earth the acceleration $g = \mu / R^2$.

$\therefore \quad \mu = R^2 g$

$\therefore \quad V = \sqrt{\left(\frac{2R^2 g}{R}\right)} = \sqrt{(2Rg)}$

But $R = 4000$ miles $= 4000 \times 1760 \times 3$ ft. and $g = 32$ ft./sec^2

$= V = \sqrt{[2 \times 4000 \times 1760 \times 3 \times 32]}$ ft./sec

$= \frac{\sqrt{(2 \times 4000 \times 1760 \times 3 \times 32)}}{1760 \times 3}$ miles/sec

$= \sqrt{\left(\frac{2 \times 4000 \times 32}{1760 \times 3}\right)}$ miles/sec.

$= 7$ miles/sec approximately.

Hence if a body is projected from the earth's surface with a velocity exceeding 7 miles per second, it will not return to the earth.

Also we know that the velocity of the earth, say v_1 s 18.5 miles/sec. nearly. Therefore using the result (3), if it is changed to $v_2 = \sqrt{2v_1} = (185)\sqrt{2}$ miles/sec., then it will describe a parabolic path.

But $v_2 = (18.5)\sqrt{2}$ miles/sec. $= 26$ miles/sec. nearly $= (v_1 + 7.5)$ miles/sec, nearly.

Hence if a body were projected from the earth's surface in the direction of the earth's velocity with a velocity 7.5 miles/sec. more than the velocity of the earth, it would describe a parabola with the sun as its focus. But the parabola is an open curve and so the body will go to infinity and will leave the solar system.

Example:- (7) Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen to the point from rest at a distance from the sun equal to the length of the major axis.

Solution:- If V is the velocity of a patent in the elliptic path at a distance r from the sun, then $V^2 = \mu(2/r - 1/a)$. (1)

Now let the planet fall from rest at a distance $2a$ (length of the major axis of the elliptic path) from the sun. If at any time t then planet is at a distance x from the sun and v is its velocity there, then the equation of motion of the planet is $v(dv/dx) = -\mu/x^2$.

$\therefore$ $v\,dv = -(\mu/x^2)dx$.

Integrating, we get $\frac{1}{2}v^2 = \mu/x + C$, where C is a constant

But initially when $x = 2a$, $v = 0$. Therefore $C = -\mu/2a$

$\therefore$ $\frac{1}{2}v^2 = \mu/x - \mu/2a$ or $v^2 = \mu(2/x - 1/a)$. (2)

If v_1 is the velocity of the planet in this case at a distance r from the sun, then putting $v = v_1$ and $x = r$ in (2), we get $v_1^2 = \mu(2/r - 1/a)$ (3)

From (1) and (3), we observe that $V = v_1$

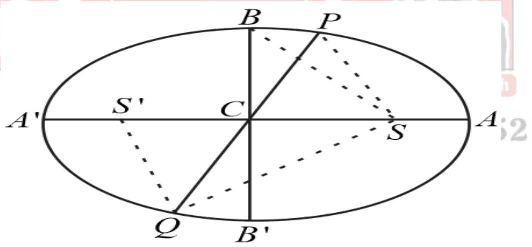
Example:- (8)(a) A particle describes an ellipse as a central orbit about the focus. Prove that the velocity at the end of the minor axis is the geometric mean between the velocities at the ends of any diameter.

Solution:- Let AA' and BB' be the major and minor axes of the ellipse.

Let S, S' be the foci and PQ any diameter of the ellipse.

The velocity v of the particle at any point of the ellipse a distance r from the focus S is given by $v^2 = \mu(2/r - 1/a)$ (1)

Where $2a$ is the length of the major axis of the ellipse.



We have $SB + S'B = 2a$ and $SB = S'B$.

$\therefore$ $SB = a$

Let V, V_1 and V_2 be the velocities of the particle at the points B, P and Q respectively. Then at $B, r = SB = a, v = V$; at P, $r = SP, v = V_1$; and at $Q, r = SQ, v = V_2$.

$\therefore$ From (1), we have $V^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a}$ (2)

$V_1^2 = \mu\left(\frac{2}{SP} - \frac{1}{a}\right)$ and $V_2^2 = \mu\left(\frac{2}{SQ} - \frac{1}{a}\right)$

Now $V_1^2 V_2^2 = \mu^2\left(\frac{2}{SP} - \frac{1}{a}\right)\left(\frac{2}{SQ} - \frac{1}{a}\right)$

$$= \mu^2\left[\frac{4}{SP.SQ} - \frac{2}{a}\left(\frac{1}{SP} + \frac{1}{SQ}\right) + \frac{1}{a^2}\right]$$

$$= \mu^2\left[\frac{4}{SP.SQ} - \frac{2}{a}\left(\frac{SQ + SP}{SP.SQ}\right) + \frac{1}{a^2}\right] \quad (3)$$

We have $SP + SQ = QS' + SQ$ $\quad [\because QS' = SP]$

$= 2a$ [by a property of the ellipse]

Substituting in (3), we have $V_1^2 \, V_2^2 = \mu^2 \left[\dfrac{4}{SP.SQ} - \dfrac{2}{a} . \dfrac{2}{SP.SQ} + \dfrac{1}{a^2} \right] = \dfrac{\mu^2}{a^2}$

Or $V_1 V_2 = \mu / a$ or $V_1 V_2 = V^2$ [from (2)]

Or $V = \sqrt{(V_1 V_2)}$.

Hence V (i.e. the velocity at the end of the minor axis) is equal to the geometric mean between V_1 and V_2 (i.e. the velocities, at the ends of any diameter)

Example:- (b) A particle describes as ellipse under a force μ / r^2 to a focus. Show that the velocity at the end of the minor axis is a geometric mean between the greatest and the least velocities.

Solution:- Proceed as in ex. 8 (a)

Example:- (9) A particle describes an ellipse under a force to the focus S. When the particle is at one extremity of the minor axis, its kinetic energy is doubles without any change in the direction of motion. Prove that the particle proceeds to describes a parabola.

Solution:- Refer figure of Ex. 8

The velocities v and V at a distance r from the focus S in an elliptic and parabolic path are respectively given by $v^2 = \mu \left(\dfrac{2}{r} - \dfrac{1}{a} \right)$ (1)

And $V^2 = \dfrac{2\mu}{r}$ (2)

If v_1 is the velocity of the particle (describing the elliptic path) at the point B (an end of the minor axis), then from (1), we have $v_1^2 = \mu \left(\dfrac{2}{SB} - \dfrac{1}{a} \right) = \mu \left(\dfrac{2}{a} - \dfrac{1}{a} \right)$ $\quad [\because SB = a]$

$$= \frac{\mu}{a} \quad (3)$$

If v_2 is the velocity of the particle when its kinetic energy is doubled at B, then $\dfrac{1}{2} m v_2^2 = 2 \left(\dfrac{1}{2} m v_1^2 \right)$ where m is the mass of the particle.

$\therefore \quad v_2^2 = 2 v_1^2$ or $v_2^2 = 2\mu / a$ or $v_2^2 = 2\mu / SB$

Also from (2) the velocity at B for a parabolic path is given by $V^2 = 2\mu / SB$.

Since $v_2^2 = V^2$ i.e. $v_2 = V$, therefore the subsequent path of the particle at B is a parabola.

Example:- (10) A particle moves with a central acceleration μ /(distance)^2; it is projected with velocity V at a distance R. Show that its path is rectangular hyperbola if the angle of projection is $\sin^{-1} \left[\mu / \left\{ VR \sqrt{\left(V^2 - 2\mu / R \right)} \right\} \right]$.

Solution:- If the particle describes a hyperbola under the central acceleration μ /(distance)^2, then the velocity v of the particle at a distance r from the centre of force is given by $v^2 = \mu(2/r + 1/a)$ (1)

Where $2a$ is the transverse axis of the hyperbola.

Since the particle is projected with velocity V at a distance R, therefore from (1), we have

$$V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right) \text{ or } \frac{\mu}{a} = V^2 - \frac{2\mu}{R} \qquad (2)$$

If $\propto$ is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$, we have $h = Vp = VR\sin\propto$. (3)

[$\because p = r\sin\phi$ and initially $r = R, \phi = \propto$]

$$\text{Also } h = \sqrt{(\mu l)} = \sqrt{\{\mu.(b^2/a)\}} = \sqrt{(\mu a)} \qquad (4)$$

[$\because$ $b = a$ for a rectangular hyperbola]

From (3) and (4), we have $VR = \sin\propto = \sqrt{(\mu a)}$ or $\sin\propto = \dfrac{\sqrt{(\mu a)}}{VR} = \dfrac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \dfrac{\mu}{VR\sqrt{(\mu/a)}}$.

Substituting for μ/a from (2), we have $\sin\propto = \mu/\{V^2 - 2\mu/R\}$ or

$\propto = \sin^{-1}\left[\mu/\left\{VR\sqrt{(V^2 - 2\mu)/R}\right\}\right]$ which is the required angle of projection.

ASSIGNMENT TO IMPROVE

Que(1):- Find the law of force towards the pole under which the following curves are described.

(i) $au = e^{n\theta}$ and (ii) $r = ae^{\theta\cot\alpha}$

Solution:- (i) we have $au = e^{n\theta}$ (1)

Differentiating w.r.t. 'θ', we have $\dfrac{du}{d\theta} = \dfrac{n}{a}e^{n\theta} = nu$ and $\dfrac{d^2u}{d\theta^2} = n\dfrac{du}{d\theta} = n.\, nu = n^2u$

Referred to the centre of force as pole, the differential equation of a central orbits is $\dfrac{P}{h^2u^2} = u + \dfrac{d^2u}{d\theta^2}$, where P is the central acceleration assumed to be attractive.

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\left(v + n^2u\right) = h^2\left(1+n^2\right)u^3$$

$$= \frac{h^2\left(1+n^2\right)}{r^3} \qquad [\because u = 1/r]$$

$\therefore$ $P \propto 1/r^3$ i.e. the force varies inversely as the cube of the distance from the pole. Also the positive value of P indicates that the force is attractive i.e. is direction the pole.

(ii) We have $r = ae^{\theta\cot\alpha}$ or $\dfrac{1}{u} = ae^{\theta\cot\alpha}$, [$\because$ $r = 1/u$]

$$\therefore \quad u = \frac{1}{a}e^{-\theta\cot\alpha}$$

Differentiating w.r.t. 'θ', we have $\dfrac{du}{d\theta} = -\dfrac{\cot\alpha}{a}e^{-\theta\cot\alpha} = -u\cot\alpha$ and

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\cot\alpha = -(-u\cot\alpha)\cot\alpha = u\cot^2\alpha$$

The differential equation of the central orbit is $\dfrac{P}{h^2u^2}=u+\dfrac{d^2u}{d\theta^2}$

$\therefore \qquad P=h^2u^2\left(u+\dfrac{d^2u}{d\theta^2}\right)=h^2u^2\left(u+u\cot^2\alpha\right)=h^2\left(1+\cot^2\alpha\right)u^3$

$=\dfrac{h^2\operatorname{cosec}^2\alpha}{r^3}$

$\therefore \qquad P\propto 1/r^3$ i.e. the force varies inversely as the cube of the distance from the pole. Also the positive value of P indicates that the force is attractive.

Que(2):- Find the law of force towards the pole under which the following curves are described.

(i) $a=r\cos n\theta$ and (ii) $a=r\tan h\left(\theta/\sqrt{2}\right)$.

Solution:- (i) The equation of the curve is $a=r\cosh n\theta=(1/u)\cos h n\theta$

or $u=(1/a)\cos h n\theta$ (1)

Differentiating, $\dfrac{du}{d\theta}=\dfrac{n}{a}\sin h n\theta$ and $\dfrac{d^2u}{d\theta^2}=\dfrac{n^2}{a}\cos h n\theta$

The differential equation of the central orbit is $\dfrac{P}{h^2u^2}=u+\dfrac{d^2u}{d\theta^2}$.

$\therefore \qquad P=h^2u^2\left(u+\dfrac{d^2u}{d\theta^2}\right)=h^2u^2\left(u+\dfrac{n^2}{a}\cos h n\theta\right)=h^2u^2\left(u+n^2u\right)$

[Substituting for $\cos h n\theta$ from (1)]

$=h^2\left(1+n^2\right)u^3=\dfrac{h^2\left(1+n^2\right)}{r^3}$.

$\therefore \qquad P\propto 1/r^3$ i.e. the force varies inversely as the cube of the distance from the pole.

(ii) The equation of the curve is $a=r\tan h\left(\theta/\sqrt{2}\right)=(1/u)\tan h\left(\theta/\sqrt{2}\right)$

Or $u=(1/a)\tan h\left(\theta/\sqrt{2}\right)$. (1)

Differentiating $\dfrac{du}{d\theta}=\dfrac{1}{a\sqrt{2}}\sec h^2\left(\theta/\sqrt{2}\right)$

$\dfrac{d^2u}{d\theta^2}=\dfrac{1}{a\sqrt{2}}.\,2\sec h\left(\theta/\sqrt{2}\right).\left\{-\dfrac{1}{2}\sec h\left(\theta/\sqrt{2}\right)\tan h\left(\theta/\sqrt{2}\right)\right\}$

$=-\dfrac{1}{a}\sec h^2\left(\theta/\sqrt{2}\right)\tan h\left(\theta/\sqrt{2}\right)=-u\sec h^2\left(\theta/\sqrt{2}\right)/$

The differential equation of the central orbits is $\dfrac{P}{h^2u^2}=u+\dfrac{d^2u}{d\theta^2}$.

$\therefore \qquad P=h^2u^2\left(u+\dfrac{d^2u}{d\theta^2}\right)$

$=h^2u^2\left[u-u\sec h^2\left(\theta/\sqrt{2}\right)\right]=h^2u^2\left[1-\sec h^2\left(\theta/\sqrt{2}\right)\right]$

$=h^2u^3\tan h^2\left(\theta/\sqrt{2}\right)$ $\qquad [\because \sec h^2\theta=1-\tan h^2\theta]$

$=h^2u^3(au)^2$ $\qquad$ [From (1)]

$$=h^2a^2u^2=\frac{h^2a^2}{r^5}.$$

$\therefore$ $P\propto 1/r^5$ i.e. the force varies inversely as the 5th power of the distance from the pole.

Que(3):- A particle describes the curve $r=a\sin n\theta$ under a force P to the pole. Find the law of force.

Solution:- The equation of the curve is $r=a\sin n\theta$

Or $u=\frac{1}{r}=\frac{1}{a}\operatorname{cosec} n\theta$ (1)

Differentiating, $\frac{du}{d\theta}=-\frac{n}{a}\operatorname{cosec} n\theta\cot n\theta=-nu\cot n\theta$ and

$$\frac{d^2u}{d\theta^2}=n^2u\operatorname{cosec}^2 n\theta-n\frac{du}{d\theta}\cot n\theta$$

$$=n^2u\operatorname{cosec}^2 n\theta-n.(-nu\cot n\theta)\cot n\theta$$

$$=n^2u^2\operatorname{cosec}^2 n\theta+n^2u\cot^2 n\theta.$$

The differential equation of the central orbit is $\frac{P}{h^2u^2}=u+\frac{d^2u}{d\theta^2}$.

$$\therefore \quad P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=h^2u^2\left(u+n^2u\operatorname{cosec}^2 n\theta+n^2u\cot^2 n\theta\right)$$

$$=h^2u^3\left[1+n^2\operatorname{cosec}^2 n\theta+n^2\left(\operatorname{cosec}^2 n\theta-1\right)\right]$$

$$=h^2u^3\left[2n^2\operatorname{cosec}^2 n\theta-\left(n^2-1\right)\right]$$

$$=h^2u^3\left[2n^2(au)^2-\left(n^2-1\right)\right]$$

[Substituting for $\operatorname{cosec} n\theta$ from (1)]

$$=h^2\left[2n^2a^2u^5-\left(n^2-1\right)u^3\right]$$

$$=h^2\left[\frac{2n^2a^2}{r^5}-\frac{\left(n^2-1\right)}{r^3}\right].$$

$$\therefore \quad P\propto\left[\frac{2n^2a^2}{r^5}-\frac{\left(n^2-1\right)}{r^3}\right].$$

Que(4):- Find the law of force towards the pole under which the following curves are described.

(iv) $r^2=2ap$, (ii) $p^2=ar$ and (iii) $b^2/p^2=(2a/r)-1$,

Solution:- (i) The equation of the curve is $r^2=2ap$.

$\therefore$ $\frac{1}{p}=\frac{2a}{r^2}$ or $\frac{1}{p^2}=\frac{4a^2}{r^4}$

Differentiating w.r.t. $'r'$, we have $-\frac{2}{p^3}\frac{dp}{dr}=-\frac{16a^2}{r^5}$.

$\therefore \qquad \dfrac{h^2}{p^3}\dfrac{dp}{dr}=\dfrac{8a^2h^2}{r^5}$ (1)

Now from the pedal equation of a central orbit, we have

$P=\dfrac{h^3}{p^3}\dfrac{dp}{dr}=\dfrac{8a^2h^2}{r^5}$ [From (1)]

$\therefore \qquad P\propto 1/r^5$ i.e. the force varies inversely as the fifth power of the distance from the pole.

(v) The equation of the curve is $p^2=ar$, which is the pedal equation of the parabola referred to the focus as the pole.

$\therefore \qquad \dfrac{1}{p^2}=\dfrac{1}{a}\dfrac{1}{r}.$

Differentiating w.r.t 'r', we get $-\dfrac{2}{p^3}\dfrac{dp}{dr}=-\dfrac{1}{a}\dfrac{1}{r^2}$

$\therefore \qquad \dfrac{h^2}{p^3}\dfrac{dp}{dr}=\dfrac{h^2}{2a}\dfrac{1}{r^2}.$ (1)

From the pedal equation of a central orbit, we have $P=\dfrac{h^2}{p^3}\dfrac{dp}{dr}=\dfrac{h^2}{2a}\dfrac{1}{r^2}$ [From (1)]

$\therefore \qquad P\propto 1/r^2$ i.e. the force varies inversely as the square of the distance from the pole.

(vi) The equation of the given central orbits is $\dfrac{b^2}{p^2}=\dfrac{2a}{r}-1$ (1)

(2) Is the pedal equation of an ellipse referred to the focus as pole.

Differentiating both sides of (1) w.r.t 'r', we get $-\dfrac{2b^2}{p^3}\dfrac{dp}{dr}=-\dfrac{2a}{r^2}$, or $\dfrac{h^2}{p^3}\dfrac{dp}{dr}=\dfrac{a}{b^2}\dfrac{h^2}{r^2}.$

$\therefore \qquad P=\dfrac{h^2}{p^3}\dfrac{dp}{dr}=\dfrac{ah^2}{b^2}\dfrac{1}{r^2}.$

Thus $P\propto 1/r^2$ i.e. the acceleration varies inversely as the square of the distance from the focus of the ellipse.

Que(5):- A particle describes the curve $r^2=a^2\cos^2\theta+b^2\sin^2\theta$ under an attraction to the origin, prove that the attraction at a distance r is $h^2\left[2\left(a^2+b^2\right)r^2-3a^2b^2\right].r^{-7}$

Solution:- The equation of the given curve is $r^2=a^2\cos^2\theta+b^2\sin\theta$ or

$\dfrac{1}{u^2}=\dfrac{a^2}{2}(1+\cos 2\theta)+\dfrac{b^2}{2}(1-\cos 2\theta)$

Or $\dfrac{1}{u^2}=\dfrac{1}{2}\left(a^2+b^2\right)+\dfrac{1}{2}\left(a^2-b^2\right)\cos 2\theta$ (1)

Differentiating w.r.t. 'θ', we have $-\dfrac{2}{u^3}\dfrac{du}{d\theta}=-\left(a^2-b^2\right)\sin 2\theta$

Or $\dfrac{du}{d\theta}=\dfrac{1}{2}\left(a^2-b^2\right)u^3\sin 2\theta$

Differentiating again w.r.t. 'θ', we have

$\dfrac{d^2u}{d\theta^2}=\dfrac{3}{2}\left(a^2-b^2\right)u^2.\dfrac{du}{d\theta}\sin 2\theta+\left(a^2-b^2\right)u^3\cos 2\theta$

$$=\frac{3}{2}\left(a^2-b^2\right)u^2.\frac{1}{2}\left(a^2-b^2\right)u^3\sin 2\theta.\sin 2\theta+\left(a^2-b^2\right)u^3\cos 2\theta$$
$$=\frac{3}{4}u^5\left(a^2-b^2\right)^2\sin^2 2\theta+\left(a^2-b^2\right)u^3\cos 2\theta$$
$$=\frac{3}{4}u^5\left(a^2-b^2\right)^2\left(1-\cos^2 2\theta\right)+u^3\left(a^2-b^2\right)\cos 2\theta$$
$$=\frac{3}{4}u^5\left(a^2-b^2\right)^2-\frac{3}{4}u^5.\left\{\left(a^2-b^2\right)\cos 2\theta\right\}^2+u^3\left(a^2-b^2\right)\cos 2\theta$$

Now from (1), $\left(a^2-b^2\right)\cos 2\theta=\frac{2}{u^2}-\left(a^2+b^2\right)$.

$$\therefore \quad \frac{d^2u}{d\theta^2}=\frac{3}{4}u^5\left(a^2-b^2\right)2-\frac{3}{4}u^5\left\{\frac{2}{u^2}-\left(a^2+b^2\right)\right\}^2+u^3\left\{\frac{2}{u^2}-\left(a^2+b^2\right)\right\}$$
$$=\frac{3}{4}u^5\left(a^2-b^2\right)^2-\frac{3}{4}u^5\left\{\frac{4}{u^4}-\frac{4}{u^2}\left(a^2+b^2\right)+\left(a^2+b^2\right)\right\}+2u-\left(a^2+b^2\right)u^3$$
$$=\frac{3}{4}u^5\left(a^2-b^2\right)^2-3u+3u^3\left(a^2+b^2\right)-\frac{3}{4}u^5\left(a^2+b^2\right)^2+2u-\left(a^2+b^2\right)u^3$$
$$=\frac{3}{4}u^5\left\{\left(a^2-b^2\right)^2-\left(a^2+b^2\right)^2\right\}+2u^3\left(a^2+b^2\right)-u$$
$$=2\left(a^2+b^2\right)u^3-3a^2b^2u^5-u$$

The differential equation of the central orbit is $\frac{P}{h^2u^2}=u+\frac{d^2u}{d\theta^2}$.

$$\therefore \quad P=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=h^2u^2\left[u+2\left(a^2+b^2\right)u^3-3a^2b^2u^5-u\right]$$
$$=h^2u^2\left[2\left(a^2+b^2\right)r^2-3a^2b^2\right]=h^2r^{-7}\left[2\left(a^2+b^2\right)r^2-3a^2b^2\right].$$

Que(6):- A particle moving with a central acceleration μ /(distance)^3 is projected from an apse at a distance a with a velocity V; show that the path is $r\cosh\left\{\frac{\sqrt{\left(\mu-a^2V^2\right)}}{aV}\theta\right\}=a$ or $r\cos\left\{\frac{\sqrt{\left(a^2V^2-\mu\right)}}{aV}\theta\right\}=a$ according as V is <or> the velocity from infinity.

Solution:- Here, the central acceleration $P=\frac{\mu}{\left(distance\right)^3}=\frac{\mu}{r^3}=\mu u^3$.

The differential equation of the path is $h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u$.

Multiplying both sides by $2\left(du/d\theta\right)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A \qquad (1)$$

Where A is a constant.

But initially when $r = a$ i.e. $u = \dfrac{1}{a}, \dfrac{du}{d\theta} = 0$ (at an apse) and $v = V$.

$\therefore$ From (1), $V^2 = h^2\left[\dfrac{1}{a^2}\right] = \dfrac{\mu}{u^2} + A$.

$\therefore$ $h^2 = a^2V^2$ and $V^2 - \dfrac{\mu}{a^2} = \dfrac{\left(V^2a^2 - \mu\right)}{a^2}$

Substituting the values of h^2 and A in (1), we have

$$a^2V^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\left(V^2a^2 - \mu\right)}{a^2} \text{ or}$$

$$a^2V^2\left(\frac{du}{d\theta}\right)^2 = -a^2V^2u^2 + \mu u^2 + \frac{\left(V^2a^2 - \mu\right)}{a^2}$$

$$= -\left(a^2V^2 - \mu\right)u^2 + \left(a^2V^2 - \mu\right)/a^2$$

$$= \left(a^2V^2 - \mu\right)\left(-u^2 + 1/a^2\right)$$

Or $a^4V^2\left(\dfrac{du}{d\theta}\right)^2 = \left(a^2V^2 - \mu\right)\left(1 - a^2u^2\right)$ (2)

If V_1 is the velocity acquired by the particle in falling form infinitely to the distance a, then

$$V_1^2 = -2\int_{\infty}^{a} P\,dr = -2\int_{\infty}^{a}\frac{\mu}{r^3}\,dr = -2\left[-\frac{\mu}{2r^2}\right]_{\infty}^{a} = \frac{\mu}{a^2}.$$

Case I:- When $V < V_1$ (velocity from infinity), we have $V^2 < V_1^2$ or $V^2 < \mu/a^2$ or $a^2V^2 < \mu$ or $\mu = a^2V^2 > 0$.

$\therefore$ From (2), we have $a^4V^2\left(\dfrac{du}{d\theta}\right)^2 = \left(\mu - a^2V^2\right)\left(a^2u^2 - 1\right)$

Or $a^2V\dfrac{du}{d\theta} = \sqrt{\left(\mu - a^2V^2\right)}.\sqrt{\left(a^2u^2 - 1\right)}$

Or $\dfrac{\sqrt{\left(\mu - a^2V^2\right)}}{aV}d\theta = \dfrac{adu}{\sqrt{\left(a^2u^2 - 1\right)}}$.

Substituting $au = z$, so that $adu = dz$, we have $\dfrac{\sqrt{\left(\mu - a^2V^2\right)}}{aV} = d\theta = \dfrac{dz}{\sqrt{\left(z^2 - 1\right)}}$.

Integrating $\dfrac{\sqrt{\left(\mu - a^2V^2\right)}}{aV}\theta + B = \cos h^{-1}z$ or $\dfrac{\sqrt{\left(\mu - a^2V^2\right)}}{aV}\theta + B = \cos h^{-1}(au)$.

But initially when $u = 1/a,\ \theta = 0$.

$\therefore$ $0 + B = \cos h^{-1}1 = 0$ or $B = 0$.

$\therefore$ $\dfrac{\sqrt{\left(\mu - a^2V^2\right)}}{aV}\theta = \cosh^{-1}(au)$

Or $au = \frac{a}{r} = \cosh\left\{\frac{\sqrt{(\mu - a^2V^2)}}{aV}\theta\right\}$

Or $r\cosh\left\{\frac{\sqrt{(\mu - a^2V^2)}}{aV}\theta\right\} = a$.

Case II:- When $V > V_1$ (velocity from infinity), we have $V^2 > V_1^2$ or $V^2 > \mu / a^2$ or $a^2V^2 - \mu > 0$.

$\therefore$ From (2), we have $a^4V^2\left(\frac{du}{d\theta}\right)^2 = (a^2V^2 - \mu)(1 - a^2u^2)$

Or $a^2V\left(\frac{du}{d\theta}\right) = \sqrt{(a^2V^2 - \mu)}.\sqrt{(1 - a^2u^2)}$

Or $\frac{\sqrt{(a^2V^2 - \mu)}}{aV}d\theta = \frac{adu}{\sqrt{(1 - a^2u^2)}}$.

Integrating $\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\theta + C = \sin^{-1}(au)$.

But initially when $u = 1/a, \theta = 0$.

$\therefore$ $0 + C = \sin^{-1}1$ or $C = \pi / 2$

$\therefore$ $\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\theta + \frac{\pi}{2} = \sin^{-1}(au)$.

Or $au = \frac{a}{r} = \sin\left\{\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\theta + \frac{\pi}{2}\right\}$

Or $a = r\cos\left\{\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\theta\right\}$.

Que(7):- A particle, acted on by a repulsive central force $\mu r / (r^2 - 9c^2)^2$, is projected from an apse at a distance c with velocity $\sqrt{(\mu / 8c^2)}$. Find the equations of its path and show that the time to the cusp is $\frac{4}{3}\pi c^2\sqrt{(2/\mu)}$.

Solution:- Considering the particle of unit mass, the central acceleration $P = \frac{-\mu r}{(r^2 - 9c^2)^2}$

(Negative sign is taken because the force is repulsive).

The differential equation of the path in **Pedal form** is $\frac{h^2}{p^3}\frac{dp}{dr} = P = -\frac{\mu r}{(r^2 - 9c^2)^2}$

Or $= -\frac{2h^2}{p^3}dp = \frac{2\mu r\, dr}{(r^2 - 9c^2)^2} = 2\mu r(r^2 - 9c^2)^{-2}\, dr$.

Integrating $v^2 = \frac{h^2}{p^2} = -\frac{\mu}{\left(r^2 - 9c^2\right)} + A$ (1)

Where A is a constant.

But the particle is projected from an apse at distance c. Also at an apse $p = r$. Therefore initially $p = r = c$ and $v = \sqrt{\left(\mu / 8c^2\right)}$.

$\therefore$ From (1), we have $\frac{\mu}{8c^2} = \frac{h^2}{c^2} = -\frac{\mu}{\left(c^2 - 9c^2\right)} + A$.

$\therefore$ $h^2 = \mu / 8$ and $A = \frac{\mu}{8c^2} - \frac{\mu}{8c^2} = 0$

Substituting the values of h^2 and A in (1), we have $\frac{\mu}{8p^2} = -\frac{\mu}{\left(r^2 - 9c^2\right)}$ or $8p^2 = 9c^2 - r^2$ (2)

Which is the pedal equation of the path and is a three-cusped hypocycloid.

Second Part:- Now we are to find the time to reach the cusp. At the cusp, we have $p = 0$. So it is required to find the from $p = c$ to $p = 0$.

We know that in a central orbit $v = \frac{ds}{dt} = \frac{h}{p}$.

$\therefore$ $h\,dt = p\,ds$ or $hdt = P\frac{ds}{dr}.dr$

But $dr / ds = \cos\phi$.

$\therefore$ $h\,dt = p.\frac{1}{\cos\phi}dr = \frac{p\,dr}{\sqrt{\left(1 - \sin^2\phi\right)}} = \frac{p\,dr}{\sqrt{\left\{1 - \left(p^2 / r^2\right)\right\}}}$.

$$= \frac{pr\,dr}{\sqrt{\left(r^2 - p^2\right)}} = \frac{p(-8p)\,dp}{\sqrt{\left(9c^2 - 8p^2 - p^2\right)}}$$

[$\because$ from (2), $-r\,dr = 8p\,dp$]

$$= \frac{-8p^2 dp}{3\sqrt{\left(c^2 - p^2\right)}}.$$

Let t_1 be the required time to the cusp. Then integrating from $p = c$ to $p = 0$, we get

$$ht_1 = -\frac{1}{3}\int_c^0 \frac{8p^2 - dp}{\sqrt{\left(c^2 - p^2\right)}} = \frac{8}{3}\int_0^c \frac{p^2 dp}{\sqrt{\left(c^2 - p^2\right)}}$$

$$= \frac{8}{3}\int_0^{\pi/2} \frac{c^2 \sin^2 z}{c\cos z} c\cos z\,dz$$

[putting $p = c\sin z$, so that $dp = c = \cos z\,dz$]

$$= \frac{8}{3}c^2 \int_0^{\pi/2} \sin^2 z\,dz = \frac{8}{3}c^2 \,.\frac{1}{2}\times\frac{\pi}{2} = \frac{2\pi c^2}{3}.$$

$\therefore$ $t_1 = \frac{2\pi c^2}{3h} = \frac{2\pi c^2}{3}.\sqrt{\left(\frac{8}{\mu}\right)}$

$$= \frac{4\pi c^2}{3}\sqrt{\left(\frac{2}{\mu}\right)}.$$

Que(8):- A particle is moving with central acceleration $\mu\left(r^5 - c^4 r\right)$ being projected from an apse at a distance c with velocity $c^3\left(\sqrt{2\mu/3}\right)$, show that its path is the curve $x^4 + y^4 = c^4$.

Solution:- Here the central acceleration $P = \mu\left(r^5 - c^4 r\right) = \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right)$.

$\therefore$ The differential equation of the path is $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u^5} - \frac{c^4}{u}\right) = \mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)$

Multiplying both sides by $2\left(du/d\theta\right)$ and then integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right) + A \qquad (1)$$

Where A is a constant.

But initially, when $r = c$ i.e., $u = 1/c$, $du/d\theta = 0$ (at an apse) and $v = c^3\sqrt{(2\mu/3)}$.

$\therefore$ From (1), we have $\frac{2\mu c^6}{3} = h^2 . \frac{1}{c^2} = \mu\left(-\frac{c^6}{3} + c^6\right) + A$

$\therefore$ $h^2 = \frac{2}{3}\mu c^8, A = 0$

Substituting the value of h^2 and A in (1), we have $\frac{2}{3}\mu c^8\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right)$

Or $c^8\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2u^6} + \frac{3c^4}{2u^2} - c^8u^2 = \frac{1}{u^6}\left[-\frac{1}{2} + \frac{3}{2}c^4u^4 - c^8u^8\right]$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^8u^8 - \frac{3}{2}c^4u^4\right)\right] = \frac{1}{u^6}\left[\frac{1}{2} - \left(c^4u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]$$

$\therefore$ $c^4u^3\frac{du}{d\theta} = \sqrt{\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]}$ or $d\theta = \frac{c^4u^3\,du}{\sqrt{\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]}}$

Putting $c^4u^4 - \frac{3}{4} = z$, so that $4c^4u^3\,du = dz$, we have $4\,d\,\theta = \frac{dz}{\sqrt{\left[\left(\frac{1}{4}\right)^2 - z^2\right]}}$

Integrating $4\theta + B = \sin^{-1}\left(\dfrac{z}{\frac{1}{4}}\right) = \sin^{-1}(4z)$ where B is a constant or b

$4\theta + B = \sin^{-1}\left(4c^4u^4 - 3\right)$.

But initially when $u = 1/c$, $\theta = 0$. $\therefore B = \sin^{-1}1 = \pi/2$.

$\therefore \quad 4\theta + \dfrac{1}{2}\pi = \sin^{-1}\left(4c^4u^4 - 3\right)$

Or $\quad \sin\left(\dfrac{1}{2}\pi + 4\theta\right) = 4c^4u^4 - 3$

Or $\quad \cos 4\theta = 4c^4u^4 - 3$

Or $\quad 4c^4/r^4 = [3 + \cos 4\theta]$

Or $\quad 4c^4 = r^4\left[3 + \left(2\cos^2 2\theta - 1\right)\right] = 2r^4\left[1 + \cos^2 2\theta\right]$

$= 2r^4\left[\left(\cos^2\theta + \sin^2\theta\right)^2 + \left(\cos^2\theta - \sin^2\theta\right)^2\right]$

$= 4r^4\left(\cos^4\theta + \sin^4\theta\right)$

$\therefore \quad c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$

Or $c^4 = x^4 + y^4$, $\quad [\because x = r\cos\theta \text{ and } y = r\sin\theta]$

Which is the required equation of the path.

Que(9):- If the law of force be $\mu\left(u^4 - \dfrac{10}{9}au^5\right)$ and the particle be projected from an apse at a distance $5a$ with a velocity equal to $\sqrt{(5/7)}$ of that in a circle at the same distance, show that the orbit is the limacon $r = a(3 + 2\cos\theta)$.

Solution:- Here the central acceleration $P = \mu\left(u^4 - \dfrac{10}{9}au^5\right) = \mu\left(\dfrac{1}{r^4} - \dfrac{10a}{9r^5}\right)$

If V is the velocity for a circle at a distance $5a$, then

$$\frac{V^2}{5a} = [P]_{r=5a} = \mu\left[\frac{1}{(5a)^4} - \frac{10a}{9(5a)^5}\right] = \frac{7\mu}{9(5a)^4}$$

$$\therefore \quad V = \sqrt{\left[\frac{7\mu}{9.(5a)^3}\right]}$$

If v_1 is the velocity of projection of the particle, then

$$v_1 = \sqrt{\left(\frac{5}{7}\right)}\,.\,V = \sqrt{\left(\frac{5}{7}\right)}.\sqrt{\left(\frac{7\mu}{9(5a)^3}\right)} = \sqrt{\left(\frac{\mu}{225a^3}\right)}.$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(u^4 - \frac{10}{9}au^5\right) = \mu\left(u^2 - \frac{10}{9}au^3\right)$$

Multiplying both sides by $2(du/d\theta)$ and then integrating, we have

$$v^2 = {}^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 - \frac{5}{9}au^4\right) + A \qquad (1)$$

Where A is a constant.

But initially, when $r = 5a$ i.e. $u = \frac{1}{5a}, \frac{du}{d\theta} = 0$ and $v^2 = \frac{\mu}{225a^3}$

$\therefore$ From (1), we have $\frac{\mu}{225a^3} = h^2\left(\frac{1}{5a}\right)^2 = \mu\left[\frac{2}{3}\left(\frac{1}{5a}\right)^3 - \frac{5a}{9}\left(\frac{1}{5a}\right)^4\right] + A$

$\therefore$ $h^2 = \frac{\mu}{9a}, A = 0$

Substituting the values of h^2 and A in (1), we have $\frac{\mu}{9a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 - \frac{5a}{9}u^4\right)$

Or $\left(\frac{du}{d\theta}\right)^2 = 6au^3 - 5a^2u^4 - u^2$.

Putting $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we have $\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{6a}{r^3} - \frac{5a^2}{r^4} - \frac{1}{r^2}$

Or $\left(\frac{dr}{d\theta}\right)^2 = 6ar - 5a^2 - r^2 = -5a^2 - (r^2 - 6ar)$

$= -5a^2 - (r-3a)^2 + 9a^2 = 4a^2 - (r-3a)^2$.

$\therefore$ $\frac{dr}{d\theta} = \sqrt{\left[(2a)^2 - (r-3a)^2\right]}$

Or $d\theta = \frac{dr}{\sqrt{\left[(2a)^2 - (r-3a)^2\right]}}$

Integrating $\theta + B = \sin^{-1}\left(\frac{r-3a}{2a}\right)$, where B is a constant.

But initially when $r = 5a$, $\theta = 0$. $\quad \therefore B = \sin^{-1}1 = \pi/2$

$\theta + \frac{1}{2}\pi \sin^{-1}\left(\frac{r-3a}{2a}\right)$ or $\sin\left(\frac{1}{2}\pi + \theta\right) = \frac{r-3a}{2a}$

$r - 3a = 2a\cos\theta$ or $r = a(3 + 2\cos\theta)$, which is the required equation of the path.

Que(10.1):- A particle is projected from an apse at a distance a with the velocity from infinity being μu^7; show that the equation to its path is $r^2 = a^2\cos 2\theta$.

Solution:- Proceed as in before example.

Here $n = 2$.

Que(10.2):- A particle is projected from an apse at a distance a with velocity of projection $\sqrt{\mu}/\left(a^2\sqrt{2}\right)$ under the action of a central force μu^5. Prove that the path is circle $r = a\cos\theta$.

Solution:- Proceed as in before example

Here $n = 1$.

Que(10.3):- If the central force varies as the cube of the distance from a fixed point then find the orbit.

Solution:- We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is $\frac{h^2}{p^3}\frac{dp}{dr} = P$ (1)

Where P is the central acceleration assumed to be attractive.

Here $P = \mu r^3$, putting $P = \mu r^3$ in (1), we get $\frac{h^2}{p^3}\frac{dp}{dr} = \mu r^3$ or $\frac{h^2}{p^3}dp = \mu r^3\, dr$

Or $-2\frac{h^2}{p^3}dp = -2\mu r^3\, dr$.

Integrating both sides, we get $v^2 = \frac{h^2}{p^2} = -\frac{\mu r^4}{2} + C$ (2)

Let $v = v_0$ when $r = r_0$

Then $v_0^2 = -\frac{\mu r_0^4}{2} + C$ or $C = v_0^2 + \frac{\mu r_0^4}{2}$

Putting this value of C in(2), the pedal equation of the central orbit is $\frac{h^2}{p^2} = -\frac{\mu r^4}{2} + v_0^2 + \frac{\mu r_0^4}{2}$.

Que(11):- A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle a, show that the equation to its path is $r\cos\left(\theta/\sqrt{2}\right) = a$.

Solution:- Here the central acceleration varies inversely as the cube of the distance i.e. $P = \mu / r^3 = \mu u^3$, where μ is a constant.

If V is the velocity for a circle of radius a, then $\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$ or $V = \sqrt{\left(\mu / a^2\right)}$

$\therefore$ The velocity of projection $v_1 = \sqrt{2}V = \sqrt{\left(2\mu / a^2\right)}$.

The differential equation of the path is $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u$.

Multiplying both sides by $2\left(du / d\theta\right)$ and integrating, we have

$$v^2 =^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \quad (1)$$

Where A is a constant.

But initially when $r = a$ i.e. $u = 1/a$, $du/d\theta = 0$ (at an apse), and $v = v_1 = \sqrt{\left(2\mu / a^2\right)}$.

$\therefore$ From (1), we have $\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$

$\therefore$ $h^2 = 2\mu$ and $A = \mu / a^2$

Substituting the values of h^2 and A in (1), we have $2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$

Or $2\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}+u^2-2u^2=\frac{1-a^2u^2}{a^2}$

Or $\sqrt{2}a\frac{du}{d\theta}=\sqrt{(1-a^2u^2)}$ or $\frac{d\theta}{\sqrt{2}}=\frac{adu}{\sqrt{(1-a^2u^2)}}$

Integrating $\left(\theta/\sqrt{2}\right)+B=\sin^{-1}(au)$, where B is a constant.

But initially, when $u=1/a$, $\theta=0$. $\therefore B=\sin^{-1}1=\frac{1}{2}\pi$.

$\therefore$ $\left(\theta/\sqrt{2}\right)+\frac{1}{2}\pi=\frac{1}{2}\sin^{-1}(au)$ or $au=a/r=\sin\left\{\frac{1}{2}\pi+\left(\theta/\sqrt{2}\right)\right\}$ or $a=r\cos\left(\theta/\sqrt{2}\right)$,

which is the required equation of the path.

Que(12):- A particle moving under a constant force from a centre is projected at a distance a from the centre in a direction perpendicular to the radius vector with velocity acquired in falling to the point of projection from the centre show that its path is $(a/r)^3=\cos^2\left(\frac{3}{2}\theta\right)$.

Also show that the particle will ultimately move in a straight line through the origin in the same way as if its path had always been this line. If the velocity of projection be double that in the previous case show that the path is $\frac{\theta}{2}=\tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)}-\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)}$

Solution:- Since the particle move under a constant force directed away from a centre, therefore the central acceleration $P=-f$, where f is a constant.

While falling in a straight line from the centre of force to the point of projection, if v is the velocity of the particle at a distance r from the centre of force, then $v\frac{dv}{dr}=f$ or $v\,dv=f\,dr$

Let V be the velocity of the particle acquired in falling from the $\int_0^V v\,dv=\int_0^a f\,dr$ or

$\frac{V^2}{2}=af$ or $V=\sqrt{(2af)}$.

Therefore the particle is projected from a distance a with velocity $\sqrt{(2af)}$ in a direction perpendicular to the radius vector.

The differential equation of the path is $h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P^2}{u^2}-\frac{f}{u^2}$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2f}{u}+A \quad (1)$$

Where A is a constant.

But initially, when $r=a$ i.e. $u=1/a$, $du/d\theta=0$ (since the particle is projected perpendicular to the radius vector), and $v=V=\sqrt{(2af)}$.

$\therefore$ From (1), $2af=h^2\left[\frac{1}{a^2}\right]=2fa+A$.

$\therefore$ $h^2=2fa^3$ and $A=0$.

Substituting the values of h^2 and A in (1), we have $2fa^3\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2f}{u}$

Or $a^3\left(\frac{du}{d\theta}\right)^2=-a^3u^2+\frac{1}{u}=\frac{1-a^3u^3}{u}$ or $a^{3/2}\frac{du}{d\theta}=\frac{\sqrt{(1-a^3u^3)}}{u^{1/2}}$

Or $d\theta=\frac{a^{3/2}u^{1/2}du}{\sqrt{(1-a^3u^3)}}$

Substituting $a^{3/2}u^{3/2}=z$, so that $\frac{3}{2}a^{3/2}u^{1/2}du=dr$, we have $\frac{3}{2}d\theta=\frac{dz}{\sqrt{(1-z^2)}}$.

Integrating $\frac{3}{2}\theta+B=\sin^{-1}(z)=\sin^{-1}\left(a^{3/2}u^{3/2}\right)$, where B is a constant.

But initially when $u=1/a,\ \theta=0$ $\qquad\therefore\ B=\sin^{-1}1=\frac{1}{2}\pi$.

$\therefore\qquad \frac{3}{2}\theta+\frac{1}{2}\pi=\sin^{-1}\left(a^{3/2}u^{3/2}\right)$

Or $a^{3/2}u^{3/2}=\sin\left(\frac{1}{2}\pi+\frac{3}{2}\theta\right)=\cos\frac{3}{2}\theta$

Or $a^{3/2}/r^{3/2}=\cos\left(\frac{3}{2}\theta\right)$

Or $(a/r)^3=\cos^2\left(\frac{3}{2}\theta\right)$. (2)

This is the required equation of the path.

Second Part:- Now as $r\to\infty$, $\cos\left(\frac{3}{2}\theta\right)\to 0$ i.e. $\frac{3}{2}\theta\to\frac{1}{2}\pi$ i.e. $\theta\to\pi/3$.

Hence the particle ultimately moves in a straight line through the origin, inclined at an angle $\theta=\pi/3$, in the same way as if its path had always been this line.

Third Part:- If the velocity of projection of the particle is double of that in the previous case, then the initial conditions are $r=a, u=1/a,\ du/d\theta=0$, and $v=2V=2\sqrt{(2af)}$

$\therefore\qquad$ From (1), we have $8af=h^2\left[\frac{1}{a^2}\right]=2fa+A$.

$\therefore\qquad h^2=8a^3f$ and $A=6af$.

Substituting these value of h^2 and A in (1), we have $8a^3f\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2f}{u}+6af$

Or $4a^3\left(\frac{du}{d\theta}\right)^2=-4a^3u^2+\frac{1}{u}+3a$.

Putting $u=\frac{1}{r}$, so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$, we have $4a^3\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=-\frac{4a^3}{r^2}+r+3a$

Or $4a^3(dr/d\theta)^2=r^5+3ar^4-4a^3r^2=r^2\left(r^3+3ar^2-4a^3\right)$

$$= r^2\left[r^2(r-a)+4ar(r-a)+4a^2(r-a)\right]$$
$$= r^2(r-a)\left(r^2+4ar+4a^2\right)=r^2(r-a)(r+2a)^2$$
Or $2a^{3/2}(dr/d\theta)=r(r+2a)\sqrt{(r-a)}$

Or $\dfrac{d\theta}{2}=\dfrac{a^{3/2}dr}{r(r+2a)\sqrt{(r-a)}}$

Substituting m $r-a=z^2$, so that $dr=2z\,dz$, we have $\dfrac{d\theta}{2}=\dfrac{2a^{3/2}z\,dz}{(z^2+a)(z^2+3a).z}$

Or $\dfrac{d\theta}{2}=\sqrt{a}\,/\left[\dfrac{1}{z^2+a}-\dfrac{1}{z^2+3a}\right]dz$.

Integrating $\dfrac{\theta}{2}+B=\sqrt{a}\,.\left[\dfrac{1}{\sqrt{a}}\tan^{-1}\dfrac{z}{\sqrt{a}}-\dfrac{1}{\sqrt{(3a)}}\tan^{-1}\dfrac{z}{\sqrt{(3a)}}\right]$, where B is a constant.

Or $\dfrac{\theta}{2}+B=\tan^{-1}\sqrt{\left(\dfrac{r-a}{a}\right)}-\dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\dfrac{r-a}{3a}\right)}$

But initially when $r=a,\ \theta=0$, $\quad\therefore\ B=0$

$\therefore\quad \dfrac{\theta}{2}=\tan^{-1}\sqrt{\left(\dfrac{r-a}{a}\right)}-\dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\dfrac{r-a}{3a}\right)}$, which is the required equation of the path.

Que(14):- A particle moves with a central acceleration μ / (distance)^5 and projected from the apse at a distance a with a velocity equal to n times that which would be acquired in falling from infinity. Show that the other apsidal distance is $a/\sqrt{(n^2-1)}$.

If $n=1$ and particle be projected in any direction, show that the path is a passing through the centre of force.

Solution:- Here, the central acceleration $P=\dfrac{\mu}{(distance)^5}=\dfrac{\mu}{r^5}=\mu u^5$.

Let V be the velocity from infinity to a distance a from the centre under the same acceleration. Then as in $V^2=-2\int_\infty^a P\,dr=-2\int_\infty^a\dfrac{\mu}{r^5}dr=-2\left[\dfrac{\mu}{-4r^4}\right]_\infty^4=\dfrac{\mu}{2a^4}$

$\therefore\quad V=\sqrt{\left(\mu/2a^4\right)}$

The differential equation of the path is $h^2\left[u+\dfrac{d^2u}{d\theta^2}\right]=\dfrac{P}{u^2}=\dfrac{\mu u^5}{u^2}=\mu u^3$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have $h^2\left[u^2+\left(\dfrac{du}{d\theta}\right)^2\right]=\dfrac{2\,\mu u^4}{4}+A$, where A is a constant.

Or $v^2=h^2\left[u^2+\left(\dfrac{du}{d\theta}\right)^2\right]=\dfrac{\mu u^4}{2}+A$ (1)

But initially, when $r=a$ i.e. $u=1/a$, $du/d\theta=0$ (at an apse) and $v=nV=n\sqrt{\left(\mu/\sqrt{2a^4}\right)}$

∴ From (1), we have $\frac{n^2\mu}{2a^4}=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{2a^4}+A.$

∴ $h^2=\frac{n^2\mu}{2a^2}$ and $A=\frac{\left(n^2-1\right)\mu}{2a^4}$

Substituting the values of h^2 and A in (1), we have $\frac{n^2\mu}{2a^2}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu u^4}{2}+\frac{\left(n^2-1\right)\mu}{2a^4}$

Or $\left(\frac{du}{d\theta}\right)^2=\frac{1}{n^2a^2}\left[a^4u^4-a^2n^2u^2+\left(n^2-1\right)\right]$

At an apse, we have $du/d\theta=0$. Therefore the apsidal distances are given by ` $0=\left(1/n^2a^2\right)\left[a^4u^4-a^2n^2a^2+\left(n^2-1\right)\right]$

Or $a^4u^4-a^2n^2u^2+\left(n^2-1\right)=0$

Or $\frac{a^4}{r^4}-\frac{a^2n^2}{r^2}+\left(n^2-1\right)=0$ $\left[\because u=\frac{1}{r}\right]$

Or $\left(n^2-1\right)r^4-a^2n^2r^2+a^4=0$, which is quadratic equation in r^2.

If r_1^2 and r_2^2 are its roots, then $r_1^2r_2^2=a^4/\left(n^2-1\right)$

Or $r_1r_2=a^2/\sqrt{\left(n^2-1\right)}$ (2)

But the first apsidal distance say r_1, is a.

∴ From (2), $ar_2=a^2/\sqrt{\left(n^2-1\right)}$ i.e. the second apsidal distance $r_2=a/\sqrt{\left(n^2-1\right)}$.

Second Part:- When $n=1$ and the particle is projected in any direction, say at an angle α to radius vector, then at the point of projection, we have $\phi=\alpha, p=r\sin\phi=a\sin\alpha$ and so $\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2=\frac{1}{(a\sin\alpha)^2}$.

Thus in this case initially when $r=a$ i.e. $u=1/a$, we have $v=V=\sqrt{\left(\mu/2a^4\right)}$ and $u^2+\left(du/d\theta\right)^2=1/\left(a^2\sin^2\alpha\right)$

∴ From (1), we have $\frac{\mu}{2a^4}=\frac{h^2}{\left(a^2\sin^2\alpha\right)}=\frac{\mu}{2a^4}+A.$

∴ $h^2=\left(\mu\sin^2\alpha\right)/\left(2a^2\right)$ and $A=0$

Substituting the values of h^2 and A in (1), we have $\frac{\left(\mu\sin^2\alpha\right)}{2a^2}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{\mu u^4}{2}$ or

$u^2+\left(\frac{du}{d\theta}\right)h2=\frac{a^2u^4}{\sin^2\alpha}$ or $\left(\frac{du}{d\theta}\right)^2=\frac{a^2u^4}{\sin^2\alpha}-u^2$.

Putting $u=\frac{1}{r}$, so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$, we have $\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{a^2}{r^4\sin^2\alpha}-\frac{1}{r^2}$

Or $\left(\frac{dr}{d\theta}\right)^2=a^2\operatorname{cosec}^2\alpha-r^2$ or $\frac{dr}{d\theta}=\sqrt{\left(a^2\operatorname{cosec}^2\alpha-r^2\right)}$ or $d\theta=\frac{dr}{\sqrt{\left(a^2\operatorname{cosec}^2\alpha-r^2\right)}}$

Integrating $\theta+B=\sin^{-1}\left(\frac{r}{a\operatorname{cosec}\alpha}\right)$, where B is a constant.

Initially when $r=a$, let $\theta=0$. Then $B=\sin^{-1}(\sin\alpha)=\alpha$.

$\therefore$ $\theta+\alpha=\sin^{-1}\{r/a(\operatorname{cosec}\alpha)\}$

Or $r=(a\operatorname{cosec}\alpha)\sin(\theta+\alpha)$

Or $r=(a\operatorname{cosec}\alpha)\cos\left\{\frac{1}{2}\pi-(\theta+\alpha)\right\}$

Or $r=(a\operatorname{cosec}\alpha)\cos\left\{(\theta+\alpha)-\frac{1}{2}\pi\right\}$

Or $r=(a\operatorname{cosec}\alpha)\cos\left\{\theta-\left(\frac{1}{2}\pi\right)-\alpha\right\}$

Or $r=(a\operatorname{cosec}\alpha)\cos(\theta-\beta)$, where $\beta=\frac{1}{2}\pi-\alpha$.

This represents a circle of diameter $a\operatorname{cosec}\alpha$ and pole on its circumference. Hence the path of the particle is a circle through the centre of force.

Que(15):- In a central orbit the force is $\mu u^3\left(3+2a^2u^2\right)$; if the particle be projected at a distance a with a velocity $\sqrt{\left(5\mu/a^2\right)}$ in a direction making an angle $\tan^{-1}\left(\frac{1}{2}\right)$ with the radius vector, show that the equation to the path is $r=a\tan\theta$.

Solution:- Here, the central acceleration $P=\mu u^3\left(3+2a^2u^2\right)$

The differential equation of the path is

$$h^2\left[u^2+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}\left(3+2a^2u^2\right)=\mu\left(3u+2a^2u^3\right)$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left(\frac{3u^2}{2}+\frac{2a^2u^4}{4}\right)+A$$ where A is a constant.

Or $v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(3u^2+a^2u^4\right)+A$ (1)

But initially when $r=a$ i.e. $u=1/v=\sqrt{\left(5\mu/a^2\right)}$, $\phi=\tan^{-1}(1/2)$

Or $\tan\phi=1/2$ or $\sin\phi=1/\sqrt{5}$ or $p=r\sin\phi=a/\sqrt{5}$ or $1/p^2=u^2+(du/d\theta)^2=5/a^2$

$\therefore$ From (1), we have $\dfrac{5\mu}{a^2} = h^2 . \dfrac{5}{a^2} = \mu\left(\dfrac{3}{a^2} + \dfrac{a^2}{c^4}\right) + A$.

$\therefore$ $h^2 = \mu$ and $A = \mu / a^2$.

Substituting the values of h^2 and A in (1), we have $\mu\left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right] = \mu\left(3u^2 + a^2u^4\right) + \dfrac{\mu}{a^2}$

or $\left(\dfrac{du}{d\theta}\right)^2 = 2u^2 + a^2u^4 + \dfrac{1}{a^2} = \dfrac{1}{a^2}\left(2a^2u^2 + a^4u^4 + 1\right)$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$,we have $\left(-\dfrac{1}{r^2}\dfrac{du}{d\theta}\right)^2 = \dfrac{1}{a^2}\left(\dfrac{2a^2}{r^2} + \dfrac{a^4}{r^4} + 1\right)$

Or $\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{1}{a^2}\left(2a^2r^2 + a^4 + r^4\right) = \dfrac{1}{a^2}\left(a^2 + r^2\right)^2$ or $\dfrac{dr}{d\theta} = \dfrac{1}{a}\left(r^2 + a^2\right)$ or $d\theta = \dfrac{a\,dr}{\left(r^2 + a^2\right)}$

Integrating, $\theta + B = \tan^{-1}(r/a)$, (2)

Where B is a constant.

But initially when $r = a$, let $\theta = \pi/4$.

Then $\dfrac{1}{4}\pi + B = \tan^{-1} 1 = \dfrac{1}{4}\pi$, so that $B = 0$.

Putting $B = 0$ in (2), we get $\theta = \tan^{-1}(r/a)$ or $r = a\tan\theta$, is the required equation of the path.

Que(16):- A particle moves with a central acceleration $\mu\left(u^5 - \dfrac{1}{8}a^2u^7\right)$; it is projected at a distance a with a velocity $\sqrt{(25/7)}$ times the velocity for a circle at that distance and at an inclination $\tan^{-1}(4/3)$ to the radius vector, show that its path is the curve. $4r^2 - a^2 = 3a^2/(1-\theta)^2$.

Solution:- Here, the central acceleration $P = \mu\left(u^5 - \dfrac{1}{8}a^2u^7\right) = \mu\left(\dfrac{1}{r^5} - \dfrac{a^2}{8r^7}\right)$

If V is the velocity for a circle at a distance a under the same acceleration, then $\dfrac{V^2}{a} = [P]_{r=a} = \mu\left(\dfrac{1}{a^5} - \dfrac{a^2}{8a^7}\right) = \dfrac{7\mu}{8a^5}$

$\therefore$ $V^2 = 7\mu/8a^4$ or $V = \sqrt{(7\mu/8a^4)}$

$\therefore$ Velocity of projection of the particle $= \sqrt{(25/7)}\,.V = \sqrt{(25/7)}\sqrt{(7\mu/8a^4)} = \sqrt{(25\mu/8a^4)}$

The differential equation of the path is $h^2\left[u + \dfrac{d^2u}{d\theta^2}\right] = \dfrac{P}{u^2} = \dfrac{\mu}{u^2}\left(u^5 - \dfrac{1}{8}a^2u^7\right)$ or

$h^2\left[u + \dfrac{d^2u}{d\theta^2}\right] = \mu\left(u^3 - \dfrac{1}{8}a^2u^5\right)$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{u^4}{2} - \frac{a^2u^6}{24}\right) + A \qquad (1)$$

But initially when $r = a$ i.e. $u = 1/a, v = \sqrt{(25\mu/8a^4)}$, $\phi = \tan^{-1}(4/3)$ or $\tan\phi = 4/3$ or $\sin\phi = 4/5$ or $p = r\sin\phi = 4a/5$ or $1/p^2 = u^2 + (du/d\theta)^2 = 25/(16a^2)$.

Substituting the above initial conditions in (1), we get

$$\frac{25\mu}{8a^4} = h^2 . \frac{25}{16a^2} = \mu\left(\frac{1}{2a^4} - \frac{a^2}{24a^6}\right) + A$$

$$\therefore \qquad h^2 = \frac{2\mu}{a^2} \text{ and } A = \frac{25\mu}{8a^4} - \frac{11\mu}{24a^4} = \frac{8\mu}{3a^4}$$

Substituting the values of h^2 and A in (1), we have

$$\frac{2\mu}{a^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{u^4}{2} - \frac{1}{24}a^2u^6\right) + \frac{8\mu}{3a^4} \text{ or } u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{4} - \frac{a^4u^6}{48} + \frac{4}{3a^2} \text{ or}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{4} - \frac{a^4u^6}{48} + \frac{4}{3a^2} - u^2$$

$$= \frac{1}{48a^2}\left(64 - 48a^2u^2 + 12a^4u^4 - a^6u^6\right) = \frac{1}{48a^2}\left(4 - a^2u^2\right)^3$$

Putting $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{du}{d\theta}$, we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{48a^2}\left(4 - \frac{a^2}{r^2}\right)^3 = \frac{1}{48a^2r^6}\left(4r^2 - a^2\right)^3 \text{ or } \left(\frac{dr}{d\theta}\right)^2 = \frac{1}{48a^2r^2}\left(4r^2 - a^2\right)^3 \text{ or}$$

$$\frac{dr}{d\theta} = \frac{1}{4\sqrt{3}ar}\left(4r^2 - a^2\right)^{3/2} \text{ or } d\theta = \frac{4\sqrt{3}ar\,dr}{\left(4r^2 - a^2\right)^{3/2}} = \left(\frac{\sqrt{3}}{2}a\right)\left(4r^2 - a^2\right)^{-3/2}(8r)\,dr .$$

Integrating $\theta + B = \left(\frac{\sqrt{3}}{2}a\right)\frac{\left(4r^2 - a^2\right)^{-1/2}}{-1/2}$ or $\theta + B = \frac{-a\sqrt{3}}{\sqrt{\left(4r^2 - a^2\right)}}$ (2)

But initially when $r = a$ let $\theta = 0$. Then $0 + B = -1$ $B = -1$

Putting $B = -1$ in (2), we get

$$\theta - 1 = \frac{-a\sqrt{3}}{\sqrt{\left(4r^2 - a^2\right)}} \quad \text{or} \quad 1 - \theta = \frac{a\sqrt{3}}{\sqrt{\left(4r^2 - a^2\right)}} \quad \text{or} \quad \sqrt{\left(4r^2 - a^2\right)} = \frac{a\sqrt{3}}{1 - \theta} \text{ or}$$

$4r^2 - a^2 = \frac{3a^2}{(1-\theta)^2}$ which is the required path.

Que(17):- A particle moves with a central acceleration $\mu\left(3u^3+a^2u^5\right)$ being projected from a distance a at an angle 45° with a velocity equal to that in a circle at the same distance. Prove that the time to the centre of force is $\frac{a^2}{\sqrt{(2\mu)}}\left(2-\frac{1}{2}\pi\right)$.

Solution:- Here the central acceleration $P=\mu\left(3u^3+a^2u^5\right)=\mu\left(\frac{3}{r^3}+\frac{a^2}{r^5}\right)$. $\quad [\because u=1/r]$

If V is the velocity in a circle at a distance a under the same acceleration, then $\frac{V^2}{a}=[P]_{r=a}=\mu\left(\frac{3}{a^3}+\frac{a^2}{a^5}\right)$ or $V^2=\frac{4\mu}{a^2}$ or $V=2\sqrt{\mu/a}$.

The differential equation of the path is $h^2\left[u^2+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left(3u^3+a^2u^5\right)=\mu\left(3u+a^2u^3\right)$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)\right]=\mu\left(3u^2+a^2u^5\right)+A \qquad (1)$$

Where A is a constant.

But initially when $r=a$ i.e. $u=1/a,\ v=2\sqrt{\mu/a},\ \phi=45^\circ$, $p=r\sin\phi=a\sin\frac{1}{4}\pi=a/\sqrt{2}$ so that $1/p^2=u^2+(du/d\theta)^2=2/a^2$.

$\therefore$ From (1), we have $\frac{4\mu}{a^2}=h^2.\frac{2}{a^2}=\mu\left(\frac{3}{a^2}+\frac{a^2}{2a^4}\right)+A$.

$\therefore$ $h^2=2\mu$ and $A=\mu/2a^2$.

Substituting the values of h^2 and A in (1), we have

$$2a\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(3u^2+\frac{a^2}{2}u^4\right)+\frac{\mu}{2a^2} \quad \text{or } 2\left(\frac{du}{d\theta}\right)^2=u^2+\frac{a^2}{2}u^4+\frac{1}{2a^2}.$$

Putting $u=\frac{1}{r}$, so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$, we have $\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}+\frac{a^2}{2r^4}+\frac{1}{2a^2}$ or

$$4a^2\left(\frac{dr}{d\theta}\right)^2=2a^2r^2+a^4+r^4=\left(r^2+a^2\right)^2 \text{ or } \frac{dr}{d\theta}=-\frac{r^2+a^2}{2a}. \qquad (2)$$

[Negative sign is taken because r decrease when θ increases. See figure in before example]

We have $h=r^2\frac{d\theta}{dr}=r^2\frac{d\theta}{dr}.\frac{dr}{dt}$ or $\sqrt{(2\mu)}=-r^2.\frac{2a}{\left(r^2+a^2\right)}.\frac{dr}{dt}$

[Substituting for h and $dr/d\theta$]

Or $dt=-\frac{2a}{\sqrt{(2\mu)}}.\frac{r^2dr}{\left(r^2+a^2\right)}$.

Integrating between the limits $r=a$ to $r=0$, the required time t_1 from the distance a to the centre of force is given by $t_1 = -\frac{2a}{\sqrt{(2\mu)}}\int_{r=a}^{0}\frac{r^2 dr}{r^2+a^2} = -\frac{2a}{\sqrt{(2\mu)}}.\int_a^0\left(1-\frac{a^2}{r^2+a^2}\right)dr$

$$= -\frac{2a}{\sqrt{(2\mu)}}.\left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]_a^0$$

$$= -\frac{2a}{\sqrt{(2\mu)}}\left[\left\{0-a\tan^{-1}0\right\}-\left\{a-a\tan^{-1}(a/a)\right\}\right]$$

$$= \frac{2a}{\sqrt{(2\mu)}}\left[a-a.\frac{1}{4}\pi\right] = \frac{a^2}{\sqrt{(2\mu)}}\left[2-\frac{1}{2}\pi\right].$$

Que(18):- A particle moves with central acceleration $\left(\mu u^2+\lambda u^3\right)$ and the velocity of projection at distance R is V; show that the particle will ultimately go off to infinity if $V^2 > \frac{2\mu}{R}+\frac{\lambda}{R^2}$.

Solution:- Here, the central acceleration $P = \mu u^2 + \lambda u^3$.

The differential equation of the path is $h^2\left\{u+\frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \frac{1}{u^2}\left(\mu u^2+\lambda u^3\right) = \mu+\lambda u$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2\mu u + \lambda u^2 + A \qquad (1)$$

But initially when $r=R$ i.e. $u=1/R,\ v=V$

$\therefore$ From (1), we have $V^2 = \frac{2\mu}{R}+\frac{\lambda}{R^2}+A$ or $A = V^2 - \frac{2\mu}{R}-\frac{\lambda}{R^2}$ (2)

Hence the equation (1) is $h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2\mu u+\lambda u^2+A$, where A is given by the equation (2) or $h^2\left(\frac{du}{d\theta}\right)^2 = \left(\lambda-h^2\right)u^2+2\mu u+A$

$$= \left(\lambda-h^2\right)\left\{u^2+\frac{2\mu}{\left(\lambda-h^2\right)}+\frac{A}{\left(\lambda-h^2\right)}\right\}$$

$$= \left(\lambda-h^2\right)\left\{\left(\frac{\mu}{\left(\lambda-h^2\right)}\right)^2\left(\frac{A}{\lambda-h^2}-\frac{\mu^2}{\left(\lambda-h^2\right)^2}\right)\right\}$$

Or $h\left(\frac{du}{d\theta}\right) = \sqrt{\left(\lambda-h^2\right)}.\left\{\left(\mu+\frac{\mu}{\left(\lambda-h^2\right)}\right)^2+\left(\frac{A}{\left(\lambda-h^2\right)}-\frac{\mu^2}{\left(\lambda-h^2\right)^2}\right)\right\}^{1/2}$

Or $d\theta = \dfrac{h}{\sqrt{(\lambda - h^2)}} \cdot \dfrac{du}{\left\{\left(u + \dfrac{\mu}{(\lambda - h^2)}\right)^2 + \left(\dfrac{A}{(\lambda - h^2)} - \dfrac{\mu^2}{(\lambda - h^2)^2}\right)\right\}^{1/2}}$

Integrating,

$$\theta + B = \frac{h}{\sqrt{(\lambda - h^2)}} . \log\left[\left(u + \frac{\mu}{(\lambda - h^2)}\right) + \left\{\left(u + \frac{\mu}{(\lambda - h^2)}\right)^2 + \left(\frac{A}{(\lambda - h^2)} - \frac{\mu^2}{(\lambda - h^2)^2}\right)\right\}^{1/2}\right]$$

(3)

Ultimately means when $r \to \infty$. So that particle will ultimately go off to infinity if θ is real when $r \to \infty$ i.e. when $u \to 0$.

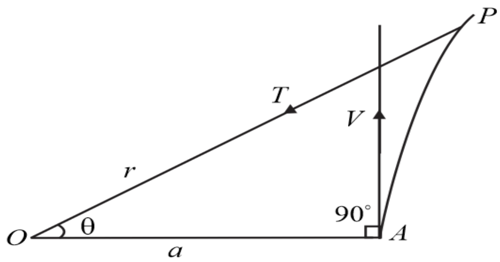
Now when $u = 0$, the equation (3) becomes $\theta + B = \dfrac{h}{(\lambda - h^2)} . \log\left[\dfrac{\mu}{(\lambda - h^2)} + \left(\dfrac{A}{\lambda - h^2}\right)^{1/2}\right]$

(4)

Assuming $\lambda > h^2$, we see that the equation (4) always gives a real value of θ provided A is positive. Therefore the particle will ultimately go off to infinity if $A > 0$ i.e. if $V^2 - \dfrac{2\mu}{R} - \dfrac{\lambda}{R^2} > 0$ [using (2)] i.e. if $V^2 > \dfrac{2\mu}{R} + \dfrac{\lambda}{R^2}$.

Que(19):- A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg, it is projected from an apse at a distance a with velocity $\sqrt{(2pgha)}$ show that the other apsidal distance is given by the equation $nr^2(r-a) - 2pha(r+a) = 0$.

Solution:- Let a particle of mass m be attached to a fixed point O by an elastic string of natural length a. initially the particle is at A such that $OA = a$ and is projected perpendicular to OA with velocity $V = \sqrt{(2pgh)}$. Let P be the position of the particle at any time t, where $OP = r$ and $\angle AOP = \theta$.



The only force acting on the particle at P in the plane of motion is the tension T in the string OP and is always directed towards the fixed centre O. So the path of the particle is a central orbit. By Hooke's law, the tension T in the string OP is given by $T = \lambda \dfrac{OP - a}{a} = nmg \dfrac{r - a}{a}$.

$[\because \lambda = nmg]$

$\therefore$ $P =$ the central acceleration of the particle at P.

[$\because$ acceleration $= \dfrac{force}{mass}$]

$$= ng\left(\frac{r-a}{a}\right) = \frac{ng}{a}\left(\frac{1}{u}-a\right) \qquad \left[\because r = \frac{1}{u}\right]$$

The differential equation of the path of the particle is

$$H^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{1}{u^2}\cdot\frac{ng}{a}\left(\frac{1}{u}-a\right) = \frac{ng}{a}\left(\frac{1}{u^3}-\frac{a}{u^2}\right).$$

Here the letter H has been used because the latter h is given in the problem.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = H^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{ng}{a}\left(-\frac{1}{u^2}+\frac{2a}{u}\right) + A \qquad (1)$$

Where A is a constant.

But initially at the apse A, $r = a, u = 1/a, du/d\theta = 0$, $v = \sqrt{(2pgh)}$.

Applying these initial conditions to (1), we have $2pgh = \dfrac{H^2}{a^2} = \dfrac{ng}{a}\left(-a^2 + 2a^2\right) + A$.

$\therefore$ $H^2 = 2pgha^2$ and $A = 2pgh - nga$.

Substituting the values of H^2 and A in A(1), we have

$$2pgha^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{ng}{a}\left(-\frac{1}{u^2}+\frac{2a}{u}\right) + 2pgh - nag \qquad (2)$$

Now at an apse, $du/d\theta = 0$. Therefore putting $du/d\theta = 0$ in (2), the apsidal distances are given by the equation $2pgha^2u^2 = \dfrac{ng}{a}\left(-\dfrac{1}{u^2}+\dfrac{2a}{u}\right) + 2pgh - nga$

or $\dfrac{2pha^2}{r^2} = \dfrac{n}{a}\left(-r^2 + 2ar\right) + 2ph - na \qquad \left[\because \dfrac{1}{u} = r\right]$

or $2pha^3 = nr^2\left(-r^2 + 2ar\right) + (2ph - na)ar^2$

or $2pha^3 - 2phar^2 + nr^2\left(r^2 - 2ar\right) + na^2r^2 = 0$

or $2pha\left(a^2 - r^2\right) + nr^2\left(r^2 - 2ar + a^2\right) = 0$

or $nr^2(r-a)^2 - 2pha(r-a)(r+a) = 0$

or $(r-a)\{nr^2(r-a) - 2pha(r+a)\} = 0$.

But $r - a = 0$ gives the first apsidal distance $r = a$. Therefore the other apsidal distance is given by the equation $nr^2(r-a) - 2pha(r+a) = 0$.

Que(20):- A particle is attached to a fixed point on a smooth horizontal plane by an elastic string of natural length a. Initially the particle is at rest on the plane with the string just taut and it is projected horizontally in a direction perpendicular to the string with a kinetic energy equal to the potential energy of the string when its extension is $3a/\sqrt{2}$. Prove that the second apsidal distance is equal to $3a$.

Solution:- By Hooke's law, the tension in the string when its extension is $3a/\sqrt{2}$.

$= \lambda . \dfrac{3a\sqrt{2}}{a} = \dfrac{3\lambda}{\sqrt{2}}$, where λ is the modulus of elasticity of the staring.

We know that the potential energy of an elastic string in any stretched position $= \dfrac{1}{2}$(initial tension + final tension) $\times$ extension.

$\therefore$ The potential energy of the string when its extension is $3a/\sqrt{2} = \dfrac{1}{2}\left[0 + \dfrac{3\lambda}{\sqrt{2}}\right] \times \dfrac{3a}{\sqrt{2}} = \dfrac{9a\lambda}{4}$

.

[Note that the initial tension is zero]

If V is the velocity of projection of the particle, then its kinetic energy at that time $= \dfrac{1}{2}mV^2$

.

According to the question $\dfrac{1}{2}mV^2 = \dfrac{9a\lambda}{4}$ or $V^2 = \dfrac{9a\lambda}{2m}$ or $V = \sqrt{\left(\dfrac{9a\lambda}{2m}\right)}$.

Now suppose the particle is initially at A, where $OA = a =$ natural length of the string. [Refer figure of before example]

The particle is projected from A perpendicular to OA with velocity $V = \sqrt{(9a\lambda/2m)}$. Let P be the position of the particle at any time t, where $OP = r$. The only force acting on the particle at P in the plane of motion is the tension T in the string OP and is always directed towards the fixed centre O. By Hooke's law, $T = \lambda \dfrac{OP - a}{a} = \lambda \dfrac{r-a}{a}$.

$\therefore$ $P =$ the central acceleration of the particle at the point P

$$= \frac{T}{m} = \frac{\lambda}{am}(r-a) = \frac{\lambda}{am}\left(\frac{1}{u} - a\right).$$

The differential equation of the particle is $h^2\left[u^2 + \dfrac{d^2u}{d\theta^2}\right] = \dfrac{P}{u^2} = \dfrac{\lambda}{am}\left(\dfrac{1}{u^3} - \dfrac{a}{u^2}\right)$.

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\lambda}{am}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + A \qquad (1)$$

Now the point A is an apse. So initially at $A, r = a, u = 1/a, du/d\theta = 0, v = \sqrt{(9a\lambda/2m)}$.

$\therefore$ From (1), we have $\dfrac{9a\lambda}{2m} = h^2 . \dfrac{1}{a^2} = \dfrac{\lambda}{am}\left(-a^2 + 2a^2\right) + A$.

$\therefore$ $h^2 = \dfrac{9a^3\lambda}{2m}, A = \dfrac{7a\lambda}{2m}$.

Substituting the values of h^2 and A in (1), we get

$$\frac{9a^3\lambda}{2m}\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \frac{\lambda}{am}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + \frac{7a\lambda}{2m}.$$

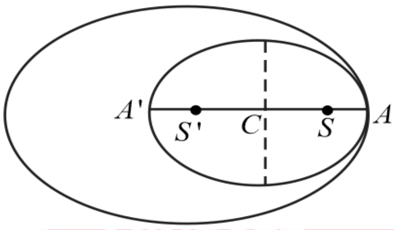
Putting $du/d\theta = 0$, the apsidal distances are given by $\frac{9}{2}a^3u^2 = -\frac{1}{au^2} + \frac{2}{u} + \frac{7a}{2}$, or

$\frac{9a^3}{2r^2} = -\frac{r^2}{a} + 2r + \frac{7a}{2}$ or $9a^4 - 7a^2r^2 - 4ar^3 + 2r^4 = 0$ or $2r^4 - 4ar^3 - 7a^2r^2 + 9a^4 = 0$ or $(r-a)(r-3a)(2r^2 + 4ar + 3a^2) = 0$.

Here $r = a,\ r = 3a$ are $+ive$ real roots. But $r = a$ is the given apsidal distance. Therefore $r = 3a$ is the other apsidal distance.

Que(21):- A body is describing an ellipse of eccentricity e under the action of a force tending to a focus and when at the nearer apse the centre of force transferred to the other focus. Prove that the eccentricity of the new orbit is $e(3+e)/(1-e)$.

Solution:- Let S and S' be the foci of an ellipse of eccentricity e and major axis of length $2a$ described by body under the actin of a force tending to the focus S. When S is the centre of force, A is the nearer apse.



The velocity v of the body at a distance r from S is given by $v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$. (1)

If V is the velocity of the body at A, when S is the centre of force, then from (1), we have

$$V^2 = \mu\left[\frac{2}{SA} - \frac{1}{a}\right] \qquad [\because at\ A, r = SA]$$

But $SA = CA - CS = a - ae = a(1-e)$ (2)

$$\therefore \quad V^2 = \mu\left[\frac{2}{a(1-e)} - \frac{1}{a}\right] = \frac{\mu(1+e)}{a(1-e)} \qquad (3)$$

When the body is at A and the centre of force is transferred to the other focus S', the body will described a new elliptic orbit with the centre of force S'as a focus. Since the velocity of the body at A is not changed, therefore if $2a'$ is the length of the major axis of the new ellipse, then the velocity V at A is given by $V^2 = \mu\left[\frac{2}{S'A} - \frac{1}{a'}\right] = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a'}\right]$ (4)

$$[\because S'A = CS' + CA = ae + a = a(1+e)]$$

From (3) and (4), we have $\frac{\mu(1+e)}{a(1-e)} = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a'}\right]$ or $\frac{(1+e)}{a(1-e)} = \frac{2}{a(1+e)} - \frac{1}{a'}$ (5)

Since the direction of the velocity of the body at A, is also not changed therefore for the new elliptic orbit also the point A is an apse. If e' is the eccentricity of the new ellipse, then corresponding to the result (2) for the original ellipse, we have for the new ellipse $S'A = a'(1-e')$.

But $S'A = a(1+e)$, from the original ellipse, as mentioned above.

$\therefore \quad a(1+e)=a'(1=e')$ or $a'=a(1+e)/(1-e')$.

Substituting this value of a'in (5), we have $\dfrac{(1+e)}{a(1-e)}=\dfrac{2}{a(1+e)}-\dfrac{(1-e')}{a(1+e)}$ or

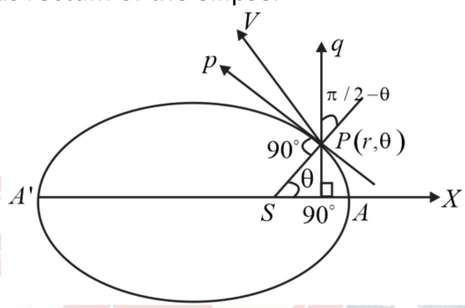
$$\frac{1+e}{a(1-e)}=\frac{2-(1-e')}{a(1+e)} \text{ or } 1+e'=\frac{(1+e)^2}{(1-e)}$$

or $e'=\dfrac{(1+e)^2}{(1-e)}-1=\dfrac{(1+e)^2-(1-e)}{(1-e)}=\dfrac{3e+e^2}{(1-e)}=\dfrac{e(3+e)}{1-e}$.

Que(22):- Show that the velocity of a particle moving in an ellipse about a centre of force in the focus in the focus is compounded of two constant velocities μ/h perpendicular to the radius and $\mu e/h$ perpendicular to the major axis.

Solution:- Referred to the focus S (i.e. the centre of force) as pole, let the equation of the ellipse orbit be $l/r=1+e\cos\theta$ (1)

Where l is the semi-latus rectum of the ellipse.



Let $P(r,\theta)$ be the position of the particle ay any time t. The resultant velocity v of the particle at P is along the tangent to ellipse at P. Suppose the velocity v is the resultant of two velocities p and q where p is perpendicular to the radius vector SP and q is perpendicular to the major axis AA'. Resolving the velocities p and q at P along and perpendicular to the radius vector SP, we have the radial velocities

$$dr/dt=q\cos\left(\frac{1}{2}\pi-\theta\right)=q\sin\theta \quad (2)$$

And the transverse velocity $r(d\theta/dt)=p+q\sin\left(\dfrac{1}{2}\pi-\theta\right)=p+q\cos\theta$ (3)

From (2), $q=\dfrac{1}{\sin\theta}\dfrac{dr}{dt}$ (4)

Differentiating both sides of (1) w.r.t 't' we have $-\dfrac{1}{r^2}\dfrac{dr}{dt}=-e\sin\theta\dfrac{d\theta}{dt}$.

$$\therefore \quad \frac{dr}{dt}=\frac{e}{l}\sin\theta r^2\frac{d\theta}{dt}$$

$$=\frac{eh}{l}\sin\theta \qquad [\because \text{ in a central orbit, } r^2\frac{d\theta}{dt}=h]$$

Substituting the value of dr/dt in (4), we get $q=\dfrac{1}{\sin\theta}.\dfrac{eh}{l}\sin\theta=\dfrac{eh}{l}$

$$= \frac{eh}{\left(h^2/\mu\right)} \qquad \left[\because h^2 = \mu l\right]$$

$= e\,\mu =$ constant.

This given one desired result.

Again from (3), we have $p = r\frac{d\theta}{dt} - q\cos\theta$

$$= \frac{h}{r} - \frac{e\mu}{h}\cos\theta \qquad \left[\because r^2\frac{d\theta}{dt} = h \;\; and \;\; q = \frac{e\mu}{h}\right]$$

$$= \frac{h}{r} + \frac{\mu}{h}\left(\frac{l}{r} - 1\right) \qquad \left[\because \;\; from(1), e\cos\theta = \frac{l}{r} - 1\right]$$

$$= \frac{h}{r} - \frac{\mu l}{hr} + \frac{\mu}{h}$$

$$= \frac{h}{r} - \frac{h^2}{hr} + \frac{\mu}{h} \qquad \left[\because h^2 = \mu\, l\right]$$

$= \mu / h =$ constant.

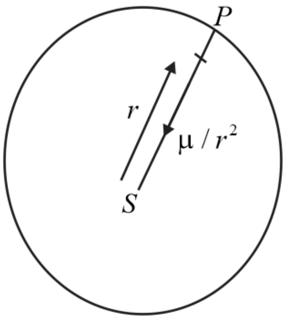
This given the other desired result.

Que(22):- If a planet were suddenly stopped in its orbit suppose circular, show that would fall into the sun in a time which is $\sqrt{2/8}$ times the period of the planet's revolution.

Solution:- Let a planet describing a circular path of radius a and centre S (the sum) be stopped suddenly at the point P of its path. Then it will begin to move towards S along the straight line PS under the acceleration μ (distance)^2

If Q is the position of the planet at time t such that $SQ = r$, then the acceleration at Q is μ / r^2 directed towards S.

$\therefore$ The equation of motion of the planet at Q is $v\dfrac{dv}{dr} = -\dfrac{\mu}{r^2}$ ($-ive$ sign is taken as the acceleration at Q is in the direction of r decreasing) or $v\,dv = -\dfrac{\mu}{r^2}dr$



Integrating $\dfrac{v^2}{2} = \dfrac{\mu}{r} + A$ where A is a constant.

But at $P, r = SP = a$ and $v = 0$.

[Note that the planet begins to move along PS with zero velocity at P]

$\therefore$ $0 = \dfrac{\mu}{a} + A$ or $A = -\dfrac{\mu}{a}$.

$\therefore$ $\dfrac{v^2}{2} = \dfrac{\mu}{r} - \dfrac{\mu}{a} = \dfrac{\mu(a-r)}{ar}$ or $v = \dfrac{dr}{dt} = -\sqrt{(2\mu / a)}.\sqrt{\left(\dfrac{a-r}{r}\right)}$ ($-ive$ sign is taken because r decreases as t increases) or $dt = -\sqrt{\left(\dfrac{a}{2\mu}\right)}.\sqrt{\left(\dfrac{r}{a-r}\right)}dr$ (1)

If t_1 is the time taken by the planet from P to S then integrating (1), we have

$$\int_0^{t_1} dt = -\sqrt{\left(\frac{a}{2\mu}\right)} \int_{r=a}^{0} \sqrt{\left(\frac{r}{a-r}\right)} dr$$ or

$$t_1 = \sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} \sqrt{\left(\frac{a\cos^2\theta}{a - a\cos^2\theta}\right)}.2a\cos\theta\sin\theta\, d\theta$$

Putting $r = a\cos^2\theta$ so that $dr = -2a\cos\theta\sin\theta\, d\theta$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} 2\cos^2\theta\, d\theta = a\sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi 2} = \frac{\pi a^{3/2}}{2\sqrt{(2\mu)}}.$$

But the time period T of the planet's revolution is given by $T = \dfrac{2\pi a^{3/2}}{\sqrt{\mu}}$

$\therefore$ $\dfrac{t_1}{T} = \dfrac{1}{4\sqrt{2}} = \dfrac{\sqrt{2}}{8}$ or $t_1 = \left(\sqrt{2/8}\right)T$ i.e. the time taken by the planet from P to S is $\sqrt{2/8}$ times the period of the planet's revolution.

Que(23):- A comet describing a parabola about the sun, when nearest to it suddenly breaks up, without gain or loss of kinetic energy into two equal portions one of which describes a circle; prove that the other will describe a hyperbola of eccentricity 2.

Solution:- Let a comet describe a parabola of latus rectum $4a$ about the sum S as the focus. The velocity v of the comet at a distance r from the sun (i.e. the focus) is given by $v^2 = 2\mu / r$ (1)

If V is the velocity of the comet at he point nearest to the sun i.e. at the vertex A, then from (1), we have $V^2 = 2\mu / a$.

[$\because$ $r = SA = a$ at the vertex A]

Let m be the mass of the comet which breaks into two equal part $m/2$ and $m/2$ at A and let their velocities at A be v_1 and v_2. It is given that on account of explosion of mass there is no loss or gain of kinetic energy. Therefore $\frac{1}{2}mV^2 = \frac{1}{2}\left(\frac{1}{2}mv_1^2\right) + \frac{1}{2}\left(\frac{1}{2}m_2^2\right)$ or

$v_1^2 + v_2^2 = 2V^2 = 2.(2\mu / a)$ from (2)

Or $v_1^2 + v_2^2 = 4\mu / a$ (3)

If the portion of the comet with velocity v_1 at A describes a circle about S we have $\frac{v_1^2}{a} = \frac{\mu}{a^2}$

or $v_1^2 = \frac{\mu}{a}$ (4)

Substituting this value of v_1^2 in (3), we have $v_2^2 = \frac{4\mu}{a} - \frac{\mu}{a} - \frac{3\mu}{a}$

Now $v_2^2 = \frac{3\mu}{a} > \frac{2\mu}{a}$ i.e. at $A, v_2^2 > \frac{2\mu}{r}$

Therefore the portion of the comet with velocities v_2 at A describes a hyperbola of transverse axis $2a_1$ (say).

In a hyperbola orbit with transverse axis of length $2a_1$ the velocity v at a distance r from the focus (i.e. the sun) is given by $v^2 = \mu\left[\frac{2}{r} + \frac{1}{a_1}\right]$

Here at the point A, $v = v_2$ and $r = a$

$\therefore$ $v_2^2 = \mu\left[\frac{2}{a} + \frac{1}{a_1}\right]$ or $\frac{3\mu}{a} = \mu\left[\frac{2}{a} + \frac{1}{a_1}\right]$ or $\frac{1}{a} = \frac{1}{a_1}$ or $a_1 = a$

Thus the length of the transverse axis of the hyperbola described is $2a$. Let $2b$ be the length of the conjugate axis of this hyperbola and e be its eccentricity.

Now we know that in a hyperbola orbit $vp = h = \sqrt{(\mu l)}$

Here at the point $A, v = v_2$ and $p = a$

$\therefore$ $v_2 a = \sqrt{(\mu l)} = \sqrt{\{\mu(b^2/a)\}}$ $\left[\because l = b^2/a\right]$

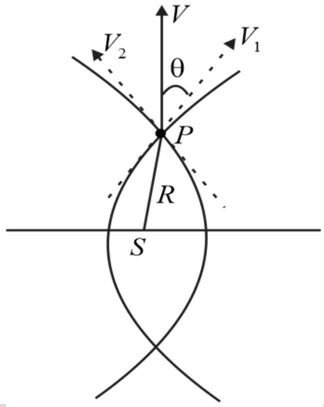
Or $\sqrt{\left(\frac{3\mu}{a}\right)}.a = \sqrt{\left\{\frac{\mu a^2(e^2-1)}{a}\right\}}$ [$\because$ for the hyperbola $b^2 = a^2(e^2-1)$]

Or $3 = e^2 - 1$ or $e^2 = 4$ or $e = 2$.

Que(24):- Two particles of mass m_1 moving in coplanar parabolas round the sun, collide at right angles and coalesces when their common distance from the sun is R. Show that the subsequent path of the combined particle is an ellipse of major axis $\dfrac{(m_1+m_2)^2 R}{2m_1 m_2}$

Solution:- Let the two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun S (as focus) collide at right angles at the common point P at a distance R from S.

The velocity v of a particle in a parabolic path at a distance r from the focus S is given by $v^2 = 2\mu / r$.



Let v_1 and v_2 be the velocities of m_1 and m_2 respectively at the time of collision at P. Then $v_1^2 = \dfrac{2\mu}{R}$ and $v_2^2 = \dfrac{2\mu}{R}$

$\therefore \quad v_2^2 = v_1^2 \qquad (1)$

Let the two particles coalesce into a single body of mass (m_1+m_2) after collision at P and let this single mass move with velocity V at an angle with the direction of the velocity v_1 of the mass m_1.

By the principle of conservation of momentum in the direction of v_1 and perpendicular to it (i.e. along the direction of v_2), we have $(m_1+m_2)V\cos\theta = m_1 v_1 + m_2 0$ and $(m_1+m_2)V\sin\theta = m_1.0 + m_2 v_2$

Squaring and adding, we have $(m_1+m_2)^2 V^2 = m_1^2 v_1^2 + m_2^2 v_2^2 = m_1^2 + v_1^2 + m_2^2 v_1^2$

$\left[\because\ v_2^2 = v_1^2\right]$

Or $V^2 = \dfrac{(m_1^2+m_2^2)v_1^2}{(m_1+m_2)^2} \qquad (2)$

Since $(m_1+m_2)^2 > m_1^2 + m_2^2$, therefore $V^2 < v_1^2$ i.e. $V^2 < \dfrac{2\mu}{R}$.

Therefore the path of combined body after collision at P is an ellipse with S as focus. If $2a_1$ is the length of the major axis of this ellipse, then the velocity at any point of this elliptic orbit at a distance r from S is given by $v^2 = \mu\left[\dfrac{2}{r} - \dfrac{1}{a_1}\right]$

But at the point P is elliptic orbit, $r = R$ and $v = V$.

$$\therefore \quad V^2 = \mu\left[\frac{2}{r} - \frac{1}{a_1}\right] \qquad (3)$$

From (2) and (3), we have $\dfrac{\left(m_1^2 + m_2^2\right)v_1^2}{\left(m_1 + m_2\right)^2} = \mu\left[\dfrac{2}{R} - \dfrac{1}{a_1}\right]$ or $\dfrac{m_1^2 + m_2^2}{\left(m_1 + m_2\right)}.\dfrac{2\mu}{R} = \mu\left[\dfrac{2}{R} - \dfrac{1}{a_1}\right]$

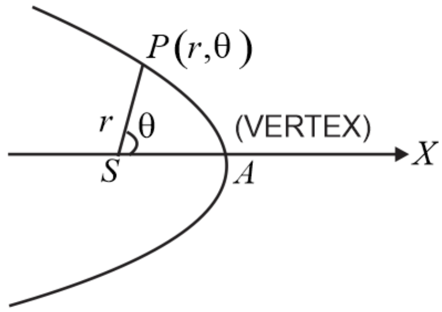
$$\left[\because\ v_1^2 = \frac{2\mu}{R}\right]$$

Or $\dfrac{1}{a_1} = \dfrac{2}{R} - \dfrac{\left(m_1^2 + m_2^2\right)}{\left(m_1 + m_2\right)^2}.\dfrac{2}{R} = \dfrac{2}{R}\left[1 - \dfrac{\left(m_1^2 + m_2^2\right)}{\left(m_1 + m_2\right)^2}\right] = \dfrac{2}{R}.\dfrac{2m_1m_2}{\left(m_1 + m_2\right)^2}$.

$\therefore \quad 2a_1 = \dfrac{\left(m_1 + m_2\right)^2}{2m_1m_2}.R$, which is the required length of the major axis.

Que(25):- Prove that in a parabolic orbit the time taken to move from the vertex to a point distance r from the focus is $\dfrac{1}{3\sqrt{\mu}}(r+1)\sqrt{(2r-l)}$, where $2l$ is the latus rectum.

Solution:- For figure refer $7, page 20



The polar equation of a parabola of latus rectum $2l$ referred to the focus S as the pole and the axis SA, where A is the vertex , as the initial line $l/r = 1 + \cos\theta = 2\cos^2\frac{1}{2}\theta$

Or $r = \dfrac{1}{2}l\sec^2\dfrac{1}{2}\theta \qquad (1)$

But we have $r^2\left(d\theta/dt\right) = h$

$$\therefore \quad dt = \frac{r^2}{h}d\theta = \frac{\frac{1}{4}l^2\sec^4\frac{1}{2}\theta}{\sqrt{(\mu l)}}d\theta \qquad \left[\because\ h^2 = \mu l\right]$$

$$= \frac{1}{4}\left(l^{3/2}/\sqrt{\mu}\right)\sec^4\frac{1}{2}\theta d\theta$$

Integrating the time taken from the vertex (i.e. $\theta = 0$) to the point $P(r,\theta)$ is given by

$$t = \frac{1}{2}\frac{l^{3/2}}{\sqrt{\mu}}\int_0^\theta \sec^4\frac{1}{2}\theta d\theta$$

$$= \frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\int_0^\theta\left(1 + \tan^2\frac{1}{2}\theta\right)\sec^2\frac{1}{2}\theta\, d\theta$$

$$=\frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\int_0^{\theta}\left[\sec^2\frac{1}{2}\theta+2\left(\tan^2\frac{1}{2}\theta\right)\left(\frac{1}{2}\sec^2\frac{1}{2}\theta\right)\right]d\theta$$

$$=\frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\left[2\tan\frac{1}{2}\theta+2.\frac{1}{3}\tan^3\frac{1}{2}\theta\right]_0^{\theta}$$

$$=\frac{1}{2}\left(l^{3/2}/\sqrt{\mu}\right)\left[\tan\frac{1}{2}\theta+\frac{1}{3}\tan^3\frac{1}{2}\theta\right]$$

$$=\frac{1}{6}\left(l^{3/2}/\sqrt{\mu}\right)\tan\frac{1}{2}\theta.\left(3+\tan^2\frac{1}{2}\theta\right)$$

But from (1), $\sec^2\frac{1}{2}\theta=2r/l$.

$\therefore\quad 1+\tan^2\frac{1}{2}\theta=2r/l$ or $\tan^2\frac{1}{2}\theta=(2r/l)-1=(2r-l)/l$.

$$\therefore\quad t=\frac{1}{6}\left(l^{3/2}/\sqrt{\mu}\right).\sqrt{\{(2r-l)/l\}}.\{3+(2r-l)/l\}$$

$$=\frac{1}{6}\frac{l^{3/2}\sqrt{(2r-l)}.(2l+2r)}{\sqrt{\mu}\,l^{3/2}}=\frac{1}{3\sqrt{\mu}}(r+l)\sqrt{(2r-l)}.$$

DYNAMICS
UPSC PREVIOUS YEARS QUESTIONS CHAPTERWISE(UPSC CSE/IFoS)

1.KINEMATICSIN TWO DIMENSIONS
2.RECTILINEAR MOTION
SIMPLE HARMONIC MOTION
PLANATORY MOTION
3.CONSTRAINED MOTION
MOTION IN A RESISTING MEDIUM
PROJECTILES

4.CENTRAL ORBITS
5.WORK, ENERGY AND IMPULSE

KINEMATICS IN TWO DIMENSIONS

Q1. If the radial and transverse velocities of a particle are proportional to each other, then prove that the path is an equiangular spiral. Further, if radial acceleration is proportional to transverse acceleration, then show that the velocity of the particle varies as some power of the radius vector.

[5c 2020 IFoS]

Q2. A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet? **[6a 2016 IFoS]**

Q3. A particle is acted on a force parallel to the axis of *y* whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to *x*-axis at the point where $y = a$. Prove that it will describe a catenary. **[8d 2016 IFoS]**

Q4. A particle is acted on by a force parallel to the axis of *y* whose acceleration (always towards the axis of *y*) is μy^{-2} and when $y = a$, it is projected parallel to the axis of *x* with velocity $\sqrt{\frac{2\mu}{a}}$. Find the parametric equation of the path of the particle. Here μ is a constant. **[8b UPSC CSE 2014]**

Q5. The velocity of a train increases from 0 to *v* at a constant acceleration f_1, then remains constant for an interval and again decreases to 0 at a constant retardation f_2. If the total distance described is *x*, find the total time taken. **[5c UPSC CSE 2011]**

CHAPTER-2 RECTILINEAR MOTION

Q1. **UPSC CSE 2023** A particle is moving under Simple Harmonic Motion of period *T* about a centre *O*. It passes through the point *P* with velocity *v* along the direction *OP* and *OP* = *p*. Find the time that elapses before the particle returns to the point *P*. What will be the value of *p* when the elapsed time is $\frac{T}{2}$? **(10)**

Q2. A particle moving along the *y*-axis has an acceleration Fy towards the origin, where F is a positive and even function of *y*. The periodic time, when the particle vibrates between $y = -a$ and $y = a$, is T. Show that

$$\frac{2\pi}{\sqrt{F_1}} < T < \frac{2\pi}{\sqrt{F_2}}$$

where F_1 and F_2 are the greatest and the least values of F within the range $[-a, a]$. Further, show that when a simple pendulum of length l oscillates through 30° on either side of the vertical line, T lines between $2\pi\sqrt{l/g}$ and $2\pi\sqrt{l/g}\sqrt{\pi/3}$. **[7c UPSC CSE 2019]**

Q3. A particle moving with simple harmonic motion in a straight line has velocities v_1 and v_2 at distance x_1 and x_2 respectively from the centre of its path. Find the period of its motion.

[6b UPSC CSE 2018]

Q4. If the velocities in a simple harmonic motion at distances a, b and c a fixed point on the straight line which is not the centre of force, are *u*, *v* and *w* respectively, show that the periodic time T is given by

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$ **[5c 2018 IFoS]**

Q5. Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and C_2 and C_2 are the statistical extension due to these weights and g is the acceleration due to gravity. Show that $g = \frac{4\pi^2(C_1 - C_2)}{T_1^2 - T_2^2}$. **[6b 2018 IFoS]**

Q6. A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position $P(OP = b)$ with velocity *v* in the direction OP. Prove that the time that elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$. **[5c 2017 IFoS]**

Q7. A body moving under SHM has an amplitude '*a*' and time period 'T'. If the velocity is trebled, when the distance from mean position $\frac{2}{3}a$, the period being unaltered, find the new amplitude.

[5c UPSC CSE 2015]

Q8. A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length *a* and then let go. Find the time taken by the particle to return to the starting point. **[5b 2015 IFoS]**

Q9. A particle is performing a simple harmonic motion (S.H.M.) of period T about a centre O with amplitude a and it passes through a point P, where $OP = b$ in the direction OP. Prove that the time which elapses before it returns to P to $\frac{T}{\pi}\cos^{-1}\left(\frac{b}{a}\right)$. **[5c UPSC CSE 2014]**

Q10. A body is performing S.H.M. in a straight line OPQ. Its velocity is zero at points P and Q whose distances from O are *x* and *y* respectively and its velocity is *v* at the mid-point between P and Q. Find the time of one complete oscillation. **[5c UPSC CSE 2013]**

Q11. A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P, where $OP = b$ with velocity v in the direction of OP. Find the time which elapses before it returns to P. **[5b 2013 IFoS]**

Q12. A particle is thrown over a triangle from one end of horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that, $\tan\theta = \tan\theta_1 + \tan\theta_2$. **[5d 2010 IFoS]**

MOTION THROUGH RESISTING MEDIUM

Q1. A particle of mass *m* is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time *t*. **[7c 2015 IFoS]**
Q2. A particle is projected vertically upwards with a velocity *u*, in a resisting medium which produces a retardation kv^2 when the velocity is *v*. Find the height when the particle comes to rest above the point of projection. **[7c 2013 IFoS]**
Q3. A particle is projected with a velocity *v* along a smooth horizontal plane in a medium whose resistance per unit mass is double the cube of the velocity. Find the distance it will describe in time *t*.

[8c 2013 IFoS]

CONSTRAINED MOTION

Q1. A particle of mass *m*, which is attached to one end of a light string whose other end is fixed at a point O, describes a circular motion in a horizontal plane about the vertical axis through O. Prove that the particle moves in a conical pendulum only if $g < l\omega^2$, where l is the length of the string and ω being angular velocity.
Further, a particle of mass *m* is attached to the middle of a light string of length $2l$, one end of which is fastened to a fixed point and the other end to a smooth ring of mass M which slides on a smooth vertical rod. If the particle describes a horizontal circle with uniform angular velocity ω about the rod, then prove that the inclination of both portions of the string to the vertical is
$\cos^{-1}\left\{\frac{(m+2M)g}{ml\omega^2}\right\}$. [6b]**IFoS 2022**

Q2. A heavy particle hangs by an inextensible string of length *a* from a fixed point and is then projected horizontally with a velocity $\sqrt{2gh}$. If $\frac{5a}{2} > h > a$, then prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$ from the point of projection. Also, prove that the greatest height ever reached by the particle above the point of projection is $\frac{(4a-h)(a+2h)^2}{27a^2}$.
[7]UPSC CSE 2021

Q3. A fixed wire is in the shape of the cardiod $r = a(1+\cos\theta)$, the initial line being the downward vertical. A small ring of mass *m* can slide on the wire and is attached to the point $r = 0$ of the cardiod by an elastic string of natural length *a* and modulus of elasticity 4 mg. The string is released from rest when the string is horizontal. Show by using the laws of conservation of energy that $a\theta^2(1+\cos\theta) - g\cos\theta(1-\cos\theta) = 0$, *g* being the acceleration due to gravity.

[5c UPSC CSE 2017]

Q4. A particle is free to move on a smooth vertical circular wire of radius *a*. At time $t = 0$ it is projected along the circle from its lowest point A with velocity just sufficient to carry it to the highest point B. Find the time T at which the reaction between the particle and the wire is zero.

[7c UPSC CSE 2017]

Q5. A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half. **[6b UPSC CSE 2010]**

PROJECTILES

Q1. 6(b) **UPSC CSE 2023** When a particle is projected from a point O_1 on the sea level with a velocity v and angle of projection θ with the horizon in a vertical plane, its horizontal range is R_1. If it is further projected from a point O_2, which is vertically above O_1 at a height h in the same vertical plane, with the same velocity v and same angle θ with the horizon, its horizontal range is R_2. Prove that $R_2 > R_1$ and ($R_2 = R_1$): R_1 is equal to $\frac{1}{2}\left\{\sqrt{\left(1+\frac{2gh}{v^2\sin^2\theta}\right)}-1\right\}:1$ **(15)**

Q2. 8(b) **UPSC CSE 2023** A particle is projected from an apse at a distance $\sqrt{c}$ from the centre of force with a velocity $\sqrt{\frac{2\lambda}{3}c^3}$ and is moving with central acceleration $\lambda(r^5-c^2r)$. Find the path of motion of this particle. Will that be the curve $x^4+y^4=c^2$? **(20)**

Q3. 5(d) A projectile is fired from a point O with velocity $\sqrt{2gh}$ and hits a tangent at the point $P(x,y)$ in the plane, the axes OX and OY being horizontal and vertically downward lines through the point O, respectively. Show that if the two possible directions of projection be at right angles, then $x^2=2hy$ and then one of the possible directions of projection bisects the angle POX. **UPSC CSE 2022**

Q4. 8(b) Describe the motion and path of a particle of mass m which is projected in a vertical plane through a point of projection with velocity u in a direction making an angle θ with the horizontal direction. Further, if particles are projected from that point in the same vertical plane with velocity $4\sqrt{g}$, then determine the locus of vertices of their paths. **UPSC CSE 2021**

Q5. 5(c) A particle is projected in a direction making an angle α with the horizon. It passes through the two points (x_1,y_1) and (x_2,y_2). Prove that

$$\tan\alpha=\frac{y_1R}{Rx_1-x_1^2}=\frac{x_2^2y_1-x_1^2y_2}{x_1x_2(x_2-x_1)},$$

where R denotes the horizontal range. **IFoS 2021**

Q6. A short projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h metres above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection.

[8b 2019 IFoS]

Q7. A particle projected from a given point on the ground just clears a wall of height h at a distance d from the point projection. If the particle moves in a vertical plane and if the horizontal range is R, find the elevation of the projection. **[1e UPSC CSE 2018]**

Q8. From a point in a smooth horizontal plane, a particle is projected velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cos\beta = 2\tan(\alpha - \beta)$. **[5d 2016 IFoS]**

Q9. A particle is projected from the base of a hill whose slope is that of right circular cone, whose axis is vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If the semivertical angle of the cone is 30°, h is height, determine the initial velocity u of the projection and its angle of projection. **[7b UPSC CSE 2015]**

Q10. A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is $\dfrac{2u}{g\sqrt{(1+3\sin^2\beta)}}$, range on the plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$ and the vertical height of the point struck is $\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$ above the point of projection. **[6c 2012 IFoS]**

Q11. A projectile aimed at a mark which is in the horizontal plane through the point of projection, falls x meter short of it when the angle of projection is α and goes y meter beyond when the angle of projection is β. If the velocity of projection is assumed same in all cases, find the correct angle of projection. **[5d UPSC CSE 2011]**

Q12. If v_1, v_2, v_3 are the velocities at three points A, B, C of the path of a projectile, where the inclinations to the horizon are $\alpha, a-\beta, a-2\beta$ and if t_1, t_2 are the times of describing the arcs AB, BC respectively, prove that

$v_3 t_1 = v_1 t_2$ and $\dfrac{1}{v_1} + \dfrac{1}{v_3} = \dfrac{2\cos\beta}{v_2}$. **[5d UPSC CSE 2010]**

CENTRAL ORBITS

Q1. If a planet, which revolves around the Sun in a circular orbit, is suddenly stopped in its orbit, then find the time in which it would fall into the Sun. Also, find the ratio of its falling time to the period of revolution of the planet. [5d **UPSC CSE 2021]**

Q2. A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant 'b' from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre? **[6b 2020 IFoS]**

Q3. Find the law of force for the orbit $r^2 = a^2\cos 2\theta$ (the pole being the centre of the force).

[6b 2019 IFoS]

Q4. A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\left(\dfrac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\dfrac{8}{15}\sqrt{\dfrac{6}{\mu}}$.

[8b 2017 IFoS]

Q5. A particle moves with a central acceleration which varies inversely as the cube of the distance. If it is projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, then find the equation to the path. **[5e UPSC CSE 2016]**

Q6. A particle moves in a straight line. Its acceleration is directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then find time, the particle will arrive at O. **[8c UPSC CSE 2016]**

Q7. A mass starts from rest at a distance 'a' from the centre of force which attracts inversely as the distance. Find the time of arriving at the centre. **[6d UPSC CSE 2015]**

Q8. A particle moves in a plane under a force, towards a fixed centre, proportional to the distance. If the path of the particle has two apsidal distance $a, b\left(a > b\right)$, then find the equation of the path.

[8b UPSC CSE 2015]

Q9. A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path. **[8c 2015 IFoS]**

Q10. A particle whose mass is m, is acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

[5c 2014 IFoS]

Q11. A particle moves with an acceleration

$\mu\left(x+\frac{a^4}{x^3}\right)$

towards the origin. If it starts from rest at a distance a from the origin, find its velocity when its distance from the origin is $\frac{a}{2}$. **[5d UPSC CSE 2012]**

Q12. A particle is moving with central acceleration $\mu\left[r^5 - c^4 r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)}c^3$, show that its path is a curve, $x^4 + y^4 = c^4$.

[7a 2012 IFoS]

Q13. A particle of mass m moves on straight line under an attractive force mn^2x towards a point O on the line, where x is the distance from O. If $x = a$ and $\frac{dx}{dt} = u$ when $t = 0$, find $x(t)$ for any time $t > 0$. **[7c(ii) UPSC CSE 2011]**

Q14. A particle moves with a central acceleration $\mu\left(r^5 - 9r\right)$, being projected from an apse at a distance $\sqrt{3}$ with velocity $3\sqrt{(2\mu)}$. Show that its path is the curve $x^4 + y^4 = 9$.

[7b UPSC CSE 2010]

Q15. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is, $\sin^{-1}\left[\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}\right]$. **[7b 2010 IFoS]**

KEPLER'S LAWS OF PLANETARY MOTION

Q1. A particle starts at a great distance with velocity V. Let *p* be the length of the perpendicular from the centre of a star on the tangent to the initial path of the particle. Show that the least distance of particle from the centre of the star is λ, where $V^2\lambda = \sqrt{\mu^2 + p^2V^4} - \mu$. Here μ is a constant.

[7c UPSC CSE 2020]

Q2. Prove that the path of a planet, which is moving so that its acceleration is always directed to a fixed point (star) and is equal to $\frac{\mu}{(\text{distance})^2}$, is a conic section. Find the conditions under which the path becomes (i) ellipse, (ii) parabola and (iii) hyperbola. **[8b UPSC CSE 2019]**

Q3. A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that the velocity then is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis, *e* is the eccentricity and T is the periodic time. **[7b 2017 IFoS]**

Q4. The apses of a satellite of the Earth at distance r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2. **[5c 2011 IFoS]**

WORK, POWER & ENERGY

Q1. 5(d) A person is drawing water from a well with a light bucket which leaks uniformly. The bucket weighs 50 kg when it is full. When it arrives at the top, half of the water remains inside. If the depth of the water level in the well from the top is 30 m, then find the work done in raising the bucket to the top from the water level. **IFoS 2022**

Q2. A light rigid rod ABC has three particles each of mass *m* attached to it at A, B and C. The rod is struck by a blow P at right angles to it at a point distant from A equal to BC. Prove that the kinetic energy set up is $\frac{1}{2}\frac{P^2}{m}\frac{a^2-ab+b^2}{a^2+ab+b^2}$, where $AB = a$ b and $BC = b$. **[5e UPSC CSE 2020]**

Q3. A four-wheeled railway truck has a total mass M, the mass and radius of gyration of each pair of wheels and axle are *m* and *k* respectively, and the radius of each wheel is *r*. Prove that if the truck is propelled along a level tack by a force P, the acceleration is $\frac{P}{M+\frac{2mk^2}{r^2}}$, and find the horizontal force exerted on each axle by the truck. The axle friction and wind resistance are to be neglected.

[8c UPSC CSE 2020]

Q4. The force of attraction of a particle by the earth is inversely proportional to the square of its distance from the earth's centre. A particle, whose weight on the surface of the earth is W, falls to the surface of the earth from a height $3h$ above it. Show that the magnitude of work done by the earth's attraction force is $\frac{3}{4}hW$, where h is the radius of the earth. **[5d UPSC CSE 2019]**

Q5. A spherical shot of W gm weight and radius r cm, lies at the bottom of cylindrical bucket of radius R cm. The bucket is filled with water up to a depth of h cm $(h \geq 2r)$. Show that the minimum amount of work done in lifting the shot just clear of the water must be $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ cm gm. W' gm is the weight of water displaced by the shot.

[8a UPSC CSE 2017]

Q6. An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2-\frac{1}{2}\right)$.

[6b 2014 IFoS]

Q7. A particle of mass 2.5 kg hangs at the end of a string, 0.9 m long, the other end of which is attached to a fixed point. The particle is projected horizontally with a velocity 8 m/sec. Find the velocity of the particle and tension in the string when the string is (i) horizontal (ii) vertically upward.

[7a UPSC CSE 2013]

Q8. A heavy ring of mass m, slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant a from the rod and has a mass $M(>m)$ fastened to its other end. Show that if the ring be dropped from a point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2Mma}{M^2-m^2}$ before coming to rest. **[7a UPSC CSE 2012]**

Q9. A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$. **[5c 2012 IFoS]**

Q10. A mass of 560 kg. moving with a velocity of 240 m/sec strikes a fixed target and is brought to rest in $\frac{1}{100}$ sec. Find the impulse of the blow on the target and assuming the resistance to be uniform throughout the time taken by the body in coming to rest, find the distance through which it penetrates.

[7a UPSC CSE 2011]

Q11. After a ball has been falling under gravity for 5 seconds it passes through a pane of glass and loses half its velocity. If it now reaches the ground in 1 second, find the height of glass above the ground. **[7c(i) UPSC CSE 2011]**

Q12. The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$\vec{r}=\left(t^2-2t\right)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}$

At time $t=1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. **[8d 2011 IFoS]**