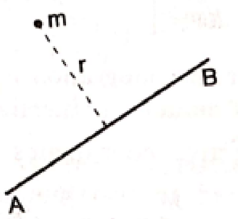
Ch. -1: Moment of Inertia

Some basic terms and their meanings.

Rigid Body: A rigid body is the system of particles such that the mutual distance of every pair of specified particles in it is invariable and the body does not expand or contract or change its shape in any way. i.e. the rigid body has invariable size and shape and the distance between any two particles remains always same.

Moment of inertia of a particle: Consider a particle of mass m and a line a line AB, then the moment of inertia of the particle of mass m about the line AB is defined as $I = mr^2$, where r is the perpendicular distance of the particle from the line.

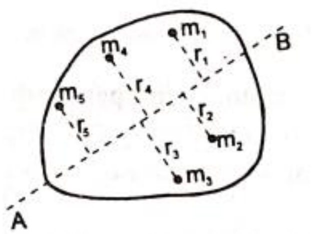


Moment of inertia of a system of particles:

Let there be a number of particles m_1, m_2, m_3, …..m_p, and let r_1, r_2, r_3 …….r_p be the perp. distances of these masses from the given line AB, then the moment of inertia of the system is defined as

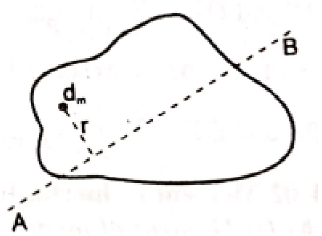
$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...m_p r_p^2$$

$$= \sum_{p=1}^{n} m_p r_p^2$$



Moment of inertia of a continuous distribution of mass: Consider a rigid body and let dm be mass of the elementary portion of the body which is at a perpendicular distance r from the given line AB, then the moment of inertia of the whole body is defined as $I = \int r^2 dm$,

where the integration is taken over the whole body.



Radius of Gyration: The moment of inertia of a system of particles about the line AB is

$$=\sum_{p=1}^{n} m_p r_p^2$$

Let the total mass of the system of particles be M, then

$M=\sum_{p=1}^{n} m_p$ and further define a quantity K such that

$$I = MK^2 \Rightarrow K^2 = \left(\frac{I}{M}\right) = \frac{\sum_{p=1}^{n} m_p r_p^2}{\sum_{p=1}^{n} m_p}$$

Then K is called the **radius of gyration** of the system about AB. In the case of continuous mass distribution, we similarly have

$$K^2 = \left(\frac{I}{M}\right) = \left[\frac{\int r^2 dm}{\int dm}\right]$$

Where the integration is taken for the whole body.

Product of inertia: If (x_1, y_1), (x_2, y_2), (x_3, y_3), ………(x_p, y_p), be the respective coordinates of the particles of masses m_1, m_2, m_3, …….m_p , referred to two mutually perpendicular lines OX and OY, then the product of inertia of the system of particles with respect to the lines OX and OY, is defined as,

$$P = m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3 ..... + m_px_py_p = \sum_{p=1}^{n} m_p x_p y_p$$

If mutually perpendicular axes OX, OY, OZ be taken in space and $(x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_p, y_p, z_p)$ be the respective co-ordinates of the particles of masses m_1, m_2, …….m_p, then we have, product of inertia of the system with respect to the axes OX and OY $=\sum_{p=1}^{n} m_p x_p y_p$

Product of inertia of the system with respect to the axes OY and OZ $=\sum_{p=1}^{n} m_p y_p z_p$

Product of inertia of the system with respect to the axes OZ and OX = $= \sum_{p=1}^{n} m_p x_p z_p$

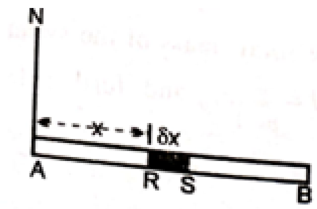
Moment of inertia in some simple cases.

(a) (i) Moment of inertia of a *rod of length 2a and mass M* about a line *through one of its extremities* perp. To its length.

Consider an element RS of breadth δx of the rod AB at distance x from the line AN, where AN is perp. To AB, M.I. of the element RS about AN $= \frac{M}{2a}\delta x\, x^2$ where $\left(\frac{M}{2a}\right)\delta x =$ mass of the element.

$\therefore$ M.I. of the whole rod

$$= \int_0^{2a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = M \cdot \frac{4a^2}{3}$$

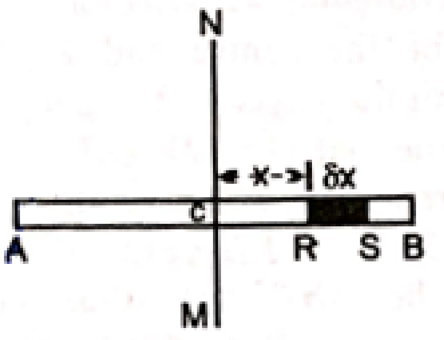


(ii) Moment of inertia of a *rod of length 2a and of mass M* about a line through its centre *perpendicular to its length.*

Consider an element RS of breadth δx at distance x from the centre C.

$\therefore$ M.I. of the element RS about NCM is= $\frac{M}{2a}\delta x \cdot x^2$

$\Rightarrow$ M.I. of the whole rod about MN $= \int_{-a}^{a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = M \cdot \frac{a^2}{3}$



(b) Rectangular Lamina.

(i) Moment of inertia of a *rectangular lamina* about a line *through its centre and parallel to one of its edges.*

Consider the strip RSPQ of breadth δx of the rectangular lamina ABCD such that AB = 2a and AD = 2b.

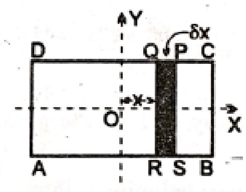
Let M be the mass of the rectangular lamina. Then mass per unit area = $\dfrac{M}{4ab} = \rho$ (say).

$\therefore$ Mass of the strip RSPQ = $2b\,\delta x\,\rho = \dfrac{M}{4ab}(2b\cdot\delta x)$

Now using [A case (ii)], we get M.I. of the strip about

$$OX = \frac{M}{4ab}2b\,\delta x\left(\frac{b^2}{3}\right) = \frac{M}{2a}\cdot\frac{b^2}{3}\delta x$$

$\therefore$ M.I. of the rectangular lamina about $OX = \displaystyle\int_{-a}^{a}\frac{M}{2a}\cdot\frac{b^2}{3}dx = \frac{1}{3}Mb^2$



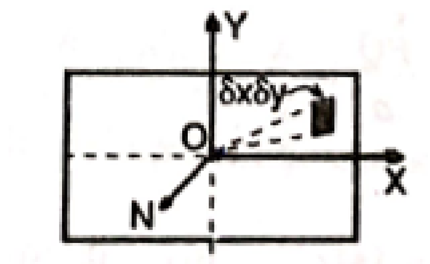
Similarly M.I. of the rectangular lamina – about OY is $\dfrac{1}{3}Ma^2$

(ii) Moment of inertial of a *rectangular lamina* about a line *through its centre* and *perp. To its plane.*

Consider an elementary area $\delta x\delta y$ of the lamina at a distance $\sqrt{x^2+y^2}$ from O. Mass of the elementary area $= \dfrac{M}{4ab}\delta x\cdot\delta y$.

M.I. of this elementary area about the line ON through O and perpendicular to the plane of the rectangular lamina $= \dfrac{M}{4ab}\delta x\cdot\delta y\left(x^2+y^2\right)$

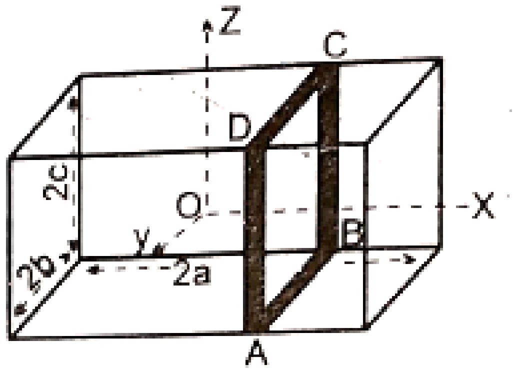
$\therefore$ M.I. of the rectangular lamina about ON is



$$= \frac{M}{4ab}\int_{x=-a}^{a}\int_{y=-b}^{b}\left(x^2+y^2\right)dx\,dy = \frac{M}{4ab}4\int_{x=0}^{a}\int_{y=0}^{b}\left(x^2+y^2\right)dx\,dy$$

$$= \frac{M}{ab}\left[\frac{1}{3}bx^3+\frac{1}{3}b^3x\right]_0^a = \frac{M}{3}\left(a^2+b^2\right)$$

(iii) Rectangular Parallelopiped: Let O be the centre and 2a, 2b, 2c the lengths of the edges of the parallelopiped and further let OX, OY, OZ, be the axes of reference, parallel to the edges of lengths 2a, 2b and 2c respectively. Divide the parallelopiped into thin rectangular slices perp. to OX, ABCD being one such slice at a distance x. Let the width of the slice be δx.



$\therefore$ .M.I. of the rectangular slice about OX

$$= \text{mass } \times \frac{b^2+c^2}{3} = 2b \cdot 2c \cdot \rho \cdot \delta x \frac{b^2+c^2}{3} = 4bc\rho \frac{b^2+c^2}{3} \delta x$$

[mass of the slice ABCD = 2b 2c $\delta x\rho$]

$\Rightarrow$ M.I. of the parallelopiped about OX

$$= 4bc\rho \frac{b^2+c^2}{3} \int_{-a}^{a} dx = 8abc\rho \frac{b^2+c^2}{3}$$

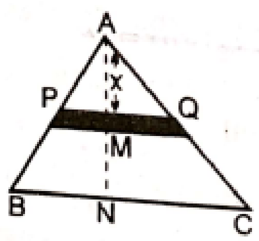
$$= M\frac{b^2+c^2}{3}$$ [mass of the parallelopiped = 2a 2b.2cρ = 8abc ρ]

Similarly, M.I. of the parallelopiped about OY $= M\frac{c^2+a^2}{3}$ and M.I. of the parallelopiped about

OZ $= M\frac{a^2+b^2}{3}$

Note: M.I. of the cube of side 2a about any of its axis is $\frac{2}{3}Ma^2$.

(c) Moment of inertia of a *uniform triangular lamina about one side*. Let us divide the lamina ABC by strips parallel to BC. Let PQ be one of such strips of breadth δx at distance x from A and let p be the length of perpendicular AN.



Now $\frac{PQ}{a}=\frac{x}{p}$ i.e. $\frac{PQ}{a}=\frac{xa}{p}$. If M is the mass of the triangular lamina, then mass per unit

$$\text{Area} = \left[\frac{M}{\left(\frac{1}{2}a\cdot p\right)}\right]=\rho(\text{say}).$$

$$\therefore \text{ Mass of the strip } = \frac{M}{\frac{1}{2}ap}PQ\,\delta x=\frac{2M}{p^2}x^2\,\delta x$$

$$\therefore \text{ M.I. of the strip about BC } = \frac{2M}{p^2}x\,\delta x(p-x)^2$$

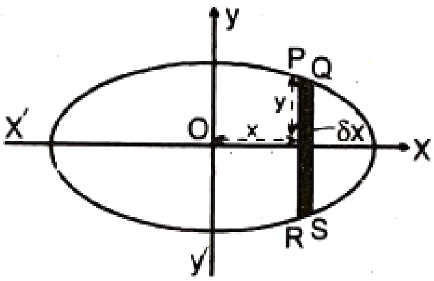
$$\therefore \text{ M.I. of the triangle about BC } = \frac{2M}{p^2}\int_0^p (p-x)^2\,x\,dx=\frac{1}{6}Mp^2$$

(D) Elliptic disc: Moment of inertia of *an elliptic disc about its major axis.*

Let PRSQ be an elementary strip of breadth δx at a distance x from O, where O is the centre of the disc. M.I. of the strip about.

$OX=2y\cdot\delta x\cdot\rho\cdot\frac{y^2}{3}$, where ρ is the mass per unit area.

$\therefore$ M.I. of the elliptic lamina about OX



$$=\int_{-a}^{a}2y\rho\cdot\frac{y^2}{3}dx=\frac{2\rho}{3}\int_{-a}^{a}y^3dx$$

$$=\frac{2\rho}{3}b^3\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx$$

$$=\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow y=b\left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}}\right]$$

Put x = a sinϕ, so that dx = a cosϕ dϕ.

$\therefore$ M.I. of the elliptic lamina about OX

$$= \frac{2\rho b^3}{3}\int_{-\pi/2}^{\pi/2}\cos^3\phi\cdot a\cos\phi\, d\phi = \frac{4\rho b^3 a}{3}\int_{0}^{\pi/2}\cos^4\phi\, d\phi$$

$$= \frac{4\rho b^3 a}{3}\cdot\frac{3\pi}{16} = \frac{\pi a b^3\rho}{4}$$

Again mass of the elliptic lamina

$$M = \rho\int_{-a}^{a} 2y\, dx = 2\rho b\int_{-a}^{a}\left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}} dx$$

$$= 2\rho b\int_{-\pi/2}^{\pi/2}\cos\phi\cdot a\cos\phi\, d\phi = \pi a b\rho \Rightarrow \rho = \frac{M}{\pi a b}$$

Hence from (1), M.I. of the elliptic lamina about OX i.e. about major

axes $= \frac{\pi a b^3}{4}\cdot\frac{M}{\pi a b} = \frac{1}{4}Mb^2$

Similarly M.I. of the elliptic lamina about OY i.e. about minor axes $= \frac{1}{4}Ma^2$

(e) Hoop or Circumference of a circle.

(i) Moment of inertia of a *hoop about a diameter*.

Consider an element PQ of the hoop and let it subtend an angle $\delta\theta$ at its centre O i.e.

$P\hat{O}Q = \delta\theta$ where $P\hat{O}X = \theta$

By the figure it is obvious that arc PQ= a $\delta\theta$, where a is the radius of the hoop.

Now M.I. of the element PQ about OX = (a $\delta\theta$,) $\rho\cdot a^2\sin^2\theta$,

where ρ is the mass per unit length of the hoop.

M.I. of the hoop about OX= $\int_{\theta=0}^{2\pi}(a\, d\theta)\rho\cdot a^2\sin^2\theta$

$$= \frac{Ma^2}{4\pi}\int_{0}^{2\pi}(1-\cos 2\theta)\, d\theta = \frac{1}{2}Ma^2$$

(ii) Moment of inertia of a hoop about a line through its centre and perp. to its plane. M.I. of the hoop about a line through O and perp. to its plane

$$= (a\cdot\delta\theta)\rho\cdot OP^2 = \frac{M}{2\pi}a^2\delta\theta \qquad (\because OP = a,\ M = 2\pi a\rho)$$

$\therefore$ M.I. of the hoop about a line through O and perp. to its plane

$$= \frac{Ma^2}{2\pi}\int_0^{2\pi} d\theta = \frac{Ma^2}{2\pi}[\theta]_0^{2\pi} = Ma^2$$

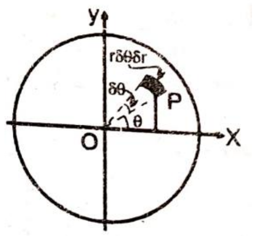
(f) Circular Disc.

(i) Moment of inertia of a *circular disc* of radius a *about its diameter*.

Consider an element $r\delta\theta\delta r$ of the disc at P such that OP makes an angle θ with the axis OX.

The perp. distance of P from OX is $r\sin\theta$.

Let ρ be the mass per unit area of the disc. M.I. of this element about OX = $r\,\delta\theta\,\delta r\cdot\rho\cdot(r\sin\theta)^2$.



$\therefore$ M.I. of this element about the diameter OX

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\rho\sin^2\theta\,d\theta\,dr = \frac{M}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\sin^2\theta\,d\theta\,dr \qquad (\because M = \pi a^2\rho)$$

$$= \frac{M}{2\pi a^2}\int_0^a r^3\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} dr = \frac{Ma^2}{4}$$

(ii) Moment of inertia of a *circular disc* of radius a about a *line through its centre perp* to its plane.

M.I. of the element $r\delta\theta\,\delta r$ about a line through O and perp. To the plane of the disc.

$$= (r\delta\theta\,\delta r)\cdot\rho\cdot OP^2 = \frac{Mr^3}{\pi a^2}d\theta\,dr \qquad (\because \pi a^2\rho = M)$$

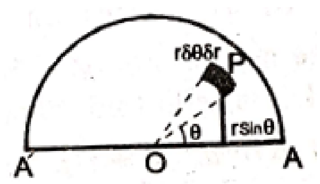
M.I. of the circular disc about a line through O and perp. To the plane of the disc.

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\frac{Mr^3}{\pi a^2}d\theta\,dr = \frac{Ma^2}{2}$$

(g) Solid Sphere.

If a semi-circular area is revolved about its bounding diameter then the solid so generated is called sphere. Now consider an element of area $r\delta\theta\,\delta r$ at P such that OP = r and makes an angle θ with the diameter.

When this area is revolved about the diameter A 'A, it will generate a ring of cross-section $r\delta\theta\,\delta r$ and radius $r\sin\theta$.



$\therefore$ Mass of this elementary ring= $2\pi r \sin\theta . r\delta\theta\ \delta r . \rho$

M.I. of the elementary ring about A 'A

$= (2\pi r \sin\theta . r\delta\theta\ \delta r . \rho)\ (r \sin\theta)^2$ **[see (e), (ii)]**

$= 2\pi\ \rho r^4 (\sin^3\theta)\ \delta\theta\ \delta r.$

M.I. of the solid sphere about the diameter A 'A

$$= 2\pi\rho \int_0^a \int_0^a r^4 \sin^3\theta\, d\theta\, dr = 4\pi\rho \int_0^{\pi/2} \int_0^a r^4 \sin^3\theta\, d\theta\, dr$$

$$= 4\pi\rho \left[\frac{r^5}{5}\right]_0^a \cdot \frac{2}{3} = 4\pi\rho \cdot \frac{a^5}{5} \cdot \frac{2}{3} = \frac{8\pi a^5 \rho}{15} = I \text{ Say}$$

But mass of the sphere, $M = \frac{4}{3}\pi a^3 \rho \Rightarrow \rho = \frac{3M}{4\pi a^3}$ ...(1)

$$= I = \frac{8}{15}\pi a^5 \cdot \frac{3M}{4\pi a^3} = \frac{2}{3}\left(Ma^2\right)$$

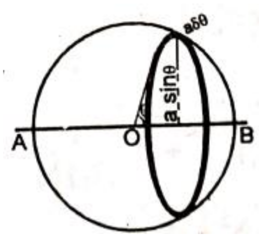
(h) Hollow sphere.

If semicircular arc is revolved about its diameter, then the surface so formed is known as hollow sphere. Consider an elementary arc $a\delta\theta$.

This arc $a\delta\theta$ will generate a circular ring of radius $a\sin\theta$ when revolved about the diameter AB.

Now mass of the elementary ring = $2\pi a \sin\theta . \delta\theta . \rho$.

M.I. of the elementary ring about AB= $(2\pi a \sin\theta . \delta\theta . \rho .) . a^2 \sin^2\theta$



$= 2\pi a^4 \rho \sin^3\theta\, \delta\theta$ **[see (e)…(iii)]**

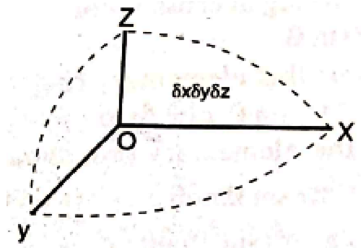
$\Rightarrow$ M.I. of the hollow sphere about the diameter AB

$$= 2\pi a^4 \rho \int_0^{\pi} \sin^3\theta\, d\theta = 2\pi a^4 \cdot \frac{M}{4\pi a^2} \int_0^{\pi} \sin^3\theta\, d\theta \qquad \left(\because M = 4\pi a^2 \rho\right)$$

$$= \frac{Ma^2}{2} \cdot 2 \int_0^{\pi/2} \sin^3\theta\, d\theta = Ma^2 \int_0^{\pi/2} \sin^3\theta\, d\theta = Ma^2 \cdot \frac{2}{3}$$

(i) Ellipsoid. Consider an elementary volume $\delta x\ \delta y\ \delta z$ in the positive octant of the ellipsoid $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)=1$. Let ρ be mass per unit volume then mass of the elementary volume

$$= \rho \cdot (\delta x \cdot \delta y \cdot \delta z)$$



Distance of this element from OX = $\sqrt{(y^2+z^2)}$

$\therefore$ M.I. of the ellipsoid about OX

$= 8\iiint \rho\, dx\, dy\, dz \left(y^2+z^2\right)$, the integration being taken over the positive octant of the ellipsoid and $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)\le 1$. Putting $\left(\frac{x^2}{a^2}\right)=u, \left(\frac{y^2}{b^2}\right)=v, \left(\frac{z^2}{c^2}\right)=w$ we get,

$$x = au^{\frac{1}{2}},\ dx = \frac{1}{2}au^{\frac{-1}{2}}du;\ y = bv^{\frac{1}{2}},\ dy = \frac{1}{2}bv^{-\frac{1}{2}}dv;$$

$$z = cw^{\frac{1}{2}}, dz = \frac{1}{2}cw^{-\frac{1}{2}}dw$$

Now, M.I. of the ellipsoid about OX

$$= 8\iiint \frac{\rho}{8} abc\left(b^2v+c^2w\right)u^{-\frac{1}{2}}v^{-\frac{1}{2}}w^{-\frac{1}{2}}\,du\,dv\,dw \text{ where } u + v + w \le 1$$

$$= abc\rho\iiint\left(b^2u^{-\frac{1}{2}}v^{-\frac{1}{2}}w^{-\frac{1}{2}}+c^2u^{-\frac{1}{2}}v^{-\frac{1}{2}}w^{-\frac{1}{2}}\right)du\,dv\,dw$$

$$= abc\rho\iiint\left(b^2u^{\frac{1}{2}-1}v^{\frac{3}{2}-1}w^{\frac{1}{2}-1}+c^2u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{3}{2}-1}\right)du\,dv\,dw$$

$$= abc\rho\cdot\left[b^2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\frac{7}{2}}+c^2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}\right] \text{ (using Dirichlet's theorem)}$$

$$= abc\rho\left(b^2+c^2\right)\frac{\pi}{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{4abc\rho\pi}{3}\cdot\frac{b^2+c^2}{5}$$

$$= M\cdot\frac{b^2+c^2}{5} \qquad \left(\text{where } M = \frac{4}{3}\pi abc\rho\right)$$

(j) Right circular cylinder.

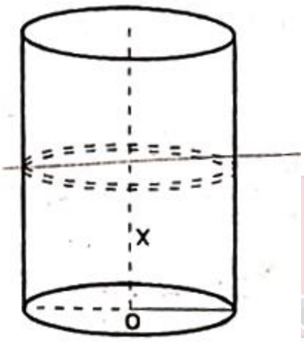
Let there be a right circular cylinder of radius a and height h.

Consider a circular disc of thickness δx at a distance x from O the centre of base.

Mass of the disc = $\pi a^2\,\delta x.\,\rho$, where ρ is mass per unit volume.

$\therefore$ M.I. of the disc about the axes perp. to the plane of the disc

$$= \pi a^2\delta x\rho\cdot\frac{1}{2}a^2 \qquad \textbf{[see f (ii)]}$$



M.I. of the cylinder = $\dfrac{\pi a^4\rho}{2}\displaystyle\int_0^h dx = \dfrac{\pi a^4 h\rho}{2} = \dfrac{1}{2}Ma^2 \qquad \left[\because \pi a^2 h\rho\right]$

Easy to Remember for Exam: The following table shows the moments of inertia of *various rigid bodies* considered above. In all cases it is assumed that the body has *uniform density.*

Sr.No.	Rigid Body	Moments of inertia
1.	Uniform rod of length 2a and mass M.	
(i)	About an axis perp. to the rod through the centre of mass.	$\frac{1}{3}Ma^2$
(ii)	About a line perp. to the rod through an end.	$\frac{4}{3}Ma^2$
2.	Rectangular plate of sides 2a, 2b and mass M	
(i)	About an axis perp. to the plate through the centre of mass.	$\frac{M}{3}\left(a^2+b^2\right)$
(ii)	About a line through centre parallel to the side 2a.	$\frac{1}{3}Mb^2$
3.	Rectangular parallelopiped of edges 2a, 2b, 2c. About a line parallel to the edge 2a, through the centre	$\frac{M}{3}\left(b^2+c^2\right)$
4.	Circular plate of radius a and mass M.	
(i)	About its diameter.	$\frac{1}{4}Ma^2$
(ii)	About a line perp. to the plate through the centre.	$\frac{1}{2}Ma^2$

5.	Elliptic disc of axes 2a and 2b and mass M.	
(i)	About the axis 2a.	$\frac{1}{4}Mb^2$
(ii)	About a line perp. to the disc through its centre.	$\frac{M}{4}\left(a^2+b^2\right)$
6.	Circular ring of radius a and mass M.	
(i)	About a diameter.	$\frac{1}{2}Ma^2$
(ii)	About a line perp. to the plate through the centre.	Ma^2
7.	Solid sphere of radius a and mass M. About a diameter.	$\frac{2}{5}Ma^2$
8.	Hollow sphere of radius a and mass M. About a diameter (thickness negligible)	$\frac{2}{3}Ma^2$
9.	Ellipsoid of axes 2a, 2b and 2c. about the axis 2a.	$\frac{M}{5}\left(b^2+c^2\right)$

Routh's Rule:

For remembering the *moment of inertia of symmetric rigid bodies*. M.I. *about an axis of symmetry*

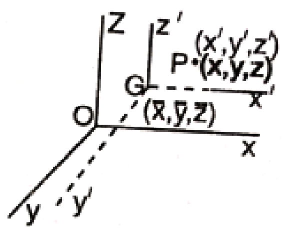
$$= \text{Mass} \times \frac{\text{Sum of the squares of perp. semi axes}}{3,\ 4 \text{ or } 5}$$

The denominator is 3, 4 or 5 according as the body is rectangular (including rod) elliptical (including circular) or ellipsoid (including sphere). (using Dirichlet's theorem)

Theorem of Parallel Axes

The Moment of Inertia and The Products of inertia about axes through the centre of gravity are given, to find the moments and products of inertia about parallel axes.

Let OX, OY, OZ be a set of co-ordinate axes through any point O, parallel to a set of co-ordinate axes GX', GY', GZ' through G, the centre of gravity. Let ($(\bar{x}, \bar{y}, \bar{z})$) be the co-ordinates of G with regard to co-ordinate axes OX, OY, OZ.



Let the co-ordinates of any element of mass m situated at the point P with regard to axes OX, OY, OZ be (x, y, z) and with regard to parallel axes through G be (x', y', z')

$$\therefore\ x=\bar{x}+x',\ y=\bar{y}+y',\ z=\bar{z}+z'$$

M.I. of the body about OX = $\sum m\left(y^2+z^2\right)$

$$=\sum m\left[(\bar{y}+y')^2+(\bar{z}+z')^2\right]$$

$$=\sum m\left[\left(\bar{y}^2+\bar{z}^2+2y'\bar{y}+2z'\bar{z}+y'^2+z'^2\right)\right]$$

$$= \sum m\left(\bar{y}^2 + \bar{z}^2\right) + \sum m\left[\left(y'^2 + z'^2\right) + 2\bar{y}\sum my' + 2\bar{z}\sum mz'\right]$$

Now referred to G as origin, the co-ordinates of the centre of the gravity of the body.

$$= \frac{\sum mx'}{\sum m} = 0, \frac{\sum my'}{\sum m} = 0, \frac{\sum mz'}{\sum m} = 0$$

$$\therefore \sum mx' = 0, \sum my' = 0, \sum mz' = 0$$

Hence M.I. of the body about OX = $\sum m\left(\bar{y}^2 + \bar{z}^2\right) + \sum m\left(y'^2 + z'^2\right)$

= $\left(\bar{y}^2 + \bar{z}^2\right)\sum m +$ M.I. about GX'

= $M\left(\bar{y}^2 + \bar{z}^2\right) +$ M.I. about GX'

= M.I. of mass M placed at G about OX + M.I. about GX'.

Again product of inertia of the body about OX and OY.

$$= \sum m\,xy = \sum m(x' + \bar{x})(y' + \bar{y})$$

$$= \sum m\,x'y' + \sum mx'\bar{y} + \sum m\bar{x}y' + \sum m\bar{x}\bar{y} + \sum mx'y' + \bar{y}\sum mx' + \bar{x}\sum my' + \bar{x}\bar{y}\sum m$$

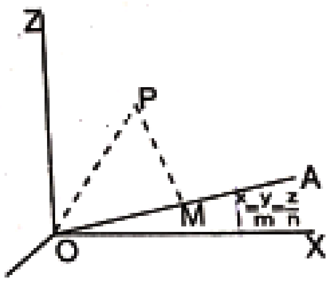
= $\sum m\,x'y' + M\bar{x}\bar{y}$ = The product of inertia about (GX' + GY') + Product of inertia of mass M placed at G about the axes OX and OY.

Moment of Inertia about a line: To find the moment of inertia about any axis through the meeting point of three perp. Axes, the moments and products of inertia about these three axes being known.

Proof: Let OX, OY, OZ be a set of three mutually perp. Axes.

Let A = M.I. about OX,

B = M.I. about OY, C = M.I. about OZ,



D = Product of inertia w.r.t. axes of y and z. E = Product of inertia w.r.t. axes of z and x and F = Product of the inertia w.r.t. axes of x and y. Now if m' is the mass of the element at P whose co- ordinates are (x, y, z), then we easily have

$$A=\sum m'\left(y^2+z^2\right), B=\sum m\left(x^2+z^2\right); C=\sum m'\left(x^2+y^2\right);\ D=\sum m'yz,\ E=\sum m'zx$$

$$F=\sum m'xy$$

Let OA be a line with direction cosines (l, m, n). From P draw PM $\perp$ to OA, then $PM^2 = OP^2 - OM^2$

$= (x^2 + y^2 + z^2) - (lx + my + nz)$

$$\left[\because \quad OP=\left(x^2+y^2+z^2\right), ON=\left(lx+my+nz\right)\right]$$

$$= x^2\left(1-l^2\right)+y^2\left(1-m^2\right)+z^2\left(1-n^2\right)-2mnyz-2\ln zx-2lmxy$$

$$= x^2\left(m^2+n^2\right)+y^2\left(l^2+n^2\right)+z^2\left(l^2+m^2\right)-2mnyz-2\ln zx-2lmxy$$

$$\left[\text{using } l^2+m^2+n^2=1\right]$$

$$= l^2\left(y^2+z^2\right)+m^2\left(x^2+z^2\right)+n^2\left(x^2+y^2\right)-2mnyz-2\ln zx-2lmxy$$

$\therefore$ Moment of inertia of the body about OA,

$$=\sum m'PM^2 = l^2\sum m'\left(y^2+z^2\right)+m^2\sum m'\left(x^2+z^2\right)+n^2\sum m'\left(x^2+y^2\right)-2mn\sum m'yz$$

$$-2\ln\sum m'zx-2lm\sum m'xy$$

$I = Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 2Flm$

Moment of Inertia of Heterogeneous Bodies:

In the case of a heterogeneous body whose boundary is a surface of uniform density, the method of differentiation can be successfully used in finding the moment of inertia of the body, the method is as follows:

(i) Suppose the M.I. of a homogeneous solid body of density ρ is known

(ii) Let this M.I. be expressed as a function of single parameter α (say) i.e.

M.I. = $\rho\phi(\alpha)$.

Then the M.I. of a shell which is considered to be made of a layer of a uniform density

$\rho = \rho\,\phi'(\alpha)\,d\alpha.$ …(1)

In case the density is not uniform and the variable density is given to be σ then we have,

M.I.= $\int \sigma\phi'(\alpha)d\alpha$ …(2)

D’alemberts principle

Motion of a particle. The motion of a single particle under the action of given forces is: determined by the Newton’s second law of motion, which sates that the rate of change of momenutm in any dirction is proportionate to the applied force in the direction.

From this law it is deduced that P = mf where f is the acceleration of particle m in the direction of the force P.
Here mf is called the *effective force* and P the *applied force*.
If (x, y, z) be the co-ordinates of a moing particle of mass m at any time t referred to three rectangular axes fixed in spce anf X, Y, Z, be the components of the forces acting on the particle in directions parallel ot the axes of x, y, z respectively,

Exam Point: the motion is found by solving the following three simultaneous equations:

$$m\ddot{x} = X, m\ddot{y} = Y, m\ddot{z} = Z$$

Motion of a rigid body.
Explanation: If the rigid body is considered as the collection of material particles. we can write the equation of motion of all particles according to the above law but here the external forces include, over and above the applied forces, the mutual actions between the particles. As regards mutual actions between any two particles we assume that (1).

The mutual action between two particles is along the line which joins them (2). The action and reaction beetwen them are equal and opposite. In order to find the motion of a rigid body or bodies, D' Alembert gave a method by which all the necessary equations may be obtained of the body. In doing so only the following consequence of the laws of motion is kept in view:

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

Impressed and effective forces.
Impressed forces. The external forces acting on a rigid body are termed as impressed forces *e.g.* weight of the body.
If the body is tied to the string, then tension in the string is the impressed force on the body.

Effective forces. When a rigid body is in motion, each particle of it is acted upon by the external impressed forces and also by the molecular reactions of the other particles. If we assume that particle is separated from the rest of the body, and all these forces are removed, *there is some force which would* *make it to move in the same direction as before*. This force is termed as *effective force* on the particle, it is the resultant of the impressed and molecular forces on the particle.

D'Alemberts Principle. *The reversed effective forces acting on each particle of the body and the external forces of the system are in equilibrium.*

Let (x,y,z) be the co-ordinates of a particle of mass m, a rigid body of at any time t.
Let f be the resultant of its component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$, so that the effective force on m is mf.
Let F be the resultant of the impressed forces on m and R be the resultant of mutual actions, then mf is resultant of F and R.
In case mf is reversed, the mf (reversed), F and R are in equilibrium. So for all the other particles of the body. Thus the reversed-effective forces $\Sigma(mf)$ acting on each particle of the body, the external forces (ΣF) and the internal actions and reactions (ΣR) of the rigid body form a system of forces in equilibrium. But ΣR *i.e.* the internal actions and reactions of the body are itself in equilibrium i.e. $\Sigma R=0$ Hence the forces ΣF and Σmf (reversed are in equilibrium
i.e. $\mathbf{\Sigma - (\mathit{mf}) + \Sigma \mathit{F} = 0}$
Hence the reversed effective forces acting at each point of the system and the impressed (external) forces on the system are equilibriu.

Note. This principle reduces the dynamical proble to the statical one.

Vector Method:
Consider a rigid body in motion.
Let at any time t, r be the position vector of a particle of mass m
and F and R be the external and internal forces respetively acting on it.

Now by Newton's second law $m(d^2r/dt^2) = \mathbf{F} + \mathbf{R}$ or $\mathbf{F} + \mathbf{R} - m(d^2r/dt^2) = 0$
i.e. the three forces, namely $\mathbf{F}$, $\mathbf{R}$ and $-m(d^2r/dt^2)$ are in equlibrium.

Now applying the same argument to every particle of the rigid body,

the force $\mathbf{\Sigma F}$, $\mathbf{\Sigma R}$ and $\sum\left(-m\dfrac{d^2r}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles.

Since the internal forces acting on the rigid body form pairs of equal and opposite forces, thus their vector sum must be zero
i.e. $\mathbf{\Sigma R = 0}$
$\Rightarrow$ The forces $\mathbf{\Sigma F}$ and $\mathbf{-\Sigma m\,(d^2r/dt^2)}$ are in equilibrium. This proves the D′ Alembert's Principle.

Angular momentum of a system of particles:
If r be the position vector of a particle of mass m relative to a point O, then the vector sum

$H = \Sigma r \times mv = \Sigma mr \times v$; is called *angular momentum (or moment of momentum)* of the system *about O*.

General equation of motion:
To deduce the general equation of motion of rigid body form D′ Alembert's Principle,

Cartesian method. Let (*x*, *y*, *z*) be the coordinate of a particle of mass *m* at any time *r* referred to a set of rectangular axes fixed in space. Let X, Y, Z represent the components, parallel to the axes of the external force acting of it.

By D′ Alembert's Principle of the forces

$X - m\ddot{x}, Y - m\ddot{y}, Z - m\ddot{z}$

Together with similar forces acting on each particle of the body will be in equilibrium.

Hence *as in statics* the six conditions of equilibrium are

$$\sum(X - m\ddot{x}) = 0, \sum(Y - m\ddot{y}) = 0, \sum(Z - m\ddot{z}) = 0.$$

$$\sum[y(Z - m\ddot{z}) - z(Y - m\ddot{y})] = 0$$

$$\sum[z(X - m\ddot{x}) - x(Z - m\ddot{z})] = 0$$

and $\sum[x(Y - m\ddot{y}) - y(X - m\ddot{x})] = 0$

Where summations are to be taken over all the particle of the body.

These equations give

$$\sum m\ddot{x} = \sum X, \sum m\ddot{y} = \sum Y, \sum m\ddot{z} = \sum Z$$

$$\sum m(y\ddot{z} - z\ddot{y}) = \sum(yZ - zY)$$

$$\sum m(y\ddot{x} - x\ddot{z}) = \sum(zX - xZ)$$

and $\sum m(x\ddot{y} - y\ddot{x}) = \sum(xY - yX)$

These are the **six equations of motion of any rigid body: Exam Point**

The first three equations can be written as

$$\frac{d}{dt}\sum m\dot{x} = \sum X, \frac{d}{dt}\sum m\dot{y}. = \sum Y, \frac{d}{dt}\sum m\dot{z}. = \sum Z$$

and the other three equations are written as

$$\frac{d}{dt}\sum m(y\dot{z} - z\dot{y}) = \sum(yZ - zY)$$

$$\frac{d}{dt}\sum m(y\dot{x} - x\dot{z}) = \sum(zX - xZ)$$

$$\frac{d}{dt}\sum m(x\dot{y} - y\dot{x}) = \sum(xY - yX)$$

Vector Method:

At time *t* let *r* be the position vector of a particle mass *m* and *F* be the external force acting on it, then by D′Alembert's Principle

$$\Sigma\left(-m\frac{d^2r}{dt^2}\right) + \Sigma F = 0 \text{ or } \Sigma m\frac{d^2r}{dt^2} = \Sigma F \qquad \ldots(1)$$

Taking cross product by r, we get

$$\Sigma \mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{r} \times \mathbf{F} \qquad \ldots(2)$$

Equations (1) and (2) are in general, vector equations of motion of a rigid body.

Again $r = x\,i + y\,j + z\,k$ …(3) and $F = Xi + Yj + Zk$ …(4)

where *X*, *Y*, *Z* are the components of *F*.

From (3) $\left(d^2r/dt^2\right)=\left(d^2x/dt^2\right)i+\left(d^2y/dt^2\right)j+\left(d^2v/dt^2\right)k$ ...(5)

Putting for r, F and $\left(d^2\mathbf{r}/dt^2\right)$ from (3), (4) and (5) respectively in (1) and (2), we get

$\Sigma m\left[\left(d^2x/dt^2\right)i+\left(d^2y/dt^2\right)j+\left(d^2z/dt^2\right)k\right]=\Sigma(Xi+Yj+Zk)$

and $\Sigma m(xi+yj+zk)\times\left[\left(d^2x/dr^2\right)i+\left(d^2y/dt^2\right)j+\left(d^2z/dr^2\right)k\right]$

$=\Sigma[(xi+yj+zk)\times(Xi+Yj+Zk)]$

Equating the coefficients of ***i, j, k***, we get the six conditions of equilibrium as obtained earlier.

Linear Momentum:

The linear momentum in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.

Let $(\bar{x},\bar{y},\bar{z})$ be the co-ordinates of the C.G. of the system and M the whole mass, then

$M\,\bar{x}-\Sigma mx, M\,\bar{y}=\Sigma my \text{ and } M\,\bar{z}=\Sigma mz$

Differentiating these relations, we get

$M\,\bar{x}=\Sigma mx$ etc. Hence the result.

Motion of the center of inertia:

To prove that the centre of ineria (C.G.) of a body moves as if the whole mass of the body were collected at it, and as if all the exteral forces were acting at it in directions parallel to those in which they act.

Let $(\bar{x},\bar{y},\bar{z})$ be the co-ordinates of the C.G. of the body of mass M then

$M\,\bar{x}=\Sigma mx$, so that $M\,\ddot{\bar{x}}=\Sigma m\ddot{x}.$

But from the general equation of motion, we have $\Sigma mx=\Sigma X$

Therefore,

$M\,\ddot{\bar{x}}=\Sigma X$ ...(1)

Similarly we have, $M\,\ddot{\bar{y}}=\Sigma Y$ .....(2) and $M\,\ddot{\bar{z}}=\Sigma Z$ ...(3)

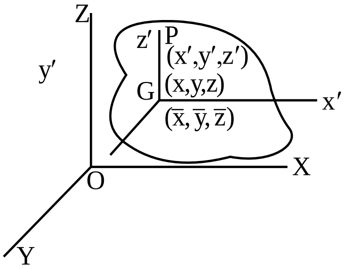
The equation (1) is the equation of motion of a particle of mass M (placed at the centre of inertia) acted on by a force ΣX parallel to the original directions of the forces on different particles.

Similarly the equations (2) and (3) can be interpreted.

Motion relative to centre of interia:

The motion of a body about its center of inertia is the same as it would be if the centre of inertia were fired and the same forces acted on the body.

Let $(\bar{x},\bar{y},z)$ be the co-ordinates of the centre of gravity G of the body with reference to the rectangular axes through a fixed point, say O.



Let $\left(x^{'},y^{'},z^{'}\right)$ be the coordinates of a particle of mass m with G (centre of inertia) as original axes parallel to the original axes and (x, y, z) be its coordinates with reference to original axes.

Then $x=\bar{x}+x^{'}, y=\bar{y}+y^{'}$ and $z=\bar{z}+z^{'}$

Now consider the fourth equation of the general equation of motion of rigid body,

$\Sigma m(y\ddot{z}-z\ddot{y})=\mathbf{\Sigma}(yZ-zY).$ ...(1)

If r is position vector of any particle of mass m of the system relative to a point O, the original of vectors then the point with position vector $\bar{r}=(\sum mr/\sum m)$ is defined as the centroid of the system.

Again, $(y\ddot{z}-z\ddot{y})=(\bar{y}+y')\left(\ddot{\bar{z}}+\ddot{z}'\right)-(\bar{z}+z')\left(\ddot{\bar{y}}+\ddot{y}'\right)$

$\left(\because \ddot{x}=\ddot{\bar{x}}+\ddot{x}'etc.\right)$

Therefore, from (1), we get

$\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma m\bar{y}\,\ddot{\bar{z}}+\Sigma m\bar{y}\ddot{z}^{'}+\Sigma my'\ddot{\bar{z}}+\Sigma my'\cdot\ddot{z}' \quad -\Sigma m\bar{z}\,\ddot{\bar{y}}-\Sigma m\bar{z}\,\ddot{y}'-\Sigma mz'\ddot{\bar{y}}-\Sigma mz'\ddot{y}'$ ...(2)

As G (the centre of inertia) is the origin of coordinates w. r. t. the new axis.

$\therefore \Sigma mx'=\Sigma my'=\Sigma mz'=0 \quad \left(\because \dfrac{\Sigma mx'}{\Sigma m}=0\,\text{etc.}\right)$

Therefore $\Sigma m\ddot{x}'=0=\Sigma m\ddot{y}'=\Sigma m\ddot{z}'$, also $\Sigma m=M=$ total mass of the body. Again $\bar{x},\bar{y},\bar{z}$ and their differential coefficients are common to all particles of the body, so we can take them outside the sigma sign.

Hence equation (2)

$\Rightarrow \Sigma m(y\ddot{z}-z\ddot{y})=M\,\bar{y}\,\ddot{\bar{z}}-M\,\bar{z}\,\ddot{\bar{y}}+\Sigma m\left(y'\ddot{z}'-\ddot{z}'y'\right)$

$\therefore$ Equation (1) becomes

$M\,\bar{y}\,\ddot{\bar{z}}-M\,\bar{z}\,\ddot{\bar{y}}+\Sigma m\left(y'\ddot{z}'-z'\ddot{y}'\right)=\Sigma\{\left(\bar{y}+y'\right)Z-\left(\bar{z}+z'\right)Y\}=\Sigma\,\bar{y}\,Z+\Sigma y'Z-\Sigma\,\bar{z}Y-\Sigma z'Y.$

we know that $M\,\ddot{\bar{z}}=\Sigma Z, M\,\ddot{\bar{y}}=\Sigma Y.$

Hence $\Sigma m(y'\ddot{z}'-z'\ddot{y}')=\Sigma(y'Z-z''Y).$

Similarly, we get other two equations.

But these equations are the same as would have been obtained had we regarded the C.G. to be a fixed point and same forces acted on the body.

Note. 1. The two important properties discussed above, are called the principle of conservation of motion of translation and rotation and together called the principle of independence of translation and rotation.

Note. 2. The motion of the C.G. is the same as if the whole mass collected at the point and is therefore independent of rotation.

Note 3. The motion round the C.G. is the same as if that point were fixed and is therefore independent of the motion of that point.

Impulsive Forces:

When the forces acting on a body are very large and act for a very short time, then their effects are measured by impulses.

Let a particle of mass '*m*' be acted upon by a force F always in the same direction, the equation of motion is m $(dv/dt) = F$. … (1)

where v is the velocity of the particle at time t.

If t be the time during which the force F acts and v_1, v_2 be the velocities before and after the action of the force, then on integrating (1), we have

$$m(v_2 - v_1) = \int_0^{\tau} F\,dt \quad \ldots\ldots.(2)$$

Now if F increases indefinitely while τ decreases indefinitely, then the integral on the right hand side of (2) may have a definite finite limit.

Let this finite limit be I then equation (2) may be written as

$$m(v_2 - v_1) = I \quad \ldots\ldots..(3)$$

The velocity during the time τ has increased or decreased from v_1 to v_2. Supposing that the velocity have remained finite, let v be the greatest velocity during the interval. Then the space described is less than $v\tau$. Since $v\tau \to 0$ as $\tau \to 0$, hence we conclude that the particle has not moved during the action of the force F. It could not have time to move, but its velocity has been changed from v_1 to v_2.

Thus in the case of finite forces which act on a body for indefinitely short time, the change of place is zero and the change of velocity is the measure of these forces. A force so measured is called an impulse. We can define impulse as the limit of a force which is indefinitely greater but acts only for an indefinitely short time e.g. the below of a hammer is a force of this kind. In fact an impulsive force is measured by the whole momentum generated by the impulse.

Note- When impulsive force acts, the finite forces acting on the body may be neglected in calculating the effect.

Let F be the impulsive force and f a finite force acting simultaneously on the body.

Then, $m(v_1 - v_2) = \int_0^{\tau} F\,dt + \int_0^{\tau} f\,dt = P + f\tau.$

But since $f\tau \to 0$ as $\tau \to 0$, f may be neglected in forming the equations.

Note- Application of D' Alembert's principle to impulsive forces, general equation of motion.

Scalar Method.

let u, v, w be the velocities parallel to co-ordinate axes before the action of impulsive forces and u', v', w' be the velocities after the action of these forces.

Let X', Y', Z' be the resolved parts of the impulsive forces parallel to the axes.

Then, from $\Sigma m\ddot{x} = \Sigma X$,

on integrating with respect to t, we get

$$\left[\Sigma m\frac{dx}{dt}\right]_0^\tau = \int_0^\tau \Sigma X\,dt = \Sigma\int_0^\tau Xdt = \Sigma X'$$

$$\Sigma m\left(u' - u\right) = \Sigma X'.$$

Similarly, $\Sigma m\left(v' - v\right) = Y'$ and $\Sigma m\left(w' - w\right) = \Sigma Z'$

Observation-Thus the change in the momentum parallel to any of the axes of the whole mass M. supposed collected at the centre of inertia and moving with it is equal to the impulse of the external forces parallel to the corresponding axis. Again we have the moment equation

$$\Sigma m(y\ddot{z} - z\ddot{y}) = ?m(yZ - zY)$$

Integrating this we have $\left[\Sigma m\left(y\dot{z} - z\dot{y}\right)\right]_0^\tau = \Sigma\left[y\int_0^\tau Zdt - z\int_0^\tau Ydt\right]$

Since the interval τ is so short that the body has not moved during this period, we may take x,y,z as constants, thus the above equation becomes

$$\Sigma m\left\{y\left(w' - w\right) - z\left(v' - v\right)\right\} = \Sigma\left(yZ' - zY'\right)$$

Similarly, we have other two equations

$$\Sigma m\left\{x\left(v' - v\right) - y\left(u' - u\right)\right\} = \Sigma\left(xY' - yX'\right)$$

and $\Sigma m\left\{z\left(u' - u\right) - x\left(w' - w\right)\right\} = \Sigma\left(zX' - xZ'\right)$

Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

PREVIOUS YEARS QUESTIONS

CHAPTER 1. MOMENT OF INERTIA

Q1. Find the moment of inertia of a right circular solid cone about one of its slant sides (generator) in terms of its mass M, height h and the radius of base as a.

[6C UPSC CSE 2022]

Q1. Prove that the moment of inertia of a triangular lamina ABC about any axis through A in its plane is $\frac{M}{6}\left(\beta^2 + \beta\gamma + \gamma^2\right)$

where M is the mass of the lamina and β, γ are respectively the length of perpendiculars from B and C on the axis. **[5e UPSC CSE 2020]**

Q2. Show that the moment of inertia of an elliptic area of mass M and semi-axis a and b about a semi-diameter of length r is $\frac{1}{4} M \frac{a^2 b^2}{r^2}$. Further, prove that the moment of inertia about a tangent is $\frac{5M}{4} p^2$, where p is the perpendicular distance from the centre of the ellipse to the tangent.

[5e UPSC CSE 2017]

Q3. A uniform rectangular parallelepiped of mass M has edges of lengths $2a, 2b, 2c$. Find the moment of inertia of this rectangular parallelepiped about the line through its centre parallel to the edge of length $2a$. **[5c 2017 IFoS]**

Q4. Calculate the moment of inertia of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) relative to the x-axis

(ii) relative to the y-axis and

(iii) relative to the origin. **[5e 2016 IFoS]**

Q5. Find the moment of inertia of a right solid cone of mass M, height h and radius of whose base is a, about its axis. **[8a 2016 IFoS]**

Q6. Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with mass m about the OZ-axis. **[5e UPSC CSE 2015]**

Q7. Find the moment of inertia of a uniform mass M of a square shape with each side a about its one of the diagonals. **[7b 2015 IFoS]**

Q8. Show that the moment of inertia of a uniform rectangular mass M and sides $2a$ and $2b$ about a diagonal is $\frac{2Ma^2b^2}{3\left(a^2+b^2\right)}$. **[6b 2014 IFoS]**

Q9. Four solid spheres A, B, C and D, each of mass m and radius a, are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about a diagonal of the square. **[5e UPSC CSE 2013]**

Q10. A pendulum consists of a rod of length $2a$ and mass m; to one end of which a spherical bob of radius $a/3$ and mass 15 m is attached. Find the moment of inertia of the pendulum:

(i) about an axis through the other end of the rod and at right angles to the rod.

(ii) about a parallel axis through the centre of mass of the pendulum.

[Given: The centre of mass of the pendulum is $a/12$ above the centre of the sphere.]

[8a UPSC CSE 2012]

Q11. Let a be the radius of the base of a right circular cone of height h and mass M. Find the moment of inertia of that right circular cone about a line through the vertex perpendicular to the axis.

[5e UPSC CSE 2011]

Q12. From a uniform sphere of radius a, a spherical sector of vertical angle 2α is removed. Find the moment of inertia of the remainder mass M about the axis of symmetry. **[8a 2011 IFoS]**

Q13. A uniform lamina is bounded by a parabolic arc of latus rectum $4a$ and a double ordinate at a distance b from the vertex.

If $b = \frac{a}{3}\left(7 + 4\sqrt{7}\right)$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there. **[5e UPSC CSE 2010]**

Q14. Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same. **[5d 2010 IFoS]**

Ch. -2: D’alemberts Priniple & Motion about a fixed axis

Motion of a particle. The motion of a single particle under the action of given forces is: determined by the Newton’s second law of motion, which sates that the rate of change of momenutm in any dirction is proportionate to the applied force in the direction.

From this law it is deduced that P = *mf* where *f* is the acceleration of particle *m* in the direction of the force *P*.
Here *mf* is called the *effective force* and *P* the *applied force.*
If (*x, y, z*) be the co-ordinates of a moing particle of mass *m* at any time *t* referred to three rectangular axes fixed in spce anf *X, Y, Z,* be the components of the forces acting on the particle in directions parallel ot the axes of *x, y, z* respectively,

Exam Point: the motion is found by solving the following three simultaneous equations:

$$m\ddot{x} = X, m\ddot{y} = Y, m\ddot{z} = Z$$

Motion of a rigid body.
Explanation: If the rigid body is considered as the collection of material particles. we can write the equation of motion of all particles according to the above law but here the external forces include, over and above the applied forces, the mutual actions between the particles. As regards mutual actions between any two particles we assume that (1).

The mutual action between two particles is along the line which joins them (2). The action and reaction beetwen them are equal and opposite. In order to find the motion of a rigid body or bodies, D' Alembert gave a method by which all the necessary equations may be obtained of the body. In doing so only the following consequence of the laws of motion is kept in view:

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

Impressed and effective forces.
Impressed forces. The external forces acting on a rigid body are termed as impressed forces *e.g.* weight of the body.
If the body is tied to the string, then tension in the string is the impressed force on the body.

Effective forces. When a rigid body is in motion, each particle of it is acted upon by the external impressed forces and also by the molecular reactions of the other particles. If we assume that particle is separated from the rest of the body, and all these forces are removed, *there is some force which would make it to move in the same direction as before*. This force is termed as *effective force* on the particle, it is the resultant of the impressed and molecular forces on the particle.

D'Alemberts Principle. *The reversed effective forces acting on each particle of the body and the external forces of the system are in equilibrium.*

Let (x,y,z) be the co-ordinates of a particle of mass m, a rigid body of at any time t.
Let f be the resultant of its component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$, so that the effective force on m is mf.
Let F be the resultant of the impressed forces on m and R be the resultant of mutual actions, then mf is resultant of F and R.
In case mf is reversed, the mf (reversed), F and R are in equilibrium. So for all the other particles of the body. Thus the reversed-effective forces $\Sigma(mf)$ acting on each particle of the body, the external forces (ΣF) and the internal actions and reactions (ΣR) of the rigid body form a system of forces in equilibrium. But ΣR *i.e.* the internal actions and reactions of the body are itself in equilibrium i.e. $\Sigma R=0$ Hence the forces ΣF and Σmf (reversed are in equilibrium
i.e. $\boldsymbol{\Sigma - (mf) + \Sigma F = 0}$
Hence the reversed effective forces acting at each point of the system and the impressed (external) forces on the system are equilibriu.

Note. This principle reduces the dynamical proble to the statical one.

Vector Method:
Consider a rigid body in motion.
Let at any time t, r be the position vector of a particle of mass m
and F and R be the external and internal forces respetively acting on it.

Now by Newton's second law $m(d^2r/dt^2) = \boldsymbol{F} + \boldsymbol{R}$ or $\boldsymbol{F} + \boldsymbol{R} - m(d^2r/dt^2) = 0$
i.e. the three forces, namely $\boldsymbol{F}, \boldsymbol{R}$ and $-m(d^2r/dt^2)$ are in equlibrium.

Now applying the same argument to every particle of the rigid body,

the force $\boldsymbol{\Sigma F}, \boldsymbol{\Sigma R}$ and $\sum\left(-m\frac{d^2r}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles.

Since the internal forces acting on the rigid body form pairs of equal and opposite forces, thus their vector sum must be zero
i.e. $\boldsymbol{\Sigma R = 0}$
$\Rightarrow$ The forces $\boldsymbol{\Sigma F}$ and $\boldsymbol{-\Sigma m\,(d^2r/dt^2)}$ are in equilibrium. This proves the D′ Alembert's Principle.

Angular momentum of a system of particles:
If r be the position vector of a particle of mass m relative to a point O, then the vector sum

$H = \Sigma r \times mv = \Sigma mr \times v$; is called *angular momentum (or moment of momentum)* of the system *about O*.

General equation of motion:
To deduce the general equation of motion of rigid body form D′Alembert's Principle,

Cartesian method. Let (x, y, z) be the coordinate of a particle of mass m at any time r referred to a set of rectangular axes fixed in space. Let X, Y, Z represent the components, parallel to the axes of the external force acting of it.

By D′ Alembert's Principle of the forces

$X - m\ddot{x}, Y - m\ddot{y}, Z - m\ddot{z}$

Together with similar forces acting on each particle of the body will be in equilibrium.
Hence *as in statics* the six conditions of equilibrium are

$$\sum(X - m\ddot{x}) = 0, \sum(Y - m\ddot{y}) = 0, \sum(Z - m\ddot{z}) = 0.$$

$$\sum[y(Z - m\ddot{z}) - z(Y - m\ddot{y})] = 0$$

$$\sum[z(X - m\ddot{x}) - x(Z - m\ddot{z})] = 0$$

and $\sum[x(Y - m\ddot{y}) - y(X - m\ddot{x})] = 0$

Where summations are to be taken over all the particle of the body.
These equations give

$$\sum m\ddot{x} = \sum X, \sum m\ddot{y} = \sum Y, \sum m\ddot{z} = \sum Z$$

$$\sum m(y\ddot{z} - z\ddot{y}) = \sum(yZ - zY)$$

$$\sum m(y\ddot{x} - x\ddot{z}) = \sum(zX - xZ)$$

and $\sum m(x\ddot{y} - y\ddot{x}) = \sum(xY - yX)$

These are the **six equations of motion of any rigid body: Exam Point**

The first three equations can be written as

$$\frac{d}{dt}\sum m\dot{x} = \sum X, \frac{d}{dt}\sum m\dot{y}. = \sum Y, \frac{d}{dt}\sum m\dot{z}. = \sum Z$$

and the other three equations are written as

$$\frac{d}{dt}\sum m(y\dot{z} - z\dot{y}) = \sum(yZ - zY)$$

$$\frac{d}{dt}\sum m(y\dot{x} - x\dot{z}) = \sum(zX - xZ)$$

$$\frac{d}{dt}\sum m(x\dot{y} - y\dot{x}) = \sum(xY - yX)$$

Vector Method:

At time t let r be the position vector of a particle mass m and F be the external force acting on it, then by D′Alembert's Principle

$$\Sigma\left(-m\frac{d^2r}{dt^2}\right) + \Sigma F = 0 \text{ or } \Sigma m\frac{d^2r}{dt^2} = \Sigma F \qquad \ldots(1)$$

Taking cross product by r, we get

$$\Sigma \mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{r} \times \mathbf{F} \qquad \ldots(2)$$

Equations (1) and (2) are in general, vector equations of motion of a rigid body.

Again $r = x\,i + y\,j + z\,k$ …(3) and $F = Xi + Yj + Zk$ …(4)

where X, Y, Z are the components of F.

From (3) $\left(d^2r/dt^2\right) = \left(d^2x/dt^2\right)i + \left(d^2y/dt^2\right)j + \left(d^2v/dt^2\right)k$ …(5)

Putting for r, F and $\left(d^2\mathbf{r}/dt^2\right)$ from (3), (4) and (5) respectively in (1) and (2), we get

$$\Sigma m\left[\left(d^2x/dt^2\right)i + \left(d^2y/dt^2\right)j + \left(d^2z/dt^2\right)k\right] = \Sigma(Xi + Yj + Zk)$$

and $\Sigma m(xi+yj+zk)\times\left[\left(d^2x/dr^2\right)i+\left(d^2y/dt^2\right)j+\left(d^2z/dr^2\right)k\right]$

$=\Sigma[(xi+yj+zk)\times(Xi+Yj+Zk)]$

Equating the coefficients of ***i, j, k***, we get the six conditions of equilibrium as obtained earlier.

Linear Momentum:

The linear momentum in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.

Let $(\bar{x},\bar{y},\bar{z})$ be the co-ordinates of the C.G. of the system and M the whole mass, then

$M\,\bar{x}-\Sigma mx, M\,\bar{y}=\Sigma my \ and \ M\,\bar{z}=\Sigma mz$

Differentiating these relations, we get

$M\,\bar{x}=\Sigma mx$ etc. Hence the result.

Motion of the center of inertia:

To prove that the centre of ineria (C.G.) of a body moves as if the whole mass of the body were collected at it, and as if all the exteral forces were acting at it in directions parallel to those in which they act.

Let $(\bar{x},\bar{y},\bar{z})$ be the co-ordinates of the C.G. of the body of mass M then

$M\,\bar{x}=\Sigma mx$, so that $M\,\ddot{\bar{x}}=\Sigma m\ddot{x}.$

But from the general equation of motion, we have $\Sigma mx=\Sigma X$

Therefore,

$M\,\ddot{\bar{x}}=\Sigma X\ \ldots(1)$

Similarly we have, $M\,\ddot{\bar{y}}=\Sigma Y\ \ldots..(2)$ and $M\,\ddot{\bar{z}}=\Sigma Z\ \ldots(3)$

The equation (1) is the equation of motion of a particle of mass M (placed at the centre of inertia) acted on by a force ΣX parallel to the original directions of the forces on different particles.

Similarly the equations (2) and (3) can be interpreted.

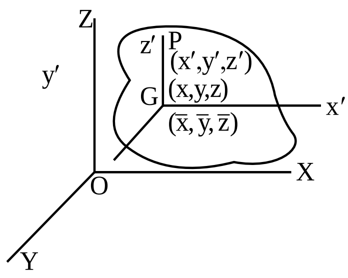
Motion relative to centre of interia:

The motion of a body about its center of inertia is the same as it would be if the centre of inertia were fired and the same forces acted on the body.

Let $(\bar{x},\bar{y},z)$ be the co-ordinates of the centre of gravity G of the body with reference to the rectangular axes through a fixed point, say O.

Let $\left(x',y',z'\right)$ be the coordinates of a particle of mass m with G (centre of inertia) as original axes parallel to the original axes and (x, y, z) be its coordinates with reference to original axes.

Then $x=\bar{x}+x', y=\bar{y}+y'$ and $z=\bar{z}+z'$



Now consider the fourth equation of the general equation of motion of rigid body,

$\Sigma m(y\ddot{z}-z\ddot{y})=\mathbf{\Sigma}(yZ-zY).\qquad\ldots(1)$

If r is position vector of any particle of mass m of the system relative to a point O, the original of vectors then the point with position vector $\bar{r} = (\sum mr / \sum m)$ is defined as the centroid of the system.

Again, $(y\ddot{z} - z\ddot{y}) = (\bar{y} + y')(\ddot{\bar{z}} + \ddot{z}') - (\bar{z} + z')(\ddot{\bar{y}} + \ddot{y}')$

$(\because \ddot{x} = \ddot{\bar{x}} + \ddot{x}' etc.)$

Therefore, from (1), we get

$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma m\bar{y}\ddot{\bar{z}} + \Sigma m\bar{y}\ddot{z}' + \Sigma my'\ddot{\bar{z}} + \Sigma my' \cdot \ddot{z}' \;\; - \Sigma m\bar{z}\ddot{\bar{y}} - \Sigma m\bar{z}\ddot{y}' - \Sigma mz'\ddot{\bar{y}} - \Sigma mz'\ddot{y}' \ldots(2)$

As G (the centre of inertia) is the origin of coordinates w. r. t. the new axis.

$\therefore \Sigma mx' = \Sigma my' = \Sigma mz' = 0 \quad \left(\therefore \frac{\Sigma mx'}{\Sigma m} = 0 \text{ etc.}\right)$

Therefore $\Sigma m\ddot{x}' = 0 = \Sigma m\ddot{y}' = \Sigma m\ddot{z}'$, also $\Sigma m = M =$ total mass of the body. Again $\bar{x}, \bar{y}, \bar{z}$ and their differential coefficients are common to all particles of the body, so we can take them outside the sigma sign.

Hence equation (2)

$\Rightarrow \Sigma m(y\ddot{z} - z\ddot{y}) = M\,\bar{y}\,\ddot{\bar{z}} - M\,\bar{z}\,\ddot{\bar{y}} + \Sigma m(y'\ddot{z}' - \ddot{z}'y')$

$\therefore$ Equation (1) becomes

$M\,\bar{y}\,\ddot{\bar{z}} - M\,\bar{z}\,\ddot{\bar{y}} + \Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma\{(\bar{y} + y')Z - (\bar{z} + z')Y\} = \Sigma\bar{y}Z + \Sigma y'Z - \Sigma\bar{z}Y - \Sigma z'Y.$

we know that $M\,\ddot{\bar{z}} = \Sigma Z, M\,\ddot{\bar{y}} = \Sigma Y.$

Hence $\Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma(y'Z - z''Y).$

Similarly, we get other two equations.

But these equations are the same as would have been obtained had we regarded the C.G. to be a fixed point and same forces acted on the body.

Note. 1. The two important properties discussed above, are called the principle of conservation of motion of translation and rotation and together called the principle of independence of translation and rotation.

Note. 2. The motion of the C.G. is the same as if the whole mass collected at the point and is therefore independent of rotation.

Note 3. The motion round the C.G. is the same as if that point were fixed and is therefore independent of the motion of that point.

Impulsive Forces:

When the forces acting on a body are very large and act for a very short time,

then their effects are measured by impulses.

Let a particle of mass 'm' be acted upon by a force F always in the same direction, the equation of motion is m $(dv/dt) = F$. … (1)

where v is the velocity of the particle at time t.

If t be the time during which the force F acts and v_1, v_2 be the velocities before and after the action of the force, then on integrating (1), we have

$$m(v_2 - v_1) = \int_0^\tau F\,dt \;\ldots\ldots(2)$$

Now if F increases indefinitely while τ decreases indefinitely, then the integral on the right hand side of (2) may have a definite finite limit.

Let this finite limit be I then equation (2) may be written as

$$m\,(v_2 - v_1) = I \;\ldots\ldots(3)$$

The velocity during the time τ has increased or decreased from v_1 to v_2. Supposing that the velocity have remained finite, let v be the greatest velocity during the interval. Then the space described is less than $v\tau$. Since $v\tau \to 0$ as $\tau \to 0$, hence we conclude that the particle has not moved during the action of the force F. It could not have time to move, but its velocity has been changed from v_1 to v_2.

Thus in the case of finite forces which act on a body for indefinitely short time, the change of place is zero and the change of velocity is the measure of these forces. A force so measured is called an impulse. We can define impulse as the limit of a force which is indefinitely greater but acts only for an indefinitely short time e.g. the below of a hammer is a force of this kind. In fact an impulsive force is measured by the whole momentum generated by the impulse.

Note- When impulsive force acts, the finite forces acting on the body may be neglected in calculating the effect.

Let F be the impulsive force and f a finite force acting simultaneously on the body.

Then, $m(v_1 - v_2) = \int_0^\tau F\,dt + \int_0^\tau f\,dt = P + f\tau.$

But since $f\tau \to 0$ as $\tau \to 0$, f may be neglected in forming the equations.

Note- Application of D' Alembert's principle to impulsive forces, general equation of motion.

Scalar Method.

let u, v, w be the velocities parallel to co-ordinate axes before the action of impulsive forces and u', v', w' be the velocities after the action of these forces.

Let X', Y', Z' be the resolved parts of the impulsive forces parallel to the axes.

Then, from $\Sigma m \ddot{x} = \Sigma X$,

on integrating with respect to t, we get

$$\left[\Sigma m \frac{dx}{dt}\right]_0^\tau = \int_0^\tau \Sigma X\,dt = \Sigma \int_0^\tau X dt = \Sigma X'$$

$$\Sigma m\left(u^{'}-u\right)=\Sigma X^{'}.$$

Similarly, $\Sigma m\left(v^{'}-v\right)=Y^{'}$ and $\Sigma m\left(w^{'}-w\right)=\Sigma Z^{'}$

Observation-Thus the change in the momentum parallel to any of the axes of the whole mass M. supposed collected at the centre of inertia and moving with it is equal to the impulse of the external forces parallel to the corresponding axis. Again we have the moment equation

$$\Sigma m(y\ddot{z}-z\ddot{y})=?m(yZ-zY)$$

Integrating this we have $\left[\Sigma m\left(y\dot{z}-z\dot{y}\right)\right]_0^{\tau}=\Sigma\left[y\int_0^{\tau}Zdt-z\int_0^{\tau}Ydt\right]$

Since the interval τ is so short that the body has not moved during this period, we may take x,y,z as constants, thus the above equation becomes

$$\Sigma m\left\{y\left(w^{'}-w\right)-z\left(v^{'}-v\right)\right\}=\Sigma\left(yZ^{'}-zY^{'}\right)$$

Similarly, we have other two equations

$$\Sigma m\left\{x\left(v^{'}-v\right)-y\left(u^{'}-u\right)\right\}=\Sigma\left(xY^{'}-yX^{'}\right)$$

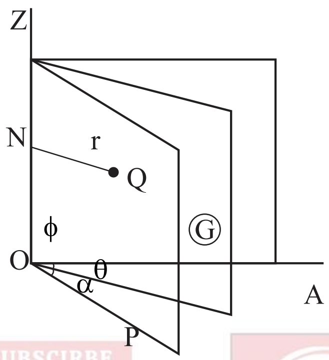
and $\Sigma m\left\{z\left(u^{'}-u\right)-x\left(w^{'}-w\right)\right\}=\Sigma\left(zX^{'}-xZ^{'}\right)$

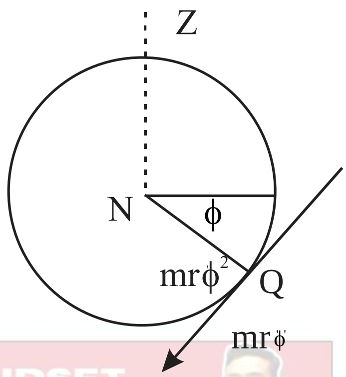
Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

Motion about a fixed axis

A rigid body is rotating about a fixed axis. To find the moment of the effective forces about the axis of rotation.

Let the axis of rotation be *OZ*, perpendicular to the plane of the paper. Take a plane *AOZ* through *OZ* and fixed in space, cutting the plane of the paper along *OA*. Let this plane be taken as the plane of reference. Let be the angle, which another plane *ZOG* through the axis fixed in the body makes with the plane *AOZ*.





Take a particle of mass m at *Q* and let the plane through *OZ* and *Q* cut the plane of the paper along *OP*. Let the angle between *ZOP* and *ZOG* be α. When body rotates about *OZ*; α remains constant. Let the angle between the plane ZOP and the plane ZOA be ϕ. Now

$\theta + \alpha = \phi \;\; \therefore \dot{\theta} = \dot{\theta}$ and $\ddot{\theta} = \ddot{\phi}$

The accelerations of the particle of mass *m* are

$r\dot{\phi}^2$ and $r\ddot{\phi}$ along *QN* and perpendicular to *QN* respectively.

Therefore effective forces on the particles are $mr\dot{\phi}^2$ and $mr\ddot{\phi}$ in the above said directions. Again

$r\dot{\phi}^2 = r\dot{\theta}^2$ and $r\ddot{\phi} = r\ddot{\theta}$

The moment of the force $mr\dot{\phi}^2$ about *OZ* is zero and moment of the force $mr\ddot{\phi}$ about *OZ* (& *NZ*) is $r.\, mr\ddot{\phi} = mr^2\ddot{\phi} = mr^2\ddot{\theta}$

Exam Point- Hence **the moment of the effective forces of the whole body about *OZ* is**

$\Sigma mr^2\ddot{\theta} = \ddot{\theta}\Sigma mr^2 = Mk^2\ddot{\theta}$, **where *k* is the radius of gyration of the body about *OZ*.**

Moment of momentum about the axis of rotation.

Velocity of the particle m is $r\dot{\phi}$ perpendicular to *QN*.

Therefore the moment of momentum of the particle about *OZ* is $mr^2\dot{\phi}$ or $mr^2\dot{\theta}$.

Hence the moment of momentum of the whole body about OZ

is $\Sigma mr^2 \dot{\phi} = \left(\Sigma mr^2\right)\dot{\theta} = \dot{\theta}\Sigma mr^2 = Mk^2\dot{\theta}$

Kinetic Energy: The kinetic energy of the particles of mass m is $\frac{1}{2}mr^2\dot{\phi}^2$

Hence K.E. of the whole body is

$$\Sigma\frac{1}{2}mr^2\dot{\phi}^2 = \Sigma\frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}\dot{\theta}^2\,\Sigma mr^2 = \frac{1}{2}Mk^2\dot{\theta}^2.$$

Equation of motion:

The impressed forces include besides the external forces, the reactions on the axis of rotation OZ. We take moment about OZ, so that this reaction could be avoided i.e. the moment of the effective forces about OZ will be equal to the moment of the external forces about OZ.

Thus $Mk^2\ddot{\theta} = L$,

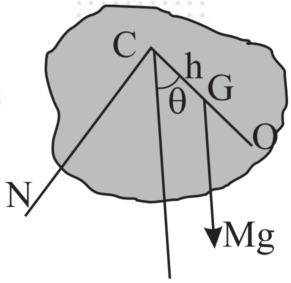
where ***L*** represents the moment of all external forces about OZ.

Above equation is called the equation of motion of the body.

In the case of impulsive forces if ω_1 and ω_2 be angular velocities of the body just before and just after the action of the impulses, ***L*** the moment of the impulses then we of the impulses then we have $Mk^2(\omega_2 - \omega_1) = L$.

The Compound Pendulum:

To determine the motion of a body acted on by the force of gravity only and moving about a fixed horizontal axis.



Let us take plane of the paper as the plane through the centre of gravity G of the body and perpendicular to the fixed axis.

Let the plane meet the axis in C.

Let θ be the angle between the vertical and CG *i.e.* θ is the angle between a plane fixed in space and a plane fixed in the body.

Let $CG = h$. The forces on the body are:

(i) its weight $M\,g$ acting downward through G.

(ii) the reaction at C of the fixed axis to eliminate this reaction.

We take moments about the fixed axis to eliminate this reaction.

The equation of motion is $Mk^2\ddot{\theta} = -M\ gh\ sin\theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\sin\theta = -\frac{gh}{k^2}\theta,\ (\theta\,\text{being small})$$

Equation (1) shows that motion is S.H.M. Hence the time of complete oscillation of compound pendulum is $2\pi\sqrt{\left(\frac{k^2}{gh}\right)}$.

Simple Equivalent Pendulum. We know that equation of motion of a particle of any mass suspended by a string of length l is $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\theta\left(\theta \text{ being small}\right)$

The time of complete oscillation is $2\pi\sqrt{\left(\frac{l}{g}\right)}$.

If $2\pi\sqrt{\left(\frac{l}{g}\right)} = 2\pi\sqrt{\left\{\frac{k^2}{(gh)}\right\}}$, then $l = \left(\frac{k^2}{h}\right)$.

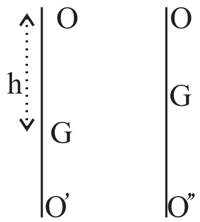
This length $\left(\frac{k^2}{h}\right)$ *in the case of a compound pendulum is called the length of the simple equivalent pendulum.*

Centre of Suspension:

Through C, if a line be drawn perpendicular to the axis of rotation cutting it at C, then $\boldsymbol{C}$ is called the Centre of suspension.

Centre of Oscillation. If $\boldsymbol{O}$ is the point on $\boldsymbol{CG}$ produced such that $CO = l = \frac{k^2}{h}$ (the length of the simple equivalent pendulum) then the point $\boldsymbol{O}$ is called the centre of oscillation.

Showing that the centres of suspension and oscillation are convertible



Let us take $\boldsymbol{O}$ and O' as the centre of suspension and oscillation 'respectively $\therefore OO' = \frac{k^2}{h}$ Where $OG = h$, and K is radius of gyration of the body about the axis through O. Now if K is the radius of gyration of the body about an axis through G parallel to the axis of rotation, then $Mk^2 = MK^2 + M.OG^2$

$$\Rightarrow Mk^2 = MK^2 + Mh^2 \Rightarrow k^2 = K^2 + h^2$$

$$\therefore OO' = \frac{K^2 + h^2}{h} = \frac{K^2 + OG^2}{OG}$$

$$\Rightarrow OO'.OG = K^2 + OG^2 \Rightarrow K^2 = OG\left(OO' - OG\right) = OG.O'G. \qquad \ldots(1)$$

Let O'' be the centre of oscillation when the body rotates about a parallel axis through O'. We can show as above that $K^2 = O'G.O''G$

From (1) and (2), we observe that O'' is simply the point O. Thus if the body were suspended from a parallel axis through O', O is the centre of oscillation. This proves the theorem.

Minimum time of oscillation of a compound pendulum.

If K is the radius of gyration of the body about an axis through G parallel to the axis of rotation, then $k^2 = K^2 + h^2$.

Therefore length of the simple equivalent pendulum is $l = \frac{k^2}{h} = \frac{K^2 + h^2}{h} = \frac{K^2}{h} + h.$

The time of oscillation of a compound pendulum will be least when the length of the simple equivalent pendulum is minimum. For that

$$\frac{dl}{dh} = \frac{d}{dh}\left(h + \frac{K^2}{h^2}\right) = 0 \Rightarrow 1 - \frac{K^2}{h^2} = 0 \Rightarrow h = K.$$

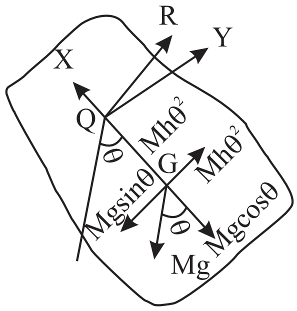
The length of simple equivalent pendulum in this case

$$l = \frac{K^2 + h^2}{h} = \frac{K^2 + K^2}{K} = 2K.$$

In case $h = 0$ or ∞ i.e. if the axis of suspension either passes through G or be at infinite, the corresponding simple equivalent pendulum is of infinite length, thus the time of oscillation is infinite.

Reactions of the axis of rotation.

A body moves about a fixed axis under the action of forces and both the body and the forces are symmetrical with respect to the plane through the C.G. perpendicular to the axis, find the reactions of the axis of rotation.



Let O be the point where the plane through G perpendicular to the axis of rotation meets this axis. By symmetry the actions on the axis reduce to a single force at O, the centre of suspension.

Let the components of this single force be X and Y along and perpendicular to GO respectively.

Now G describes a circle round O as centre, its acceleration along and perpendicular to GO are $h\dot{\theta}^2$ and $h\ddot{\theta}$.

Equations of motion of C.G. are

$$Mh\dot{\theta}^2 = X - Mg\cos\theta \text{ ……..(1)}$$

$$Mh\ddot{\theta}^2 = Y - Mg\sin\theta \text{ … (2)}$$

By taking moments about $O, Mk^2\ddot{\theta} = -Mgh\,sin\theta \ldots$ (3)

where $\boldsymbol{k}$ is the radius of gyration about the axis.

Y is obtained by eliminating $\ddot{\theta}$ from (2) and (3), By integrating (3) and determining the constant from the initial conditions, and then from we can find X.

Resultant reaction $R = \sqrt{\left(X^2 + Y^2\right)}$ and $\tan\phi = \left(\dfrac{X}{Y}\right)$ where $\boldsymbol{\phi}$ is the angle which the direction of R makes with GO.

Note: On resolving X and Y horizontally and vertically,

The horizontal reaction $= X\sin\theta - Y\cos\theta$ Vertical reaction $= X\cos\theta + Y\sin\theta$

Motion about a fixed axis: Impulsive forces.

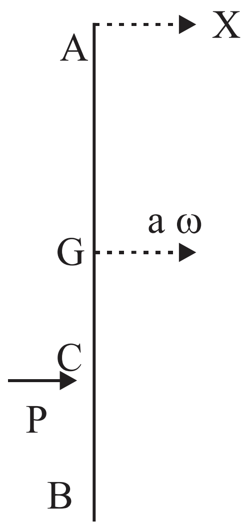
Consider a rigid body under the effect of impulsive forces. Let $\boldsymbol{\omega}$ and $\boldsymbol{\omega'}$ be the angular velocities about the axis just before and just after the action of impulsive forces. Now change in moment of momentum about the axis $= Mk^2\left(\omega' - \omega\right)$. Also let L the moment of external impulses about the axis of rotation, then we have $Mk^2\left(\omega' - \omega\right) = L$ (since change in moment of momentum of the body about the axis is equal to the moment of the impulsive forces about it).

Centre Of Percussion:

If a body, rotating about a given axis, is so struck that there is no impulsive pressure on the axis, then any point on the line of action of the force is called a centre of percussion. If the line of action of the blow is known, the axis about which the body begins to turn is called the axis of *spontaneous rotation.* Obviously this combines with the position of the fixed axis in the first case.

Centre of Percussion of a rod:

Consider a rod AB of length $2b$. Let it be suspended freely from one end A. Let a horizontal blow of impulse P be applied to it at the point C where $AC = x$.



If X is the impulsive action at A and $\boldsymbol{\omega}$ the angular velocity communicated to the rod, then the equations of motion are

$$Mk^2\omega = P_x \quad \left(moment\ eq^n\right) \qquad \ldots (1)$$

$$M(a\omega - 0) = P + X \qquad \ldots (2)$$

where $\boldsymbol{a\omega}$ is the velocity with which G moves.

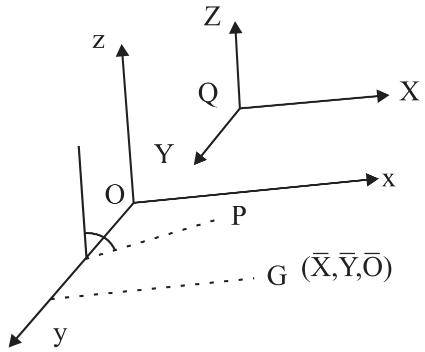
Now if the blow has been given through the centre of percussion then $X = 0$ and equation (2) becomes $Ma\omega = P$.

Substituting this value of $\boldsymbol{P}$ in (1), we get

$x = \dfrac{k^2}{a} =$ length of the equivalent simple pendulum.

General Case of Centre of Percussion:

Let us take the fixed axis as the axis of y. Also let centre of gravity G lie in the y-plane, so that coordinates of G are $(\bar{x}, \bar{y}, 0)$.



If Q is the point where the blow is applied then take a plane through Q and perp. to xy-plane as the xz-plane so that coordinates of Q may be $(\xi, 0, \zeta)$. Now consider any other point P of mass m of the body at a distance r from Oy at any angle θ with z-axis. The coordinates of P will be $x = r\sin\theta, y = const., z = r\cos\theta.$

If before the blow, angular velocity is $\boldsymbol{\omega}$ and the velocity component along the axes are $\boldsymbol{u, v, w}$ respectively, then we have

$$\dot{\boldsymbol{x}} = \boldsymbol{u} = \boldsymbol{r}\cos\boldsymbol{\theta}, \dot{\boldsymbol{\theta}} = \boldsymbol{z\omega}, \dot{\boldsymbol{y}} = \boldsymbol{v} = \boldsymbol{0}, \dot{\boldsymbol{z}} = \boldsymbol{w} = -\boldsymbol{r}\sin\boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = -\boldsymbol{x\omega}.$$

If after the blow, the angular velocity is $\boldsymbol{\omega}$ and velocity component along the axes becomes as u', v', w', then $u' = z\omega', v' = 0; w' = -x\omega'$.

If X, Y, Z are the components of the blow at the point Q, then equations of motion will be

$$X = \Sigma m(u' - u) = \Sigma mz(\omega' - \omega) = (\omega' - \omega)\Sigma mz$$

$$= (\omega' - \omega)\bar{z}\Sigma m = M(\omega' - \omega)\bar{z} = 0\left(since\, \bar{z} = 0\right) \quad \ldots (1)$$

$$Y = \Sigma m(v' - v) = 0\left(since\ v' = 0\ and\ v = 0\right) \quad \ldots (2)$$

$$Z = \Sigma m(w' - w) = -(\omega' - \omega)\Sigma mx = -(\omega' - \omega)\bar{x}\Sigma m = -M(\omega' - \omega)\bar{x} \quad \ldots (3)$$

$$-Y\zeta = \Sigma m\left\{y(w' - w) - z(v' - v)\right\} = -(\omega' - \omega)\Sigma mxy = -(\omega' - \omega)F \quad \ldots (4)$$

$$\Rightarrow F = 0 \quad (\because Y = 0)$$

$$\zeta X - \xi Z = \Sigma m\left\{z(u' - u) - x(w' - w)\right\} = -(\omega' - \omega)\Sigma m(z^2 + x^2) = Mk^2(\omega' - \omega) \quad \ldots (5)$$

[M K^2 is the M.I. of the body about y-axis]

$$Y = \Sigma m\left\{x(v' - v) - y(u' - u)\right\} = -(\omega' - \omega)\Sigma mzx = -(\omega' - \omega)D \;\Rightarrow D = 0\ [\because Y = 0] \quad \ldots (6)$$

Thus we get $X = 0$, $Y = 0$, which implies that blow has no components parallel to the axes of x and y. Hence the blow must be perp. to xy- plane which contains the fixed axis and the instantaneous position of the centre of gravity. Also we see that $F = 0$ and $D = 0$ which implies that the y-axis which is also the axis of the body is a principal axis at the point where the plane through the line of action of the blow perp. to the fixed axis cuts it. This is a necessary condition for the existence of the centre of percussion. So if the fixed axis is not a principal axis at some point, then there is no centre of percussion.

Using equation (3) and (5), we get $\xi = \dfrac{k^2}{x}$ … (7)

The obvious conclusion from the relation (7) is that the distance of the centre of percussion from the fixed axis is the same as that of the centre of oscillation.

Points to remember in finding out the centre of percussion of a body for fixed axis.

(i) Find the point where the fixed axis is principal axis.

(ii) Take a distance $\frac{k^2}{x}$.

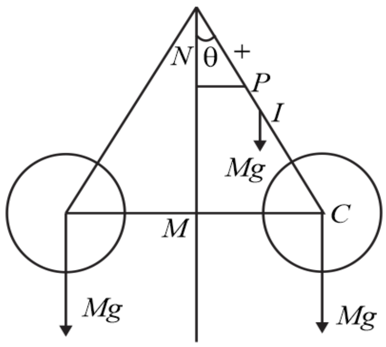
(iii) Draw an axis perp. to the plane containing the fixed axis and C.G. at a distance $\frac{k^2}{x}$ below the point where fixed axis is principal axis.

(iv) Any point on this line is a centre of percussion of the body for the fixed axis.

Examples

Example 1:- Two uniform spheres, each of mass M and radius a, are filmy fixed to the ends of two uniform thin rods. Each of mass m and length l, and the other ends of the rods are freely hinged to a point O. The whole system revolves as in the Govemor of steam-Engine, about a vertical line through O with the angular velocity ω. Show that when motion is steady, the rods are inclined to the vertical at an angle θ given by the equation $\cos\theta = \frac{g}{\omega^2} \cdot \frac{M(l+a)+\frac{1}{2}ml}{M(l+a)^2+\frac{1}{3}ml^2}$.

Solution:- Take an element δx in one of the rods at a distance x from O. Let PN, CM be the perpendiculars on the vertical line through O. Here C is the centre of one of the spheres.



The reversed effective force on the rod at P is $\delta x \frac{m}{l}\omega^2 x\sin\theta$ along NP and the revered effective force on the sphere is $M\omega^2(a+l)\sin\theta$ along MC, On taking moments about O for the system of a rod and a sphere on one side of the vertical OM, we have

$$\Sigma\left\{\delta x(m/l)\omega^2 x\sin\theta x\cos\theta\right\}+M\omega^2(a+l)\sin\theta(a+l)\cos\theta$$

$$=Mg(a+l)\sin\theta+mg(l/2)\sin\theta$$

Or $$\int_0^1 \frac{m}{l}\omega^2\sin\theta\cos\theta x^2dx+M\omega^2(a+l)^2\sin\theta\cos\theta$$

$$=Mg(a+l)\sin\theta+\frac{1}{2}mgl\sin\theta$$

Or $\quad \omega^2\left\{\frac{1}{3}ml^2+M\left(a+l\right)^2\right\}\cos\theta = g\left\{\frac{1}{2}ml+M\left(l+a\right)\right\}$

Or $\quad \cos\theta = \dfrac{g}{\omega^2}\dfrac{\frac{1}{2}ml+M\left(l+a\right)}{\frac{1}{3}ml^2+M\left(l+a\right)^2}$

Example 2:- A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ is fired with such a charge that he relative velocity of the ball and cannon at the moment when it leaves the cannon is u. Show that the cannon will recoil a distance $\left(\dfrac{mu}{M+m}\right)^2\dfrac{1}{2\mu g}$ along the plane, m being the mass of the ball.

Solution:- Let I be the impulse between the cannon and the ball and V, v be their velocities. Since their relative velocity is u, we have $V+v=u$ (1)

And $mv = I = MV$. (2)

From (1) and (2), we have $(MV/m)+V=u$ or $V=\{mu/(m+M)\}$.

Again on the rough plane, for the cannon the equation is $M\ddot{x}=-\mu R=-\mu Mg$, where x is the distance cannon has moved.

$\therefore \quad \ddot{x}=-\mu g$, Multiplying by $2\dot{x}$ and integrating, we get

$$\dot{x}^2=-2\mu gx+C$$

When $x=0, \dot{x}=V$, so that $C=V^2$, $\dot{x}^2=V^2-2\mu gx$ when the cannon comes to rest $\dot{x}=0$
,

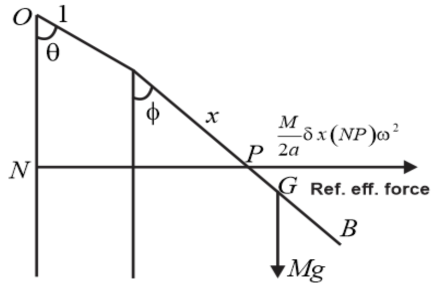
$\therefore \quad x=\left(V^2/2\mu g\right)$ or $x=\left(mu/M+m\right)^2\left(1/2\mu g\right)$

$$\left[\because\ V=\left(mu/M+m\right)\right]$$

Example 3:- A rod of length $2a$, is suspended by a string of length l, attached to one end if the string and rod revolve about the vertical with uniform angular velocity, and their inclination to the vertical be θ and ϕ respectively, show that $\dfrac{3l}{a}=\dfrac{\left(4\tan\theta-3\tan\phi\right)\sin\phi}{\left(\tan\phi-\tan\theta\right)\sin\theta}$.

Solution:- Take a small element δx of the rod AB at a distance x from A. Let be the uniform angular velocity of the rod. Mass of the element $=\dfrac{M}{2a}\delta x$. Its reversed effective force $=\dfrac{M}{2a}\delta xNP\,\omega^2$, along NP.

$$=\frac{M}{2a}\delta x\left(l\sin\theta+x\sin\phi\right)\omega^2 \text{ along } NP.$$



The external forces on the rod are (1) the tension T of the string and (2) of the weight Mg of the rod.
Resolving horizontally, vertically, and taking moments about A, we have

$$T\sin\theta = \frac{M}{2a}\omega^2 \Sigma NP\ \delta x = \frac{M}{2a}\omega^2 \int_0^{2a} (l\sin\theta + x\sin\phi)\,dx$$

$$= \frac{M}{2a}\omega^2 \left(2al\sin\theta + 2a^2\sin\phi\right) \qquad \text{[Horizontally]} \qquad (1)$$

$$T\cos\theta = Mg \qquad \text{[Vertically]} \qquad (2)$$

and $M\,ga\sin\phi = \dfrac{M}{2a}\omega^2\,\Sigma\,NP.\ \delta x.x\cos\phi$ [Moment equation]

$$= \frac{M}{2a}\omega^2 \int_0^{2a} (l\sin\theta + x\sin\phi)\,x\cos\phi\,dx$$

$$= \frac{M}{2a}\omega^2 \left(l\sin\theta\cos\phi\ 2a^2 + \frac{8a^3}{3}\sin\phi\cos\phi \right)$$

Or $\qquad \omega^2 = \dfrac{3g\sin\phi}{(3l\sin\theta + 4a\sin\phi)\cos\phi} \qquad (3)$

Dividing (1) by (2), we have $\dfrac{\sin\theta}{\cos\theta} = \omega^2 \dfrac{(l\sin\theta + a\sin\phi)}{g} \qquad (4)$

Putting value of ω^2 from (3) in (4), we get

$$\frac{\sin\theta}{\cos\theta} = \frac{3\sin\phi(l\sin\theta + a\sin\phi)}{(3l\sin\theta + 4a\sin\phi)\cos\phi}$$

Or $\qquad \sin\theta\cos\phi(3l\sin\theta + 4a\sin\phi) = \sin\phi\cos\theta(l\sin\theta + a\sin\phi)$

Or $\qquad 3l\sin\theta(\sin\theta\cos\phi - \sin\phi\cos\theta)$

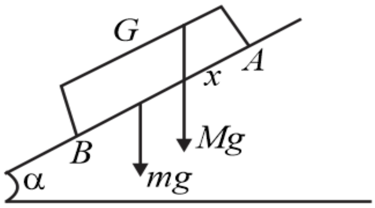
$$= a\sin\phi(3\sin\phi\cos\theta - 4\sin\theta\cos\phi)$$

Or $\qquad \dfrac{3l}{a} = \dfrac{\sin\phi(3\sin\phi\cos\theta - 4\sin\theta\cos\phi)}{\sin\theta(\sin\theta\cos\phi - \sin\phi\cos\theta)}$

$$= \frac{\sin\phi(3\tan\phi - 4\tan\theta)}{\sin\theta(\tan\theta - \tan\phi)} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$$

Example 4:- A plank, of mass m and length $2a$, is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man; of mass M, staring from the upper end walks downs the plank so that it does not move, show that he will reach the other end in time. $\left[\dfrac{4Ma}{(m+M)g\sin\alpha}\right]^{1/2}$

Solution:- Suppose that the man has come down a distance x in times t, starting from the end A of the plank. Since the plank does not move, its centre is fixed. If $\bar{x}$ be the distance of the C.G. of the system from A, then $(M+m)\bar{x} = am + Mx$.



The gives $(M+m)\ddot{\bar{x}} = M\ddot{x}$ (1)

Again the motion of the C.G. of the system is given by $(M+m)\ddot{\bar{x}} =$ Ext. forces acting parallel to the plank $= \Sigma X = (m+M)g\sin\alpha$ (2)

From (1) and (2), we get

$M\ddot{x} = (m+M)g\sin\alpha$ or $\ddot{x} = \dfrac{(m+M)g\sin\alpha}{M}$

Integrating twice and applying the condition that when we have

$x = \dfrac{(m+M)g\sin\alpha}{M}.\dfrac{1}{2}t^2$

Putting $x = 2a$, we get the time to reach the other end as $\left[\dfrac{4Ma}{(m+M)g\sin\alpha}\right]^{1/2}$

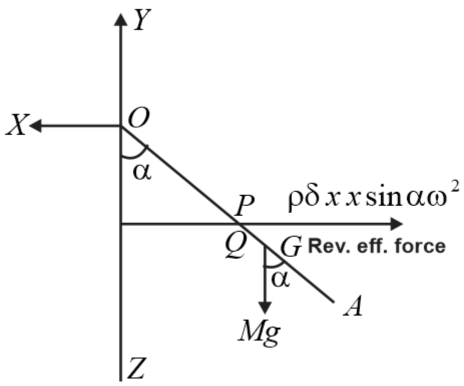
Example 5:- A uniform rod OA, of length $2a$, free to turn about its end O, resolves with uniform angular velocity ω about a vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or $\cos^{-1}(3g/4a\omega^2)$.

Solution:- Consider a small element $PQ = \delta x$ at a distance x from O. The point P will move in a horizontal circle whose radius is $PL = x\sin\alpha$. Here only effective force on the element PQ is $\rho\,\delta x\,PL\omega^2 = \rho\delta x.x\sin\alpha\,\omega^2$, where ρ is the density of the rod and angular velocity ω is constant. Reversing the effective force and taking moment about O, we have

$\Sigma(\rho\,\delta x.x\sin\alpha\,\omega^2)x\cos\alpha = M\,g\,a\sin\alpha$ or $\rho\,\omega^2\sin\alpha\cos\alpha\int_0^{2a} x^2\,dx = M\,g\,a\sin\alpha$ or

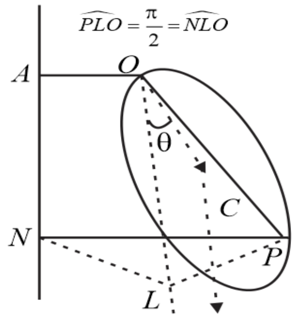
$(M/2a)\omega^2\sin\alpha\cos\alpha(8a^3/3) = M\,ga\sin\alpha$

$(\because\ 2a\rho = M)$ or $\sin\alpha\left(g - \dfrac{4a\omega^2\cos\alpha}{3}\right) = 0$. it implies either $\sin\alpha = 0$ i.e. $\alpha = 0$ or $\cos\alpha = (3g/4a\omega^2)$ i.e. $x = \cos^{-1}(3g/4a\omega^2)$



Example 6:- A thin circular disc of mass M and radius a, can tum freely about a thin axis OA. Which is perp. To its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity ω about its end A. Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}\left(g/a\omega^2\right)$ unless $\omega^2 < g/a$ and then θ is zero.

Solution:- Consider the circular disc in the vertical plane so that the axis OA about which it turns is horizontal. When the axis OA moves horizontally round A, the disc will be raised in its vertical plane and its radius OC makes an angle θ with the vertical. Consider an element δm at P. Let PL be perpendicular to the vertical through O and LN be perpendicular from L to the vertical through A so that PN is perpendicular to AN. Now P describes a circle of radius PN with a constant angular velocity ω about N. Thus the reversed effective force along NP is $\delta m NP\omega^2$.



Again $\overrightarrow{NP} = \overrightarrow{NL} + \overrightarrow{LP}$

$\therefore$ $\delta m\,.\omega^2\,\overrightarrow{NP} = \delta m\,.\omega^2\,\overrightarrow{NL} + \delta m\,.\omega^2\,\overrightarrow{LP}$ i.e. the force $\delta m\,\omega^2\,\overrightarrow{NP}$ is equivalent to forces $\delta m\,\omega^2 LP$ and other $\delta m\,\omega^2 NL$ along NL. The external forces on the disc are its weight Mg and the reaction at O.

By D' Alemberts Principle, Rev. effective forces along with external forces form the system in equilibrium. Hence moment of Rev. effective forces + moment of external forces = 0 i.e. moment of effective forces about OA = moment of external forces (1).

In order to avoid reaction at O, we take moment about the line OA. Since NL and OA lie in one plane (they are parallel also) the shortest distance between them is zero.

$\therefore$ Moment of the forces $\delta m\omega^2 \times NL$ about OA is zero. Further the shortest distance between OA and LP is OL and the shortest distance between OA and the vertical through C is a $\sin\theta$. Hence moment of the force $\delta m\,\omega^2 LP$ about OA is given by $\delta m\,\omega^2 LP \times OL$. Taking moments about OA, we get $Mga\sin\theta = \Sigma\delta m\omega^2 LP \times OL$ or $aMg\sin\theta = \omega^2\Sigma\left(\delta m\,LP.OL\right)$. But $\Sigma\left(\delta m\ LP\,OL\right) =$ product of inertia of the disc about OL and horizontal line through

$O =$ product of inertia about the parallel lines through $C + Mx'y'$. Where x', y' are the co-ordinates of C with respect of the vertical and horizontal through O.

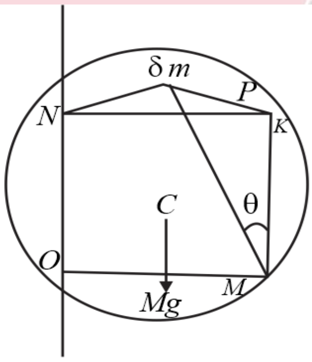
$= 0 = Ma^2 \sin\theta \cos\theta$

$\Rightarrow$ $aMg \sin\theta = \omega^2 Ma^2 \sin\theta \cos\theta \Rightarrow \sin\theta = 0$ or $\cos\theta = \left(g / a\omega^2\right)$, where $a\omega^2 > g$

But $\omega^2 < \left(g/a\right) \Rightarrow \cos\theta > 1$, which is impossible and hence in this case $\cos\theta = 1$ i.e. $\theta = 0$.

Example 7:- A thin heavy disc can tum freely about an axis in its own plane, and this axis revolves horizontally with a uniform angular velocity ω about a fixed point on itself. Show that the inclination θ of the plane of the disc to the vertical is given by $\cos\theta = \left(gh / k^2\omega^2\right)$ where h is the distance of the centre of inertia of the disc from the axis and k is the radius of the gyration of the disc about the axis. If $\omega^2 < gh < k^2$, prove that the plane of the disc is vertical.

Solution:- Let OM be the horizontal axis in the plane of the disc which, rotates about O so that the vertical line ON is the axis of rotation of the system. Consider an element of mass δm, at P. Draw PN perpendicular to this vertical axis ON then effective force for δm is $\delta m \omega^2 PN$. Here PN is not in the plane of the disc. From P draw PM perpendicular to OM, here PM is in the plane of the disc. Through N draw NK perpendicular to OM and from P draw PK perpendicular to NK so that PK is perpendicular to KM, thus if $\angle PMK = \theta, \theta$ is the inclination of the disc to the vertical, KM being vertical.



Again $\overrightarrow{PN} = \overrightarrow{PK} + \overrightarrow{KN}$. Therefore, $\delta m \omega^2 \overrightarrow{PN} = \delta m \omega^2 \overrightarrow{PK} + \delta m \omega^2 \overrightarrow{KN}$. Thus the effective force on δm are $\delta m \omega^2 \overrightarrow{PK}$ and $\delta m \omega^2 \overrightarrow{KN}$. Since KN is parallel to OM, the moment of the force $\delta m \omega^2 \overrightarrow{KN}$ about OM will be zero and the moment of $\delta m \omega^2 \overrightarrow{PK}$ about OM is $\delta m \omega^2 PK.KM$. The OM, we get $Mgh \sin\theta = \Sigma \delta m \omega^2 PK\, KM$

$= \Sigma \delta m \omega^2 \left(PM \sin\theta\right)\left(PM \cos\theta\right) = \omega^2 \sin\theta \cos\theta \Sigma \delta m PM^2$

But $\Sigma \delta m PM^2 =$ M.I. of the disc about $OM = Mk^2$, where k is the radius of gyration

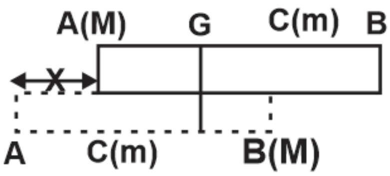
$\therefore$ $Mgh \sin\theta = \omega^2 \sin\theta \cos\theta Mk^2$

Hence either $\sin\theta = 0$ i.e. $\theta = 0$ or $\cos\theta = \dfrac{gh}{\omega^2 k^2}$.

If $\omega^2 < \dfrac{gh}{k^2}$, as in that case $\cos\theta > 1$ the only possible value of θ is zero and then plane of the disc is vertical.

Example 8:- A rough uniform board, of mass m and length $2a$, rests on a smooth horizontal plane, and a boy, of mass M, walks on it from one end to the other, show that the distance though which the board moves in this time is $2Ma/(m+M)$.

Solution:- Here the weight of the boy and the board are downwards, the actions and reactions between the boy and the board vanish for the system. The reaction of the smooth plane is acting vertically upwards. Thus there are no external forces on the system in the horizontal direction. Thus by D' Alembert's Principle the C.G. of the system does not move. As the boy goes to left, the board comes to the right.



Let $\bar{x}$ be the distance of the C.G. of the system and x be the distance through which the board moves, when the boy goes from one end to the other.

Now in the initial position, $(M+m)\bar{x} = M2a + ma$

In the final position, $(M+m)\bar{x} = M\,x + m(a+x)$

Therefore, $M\,.\,2a + ma = M\,x + m(a+x)$

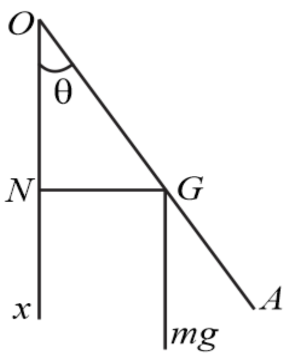
Or $\quad x = 2M\,a/(M+m)$.

Motion about a Fixed Axis

Example:- A straight uniform rod can turn freely about one end O, hangs from O vertical. Find the least angular velocity with which it must begin to move so that it may perform complete revolution in a vertical plane.

Solution:- Let the rod OA at any instant t make an angle θ with initial vertical position OX. Let G be the centre of gravity and GN perpendicular to OX. Let $OA = a$ and mass of the rod be m. The equation of motion is

$$mk^2\,\ddot{\theta} = -m\,g\left(\frac{a}{2}\right)\sin\theta$$



$\because\quad$ Moment of effective forces about the axis of rotation $= mk^2\,\theta$ and moment of external forces about the axis of rotation $= -m\,g\,(a/2)\sin\theta$

$\Rightarrow \qquad 2a\ddot{\theta} = -3g\sin\theta + C \qquad (1)$

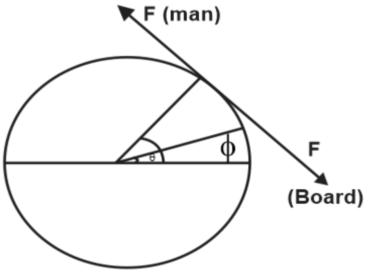
Let $\dot{\theta} = \omega$ when $\theta = 0$ $\qquad \therefore \; a\,\omega^2 = 3g + C \qquad (2)$

Hence from (1) and (2), we get $a\dot{\theta} = a\,\omega^2 - 3g(1-\cos\theta)$ we required that $\dot{\theta} = 0$ when $\theta = \pi$

$\therefore \qquad 0 = a\omega^2 - 6g \Rightarrow \omega = \sqrt{(6g/a)}$

Example:- A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through the centre. A man whose weight is equal to that of the board walks on and around it at the edge, when he has completed the circuit, what will be his position in space.

Solution:- Let any time t, θ and ϕ be the angles described by the board and man respectively and let F be the action between the feet of the man and the board. Equation of motion for the man is $ma\ddot{\phi} = F$ $\qquad (1)$



Equation of motion for the board is $mk^2\ddot{\theta} = -Fa$ $\qquad (2)$

On eliminating F between (1) and (2), we g et

$$a^2\ddot{\phi} + k^2\dot{\theta} = 0 \Rightarrow \ddot{\phi} + \ddot{\theta} = 0 \quad \left(\because k^2 = \frac{a^2}{2}\right)$$

Integrating twice the above equation and considering that initially both man and the board were at rest, we get $2\phi = \theta = 0$.

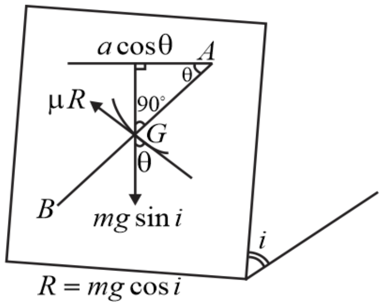
Therefore, when $\phi - \theta = 2\pi$ (after completing the circuit)

We get, $3 = 2\pi \Rightarrow \phi = 2\pi/3$.

This is the angle in space described by the man.

Example:- A uniform rod AB is freely movable on a rough inclined plane whose inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pint fixed through the end A: the bar is held in the horizontal position in the plane and allowed to fall from this position, if θ be the angle through which it falls from rest show that $(\sin\theta/\theta) = \mu\cot i$.

Solution:- Let any instant t, the position of the rod be AB, making an angle θ with the initial horizontal position. The external forces acting on the rod, perpendicular to the plane, are the normal reaction R and resolved part of its weight i.e. $mg\cos i$. External forces acting on the rod in the plane are, (i) the resolved part of its weight, $mg\sin i$ acting down the line of greatest slope through G (centre of gravity). (ii) the friction $\mu R = \mu mg\cos i$ acting perpendicular to AB through G; (iii) the reaction at A. We take moments about A to avoid reaction, so



$$mk^2\ddot{\theta} = mg\sin i . a\cos\theta - \mu mg\cos i\, a$$

$\Rightarrow$ $\quad k^2\ddot{\theta} = ga(\sin i\cos\theta - \mu\cos i)$, where $2a$ is length of the rod.

Multiplying the above equation by $2\dot{\theta}$ and integrating, we get

$$k^2\dot{\theta}^2 = 2a\,g\sin i\sin\theta - 2\mu\, ag\,\theta\cos i + D$$

When $\theta = 0,\ \dot{\theta} = 0 \quad \therefore D = 0$. Hence $k^2\ \dot{\theta} = 2ag\sin i\sin\theta - 2\,\mu a\,g\,\theta\cos i$

Rod will come to rest when $\dot{\theta} = 0$

$\therefore$ $\quad 0 = 2ag\sin i\sin\theta - 2\,\mu ag\cos i \Rightarrow (\sin\theta/\theta) = \mu\cot i$

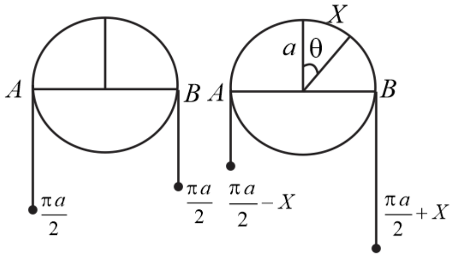
Example:- A uniform vertical circular plate of radius a, is capable of revolving about a smooth horizontal axis through its centre; a rough perfectly flexible chain, whose mass is equal to that of the plate and whose length is equal to its circumference hangs over its rim in equilibrium , if one end be slightly displaced show that the velocity of the chain when the other end reaches the plate is $\left(\dfrac{\pi ag}{6}\right)^{1/2}$

Solution:- Let x be the distance described in time t. Let v be the velocity of the string and θ be the angular velocity of the plane, then $v = \dot{x} = a\dot{\theta}$. Let m be the mass of the plate and that of string, then

$K.E.$ of the string $= \dfrac{1}{2}mv^2$. $K.E.$ of the plate $= \dfrac{1}{2}mk^2\dot{\theta}^2 = \dfrac{1}{2}mk^2\dfrac{v^2}{a^2}$

$$= \frac{1}{2}m\frac{a^2}{2}.\frac{v^2}{a^2} = \frac{1}{4}mv^2$$

$$\left[\because\ k^2 = \frac{a^2}{2}\right]$$



Hence, the total $K.E.$ generated $= \dfrac{1}{2}mv^2 + \dfrac{1}{4}mv^2 = \dfrac{3}{4}mv^2$

At time t, length of the string hanging to the right is $\left(\frac{\pi a}{2}+x\right)$ and hanging to the left is $\left(\frac{\pi a}{2}-x\right)$, the weights of these two portion are respectively, $\frac{mg}{2\pi}\left(\frac{\pi a}{2}+x\right)$ and $\frac{mg}{2\pi}\left(\frac{\pi a}{2}-x\right)$.

The depths of the C.G.'s of these portions below AB are $\frac{1}{2}\left(\frac{\pi a}{2}+x\right)$ and $\frac{1}{2}\left(\frac{\pi a}{2}-x\right)$

Hence when x is the displacement, work function on the right is

$$W_1=\frac{mg}{2\pi a}\left(\frac{\pi a}{2}+x\right).\frac{1}{2}\left(\frac{\pi a}{2}+x\right)$$

Work function of the left is $W_2=\frac{mg}{2\pi a}\left(\frac{\pi a}{2}-x\right).\frac{1}{2}\left(\frac{\pi a}{2}-x\right)$

$\therefore$ Total work function $W=W_1+W_2=\frac{mg}{4\pi a}\left(\frac{\pi a}{2}+x\right)^2+\frac{mg}{4\pi a}\left(\frac{\pi a}{2}-x\right)^2$ (1)

In the initial position i.e. when $x=0$

$$W_0=2\frac{mg}{4\pi a}\frac{\pi^2 a^2}{4}=\frac{1}{8}mg\pi a \qquad \text{[From (1)]}$$

Hence total work done $=W-W_0=\frac{mg}{4\pi a}\left[\left(\frac{\pi a}{2}+x\right)^2+\left(\frac{\pi a}{2}-x\right)^2-\frac{1}{2}\pi^2 a^2\right]=\frac{mgx^2}{2\pi a}$

Therefore energy equation gives $\frac{3}{4}mv^2=\frac{mgx^2}{2\pi a}\Rightarrow v^2=\frac{2gx^2}{3\pi a}$.

When $x=\frac{\pi a}{2}$ (i.e. when other end reaches the plate)

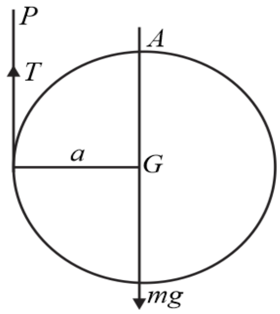
$$v^2=\frac{1}{6}\pi a g\Rightarrow v=\left(\frac{1}{6}\pi ag\right)^{1/2}$$

Example:- One end of a light string is fixed to a point of the rim of a uniform circular disc of radius a and mass m and the string is wounded several times round the rim. The free end is attached to a fixed point and the disc is held so that the part of the string not in contact with it, is vertical. If the disc be let go, find the acceleration and the tension of the string.

Solution:- Let the free end be attached to the fixed point P. Let A be the initial position of the centre of gravity G. Let T be the tension of the string. There being no horizontal force the C.G. will move vertically downward. Let x be the distance moved by G in time t and during this period, θ be the angle turned through some radius.

$$\therefore \qquad mg-T=m\ddot{x} \qquad (1)$$

$$\text{And} \qquad Ta=mk^2\ddot{\theta}=m\frac{a^2}{2}\ddot{\theta} \qquad (2)$$



Again $x = a\theta$, $\therefore \ \ddot{x} = a\,\ddot{\theta}$

On eliminating T and $\ddot{\theta}$ from (1), (2) and (3), we get $m\,ga = m\,a\,\ddot{x} + m\frac{a}{2}\ddot{x} \Rightarrow \ddot{x} = \frac{2g}{3}$

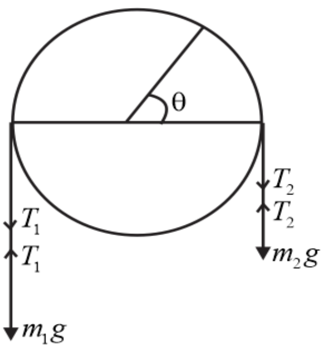
Substituting this value in (1), we get $T = \frac{1}{3}m\,g$

Example:- Two unequal masses m_1 and $m_2\left(m_1 > m_2\right)$ are suspended by a light string over a circular pulley of mass M and radius a. There is no slipping and the friction of axis may b e neglected. If f be the acceleration: show that this is constant, and if k^2 be the radius of gyration of the pulley about the axle, show that $k^2 = \frac{a^2}{M\,f}\left[\left(g - f\right)m_1 - \left(g + f\right)m_2\right]$

Solution:- Let in time t, m_1 move a distance x downwards and m_2 move a distance x upwards. Let θ be the angle through which the pulley has rotated in time t. Since $x = a\theta$, $\therefore \ddot{x} = a\ddot{\theta}$.

Equations of motion of m_1 and m_2 are $m_1\ddot{x} = m_1 g - T_1$ (1)

and $m_2\ddot{x} = T_2 - m_2 g$. (2)



Equation of motion of the pulley is $M\,k^2\,\ddot{\theta} = T_1 a - T_2 a$

(Moment is taken about the axle)

$$\Rightarrow \qquad M\frac{k^2}{a^2}\ddot{x} = T_1 - T_2 \qquad \left(\because \ddot{\theta} = \frac{\ddot{x}}{a}\right) \qquad (3)$$

Adding (1), (2) and (3), we get $\ddot{x}\left(m_1 + m_2 + M\frac{k^2}{a^2}\right) - m_1 g = m_2 g$

$\Rightarrow \quad \ddot{x} = f = \dfrac{(m_1 - m_2)g}{m_1 + m_2 + M\dfrac{k^2}{a^2}}$, which is constant.

From above we get $f(m_1 + m_2) + \dfrac{Mk^2}{a^2} f = (m_1 - m_2)g$

$\Rightarrow \quad k^2 = \dfrac{a^2}{Mf}\left[(m_1 - m_2)g - (m_1 + m_2)f\right] = \dfrac{a^2}{Mf}\left[(g-f)m_1 - (g+f)m_2\right]$

Pressure on the pulley $= T_1 + T_2$.

Again on subtracting (1) from (2), we get $(m_2 - m_1)\ddot{x} = T_2 + T_1 - (m_1 + m_2)g$

$\Rightarrow \quad T_2 + T_1 = (m_2 - m_1)\ddot{x} + (m_1 + m_2)g = (m_2 - m_1)f + (m_1 + m_2)g$.

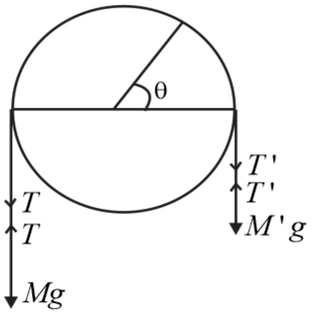
Example:- Fine string has two masses M and M' tied to its ends and passes over a rough pulley, of mass m, whose centre is fixed. If the string does not slip over the pulley, show that M will descend with acceleration $\dfrac{M - M'}{M + M' + (mk^2/a^2)}g$ where a is the radius and k the radius of gyration of the pulley. If pulley be not sufficient rough to prevent sliding, and M be the descending mass, show that its acceleration is $\dfrac{M - M'e^{\mu\pi}}{M + M'e^{\mu\pi}}g$ and that pulley will now spin with an angular acceleration equal to $\dfrac{2MM'ga(e^{\mu\pi} - 1)}{mk^2(M + M'e^{\mu\pi})}$.

Solution:- First part, when the pulley is rough enough to prevent sliding proceeding like Ex.6 the equations of motion of masses and pulley are

$$M\ddot{x} = Mg - T \qquad (1)$$

And $M'\ddot{x} = T' - M'g$ (2)

And moment of effective forces about the axis of rotation $= mk^2\ddot{\theta} = (T - T')a$ (3)



Again $x = a\theta$, $\ddot{x} = a\ddot{\theta}$

$\therefore \quad mk^2 \dfrac{\ddot{x}}{a^2} = T - T'$ (4)

Adding (1), (2) and (4), we $\ddot{x}\left[M + M' + (mk^2/a^2)\right] = (M - M')g$

$\Rightarrow$ Acceleration $\ddot{x} = \dfrac{(M-M')g}{M+M'+(mk^2/a^2)}$.

Second Part:- When the pulley is not sufficiently rough to prevent sliding, then we can not take $x = a\theta$. In this case, statics, we have $T = T'e^{\mu\pi}$

(5)

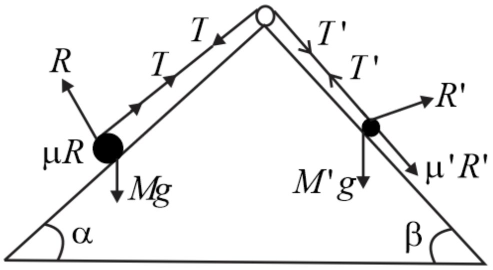
Solving (1), (2) and(5), we have $T = \dfrac{2MM'g\,e^{\mu\pi}}{M+M'e^{\mu\pi}} T' = \dfrac{2MM'g}{M+M'e^{\mu\pi}}$ and

$\ddot{x} = \dfrac{M-M'e^{\mu\pi}}{M+M'e^{\mu\pi}} g$.

Further putting above values of T and T' in (3), we get $\ddot{\theta} = \dfrac{2ga(e^{\mu\pi}-1)}{mk^2} \cdot \dfrac{MM'}{M+M'e^{\mu\pi}}$

Example:- Two unequal masses, M and M' rest on two rough planes inclined at an angles α and β to the horizon: they are connected by a fine string passing over a small pulley, of mass m and radius a, which is placed at the common vertex of the two planes; show that the acceleration of either mass is $\dfrac{g[m(\sin\alpha-\mu\cos\alpha)-M'(\sin\beta+\mu'\cos\beta)]}{M+M'+(mk^2/a^2)}$ where μ and μ' are the coefficients of friction k is the radius of gyration of the pulley about its axis and M is the mass which moves downwards.

Solution:- Suppose in time t, the mass M moves a distance x downwards, and also M moves a distance x upwards. Let the pulley turn through an angle θ. In the same time t.



$\therefore$ $x = a\theta$, $\ddot{x} = a\ddot{\theta}$. The equations of motion of the masses M and M' are

$$M\ddot{x} = Mg\sin\alpha - Mg\mu\cos\alpha - T \qquad (1)$$

$$M'\ddot{x} = T' - M'g\sin\beta - M'g\mu\cos\beta \qquad (2)$$

Equation of motion of pulley is $mk^2\ddot{\theta} = (T-T')a$

$$\Rightarrow \quad \frac{mk^2\ddot{x}}{} = T - T' \qquad \left(\because \ddot{\theta} = \frac{\ddot{x}}{a}\right) \qquad (3)$$

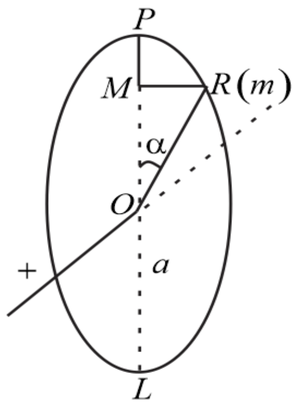
Adding (1), (2) and (3), we get

$$\left(\frac{mk^2}{a^2} + M + M'\right)\ddot{x} = g[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta - \mu'\cos\alpha)]$$

$$\Rightarrow \quad \ddot{x} = \frac{g\left[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta + \mu'\cos\beta)\right]}{M + M' + \frac{mk^2}{a^2}}$$

Example:- A uniform circular disc is free to turn about a horizontal axis through its centre perpendicular to its planes. A particle of masses attached to a point in the edge of the dis. If the motion starts from the position in which radius to the particle makes an angle α with the upward vertical, find the angular velocity when m is in its lowest position. Take the mass of the disc as M .

Solution:- The circular disc is turning about the fixed horizontal axis OX , through its centre O. Let ω be the angular velocity when m is in its lowest position. Say L then energy principle gives. Change in K.E. = work done by forces.



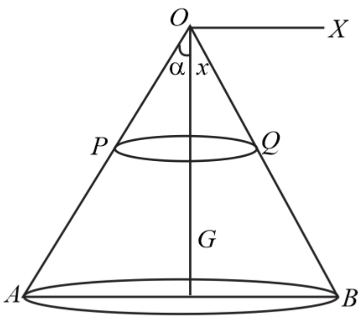
$$\Rightarrow \left(\frac{1}{2}ma^2\omega^2 + \frac{1}{2}M\frac{a^2}{2}\omega^2\right) - 0 = mg(a + a\cos\alpha) \quad \text{or} \quad a\omega^2(2m+M) = 4g(1+\cos\alpha) \quad \text{or}$$

$$\omega = 2\frac{\sqrt{2}}{\sqrt{(2m+M)}}\sqrt{(g/a)}\cos\frac{\alpha}{2}$$

Remark: The weight of the disc does not work as its C.G. is fixed.

Example:- A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\frac{1}{5}h\left(4+\tan^2\alpha\right)$

Solution:- Let OX be the horizontal axis through the vertex O. Let us take a circular disc PQ of thickness δx at distance x from O. Moment of Inertia of disc about OX $=\left(\rho\pi x^2\tan^2\alpha\,\delta x\right)\left(\frac{1}{4}x^2\tan^2\alpha + x^2\right)$.



Therefore, $M.I.$ of whole cone about OX

$$M\,k^2 = \rho\pi\tan^2\alpha\left(1+\frac{1}{4}\tan^2\alpha\right)\int_0^h x^4 dx$$

$$= \rho\pi\tan^2\alpha\left(1+\frac{1}{4}\tan^2\alpha\right)\frac{1}{5}h^5$$

$$= \frac{1}{20}\rho\pi\tan^2\alpha\left(\tan^2\alpha+4\right)h^5$$

$$= \frac{3}{20}M\left(\tan^2\alpha+4\right)h^2 \qquad \left(\because M = \frac{1}{3}\pi\, h^3\tan^2\alpha\,\rho\right)$$

$\therefore$ $\quad k^2 = \dfrac{3}{20}\left(\tan^2\alpha+4\right)h^2$. Again $OG = \dfrac{3}{4}h$.

Therefore the length of the simple equivalent pendulum i.e. $l = \dfrac{k^2}{OG} = \dfrac{1}{5}\left(\tan^2\alpha+4\right)h$

Example:- A solid homogeneous cone of height h and semi-vertical angle α oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is $\dfrac{1}{5}h\left(2+3\tan^2\alpha\right)$

Solution:- Referring to the fig. of the example 10. We observe that M.I. of the cone about AB

$$= \int_0^h \rho\pi x^2\tan^2\alpha\, dx\left[\frac{x^2\tan^2\alpha}{4}+(h-k)^2\right]$$

$$= \int_0^h \frac{\rho\pi\tan^2\alpha}{4}\left[x^4\tan^2\alpha+4x^2(h-x)^2\right]dx$$

$$= \frac{\rho\pi\tan^2\alpha}{4}\int_0^h\left(x^4\tan^2\alpha+4x^4-8hx^2+4h^2x^2\right)dx$$

$$= \frac{1}{4}\rho\pi\tan^2\alpha\left(\frac{h^5}{5}\tan^2\alpha+4\frac{h^5}{5}-8h\frac{h^4}{4}+4h^2\frac{h^3}{3}\right)$$

$$= \frac{1}{4}\rho\pi\tan^2\alpha\, h^5\left[\frac{1}{5}\tan^2\alpha+\frac{2}{15}\right] = \frac{1}{60}\rho\pi h^4\tan^2\alpha\left[3\tan^2\alpha+2\right]$$

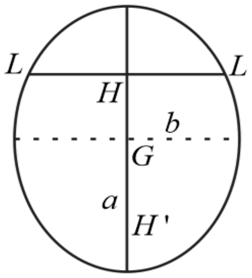
$$= \frac{1}{20}M\,h^2\left(3\tan^2\alpha+2\right), \text{ since } M = \frac{1}{3}\pi h^3\tan^2\alpha\,.\,\rho$$

$\therefore$ $\quad M\,k^2 = \dfrac{1}{20}M\,h^2\left(3\tan^2\alpha+2\right) \Rightarrow k^2 = \dfrac{1}{20}h^2\left(3\tan^2\alpha+2\right)$, where k is the radius of gyration of cone about AB. Hence length of the simple equivalent pendulum

$$= \frac{k^2}{distance\, of\ G\ from\, AB} = \frac{k^2}{(h/4)} = \frac{1}{5}h\left(3\tan^2\alpha+2\right)$$

Example:- An elliptical lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum possess through the centre of oscillation, prove that the eccentricity is $\dfrac{1}{2}$.

Solution:- When one of the focii say H, is the centre of suspension then the other focus H' is the centre of oscillation. LHL' is the latus rectum (horizontal axis) about which the elliptic lamina oscillates.



The length of simple equivalent pendulum $t = HH' = 2ae$ (1)

Also $HG = ae$ and $M\,k^2 =$ Moment of Inertia of the body about the axis of the rotation.

$$LHL' = M\left[\frac{a^2}{4} + a^2e^2\right] \Rightarrow k^2 = \frac{1}{4}a^2\left(1+4e^2\right) \qquad (2)$$

$$\therefore \qquad l = \frac{k^2}{HG} = \frac{1}{4}\frac{a^2\left(1+4e^2\right)}{a\,e}$$

Form (1) and (2), we get $2ae = \dfrac{1}{4}\dfrac{a^2\left(1+4e^2\right)}{ae}$

$$\Rightarrow \qquad 8e^2 = 1+4e^2 \Rightarrow 4e^2 = 1 \Rightarrow e = \frac{1}{2}.$$

Example:- A uniform elliptic board swings about a horizontal axis at right angles to the plane of the board and passing through one focus. If the centre of oscillation be the other focus prove that its eccentricity is $\sqrt{(2/5)}$

Solution:- Refer fig. before example here $M\,k^2 = M\left[\frac{1}{4}\left(a^2+b^2\right)+a^2\,e^2\right]$

$\therefore$ Length of simple equivalent pendulum $l = \dfrac{k^2}{HG} \Rightarrow \dfrac{k^2}{ae} = \dfrac{1}{4ae}\left(a^2+b^2+4a^2e^2\right)$ (1)

Also $l = 2ae \ \therefore\ 2ae = \dfrac{1}{4ae}\left(a^2+b^2+4a^2+e^2\right)$

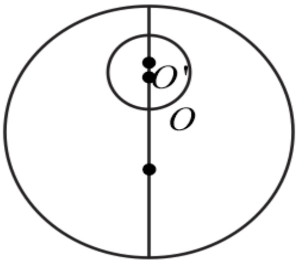
$$\Rightarrow \qquad 8a^2e^2 = a^2+b^2+4a^2e^2 = a^2+\left(1+e^2\right)a^2+4a^2e^2$$

$$\Rightarrow \qquad 5a^2e^2 = 2a^2 \Rightarrow e = \sqrt{(2/5)}.$$

Example:- A flat circular disc of radius a has a hole in it of radius b whose centre is at a distance c from the centre of the disc $(c < a-b)$. The disc is free to oscillates in a vertical plane about a smooth horizontal circular rod of radius b passing through the hole. Show that the length of the equivalent pendulum is $c + \dfrac{1}{2}\dfrac{a^4-b^4}{a^2c}$

Solution:- Let O' be the centre of the hole in the disc whose centre is O. $OO' = c$ (given). The disc is oscillated in a vertical plane about a smooth horizontal circular rod of radius b passing through O'.

If h be the depth of C.G. of the body from O', then $h = \dfrac{\rho\pi a^2c - \rho\pi b^2.\,0}{\rho\pi b^2 - \rho\pi b^2} = \dfrac{a^2c}{a^2-b^2}$



Let k be the radius of gyration about the axis of rotation, then we have

$$\left(\rho\pi a^2-\rho\pi b^2\right)=\rho\pi a^2\left(\frac{a^2}{2}+c^2\right)-\rho\pi b^2.\frac{b}{2}$$

$$\Rightarrow \qquad k^2=\frac{a^2+2a^2c^2-b^4}{2\left(a^2-b^2\right)}$$

$$\therefore \qquad l=\frac{k^2}{h}=\left[\frac{a^2+2a^2c^2-b^4}{2\left(a^2-b^2\right)}\right]/\left[\frac{a^2c}{\left(a^2-b^2\right)}\right]$$

$$=\frac{a^4+2a^2c^2-b^4}{2a^2c}=c+\frac{1}{2}\frac{a^4-b^4}{a^2c}.$$

Example:- A bent lever, whose ams are of length a and b, the angle between them being α, makes small oscillations in its own plane about the fulcrum, show that the length of the corresponding simple pendulum is $\frac{2}{3}\frac{a^3+b^3}{\sqrt{\left(a^4+2a^2b^2\cos\alpha+b^4\right)}}$

Solution:- Let G_1 and G_2 be the centre of gravity of the arms OA and OB of the lever. Let $OA=a$ and $OB=b$. Also let OA be the axis of x and a perpendicular line OY the axis of y. Then the co-ordinates of G_1 and G_2 will be $\left(\frac{1}{2}a,0\right)$ and $\left(\frac{1}{2}b\cos\alpha,\frac{1}{2}b\sin\alpha\right)$ respectively.

Now if $\left(\bar{x},\bar{y}\right)$ is the C.G. of the lever, then $\bar{x}=\dfrac{a\omega.\frac{1}{2}a+b\omega.\frac{1}{2}b\cos\alpha}{a\omega+b\omega}=\dfrac{1}{2}\dfrac{a^2+b^2\cos\alpha}{a+b}$;

where ω is the weight of unit length of the rod.

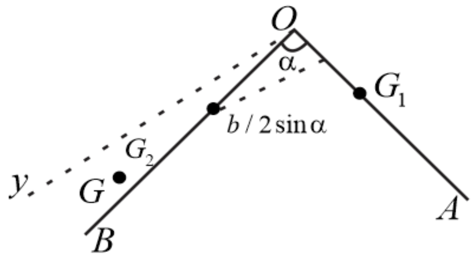
$$\bar{y}=\frac{a\omega.0+b\omega.\frac{1}{2}b\sin\alpha}{a\omega+b\omega}=\frac{1}{2}\frac{b^2\sin\alpha}{a+b}$$

Also the distance of C.G. $\left(\bar{x},\bar{y}\right)$ from $O(0,0)$ is

$$\sqrt{\left(\overline{x^2}+\overline{y^2}\right)}=\frac{1}{2(a+b)}\sqrt{\left\{a^4+2a^2b^2\cos\alpha+b^4\right\}}$$

Now if k is the radius of gyration about the axis of rotation through O, then we have

$$(a+b)\omega k^2=a\omega.\frac{4}{3}\left(\frac{1}{2}a\right)^2+b\omega.\frac{4}{3}\left(\frac{1}{2}b\right)^2\Rightarrow k^2=\frac{a^3+b^3}{3(a+b)}.$$



Hence the length of the simple pendulum

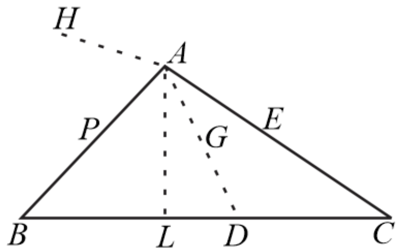
$$= \frac{k^2}{Dist.of\ C.G.of\ the\ lever\ from\ O}$$

$$= \frac{1}{3}\frac{a^3+b^3}{a+b}\cdot\frac{2(a+b)}{\left(a^4+2a^2b^2\cos\alpha+b^4\right)^{1/2}}$$

$$= \frac{2}{3}\frac{a^3+b^3}{\left(a^4+2a^2b^2\cos\alpha+b^4\right)^{1/2}}$$

Example:- A uniform triangular lamina can oscillate in its own plane about the angle A. Prove that the length of the simple equivalent pendulum is $\dfrac{3\left(b^2+c^2\right)-a^2}{4\sqrt{\left\{2\left(b^2+c^2\right)-a^2\right\}}}$

Solution:- Let AH be perpendicular to the plane of the lamina so that it oscillates in its own plane about AH. Instead of the triangular lamina of mass m, we can have three perpendicular each of mass $\frac{1}{3}m$ placed at the mid points D, E, F of the sides respectively. Distance of D from AH is.



$$AD = \left[AL^2+LD^2\right]^{1/2} = \left[AL^2+(BD-BL)^2\right]^{1/2}$$

$$= \left[AL^2+BD^2+BL^2-2BD.BL\right]^2$$

$$= \left[\left(AL^2+BL^2\right)+BD^2-2BD.BL\right]^{1/2}$$

$$= \left[\left(AB^2\right)^2+\left(\frac{1}{2}BC\right)^2-2\left(\frac{1}{2}BC\right).\ AB\cos B\right]^{1/2}$$

$$= \left(c^2+\frac{a^2}{4}-ac\cos B\right)^{1/2} = \left(c^2+\frac{a^2}{4}-ac.\frac{a^2+c^2-b^2}{2ac}\right)^{1/2}$$

$$= \left(\frac{2b^2+2c^2-a^2}{4}\right)^{1/2}$$

Distance of E from $AH = EA = b/2$

Distance of F from $AH = FA = c/2$

M.I. of the triangle about AH

$$= \frac{1}{3}m\left(\frac{2b^2+2c^2-a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}\right) = \frac{1}{12}m\left(3b^2+3c^2-a^2\right)$$

$$\therefore \qquad mk^2 = \frac{m}{12}\left[3b^2+3c^2-a^2\right] \Rightarrow k^2 = \frac{3b^2+3c^2-a^2}{12}$$

Hence length of the simple equivalent pendulum

$$= \frac{k^2}{Dist.of\ C.G.\ from\ AH} = \frac{k^2}{AG} = \frac{k^2}{\frac{2}{3}AD}$$

$$= \frac{k^2}{\frac{2}{3}\cdot\frac{1}{2}\left(2b^2+2c^2a^2\right)^{1/2}} = \frac{3k^2}{\sqrt{\left(2b^2+2c^2-a^2\right)}}$$

$$= \frac{3\left(3b^2+3c^2-a^2\right)}{12\sqrt{\left(2b^2+2c^2-a^2\right)}} = \frac{3\left(b^2+c^2\right)-a^2}{4\sqrt{\left(2b^2+2c^2\right)-a^2}}$$

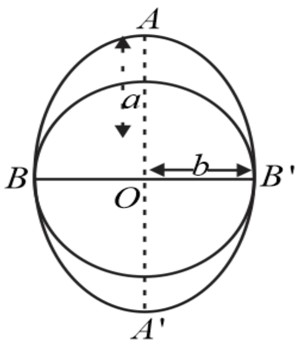
Example:- An ellipse of axis a, b and a circle of radius b are cut from the same sheet of thin uniform metal and are superposed and fixed together with their centres coincident. The figure is free to move in its own vertical plane about one end of the major axis. Show the length of the equivalent simple pendulum is $\frac{5a^2-ab+2b}{4a}$

Solution:- Mass of the circle $\pi b^2 \rho$.

Mass of the ellipse $\pi ab\rho$, where ρ is the mass of the sheet per unit area.

Mass of the system $\pi b^2\rho + \pi ab\rho$

Now taking k to be the radius of gyration of the body about a line through A perpendicular to lamina, we have $\left(\pi ab+\pi b^2\right)\rho k^2$



$$= \pi b^2\rho.\left(a^2+\frac{b^2}{2}\right)+\pi ab\rho\left(a^2+\frac{a^2+b^2}{4}\right)$$

$$\Rightarrow \qquad 4\pi\, b(a+b)\rho k^2 = \pi b^2\rho\left(4a^2+2b^2\right)+\pi ab\rho\left(5a^2+b^2\right)$$

$$\Rightarrow \qquad k^2 = \frac{b\left(2b^2+4a^2\right)+a\left(5a^2+b^2\right)}{4(a+b)} = \frac{5a^3+4a^2b+ab^2+2b^3}{4(a+b)}$$

$$=\frac{5a^2(a+b)-ab(a+b)+2b^2(a+b)}{4(a+b)}$$

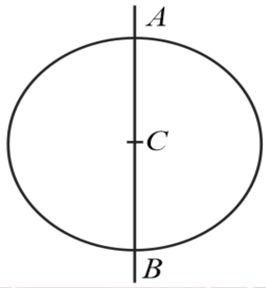
$$=\frac{(a+b)\left(5a^2+ab+2b^2\right)}{4(a+b)}=\frac{1}{4}\left(5a^2-ab+2b^2\right)$$

Hence length of the equivalent simple pendulum

$$=\frac{k^2}{Dist.of\ C.G.of\ the\, system\ from\ A}=\frac{k^2}{a}=\frac{5a^2+ab+2b^2}{4a}$$

Example:- A uniform rod of mass m and length $2a$ can oscillate about a horizontal axis through one end. A circular disc of mass $24m$ and radius $\frac{1}{3}a$ can have its centre clamped to any point of the rod and its plane contains the axis of rotation. Show that for oscillations under gravity the length of the simple equivalent pendulum lies between $(a/2)$ and $2a$.

Solution:- Let AB be the rod axis of rotation pass through A. Let the centre C of the disc, be clamped at a distance x from A.



The distance of C.G. of the system i.e. of the rod and the disc together, $h=\frac{ma+24m.x}{m+24m}=\frac{2a^2+24x^2}{25}$

then if k is the radius of gyration then

$$(m+24m)k^2=m\frac{4}{3}a^2+24m\times\left[\left(\frac{1}{4}.\frac{a}{3}\right)^2+x^2\right]$$

$$\Rightarrow \qquad k^2=\frac{4a^2+2a^2+72x^2}{3\times 2}=\frac{2a^2+24x^2}{25}$$

Hence length of the simple equivalent pendulum

$$l=\frac{k^2}{h}=\left(\frac{2a^2+24x^2}{25}\right)/\left(\frac{a+24x}{25}\right)\Rightarrow l=\frac{2a^2+24x^2}{a+24x} \qquad (1)$$

For maximum of minimum of l . $\frac{d\,l}{d\,x}=0$

$$\Rightarrow \qquad \frac{d\,l}{d\,x}=\frac{48x(a+24x)-24\left(2a^2+24x^2\right)}{(a+24\,x)^2}=0$$

$$\Rightarrow \qquad \left(24x^2+2ax-2a^2\right)=0\Rightarrow 24x^2+8ax-6ax-2a^2=0$$

$$\Rightarrow \qquad 8x(3x+a)-2a(3x+a)=0\ \Rightarrow (3x+a)(8x-2a)=0$$

$\Rightarrow \qquad x=\frac{a}{4}$ or $x=-\frac{a}{3}$. Since $x\neq -\frac{a}{3}$ we have $x=\frac{a}{4}$

When $x=\frac{a}{4}$, we get $l=\frac{a}{2}$.

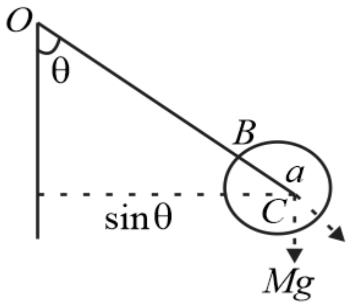
The other extreme value of $l\left(i.e. 2a\right)$ is given by putting $x=0$ or $x=2a$ in (1). Hence the length of the simple equivalent pendulum lies between $\frac{a}{2}$ and $2a$.

Example:- A sphere of radius a is suspended by a fine wire from a fixed point at a distance l from its centre. Show that the time a small oscillation is given by $2\pi\left(\frac{5t^2+2a^2}{5\lg}\right)^{1/2}\left[1+\frac{1}{4}\sin^2\left(\frac{\alpha}{2}\right)\right]$ where α represents the amplitude of the vibration.

Solution:- Suppose that the axis of rotation is passing, through O, where $OC=l$. Moment of inertia of sphere of mass M about the axis of rotation is $M=\left(\frac{2}{5}a^2+l^2\right)$. Equation of motion is

$$M\left(\frac{2}{5}a^2+l^2\right)\ddot{\theta}=-M\,gl\sin\theta$$

$$\Rightarrow \qquad \ddot{\theta}=-\frac{5\,gl}{2a^2+5l^2}\sin\theta$$



Integrating, we get $\dot{\theta}^2=\frac{10\,gl}{2a^2+5l^2}\cos\theta+\lambda$ (1)

Let when $\theta=\alpha,\ \dot{\theta}=0$

Hence (1) reduces to $\dot{\theta}^2=\frac{10gl}{2a^2+5l^2}\left(\cos\theta-\cos\alpha\right)$

$$\Rightarrow \qquad \frac{d\theta}{dt}=-\sqrt{\left(\frac{10gl}{5a^2+5l^2}\right)}\sqrt{\left(\cos\theta-\cos\alpha\right)}$$

($\because$ Sphere is coming in the direction of θ decreasing)

$$=-\sqrt{\left(\frac{10\,gl}{2a^2+5l^2}\right)}\sqrt{\left(1-2\sin^2\frac{\theta}{2}-1+2\sin^2\frac{\alpha}{2}\right)}$$

$$=-\sqrt{\left(\frac{10gl}{2a^2+5l^2}\right)}\ \sqrt{2}.\sqrt{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)}.$$

It t is be the time from one extreme to the lowest point, then

$$t=-\frac{1}{\sqrt{2}}\sqrt{\left(\frac{2a^2+5l^2}{10\,gl}\right)}\int_{\alpha}^{\theta}\frac{d\theta}{\left(\sin^2(\alpha/2)-\sin^2(\theta/2)\right)}$$

$$=\frac{1}{\sqrt{2}}\sqrt{\left(\frac{2a^2+5l^2}{10\,gl}\right)}\int_{0}^{\alpha}\frac{d\,\theta}{\sqrt{\left\{\sin^2(\alpha/2)-\sin^2(\theta/2)\right\}}}$$

Putting $\sin(\theta/2)=\sin(\alpha/2)\sin\phi$, i.e. $\frac{1}{2}\cos\frac{\theta}{2}\,d\theta=\sin\frac{1}{2}\alpha\cos\phi\,d\phi$,we get

$$t=\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_{0}^{\pi/2}\frac{d\,\phi}{\cos(\theta/2)}$$

$$=\sqrt{\left(\frac{2a^2+5l^2}{5\,gl}\right)}\int_{0}^{\pi/2}\frac{d\,\phi}{\left\{\left(1-\sin^2\frac{\alpha}{2}.\sin^2\phi\right)\right\}}$$

$$=\sqrt{\left(\frac{2a^2+5l^2}{5\,gl}\right)}\int_{0}^{\pi/2}\left[1+\frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi+....\right]d\,\phi$$

$$\left[\because(1-x)^{1/2}=1+\frac{1}{2}x+.....\right]$$

$$=\sqrt{\left(\frac{2a^2+5l^2}{5\,gl}\right)}\left[\frac{\pi}{2}+\frac{1}{2}\sin^2\frac{\alpha}{2}.\frac{\pi}{4}+....\right]$$

$$\because\int_{0}^{\pi/2}\sin^2\phi\,d\,\phi=(\pi/4)$$

$=(\pi/2)\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}\right]$ neglecting higher powers of $\sin\frac{\alpha}{2}$, since α is small.

$\therefore$ Time for one small oscillation is $4t=2\pi\sqrt{\left(\frac{2a^2+5l^2}{5\,gl}\right)}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}\right]$

Example:- There equal particles are attached to a weightless rod at equal distances a apart. The system is suspended and is free to tum about a point of the rod distance x from the middle particle. Find the time of a small oscillation and show that particles each of mass $x=82a$ nearly.

Solution:- Let the three particles each of mass m, be attached to the rod at the points A, B and C such that $AB=BC=a$.

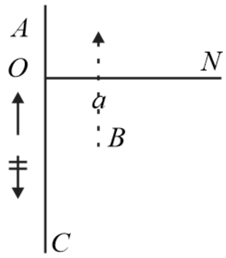
Again let the system rotate about ON such that $OB=x$. Then M.I. of the three particles about ON

$$=m(a-x)^2+mx^2+m(a+x)^2$$

$$\Rightarrow\quad 3mk^2=m(a-x)^2+mx^2+m(a+x)^2$$

$\Rightarrow\quad k^2=\frac{3x^2+2a^2}{3}$, where k is the radius of gyration of the system about ON. Now if l is the length of the equivalent pendulum then we have

$$l=\frac{k^2}{Dist.of\ C.G.of\ the\,system\ from\,O}=\frac{k^2}{x}$$



$$=\frac{3x^2+2a^2}{3x}=x+\frac{2a^2}{3x}$$

$$\therefore \qquad \frac{dl}{dx}=1-\frac{2a^2}{3x^2}$$

For maximum or minimum of l, we have $\frac{dl}{dx}=0$ i.e.

$1-\frac{2a^2}{3x^2}=0 \Rightarrow x=\frac{a}{3}\sqrt{6=.816a=.82a}$ nearly.

Further $\frac{d^2l}{dx^2}=\frac{4a^2}{3x^3}$, which is positive for $x=.82a$

Hence minimum value of l is given by $x=.82a$

Example:- Find the time of oscillation of compounded pendulum consisting of a rod of mass m and length a, carrying at one end a sphere of mass m_1 and diameter $2b$, the other end of the rod being fixed.

Solution:- Let $OA=a$ be the rod of mass m, and a sphere of mass m_1 be attached to it at A.

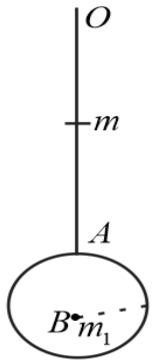
If h is the distance of the C.G. of the system from O, then

$$h=\frac{m.\frac{a}{2}+m_1(a+b)}{m+m_1} \qquad (1)$$

Also if k is the radius of gyration of the system about the axis through O, we have

$$(m+m_1)k^2=m.\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]$$

$$\Rightarrow \qquad k^2=\frac{m\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]}{m+m_1}$$



Hence length of equivalent simple pendulum

$$=\frac{k^2}{h}=\frac{m\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]}{m\frac{a}{2}+m_1(a+b)}\cdot\frac{m+m_1}{m\frac{a}{2}+m_1(a+b)}$$

$$=\frac{m\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]}{m\frac{a}{2}+m_1(a+b)}$$ and the time of complete oscillation is

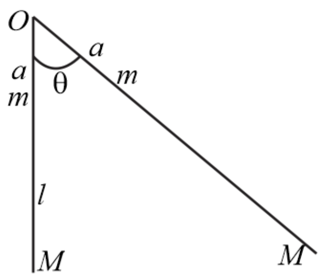
$$=2\pi\left(\frac{k^2}{gh}\right)^{1/2}=\frac{2\pi}{\sqrt{8}}\left[\frac{m\frac{a^2}{3}+m_1\left\{\frac{2}{5}b^2+(a+b)^2\right\}}{m\frac{a}{2}+m_1(a+b)}\right]^{1/2}$$

Example:- A simple circular pendulum is formed of a mass M suspended from a fixed point by a weightless wire of length l, if a mass m, very small compared with M, be knotted on to the wire at a distance from the point of suspension, show that the time of small vibration of the pendulum is approximately diminished by $\frac{m}{2M}\cdot\frac{a}{l}\left(1-\frac{a}{l}\right)$ of itself.

Solution:- Let t be the period of simple pendulum before knotting the mass m, then $t=2\pi\sqrt{\left(\frac{l}{g}\right)}$

Let k be the radius of gyration when mass m is attached to the wire at a distance a from the point of suspension O.

Then $(m+M)k^2=Ml^2+ma^2$ or $k^2=\frac{Ml^2+ma^2}{M+m}$



Distance of C.G. of the system from O is $h=\dfrac{Ml+ma}{M+m}$

If t' be period for the compound pendulum consisting of masses M and m, then

$$t'=2\pi\sqrt{\left(\frac{k^2}{gh}\right)}=2\pi\left[\frac{l}{g}\left(\frac{ml^2+ma^2}{M+m}\cdot\frac{M+m}{Ml+ma}\right)\right]^{1/2}$$

$$=2\pi\left(\frac{Ml^2+ma^2}{g(Ml+ma)}\right)^{1/2}=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{ma^2}{Ml^2}\right]^{1/2}\left[1+\frac{ma}{Ml}\right]^{-1/2}$$

$$=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{ma^2}{2Ml^2}\right]\left[1-\frac{ma}{2Ml}\right] \text{ neglecting higher powers of } \frac{m}{M}.$$

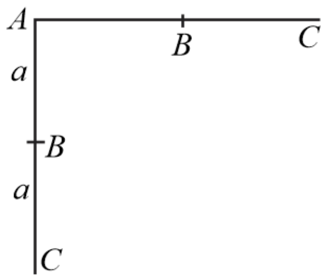
$$=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1-\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)\right]=t\left[1-\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)\right]$$

$$\Rightarrow \qquad t-t'=\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)t.$$

Example:- A weightless straight rod ABC of length $2a$ is movable about the end A which is fixed and carries two particles of the same mass, one fastened to the middle point B and the other to the end C of the rod. If the rod be held in a horizontal position and then let go, show that its angular velocity when vertical is $\left(\dfrac{6g}{5a}\right)^{1/2}$ and that $\dfrac{5a}{3}$ is the length of the simple equivalent pendulum.

Solution:- Let v, v' be the velocities of the masses at B and C when in vertical position. Let ω be the angular velocity of the rod in this position.

Then we have energy equation as $\dfrac{1}{2}mv^2+\dfrac{1}{2}mv'^2=mg.a+mg.2a$



Also $v=a\omega$ and $v'=2a\omega$

$$\therefore \qquad \frac{1}{2}m\left(a^2+4a^2\right)\omega^2=mga+2mga$$

$$\Rightarrow \qquad \omega=\left(\frac{6g}{5a}\right)^{1/2}$$

Again $(m+m)k^2=ma^2+m(2a)^2 \Rightarrow k^2=\dfrac{5a^2}{2}$

Distance of C.G. from A

$$h = \frac{m.a + m.2a}{m+m} = \frac{3a}{2} \quad \therefore\ l = \frac{k^2}{h} = \frac{\frac{5a^2}{2}}{\frac{3a}{2}} = \frac{5a}{3}$$

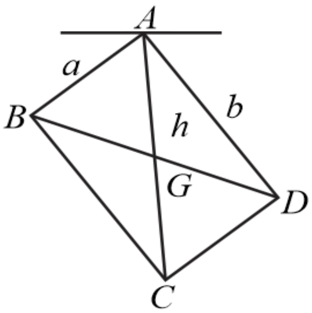
Example:- A rectangular plate swings in a vertical plane about one of its comers. If its period is one second, find the length of the diagonal.

Solution:- Let k be the radius of gyration of the plane about the axis, through A and perpendicular to its plane; then we have $mk^2 = m\frac{a^2+b^2}{4.3} + mh^2$ [by parallel axis theorem]

$$= \frac{mh^2}{3} + mh^2 = \frac{4mh^2}{3} \Rightarrow k^2 = \frac{4h^2}{3}$$

$BG = GD$. Further, distance of C.G. from A

$$= AG = h = \frac{1}{2}\sqrt{\left(a^2+b^2\right)}$$



$\therefore$ Period $= 2\pi\sqrt{\left(\frac{k^2}{hg}\right)} = 2\pi\sqrt{\left(\frac{4h^2}{3gh}\right)} = 4\pi\sqrt{\left(\frac{h}{3g}\right)}$

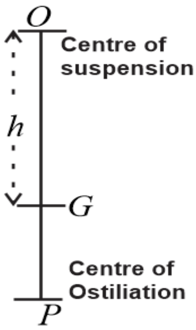
But period $=1 \Rightarrow 4\pi\sqrt{\left(\frac{h}{3g}\right)} = 1$ or $h = \frac{3g}{16\pi^2}$

$\therefore$ Length of the diagonal $= 2h = \frac{3g}{8\pi^2}$.

Example:- A pendulum is supported at O, and P is the centre of oscillation. Show that if an additional weight is rigidly attached at P, the period of oscillation is unaltered.

Solution:- Let m be the mass of the body forming the compound pendulum and let h be the depth of its C.G. below the point of suspension O. Also let k be its radius of gyration about the horizontal axis through O; then we easily obtain $OP = \left(k^2/h\right)$

$\Rightarrow$ Period of Oscillation $= 2\pi\sqrt{\left(\frac{k^2/h}{g}\right)} = T$, say



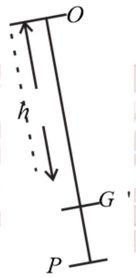
Let an additional weight M be knotted at P, then if k' is the radius of gyration about the horizontal axis through O, we immediately have $(M+m)k'^2 = mk'^2 + M.OP^2$

$$= mk^2 + M\left(\frac{k^2}{h}\right)^2 = k^2\left(m + M\frac{k^2}{h^2}\right) \qquad (1)$$

And by well-known C.G. formula

$$(M+m)h' = mh + M.OP = mh + M.\frac{k^2}{h} = h\left(m + M.\frac{k^2}{h^2}\right) \qquad (2)$$

(1) And (2) $\Rightarrow \dfrac{k'^2}{h^2} = \dfrac{k^2}{h} \Rightarrow T'$



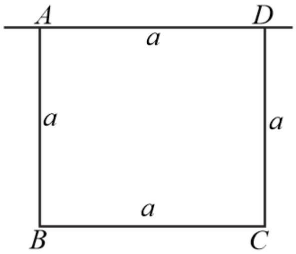
i.e. $2\pi\sqrt{\left(\dfrac{k'^2/h'}{g}\right)} = 2\pi\sqrt{\left(\dfrac{k^2/h}{g}\right)} = T$

$\Rightarrow$ Period of oscillation is unaltered.

Example:- Three uniform rods AB, BC, CD each of length a, are freely jointed at B and C and suspended from the points A and D which are in the same horizontal line and a distance a apart. Prove that when the rods move in a vertical plane, the length of simple equivalent pendulum is $\dfrac{5a}{6}$.

Solution:- The system from a compound pendulum horizontal AD. The figure is self-explanatory. Let m be the mass of the each rod.

Let h be the depth of C.G. of the system from AD and k the radius of gyration of the system about the horizontal axis AD, then we easily obtain $3mk^2 =$ sum of the moments of inertia of three rods about AD.



$$= m\frac{a^2}{3} + m\frac{a^2}{3} + m\,a^2 = \frac{5\,ma^2}{3} \Rightarrow k^2 = \left(5a^2/9\right) \text{ and}$$

$$h = \frac{\left(m\dfrac{a}{2} + m\dfrac{a}{2} + ma\right)}{3m} = \frac{2ma}{3m} = \frac{2a}{3} \qquad (2)$$

$\Rightarrow$ $$\left(k^2/h\right) = \left(5a^2/9\right)/\left(2a/3\right) = \frac{5a}{6}$$

$\Rightarrow$ Length of simple equivalent pendulum $= \dfrac{5a}{6}$.

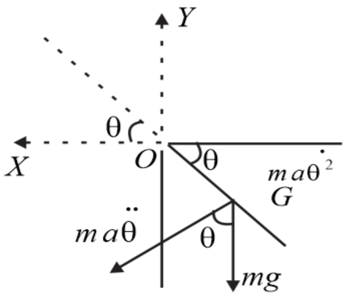
Example:- A thin uniform rod has one end attached to a smooth hinge and is allowed to fall a horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle of 45° to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.

Solution:- Let $OA = 2a$, and the rod make an angle θ with the horizontal after time t. Equations of motion of G of along and perpendicular to GO are

$$ma\dot{\theta} = Y\sin\theta + X\cos\theta - mg\sin\theta \qquad (1)$$

$$ma\ddot{\theta} = -Y\cos\theta + X\sin\theta - mg\cos\theta \qquad (2)$$

Since $k^2 = a^2 + \frac{a^2}{3} = \frac{4}{3}a^2$



$\therefore$ moment equation about O is

$$m,\frac{4}{3}a^2\ddot{\theta} = mg.a = \cos\theta \Rightarrow \ddot{\theta} = \frac{3g}{4a}\cos\theta \qquad (3)$$

Integrating (3), we get $\dot{\theta}^2 = \frac{3g}{2a}\sin\theta + C$ when $\theta = 0,\ \dot{\theta} = 0$ $\therefore C = 0$, $\therefore \dot{\theta}^2 = \frac{3g}{2a}\sin\theta$

Putting this value of $\dot{\theta}^2$ in (1), we get

$$\frac{3}{2}mg\sin\theta = Y\sin\theta + X\cos\theta - mg\sin\theta$$

$$\Rightarrow \quad Y\sin\theta + X\cos\theta = \frac{5}{2}mg\sin\theta \qquad (4)$$

With the help of (3), the equation (2) becomes as

$$-Y\cos\theta + X\sin\theta + mg\cos\theta = \frac{3mg}{4}\cos\theta \qquad (5)$$

Multiplying (4) by $\cos\theta$ and (5) by $\sin\theta$ and adding, we get

$$X = \left(\frac{5}{2} - \frac{1}{4}\right)mg\sin\theta\cos = \frac{9}{4}mg\sin\theta\cos\theta = \frac{9}{8}mg\sin 2\theta$$

Similarly, we have $Y = mg\left(\frac{5}{2}\sin^2\theta + \frac{1}{4}\cos^2\theta\right)$

We observe that X is maximum when $\sin 2\theta = 1$ i.e. when $2\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{4}$

When $\theta = (\pi/4)$, we have $Y = mg\left[\frac{5}{2}\sin^2(\pi/4) + \frac{1}{4}\cos^2(\pi/4)\right]$

$$= mg\left[\frac{5}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2}\right] = \frac{11}{8}mg = \frac{11}{8} \text{ times the weight of the rod.}$$

Example:- A heavy homogenous cube of weight W, can swing about an edge which is horizontal, it starts from rest being displaced from its unstable position of equilibrium. When the perpendicular from the centre of gravity upon the edge has tumed through an angle θ, show that the components of the action at the hinge along and at right angles to this perpendicular are $\frac{1}{2}W(3-5\cos\theta)$ and $\frac{1}{4}W\sin\theta$.

Solution:- Let G_0 be the initial position of C.G. and G be the position of C.G. when the edge has turned through an angle θ.

$$OG_0 = OG = \sqrt{\left(OL^2 + LG_0^2\right)} = \sqrt{\left(a^2 + a^2\right)} = a\sqrt{2}$$

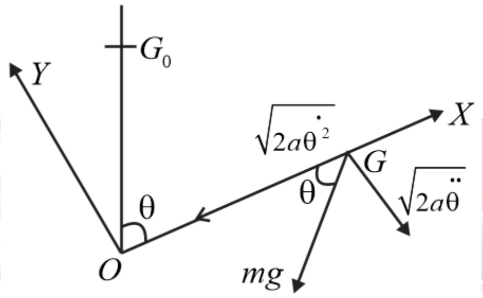
Where $2a$ is the length of the edge.

Equation of motion of G along and perpendicular to GO are

$$Ma\sqrt{2\dot{\theta}^2} = mg\cos\theta - X \qquad (1)$$

And $ma\sqrt{2\ddot{\theta}} = mg\sin\theta - Y \qquad (2)$

Where X, Y are the components of the reaction of the axis in this position.

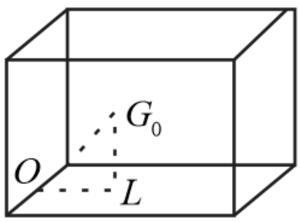


Moment equation about O is $mk^2\ddot{\theta} = mg\,a\sqrt{2}\sin\theta$

$$\Rightarrow \qquad m\left(2a^2 + \frac{2}{3}a^2\right)\ddot{\theta} = \sqrt{2amg\sin\theta} \quad \Rightarrow \ddot{\theta} = \frac{3}{8}\cdot\frac{\sqrt{2}}{a}g\sin\theta \qquad (3)$$

Integrating, we get $\dot{\theta}^2 = -\frac{3}{4a}\sqrt{2g\cos\theta + C}$

Initially $\dot{\theta}^2 = \frac{3\sqrt{2g}}{4a}(1-\cos\theta) \qquad (4)$



From (1) and (4), we have $\frac{3}{2}mg(1-\cos\theta) = mg\cos\theta - X$

$$\Rightarrow \qquad X = mg\left(\frac{3}{2}\cos\theta + \cos\theta - \frac{3}{2}\right) = \frac{mg}{2}(5\cos\theta - 3) = -\frac{mg}{2}(3 - 5\cos\theta)$$

$$= -\frac{1}{2}W(3-5\cos\theta) \qquad [\because mg = W]$$

Where negative sign of X shows its opposite direction.

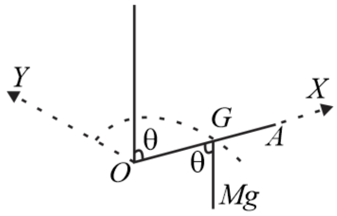
From (2) and (3), we have $\frac{3}{4}mg\sin\theta = mg\sin\theta - Y$

$$\Rightarrow \qquad Y = mg\sin\theta - \frac{3}{4}mg\sin\theta = \frac{1}{4}mg\sin\theta .$$

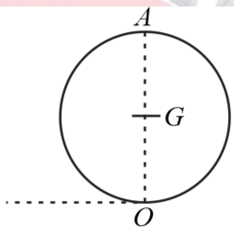
Example:- A circular area can tum freely about a horizontal axis which passes through a point O of its circumference and is perpendicular to its plane. If motion commences when the diameter through O is vertically above, show that when the diameter has tumed through an angle θ the components of the strain at O along and perpendicular to this diameter are respectively $\frac{1}{3}W(7\cos\theta - 4)$ and $\frac{1}{3}W\sin\theta$

Solution:- Initially when the diameter through O is vertically above O

M.I. of the dis about an axis through O perpendicular to the disc $= M\frac{a^2}{2} + Ma^2$



$$= \frac{3Ma^2}{2}$$



If k is the radius of gyration, then $Mk^2 = \frac{3Ma^2}{2} \Rightarrow k^2 = \frac{3a^2}{2}$

After time t, let the diameter OA makes an angle θ with the vertical. In this position we will have $Mk^2\frac{d^2\theta}{dt^2} = M\ gh\sin\theta$ where $h =$ distance of C.G. of the disc from $O = a$.

$$\therefore \qquad Mk^2 = \frac{d^2\theta}{dt^2} = M\ ga\sin\theta \Rightarrow \frac{3a^2}{2}\frac{d^2\theta}{dt^2} = ga\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} = \frac{2a}{3a}\sin\theta .$$

Multiplying by $2\dot{\theta}$ on both sides and integrating it, we get $(d\theta/dt)^2 = -\frac{4g}{3a}\cos\theta + c$.

Initially $\theta = 0, (d\theta/dt) = 0$. $\therefore\ 0 = -\frac{4a}{3a} + a \Rightarrow c = \frac{4g}{3a}$

Hence $\left(\frac{d\theta}{dt}\right)^2 = \frac{4a}{3a}(1-\cos\theta)$ (2)

Now considering the motion of C.G., we have $M\,a\left(\dfrac{d\theta}{dt}\right)^2 = M\,g\cos\theta - X$ (3)

And $M\,a\dfrac{d^2\theta}{dt^2} = M\,g\sin\theta - Y$ (4)

Where X, Y are the components of the reaction and perpendicular to GO.

Solving equation (3), we get $X = Mg\cos\theta - Ma\dfrac{4g}{3a}(1-\cos\theta) \Rightarrow X = \dfrac{Mg}{3}(7\cos\theta - 4)$

$= \dfrac{1}{3}W(7\cos\theta - 4)$ (5)

Similarly, solving equations (1) and (4), we get

$Y = Mg\sin\theta - Ma\dfrac{2g}{3a}\sin\theta = \dfrac{Mg}{3}\sin\theta = \dfrac{1}{3}W\sin\theta$ (6)

Example:- A circular disc of weight W can tum freely about a horizontal axis perpendicular to its plane which passes through a point O on its circumference. If is starts from rest with the diameter vertically above O, show that the resultant pressure on the axis when that diameter is horizontal and vertically below O are respectively $\dfrac{1}{3}\sqrt{(17)}W$ and $\dfrac{11}{3}W$. Further prove that the axis must be able to bear at least $\dfrac{11}{3}$ times the weight of the disc.

Solution:- This equation is a particular case of the previous example.

When the diameter is horizontal $viz\ \theta = \dfrac{\pi}{2}$, we have

$X = \dfrac{W}{3}(0-4) = -\dfrac{4W}{3},\ Y = \dfrac{W}{3}$ $\left(\because \sin\dfrac{\pi}{2} = 1\right)$

Hence resultant pressure in this case $= \sqrt{\left(\dfrac{16}{9}W^2 + \dfrac{W^2}{2}\right)} = \dfrac{W}{3}\sqrt{(17)}$

When the diameter is vertically below

$\theta = \pi,\qquad \therefore X = \dfrac{W}{3}(-7-4) = -\dfrac{11W}{3},\ Y = \dfrac{1}{3}W\sin\pi = 0$

Resultant pressure in this case $= \left\{\left(\dfrac{11W}{3}\right)^2 + 0\right\}^{1/2} = \dfrac{11}{3}W$ in general, we have

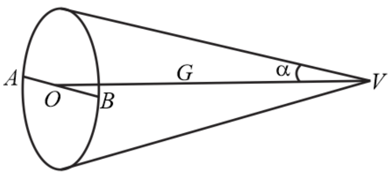
$$\sqrt{(X^2+Y^2)} = \left[\left\{\frac{W}{3}(7\cos\theta - 4)\right\}^2 + \left\{\frac{W}{3}\sin\theta\right\}^2\right]^{1/2}$$

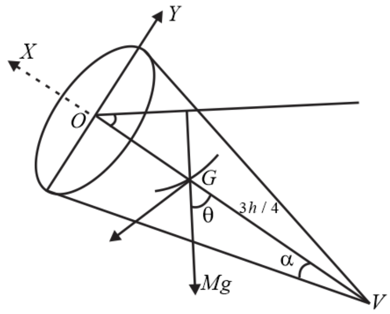
$$= \left\{\frac{W^2}{9}\left(48\cos^2\theta - 56\cos\theta + 17\right)\right\}^{1/2}$$

This is maximum when $\theta = \pi$ and its value is $\dfrac{11}{3}W$, which implies that the maximum pressure, that the axis must be able to bear is at least $\dfrac{11}{3}$ times the weight of the disc.

Example:- A right cone of angle 2α can tum freely about an axis passing through the centre of its base and perpendicular to the axis, if the cone starts from rest with its axis horizontal, show that when the axis is vertical, the thrust on the fixed axis is to the weight of the cone as $1+\frac{1}{2}\cos^2\alpha : 1-\frac{1}{3}\cos^2\alpha$.

Solution:- Let initially the cone be as shown in fig. (i) . After any time t, let the cone take the position as shown in fig. (ii). If the height of the cone i.e. $OV = h$ then $OG = \frac{1}{4}h$ where G denotes the centre of gravity of the cone.





Now since the C.G. of the cone i.e. point G is describing a circle of radius $h/4$, the equations of motion of G are

$$M.\frac{1}{4}h\ddot{\theta} = X - Mg\sin\theta \tag{1}$$

$$M.\frac{1}{4}h\dot{\theta}^2 = Mg\cos\theta - Y \tag{2}$$

Where X and Y denote the components of reaction at O along and perpendicular to OX.

Taking moments about O, we have $Mk^2\ddot{\theta} = Mg\frac{-1}{4}h\cos\theta$ (3)

Also $Mk^2 = M.I.$ of the cone about $AB = M.\frac{1}{20}\left(2h^2+3h^2\tan^2\alpha\right)$

$$\Rightarrow \quad k^2 = \frac{h^2}{20}\left(2+3\tan^2\alpha\right) \tag{4}$$

Substituting this value of k^2 in (3), we gt

$$h\ddot{\theta} = \frac{5}{2+3\tan^2\alpha}g\cos\theta \tag{5}$$

Multiplying both sides by $2\dot{\theta}^2 = \frac{10g}{2+3\tan^2\alpha}\sin\theta + C$

Initially $\dot{\theta} = 0$, when $\theta = 0$, giving there by the constant $C = 0$

Therefore, we have $h\dot{\theta}^2 = \frac{10g}{2+3\tan^2\alpha}\sin\theta$ (6)

Using (6) in (1), we get $M.\frac{1}{4}\frac{10g}{2+3\tan^2\alpha}\sin\theta = X - Mg\sin\theta$

$$\Rightarrow \quad X = Mg\sin\theta\left(\frac{9+6\tan^2\alpha}{4+6\tan^2\alpha}\right)$$

Also using (5) in (2), we get $Y = M\,g\cos\theta\left[\dfrac{3+6\tan^2\alpha}{8+12\tan^2\alpha}\right]$

When the axis is vertical i.e. when $\theta = \pi/2$, we have $X = M\,g\left(\dfrac{9+6\tan^2\alpha}{4+6\tan^2\alpha}\right)$, $Y=0$

$\therefore$ Resultant pressure $=\sqrt{\left(X^2+Y^2\right)} = X = Mg\left(\dfrac{9+6\tan^2\alpha}{4+6\tan^2\alpha}\right) = M\,g\left(\dfrac{9\cos^2\alpha+6\sin^2\alpha}{4\cos^2\alpha+6\sin^2\alpha}\right)$

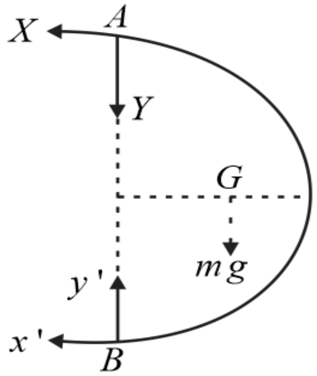
$$\Rightarrow \quad \frac{X}{Mg} = \frac{6+3\cos^2\alpha}{6-2\cos^2\alpha} = \frac{1+\frac{1}{2}\cos^2\alpha}{1-\frac{1}{3}\cos^2\alpha}$$

Note:- If $2\alpha = \pi/2$ then in that case, we have

$$\frac{X}{Mg} = \frac{1+\frac{1}{2}\cos^2(\pi/4)}{1-\frac{1}{3}\cos^2(\pi/4)} = \frac{1+\frac{1}{4}}{1-\frac{1}{6}} = \frac{3}{2}$$

Example:- A uniform semi-circular arc, of mass m and radius a, is fixed at its ends to two points in the same vertical line and is rotating with constant angular velocity ω. Show that the horizontal thrust on the upper end is $m\dfrac{g+\omega^2 a}{\pi}$

Solution:- Let the uniform semi-circular arc with centre at O rotate about AB with constant angular velocity ω. If G is the C.G. of the arc, then $OG = \dfrac{2a}{\pi}$ As the arc rotate, the point G will describe a circle of radius $\dfrac{2a}{\pi}$ about the point O



Let X and Y be the horizontal and vertical components of reactions at the point A and X' and Y' the horizontal and vertical reactions at the lower end B. Now since the arc is rotating with constant angular velocity ω about AB, the only effective force on it is $m\dfrac{2a}{\pi}\omega^2$ along GO

Taking moments about the point B, we have $m\dfrac{2a}{\pi}\omega^2 a = -mg\dfrac{2a}{\pi} + X.\,2a$

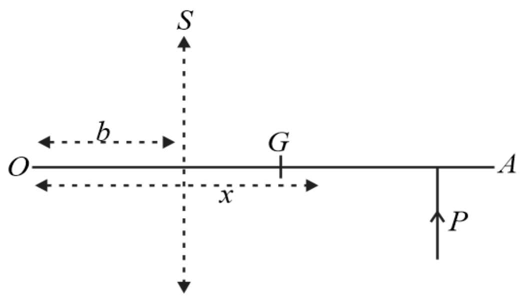
[$\because$ Moment of the effective forces $=$ moment of external forces]

$$\Rightarrow \qquad X = w\frac{\left(g + a\omega^2\right)}{\pi}.$$

Example:- A uniform rod OA of mass M and length $2a$ rests on a smooth table and is free to tum about a smooth pivot at its and O; in contact with it at distance b from O is an inelastic particle of mass m, a horizontal blow of impulse P, is given to the rod at a distance x from O in a direction perpendicular the impulsive action at O and on the particle.

Solution:- Let OA be the rod of length $2a$ and let a horizontal blow of impulse P be given at a distance x from O. Further let S be the impulse of the action between the rod and inelastic particle of mass m Then the moment equation about A is $M\frac{4}{3}a^2 = Px - Sb$.

But $S = mb\omega$. (since velocity $b\omega$ is generated in mass m by the impulse S)



$$\therefore \qquad M\frac{4}{3}a^2\omega = Px - mb^2\omega$$

$$\Rightarrow \qquad \omega = \frac{Px}{\frac{4}{3}Ma^2 + mb^2} \text{ and } S = \frac{mPbx}{\frac{4}{3}Ma^2 + mb^2}.$$

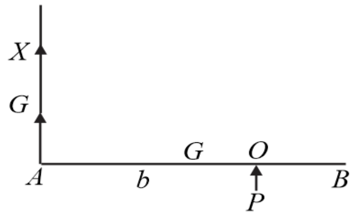
Now since the change in the motion of C.G. of the rod is the same as if all the impulsive forces were applied there, so $Ma\omega = P - S - X$ where X is the impulsive action at O

$$\therefore \qquad X = P - (Ma + mb)\omega = P\left[1 - (mb + Ma)x\right] / \left(M\frac{4}{3}a^2 + mb^2\right)$$

Example:- A rod, of mass m and length $2a$, which is capable of free motion about one end A falls from a vertical position and when it is horizontal strikes a fixed inelastic obstacle at a distance b from the end A. Show that the impulse of the blow is $m\frac{2a}{b}\sqrt{(2ga/3)}$ and that the impulse of the reaction at A is $m\sqrt{(3ga/2)}\left[1 - \frac{4a}{3b}\right]$ vertically upwards.

Solution:- If ω is the angular velocity just before striking the obstacle then we have the energy equation as $\frac{1}{2}m.\frac{4}{3}a^2\omega^2 - 0 = mga$

[Change of K.E. = work done] $\quad \therefore \omega = \sqrt{(3g/2a)}$



Let the rod AB strike the inelastic obstacle at O such that $AO = b$ and the impulse of the blow be P and the impulsive reaction at A be X. Since the rod reduces to rest after striking the obstacle, therefore we get on taking moment about A.

$$m\frac{4}{3}a^2\left(0-\omega\right) = -Pb$$

$$\Rightarrow \qquad P = \frac{4ma^2\omega}{3b} = \frac{2a}{b}m \,.\, \sqrt{\left(2ga/3\right)}$$

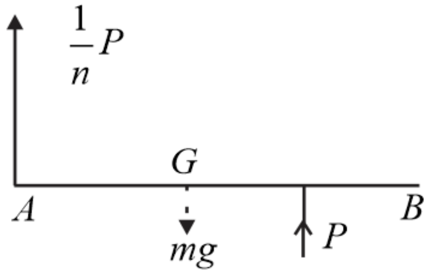
Also for G, we have $m\left(0-a\,\omega\right) = -P - X \Rightarrow X = m\sqrt{\left(3ga/2\right)}\left[1-\frac{4a}{3b}\right]$

Example:- A uniform beam AB can tum about its end A is the equilibrium; find the point of its length where a blow must be applied to it so that the impulses at A may be in each case $\frac{1}{n}th$ of that of the blow.

Solution:- Let AB be the uniform rod of mass m and length $2a$. Let an impulse P and applied at a distance x from A so as to produce an impulsive action $\frac{1}{n}P$ at A. If the angular velocity produced is ω, then the equation of motion are

$$mk^2\,\omega = P\,x \Rightarrow m\frac{4}{3}a^2\omega = P\,x \qquad (1)$$

And $$m\,a\,\omega = P + \frac{1}{n}P = \frac{n+1}{n}\,P \qquad (2)$$

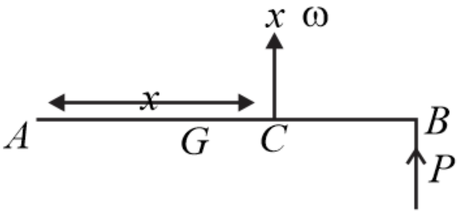


Eliminating P from these two equations, we get $x = \frac{4}{3}\left(\frac{n+1}{n}\right)a$.

Note:- If the direction of the impulsive action is opposite to that as shown in the fig. then in that case we will have $x = \frac{4}{3}\left(\frac{n-1}{n}\right)a$.

Example:- A rod of mass $x = \frac{4}{3}\left(\frac{n-1}{n}\right)a$ s lying in a horizontal table and has one end fixed; a particle of mass M is in contact with it. The rod receives a horizontal blow at its free end; find the position of the particle so that it may start moving with the maximum velocity. In this case show that the kinetic energies communicated to the rod and mass are equal.

Solution:- Let AB be the rod, the end A of which is fixed. Let an impulse P be applied to the rod at the and B so as to give an angular velocity ω, if the particle of mass M is at C where $AC = x$ then the velocity V acquired by the particle will be $V = x\omega$. Thus we get the moment equation as $nM\,\frac{4}{3}a^2\,\omega + M\,x\omega . x + P.\,2a$



$$\Rightarrow \qquad \omega = \frac{2aP}{M\,\frac{4}{3}\left(a^2 n + x^2\right)}. \qquad \therefore\ V = x\omega = \frac{2a\,Px}{M\,\frac{4}{3}\left(na^2 + x^2\right)}$$

For maximum V, we must have $\frac{dV}{dx} = 0 \Rightarrow \frac{2aP}{M}\left[\frac{\frac{4}{3}ma^2 + x^2 - 2x^2}{\left(\frac{4}{3}na^2 + x^2\right)^2}\right] = 0$

$$\Rightarrow \frac{4}{3}na^2 - x^2 = 0 \Rightarrow x = 2a\sqrt{(n/3)}$$

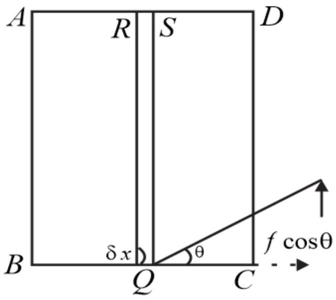
Also K.E. of the rod $= \frac{1}{2}nM\,.\frac{4}{3}nM.\,\frac{4}{3}a^2\omega^2 = \frac{2}{3}nM\,a^2\omega^2$ (1)

And K.E. of the particle $= \frac{1}{2}M\,x^2\,\omega^2 = \frac{1}{2}M\,\frac{4a^2 n}{3}\omega^2 = \frac{2}{3}M\,a^2\,\omega^2$ (2)

From (1) and (2), we observe that kinetic energies of the rod and mass are equal.

Example:- The door of a railway carriage stands upon at right angles to the length of the train when the latter starts to move with an acceleration f; then door being supposed to the smoothly hinged to the carriage and to be uniform and of breadth $2a$. Show that its angular velocity when it has tumed through an angle θ is $\sqrt{\left(\frac{3f}{2a}\sin\theta\right)}$

Solution:- Let $ABCD$ be the door which can rotate about AB. If the train moves with acceleration f, then every element of the door will have the same acceleration f parallel to the rails. Now consider an elementary strip $PQRS$ at a distance x from AB. Mass of the strip $= \frac{M}{2a}\delta x$, where M is the mass of the door. Hence moment equation about AB give $M\,\frac{4}{3}a^2\,\ddot{\theta} = \int_0^{2a}\frac{m}{2a}dx\,f\cos\theta . x = m\,a\,f\cos\theta$



$\Rightarrow \qquad \ddot{\theta} = \dfrac{3f}{4a}\cos\theta$

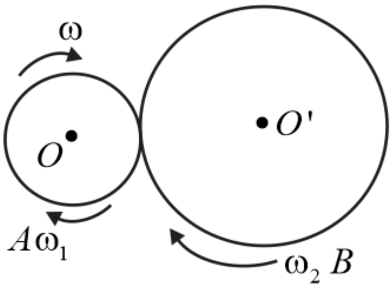
Multiplying both sides by $2\dot{\theta}$ and integrating it, we get $\dot{\theta}^2 = \dfrac{3f}{2a}\sin\theta + \lambda$. Initially $\dot{\theta} = 0$ when $\theta = 0 \qquad \therefore \lambda = 0$

Hence $\dot{\theta} = \sqrt{\left(\dfrac{3f}{2a}\sin\theta\right)}$

Example:- Two wheels on spindles in fixed bearings suddenly engage so that their angular velocities become inversely proportional to their radii and in opposite directions. One wheel, of radius a and moment of inertia I_1 has angular velocity ω initially, the other of radius b and moment of inertia I_2 is initially at rest. Show that their new angular velocities are $\dfrac{I_1 b^2}{I_1 b^2 + I_2 a^2}\omega$ and $\dfrac{I_1 ab\omega}{I_1 b^2 + I_2 a^2}$

Solution:- Let A and B be the two wheels. The wheel A is of radius a and moment of inertia I_1 whereas the wheel B is of radius b and moment of inertia I_2. Initially A was rotating with angular velocity ω and the wheel B at rest. Now let ω_1 ad ω_2 be the angular velocity of A and B after the impact. Since the velocity of the point of contact is the same for each wheel, we have $a\omega_1 = b\omega_2$

(1)



Also $I_1(\omega - \omega_1) = R \times a$ (for the wheel A) (2)

$I_2(\omega - 0) = R \times b$ (for the wheel B) (3)

Where R is the impulsive force.

From the last two equations, we get $I_1(\omega - \omega_1)b = I_2 a\omega_2$ (4)

Now substituting the value of ω_2 from (1) in (4), we get

$$\omega_1 = \frac{I_1 b^2}{I_1 b^2 + I_2 a^2}\omega$$

Substituting the value of ω_1 in (1), we have $\omega_2 = \dfrac{I_1 a b}{I_1 b^2 + I_2 a^2}\omega$

Example:- A pendulum is constructed of a solid sphere of mass M and radius a which is attached to the end of a rod of mass m and length b. Show that there will be no strain on this axis if the pendulum be struck at a distance $\left[M.\left\{\frac{2}{5}a^2 + (a+b)^2\right\} + \frac{1}{3}mb^2\right] + \left[M(a+b) + \frac{1}{2}mb\right]$ from the axis

Solution:- Let $OA = b$ be the rod fixed at the point O. Let a sphere of radius a and mass M be attached to the other end A of the rod.

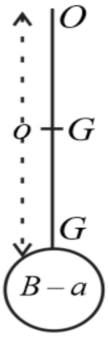
Distance of the C.G. of the pendulum from O

$$h = \frac{m(b/2) + M(b+a)}{m+M} \qquad (1)$$

Let k be the radius of gyration of the pendulum about O, then we have

$$(m+M)k^2 = M\left[(b+a)^2 + \frac{2}{5}a^2\right] + \frac{4}{3}m\left(\frac{b}{2}\right)^2$$

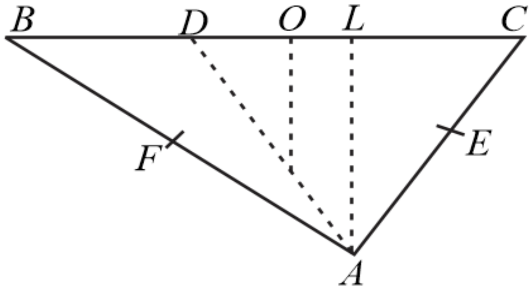
$$\Rightarrow \quad k^2 = \frac{M}{m+M}\left[(b+a)^2 + \frac{2}{5}a^2\right] + \frac{4m}{3(m+M)}\left(\frac{k}{2}\right)^2$$



$\therefore$ Distance of centre of percussion from $O = \dfrac{k^2}{h}$

$$= \frac{M\left[\frac{2}{5}a^2 + (a+b)^2\right] + \frac{1}{3}mb^2}{\frac{1}{2}mb + M(a+b)}$$

Example:- Find the centre of percussion of a triangle ABC which is free to move about its side BC.
Solution:- To find the point where BC is a principle axis. Let us proceed like this. Draw AD, the median and AL the perpendicular from A on BC. Let O be the mid-point of DL. Then by the elementary knowledge of M.I. and P.I BC is a principle axis is point O. Let the mass of the ΔABC be m. The triangle of mass m is kinetically equivalent to the particles each of mass $\frac{m}{3}$ placed at the mid point D, E and F. Let $AL = P$, then



$$mk^2 = \frac{m}{3}\left(\frac{1}{2}p\right)^2 + \frac{m}{3}\left(\frac{1}{2}p\right)^2 + \frac{m}{3}(0) = \frac{1}{6}m\,p^2 \Rightarrow k^2 = \frac{1}{6}p^2$$

But the depth of C.G. below $BC = h = \frac{1}{3}p$.

Hence depth of the centre of percussion below BC along a vertical through $O = \left(k^2/h\right) = \frac{1}{2}p$.

Particular Case:- If the triangle ABC is an equilateral triangle, then the point D and O coincide. In this case $k^2 = \frac{1}{6}p^2$. $h = \frac{1}{3}p$

Hence the depth of the centre of percussion below BC along the median bisecting BC is $= \frac{k^2}{h} = \frac{1}{2}p$

Example:- Find how an equilateral lamina must be struck that it may commences to rotate about a side.
Solution:- refer fig. above example. The triangle ABC rotate about the side BC. The blow should be given at the centre of percussion when BC is the axis of rotation of the lamina. Here BC is the principle axis of triangle at its middle point (Points D, O, L will coincide)

Again $k^2 = \frac{1}{6}p^2$; $h = \frac{1}{3}p$ where p is the height of the triangle.

$\therefore$ Depth of the centre of percussion below BC along the median bisecting BC is $\frac{k^2}{h}$ i.e. $\frac{1}{2}p$.

Hence the blow should be given at the middle point of the median bisecting the side about which the lamina rotates.

Example:- Find the position of the centre of percussion of a sector of a circle axis in the plane of the sector, perpendicular to its symmetrical radius and passing through the centre of the circle.
Solution:- Consider the sector AOB of a circle of radius a. Let $\angle AOB = 2\alpha$

Let a line OY perpendicular to the plane of the sector be the axis of rotation.

Then M.I. of the sector AOB about $OY = 2\int_0^{\alpha}\int_0^{\alpha} \rho r^2 \cos^2\theta \, r \, d\theta \, dr$

$$= \rho \, . \frac{a^4}{4}\int_0^{\alpha}(1+\cos 2\theta)\, d\theta = \rho . \frac{a^2}{4}(\alpha + \sin\alpha \cos\alpha)$$

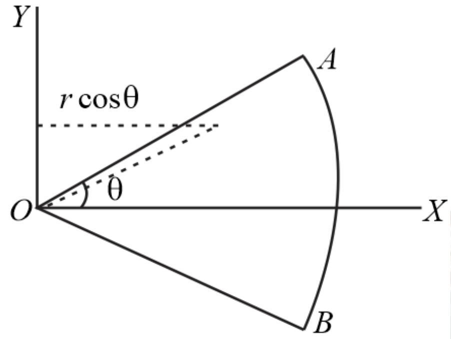
$= \dfrac{M a^2}{4\alpha}(\alpha - \sin\alpha \, \cos\alpha)$, since mass of the sector $M = \rho . \alpha a^2$

$$\therefore \qquad M k^2 = \frac{M a^2}{4\,\alpha}(\alpha + \sin\alpha\cos\alpha) \Rightarrow k^2 = \frac{a^2}{4\alpha}(\alpha + \sin\alpha \cos\alpha)$$

Distance of C.G. from $O = h = \dfrac{2a}{3} . \dfrac{\sin\alpha}{\alpha}$

Hence distance of centre of percussion from O

$$= \frac{k^2}{h} = \frac{3a}{8}\left(\frac{\alpha + \sin\alpha\cos\alpha}{\sin\alpha}\right)$$



Chapter-3: Motion in 2 D

Dynamical Equations of Motion. To determine dynamical equations of motion in two dimensions when the forces acting on the body are finite. The motion of a rigid body consists of two independent motions viz.,

(i) the motion of centre of gravity, and

(ii) the motion about the centre of gravity.

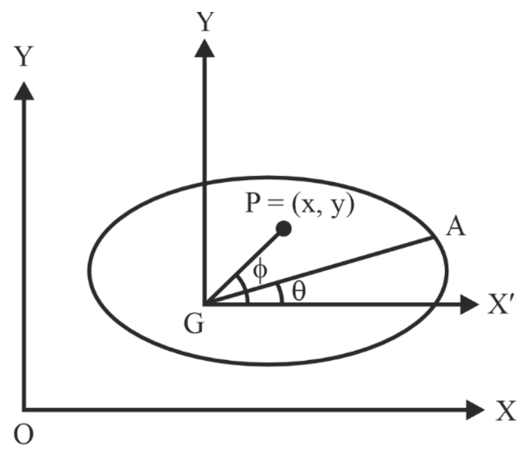
Motion of Centre of Gravity.

Cartesian Method

Cartesian Method

Motion of C.G. states that the motion of centre of gravity is such that the total mass M of the rigid body is allowed to act at the C.G. and all the external forces are transferred parallel to themselves to act at the C.G. of the body.

Consider a particle m of the rigid body at point P whose coordinates referred to two axes fixed in space of two dimensions, OX, OY are (x, y). Now the effective forces acting on the particle are $m\ddot{x}$ and $m\ddot{y}$, let X, Y be the components of the external forces acting at P. By D' Alembert's principle $(X - m\ddot{x}), (Y - m\ddot{y})$ together with similar forces acting on all other particles of the body form a system in statical equilibrium, thus we have



$$\Sigma(X - m\ddot{x}) = 0, \Sigma(mY - m\ddot{y}) = 0$$

and $$\Sigma[x(Y - m\ddot{y}) - y((X - m\ddot{x})] = 0$$

and $$\left.\begin{aligned}&\Rightarrow \Sigma m\ddot{x} = \Sigma X,\ \Sigma m\ddot{y} = \Sigma Y \\ &\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX)\end{aligned}\right\} \qquad \ldots(1)$$

Let (x_G, y_G) be the co-ordinates of the centre of gravity refered to axes OX and OY and (x', y') be the co-ordinates of the point P referred to parallel axes GX' and GY' through G.

$\therefore x = x_G + x', y = y_G + y$

then $Mx_G = \Sigma mx, My_G = \Sigma my(\text{ where } \Sigma m = M)$

$\Rightarrow \ddot{x}M_G = \Sigma m\ddot{x}$ and $M\ \ddot{y}_G = \Sigma m\ddot{y}$.

Thus the first two equations of (1) reduces to

$M\ddot{x}_G = \Sigma X$ and $M\ddot{y}_G = \Sigma Y$

Motion Relative to Centre of gravity …(2)

Third equation of (1) gives

$$\Sigma m\left[(x_G + x')(\ddot{y}_G + \ddot{y}') - (y_G + y')(\ddot{x}_G + \ddot{x}')\right]$$

$$= \Sigma\left[(x_G + x')Y - (y_G + y')X\right]$$

Or $$(x_G\ddot{y}_G - y_G\ddot{x}_G)\Sigma m + x_G\Sigma m\ddot{y}' + \ddot{y}_G\Sigma mx'$$

$$-y_G\Sigma m\ddot{x}' - \ddot{x}_G\Sigma my' + \Sigma m(x'\ddot{y}' - y'\ddot{x}')$$

$$= x_G\Sigma Y - y_G\Sigma X + \Sigma(x'Y - y'X)$$

Where $\Sigma m = M$ …(3)

By (2), first term on L.H.S. of (3) cancels the first two terms on the R.H.S. of (3).

Again $\dfrac{\Sigma mx'}{\Sigma m}$ and $\dfrac{\Sigma my'}{\Sigma m}$ give the coordinates of G with respect to axes GX' and GY'

i.e. $\Sigma \cdot mx' = 0, \Sigma my' = 0 \Rightarrow \Sigma m\ddot{x}' = 0, \Sigma m\ddot{y}' = 0$

Thus (3) reduces to

$$\Sigma m(x'\ddot{y}' - y'\ddot{x}') = \Sigma(x'Y - y'X) \qquad \ldots(4)$$

$$\frac{d}{dt}\Sigma m(x'\dot{y}' - y'\dot{x}') = \Sigma(x'Y - y'X) \qquad \ldots(5)$$

Let GA be a line fixed on the body which makes an angle θ with GX Let $GP = r$ and $\angle PGX' = \phi$.

$\phi = \theta + \angle AGP$.

Since the body turns about $G, \angle AGP$ remains constant.

θ

Again the velocity of m at point P is $r\dot{\phi}$ perpendicular the GP, its moment about G is $\dot{r}\dot{\phi}.\ r = r^2\dot{\phi}$

$\therefore \Sigma m\left(x'\dot{y}' - y'\dot{x}'\right) = \Sigma m r^2\dot{\phi}$

Or $\quad \Sigma m\left(x'\dot{y}' - y'\dot{x}'\right) = \Sigma m r^2\dot{\theta} = \dot{\theta} \quad \Sigma m r^2 = Mk^2\dot{\theta}$

where Mk^2 is the moment of inertia of the body about G. Hence equation (5) may be put as

$$\frac{d}{dt}\left(Mk^2\theta\right) = \Sigma\left(x'Y - y'X\right) \text{ or } Mk^2\ddot{\theta} = L \qquad \ldots(6)$$

where L is the moment of the external forces about G.

Thus the equations of motion of the body are $M\,\ddot{x}_g = \Sigma X, M\,\ddot{y}_g = \Sigma Y$ and are known as equations of motion of the centre of gravity

And $\quad \Sigma m\left(x\ddot{y}' - y\ddot{x}'\right) = \Sigma\left(x'Y - y'X\right)$

known as equation of motion about the centre of gravity, this can also be put as $Mk^2\ddot{\theta} = L$

where L is the moment of external forces about G.

This states that the sum of the moments, of the effective forces about the centre of gravity G, is equal to the sum of the moments of the external forces about G.

Vector Method.

Let r_G be the position vector of the C.G. and F the external forces acting at any particle m of the body, then we have $M\dfrac{d^2 r_G}{dt^2} = \Sigma F$.

But $\quad r_G = x_G i + y_G j$ and $F = Xi + Yj$

where (x_g, y_g) are the co-ordinates of C.G. and X, Y the components of the forces F parallel of the axes.

$\therefore$ (1) gives; $M\left[\frac{d^2x_G}{dt^2}i+\frac{d^2y_G}{dt^2}j\right]=\Sigma(Xi+Yj)$.

Equating coefficients of i and j on both sides, we get

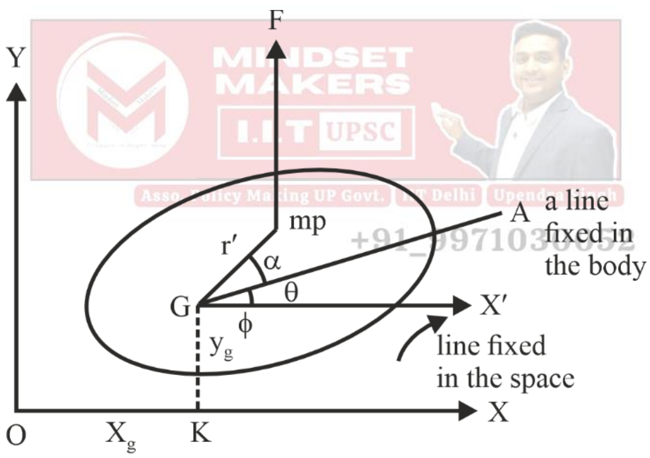
$$M\frac{d^2x_G}{dt^2}=\Sigma X \qquad \ldots(2) \quad \text{and} \quad M\frac{d^2y_G}{dt^2}=\Sigma Y \qquad \ldots(3)$$

These are the equations of the centre of gravity.

Let $\mathbf{r}'$ be the position vector, of the particle *m* at *P*, relative to *G*, and *F* the external forces acting on it, then we have

$$\Sigma r'\times\frac{d^2r'}{dt^2}=\Sigma r'\times F\Rightarrow\frac{d}{dt}\Sigma mr'\times\frac{dr'}{dt}=\Sigma r'\times F \qquad \ldots(4)$$

Now let θ be the angle that a line *GA* fixed in the body makes with a line *GB* fixed in the space, and let ϕ be the angle which the line joining *P* to *G* makes with the line *GB* (fixed in the space), then as obvious from the adjoining figure, we have $\phi=\theta+\angle AGP=\theta+\alpha$.



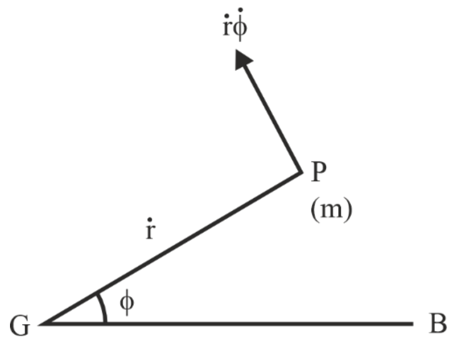
$\therefore \frac{d\phi}{dt}=\frac{d\theta}{dt}$, $\qquad [\because \angle AGP=\alpha$ is constant]

Let $Gm=r'$

$\therefore$ The velocity of *m* relative to *G*

$=r'\frac{d\phi}{dt}$ in a direction perpendicular to r' in the plane *AGP*.

If $\hat{e}_1,\hat{e}_2$ be the unit vectors along and perpendicular to r' in the *AGP* plane, then we have $r'=r'\hat{e}_1$ and $\frac{dr'}{dt}=r'\frac{d\phi}{dt}\hat{e}_2$



$$\Rightarrow \Sigma mr' \times \frac{dr}{dt} = \Sigma m(r'\hat{e}_1) \times r' \frac{d\phi}{dt}\hat{e}_2$$

$$= \Sigma mr'^2 \frac{d\theta}{dt}\hat{e}_1 \times \hat{e}_2 \quad \left[\because \frac{d\phi}{dt} = \frac{d\theta}{dt} + \frac{d\alpha}{dt} = \frac{d\theta}{dt}\right]$$

$= \frac{d\theta}{dt}\Sigma mr'^2\hat{n}$ where $\hat{n}$ is the unit vector normal to the plane AGP.

$$= \frac{d\theta}{dt}\left(\Sigma mr'^2\right)\hat{n}$$

$= \frac{d\theta}{dt}\left(Mk^2\right)n$ where k is the radius of gyration of the body about $G = \left(Mk^2\frac{d\theta}{dt}\right)\hat{n}.$

Also we have, moment of the forced F about $G = \Sigma r' \times F$

$= \Sigma p'F\hat{n}$ where p is the lwength of the perpendicular from G upon the direction of the force F

$\therefore$ Equation (4) reduces to $\frac{d}{dt}\left(Mk^2\frac{d\theta}{dt}\right)\hat{n} = \left(\Sigma p'F\right)\hat{n}$ ...(5)

Equating coefficients of $\hat{n}$ on both sides, we get

$\frac{d}{dt}\left(Mk^2\frac{d\theta}{dt}\right) = \Sigma p'F$ ...(6) $\Rightarrow Mk^2\frac{d^2\theta}{dt^2} = \Sigma p'F$ ...(7)

Let (x', y') be the co-ordinates of P relative to G and X, Y the components of F in the directions of the axes, scalar moment, of the force F about G is $p'F$ which is equivalent to $(x'Y - y'X)$

$\therefore$ (6th) equation may be written as $\frac{d}{dt}\left(Mk^2\frac{d\theta}{dt}\right) = \Sigma\left(x'Y - y'X\right)$

$\Rightarrow Mk^2\frac{d^2\theta}{dt^2} = \Sigma\left(x'Y - y'X\right).$ ...(8)

Equations (2), (3) and (7) are the dynamical equations of motion of rigid body moving in two dimensions, under finite forces.

3.02. Kinetic Energy. When a body is moving in two dimensions, then to express the kinetic energy in terms of the motion of the centre of inertia and the motion relative to the centre of inertia.

At any time *t*, let r_G be the position vector of the centre of gravity of G of the rigid body, referred to an origin O; and let r be the position vector of a particle *m*, referred to an origin *O*, then we have

$$r = r_G + r'$$

where r' is the *p.v* of the particle of mass *m* w.r.t. C. G. Now let *T* be the kinetic energy' of the body, then we get

$$T = \frac{1}{2}\Sigma m\dot{r}^2 \qquad \ldots(1) \qquad = \frac{1}{2}\Sigma m\left(\dot{r}_G + \dot{r}'\right)^2$$

$$= \frac{1}{2}\Sigma m\dot{r}_G^{\ 2} + \frac{1}{2}\Sigma m\dot{r}'^2 + \Sigma m\dot{r}_G \cdot \dot{r}'$$

$$= \frac{1}{2}\dot{r}_G^{\ 2}\Sigma m + \frac{1}{2}\Sigma m\dot{r}'^2 + \dot{r}_G \cdot \Sigma m\dot{r}'$$

But $\dfrac{\Sigma mr'}{\Sigma m} = 0,$

[$\because$ r' is the position vector of the centroid relative to the centroid itself.]

$\therefore \qquad \Sigma mr' = 0,\ and\ so\, \Sigma m\dot{r}' = 0,$

$$\therefore \qquad T = \frac{1}{2}M\dot{r}_g^{\ 2} + \frac{1}{2}\Sigma m\dot{r}'^2 \ [\because \quad \Sigma m = M] \qquad \ldots(2)$$

Another form. Let v_G be the velocity of centre of gravity and let $\hat{e}_2$ be the unit vector perpendicular to the direction of r' then we readily obtain

$$v_G = \frac{dr_g}{dt} = \dot{r}_G$$

And $\quad \dot{r}'^2 = \left(r'\dfrac{d\phi}{dt}\hat{e}_2\right)^2 = r'^2\left(\dfrac{d\theta}{dt}\right)^2 \left[\therefore \dfrac{d\phi}{dt} = \dfrac{d\theta}{dt}\, and\ \hat{e}_2^2 = \hat{e}_2 \cdot \hat{e}_2 = 1\right]$

$$\therefore (2) \Rightarrow T = \frac{1}{2}M\mathbf{v}_G^{\ 2} + \frac{1}{2}\Sigma mr^2\left(\frac{d\theta}{dt}\right)^2$$

$$=\frac{1}{2}Mv_G{}^2+\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\Sigma mr'^2 \qquad \left[\because \quad v_G^2=v_G^2\right]$$

$$=\frac{1}{2}Mv_G{}^2+\frac{1}{2}Mk^2\left(\frac{d\theta}{dt}\right)^2 \qquad \ldots(3)$$

where k is the radius of gyration of the body about the centre of intertia.

Hence equation (3) expresses that;

The total kinetic energy of a rigid body moving in two dimensions is equal to the kinetic energy of a particle of mass M placed at the centre of inertia and moving with it together with the kinetic energy of the body relative to the centre of inertia.

Equation (3) can also be put as

K.E. of the body = (K.E. due to translation) + (K.E. due to rotation) ...(4)

3.03. Moment of the Momentum. *To find the moment of momentum of the body about the fixed origin O, when the body is moving in two dimensions.*

At any time *t*, let r_G be the position vector of the centre of gravity *G* of the body referred the origin *O*, and let r be the position vector of a particle of mass *m*, referred to the origin *O*,

then we have $r=r_G+r'$; where r' is the position vector of the particle of mass *m* w.r.t. *G*.

Now let H be the moment of momentum (or angular momentum) of the body about *O*, then we have $=\Sigma r\times m\dot{r}$

$$=\Sigma m\times\dot{r}=\Sigma m\left(r_G+r'\right)\times\left(\dot{r}_G+\dot{r}'\right)$$

$$=\Sigma mr_G\times\dot{r}_G+\Sigma mr_G\times\dot{r}'+\Sigma mr'\times\dot{r}_G+\Sigma mr'\times\dot{r}'$$

$$=r_G\times\dot{r}_G\Sigma m+r_G\times\Sigma m\dot{r}'+\left(\Sigma mr'\right)\times\dot{r}_G{}'+\Sigma mr'\times\dot{r}'$$

$$=r_G\times\dot{r}_G\Sigma m+r_G\times\Sigma m\dot{r}'+\left(\Sigma mr'\right)\times\dot{r}_G{}'+\Sigma mr'\times\dot{r}' \qquad \ldots(1)$$

But $\frac{\Sigma mr'}{\Sigma m}=0$, being position vector of C.G. relative to C.G.

$\therefore \Sigma mr'=0\,and\;so\,\Sigma m\dot{r}'=0$

$\therefore (1)\Rightarrow H=r_G\times\dot{r}_G\,\Sigma m+\Sigma mr'\times\dot{r}'$

$$= r_G \times M\, r_G + \Sigma r' \times m\dot{r}' \qquad [\because \quad \Sigma m = M]$$

$$= r_G \times M v_G + \Sigma r' \times m\dot{r}' \qquad \ldots(2)$$

Another form. Let $\hat{n}$ be the unit vector parallel to H, then we get

$$r_G \times M v_G = M r_G \times v_G$$

$$= (M v_G\, p)\hat{n}$$

[using the definition of moment; p is the length of the perpendicular from the origin O on the direction of the velocity v_g of centre of gravity].

But we have $\Sigma r' \times m\dot{r}' = \left(Mk^2 \dfrac{d\theta}{dt}\right)\hat{n}$ [3.01]

and $\quad H = H\hat{n}$

$$\therefore (2) \Rightarrow H\hat{n} = M v_g\, p\hat{n} + Mk^2 \frac{d\theta}{dt}\hat{n}$$

Equating coefficients of $\hat{n}$ on both sides, we get

$$H = M v_g\, p + Mk^2 \frac{d\theta}{dt} \qquad \ldots(3)$$

This equation expresses that the moment of momentum (or angular momentum) of a rigid body about a fixed point O is equal to the angular momentum about O of a single particle of mass M (equal to mass of the body concentrated at its $C.G.$ and moving with the centroid's. velocity), together with the angular momentum of the body in motion relative to the $C.G.$

Equation (3) can also be written as.

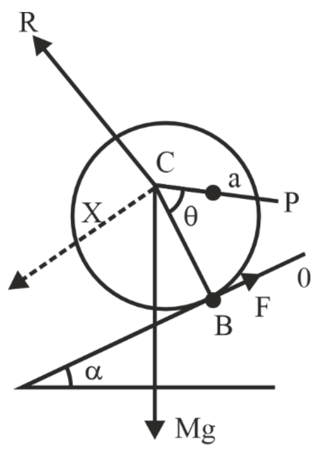
Angular momentum of the rigid body

= Angular momentum of centre of inertia

+ angular momentum relative to the centre of inertia.

Category-1

A uniform sphere rolls down an inclined plane, rough enough to prevent any sliding; to discuss the motion.



Initially, the sphere was at rest with its points P in contact with O.

During the motion, after any time t, let the centre "C" of the sphere describes a distance x on the inclined plane and θ is the angle through which the sphere turns.

Thus CP a line fixed in the body, makes an angle θ with the normal to the plane, a line fixed in the space.

Let F be the frictional force and R the normal reaction at the point of contact B,

then equations of motion of $C.G.$ of the body are

$$M\frac{d^2x}{dt^2} = Mg\sin\alpha - F \qquad \ldots(1)$$

Since there is no motion perpendicular to the plane, we have

$$M\ddot{y} = 0 = Mg\cos\alpha - R \quad \text{or} \quad Mg\cos\alpha = R. \qquad \ldots(2)$$

Also equation of motion about the centre of gravity is

$$Mk^2\frac{d^2\theta}{dt^2} = F.a \qquad \ldots(3)$$

Since there is no sliding, so we have OB = arc PB

$$\Rightarrow x = a\theta, \dot{x} = a\dot{\theta} \text{ and } \ddot{x} = a\ddot{\theta} \qquad \ldots(4)$$

$\therefore$ (3) gives $M\cdot\frac{k^2}{a^2}\frac{d^2x}{dt^2} = F\cdot a$ $\qquad [\because \ \ddot{x} = a\ddot{\theta}]$

Substituting the value of F from here in (1), we get

$$\frac{d^2x}{dt^2}\left(1+\frac{k^2}{a^2}\right)=g\sin\alpha \quad \text{or} \quad \frac{d^2x}{dt^2}=\frac{a^2 g\sin\alpha}{a^2+k^2} \qquad \ldots(5)$$

i.e. the sphere rolls down with a constant acceleration $\frac{\dot{a}^2 g\sin\alpha}{a^2+k^2}$

Equation (5) gives $\frac{dx}{dt}=\frac{a^2 g\sin\alpha}{a^2+k^2}t+C$; and C, integration constant. as $t=0$ and $\dot{x}=0$ gives $C=0$

Intergrating again, $x=\frac{1}{2}\frac{a^2 g\sin\alpha}{a^2+k^2}t^2$

because constant of intergration again vanishes as x and t vanish simultaneously.

Exam Points; Now we shall discuss various cases:

(i) If the body be a solid sphere, $k^2=\frac{2}{5}a^2$ and then equation (5) implies,

$\ddot{x}=\frac{5}{7}g\sin\alpha.$

(ii) If the body be hollow sphere, $k^2=\frac{2}{3}a^2$ $\quad \therefore \ddot{x}=\frac{3}{5}g\sin\alpha.$

(iii) If the body be circular disc, $k^2=\frac{1}{2}a^2$ $\quad \therefore \ddot{x}=\frac{2}{3}g\sin\alpha.$

(iv) If the body be circular ring, $k^2=a^2$ $\quad \therefore \ddot{x}=\frac{1}{2}g\sin\alpha.$

Pure rolling: Eliminating $\frac{d^2x}{dt^2}$ from (5), and (1), we get

$$F=Mg\sin\alpha-\frac{5}{7}Mg\sin\alpha=\frac{2}{7}Mg\sin\alpha\left(\because k^2=\frac{2a^2}{5}\right)$$

Also from (2) $R=Mg\cos\alpha.$

In order that there may be no sliding $\frac{F}{R}$ must be less than μ i.e. for pure rolling

$F<\mu R$ i.e. $\mu>\frac{F}{R}=\frac{2}{7}\tan\alpha.$

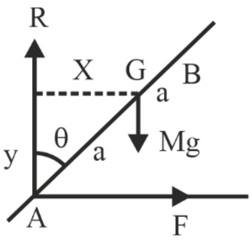
Category-1: Slipping of rods.

A uniform rod is held in a vertical position with one end resting upon a perfectly rough table and when released rotates about the end in contact with the table. To discuss the motion.

Let *AB* be the rod having length 2a and mass *M*.

Let the rod which is rotating about *A* makes an angle θ with the vertical at any time *t*.

Taking *A* point as the origin and horizontal and vertical lines as axes, the coordinate (x, y) of centre of mass *G* are given by



$x = a\sin\theta, y = a\cos\theta$

$\therefore\ \dot{x} = a\cos\theta\dot{\theta}, \dot{y} = -a\sin\theta\dot{\theta}$

and $\ddot{x} = -a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}$, $\ddot{y} = -a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}$.

Let *F* be the frictional force and R the normal reaction at *A*. Now the equation of motion of C.G. are

$$M\frac{d^2x}{dt^2} = M\left[a\cos\theta\ddot{\theta} - a\sin\theta\dot{\theta}^2\right] = F \qquad \ldots(1)$$

$$M\frac{d^2y}{dt^2} = M\left[-a\sin\theta\ddot{\theta} - a\cos\theta\dot{\theta}^2\right] = R - Mg \qquad \ldots(2)$$

Again energy of the rod $= \frac{1}{2}M\left[\left(\dot{x}^2 + \dot{y}^2 + \frac{1}{3}a^2\dot{\theta}^2\right)\right] \quad \because \quad v^2 = \dot{x}^2 + \dot{y}^2, k^2 = \frac{1}{3}a^2$

$= \frac{1}{2}m\left[(a\dot{\theta})^2 + \frac{1}{3}a^2\dot{\theta}^2\right] = \frac{2}{3}Ma^2\dot{\theta}^2$

and work done by the forces $= Mg(a - a\cos\theta)$. Hence from energy equation, we have

$$\frac{2}{3}Ma^2\dot{\theta}^2 = Mg(a - a\cos\theta) \Rightarrow \dot{\theta}^2 = \frac{3g}{2a}(1 - \cos\theta)^* \qquad \ldots(3)$$

Differentiating (3) with respect to *t*, we have $\ddot{\theta} = \frac{3g}{4a}\sin\theta$ $\qquad \ldots(4)$

Putting the values of θ and θ from (3) and (4) in (1) and (2), we get

$F = \frac{3}{4} Mg \sin\theta(3\cos\theta - 2)$ and $R = \frac{1}{4} Mg(1 - 3\cos\theta)^2$

We observe that R does not change its sign and vanishes when $\cos\theta = \frac{1}{3}$. Hence the end **A does not leave the plane.**

From the value of F, we see that F changes its sign as θ passes through the angle $\cos^{-1}\left(\frac{2}{3}\right)$; thus its direction is then reversed.

At $\cos\theta = \frac{1}{3}, R = 0$, hence the ratio $\frac{F}{R}$ **becomes infinite where** $\cos\theta = \frac{1}{3}$, hence unless the plane be infinitely rough there will be sliding at this value of θ. In practice the end A of the rod begins to slip for some value of θ less then $\cos^{-1}\left(\frac{1}{3}\right)$. The end A will slip backwards or forward according as the slipping takes place before of after the $\cos^{-1}\left(\frac{2}{3}\right)$.

we observe that R is positive for every value of α and θ. Hence the end never leaves the plane.

Category-3: ***A uniform straight rod slides down in a vertical plane its end being in contact with two smooth planes, one horizontal and the other vertical. If it started from rest at an angle α with the horizontal; to discuss the motion.***

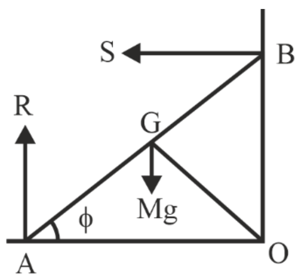
Let at any instant t, the rod makes an angle θ with the horizontal. Let R and S be the reactions at the ends A and B of the rod AB whose length is 2a and mass M.

With reference to point O as origin, the co-ordinates of G i.e. centre of gravity are
$x = a\cos\theta, y = a\sin\theta$

$\therefore \quad \ddot{x} = -a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}$,

$\quad \ddot{y} = -a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}$

The equation of motion of C.G. are $M\ddot{x} = S$



$$1 \Rightarrow M\left(-a\cos\theta\theta^2 - a\sin\theta\ddot{\theta}\right) = S \qquad \ldots(1)$$

and $\quad M\ddot{y} = R - Mg$

$$\Rightarrow M\left(-a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}\right) = R - Mg$$

Energy equation given

$$\frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}Mk^2\dot{\theta}^2 = \text{work done by the gravity.}$$

$$\Rightarrow \frac{1}{2}M\left(a^2\dot{\theta}^2 + \frac{1}{3}a^2\dot{\theta}^2\right) = Mga(\sin\alpha - \sin\theta)$$

$$\Rightarrow \dot{\theta}^2 = \left(\frac{3g}{2a}\right)(\sin\alpha - \sin\theta) \qquad \ldots(3)$$

Differentiating (3) w.r.t. t, we get $\ddot{\theta} = -\left(\frac{3g}{4a}\right)\cos\theta$ $\qquad \ldots(4)$

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ in (1) and (2), we have

$$S = M\left[-a\cos\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) - a\sin\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

$$= \frac{3}{4}Mg\cos\theta(3\sin\theta - 2\sin\alpha) \qquad \ldots(5)$$

$$R = Mg + M\left[-a\sin\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) + a\cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

$$= \frac{1}{4}Mg\left[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta\right]$$

$$= \frac{1}{4}Mg\left[1 - 6\sin\theta\sin\alpha + 9\sin^2\theta\right]$$

$$= \frac{1}{4}Mg\left[1 - \sin^2\alpha + \sin^2\alpha - 6\sin\theta\sin\alpha + 9\sin^2\theta\right]$$

$$= \frac{1}{4}Mg\left[(3\sin\theta \neg \sin\alpha)^2 + \cos^2\alpha\right] \qquad \ldots(6)$$

From (5), we observe that S = 0 when $\sin\theta = \frac{2}{3}\sin\alpha$ and S will be negative when this value of θ is reached. Hence the end B leaves the wall when $\sin\theta = \frac{2}{3}\sin\alpha$.

Again from (6), we observe that R is always positive *i.e.* the end A never leaves the plane

Further when the end B leaves the plane $\sin\theta = \frac{2}{3}\sin\alpha$ and $S = 0$ thus equations of motion (1), (2), (3) and (4) cease to hold good for further motion.

Putting $\sin\theta = \frac{2}{3}\sin\alpha$ in (3), the angular velocity of the rod now becomes $\left(\frac{g}{2a}\sin\alpha\right)^{1/2}$, this will be the initial angular velocity for the next part of the motion.

Second part of the motion.

When the end B leaves the wall, let R_1 be the normal reaction at A. Let the rod be inclined at angle ϕ to the horizontal.

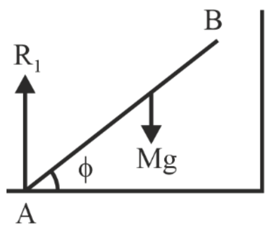
The equations of motion are

$$M\ddot{x} = 0 \qquad \ldots(1)$$

$$M\,\ddot{y} = R_1 - Mg \qquad \ldots(2)$$

and $$M\frac{a^2}{3}\ddot{\phi} = -R_1 a\cos\phi \qquad \ldots(3)$$

As $y = a\sin\phi, \quad \therefore \ddot{y} = -a\sin\phi\dot{\phi}^2 + a\cos\phi\ddot{\phi}$



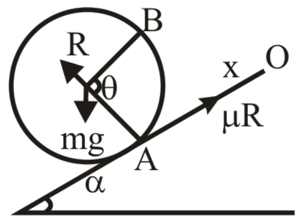
Hence from (2) and (3), we get

$$\left(\frac{1}{3} + \cos^2\phi\right)\left(\frac{d^2\phi}{dt^2}\right) - \sin\phi\cos\phi\left(\frac{d\phi}{dt}\right)^2 = -\frac{g}{a}\cos\phi \qquad \ldots(4)$$

Integrating it, we get $\left(\frac{1}{3} + \cos^2\phi\right)\left(\frac{d\phi}{dt}\right)^2 = -\frac{2g}{a}\sin\phi + C \qquad \ldots(5)$

When $\sin\phi = \frac{2}{3}\sin\alpha, \frac{d\phi}{dt} = \sqrt{\left(\frac{g}{2a}\sin\alpha\right)}$,

$$\therefore \quad \frac{g\sin\alpha}{2a}\left[\frac{1}{3}+1-\frac{4}{9}\sin^2\alpha\right] = -\frac{2g}{a}\cdot\frac{2}{3}\sin\alpha + C$$



or $$C = \frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right).$$

Hence from (5), we have

$$\left(\frac{1}{3}+\cos^2\phi\right)\left(\frac{d\phi}{dt}\right)^2 = \frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right) - \frac{2g}{a}\sin\phi. \qquad \ldots(6)$$

When $\phi = 0$ i.e. when rod reaches the horizontal plane, let its angular velocity be Ω, then

$$\Omega^2\left(\frac{1}{3}+1\right) = \frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right) \Rightarrow \Omega^2 = \frac{3g}{2a}\left(1-\frac{\sin^2\alpha}{9}\right)\sin\alpha. \qquad \ldots(7)$$

Category-4:When rolling and sliding are combined.

An imperfectly rough sphere moves from rest down a plane inclined at an angle α to the horizon, to determine the motion.

Let C be the centre of sphere whose radius is a . Let in time t the sphere have turned through an angle θ

i.e. let CB be a radius (a line fixed in the body) which was initially normal to the plane, makes an angle θ with the normal CA during this period.

Let us suppose that the friction is not sufficient to produce pure rolling therefore the sphere slides as well as turns. So the maximum friction μR acts up the plane, μ being the coefficient of friction. Let x be the distance described by the centre of gravity C parallel to the inclined plane in time t, and θ the angle through which the sphere turns.

As there is no motion perpendicular to the plane, so the C.G. of the sphere always moves parallel to the plane. The equations of motion are

$m\ddot{x} = mg\sin\alpha - \mu R$ …(1) $\qquad$ 0 = R – mg cos α …(2)

And $m\frac{2}{5}a^2\ddot{\theta} = \mu Ra$ …(3)

Form (1) and (2), we have $\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$ …(4)

Integrating (4) w. r. t. ‘t’ we get $\dot{x} = g(\sin\alpha - \mu\cos\alpha)t$ …(5)

Integrating (5) again, we get $x = g(\sin\alpha - \mu\cos\alpha)\frac{t^2}{2}$ ...(6)

Constants or integration vanish as $\dot{x} = 0$, x = 0 when t = 0

From (2) and (3), we get $a\ddot{\theta} = \frac{5}{2}\mu g\cos\alpha$

Integrating it, we get $a\dot{\theta} = \frac{5}{2}\mu g t\cos\alpha$

Integrating it again , we get $\theta = \frac{5\mu g}{4}t^2\cos\alpha$

The constants of integration varish as $\dot{\theta}$ = 0, θ = 0 when t=0.

The velocity of the point of contact A down the plane

= velocity of C, the centre of sphere, + velocity of A relative to C $= \dot{x} - a\dot{\theta}$

$$= g(\sin\alpha - \mu\cos\alpha)t - \frac{5}{2}\mu g t\cos\alpha = \frac{1}{2}g(2\sin\alpha - 7\mu\cos\alpha) \quad ...(8)$$

Equation (8) gives rise to the following three cases:

First case. If 2 sin α >7μ cos α i.e. if μ < 2/7 tan α.

In this case, velocity of the point of contact is positive for all values of t i.e. it does not vanish, hence the point of contact always slides down and the maximum friction μR acts. The sphere never rolls. The equations of motion established above govern the entire motion.

Second case. If 2 sin α = 7μ cos α i.e. if μ = 2/7 tan α

In this case velocity of the point of contact is zero for all values of t and therefore motion of the sphere is that of pure rolling throughout and the maximum friction μR is always exerted.

Third case. 2 sin a < 7μ cos α i.e. if μ > 2/7 tan a

In this case velocity of the point of contact is negative i.e. if the maximum. friction μR were allowed to act, the point of contact will slide up the plane which is impossible because that amount of friction will only act which is just sufficient to keep the point of contact at rest. Hence in this case the motion is of pure rolling from the very start and remains the same throughout and the maximum friction μR is not exerted. Therefore in this case the equations of motion established above do not hold good.

Let F be the frictional force now in play, then equations of motion are

$m\ddot{x} = mg\sin\alpha - F$ ...(9) $0 = R - mg\cos\alpha$ ...(10)

And $m\frac{2}{5}a^2\ddot{\theta} = Fa$ ...(11)

Because the point of contact is at rest, we have

$\dot{x} = a\dot{\theta} = 0 \Rightarrow \dot{x} = a\dot{\theta}$ ...(12)

From (9), (11) and (12) , we have $\ddot{x} = a\dot{\theta} = \frac{5}{7} g \sin\alpha$

Integrating above, we get $\dot{x} = a\dot{\theta} = \frac{5}{7} g t \sin\alpha$

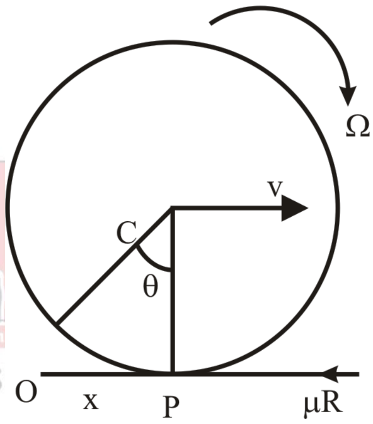
Again integrating above, we get $x = a\theta = \frac{5}{14} g t^2 \sin\alpha$ …(14)

The constants of integration vanish as $\dot{x} = 0, x = 0$ when t = 0

Category-5: ***A uniform circular disc is projected with its plane vertical along a Tough horizontal plane with a velocity v of translation and an velocity Ω about the centre. Find the motion. angular***

Case I. When $v \rightarrow, \Omega \downarrow .and\, v > a\Omega$

In this case initial velocity of the point of contact P is given by v - aΩ, hence its direction is →as v > aΩ, so the friction μR acts in the direction ←. When the centre has moved through a distance x and θ is the angle through which the disc has turned the equations of motion are given by



$m\ddot{x} = -\mu R = -\mu mg$ i.e. $\ddot{x} = -\mu g$ …(1)

$m\frac{a^2}{2}\ddot{\theta} = \mu Ra = \mu mga$ i.e. $a\ddot{\theta} = 2\mu g$ …(2)

Integrating (1) and (2) and making use of initial conditions

i.e. $t = 0, \dot{x} = v\, and\, \dot{\theta} = \Omega, we\ have\ \dot{x} = -\mu gt + v$ …(3)

and $a\theta = 2\mu gt + a\Omega$ …(4)

Now rolling commences when $\dot{x} - a\dot{\theta} = 0$. Let this happen after time t_1

Then $\dot{x} = a\dot{\theta} = -\mu g t_1 + v - 2\mu g t_1 - a\Omega = 0$ or $t_1 = \frac{v - a\Omega}{3\mu g}$.

Putting this value of t_1 in (3), we observe that at this time velocity of the centre i.e. $\dot{x} = \frac{2v + a\Omega}{3}$ …(5)

When rolling commences equations of motion reduce to

$m\ddot{x} = -F$ and $\frac{ma^2}{2}\dot{\phi} = Fa$ …(6)

Since there is no sliding, $\dot{x} = a\dot{\phi}$ or $\ddot{x} = a\ddot{\phi}$

Solving these equations, we have F = 0.

Thus we observe that no friction is required throughout the motion for pure rolling, so equations for this motion are

$m\ddot{x}=0$ i.e. $\ddot{x}=0$ and $m\frac{a^2}{2}\ddot{\phi}=0$ i.e. $a\ddot{\phi}=0$ ...(7)

Integrating (7), we get $\dot{x}=cons\tan t=\frac{2v+a\omega}{3}$, from (5).

The disc therefore continues to roll with a constant velocity $\frac{2v+a\omega}{3}$

which is less then its initial velocity.

Case II. when $v\to,\Omega\downarrow and\ v<a\Omega.$

In this case initial velocity of the point of contant is $v-a\Omega<0,$ so its direction is ←, hence friction μR acts in the direction →.

Now the equations of motion are

$m\ddot{x}=\mu R=\mu mg$ i.e. $\ddot{x}=\mu g$ ...(1)

And $m\frac{a^2}{2}\ddot{\theta}=-\mu Ra=-\mu mga$ i.e. $a\ddot{\theta}=-2\mu g$ ...(2)

Integrating these equations and making use of initial conditions to evaluate constants, we get

$\dot{x}=\mu gt+v$ ...(3) and $a\dot{\theta}=-2\mu gt+a\Omega$ ...(4)

Pure rolling commences when $\dot{x}-a\dot{\theta}=0,$ let this happen after time t_1 then from (3) and (4) ,

$\mu gt_1+v+2\mu gt_1-a\Omega=0$ or $t_1=\frac{a\Omega-v}{3\mu g}$

Putting this value of t_1 in (3) , we get $\dot{x}=\frac{2v+a\Omega}{3}$

When pure rolling begins, equations of motion are same as in **case I** by

which F= 0, so the disc continues rolling with constant velocity $=\frac{2v+a\omega}{3}$

Case III when $v\to,\Omega\uparrow$

In this case, initial velocity of the point of contact is $v+a\Omega,$ so its direction is →, hence μR acts in the direction ←.

Equations of motion are

$m\ddot{x}=-\mu R=-\mu mg$ i.e. $\ddot{x}=-\mu g$ ...(1)

And $m\frac{a^2}{2}\ddot{\theta}=-\mu Ra=-\mu mga$ i.e. $a\ddot{\theta}=-2\mu g$ ...(2)

Integrating these equations and making use of initial conditions to determine the constants, we get

$\dot{x}=-\mu gt+v$ ...(3) and $a\dot{\theta}=-2\mu gt\_a\Omega$ ...(4)

They velocity of the point of contact is $\dot{x}+a\dot{\theta}$ ($\dot{x}$ *and* $\dot{\theta}$ are in the same direction). Pure rolling begins when

$\dot{x}+a\dot{\theta}=0$, let this happen after time t_1 then from (3) and (4), we get

$$-\mu g t_1+v+\left(-2\mu g t_1+a\Omega\right)=0 \quad or \quad t_1=\frac{v+a\Omega}{3\mu g}$$

$$\dot{x}=a\dot{\theta}=\frac{2v-a\Omega}{3}$$

If 2v > aΩ the velocity of the centre is positive, so the motion is of pure rolling with uniform velocity $\frac{2v-a\Omega}{3}$

If 2v < aΩ the velocity of the centre is negative (backward). In this case we observe from equation (3) that velocity of the centre becomes zero v when $t=\frac{v}{\mu g}$ and at that time from equation (4) we observe that

$a\dot{\theta}=-2v+a\Omega$. which is positive since 2v <aΩ. Hence when 2v <aΩ, the disc begins to move backward before pure rolling begins.

In other us we say that u > aΩ, the rolling will commence before the forward motion ceases.

Category 6:

When two bodies are in contact; then to determine whether the relative motion involves sliding or rolling at the point of contact. Let P be the point of contact of a moving body placed over the other, assume that the initial velocity of the point of contact is zero. To find whether the relative motion is of sliding or rolling we make use of following two methods

In the first method, assume that the body rolls and suppose F is the force of friction sufficient to keep P (the point of contact) at rest. Hence F is unknown. Again write the equations of motion along with the geometrical equation to express the condition that the tangential velocity of the point

P is zero. Solve these equation and find F/R

In case F/R <μ, the necessary friction can be called into play to keep the point P at rest. Thus the body rolls and will remain so long as F/R ≤ μ, but when F/R >μ, the point of contact will slide. When this happens the

R equations of motion discussed before will not hold good, and we apply the following method.

In this method write the equations of motion on the supposition that the point of contact slids. i.e. the friction is μR instead of F and there is no geometrical equations. On solving these equations we find the tangential velocity of the point of contact P. In case this velocity is not zero and is in the direction opposite to the direction in which μR acts (μ has a proper sign), the body will slide at P and will remain so long as the velocity at P does not vanish, when velocity at P vanishes, we again apply the first method.

Category 7:

A sphere, of radius a whose centre of gravity G is at a distance c from its centre C, placed on a rough plane so that CG is horizontal; show that it will begin to roll or slide according as the coefficient of friction

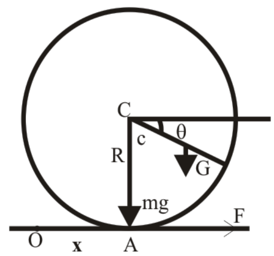
axis through G; if u is equal to this value, what happens ?

When CG is inclined at an angle θ to the horizontal, let A, the point of contact have moved through a horizontal distance x from its initial position O, and let OA = x. Assume that the sphere rolls and F be the force of friction sufficient for pure rolling. Since the motion is of pure rolling so $x = a\theta$ and the point of contact A is at rest

$\therefore \dot{x} = a\dot{\theta}$ ...(1)

The coordinates of G (the centre of gravity) with reference to O the fixed point as origin and horizontal and vertical lines through O as coordinate axes are $(x + c\cos\theta, a - c\sin\theta)$.

Equations of motion of the sphere are



$$F = M\frac{d^2}{dt^2}(x + c\cos\theta) = M\frac{d^2}{dt^2}(a\theta + c\cos\theta)$$

$$= M\left[a\dot{\theta} - c\sin\theta\,\ddot{\theta} - c\cos\theta\,\dot{\theta}^2\right] \quad ...(2)$$

$$R - Mg = m\frac{d^2}{dt^2}(a - c\sin\theta) = M\left[-c\cos\theta\,\ddot{\theta} + c\sin\theta\,\dot{\theta}^2\right] \quad ...(3)$$

And $Rc\cos\theta - F(a - c\sin\theta) = Mk^2\ddot{\theta}$ ...(4)

We only want the initial motion when $\theta = 0$ and $\ddot{\theta}$ is zero but is not zero.

The equations (2), (3), (4) then give

$F = ma\ddot{\theta}; R = mg - Mc\ddot{\theta}; Rc - Fa = Mk^2\ddot{\theta}.$ **for the initial values**

On eliminating R, and F, we get $\ddot{\theta} = \dfrac{gc}{k^2 + a^2 + c^2}$, then

$$\frac{F}{M} = g\frac{ac}{k^2 + a^2 + c^2} \text{ and } \frac{R}{M} = g\frac{k^2 + a^2}{k^2 + a^2 + c^2}; \frac{F}{M} = \frac{ac}{k^2 + a^2}$$

The sphere will roll or slide according as

$$F < or > \mu R \qquad or \quad as\ \mu > \frac{F}{R} \quad i.e.\ \mu > or < \frac{ac}{k^2 + c^2}$$

Critical case. If $\mu = \dfrac{ac}{k^2 + c^2}$ In this case we shall consider whether F/R is a little greater or little less than μR when θ is small but bot absolutely zero .

From equation (1), (2) and (3) on eliminating F and R, we get

$$\left(k^2 + a^2 + c^2 - 2ac\sin\theta\right)\ddot{\theta} - ac\cos\theta\,\dot{\theta}^2 = gc\cos\theta \quad ...(4)$$

Integrating it, we get $\left(k^2 + a^2 + c^2 - 2ac\sin\theta\right)\dot{\theta}^2 = 2gc\sin\theta$ ...(5)

As θ is small, sin θ can be replaced by θ and cos θ by unity, neglecting squares and higher powers of θ sin θ $\dot{\theta}^2$ is also neglected.

Thus (5) reduces to $\left(k^2+a^2+c^2\right)\dot{\theta}^2=2gc\theta$ …(6)

And then from (4), $\left(k^2+a^2+c^2-2ac\theta\right)=gc\left(1+\dfrac{a\dot{\theta}^2}{g}\right)$

$$=gc\left(1+\frac{2ac\theta}{k^2+a^2+c^2}\right) \qquad \text{[From (6)]}$$

Or $$\left(k^2+a^2+c^2\right)\left(1-\frac{2ac\theta}{k^2+a^2+c^2}\right)\ddot{\theta}=gc\left(1+\frac{2ac\theta}{k^2+a^2+c^2}\right)$$

Or $$\left(k^2+a^2+c^2\right)\ddot{\theta}=gc\left(1+\frac{2ac\theta}{k^2+a^2+c^2}\right)\left(1-\frac{2ac\theta}{k^2+a^2+c^2}\right)^{-1}$$

$$=gc\left(1+\frac{2ac\theta}{k^2+a^2+c^2}\right)\left(1+\frac{2ac\theta}{k^2+a^2+c^2}\right)$$

$$=gc\left(1+\frac{2ac\theta}{k^2+a^2+c^2}\right) \text{ approximately}$$

From (1) and (2), we have $\dfrac{F}{M}=\dfrac{(a-c\sin\theta)\ddot{\theta}-c\cos\theta\,\dot{\theta}^2}{g-c\cos\theta\,\ddot{\theta}+c\sin\theta\,\dot{\theta}^2}$

$$=\frac{(a-c\,\theta)\ddot{\theta}-c\,\dot{\theta}^2}{g-c\,\ddot{\theta}} \text{ neglecting } \theta^2,\theta^3 \text{ etc. and also } \sin\ \theta\dot{\theta}^2$$

$$=\frac{ac}{k^2+c^2}\left[1-c\frac{\left(3k^2-a^2\right)\theta}{a\left(k^2+a^2\right)}\right] \text{ by putting the values of } \ddot{\theta}\, and\, \dot{\theta}^2 \text{ as found above}$$

If $k^2>\dfrac{a^2}{3}$ then F/R is less than $\dfrac{ac}{k^2+a^2}$

i.e. F/R is less than μ or F < μR and the sphere rolls.

If $k^2<\dfrac{a^3}{3}$ then $\dfrac{F}{R}>\dfrac{ac}{k^2+a^2}$ i.e. $\dfrac{F}{R}>\mu\ or\ F>\mu R$ and the sphere slides.

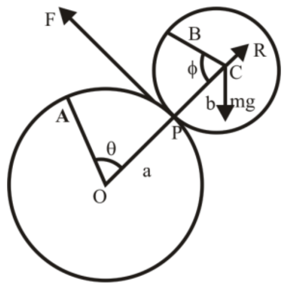
Category 8: One of the bodies fixed.

A solid homogeneous sphere, resting on the top of another fixed sphere is slightly displaced and begins to roll down it. Show that it will slip when the common normal makes with the vertical an angle given by the equation 2 sin $(\theta-\lambda)=5\sin\lambda(3\cos\theta-2)$ where λ is the angle of friction.

Also prove that the upper sphere will leave when $\theta = \cos^{-1}(10/17)$.

Sol. Let O be the centre of the fixed sphere whose highest point is A. Let CB be the position at any point t, of the radius of the upper sphere (moving sphere) which was originally vertical.

So if P is the point of contact, then arc AP = arc BP



$i.e. a\theta = b\phi, \; then \;\; a\dot{\theta} = b\dot{\phi}$ ...(1)

where a and b are the radii of the lower and upper sphere respectively, and are the angle which the common normal OC makes with the vertical and CB, a line fixed in the moving sphere respectively. Let R and F be the normal reaction and the friction acting on the upper sphere. Since C describe a circle of radius (a + b)

about O, its acceleration are (a+b) $\dot{\theta}^2$ and $(a+b)\ddot{\theta}$ along and perpendicular to CO.

Hence $m(a+b)\dot{\theta}^2 = mg\cos\theta - F$ ...(2)

And $m(a+b)\ddot{\theta} = mg\sin\theta - F$ ...(3)

Referred the O as the origin , the coordinates of the centre C are $\{(a+b)\sin\theta, (a+b)\cos\theta\}$.

This energy equation gives us

$$\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}mk^2\left(\dot{\phi} + \dot{\theta}\right)^2 = \text{work done by gravity}$$

$$= mg(a+b)(1-\cos\theta)$$

Or $$(a+b)^2\dot{\theta}^2 + \frac{2b^2}{5}\left(\frac{a+b}{5}\right)\dot{\theta}^2 = 2g(a+b)(1-\cos\theta)\left(\because \dot{\phi}\right) = \frac{a}{b}\dot{\theta}$$

Or $$\frac{7}{5}\dot{\theta}^2 = \frac{2g}{a+b}(1-\cos\theta) \; or \; \dot{\theta}^2 = \frac{10g}{7(a+b)}(1-\cos\theta) \quad ...(4)$$

Differentiating w.r.t. to 't' and dividing by $2\dot{\theta}$, we get

$$\ddot{\theta} = \frac{5g}{7(a+b)}\sin\theta \quad ...(5)$$

From (2) and (4), we get $R = mg\;\cos\theta - m\frac{10g}{7}(1-\cos\theta)$

$$= \frac{mg}{7}(17\cos\theta - 10) \quad ...(6)$$

From (5) and (3), we get $F = mg\;\sin\theta - \frac{5}{7}mg\sin\theta - \frac{2}{7}mg\;\sin\theta$

Hence the sphere will slip if F = μR

i.e. if $\frac{2}{7}\ mg\ \sin\ \theta = \tan\lambda.\frac{(17\ \cos\theta - 10)mg}{7}$

or $\quad$ if $2\sin\theta\cos\lambda = (17\cos\theta - 10)\sin\lambda$

or $\quad$ if $2(\sin\ \theta\ \cos\ \lambda - \cos\theta\sin\lambda = 5(3\ \cos\theta - 2)\sin\lambda$

or $\quad$ if $2\sin(\theta - \lambda) = 5\sin\lambda(3\ \cos\theta - 2)$

The upper sphere will leave the lower one when R = 0 , hence from (6)

(17 cos θ -10) = 0 i.e. $\theta = \cos^{-1}\left(\frac{10}{17}\right)$

When both the spheres are smooth. In this case F = 0, so the energy equation becomes

$$\frac{1}{2}m(a+b)^2\dot{\theta}^2 = mg(a+b)(1-\cos\theta)$$

i.e. $\quad \dot{\theta}^2 = \frac{2g}{(a+b)}(1-\cos\theta)$

Further equation (2) remains unchanged,

$$\Rightarrow R = mg\cos\theta - 2mg(1-\cos\theta) = mg(3\cos\theta - 2)$$

The upper sphere will leave the lower if R = 0

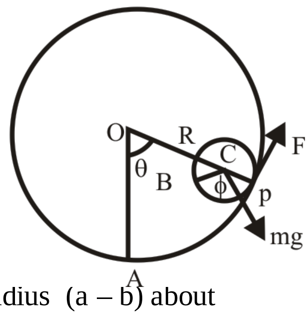
i.e. $\;$ if $mg(3\ \cos\ \theta - 2) = 0\ \ or\ \theta = \cos^{-1}\left(\frac{2}{3}\right)$.

Category 9:

A hollow cylinder, of radius a is fixed with its axis horizontal, in side it moves a solid cylinder, of radius b, whose velocity in its lowest position is given if the friction between the cylinders be sufficient to prevent any sliding, find the motion.

Let C be the centre of the moving cylinder and let ϕ be the angle which a line CB fixed in the moving cylinder makes with the vertical, a line fixed in space. Initially B coincided with A. Let a be the radius of the fixed cylinder whose centre is O and b that of the movable cylinder.

Since there is no slipping between the two cylinders therefore arc AP = arc BP



or $a\theta = (\phi + \theta)$

or $\quad b\phi = (a-b)\theta; \therefore b\phi = (a-b)\ldots(1)$

Let R and F be the normal reaction and friction at P. Since C describes a circle of radius (a – b) about O, the equation of motion of the cylinder are $m(a+b)\dot{\theta}^2 = R - mg\sin\theta \quad \ldots(2)$

And $\quad m(a-b)\ddot{\theta} = F - mg\sin\theta \qquad \ldots(3)$

The co-ordinates of C with respect to O as origin and the vertical and horizontal lines as axes through O are {(a - b) sinIO θ , (a - b) cos θ }.

(its velocity)^2 $=\left(\dot{x}^2+\dot{y}^2\right)=\left(a-b\right)^2\cos^2\theta\dot{\theta}^2+\left(a-b\right)^2\sin^2\theta\dot{\theta}^2=\left(a-b\right)^2\dot{\theta}^2$. So kinetic energy of the moving cylinder at any time 't' is

$$\frac{1}{2}mk^2\dot{\phi}^2+\frac{1}{2}m\left(a-b\right)^2\dot{\theta}^2=\frac{1}{2}m\frac{b^2}{2}\dot{\phi}^2+\frac{1}{2}m\left(a-b\right)^2\dot{\theta}^2\left(\therefore k^2=\frac{b^2}{2}\right)$$

$$\frac{1}{2}\frac{\left(a-b\right)^2}{2}\dot{\theta}^2=\frac{1}{2}m\left(a-b\right)^2\dot{\theta}^2=\frac{3}{4}m\left(a-b\right)^2\dot{\theta}^2$$

$$\{\therefore b\dot{\phi}=\left(a-b\right)\dot{\theta}\ from\left(1\right)\}$$

$\therefore$Kinetic energy at the time of projection $=\frac{3}{4}m\left(a-b\right)^2\Omega^2$

$$(\therefore\dot{\theta}=\Omega\ initially)$$

Therefore equation gives

$$\frac{3}{4}m\left(a-b\right)^2\dot{\theta}^2-\frac{3}{4}m\left(a-b\right)^2\Omega^2=-mg\left(a-b\right)\left(1-\cos\theta\right)$$

i.e. $\left(a-b\right)^2\dot{\theta}^2=\left(a-b\right)^2\Omega^2-\frac{4}{3}g\left(1-\cos\theta\right)$

Differentiating (4) w.r.t. θ and dividing be 2 θ , we get

$$f\left(a-b\right)\ddot{\theta}=-\frac{2}{3}g\ \sin\ \theta \qquad \ldots(5)$$

Again from (2) $R=mg\cos\theta+m\left(a-b\right)\dot{\theta}^2$

$$=mg\cos\theta+m\left(a-b\right)\Omega^2-\frac{3}{4}mg\left(1-\cos\theta\right)\ \text{from (4)}$$

Or $\quad R=m\left(a-b\right)\Omega^2+\frac{mg}{3}\left(7\cos\theta-4\right)$

From (3), $F=mg\sin\theta+m\left(a-b\right)\ddot{\theta}=mg\sin\theta-\frac{2}{3}mg\ \sin\theta$ from (5)

Or $\quad F=\frac{1}{3}mg\ \sin\ \theta \qquad \ldots(7)$

Case 1.

In order that the cylinder may roll down completely, R should be zero at the highest point.

i.e. R = 0 when θ = π

$$\therefore 0=m\left(a-b\right)\Omega^2+\frac{1}{3}mg\left(-7-4\right)\ or\ \Omega=\sqrt{}[\frac{11g}{3\left(a-b\right)}]$$

Case 2.

The moving cylinder will leave the fixed cylinder if

$$R = 0 \, i.e. \; m(a+b)\Omega^2 + \frac{1}{3}(7\cos\theta - 4) = 0$$

$$\cos\theta = \left[\frac{4}{3}g - (a-b)\Omega^2\right]\frac{3}{7g}$$

$$\cos\theta = \frac{1}{7g}[4g - 3(a-b)\Omega^2]$$

This gives the position when the two bodies separate.

Case 3.

If the rolling cylinder makes small oscillations about the the lowest point of the fixed cylinder, then θ is always small, hence equation (5) gives on taking for $\sin\theta$

$$\ddot{\theta} = \frac{-2g}{3(a-b)}\theta$$

Hence time of small oscillation is $\dfrac{2\pi}{\left[\dfrac{2g}{3(a-b)}\right]^{1/2}} = 2\pi\left[\dfrac{3(a-b)}{2g}\right]^{1/2}$

Example:- A uniform solid cylinder is placed with tits axis horizontal on a plane, whose inclination to the horizon is α, show that the least coefficient of function between it and the plane, so that it may roll and not slide, is $\frac{1}{3}\tan\alpha$. If the cylinder be hollow, and of small thickness, the least value is $\frac{1}{2}\tan\alpha$.

Solution:- At any time t, let the axis of the cylinder describe is distance x and θ be the angle turned Arguing as in 3.04, we have $x = a\theta$ [$\because$ there is no sliding]

Also the equations of a C.G. are given by $M\frac{d^2x}{dt^2} = M\,g\,\sin\alpha - F$ (1)

And $0 = M\,g\cos\alpha - R$
(2)

Again taking moments about the axis through G, the centre of gravity of the body, we have

$$M\,k^2\frac{d^2\theta}{d\,t^2} = F\times a \Rightarrow M\,\frac{k^2}{a}.\frac{d^2x}{dt^2} = F\times a \qquad (3)$$

Whence elimination of $M\frac{d^2x}{dt^2}$ in between (1) and (3), we get

$$\frac{k^2}{a}\left(M\,g\sin\alpha - F\right) = F\times a \Rightarrow F = \frac{k^2}{a^2+k^2}M\,g\sin\alpha \qquad (4)$$

But $R = M\,g\,\cos\alpha$ (5)

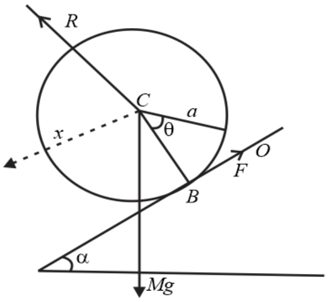
$\therefore$ For pure rolling $\mu > \frac{F}{R} = \frac{k^2}{a^2+k^2}\tan\alpha$

But when cylinder is solid, we have $k^2 = \frac{1}{2}a^2, \Rightarrow \mu > \frac{\frac{1}{2}a^2}{a^2+\frac{1}{2}a^2}\tan\alpha$

In case of hollow cylinder, we have $k^2 = a^2, \Rightarrow \mu > \frac{a^2}{a^2+a^2}\tan\alpha = \frac{1}{2}\tan\alpha$

Example:- A cylinder rolls down a smooth plane whose inclination to the horizontal is α, unwrapping, as it goes, a fine string fixed to the highest point of the plane ; fine its acceleration and the tension of the string.

Solution:- When the cylinder has rolled down a distance x along and plane, let T be the tension in the string and in this time (say t), let θ be the angle turned by the cylinder, then as the string is tight, the motion is of pure rolling i.e. arc $BP = OB \Rightarrow x = a\theta$ (1)



$\therefore$ $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$ equations of motion of the centre of gravity of the cylinder are

$$M\frac{d^2x}{dt^2} = M\,g\sin\alpha - T \qquad (2)$$

and $$M\frac{d^2y}{dt^2} = 0 = M\,g\cos\alpha - R \qquad (3)$$

Now taking moments about the centre, we have $M\,k^2\,\ddot{\theta} = T\times$ i.e. $M\,.\frac{1}{2}a^2\,\ddot{\theta} = T\times a$

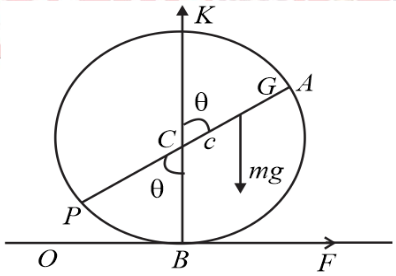
Or $\frac{1}{2}M\,\ddot{x} = T$ $\qquad$ $[\because\ \ddot{x} = a\ddot{\theta}]$ $\qquad$ (4)

$\therefore$ (5) and (2), gives $\frac{3}{2}M\,\ddot{x} = M\,g\sin\alpha$ i.e. $\ddot{x} = \frac{2}{3}g\sin\alpha$

$\Rightarrow$ $T = \frac{1}{2}m\,\ddot{x} = \frac{1}{2}M\left(\frac{2}{3}g\sin\alpha\right) = \frac{1}{3}\,M\,g\sin\alpha$

Example:- A circular cylinder, whose centre of inertia is at a distance c from axis, rolls on a horizontal plane. If it be just started from a position of unstable equilibrium. Show that the normal reaction of the plane when the centre of mass is in its lowest position is $\left[1+\frac{4c^2}{(a-c)^2+k^2}\right]$ times its weight, where k is the radius of gyration about an axis through the centre of mass.

Solution:- Initially the point of contact P of the cylinder was at O when its centre of gravity was vertically above the centre of the figure.



At any time t let the radius through G turn through an angle θ.

Referred to O as origin and horizontal and vertical line as axes, the co-ordinates (x, y) of G fiven by $x = a\theta + c\sin\theta,\ y = a + c\cos\theta$

$$[\because\ CG = c]$$

Equations of motion of C.G. are $m\frac{d^2x}{dt^2} = m\frac{d^2}{dt^2}(a\theta + c\sin\theta) = F$ $\qquad$ (1)

And $m\frac{d^2y}{dt^2} = m\frac{d^2}{dt^2}(a + c\cos\theta) = R - m\,g$ $\qquad$ (2)

Also energy equations gives $\frac{1}{2}m\left[\left(\dot{x}^2 + \dot{y}^2\right) + k^2\,\dot{\theta}^2\right] =$ work done by the forces i.e.

$$\frac{1}{2}m\left[\left(a\dot{\theta} + c\cos\theta\,\dot{\theta}\right)^2 + \left(-c\sin\theta\,\dot{\theta}\right)\right] + \frac{1}{2}mk^2\,\dot{\theta}^2 = m\,g\,(c - c\cos\theta)$$

Let ω be the angular velocity when G is in its lowest position i.e. $\dot{\theta} = \omega$ when $\theta = \pi$; thus

we have $\frac{1}{2} m\left[(a-c)^2 + k^2\right] \omega^2 = 2mgc \Rightarrow \omega^2 = \frac{4gc}{k^2 + (a-c)^2}$

Now (2) gives $R = mg - mc\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right)$

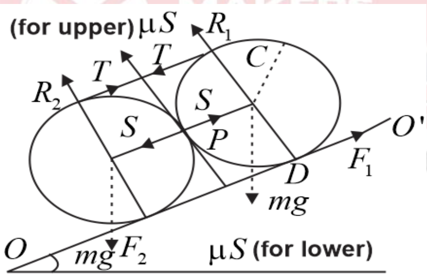
$= mg - mc\cos\pi .\ \omega^2$

(Since in the lowest position $\theta = \pi$; $\dot{\theta} = \omega$)

$= mg + mc = \frac{4cg}{k^2 + (a-c)^2} = mg\left[1 + \frac{4c^2}{k^2 + (a-c)^2}\right]$

Example:- Two equal cylinders, of mass m, are bound together by an elastic string, whose tension is T, and roll with their axes horizontal down a rough plane of inclination α. Show that their acceleration is $\frac{2}{3} g\sin\alpha\left[1 - \frac{2\mu T}{mg\sin\alpha}\right]$, where μ is the coefficient of friction between the cylinders.

Solution:- Let R_1, F_1 be the normal reaction and friction on the upper cylinder and R_2, F_2 be the normal reaction and friction on the lower cylinder due to the plane. Let S be the normal reaction between the two cylinders at P. The force μS acts away from the plane for upper cylinder and towards the plane for the lower cylinder.



At any time t let the cylinders move through a distance z along the plane, and θ be the angle turned by them $z = a\theta \Rightarrow \ddot{z} = a = a\ddot{\theta}$ (1)

Equations of motion of the upper cylinder are given by $m\ddot{z} = mg\sin\alpha + 2T - F_1 - S$ (2)

$0 = R_1 - mg\cos\alpha + \mu S$ (3)

And $mk^2\ddot{\theta} = F_1 \times a - \mu S \times a$ (4)

Whereas the equations of motion for the lower cylinder are given by

$m\ddot{z} = mg\sin\alpha - 2T - F_2 + S$ (5)

$0 = R_2 - mg\cos\alpha - \mu S$ (6)

And $mk^2\ddot{\theta} = F_2 \times a - \mu S \times a$ (7)

Comparing (4) and (7), we have $F_1 = F_2$

Subtracting (2) from (5), we have $S = 2T$ (8)

Also from (4), $F_1 = \frac{mk^2}{a}\ddot{\theta} + \mu S$, where $k^2 = a^2$

$$= \frac{1}{2}m\ddot{z} = mg\sin\alpha + 2T - \left(\frac{1}{3}m\ddot{z} + 2\mu T\right) - 2T \qquad [\because S = 2T]$$

Or $$\ddot{z} = \frac{2}{3}g\sin\alpha\left[1 - \frac{2\mu T}{mg\sin\alpha}\right]$$

Example:- A uniform rod is held at an inclination α to the horizon with one end in contact with a horizontal table whose coefficient of friction is μ, if it be then released show that it will commence to slide if $\mu < \left(\frac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}\right)$

Solution:- Let AB be the rod having length $2a$ and mass m. Let F be the force

Equation (3) can also be obtained by taking moments about G, then

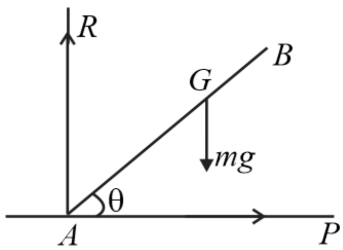
$$M\frac{a^2}{3}\ddot{\theta} = Ra\sin\theta - Fa\cos\theta = Mg\sin\theta - Ma^2\ddot{\theta} \text{ [From (1) and (2)]}$$

Or $$\ddot{\theta} = \frac{3g}{4a}\sin\theta$$

Multiplying by $2\dot{\theta}$ and integrating, we get $\dot{\theta}^2 = -\frac{3g}{2a}\cos\theta + C$

When $\theta = 0$, $\dot{\theta} = 0 \Rightarrow C = \frac{3g}{2a}$ $\therefore \dot{\theta}^2 = \frac{3g}{2a}(1-\cos\theta)$ of friction sufficient to prevent sliding and R the normal reaction. With reference to point A as the origin, the coordinated of point G i.e. C.G. ae $(a\cos\theta, a\sin\theta)$, the coordinates of point G, before the motion begins are $(a\cos\alpha, a\sin\alpha)$.

Thus the vertical distance moved by the C.G. is $(a\sin\alpha - a\sin\theta)$.



Equations of motion of C.G. are $m\frac{d^2x}{dt^2} = m\left[-a\cos\theta\,\dot{\theta} - a\sin\theta\,\ddot{\theta}\right] = F$ (1)

And $m\frac{d^2x}{dt^2} = m\left[-a\sin\theta\,\dot{\theta}^2 - a\cos\theta\,\ddot{\theta}\right] = R - mg$ (2)

The equation of energy given $\frac{1}{2}m\left[\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{3}a^2\dot{\theta}^2\right] = mg(a\sin\alpha - a\sin\theta)$

$$\Rightarrow \quad \frac{1}{2}m\left(a^2\dot{\theta}^2 + \frac{1}{3}a^2\dot{\theta}^2\right) = amg\,(\sin\alpha - \sin\theta)$$

$$\Rightarrow \quad \frac{2}{3}a^2\dot{\theta}^2 = ga(\sin\alpha - \sin\theta) \qquad \Rightarrow \dot{\theta}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta) \qquad (3)$$

Differentiating (3) w.r.t. to t, we get $\ddot{\theta} = \dfrac{-3g}{4a}\cos\theta$ (4)

Putting the value of $\dot{\theta}^2$ and $\ddot{\theta}$ from (3) and (4) in (1) and (2), we

$$F = m\left[-a\cos\theta.\frac{3g}{2a}(\sin\alpha - \sin\theta) - a\sin\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \frac{3}{4}mg\cos\theta(3\sin\theta - 2\sin\alpha) = \frac{3}{4}mg\cos\alpha\sin\alpha, \qquad \text{when } \theta = \alpha$$

and $$R = mg + m\left[-a\sin\theta.\frac{3g}{2a}(\sin\alpha - \sin\theta) + a\cos\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \frac{1}{4}mg\left[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta\right]$$

$$= \frac{1}{4}mg\left(4 - 3\cos^2\alpha\right), \text{ when } \theta = \alpha$$

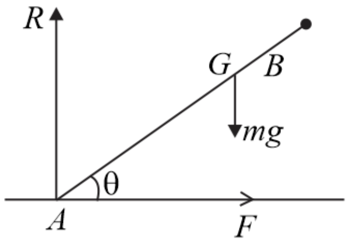
$$= \frac{1}{4}mg\left[1 + 3\left(1 - \cos^2\alpha\right)\right] = \frac{1}{4}mg\left(1 + 3\sin^2\alpha\right)$$

The end A will commence to slide if $\mu < \dfrac{F}{R}$ i.e. $\mu < \dfrac{3\sin\alpha\cos\alpha}{1 + 3\sin^2\alpha}$.

Example:- The lower end of a uniform rod, inclined initially at an angle α the horizon is placed on a smooth horizontal table. A horizontal force is applied its lower end of such a magnitude that the rod rotates in vertical plane with constant angular velocity ω. Show that when the rod is inclined at an angle θ to the horizon the magnitude of the force is $mg\cot\theta - ma\omega^2\cos\theta$ where m is the mass of the rod.

Solution:- Let the horizontal force applied at the lower end A of the rod be F. Let at any time t, θ be the angle that the rod makes with the horizontal. Since the rod rotate with a uniform angular velocity $\omega \therefore \dot{\theta} = \omega$ (constant)

$$\Rightarrow \quad \ddot{\theta} = 0 \qquad (2)$$



The equation of motion of G along the vertical

$$R - mg = m\frac{d^2}{dt^2}(a\sin\theta) = ma\left(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}\right)$$

$= -ma\sin\theta\,.\,\omega^2$ from (1) and (2) (3)

Since the end A is not fixed, the equation of horizontal motion of C.G. is not written .

Again taking moments about G, we have

$$mk^2\ddot{\theta} = Fa\sin\theta - Ra\cos\theta \;\Rightarrow F = R\cot\theta \qquad \{\because \ddot{\theta} = 0 \text{ from (2)}\}$$

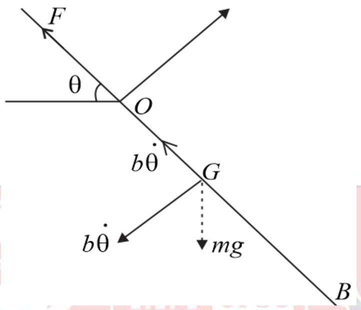
$$\Rightarrow \quad F = \left(mg - ma\sin\theta.\ \omega^2\right)\cot\theta \text{ from (3)}$$

$$\Rightarrow \quad F = mg\cot\theta - ma\,\omega^2\cos\theta$$

Example:- A rough uniform rod, of length $2a$, is placed on a rough table as right angles to its edge: if its centre of gravity by initially at distance b beyond the edge, show that the rod will being to slide when it has turned through an angle $\frac{\mu a^2}{a^2+9b^2}$ where μ is the coefficient of friction.

Solution:- Initially the rod was at right angles to the edge of the rough table, now it has turned through an angle θ. Let there be no sliding when the rod has turned through this angle. A and R be the normal reaction and the force of friction on the rod. Acceleration of G along and perpendicular to GO are respectively $b\dot{\theta}^2$ be $b\ddot{\theta}$. Equations of $mb\ddot{\theta} = mg\ \cos\theta - R$ (1)

And $mb\dot{\theta}^2 = F\ mg\sin\ \theta$ (2)



Taking moments about O, the point contact of the rod and table, we have

$$mk^2\ddot{\theta} = mg\ b\cos\theta, \;\Rightarrow m\left(b^2 + \frac{a^2}{3}\right)\ddot{\theta} = mg\,b\cos\theta$$

$$\Rightarrow \quad \ddot{\theta} = \frac{3gb}{a^2+3b^2}\cos\ \theta \qquad (3)$$

Multiplying (3) by $2\dot{\theta}$ and integrating , we get $\dot{\theta}^2 = \frac{6gb}{a^2+3b^2}\sin\theta$

The constant of integrating vanishes as initially when $\theta = 0,\ \dot{\theta} = 0$. Putting the values of $\ddot{\theta}$ and $\dot{\theta}^2$ in (1) and (2) from (3) and (4), we have

$$R = -gb\ .\frac{3bg}{a^2+3b^2}\cos\theta + mg\cos\theta = \frac{mga^2}{a^2+3b^2}\cos\theta \text{ and}$$

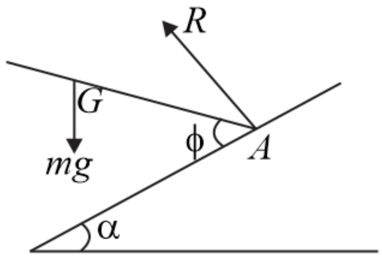
$$F = mg\sin\theta + mb\ \frac{6gb}{a^2+3b^2}\sin\theta = mg\frac{a^2+9b^2}{a^2+3b^2}\sin\theta$$

The sliding commences when $F = \mu R$ i.e. when $mg\frac{a^2+9b^2}{a^2+3b^2}\sin\theta = \mu = \frac{mg\,a^2}{a^2+3b^2}\cos\theta$ or when $\tan\theta = \frac{\mu a^2}{a^2+9b^2}$

Example:- A uniform rod of mass m, is placed at right angle to a smooth plane of inclination α, with one end in contact with it. The rod is then released. Show that when the inclination to the plane is ϕ, the reaction of the plane will $mg\dfrac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\cos\alpha$

Solution:- As there is no force acting the plane, so initially there is no motion along the plane. The C.G. i.e. point G moves perpendicular to the plane.

Let ϕ be the angle which the rod makes with the plane after time t, Taking A as the origin, the plane as x-axis and a line perpendicular to the plane as y-axis, the co-ordinates of G are $x = a\cos\phi$, $y = a\sin\phi$



Equation of motion of point G are $m\ddot{y} = m\left(a\cos\phi\,\ddot{\phi} - a\sin\phi\,\dot{\phi}^2\right) = R - mg\cos\alpha$ (1)

And $m\dfrac{a^2}{2}\dot{\phi} = -R,\ a\cos\phi$ (2)

Also from energy equation, we have $\dfrac{1}{3}ma^2\cos^2\phi\,\dot{\phi}^2 + \dfrac{1}{2}\dfrac{ma^2}{3} = \dot{\phi}^2$ = work done by gravity

$= mg\ a\cos\alpha\,(1-\sin\phi)$ or $\dot{\phi}^2 = \dfrac{6g(1-\sin\phi)}{a(1+3\cos^2\phi)}\cos\alpha$ (3)

Differentiating (3) w.r.t. t, we get

$$\dot{\phi}\ddot{\phi} = \frac{3g\cos\alpha}{a}\left[\frac{-\cos\phi}{(1+3\cos^2\phi)} + \frac{6\cos\phi\sin\phi(1-\sin\phi)}{(1+3\cos^2\phi)}\right]\dot{\phi}$$

$$= -\frac{3g}{a}\cos\alpha\left[\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)}\right]\cos\phi.\,\dot{\phi} \text{ or } \ddot{\phi} = -\frac{3g}{a}\cos\phi\cos\alpha\left[\frac{1+3(1-\sin\phi)^2}{(1+3\cos^2\phi)^2}\right]$$

Putting the value of $\ddot{\phi}$ in (2), we get $R = mg\dfrac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\cos\alpha$

Example:- A uniform rod is held nearly vertically with one end resting on an imperfectly rough plane. It is released from rest and falls forward. The inclination to the vertical at any instant is θ. Prove that

(i) If the coefficient of friction is less than a certain finite amount, the lower end of the rod will slip backwards before $\sin^2(\theta/2) = \left(\dfrac{1}{6}\right)$

(ii) However great the coefficient of friction may be, the lower end will being to slip forward at a value of $\sin^2(\theta/2)$ between $\dfrac{1}{6}$ and $\dfrac{1}{3}$

Solution:- **(i)** Proceeding in the same way as in 3.05, we get $F=\frac{3}{4}Mg\sin\theta\left(3\cos\theta-2\right)$ and $R=\frac{1}{4}mg\left(1-3\cos\theta\right)^2$. Obviously $F=0$ if $\sin\theta=0$ or $3\cos\theta-2=0$ i.e. if $\theta=0$ or $\cos\theta=\frac{2}{3}$ i.e. if $\theta=0$m or $1-2\sin^2\left(\theta/2\right)=\frac{2}{3}$ or $\sin^2\left(\theta/2\right)=\frac{1}{6}$.

The value of F is positive when θ takes all intermediate value between $\theta=0$ and $\theta=\cos^{-1}\frac{2}{3}$ and is continuous function of θ, hence between these two values of θ where F vanishes, F has a maximum value for some θ. Let F_1 be the maximum value. We observe that for $0\le\theta\le\cos^{-1}\frac{2}{3}$ the value of $R\le Mg$.

Thus there is a finite value of μ for which $F_1\ge\mu R$ and therefore for this value of μ, sliding will take place before $\cos^{-1}\frac{2}{3}$ i.e. before $\sin^2\frac{\theta}{2}=\frac{1}{6}$. Since F is positive (in the forward direction) hence the slipping will start in the backward direction.

(ii) We observe from the value of F that if $\cos\theta>3/2$, F changes its sign. i.e. the direction of the friction is reversed if $F'=-F=\frac{3}{4}mg\left(2-3\cos\theta\right)$

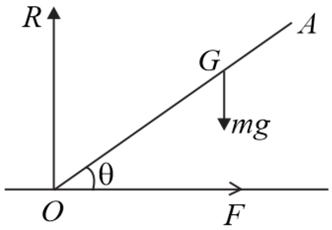
Now the slipping may start when $F'>\mu R$

i.e. when $3\sin\theta\left(2-3\cos\theta\right)>\mu\left(1-3\cos\theta\right)^2$ (1)

As θ increases from $\cos^{-1}\frac{2}{3}$ to $\cos^{-1}\frac{1}{3}$, the term on the left hand side increases while the right hand side term decreases from 1 to 0. Therefore, for some value of θ between $\cos^{-1}\frac{2}{3}$ and $\cos^{-1}\frac{1}{3}$ i.e. for $\sin^2\left(\theta/2\right)$ between $\frac{1}{6}$ and $\frac{1}{2}$ the condition (1) is satisfied and the slipping will then start in the forward direction.

Example:- A uniform rod is placed with one end in contact with a horizontal table, and is then at an inclination α to the horizon and is allowed to fall, when it becomes horizontal, show that its angular velocity is $\left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$ whether the plane is perfectly smooth or perfectly rough. Show also that the end of the rod will not leave the plane in either case.

Solution:- Let at any instant t the rod makes an angle θ with the horizontal. Let R and F be the normal reaction and friction at the instant with O as origin the co-ordinates of C.G. are $x=a\cos\theta$, $y=a\sin\theta$.



Case I:- When plane is perfectly rough and O is fixed.

Then energy equation given $\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}mk^2\dot{\theta}^2 =$ work done by gravity

$$\frac{1}{2}m\left(a^2\dot{\theta}^2+\frac{1}{3}a^2\theta^2\right)=mga\left(\sin\alpha-\sin\theta\right)$$

$$\dot{\phi}^2=\frac{3g}{2a}\left(\sin\alpha-\sin\theta\right) \qquad (1)$$

When the rod becomes horizontal i.e. when $\theta=0$, the angular velocity $\dot{\theta}=\omega$ (say) is given by $\omega^2+\frac{3g}{2a}\sin\alpha$ or $\omega=\left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$

Differentiating (1) w.r.t 't' we get $\ddot{\theta}=\frac{-3g}{4a}\cos\theta \qquad (2)$

The equation of motion of C.G> is $R-mg=m\frac{d^2}{dt^2}\left(a\sin\theta\right)=ma\left(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta}\right)$

$$\Rightarrow \quad R=mg+ma\left[-\sin\theta.\frac{3g}{2a}\left(\sin\alpha-\sin\theta\right)+\cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

[Substituting use values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (1) and (2)]

$$=\frac{1}{4}mg\left(4-6\sin\alpha\sin\theta+6\sin^2\theta-3\cos^2\theta\right)$$

$$=\frac{1}{4}mg\left[\left(1-3\sin\alpha\sin\theta\right)^2-9\sin^2\alpha\sin^2\theta+9\sin^2\theta\right]$$

$$=\frac{1}{4}mg\left[\left(1-3\sin\alpha\sin\theta\right)^2+9\sin^2\theta\left(1-\sin^2\alpha\right)\right]$$

$$=\frac{1}{4}mg\left[\left(1-3\sin\alpha\sin\theta\right)^2+9\sin^2\theta\cos^2\alpha\right]$$

This is show that R is always positive, therefore the end O of the rod never leaves the plane.

Case II:- When the plane is perfectly smooth.

In this case there is no horizontal forces, hence C.G. descends in a vertical line i.e. the only velocity of G being along the vertical direction $y=a\sin\theta$, $\dot{y}=a\cos\theta\dot{\theta}$

The energy equation give $\frac{1}{2}m\dot{y}^2+\frac{1}{2}mk^2\dot{\theta}^2 =$ work done by gravity i.e.

$$\frac{1}{2}m\left(a^2\cos^2\theta\dot{\theta}^2+\frac{1}{3}a^2\dot{\theta}^2\right)=mg\left(a\sin\alpha-a\sin\theta\right) \text{ or}$$

$\dot{\theta}^2\left(\cos^2\theta+\frac{1}{3}\right)=\left(\frac{2g}{a}\right)(\sin\alpha-\sin\theta)$, when the rod becomes horizontal i.e. when $\theta=0$,

the angular velocity $\dot{\theta}=\omega$ (say) is given by

$$\omega^2\left(1+\frac{1}{3}\right)=\frac{2g}{a}\sin\alpha \Rightarrow \omega^2=\frac{3g}{2a}\sin\alpha \Rightarrow \omega=\left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$$

This gives the required result in the case of plane being smooth.
Differentiating (1), we have

$$\ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right)-\dot{\theta}^2\sin\theta\cos\theta=-\left(\frac{g}{a}\right)\cos\theta$$

$$\Rightarrow \quad \ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right)-\sin\theta\cos\theta\left[\frac{(2g/a)(\sin\alpha-\sin\theta)}{\cos^2\theta+\frac{1}{3}}\right]=-\left(\frac{g}{a}\right)\cos\theta$$

$$\Rightarrow \quad \ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right)^2=-\left(\frac{g}{a}\right)\cos\theta\left[\sin^2\theta-2\sin\alpha\sin\theta+\frac{4}{3}\right]$$

$$=-(g/a)\cos\theta\left[(\sin\theta-\sin\alpha)^2+\frac{1}{3}+\cos^2\alpha\right] \qquad (3)$$

Again taking moments about G, we have $m\frac{a^2}{3}\ddot{\theta}=-Ra\cos\theta$ or $R=-\frac{1}{3}a\sec\theta.\, m\ddot{\theta}$

$$\Rightarrow \quad R=\frac{mg}{3}\left[\frac{(\sin\theta-\sin\alpha)^2+\frac{1}{3}+\cos^2\alpha}{\left(\cos^2\theta+\frac{1}{3}\right)^2}\right]$$ from (2) by putting the value of $\ddot{\theta}$

$$\Rightarrow \quad R=mg\left[\frac{1+3\cos^2\alpha+3(\sin\theta-\sin\alpha)^2}{(1+3\cos\theta)^2}\right]$$

We observe that R is positive for every value of α and θ. Hence the end never leaves the plane.

Example:- A heavy rod, of length $2a$ is placed in a vertical plane with its ends in contact with a rough vertical wall and an equality rough horizontal plane, the coefficient of friction being $\tan\varepsilon$. Show that it will being to slip down if its initial inclination to the vertical is grater that 2ε. Prove also that the inclination θ of the rod to the vertical at any time is given by

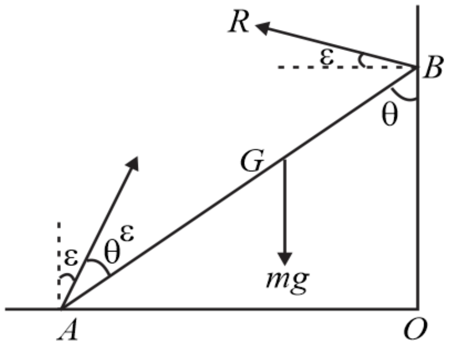
$$\ddot{\theta}\left(k^2+a^2\cos^2\varepsilon\right)-a^2\dot{\theta}^2\sin 2\varepsilon=ag\sin(\theta-2\varepsilon)$$

Solution:- Let AB be the rod of length $2a$ and mass m. When AB makes an angle θ with the vertical and let R and S be the resultant reactions at B and A respectively

Writing equation of motion of centre of mass G, we have

$$m\frac{d^2}{dt^2}(a\sin\theta)=-S\sin\varepsilon+R\cos\varepsilon \qquad (1)$$

And $m\frac{d^2}{dt^2}(a\cos\theta)=R\sin\varepsilon+S\cos\varepsilon-mg \qquad (2)$



Taking moments about G, we have $mk^2\ddot{\theta} = Sa\sin(\theta-\varepsilon) - Ra\cos(\theta-\varepsilon)$ (3)

From (2), we have $ma\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) = R\cos\varepsilon - S\sin\varepsilon$ (4)

From (2), we have $ma\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) = mg - R\sin\varepsilon - S\cos\varepsilon$ (5)

On solving equations (4) and (5), we have

$$R = mg\sin\varepsilon + ma\cos(\theta+\varepsilon)\ddot{\theta} - ma\sin(\theta+\varepsilon)\dot{\theta}^2 \qquad (6)$$

$$S = mg\cos\varepsilon - ma\sin(\theta+\varepsilon)\ddot{\theta} - ma\cos(\theta+\varepsilon)\dot{\theta}^2 \qquad (7)$$

Putting the values of R and S in (3), we have

$$mk^2\ddot{\theta} = a\sin(\theta-\varepsilon)\left[mg\cos\varepsilon - ma\sin(\theta-\varepsilon)\ddot{\theta} - ma\cos(\theta+\varepsilon)\dot{\theta}^2\right]$$

$$-a\cos(\theta-\varepsilon)\left[mg\sin\varepsilon + ma\cos(\theta+\varepsilon)\ddot{\theta} - ma\sin(\theta+\varepsilon)\dot{\theta}^2\right]$$

$$= m\,ga\sin(\theta-2\varepsilon) - ma^2\ddot{\theta}\cos 2\varepsilon + ma^2\dot{\theta}^2\sin 2\varepsilon]$$

Or $\ddot{\theta}\left(k^2 + a^2\cos 2\varepsilon\right) - a^2\dot{\theta}^2\sin 2\varepsilon = ag\sin(\theta-2\varepsilon)$, which gives θ.

If $\theta > 2\varepsilon$, it is obvious that $\ddot{\theta}$ is positive and hence the rod start slipping if $\theta > 2\varepsilon$.

Example:- A hoop is projected with velocity V down on inclined plane of inclination α, the coefficient of friction being $\mu(>\tan\alpha)$. It has initially such a backward spin Ω that after a time t_1 it starts moving uphill and continues to do so for a time t_2 after which it once more descends. The motion being in a vertical at right angles to the given inclined plane, show that $(t_1+t_2)g\sin\alpha = a\Omega - V$.

Solution:- Let C be the centre of the hoop and CB its radius (a line fixed in the body) makes an angle θ with CA which is normal to the plane (CA is a line fixed in space), after time t. Initially CB was normal to the plane. Initially the velocity of the point of contact A down the plane = velocity of centre $C+$ velocity of A relative to $C = V + a\Omega$, which is a positive quantity

Hence the point of contact slides down and friction μR acts up the plane.

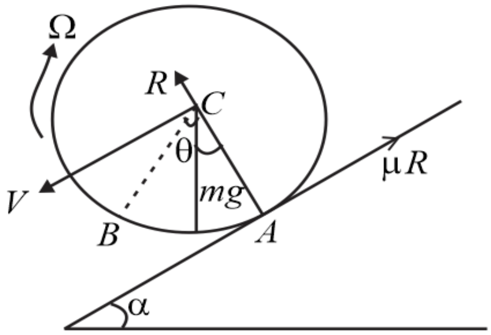
The equations of motion are $m\ddot{x} = mg\sin\alpha - \mu R$ (1)

$0 = R - mg\cos\alpha$ (2)

And $ma^2\ddot{\theta} = -\mu Ra$ (3)

From (1) and (2), we have $\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$, integrating it, we get

$\dot{x} = g(\sin\alpha - \mu\cos\alpha)t + \text{constant}$ when $\dot{x} = V,\ t = 0$. $\quad\therefore$ constant $= V$



Therefore $\dot{x} = g(\sin\alpha - \mu\cos\alpha)t + V$ (4)

From (1) and (3), we get $a\ddot{\theta} = -\mu g\cos\alpha$, integrating it, we get $a\dot{\theta} = -\mu gt\cos\alpha +$ constant; when $t = \theta,\ \dot{\theta} = \Omega$ $\quad$ Constant $= a\Omega$

Therefore $a\dot{\theta} = -\mu gt\cos\alpha + a\Omega$ (5)

The hope will cease to move downwards, when $\dot{x} = 0$ i.e. from (4),

$$t_1 = \frac{V}{g(\mu\cos\alpha - \sin\alpha)} \tag{6}$$

Obviously the velocity of the point of contact is $\dot{x} + a\dot{\theta}$, even when $\dot{x} = 0$, for the hoop to move uphill $a\dot{\theta}$ should be positive. It follows that throughout the downwards motion $\dot{x} + a\dot{\theta}$ is always positive. Therefore when moving downwards pure rolling does not take place. Thus the equations established above are true throughout the downwards motion.

Putting the value of t_1 from (6) in (5), we get

$$a\dot{\theta} = a\Omega - \frac{\mu V\cos\alpha}{(\mu\cos\alpha - \sin\alpha)}$$ since $a\dot{\theta}$ is positive, the hoop beings to move uphill.

When the hoop starts moving uphill. The initial velocity of the centre is zero and $a\dot{\theta}$ is positive with the sense of the direction as θ.

Initial velocity of the point of contact $= 0 - a\dot{\theta}$ which is negative

Thus initially the velocity of the point of contact is in the downwards direction

$m\ddot{y} = -mg\sin\alpha + \mu R$ (1)

$0 = R - mg\cos\alpha$ (2)

$ma^2\ddot{\phi} = -\mu Ra$ (3)

On eliminating R, we get $\ddot{y} = (\mu\cos\alpha - \sin\alpha)g$ and $a\ddot{\phi} = -\mu g\cos a$ integrating these two equations, with the initial condition, we get

$$\dot{y} = g(\mu\cos\alpha - \sin\alpha)t \qquad (4)$$

And $a\dot{\phi} = -\mu g t\cos\alpha + a\Omega - \dfrac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$ (5)

[$\because$ when $t = 0$, $\dot{y} = 0$, $a\dot{\phi} = a\Omega - \dfrac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$]

Rolling commences when the velocity of the point of contact is zero i.e. $\dot{y} - a\dot{\phi} = 0 \quad \Rightarrow \dot{y} = a\dot{\phi}$

$\Rightarrow \quad g(\mu\cos\alpha - \sin\alpha)\,t' = \mu g t'\cos\alpha + a\Omega - \dfrac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$

$\Rightarrow \quad gt'(2\mu\cos\alpha - \sin\alpha) = a\Omega - \dfrac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$

This gives value of t'

$\therefore$ At this time $\dot{y} = gt'(\mu\cos\alpha - \sin\alpha)$ from (4)

When Rolling commences:- Equations of motion are $m\ddot{z} = F - mg\sin\alpha$ (1)

$ma^2\ddot{\psi} = -Fa$ (2)

And $\dot{z} - a\dot{\psi} = 0$ (3)

Solving these equations, we get $F = \dfrac{1}{2}mg\,\sin\alpha$.

Since $\mu > \tan\alpha \Rightarrow \mu R > \tan\alpha\, mg\cos\alpha \Rightarrow \mu R > mg\sin\alpha$

We observe that $F < \mu R$, so the condition of pure rolling is satisfied, and hence the equation of motion holds good for the motion.

From (1), we have $m\ddot{z} = F - mg\sin\alpha = \dfrac{1}{2}mg\sin\alpha - mg\sin\alpha$ i.e. $\ddot{z} = -\dfrac{1}{2}g\sin\alpha$;

integrating it, we get $\dot{z} = -\dfrac{1}{2}gt\sin\alpha + K$ when $t = 0$, $\dot{z} = \dot{y} = gt'(\mu\cos\alpha - \sin\alpha)$,

$\therefore K = gt'(\mu\cos\alpha - \sin\alpha)$. Therefore $\dot{z} = -\dfrac{1}{2}gt\sin\alpha + gt'(\mu\cos\alpha - \sin\alpha)$

The hoop ceases to move up the hill if $\dot{z} = 0$. Let this happen after time t''

$\therefore \quad 0 = -\dfrac{1}{2}gt''\sin\alpha + gt'(\mu\cos\alpha - \sin\alpha)$

Or $t'' = 2\dfrac{(\mu\cos\alpha - \sin\alpha)t'}{\sin\alpha}$

$\therefore \quad t_2 = t' + t'' + t' + 2\dfrac{(\mu\cos\alpha - \sin\alpha)}{\sin\alpha}t' = \left(\dfrac{2\mu\cos\alpha - \sin\alpha}{\sin\alpha}\right)t'$

$$=\left(\frac{2\mu\cos\alpha-\sin\alpha}{\sin\alpha}\right)\cdot\frac{1}{g\left(2\mu\cos\alpha-\sin\alpha\right)}\left(a\Omega-\frac{\mu V\cos\alpha}{\mu\cos\alpha-\sin\alpha}\right)$$

$$=\frac{1}{g\sin\alpha}\left(a\Omega-\frac{\mu V\cos\alpha}{\mu\cos\alpha-\sin\alpha}\right)$$

Hence the total time is t_1-t_2

$$=\frac{V}{g\left(\mu\cos\alpha-\sin\alpha\right)}+\frac{1}{g\sin\alpha}\left(a\Omega-\frac{\mu V\cos\alpha}{\mu\cos\alpha-\sin\alpha}\right)$$

$$=\frac{1}{g\tan\alpha}\left[a\Omega-\frac{\mu V\cos\alpha-V\sin\alpha}{\mu\cos\alpha-\sin\alpha}\right]=\frac{1}{g\sin\alpha}\left(a\Omega-V\right)$$

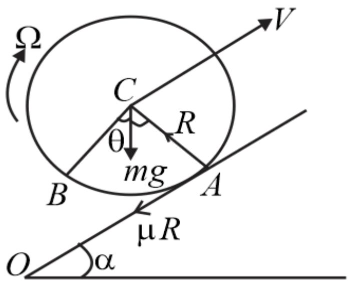
Or $\left(t_1+t_2\right)g\sin\alpha=a\Omega-V$.

Example:- A sphere, of radius a is projected up an inclined plane with a velocity V and angular velocity Ω in the sense which would cause it to roll up $V>a\Omega$, and the coefficient of friction $\frac{2}{7}\tan\alpha$; show that the sphere will cease to ascend at the end of a time $\frac{5V+2a\Omega}{5g\sin\alpha}$ where α is the inclination of the plane.

Solution:- Let C be the centre of the sphere and CB a radius which is a line fixed in the body makes an angle θ after time t with CA normal to the plane CA is a line fixed in space. Initial CB was normal to the plane. Initial velocity of the point of contact A up the plane

= Velocity of the centre $C+$ velocity of A relative to C .

$=V-a\Omega>0$ as $V>a\Omega$



Hence the friction μR acts down the plane, implying that the sphere slides as well as turns.

Equation of motion are $m\ddot{x}=mg\sin\alpha-\mu R$ (1)

$0=R-mg\cos\alpha$ (2)

And $m,\frac{2a^2}{5}\ddot{\theta}=\mu Ra$ (3)

Eliminating R from (1) and (2), we have $\ddot{x}=-g\left(\sin\alpha+\mu\cos\alpha\right)$ integrating it, we get

$\dot{x}=-g\left(\sin\alpha+\mu\cos\alpha\right)t+K$

Now when $t=0,\ \dot{x}=V,\quad\therefore K=V$

Therefore, $\dot{x}=-g\left(\sin\alpha\cos\alpha\right)t+V$ (4)

Similarly, we have $a\ddot{\theta}=\frac{5}{2}\mu gt\cos\alpha$, integrating it with initial conditions i.e. when $t=0,\ \dot{\theta}=\Omega$, we get $a\dot{\theta}=\frac{5}{2}\mu gt\cos\alpha\ +a\Omega$ (5)

The velocity of the point contact $=\dot{x}-a\dot{\theta}$. Rolling commences, say after time t_1 when $\dot{x}-a\dot{\theta}=0$ or $-g(\sin\alpha-\mu\cos\alpha)t_1+V-\frac{5\mu}{2}gt_1\cos\alpha-a\Omega=0$

Or $t_1=\dfrac{2V-2a\Omega}{g(7\mu\cos\alpha+2\sin\alpha)}$

Putting this value of $t=t_1$ in (4), we get

$$\dot{x}=V-g(\sin\alpha+\mu\cos\alpha)\left[\frac{2V-2a\Omega}{g(7\mu\cos\alpha+2\sin\alpha)}\right]$$

$$=\frac{5\mu V\cos\alpha+2a\Omega(\sin\alpha+\mu\cos\alpha)}{7\mu\cos\alpha+2\sin\alpha}=V_1\ \text{(say)}$$

When rolling beings i.e. when the point of contact has been brought to rest, let F be the friction which is sufficient for pure rolling. Because the point of contact is at rest, so friction will try to keep it at rest if possible, hence the friction F acts upwards.

Equations of motion are $m\ddot{y}=-mg\sin\alpha+F$ (1)

And $m.\frac{2a^2}{5}\ddot{\phi}=-Fa$ (2)

Since throughout the motion the point of contact is at rest so $\dot{y}-a\dot{\phi}=0$ or $\dot{y}=a\dot{\phi}$

$\Rightarrow \ddot{y}=a\ddot{\phi}$

Solving equations (1) and (2), we get $F=\frac{2}{7},mg\sin\alpha$

Again $\mu R=\mu.\ mg\cos\alpha>\frac{2}{7}\tan\alpha\ .mg\cos\alpha$ i.e. $>\frac{2}{7}mg\sin\alpha$

Therefore the condition $F<\mu R$ is satisfied.

Putting the value of F in (1), we get $\ddot{y}=-\frac{5}{7}g\sin\alpha$

Integrating it with initial conditions i.e. when $t=0$, $\dot{y}=V_1$, we get $\dot{y}=-\frac{5}{7}gt\ \sin\alpha+V_1$.

The sphere will ceases to ascend when $\dot{y}=0$, let this happen after time t_2.

$\therefore$ $0=-\frac{5}{7}gt_2\sin\alpha+V_1$ or $t_2=\dfrac{7V_1}{5g\sin\alpha}$

$\therefore$ The total time of ascent $=t_1+t_2$

$$=\frac{2V-2a\Omega}{g(7\mu\cos\alpha+2\sin\alpha)}+\frac{7}{5g\sin\alpha}\times\left\{\frac{5\mu V\cos\alpha+2a\Omega(\sin\alpha+\mu\cos\alpha)}{7\mu\cos\alpha+2\sin\alpha}\right\}$$

$$= \frac{10(V - a\Omega)\sin\alpha + 35\mu V\cos\alpha + 14a\Omega(\sin\alpha + \mu\cos\alpha)}{5g\sin\alpha(7\mu\cos\alpha + 2\sin\alpha)}$$

$$= \frac{5V(7\mu\cos\alpha + 2\sin\alpha) + 2a\Omega(7\mu\cos\alpha + 2\sin\alpha)}{5g\sin\alpha(7\mu\cos\alpha + 2\sin\alpha)}$$

$$= \frac{5V + 2a\Omega}{5g\sin\alpha}.$$

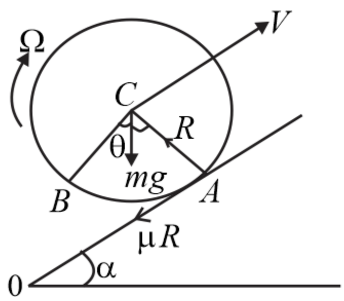
Example:- If a sphere be projected up an inclined plane, for which $\mu = \frac{1}{7}\tan\alpha$, which velocity V and an initial angular velocity Ω (in the direction in which it would roll up), and if $V > a\Omega$ show that friction acts downward at first and upwards afterword's, and prove that the whole time during which the sphere rises is $\frac{17V + 4a\Omega}{18g\sin\alpha}$

Solution:- Let C be the centre of the sphere and CB a radius which is a line fixed in the body makes an angle θ after time t with CA, the normal to the plane (CA is a line fixed in the space). Initially CB was normal to the plane.

Initial velocity of the point of contact A up the plane

$=$ Velocity of the centre $C +$ velocity of A relative of C

$= V - a\Omega > 0$, since $V > a\Omega$.



Hence the velocity of the point of contact A is up the plane, thus the friction μR acts down the plane. The sphere therefore slides as well as turns.

Equations of motion are $m\ddot{x} = -mg\sin\alpha - \mu R$ (1)

$0 = R - mg\cos\alpha$ (2)

And $m\frac{2a^2}{5}\ddot{\theta} = \mu Ra$ (3)

Eliminating R from (1) and (2), we get

$$m\ddot{x} = -mg\sin\alpha - \mu(mg\cos\alpha) = -mg\sin\alpha - \frac{1}{7}\tan\alpha .\ mg\cos\alpha$$

$$= -\frac{8}{7}mg\sin\alpha\left(\mu = -\frac{1}{7}\tan\alpha\right) \text{ or } \ddot{x} = -\frac{8}{7}g\sin\alpha$$

Similarly, we have $m\frac{2a}{5}\ddot{\theta} = \mu R = \frac{1}{7}\tan\alpha\, mg\cos\alpha = \frac{1}{7}mg\sin\alpha$ or $a\ddot{\theta} = \frac{5}{14}g\sin\alpha$ (5)

Integrating (4) and (5) with initial conditions i.e. when $t=0$, $\dot{x}=V$ and $\dot{\theta}=\Omega$, we get

$$\dot{x}=-\frac{8}{7}gt\sin\alpha+V \tag{6}$$

And $$a\dot{\theta}=\frac{5}{14}gt\sin\alpha+a\Omega \tag{7}$$

Let the velocity of the point of contact i.e. $\dot{x}-a\dot{\theta}$ be zero after time t_1 (then the point of contact is brought to rest) i.e. $\dot{x}-a\dot{\theta}=0 \Rightarrow \dot{x}=a\dot{\theta}$

$\Rightarrow \quad -\frac{8}{7}gt_1\sin\alpha+V=\frac{5}{14}gt\sin\alpha+a\Omega$ (Putting the values of $\dot{x}$ and $\dot{\theta}$)

$\Rightarrow \quad t_1=\frac{2(V-a\Omega)}{3g\sin\alpha}$. Putting this value of t_1 in (6), we get

$$\dot{x}=V-\frac{16}{21}(V-a\Omega)=\frac{5V+16a\Omega}{21}=V_1 \text{ (say)}$$

When the point of contact has been brought to rest, the pure rolling will commence if there is enough friction to keep the point of contact at rest. Let F be the force of friction sufficient for pure rolling. Equation of motion are $m\ddot{y}=-mg\sin\alpha+F$, $m\frac{2a^2}{5}\dot{\phi}=-Fa$. Also $\dot{y}-a\dot{\phi}=0$

.

Solving these equations, we get $F=\frac{2}{7}mg\,\sin\alpha$ while $\mu R=\frac{1}{7}\tan\alpha\, mg\cos\alpha=\frac{1}{7}mg\sin\alpha$.

Hence we observe that $F>\mu R$

From this we conclude that the friction required for pure rolling is more than the maximum friction that can be exerted by the plane, so the pure rolling is impossible.

Inspite of exerting the maximum friction μR upwards, the friction cannot keep the point of contact at rest. Hence the sphere as well as turns.

The equations of motion we $m\ddot{y}=-mg\sin\alpha+\mu R$ (i)

$0=R-mg\cos\alpha$ (ii) and $m\frac{2a^2}{5}\ddot{\phi}=-\mu Ra$ (iii)

i.e. $m\ddot{y}=-mg\sin\alpha+\frac{1}{7}\tan\alpha.mg\cos\alpha=-\frac{6}{7}mg\sin\alpha$

$$\dot{y}=-\frac{6}{7}gt\sin\alpha+V_1$$

The sphere will cease to ascend when $\dot{y}=0$, let this happen after times t_2.

$\therefore \quad 0=-\frac{6}{7}gt_2\sin\alpha+V_1$ or $t_2=(7V_1/6g\sin\alpha)$

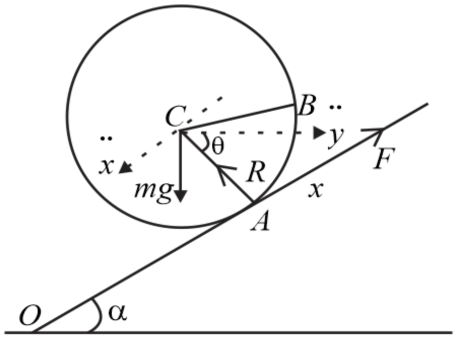
Hence the whole time of ascent $=t_1+t_2=\frac{2(V-a\Omega)}{3g\sin\alpha}+\frac{7}{6g\sin\alpha}-\left(\frac{5V+16a\Omega}{21}\right)$

$$= \frac{12(V-a\Omega)+5V+16a\Omega}{18g\sin\alpha} = \frac{17V+4a\Omega}{18g\sin\alpha}.$$

Example:- An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If y be the horizontal distance advanced by the inclined plane and x the part of the plane over by the sphere, prove that $(M+m)y = mx\cos\alpha$, and $\frac{7}{5}x - y\cos a = \frac{1}{2}gt^2\sin\alpha$ m, where α is the inclination of the plane to the horizon.

Solution:- There are two accelerations of the centre C, one $\ddot{x}$ down the plane and other $\ddot{y}$ in a horizontal direction.

The actual acceleration of C parallel to the plane $= \ddot{x} - \ddot{y}\cos\alpha$



Equations of motion of the sphere are $m\left(\ddot{x} - \ddot{y}\cos\alpha\right) = mg\sin\alpha - F$ (1)

$m\ddot{y}\sin\alpha = mg\cos\alpha - R$ (2)

And $m\frac{2a^2}{5}\ddot{\theta} = Fa$ (3)

Since it is a case of pure rolling $x = a\theta \Rightarrow \dot{x} = a\dot{\theta}$ (4)

Equations of motion of the plane is given by $M\ddot{y} = R\sin\alpha - F\cos\alpha$ (5)

From (1) and (3) on adding, we have

$\frac{7}{5}\ddot{x} - \ddot{y}\cos\alpha = g\sin\alpha$ {from (4) $\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta}$}

Integrating above, we get $\frac{7}{5}\dot{x} - \dot{y}\cos\alpha = gt\sin\alpha$

Integrating again $\frac{7}{5}x - y\cos\alpha = \frac{1}{2}gt^2\sin\alpha$

The constants of integrating vanish initially all $\dot{x}, \dot{y}, x$ and y are zero. Equations (5) is

$M\ddot{y} = R\sin\alpha - F\cos\alpha$

$= \left(mg\cos\alpha - m\ddot{y}\sin\alpha\right)\sin\alpha + \left(m\ddot{x} - m\ddot{y}\cos\alpha - mg\sin\alpha\right)\cos\alpha$

[Putting the values of F and R from (1) and (2)]

Or $M\ddot{y} = -m\ddot{y}\left(\cos^2\alpha + \sin^2\alpha\right) + m\ddot{x}\cos\alpha$

$= -m\ddot{y} + m\ddot{x}\cos\alpha$

$\Rightarrow \quad (M+m)\ddot{y} = m\ddot{x}\cos\alpha$

Integrating, we get $(M+m)\dot{y} = m\dot{x}\cos\alpha$

Again integrating, we get $(M+m)y = mx\cos\alpha$.

The contants of integrating vanish as initially $\dot{x}, \dot{y}, x$ and y are all zero.

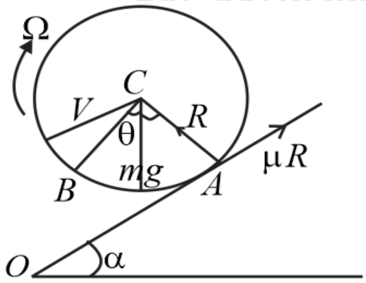
Example:- A uniform sphere, of radius a, is rotating about a horizontal diameter with angular velocity Ω and is gently on a rough plane which is inclined at an angle α to the horizontal, the sense of rotation being such as to tend to cause the sphere to move up the plane along the line of greatest slope. Show that, if the coefficient of friction be $\tan\alpha$, the centre of the sphere will remain at rest for a time $\dfrac{2a\Omega}{5g\sin\alpha}$ and will then move downwards with acceleration $\dfrac{5}{7}\sin\alpha$. If the body be a thin circular hoop instead of sphere, show that the time is $\dfrac{a\Omega}{g\sin\alpha}$ and the acceleration $\dfrac{1}{2}g\sin\alpha$.

Solution:- The sphere before being placed gently on the inclined plane rotating with an angular velocity Ω about the horizontal diameter. Hence initially the velocity of the centre is zero.

The sense of rotation at the time of placing the sphere on inclined plane is such that it tends to cause the sphere to move up the plane, that means sense of Ω is as shown in the figure. The initial velocity of the point of contact A down the plane

$=$ Velocity of the centre $C +$ velocity of A relative to C.

$= 0 + a\Omega$, which is a positive quantity.



Hence the initial velocity of the point of contact is down the plane, so the friction μR acts up the plane.

Equation of motion are $m\ddot{x} = mg\sin\alpha - \mu R$ (1)

$0 = R - mg\cos\alpha$ (2)

And $mk^2\ddot{\theta} = -\mu R$ (3)

Where $\mu = \tan\alpha$

Eliminating R from (1) and (2), we get

$m\ddot{x} = mg\sin\alpha - \tan\alpha,\ mg\cos\alpha = 0 \Rightarrow \ddot{x} = 0 \Rightarrow \dot{x} = 0$ (4)

From (2) and (3), we get (Initially when $t = 0,\ \dot{x} = 0$)

$mk^2\ddot{\theta} = -\tan\alpha\left(mg\cos\alpha\right)a = -mga\sin\alpha$ or $k^2\ddot{\theta} = -ga\sin\alpha$

Integrating it, we get $k^2\dot{\theta} = -gat\sin\alpha + k^2\Omega$

From equation (4) and (5), we observe that the centre of the sphere does not move at all, but the sphere goes on revolving.

The sphere will cease to rotate when $\dot{\theta} = 0$

$\therefore$ From (5), we get $0 = -gat\sin\alpha + k^2\Omega$ or $t = \dfrac{k^2\Omega}{ga\sin\alpha}$

For sphere $k^2 = \frac{2}{5}a^2$, and for the hoop $k^2 = a^2$, hence the sphere will remain at rest for a time $\frac{2}{5}\dfrac{a\Omega}{g\sin\alpha}$ and for the hoop this time will be $\dfrac{a\Omega}{g\sin\alpha}$.

Now when $\dot{x}$ and $a\dot{\theta}$ become zero, the velocity of the point of contact $\left(\dot{x} + a\dot{\theta}\right)$ becomes zero, therefore pure rolling may commence provided the friction is sufficient for pure rolling. Let F be the value of friction sufficient for pure rolling.

The equation of motion are $m\ddot{y} = mg\sin\alpha - F$ (i)

$mk^2\ddot{\phi} = Fa$ … (ii) and $\dot{y} - a\dot{\phi} = 0$ (iii)

As $\dot{y} - a\dot{\phi} = 0 \Rightarrow \dot{y} = a\dot{\phi} \Rightarrow \ddot{y} = a\ddot{\phi}$

Solving (i) and (ii) with the help of (iii) we get $F = \dfrac{mg\sin\alpha}{1+\left(a^2/k^2\right)}$ which obviously less than $mg\sin\alpha$.

When $F < \mu R$, the rolling continuous and the equations (i), (ii) and (iii) hold good.

From (ii) we get $k^2\ddot{\phi} = \dfrac{ag\sin\alpha}{1+\left(a^2/k^2\right)}$ or $\ddot{y} = \dfrac{ga^2\sin\alpha}{\left(a^2+k^2\right)}$ $(\because a\ddot{\phi} = \ddot{y})$

Putting $k^2 = \frac{2}{5}a^2$, $\ddot{y}$ i.e. acceleration in case of sphere is $\frac{5}{7}g\sin\alpha$

Putting $d^2 = a^2, \ddot{y}$ i.e. acceleration in case of hoop is $\frac{1}{2}g\sin\alpha$.

Example:- A homogenous sphere of radius a, rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $\dfrac{2\omega a}{7\mu g}$ and that then the sphere will roll with angular velocity $\left(2\omega/7\right)$

.

Solution:- Since the sphere is gently placed on the table, the initial velocity of the centre of the sphere is zero, while angular velocity is ω.

Initial velocity of the point contact = initial velocity of the centre $C +$ Initial velocity of the point of contact with respect to C.

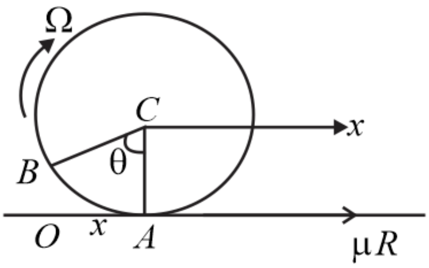
$= 0 + a\omega$ in direction $\leftarrow$

Hence the point of contact will slip in the direction $(\leftarrow)$, therefore full friction μR acts in the direction $(\rightarrow)$.

Let x e the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns, then at any time t equations of motion are,

$m\ddot{x} = \mu R$ (1)

(Here $R = mg$) and $m\dfrac{2a^2}{5}\ddot{\theta} = -\mu Ra$ (2)



Therefore, from (1) $\ddot{x} = \mu g$ and from (2) $\dfrac{2}{5}a\ddot{\theta} = -\mu g$

Integrating these equations, we get $\dot{x} = \mu gt$ (3)

And $a\dot{\theta} = -\dfrac{5}{2}\mu gt + a\omega$ (4)

Since initially when $t = 0$, $\dot{x} = 0$, $\dot{\theta} = \omega$

Velocity of the point contact $= \dot{x} - a\dot{\theta}$

Hence the point of contact will come to rest when $\dot{x} - a\dot{\theta} = 0$ i.e. when $\mu gt - \left(-\dfrac{5}{2}\mu gt + a\omega\right) = 0$ or when $t = \dfrac{2a\omega}{7\mu g}$.

Therefore, after time $\dfrac{2a\omega}{7\mu g}$ the shipping will stop and pure rolling will commence. Putting this value of t in (4), we get $\dot{\theta} = \dfrac{2\omega}{7}$ when rolling commences, the equations of motions are $m\ddot{x} = F$ (i)

$m\dfrac{2a^2}{5}\ddot{\theta} = -Fa$ (ii) and $\dot{x} - a\dot{\theta} = 0$ (iii)

From (i) and (ii) with the help of (iii), we get $ma\ddot{\theta} = F$ and $\dfrac{2}{5}ma\ddot{\theta} = -F$

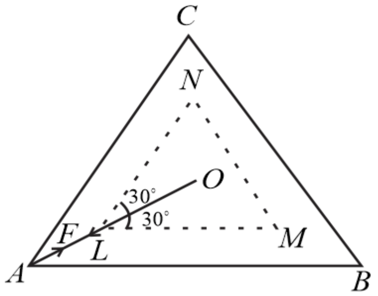
$\left(\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta}\right)$

Adding these two equations, we get $\dfrac{7}{5}ma\ddot{\theta} = 0$ or $\ddot{\theta} = 0 \Rightarrow \dot{\theta} =$ const. $= \dfrac{2\omega}{7}$.

Example:- Three uniform spheres, each of radius a and of mass m attract one another according to the law of the inverse square of the distance. Initially they are placed on a perfectly rough horizontal plane with their centres forming a triangle whose sides are each of length $4a$. Show that the velocity of their centres when they collide is $\left(\gamma\dfrac{5m}{14a}\right)^{1/2}$ where γ is the constant of gravitation.

Solution:- Let A, B and C be the points of contact of the spheres with the horizontal plane, when they are initially at rest. ABC is an equilateral triangle of side $4a$. Let O be the centre of the triangle ABC

Due to the symmetry of the attraction, the spheres will move in the way that their points of contact with the horizontal plane always form equilateral triangle.



Let L, M, N be the new positions of the points of contact with the horizontal plane after time t

Let $OL = x$

By geometry, we observe that $OL = x\dfrac{LM}{\sqrt{3}}$ $\left(\because \dfrac{1}{2}\dfrac{LM}{x} = \cos 30^\circ\right)$

Therefore, initially $x = \left(\dfrac{4a}{\sqrt{3}}\right)$ because initially the side of the triangle is $4a$.

Now when the spheres collide $x = \left(\dfrac{2a}{\sqrt{3}}\right)$ because in this case the sides of the triangle will become $2a$ (As radius of each sphere is a, so the distance between their centres will be $2a$)

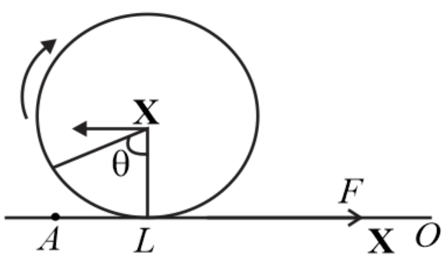
Let L be the point of contact of the first sphere with horizontal plane at time t.

Force of attraction on this sphere due to other two spheres is $= \left(\dfrac{\gamma m^2}{LM^2}\cos 30^\circ + \dfrac{\gamma m^2}{LN^2}\cos 30^\circ\right)$

in the direction LO

$$= \frac{\gamma m^2}{3x^2}.\frac{\sqrt{3}}{2} + \frac{\gamma m^2}{3x^2}.\frac{\sqrt{3}}{2}\left(LM = LN = x\sqrt{3}\right)$$

$= \dfrac{\gamma m^2}{\sqrt{3x^2}}$ in the direction LO i.e. towards x decreasing.



As the plane is perfectly rough, there is pure rolling thus the force of friction at the point of contact is F and acts opposite to the tendency of the motion of the point of contact, i.e. F acts towards x decreasing.

The equations of motion of the first sphere are $m\ddot{x} = -\left(\dfrac{\gamma m^2}{x^2\sqrt{3}}\right) - F$ (1)

$$m\left(\frac{2a^2}{5}\right)\ddot{\theta} = -Fa \tag{2}$$

Since there is no slipping, the velocity of the point of contact $\dot{x} + a\dot{\theta}$ is zero i.e.

$$\dot{x} = -a\dot{\theta} \Rightarrow \ddot{x} = -a\ddot{\theta} \tag{3}$$

From (1), (2) and (3) on eliminating F and $a\ddot{\theta}$, we have $\ddot{x} = -\dfrac{5\gamma m}{7x^2\sqrt{3}}$

Integrating, we get $\left(\dot{x}\right)^2 = \dfrac{10\gamma m}{7\sqrt{3x}} + K$

Now, when $x = \dfrac{4a}{\sqrt{3}}$, $\dot{x} = 0$, $\quad \therefore K = -\dfrac{10\gamma m}{7\sqrt{3}} \cdot \dfrac{\sqrt{3}}{4a}$

$$\therefore \quad \left(\dot{x}\right)^2 = \frac{10\gamma m}{\sqrt{3}}\left(\frac{1}{x} + \frac{\sqrt{3}}{4a}\right) \tag{4}$$

When the spheres collide i.e. when $x = \dfrac{2a}{\sqrt{3}}$; from (4), the velocity at that time is

$$\left(\dot{x}\right)^2 = \frac{10\gamma m}{7\sqrt{3}}\left(\frac{\sqrt{3}}{2a} - \frac{\sqrt{3}}{4a}\right) \text{ or } \dot{x} = \left(\gamma\frac{5m}{14a}\right)^{1/2}$$

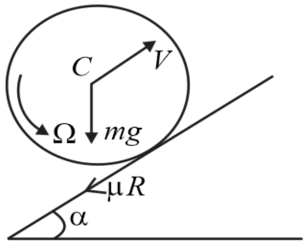
Example:- A thin napkin rings, of radius a is projected up a plane inclined at angle α to the horizontal with velocity v, and an initial angular velocity Ω in the sense which would cause the ring to move down the plane. If $v > 5a\Omega$ and $\mu = \dfrac{1}{4}\tan\alpha$, show that the ring will never roll and will cease no ascend at the end of a time $\dfrac{4(2v - a\Omega)}{9g\sin\alpha}$ and will slide back to the point of projection.

Solution:- Initial velocity of the point of contact is $v + a\Omega$, which is up the plane, hence the friction μR acts down the plane.

The equation of motion are $m\ddot{x} - mg\sin\alpha - \mu R$

$$= -mg\sin\alpha - \mu mg\cos\alpha \text{ or } \ddot{x} = -g\left(\sin\alpha + \mu\cos\alpha\right)$$

$$= -g\left(\sin\alpha + \frac{1}{4}\tan\alpha\cos\alpha\right) \text{ or } \ddot{x} = -\frac{5}{4}g\sin\alpha \tag{1}$$



And $ma^2\ddot{\theta} = -\mu Ra = -\frac{1}{4}\tan\alpha \,.\, mg\cos\alpha \,.\, a = -\frac{1}{4}mg\,a\sin\alpha$

$\therefore \quad a\ddot{\theta} = -\frac{1}{4}\,g\sin\alpha$ (2)

Integrating (1) and (2) and applying initial conditions that at $t = 0$, $\dot{x} = v$ and $\dot{\theta} = \Omega$, we get

$\dot{x} = -\frac{5}{4}g\sin\,\alpha,\, t + v$ (3)

And $\dot{\theta} = -\frac{1}{4}g\sin\alpha\,.\,t + a\Omega$ (4)

From (3), we observe that velocity of the centre is zero after time $\frac{4v}{5g\sin\alpha}$.

The velocity of the point of contact at any time $\dot{x} + a\dot{\theta}$

$= -\frac{5}{4}g\sin\alpha + v - \frac{1}{4}g\sin\alpha\,.\,t + a\Omega$ {From (3) and (4)}

$= v + a\Omega - \frac{3}{2}g\,t\sin\alpha$

Hence the point of contact will come to rest after time $\frac{2(v + a\Omega)}{3g\sin\alpha}$ $\left(\because \quad \dot{x} + a\dot{\theta} = 0\right)$

It can be seen that $\frac{2(v + a\Omega)}{3g\sin\alpha} < \frac{4v}{5g\sin\alpha}$ as $v > 5a\Omega$

$\therefore$ Pure rolling may begin before the upward motion ceases if the friction is sufficient for pure rolling.

At this time $\dot{x} = \frac{v - 5a\Omega}{6}$ which is positive and $\dot{\theta} = \frac{5a\Omega - v}{6a}$ which is $-ve\left(\because\ v > 5a\Omega\right)$ or

$\dot{\theta} = \frac{v - 5a\Omega}{6a}$ in clockwise direction

When pure rolling commences, and rotation is in the, clockwise direction, the equations of motion are $m\ddot{y} = -mg\ \sin\alpha + F$

$ma^2\ddot{\phi} = -Fa,\ \dot{y} = a\dot{\phi}$ and $\ddot{y} = a\ddot{\phi}$.

Solving these equation, we get $F = \frac{1}{2}mg\ \sin\alpha$

But $\mu R=\frac{1}{4}\tan\alpha\,.\,mg\;\cos\alpha=\frac{1}{4}m\,g\sin\alpha$; hence friction is not sufficient for pure rolling. Hence the sliding persists and pure rolling is not possible. The above equations of motion now become $m\,\ddot{y}=-mg\sin\alpha+\mu R=-mg\sin\alpha+\frac{1}{4}\tan\alpha\,.\,mg\cos\alpha$

$=-\frac{3}{4}mg\;\sin\alpha$ or $\ddot{y}=-\frac{3}{4}g\sin\alpha$ (i)

And $ma^2\ddot{\phi}=-\mu Ra=-\frac{1}{4}\tan\alpha\;.\;mg\cos\alpha\,.\,a=-\frac{1}{4}mg\,a\sin\alpha$ or $a\ddot{\phi}=-\frac{1}{4}g\sin\alpha$ (ii)

Integrating (i) and (ii) and applying the initial conditions when $t=0,\ \dot{y}=\frac{v-5a\Omega}{6}$ and $a\dot{\phi}=\frac{5a\Omega-v}{6}$, we get $\dot{y}=-\frac{3}{4}g\sin\alpha\;.\;t+\frac{v-5a\Omega}{6}$ (iii)

And $a\dot{\phi}=-\frac{1}{4}g\sin\alpha\;.\;t+\frac{5a\Omega-v}{6}$

We observe that $\dot{y}=0$ after time $\frac{2(v-5a\Omega)}{9\,g\sin\alpha}$

Putting this value of time (iv), we get $a\dot{\phi}=\frac{2(5a\Omega-v)}{9}$

Therefore, total time of upwards motion $=2\frac{(v-a\Omega)}{3g\sin\alpha}+\frac{2(v-5a\Omega)}{9g\sin\alpha}=\frac{4(2v-a\Omega)}{9g\;\sin\alpha}$

Again, when the upwards motion ceases, we have $a\dot{\phi}=\frac{2}{9}(5a\Omega-v)$ which is negative since $v>5a\Omega$, hence the ring returns.

The velocity of the point of contact.

$=$ Velocity of the centre $+$ velocity relative to the centre

$=\dot{y}-a\dot{\phi}=0-\frac{2}{9}(5a\Omega)=\frac{2}{9}(v-5a\Omega)$

$=a$ positive quantity as $v>5a\Omega$ i.e. the velocity of the point of contact is up the plane ; therefore friction μR acts downwards ; hence the equations of motion are $m\ddot{z}=mg\;\sin\alpha+\mu R=mg\sin\alpha+\frac{1}{4}\tan\alpha\;.\;mg\,.\cos\alpha$

i.e. $\ddot{z}=\frac{5}{4}g\sin\alpha$ (1)

and $ma\ddot{\psi}=-\mu Ra=-\frac{1}{4}\tan\alpha\,mg\cos\alpha.\;a$ i.e. $a\ddot{\psi}=-\frac{1}{4}g\;\sin\alpha$ (2)

Integrating (1) and (2) and applying the initial condition that when $t=0,\ \dot{z}=0$ and $a\dot{\psi}=\frac{2}{9}(5a\Omega-v)$, we get $\dot{z}=\frac{5}{4}\,gt\;\sin\alpha,\ a\dot{\psi}=\frac{1}{4}gt\sin\alpha+\frac{2}{9}(5a\Omega-v)$

Hence the velocity of the point of contact down the plane $=\dot{z}-a\dot{\psi}$

$$= \frac{5}{4} gt\sin\alpha - \left[-\frac{1}{4} gt \ \sin\alpha + \frac{2}{9}(5a\Omega - v) \right]$$

$$= \frac{2}{9}(v - 5a\Omega) + \frac{3}{2} gt\sin\alpha$$

Which is positive $(v > 5a\Omega)$; hence the ring slides back to the point of projection.

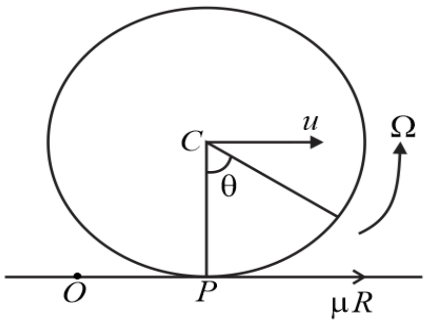
Example:- A napkin ring, of radius a , is projected forward on a rough horizontal table with a linear velocity u and a backward spin Ω which is $> \frac{u}{a}$. Find the motion and show that the ring will return to the point of projection in time $\dfrac{(\mu + a\Omega)^2}{4\mu g(a\Omega - u)}$ where μ is the coefficient of friction. What happens if $\mu > a\Omega$?

Solution:- Initially $u \rightarrow$, $\mu \uparrow$ and $u < a\Omega$. This initial velocity of the point of contact is $u + a\Omega$ and is in the direction $(\rightarrow)$. Hence the friction μR acts in the direction $(\rightarrow)$. For this forward motion, equations of motion are $m\ddot{x} = -\mu R = -\mu mg$ i.e. $\ddot{x} = -\mu g$ (1)

And $ma^2\ddot{\theta} = -\mu Ra = -\mu mg$ i.e. $a\ddot{\theta} = -\mu g$ (2)

Integrating (1) and (2) and applying the initial conditions that when $t = 0$, $\dot{x} = u$ and $\dot{\theta} = \Omega$, we get $\dot{x} = -\mu gt + u$ (3)

And $a\dot{\theta} = -\mu gt + a\Omega$ (4)



The ring ceases to move forward if $\dot{\theta} = 0$, let this happen after time t_1, then from (3) $t_1 = \frac{u}{\mu g}$

Again integrating (4) and applying the condition that when $x = 0, t = 0$, we get

$$x = -\frac{1}{2}\mu g t^2 + ut \quad (5)$$

Thus the distance traversed by the ring in time $t = \frac{u}{\mu g}$ is found by putting $t = \frac{u}{\mu g}$ In (6) i.e.

$$x = -\frac{1}{2}\mu g\left(\frac{u^2}{\mu^2 g^2}\right) + u\left(\frac{u}{\mu g}\right) = \frac{u^2}{2\mu g} \quad (6)$$

And then $a\dot{\theta} = a\Omega - u$ which is in the direction $\uparrow (\because u < a\Omega)$

Hence the ring returns.

When the ring returns

Initial velocity of the point of contact is in the direction $(\rightarrow)$ hence the friction μR will act in the direction $(\leftarrow)$.

For this motion equations are $m\ddot{y} = \mu R = \mu mg$ i.e. $\ddot{y} = \mu g$ and $ma^2\ddot{\phi} = -\mu Ra = -\mu mga$

i.e. $a\ddot{\phi} = -\mu g$ (ii)

Integrating (i) and (ii) and applying the initial condition i.e. when $t = 0, \dot{y} = 0$ and $a\dot{\phi} = a\Omega - u$, get $\dot{y} = \mu gt$ (iii) and $a\dot{\phi} = -\mu gt + a\Omega - u$ (iv)

This equations hold good unit pure rolling commence i.e. unit $\dot{y} = a\dot{\phi}$ (the velocity of the point of contact) is zero. Let this occur after time t_2 then from (iii) and (iv), we have $\mu gt_2 + \mu gt_2 - a\Omega + u = 0$ i.e. $t_2 = \dfrac{a\Omega - u}{2\mu g}$

Hence from (iii) $\dot{y} = \dfrac{a\Omega - u}{2}$ (v)

Integrating (iii) again, we get $y = \dfrac{1}{2}\mu gt^2$ $(\because at\, t = 0,\ y = 0)$

Putting $t = \dfrac{a\Omega - u}{2\mu g}$, we get $y = \dfrac{1}{2}\mu g\left(\dfrac{a\Omega - u}{2\mu g}\right)^2 = \dfrac{(a\Omega - u)^2}{8\mu g}$ (vi)

When rolling begins, equations of motion are $m\ddot{z} = F$ and $ma^2\ddot{\psi} = -Fa$.

Since there is not sliding, hence $\dot{z} = a\dot{\psi} \Rightarrow \ddot{z} = a\ddot{\psi}$.

On solving these equations, we get $F = 0$, hence no friction is required then $\ddot{z} = 0$ i.e. $\dot{z} =$ constant $= \left\{\because\ at\ t = 0,\ \dot{z} = \dfrac{a\Omega - u}{2}\ from(v)\right\}$ i.e. when pure rolling commences (in return motion) the ring continues to move with its initial constant velocity $\dfrac{a\Omega - u}{2}$.

Again the point where pure rolling commences is from the point of projection at distance $= \dfrac{u^2}{2\mu g} - \dfrac{(a\Omega - u)^2}{8\mu g}$ {from (6) and (vi)}

Therefore, the time taken to traverse this distance is $t_3 = \left\{\dfrac{u^2}{2\mu g} - \dfrac{(a\Omega - u)^2}{8\mu g}\right\} - \left(\dfrac{a\Omega - u}{2}\right)$

or $t_3 = \dfrac{u_2}{\mu g(a\Omega - u)} - \dfrac{a\Omega - u}{4\mu g}$.

Hence the total time when the ring returns to the point of projection is $t_1 + t_2 + t_3 = \dfrac{u}{\mu g} + \dfrac{a\Omega - u}{2\mu g} + \left\{\dfrac{u^2}{\mu g(a\Omega - u)} - \dfrac{a\Omega - u}{4\mu g}\right\}$

$$= \frac{u}{\mu g} + \frac{a\Omega - u}{4\mu g} + \frac{u_2}{\mu g(a\Omega - u)} = \frac{(a\Omega - u)^2}{4\mu g(a\Omega - u)}.$$

Second Part:- What happens when $u > a\Omega$?

To know this we should consider the motion in the forward direction already discussed in the beginning.

In that case velocity of the point of contact is $\dot{x} + a\dot{\theta} = (-\mu gt - u) + (-\mu gt + a\Omega)$ [From (3) and (4)] $= -2\mu gt + a\Omega + u$

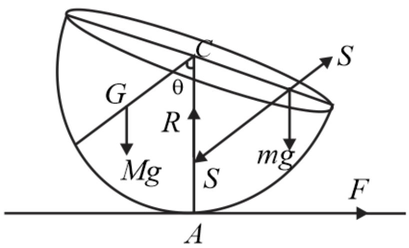
Rolling will commences when $\dot{a} + a\dot{\theta} = 0$ i.e. when $t = \dfrac{u + a\Omega}{2\mu g}$.

Again it is proved that the ring ceases to move forward after a time $\dfrac{u}{\mu g}$ for the moment of projection.

Hence the rolling commences before the forward motion has ceased i.e. if $\dfrac{u + a\Omega}{2\mu g} < \dfrac{u}{\mu g}$ i.e. $u > a\Omega$. In other words we say that $u > a\Omega$, the rolling will commence before the forward motion ceases.

Example:- A homogeneous solid hemisphere, of mass M and radius a, rests with its vertex in contact with a rough horizontal plane and a particle, of mass m, is placed on its base; which is smooth, at a distance c from the centre. Show that the hemisphere will commence to roll or slide according as the coefficient of friction is greater or less than $\dfrac{25mac}{26(M+m)a^2 + 40mc^2}$

Solution:- Let C be the centre of the base and G the centre of gravity of the hemisphere. At point P, distant c from the centre, a particle of mass m is placed. What CG is inclined at an angle θ to the vertical, let A the point of contact have moved through a horizontal distance x from its initial position O, i.e. $OA = x$. Assume that the hemisphere rolls, and the point of contact A is at rest, so $x = a\theta$, hence $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$.



The co-ordinates of G, referred to O as origin are $\left(a\theta - \dfrac{3}{8}a\sin\theta,\ a - \dfrac{3a}{8}\sin\theta\right)$

The equations of motion of the hemisphere are $F - S\sin\theta = M\dfrac{d^2}{dt^2}\left(a\theta - \dfrac{3a}{8}\sin\theta\right)$

$$= M\left[a\ddot{\theta} - \frac{3a}{8}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right)\right] \qquad (1)$$

$$R - Mg - S\cos\theta = M\frac{d^2}{dt^2}\left(a - \frac{3a}{8}\cos\theta\right)$$

$$= M\left(\frac{3a}{8}\sin\theta\,\ddot{\theta} + \frac{5a}{8}\cos\theta\,\dot{\theta}^2\right) \qquad (2)$$

Taking moments about G,

$$Sc - F\left(a - \frac{3a}{8}\cos\theta\right) - R\frac{3a}{8}\sin\theta = Mg^2\,\ddot{\theta} \qquad (3)$$

The co-ordinates of particle P re $\left(a\theta + c\cos\theta,\ a - c\sin\theta\right)$, where $GP = c$

The equation of motion of the particle is

$$S\cos\theta - mg = m\frac{d^2}{dt^2}\left(a - c\sin\theta\right) = m\left(-c\cos\theta\,\ddot{\theta} + c\sin\theta\,\dot{\theta}^2\right) \qquad (4)$$

As the initial motion is required i.e. when $\theta = 0$, $\dot{\theta} = 0$ but $\ddot{\theta} \neq 0$ we have from (1), (2), (3) and (4)

$$\left.\begin{array}{l} F = \frac{5}{8}Ma\,\ddot{\theta},\ R = Mg + S \\ Sc - \frac{5a}{8}F = Mk^2\,\ddot{\theta} \text{ and } S = mg - mc\,\ddot{\theta} \end{array}\right] \text{for the initial values.}$$

Eliminating F and S from first, third and fourth of the late for above equations, we get

$$\left(Mk^2 + \frac{25}{64}Ma^2 + mc^2\right)\ddot{\theta} = mgc \qquad (5)$$

But $Mk^2 = \frac{2}{5}Ma^2 - M\left(\frac{3a}{8}\right)^2 = \frac{83}{320}Ma^2$

Hence (5) reduces to $\left(\frac{83}{320}Ma^2 + \frac{25}{64}Ma^2 + mc^2\right)\ddot{\theta} = mgc$ or $\ddot{\theta} = \frac{20\,mgc}{13Ma^2 + 20mc^2}$. Then

$$F = \frac{5}{8}Ma.\frac{20mgc}{13Ma^2 + 20mc^2} \text{ and}$$

$$R = Mg + mg - mc.\frac{20mgc}{13Ma^2 + 20mc^2} = \frac{13Ma^2(M+m) + 20Mmc^2}{13Ma^2 + 20mc^2}g$$

$$\therefore \qquad \frac{F}{R} = \frac{25\,mac}{26(M+m)a^2 + 40mc^2}$$

The hemisphere will commence to roll or slide

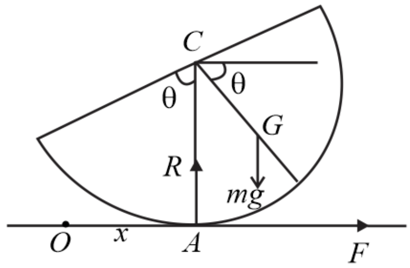
If $F <$ or $> \mu R$ i.e. If $\mu >$ or $< \frac{F}{R}$

Or $\mu >$ or $< \frac{25\,mac}{26(M+m)a^2 + 40mc^2}$

Example:- If a uniform semi-circular wire be placed in a vertical plane with one extremity on a rough horizontal plane, and the diameter through that extremity vertical, show that the semi-circle will begin to roll or slide according as μ be greater or less than $\frac{\pi}{\pi^2 - 2}$. If μ has this value, prove that the wire will roll.

Solution:- Let C be the centre of the base of the semi-circular wire and G be its centre of gravity, then $CG = \frac{2a}{\pi}$.

Let as assume that the wire rolls. When CG is inclined at an angle θ to the horizontal, let the point of contact A have moved through a distance x from its initial position O, i.e. $OA = x$. Since the motion is assumed to be of pure rolling, therefore $x = a\theta$



$\therefore \quad \dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$.

The co-ordinates of the centre of gravity G with reference to O as origin are $\left(x + \frac{2a}{\pi}\cos\theta, a - \frac{2a}{\pi}\sin\theta\right)$

Equations of motion of the wire are:

$$F = m\frac{d^2}{dt^2}\left(x + \frac{2a}{\pi}\cos\theta\right) = m\frac{d^2}{dt^2}\left(a\theta + \frac{2a}{\pi}\cos\theta\right)$$

$$= m\left(a\ddot{\theta} - \frac{2a}{\pi}\sin\theta\,\ddot{\theta} - \frac{2a}{\pi}\cos\theta\,\dot{\theta}^2\right) \qquad (1)$$

$$R - mg = m\frac{d^2}{dt^2}\left(a - \frac{2a}{\pi}\sin\theta\right) = m\left(-\frac{2a}{\pi}\cos\theta\,\ddot{\theta} + \frac{2a}{\pi\sin}\theta\,\dot{\theta}^2\right) \qquad (2)$$

$$\text{And } R\frac{2a}{\pi}\cos\theta - F\left(a - \frac{2a}{\pi}\sin\theta\right) = mk^2\ddot{\theta} \qquad (3)$$

Since we want only initial motion, when $\theta = 0$; $\dot{\theta} = 0$, but $\ddot{\theta} \neq 0$. The equations (1), (2) and (3) give us $F = ma\ddot{\theta}, R = mg - m\frac{2a}{\pi}\ddot{\theta}\,; R\frac{2a}{\pi} - Fa = mk^2\ddot{\theta}$ **For the initial values.**

On eliminating F and R between these equations, we get

$$\left(k^2 + \frac{4a^2}{\pi^2} + a^2\right)\ddot{\theta} = \frac{2a}{\pi}g \qquad (4)$$

But $mk^2 = ma^2 - m\left(\frac{2a}{\pi}\right)^2$ or $k^2 = a^2 - \frac{4a^2}{\pi^2}$

Thus (4) gives us, $\left(a^2 - \frac{4a^2}{\pi^2} + \frac{4a^2}{\pi^2} + a^2\right)\ddot{\theta} = \frac{2a}{\pi}g$ or $\ddot{\theta} = \frac{g}{\pi a}$

Then $F = ma\ddot{\theta} = ma.\frac{g}{\pi a} = \frac{mg}{\pi}$

$$R = mg - m\frac{2a}{\pi}\dot{\theta} = mg - m\frac{2a}{\pi}.\frac{g}{\pi a} = mg\left(1 - \frac{2}{\pi^2}\right) = mg\frac{\pi^2 - 2}{\pi^2}$$

$\therefore \qquad \dfrac{F}{R} = \dfrac{mg}{\pi}\cdot\dfrac{\pi^2}{mg\left(\pi^2-2\right)} = \dfrac{\pi}{\pi^2-2}$

Hence the wire will roll or slide according as

$F <$ or $> \mu R$ or $\mu >$ or $< \dfrac{F}{R}$ or $\mu >$ or $< \dfrac{\pi}{\pi^2-2}$

If μ has this value then the wire will commence to roll

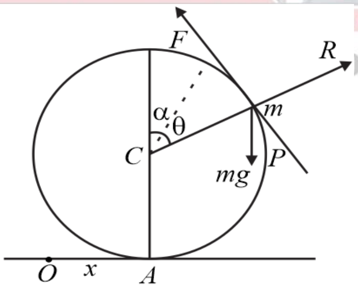
If $k^2 > \dfrac{a^2}{3}$ i.e. If $a^2 - \dfrac{4a^2}{\pi^2} > \dfrac{a^2}{3}$ i.e. if $\dfrac{2a^2}{3} > \dfrac{4a^2}{\pi^3}$

i.e. if $\pi^2 > 6$, which is true.

Hence for $\mu = \dfrac{\pi}{\pi^2-2}$, the wire rolls.

Example:- A heavy uniform sphere, of mass M, is resting on a perfectly rough horizontal plane, and a particle, of mass m, is gently placed on it at an angular distance α from its highest point. Show that the particle will at once slip on the sphere if $\mu < \dfrac{\sin\alpha\{7M+5m(1+\cos\alpha)\}}{7M\cos\alpha+5m(1+\cos\alpha)^2}$, where μ is the coefficient of friction between the sphere and the particle.

Solution:- Let C be the centre of the sphere. The horizontal plane is perfectly rough. So if the sphere rolls on the plane, the particle of mass m remains at rest placed at point P, such that CP is inclined at an angle $(\alpha+\theta)$ to the vertical. Let the distance of the point of contact A be x from the initial position O i.e. $OA = x$. Since the sphere rolls, $x = a\theta$ and the point of contact is at rest hence $\dot{x} = a\dot{\theta}$.



Let R and F be the reaction and friction at the point P

With point O as the origin and the horizontal and vertical lines through O as co-ordinates axes, the co-ordinates of point P are given by $x = a\theta + a\sin(\alpha+\theta)\ddot{\theta}, y = a + a\cos(\alpha+\theta)$

$\therefore \qquad \ddot{x} = a\ddot{\theta} + a\cos(\alpha+\theta)\ddot{\theta} - a\sin(\alpha+\theta)\dot{\theta}^2$

$\ddot{y} = -a\sin(\alpha+\theta)\ddot{\theta} - a\cos(\alpha+\theta)\dot{\theta}^2$

Equations of motion of the particle m are

$$mg\sin(\alpha+\theta) - F = \ddot{x}m\cos(\alpha+\theta) - m\ddot{y}\sin(\alpha+\theta)$$

$$= m\{a + a\cos(\alpha+\theta)\}\ddot{\theta} \qquad (1)$$

$$R - mg\cos(\alpha+\theta) = m\ddot{x}\sin(\alpha+\theta) + m\ddot{y}\cos(\alpha+\theta)$$

$$= ma\sin(\alpha+\theta)\ddot{\theta} - ma\ddot{\theta} - ma\dot{\theta}^2 \qquad (2)$$

The energy equation gives

$$\frac{1}{2}\left[M\frac{2a^2}{5}\dot{\theta}^2 + Ma^2\dot{\theta}^2 + m\left(\dot{x}^2+\dot{y}^2\right)\right] = \text{work done by gravity}$$

$$= mga\{\cos\alpha - \cos(\alpha+\theta)\}$$

i.e. $\frac{7}{10}Ma^2\dot{\theta}^2 + ma^2\{1+\cos(\alpha+\theta)\}\dot{\theta}^2 = mga\{\cos\alpha - \cos(\alpha+\theta)\}$ of

$$\left[7Ma^2 + 10ma^2\{1+\cos(\alpha+\theta)\}\right]\dot{\theta}^2 = 10mga\{\cos\alpha - \cos(\alpha+\theta)\}$$

Differentiating w.r.t to $'t'$ and dividing by $2\dot{\theta}$, we have

$$\left[7Ma^2 + 10ma^2\{1+\cos(\alpha+\theta)\}\right]\ddot{\theta} - 5ma^2\sin(\alpha+\theta)\dot{\theta}^2$$

$$= 5mga\sin(\alpha+\theta) \qquad (3)$$

As we only want initial, when $\theta = 0$, $\dot{\theta} = 0$ but $\ddot{\theta} \neq 0$ equations (1), (2) and (3) reduce to

$$F = mg\sin\alpha - ma(1+\cos\alpha)\ddot{\theta}$$

$$R = mg\cos\alpha + ma\sin a\,\ddot{\theta}$$

$$\left[7Ma^2 + 10ma^2(1+\cos\alpha)\right]\ddot{\theta} = 5mg\,a\sin\alpha$$

these equations give the initial values of F,R and $\ddot{\theta}$.

On solving these equations, we get

$$F = mg\sin\alpha - ma(1+\cos\alpha)\frac{5mga\sin\alpha}{7Ma^2 + 10ma^2(1+\cos\alpha)}$$

$$= g\sin\alpha\frac{7M + 5m(1+\cos\alpha)}{7M + 10m(1+\cos\alpha)} \quad \text{and}$$

$$R = mg\cos\alpha + ma\sin\alpha\frac{5mga\sin\alpha}{7Ma^2 + 10ma^2(1+\cos\alpha)}$$

$$= g\frac{7M\cos\alpha + 5m(1+\cos\alpha)^2}{7M + 10m(1+\cos\alpha)}$$

$$\therefore \qquad \frac{F}{R} = \sin\alpha\,\alpha\left[\frac{7M + 5m(1+\cos\alpha)}{7M\cos\alpha + 5m(1+\cos\alpha)^2}\right]$$

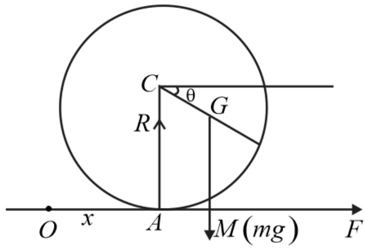
The particle will slip on the sphere if $F > \mu R$ or if $\mu < \frac{F}{R}$

i.e. if $\mu < \frac{\sin\alpha\{7M + 5m(1+\cos\alpha)\}}{7M\cos\alpha + 5m(1+\cos\alpha)^2}$

Example:- A homogeneous sphere, of mass M, is placed on an imperfectly rough table, and a particle, of mass m, is attached to the end of a horizontal diameters. Show that the sphere will begin to roll slide according as μ is greater or less than $\dfrac{5(M+m)m}{7M^2+17Mm+5m^2}$. If μ be equal to this value. Show that the sphere will begin to roll if $5m^2 < M^2 + 11Mm$.

Solution:- Let the radius of the sphere be a and mass $M.B$ is the point at which a particle of mass m is attached. Let in time t the sphere have turned through an angle θ and the point contact have moved through a distance x from its initial position O. i.e. $OA = x$.

Let G be the common centre of gravity of two masses, such that $CG = c$, then $Mc = m(a-b) \Rightarrow c(M+m) = ma$ i.e. $c - \dfrac{ma}{M+m}$ $\quad \therefore BG = a - c = \dfrac{aM}{M+m}$



Assume that the sphere rolls and F be the force of friction sufficient for pure rolling. Since the motion is of pure rolling.

$\therefore \quad x = a\theta$; and $\dot{x} = a\dot{\theta}$. Also $(M+m)k^2 = M, \dfrac{2a^2}{5} + Mc^2 + m(a-c)^2$

$$= 2\frac{Ma^2}{5} + \frac{Mm^2a^2}{(M+m)^2} + \frac{ma^2M^2}{(M+m)^2} = \frac{2Ma^2}{5} + \frac{Mma^2}{(M+m)^2}$$

$$\therefore \quad k^2 - \frac{Ma^2(2M+7m)}{5(M+m)^2} \qquad (1)$$

Referred to O as origin the co-ordinates of C.G. are $(x + c\cos\theta, a - c\sin\theta)$.

The equations of motions are

$$F(M+m)\frac{d^2}{dt^2}(x + c\cos\theta) = (M+m)\frac{d^2}{dt^2}(a\theta + c\cos\theta) \qquad (2)$$

$$= (M+m)\left[(a - c\sin\theta)\ddot{\theta} - c\cos\theta\dot{\theta}^2\right] \qquad (3)$$

And $R - (M+m)g = (M+m)\dfrac{d^2}{dt^2}\left[(a - c\sin\theta)\right]$

$$= (M+m)\left[-c\cos\theta\ddot{\theta} + c\sin\theta\dot{\theta}^2\right] \qquad (4)$$

And $Rc\cos\theta - F(a - c\sin\theta) = (M+m)k^2\ddot{\theta}$ $\qquad (5)$

As we discuss only and the initial motion when $\theta = 0$, and $\dot{\theta}$ is zero but $\ddot{\theta} \neq 0$, equations (3), (4) (5) ; become

$$\left.\begin{aligned} &F=(M+m)a\ddot{\theta} \\ &R=(M+m)g-(M+m)c\ddot{\theta} \\ &Rc-Fa=Mmk^2\ddot{\theta} \end{aligned}\right] \textbf{ For the initial values of } F, R \text{ and } \ddot{\theta}$$

Solving these equations, we get $\ddot{\theta}=\dfrac{gc}{k^2+a^2+c^2}$, putting for $\ddot{\theta}$ in above equations,

we have $\dfrac{R}{(M+m)}=\dfrac{k^2+a^2}{k^2+a^2+c^2}\ g;\dfrac{F}{(M+m)}=\dfrac{gca}{k^2+c^2+a^2}$

The sphere will commence to slide or roll according as $F >$ or $< \mu R$

i.e. if $\dfrac{gca}{k^2+a^2+c^2} >$ or $< \mu \dfrac{k^2+a^2}{k^2+c^2+a^2} - g$

i.e. if $\mu <$ or $< \dfrac{ac}{(k^2+a^2)}$

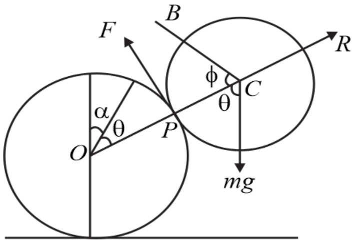
i.e. if $\mu <$ or $< \dfrac{5m(M+m)}{7M^2+17mM+5m^2}$ (putting the value of c)

Critical Case:- Suppose $\mu=\dfrac{5m(M+m)}{7M^2+17mM+5m^2}$

We have prove $k^2=\dfrac{a^2M(7M+2M)}{5(M+m)}$ in (2) and the sphere will roll if $k^2>(a^2/3)$ proved in 3.10 i.e. if $\dfrac{a^2M(7m+2M)}{5(M+m)}>(a^2/3)$ or $3M(7m+2M)>5(M+m)^2$ or $21Mm+6M^2>5(M^2+2Mm+m^2)$ i.e. $5m^2<M^2+11Mm$.

Example:- A solid sphere, resting on the top of another fixed sphere is slightly displaced and begins to roll down. If the plane through their axes makes an angle α with the vertical when first cylinder is at rest, show that it will slip when the common normal makes with the vertical an angle given by $k^2\sin\theta=\mu(k^2+3b)^2\cos\theta-2b^2\cos\alpha\}$ where b is radius of the moving sphere and k is the radius of gyration. The upper sphere will leave the fixed sphere if $\theta=\cos^{-1}\left(\dfrac{2b^2\cos\alpha}{k^2+3b^2}\right)$

Solution:- Let CB a radius fixed in the moving sphere makes an angle ϕ with the vertical, initially B coincided with A. Let R and F be the reaction and friction respectively. Since there is no slipping between the two spheres, therefore, arc $AP=arc\,BP$, i.e. $a(\theta-\alpha)=b(\phi=0)$ or $a\ddot{\theta}=b\ddot{\phi}$ or $b\ddot{\phi}=(a+b)\ddot{\theta}$



Referring to O as the origin and horizontal and vertical lines through O as co-ordinates axes, the co-ordinates of C are $x=(a+b)\sin\theta$ and $y=(a+b)\cos\theta$ (2)

The energy equation gives

$$\frac{1}{2}m\left[k^2\dot{\phi}^2+\left(\dot{x}^2+\dot{y}^2\right)\right]=mg(a+b)(\cos\alpha+\cos\theta) \quad (3)$$

Or $$\frac{1}{2}\left[k^2\dot{\phi}^2+(a+b)^2\dot{\theta}^2\right]=(a+b)(\cos\alpha-\cos\theta)$$

Or $$\frac{1}{2}\left[\frac{k^2}{b^2}(a+b)^2\dot{\theta}^2+(a+b)^2\dot{\theta}^2\right]=g(a+b)(\cos\alpha-\cos\theta)$$

Or $$\dot{\theta}^2=\frac{-2b^2g}{\left(k^2+b^2\right)(a+b)}(\cos\alpha-\cos\theta) \quad (4)$$

Differentiating (3) and dividing by $2\dot{\theta}$, we get $\ddot{\theta}=\dfrac{gb^2\sin\theta}{(a+b)\left(k^2+b^2\right)}$ (5)

As C describes a circle of radius $(a+b)$ about O, its acceleration are $(a+b)\dot{\theta}^2$ and $(a+b)\ddot{\theta}$ along and perpendicular to O. Therefore the equations of motion of the sphere are

$$m(a+b)\dot{\theta}^2=mg\cos\theta-R \quad (6)$$

And $$m(a+b)\ddot{\theta}=mg\cos\theta-F \quad (7)$$

Hence from (6) and (4), we have $R=\left[mg\cos\theta-\dfrac{2b^2mg}{\left(b^2+k^2\right)}(\cos\alpha-\cos\theta)\right]$

$=\dfrac{mg}{k^2+b^2}\left[\left(k^2+3b^2\right)\cos\theta-2b^2\cos\alpha\right]$ and from (7) and (5), we have

$$F-mg\sin\theta-\frac{mgb^2}{b^2+k^2}\sin\theta\,\frac{mgk^2\sin\theta}{\left(k^2+b^2\right)}$$

$$\therefore \quad \frac{F}{R}=\frac{k^2\sin\theta}{\left(k^2+3b^2\right)\cos\theta-2b^2\cos\alpha}$$

The sphere will slip when $F=\mu R$ i.e. if $k^2\sin\theta=\mu\left[\left(k^2-3b^2\right)\cos\theta-2b^2\cos\alpha\right]$

The upper sphere will leave the fixed sphere if $R=0$ i.e. if $\left(k^2+3b^2\right)\cos\theta=2b^2\cos\alpha$ i.e.

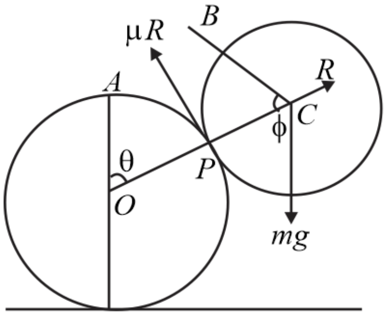
$$\theta=\cos^{-1}\left(\frac{2b^2\cos\alpha}{k^2+3b^2}\right).$$

Example:- A homogenous sphere rolls down on imperfectly rough fixed sphere starting from rest at the highest point. If the spheres separate when the line joining their centres makes an angle θ with the vertical, prove that $\cos\theta + 2\mu\sin\theta = Ae^{2\mu\theta}$ where A is the function of μ only.

Solution:- As the fixed sphere is imperfectly rough so the moving sphere rolls as well as slide on it thus friction μR acts upwards. Let a be radius of moving sphere.

Equations of motion are $mv\dfrac{dv}{ds} = mg\sin\theta - \mu R$ (1)

And $\dfrac{mv^2}{a} = mg\cos\theta - R$ (2)



Eliminating R from (1) and (2), we get $\dfrac{1}{2}\dfrac{dv^2}{ds} - \mu\dfrac{v^2}{a} = g(\sin\theta - \mu\cos\theta)$

Or $\dfrac{dv^2}{d\theta}\cdot\dfrac{d\theta}{ds} - 2\mu\dfrac{v^2}{a} = 2g(\sin\theta - \mu\cos\theta)$

Or $\dfrac{dv^2}{d\theta} - 2\mu v^2 = 2ag(\sin\theta - \mu\cos\theta)$ $\left(\because s = a\theta \Rightarrow \dfrac{ds}{d\theta} = a\right)$

Above is linear differential equations, its solution is

$$v^2 e^{-2\mu\theta} = C + 2ag\int e^{-2\mu\theta}(\sin\theta - \cos\theta)\,d\theta$$

$$= C + \frac{2age^{-2\mu\theta}}{1+4\mu^2}\left[(-2\mu\sin\theta - \cos\theta) - \mu(-2\mu\cos\theta + \sin\theta)\right]$$

Or $v^2 e^{-2\mu\theta} = C + \dfrac{2ag}{1+4\mu^2}e^{-2\mu\theta}\left[-3\mu\sin\theta - (1-2u^2)\cos\theta\right]$

Again when $\theta = 0$, $v = 0$ $\quad\therefore\ C = \dfrac{2ag}{1+4\mu^2}(1-2\mu^2)$

Therefore, $v^2e^{-2\mu\theta} = \dfrac{2ag}{1+4\mu^2}e^{-2\mu\theta}\left[-3\mu\sin\theta - (1-2u^2)\cos\theta\right]$

$$+\frac{2ag}{1+4\mu^2}(1-2\mu^2)$$

Or $v^2 = \dfrac{2ag}{1+4\mu^2}\left[-3\mu\sin\theta - (1-2u^2)\cos\theta\right] + \dfrac{2ag}{1+4\mu^2}(1-2\mu^2)e^{2\mu\theta}$

The sphere separates where $R = 0$, thus from (2), we have $v^2 = ag\cos\theta$ or

$$\frac{2ag}{1+4\mu^2}\left[-3\mu\sin\theta - (1-2u^2)\cos\theta\right] + \frac{2ag}{1+4\mu^2}(1-2\mu^2)e^{2\mu\theta} = ag\cos\theta$$

Or $2\left[-3\mu\sin\theta-\left(1-2\mu^2\right)\cos\theta\right]+2\left(1-2\mu^2\right)e^{2\mu\theta}=\left(1+4\mu^2\right)\cos\theta$

Or $6\,\mu\sin\theta+3\cos\theta=4\left(1-2\mu^2\right)e^{2\mu\theta}$

Or $\cos\theta+2\mu\sin\theta=\frac{4}{3}\left(1-2\mu^2\right)e^{2\mu\theta}$

Or $\cos\theta+2\mu\sin\theta=Ae^{2\mu\theta}$,where $A=\frac{4}{3}\left(1-2\mu^2\right)$

Example:- A rough solid circular cylinder rolls down a second rough cylinder which is fixed with its axis horizontal. If the plane through their axes make an angle α with the vertical when first cylinder is at rest, show that the bodies will separate when this angle of friction is $\cos^{-1}\left(\frac{4\cos\alpha}{7}\right)$

Solution:- Refer figure of Ex. 1

Let CB a radius fixed in the moving cylinder make an angle ϕ with the vertical, initially B coincided with A. Let c and b be the radii of fixed and moving cylinder respectively. As there is no slipping between the two cylinders, therefore arc $AP = arc\,BP$ i.e. $a(\theta-\alpha)=b(\phi-\theta)$

$\therefore \quad a\dot{\theta}=b\left(\dot{\phi}-\dot{\theta}\right)$ or $b\dot{\phi}=(a+b)\dot{\theta}$

Referring to O as the origin and horizontal and vertical lines through O as co-ordinates axes the co-ordinates of C are $\{(a+b)\sin\theta,(a+b)\cos\theta\}$

The energy equations gives $\frac{1}{2}m\left[k^2\dot{\phi}+(a+b)\dot{\theta}^2\right]=mg(a+b)(\cos\alpha-\cos\theta)$

Or $\frac{1}{2}(a+b)^2\dot{\theta}^2+(a+b)^2\dot{\theta}^2=2g(a+b)(\cos\alpha-\cos\theta)$

$$\left\{\because b\dot{\phi}=(a+b)\dot{\theta}\right\}$$

Or $\quad (a+b)\dot{\theta}^2\,\frac{4g}{3}(\cos\alpha-\cos\theta) \qquad (1)$

The centre C describes the circle of radius $(a+b)$ about O

$\therefore \quad m(a+b)\dot{\theta}^2=mg\cos\theta-R \qquad (2)$

From (1) and (2),

$R=mg\cos\theta-\frac{4mg}{3}(\cos\alpha-\cos\theta)=\frac{mg}{3}(7\cos\theta-4\cos\alpha)$

The bodies will separate when $R=0$

i.e. when $7\cos\theta-4\cos\alpha=0$ or $\cos\theta=\frac{4}{7}\cos\alpha$

or $\theta=\cos^{-1}\left(\frac{4}{7}\cos\alpha\right)$

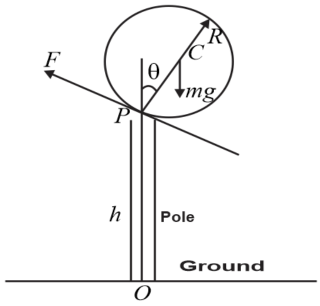
Example:- A uniform sphere of radius a is gently placed on the top of a thin vertical pole of height $h(>a)$ and then allowed to fall over. Show that however rough the pole may be the sphere will slip on the pole before it finally falls off it.

Solution:- Let OP be a fixed vertical pole of height h and a sphere is gently placed at top P and then displaced. Let us assume that friction is sufficient to keep the point of contact at rest, so the sphere turns about P without slipping.

Let at any time t the angle turned by the sphere be θ and F be the force sufficient to keep the point of contact at rest.

Equations of motion of C.G. of the sphere arc $ma\ddot{\theta} = mg\sin\theta - F$ (1)

And $ma\dot{\theta}^2 = mg\cos\theta - R$ (2)



Energy equation gives $\frac{1}{2}m\left(\frac{2a^2}{5}\dot{\theta}^2 + a^2\dot{\theta}^2\right) = mg(a - a\cos\theta)$

Or $a\dot{\theta}^2 = \frac{10}{7}g(1-\cos\theta)$ (3)

Differentiating (3), we get $a\ddot{\theta} = \frac{5}{7}g\sin\theta$ (4)

From (1) and (4), we have $F = mg\sin\theta = -ma\ddot{\theta}$ or

$F = mg\sin\theta - \frac{5}{7}mg\sin\theta = \frac{2}{7}mg\sin\theta$

From (2) and (3), we have $R = mg\cos\theta - ma\dot{\theta}^2$ or

$R = mg\cos\theta - \frac{10}{7}mg(1-\cos\theta) = \frac{1}{7}mg(17\cos\theta - 10)$

The sphere finally fall of when $R = 0$ i.e. when $17\cos\theta - 10 = 0$ or $\cos\theta = \frac{10}{17}$

Also the sphere will slip when $F \geq \mu R$ or $\mu \leq \frac{F}{R}$

Or $\mu \leq \frac{2\sin\theta}{17\cos\theta - 10}$ we observe that if μ is not negative, then $\mu = 0$ when $\theta = 0$ (i.e. when motion just begins)

And $\mu = \infty$ when $\cos\theta = \frac{10}{17}$ (i.e. when particle falls off)

Thus sphere will slip between $\theta = 0$ and $\theta = \cos^{-1}\frac{10}{17}$ if μ lies between 0 and ∞.

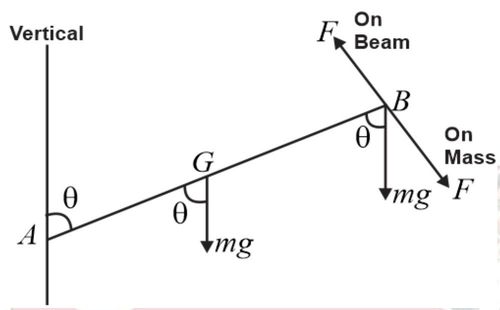
Thus we observe that however rough the pole may be, the sphere will slip on the pole before it finally falls over.

Example:- A uniform beam of mass M and length l stands upright on perfectly rough ground; on the top of it which is flat rests a weights of mass m, the coefficient of friction between the beam and the weight being μ. If the beam is allowed to fall to the ground, its inclination θ to the vertical when the weight slips is given by $\left(\frac{4}{3}M + 3m\right)\cos\theta - (M/6\mu)\sin\theta = M + 2m$

Solution:- Let at any time t, the rod AB make an angle θ with the vertical with m resting on the top B. Now, taking moments about A for the beam, we get $M.\frac{1}{3}l^2\,\theta = M\,\frac{1}{2}l\sin\theta - F.\,l$

Further equations of motion for mass m are $ml\,\ddot{\theta} = mg\sin\theta + F$ (2)

$M\,l\,\dot{\theta}^2 = mg\cos\theta - R$ (3)



Whence eliminating F between (1) and (2), we obtain

$(M + 3m)l\,\ddot{\theta} = \frac{3}{2}(M + 2m)g\sin\theta$ (4)

Again Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$(M + 3m)l\,\dot{\theta}^2 = -3(M + 2m)g\cos\theta + c$

When $\theta = 0$, $\dot{\theta} = 0 \Rightarrow c = 3g(M + 2m)$

$\Rightarrow$ $(M + 3m)l\,\dot{\theta}^2 = 3g(M + 2m)(1 - \cos\theta)$ (5)

$\Rightarrow$ $F = ml\,\ddot{\theta} - mg\sin\theta$ [using (2)]

$= \frac{3m(M + 2m)g\sin\theta}{2(M + 3m)} - mg\sin\theta$ by (4)

$\Rightarrow$ $F = \frac{mMG\sin\theta}{2(M + 3m)}$ (6)

Further $R = mg\cos\theta - l\,\dot{\theta}^2$, using (3)

$= mg\cos\theta - \frac{3mg(M + 2m)(1 - \cos\theta)}{M + 3m}$ by (5)

$$\Rightarrow \qquad R = \frac{mg\,(4M+9m)\cos\theta - 3(M+2m)}{M+3m}$$

$$\Rightarrow \qquad \frac{F}{R} = \frac{1}{2}\frac{M\sin\theta}{(4M+9m)\cos\theta - 3(M+2m)}$$

But $F = \mu R \Rightarrow \mu = F/R$ when the weight slips

$$\Rightarrow \qquad \mu = \frac{1}{2}\frac{M\sin\theta}{(4M+9m)\cos\theta - 3(M+2m)}$$

$$\Rightarrow \qquad \left(\frac{4}{3}M + 3m\right)\cos\theta - \left(\frac{M}{6\mu}\right)\sin\theta = M + 2m.$$

Example:- A circular plate rolls down the inner circumference of a rough circle under the action of gravity, the planes of both the plate and the circle being vertical. When the line joining their centres is inclined at an angle θ to the vertical, show that the friction between the bodies as $\frac{1}{3}\sin\theta$ times the weight of the plate.

Solution:- Let O be the centre of the fixed circle whose radius is a and C be the centre of circular plate that rolls down and its radius is b.

Let at any instant the radius CB (a line fixed in the body) make an angle ϕ with the vertical a line fixed in space. Initially, B coincided with A, a fixed point on fixed circle. OA is inclined at an angle α to the vertical OC.

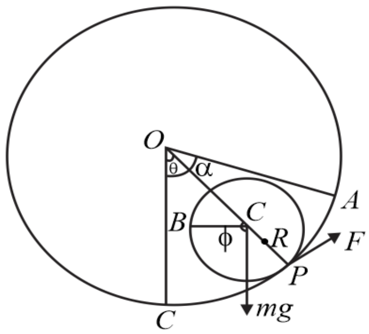
As there is no slipping between the bodies

$\therefore$ $Arc\ AP = arc\ BP$

[upper side in the figure]

i.e. $a(\alpha-\theta) = b\{2\pi - (\theta+\phi)\}$ or $-a\dot{\theta} = -b\left(\dot{\theta}+\dot{\phi}\right)$

or $b\dot{\phi} = (a-b)\dot{\theta}$ $\quad \therefore b\ddot{\phi} = (a-b)\ddot{\theta}$ (1)



Equations of motion of the plate are $m(a-b)\ddot{\theta} = F - mg\sin\theta$ (2)

And $m\frac{b^2}{2}\ddot{\phi} = -F\,.\,b$ (3)

On eliminating $\ddot{\theta}$ and $\ddot{\phi}$ from (1), (2) and (3), we get i.e. $F = \frac{\sin\theta}{3} = (mg) = \frac{\sin\theta}{3}$ (times the weight).

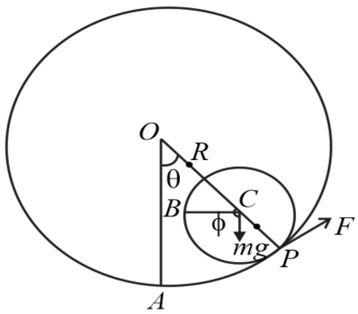
Example:- A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through their axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$

Solution:- Let θ be the angle through which the plane of axes turn and let ϕ be the angle which CB a line fixed in the moving cylinder makes with the vertical.

The outer cylinder is fixed. Equations of motion of the inner cylinder are

$$m(b-a)\ddot{\theta} = F - mg\sin\theta \qquad (1)$$

And $mk^2\ddot{\phi} = -Fa \qquad (2)$



Again there is no slipping

$\therefore$ $arc\ AP = arc\ BP$ or $b\theta = a(\theta+\phi)$ or $a\ddot{\phi} = (b-a)\ddot{\theta} \qquad (3)$

Eliminating F and $\ddot{\phi}$ between (1), (2) and (3), we get

$$m(b-a)\ddot{\theta} = -\frac{mk^2}{a}\ddot{\phi} - mg\sin\theta = -\frac{mk^2}{a^2}(b-a)\ddot{\theta} - mg\sin\theta \text{ or}$$

$$(b-a)\left(1+\frac{k^2}{a^2}\right)\ddot{\theta} = -g\sin\theta \text{ or } \ddot{\theta} = -\frac{g}{(a-b)\left(1+\frac{k^2}{a^2}\right)} \text{ as } \theta \text{ is small.}$$

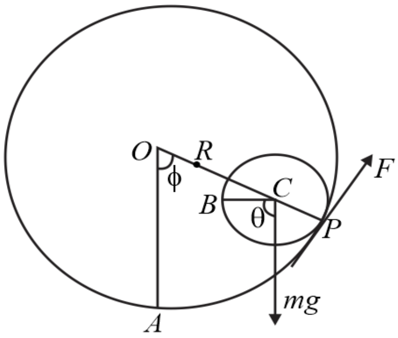
Therefore, length of the simple equivalent pendulum is $(b-a)\left(1+\frac{k^2}{a^2}\right)$.

Example:- A disc rolls on the inside of a fixed hollow circular cylinder whose axis is horizontal, the plane of the disc being vertical and perpendicular to the axis of cylinder ; if when in the lowest position, its centre is moving with a velocity $\left[\frac{8g}{3(a-b)}\right]^{1/2}$, show that the centre of the disc will describe and angle ϕ about the centre of the cylinder in time $\left[\frac{3(a-b)}{2g}\right]^{1/2}.\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)$.

Solution:- Let C be the centre of the disc and O be the centre of the fixed hollow cylinder whose radius is a. Let a line CB (fixed in the body) which was initially in a vertical position and coincided with OA makes an angle θ with the vertical.

Assume that the disc rolls, so that the $arc\ AP = arc\ BP$ or $a\phi = b(\theta+\phi)$ or $b\theta(a-b)\phi$

or $b\dot{\theta} = (a-b)\dot{\phi}$. $\qquad (1)$



Referring to O as the origin and vertical and horizontal lines through O as axes, the co-ordinates of centre C are $\{(a-b)\sin\phi,(a-b)\cos\phi\}$

Kinetic energy of the disc at any time t is

$$=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}mk^2\,\dot{\theta}^2=\frac{1}{2}m\left[(a-b)\dot{\phi}^2+\frac{b^2}{2}\dot{\theta}^2\right]$$

$$=\frac{1}{2}m\left[(a-b)^2\dot{\phi}^2+\frac{b^2}{2}\frac{(a-b)^2}{b^2}\dot{\phi}^2\right]\ \left\{\because \dot{\theta}=\left(\frac{a-b}{b}\right)\dot{\phi}\right\}$$

$$=\frac{3}{4}m(a-b)^2\dot{\phi}^2$$

It follows that the initial K.E. of the disc $=\frac{3}{4}m\frac{8}{3}g(a-b)=2mg(a-b)$

Since at $t=0$ $\dot{\phi}^2=\left[\frac{8g}{3(a-b)}\right]$

Therefore, the energy equations gives $\frac{3}{4}m(a-b)^2\dot{\phi}^2-2mg(a-b)=$ the work done by gravity $=-mg(a-b)(1-\cos\phi)$

Or $\frac{3}{4}m(a-b)^2\dot{\phi}^2=g(a-b)(1+\cos\phi)=2mg(a-b)\cos^2\frac{\phi}{2}$

Or $\dot{\phi}^2=\frac{8g}{3(a-b)}\cos\frac{\phi}{2}$ or $\frac{d\phi}{dt}=\left[\frac{8g}{3(a-b)}\right]^{1/2}\cos^2\frac{\phi}{2}$

Or $\int dt=\left[\frac{3(a-b)}{2g}\right]^{1/2}\int_0^{\phi}\sec\frac{\phi}{2}d\phi$

Or $t=\left[\frac{3(a-b)}{2g}\right]^{1/2}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)$

Example:- A solid homogenous sphere is rolling on the inside of a fixed hollow sphere, the two centres being always in the same vertical plane. Show that the smaller will make complete revaluation if, when it is in its lowest position, the pressure on it is greater that $\frac{34}{7}$ times its own weights.

Solution:- Let O be the centre of the fixed hollow sphere whose radius is a and C the centre of the moving solid sphere whose radius is b. Let CP be a radius (a line fixed in the body) makes an angle ϕ with the vertical (a line fixed in space) initially B coincided with A.

Let θ be the angle that the line of centres make with the vertical at any time t.

As there is not slipping between the two bodies, therefore, $arc\,AP = arc\,BP$ or

$a\theta = b(\phi+\theta)$ or $b\dot{\phi} = (a-b)\dot{\theta}$ (1)

C describes a circle of radius $(a-b)$ about O.

Equation of motion of the sphere is $m(a-b)\dot{\theta}^2 = R - mg\cos\theta$ (2)

Taking the horizontal and vertical lines through O as coordinates axes. Coordinates of the centre C are $\{(a-b)\sin\theta, (a-b)\cos\theta\}$

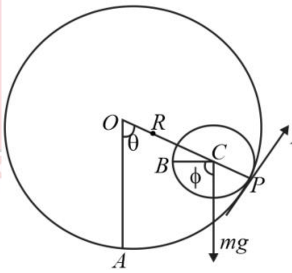
So at any time t, the (velocity)^2 of the centre $C = \left\{(a-b)\cos\theta\dot{\theta}\right\}^2$

$+\left\{-(a-b)\sin\theta\dot{\theta}\right\}^2 = (a-b)^2\dot{\theta}^2$

$\therefore$ At any time t, kinetic energy of the sphere $= \frac{1}{2}m\frac{2b^2}{5}\dot{\phi}^2 + \frac{1}{2}m(a-b)^2\dot{\theta}^2$

$= \frac{1}{2}m, \frac{2}{5}m(a-b^2)\dot{\theta}^2 + \frac{1}{2}m(a-b)^2\dot{\theta}^2$ from (1)

$= \frac{7m}{10}(a-b)^2\dot{\theta}^2$.



$\therefore$ Initially K. E. of the sphere $= \frac{7m}{10}(a-b)^2\Omega^2$ where Ω is the initial angular velocity.

Hence energy equations gives,

$= \frac{7m}{10}(a-b)^2\dot{\theta}^2 - \frac{7m}{10}(a-b)^2\Omega^2 =$ work done by the gravity

$= -mg\left[(a-b)-(a-b)\cos\theta\right]$

i.e. $(a-b)\dot{\theta}^2 = (a-b)\Omega^2 - \frac{10g}{7}(1-\cos\theta)$ (3)

Again from (2), $R = mg\cos\theta + m(a-b)\dot{\theta}^2$

$= mg\cos\theta + m(a-b)\Omega^2 - \frac{10mg}{7}(1-\cos\theta)$

The sphere will make complete revolutions if $R=0$ when $\theta=\pi$

i.e. $0 = -mg + m(a-b)\Omega^2 - \frac{20mg}{7}$ i.e. $\Omega^2 = \frac{27g}{7(a-b)}$

This gives least value of Ω for making complete revolution.

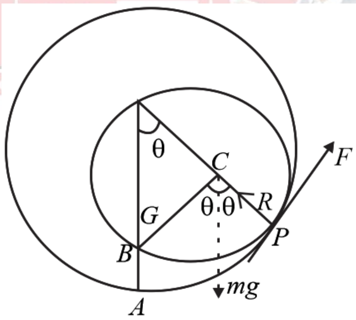
Again to know the value of R in the lowest position put $\theta = 0$ and $\dot{\theta} = \Omega$ in equation (2); then $\quad R$ (in lowest position) $= mg\cos 0 + m(a-b)\Omega^2$

$$= mg + \frac{27mg}{7} \quad \left\{ \because \Omega^2 = \frac{27g}{7(a-b)} \right\}$$

$$= \frac{34}{7}mg = \frac{37}{7} \text{ times the weight.}$$

Example:- A cylinder of radius a, lies within a rough fixed cylindrical cavity of radius $2a$. The centre of gravity of the cylinder is at a distance c from the axis, and the initial state is that of stable equilibrium at the lowest point of the cavity. Show that the smallest angular velocity with which the cylinder must be started that it may roll right round the cavity is given by $\Omega^2(a+c) = g\left\{1 + \frac{4(a+c)^2}{(a-c)^2 + k^2}\right\}$ where k is the radius of gyration about the centre of gravity.
Find also the normal reaction between the cylinder at any position.

Solution:- Let O be the centre of fixed cylindrical cavity whose radius is given $2a$, C the centre of the moving cylinder whose radius is given as a. At time t let CB (a line fixed in the moving body) makes an angle θ with the vertical (a line fixed in space). By geometry each of the other angles are also equal to θ as marked. Initially B coincided with A; it can be easily derived that B lies on the vertical line OA. Taking the horizontal and vertical lines through the fixed point O as co-ordinates axes, the coordinates of the gravity G are.



$x = (a-c)\sin\theta,\ y = (a+c)\cos\theta$,where $CG = c$

So that $\dot{x} = (a-c)\cos\theta\,\dot{\theta},\ \dot{y} = -(a+c)\sin\theta\,\dot{\theta}^2$

$\ddot{x} = (a-c)\cos\theta\,\ddot{\theta} - (a-c)\sin\theta\,\dot{\theta}^2$

$\ddot{y} = -(a+c)\sin\theta\,\ddot{\theta} - (a+c)\cos\theta\,\dot{\theta}^2$

$\therefore \qquad \dot{x}^2 + \dot{y}^2 = (a-c)^2\cos^2\theta\,\dot{\theta}^2 + (a+c)^2\sin^2\theta\,\dot{\theta}^2$

$= \left(a^2 + c^2 - 2ac\cos 2\theta\right)\dot{\theta}^2$

So at any time t, (velocity)^2 of $C = (a-c)^2\,\Omega^2$

(when $\theta = 0,\ \dot{\theta} = \Omega$)

Hence energy equation gives

$$\frac{1}{2}m\left[k^2+a^2+c^2-2ac\cos 2\theta\right]\dot{\theta}^2-\frac{1}{2}m\left[k^2+(a-c)^2\right]\Omega^2$$
$$=-mg\left[(a+c)-(a+c)\cos\theta\right] \qquad (1)$$

Equations of motion of the cylinder is $R-mg\cos\theta=-m\ddot{x}\sin\theta-m\ddot{y}\cos\theta$

$$=-m(a-c)\left(\cos\theta\ddot{\theta}-\sin\theta\dot{\theta}^2\right)\sin\theta$$
$$+m(a+c)\left(\sin\theta\ddot{\theta}+\cos\theta\dot{\theta}^2\right)\sin\theta$$
$$=m\left[c\sin 2\theta\ddot{\theta}+(a+\cos 2\theta)\dot{\theta}^2\right]$$

Or $R-mg\cos\theta=m\left[c\sin 2\theta\ddot{\theta}+(a+c\cos 2\theta)\dot{\theta}^2\right]$ (2)

The cylinder will roll round the cavity if $R=0$ when $\theta=\pi$;

Then from (2), $g(a+c)\dot{\theta}^2$ (3)

And from (1) $\left[k^2+(a-c)^2\right]\dot{\theta}^2-\left[k^2+(a-c)^2\right]\Omega^2=-4g(a+c)$ (4)

Eliminating $\dot{\theta}^2$ between (3) and (4), we have

$$\left[k^2+(a-c)^2\right]\frac{g}{a+c}-\left[k^2+(a-c)^2\right]\Omega^2=-4g(a+c)$$

Or $\left[k^2+(a-c)^2\right](a+c)\Omega=g\left[k^2+(a-c)^2+4(a+c)^2\right]$

Or $(a+c)\Omega^2=g\left[1+\frac{4(a+c)^2}{k^2+(a-c)^2}\right]$

Which is the required result.

Example:- A solid spherical ball rests in limiting equilibrium at the bottom of a fixed spherical globe whose inner surface is perfectly ough. The ball is struck a horizontal blow of such a magnitude that he initial speed of its centre is v; prove that is v lies between $\sqrt{\left\{\left(\frac{10}{7}gd\right)\right\}}$ and $\sqrt{\left\{\left(\frac{27}{7}gd\right)\right\}}$ the ball would leave the globe, d being the difference of the radii of the ball and the globe.

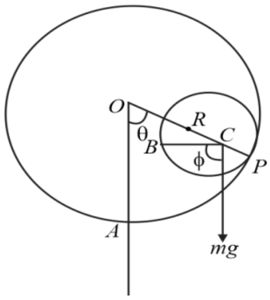
Solution:- Refer Ex. 4 and with the same figure, we have $d\theta=b(\theta+\phi)$ or $b\dot{\phi}=(a-b)\dot{\theta}=d\dot{\theta}$ where $d=a-b$. (1)

Initial velocity of the centre is given by

$\therefore$ $d^2\dot{\theta}^2=v^2$ **For initial value.** (2)

At any time t, the K.E. of the ball is given by $T_1=\frac{1}{2}m\left(\frac{2b^2}{5}\dot{\phi}^2+d^2\dot{\theta}^2\right)$

$$= \frac{1}{2} m \left(\frac{2}{5} d^2 \dot{\theta}^2 + d^2 \dot{\theta}^2 \right) = \frac{7m}{10} d^2 \dot{\theta}^2$$



At the time of projection. K. E. of the ball will be $= \frac{1}{2} m . \frac{7}{5} v^2$ using (2)

Again, energy equation gives $\frac{1}{2} m \frac{7}{5} d^2 \dot{\theta}^2 - \frac{1}{2} m \frac{7v^2}{5} = -mg \left(d - d \cos\theta \right)$

$\Rightarrow \quad d \dot{\theta}^2 = -\frac{10}{7} g \left(1 ... \cos\theta \right) + \frac{v^2}{d}$

Again centre C describes a circle of [radius a about O, so we obtain]

$md \ \dot{\theta}^2 = R - mg \cos v$ (4)

Eliminating $d \dot{\theta}^2$ between (3) and (4), we readily get

$R = mg \cos\theta - \frac{10}{7} mg \left(1 - \cos\theta \right) + m \frac{v^2}{d}$

$\Rightarrow \quad R = \frac{1}{7} mg \left[17 \cos\theta - \left(10 - \frac{7v^2}{gd} \right) \right]$

Now, the ball would leave the globe when $R = 0$

$\Rightarrow \quad 17 \cos\theta - \left(10 - \frac{7v^2}{gd} \right) = 0$

$\Rightarrow \quad \cos\theta = \frac{10gd - 7v^2}{27 \, gd} = -\frac{7v^2 - 10gd}{17 \, gd}$ (4)

But $\cos\theta$ is to be numerically less than 1 $\quad \therefore 7v^2 - 10gd < 17 \, gd$.

$\Rightarrow \quad v < \left(\left(\frac{27}{7} gd \right) \right)^{1/2}$

Again when θ is obtuse, we have $\cos\theta = -ive$

i.e. $7v^2 - 10 +$ positive i.e. $v -> \sqrt{(10gd/17)}$

i.e. $\left(\left(\frac{10 \, gd}{7} \right) \right)^{1/2} < v < \left(\left(\frac{27 \, gd}{7} \right) \right)^{1/2}$

Example:- A thin hollow cylinder of radius a and mass M is free to turn about its axis which is horizontal and a smaller cylinder of radius b and mass m rolls inside it without slipping, the axes of the two cylinders being parallel. Show that when the plane of the two axes is inclined at an angle θ to the vertical angular velocity of the large cylinder is given by

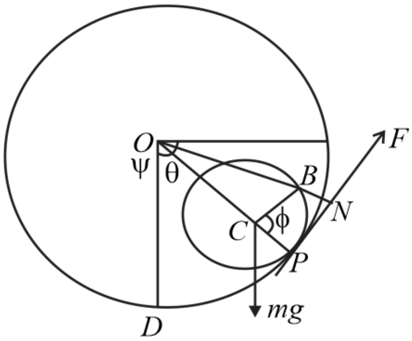
$a^2(M+m)(2M+m)\Omega^2 = 2gm^2(a-b)(\cos\theta-\cos\alpha)$ provided both the cylinder are at rest when $\theta=\alpha$.

Solution:- Let O be the centre of the outer cylinder and C the centre of the inner cylinder. The figure is the vertical section of the system through O and C.

Let CB be the line fixed in the inner cylinder and ON be the line fixed in the outer cylinder. Initially ON and CB coincided with OA i.e. initially B coincided with N.

After time t when the line OC makes an angle θ with the vertical, let ON and CB make angels ψ and ϕ with the vertical.

Since there is no slipping, $\therefore Arc\, NP = arc\ BP$



i.e. $a(\psi-\theta)=b(\phi-\theta)$ $\therefore b\phi = a\psi-(a-b)\theta$

or $b\dot{\phi}=a\dot{\psi}-(a-b)\dot{\theta}$ (1)

Considering the motion of the cylinders and taking moments about their centres of gravity, we get $mb^2\ddot{\theta}=FB$ (for smaller) (2)

And $Ma^2\ddot{\psi}=-Fa$ (for largest) (3)

From (2) and (3) we have $mb\ddot{\phi}=-Ma\ddot{\psi}$

Integrating, we get $mb\dot{\phi}=-Ma\dot{\psi}$ (4)

(initially $\dot{\phi}$ and $\dot{\psi}$ are both zero)

From (1) and (4) on eliminating $\dot{\phi}$, we get $Ma\dot{\psi}=ma\dot{\psi}-m(a-b)\dot{\theta}$ or

$a(M+m)\dot{\psi}=m(a-b)\dot{\theta}$ or $(a-b)\dot{\theta}=\dfrac{a(M+m)}{m}\dot{\psi}$ (5)

The coordinates of the centre of gravity C of the smaller cylinder with reference to O which is at rest are $\{(a-b)\sin\theta,(a-b)\cos\theta\}$.

Hence energy equation gives

$$\frac{1}{2}Ma^2\dot{\psi}^2+\frac{1}{2}m\left[b^2\dot{\phi}^2+(a-b)^2\dot{\theta}^2\right]=mg(a-b)(\cos\theta-\cos\alpha) \quad (6)$$

In (6) putting the values of $b\dot{\phi}$ and $(a-b)\dot{\theta}$ from (4) and (5) respectively we get

$$Ma^2\dot{\psi}^2+m\left[\frac{M^2}{m^2}a^2\dot{\psi}^2+\frac{(M+m)}{m^2}a^2\dot{\psi}^2\right]=2mg(a-b)(\cos\theta-\cos\alpha)$$

Or $a^2\left[M+\frac{M^2}{m}+\frac{(M+m)}{m^2}\right]\dot{\psi}^2 = 2mg(a-b)(\cos\theta-\cos\alpha)$

Or $a^2\left[M(m+M)+(M+m)^2\right]\dot{\psi}^2 = 2m^2g(a-b)(\cos\theta-\cos\alpha)$

Or $a^2(m+M)(2M+m)\dot{\psi}^2 = 2m^2g(a-b)(\cos\theta-\cos\alpha)$

Or $a^2(m+M)(2M+m)\Omega^2 = 2m^2g(a-b)(\cos\theta-\cos\alpha)$ which is the required result. If $\psi = \Omega$.

Example:- A uniform circular cylinder of mass M is free to rotate about its axis which is smooth and horizontal and about which its radius of gyration is equal to its radius. A uniform solid sphere of mass m is placed with its lowest point in contact with the highest generator of the cylinder, both sphere and cylinder being initially at rest. The sphere is then slightly disturbed and rolls down the cylinder. Show that the slipping takes place before, the sphere leaves the cylinder, and begins when $2M\sin\theta = \mu\{(17M+6m)\cos\theta-(10M+4m)\}$ where θ is the inclination to the vertical of the plane through their axes and μ the coefficient of friction.

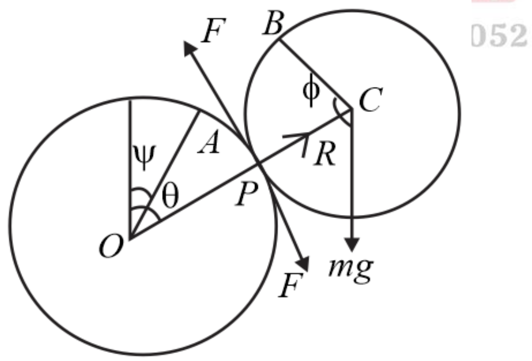
Solution:- Let O be the centre of the cylinder whose radius is a and C the centre of the sphere whose radius is b.

Let the cylinder have turned through an angle ψ to the vertical and CB a line fixed in the sphere make an angle ϕ with the vertical, a line fixed in space.

Initially B coincided with A and OA and CB were vertical.

Since there is no slipping, hence $arc\, AP = arc\, BP$.

i.e. $a(\theta-\psi) = a(\phi-\theta)$ or $b\dot{\phi}+a\dot{\psi} = (a+b)\dot{\theta}$ (1)



Considering the motion of the cylinder and the sphere respectively and taking moments about their centres, we get $Ma^2\ddot{\psi} = Fa$ (for the cylinder) ; and $m\frac{2b^2}{m}\ddot{\phi} = Fb$ (for the sphere)

$\therefore$ $Ma\ddot{\psi} = \frac{2mb}{5}\ddot{\phi}$ i.e $b\ddot{\phi} = \frac{5M}{2m}a\ddot{\psi}$

Integrating, we get $b\dot{\phi} = \frac{5M}{2m}a\dot{\psi}$ {Initially $\dot{\phi}$ and $\dot{\psi}$ zero so constant vanishe} (2)

Putting the value of $\dot{\phi}$ from (2), we get

$$\frac{5M}{2m}\,a\dot{\psi}+a\dot{\psi}=(a+b)\dot{\theta}$$

i.e. $\dfrac{5M+2m}{2m}\,a\dot{\psi}=(a+b)\dot{\theta}$ or $a\dot{\psi}=\dfrac{2m}{5M+2m}=(a+b)\dot{\theta}$

then $b\dot{\phi}=\dfrac{5M}{5M+2m}=(a+b)\dot{\theta}$

the coordinates of C, the centre of the sphere with reference to O the origin and vertical and horizontal through O as axes are $\{(a+b)\sin\theta,(a+b)\cos\theta\}$

$\therefore$ (Velocity)^2 of $C=\left\{(a+b)\cos\theta\,\dot{\theta}\right\}^2$

$$+\left\{-(a+b)\sin\theta\,\dot{\theta}\right\}^2=(a+b)^2=\dot{\theta}^2$$

Therefore energy equations gives

$$\frac{1}{2}Ma^2\dot{\psi}^2+\frac{1}{2},\left[\frac{2b^2}{5}\dot{\phi}^2+(a+b)^2\dot{\theta}^2\right]=mg\{(a+b)-(a+b)\cos\theta\}\text{ or}$$

$$\frac{1}{2}M\frac{4m^2}{(5M+2m)^2}(a+b)^2\dot{\theta}^2$$

$$+\frac{1}{2}m\left[\frac{2}{5}\cdot\frac{25M^2}{(5M+2m)^2}(a+b)^2\dot{\theta}^2+(a+b)^2\dot{\theta}^2\right]=mg(a+b)(1-\cos\theta)$$

Or $\left[\dfrac{M(5M+2m)}{(5M+2m)^2}+\dfrac{1}{2}\right](a+b)^2\dot{\theta}^2=g(1-\cos\theta)$

Or $\left(\dfrac{M}{5M+2m}+\dfrac{1}{2}\right)(a+b)\dot{\theta}^2=g(1-\cos\theta)$

Or $\dfrac{7M+2m}{5M+2m}(a+b)\dot{\theta}^2=2g(1-\cos\theta)$

Or $(a+b)\dot{\theta}^2=\dfrac{10M+4m}{7M+2m}=g(1-\cos\theta)$

Differentiating above and dividing by $2\dot{\theta}$, we get

$$(a+b)\ddot{\theta}=\frac{5M+2m}{7M+2m}g\sin\theta$$

Equations of motion are $m(a+b)\dot{\theta}^2=mg\cos\theta-R$ and $m(a+b)\ddot{\theta}=mg\sin\theta-F$

$\therefore$ $R=mg\cos\theta-m(a+b)\dot{\theta}^2=mg\cos\theta-\dfrac{10M+4m}{7M+2m}mg(1-\cos\theta)$

$$=mg\cos\theta\left(1+\frac{10M+4m}{7M+2m}\right)-\frac{10M+4m}{7M+2m}mg$$

$=\dfrac{mg}{7M+2m}\left[(17M+6m)\cos\theta-(10M+4m)\right]$ and $F=mg\sin\theta-m(a+b)\ddot{\theta}$

$$= mg\sin\theta - \frac{5M+2m}{7M+2m}mg\sin\theta = \frac{2M\,mg\sin\theta}{7M+2m}$$

$$\therefore \qquad \frac{F}{R} = \frac{2M\sin\theta}{(17M+6m)\cos\theta-(10M+4m)}$$

Slipping begins when $F=\mu R$ i.e. $2M\sin\theta = \mu\left[(17M+6m)\cos\theta-(10M+4m)\right]$

Above equation gives the value of θ, when slipping begins where $\theta<\pi$

Now $R=\frac{F}{\mu}=\frac{2M\,mg\sin\theta}{\mu(7M+2m)}$ which is obviously positive for all values of θ lying between 0 and π.

Hence the slipping begins before the sphere leaves the cylinder.

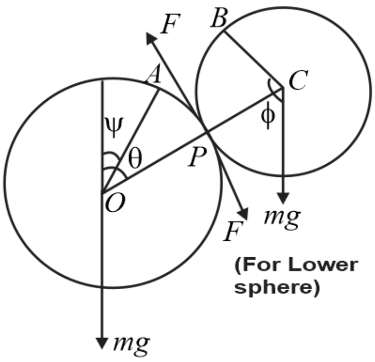
Example:- The mass of a sphere is $\frac{1}{5}$ of that of another sphere of the same material which is free to move about its centre as a fixed point, the first sphere rolls down the second from rest at the highest point, the coefficient of friction being μ. Prove that sliding will begin when the angle θ which the line of centre makes with the vertical is gives by $\sin\theta = 2\mu(5\cos\theta-3)$

Solution:- Let the mass of the lower and upper sphere be M and m respectively so that $M=5m$. The lower sphere is free to move. Let of upper sphere.

Let the lower sphere have turned through and angle ψ such that OA, a line fixed in the lower sphere make an angle ψ with the vertical and the line CB (a line fixed in the upper sphere) an angle ϕ with the vertical. Initially OA and CB were vertical and B coincided with A, OC the line joining the centres makes an angle θ with the vertical.

Since there is no slipping between the sphere, so $arc = AP = arc\,BP$ i.e. $a(\theta-\psi)=b(\phi-\theta)$

Or $a\dot\psi + b\dot\phi = (a+b)\dot\theta = c\dot\theta$ [Here $c=(a+b)$] (1)



The equation of motion for the lower sphere is $M\frac{2a^2}{5}\ddot\psi = Fa$ (2)

Equations of motion for the upper sphere are and $mc\dot\theta^2 = mg\cos\theta - R$ (3)

$mc\ddot\theta = mg\sin\theta - F$ (4)

Since C describe a circle about O of radius $(a+b)=c$.

Hence F is the friction sufficient for pure rolling.

Also $m\dfrac{2b^2}{5}\ddot{\phi}=Fb$ (5)

From (2) and (5), we have $\dfrac{a\ddot{\psi}}{m}=\dfrac{b\ddot{\phi}}{M}$ or $\dfrac{a\dot{\psi}}{m}=\dfrac{b\dot{\phi}}{M}=\dfrac{a\dot{\psi}+b\dot{\phi}}{m+M}=\dfrac{c\dot{\theta}}{m+M}$

$\therefore \quad a\dot{\psi}=\dfrac{m}{m+M}c\dot{\theta}$ and $b\dot{\phi}=\dfrac{M}{m+M}c\dot{\theta}$

Energy equations gives $\dfrac{1}{2}M\dfrac{2a^2}{5}\dot{\psi}^2+\dfrac{1}{2}m\left(\dfrac{2b^2}{5}\dot{\phi}^2+c^2\dot{\theta}^2\right)=mg\left(c-c\cos\theta\right)$

Or $\dfrac{1}{2}M\,.\dfrac{2}{5}\dfrac{m^2}{(m+M)^2}c^2\dot{\theta}^2+\dfrac{1}{2}m\left[\dfrac{2}{5}.\dfrac{M^2}{(m+M)^2}c^2\dot{\theta}^2+c^2\dot{\theta}^2\right]=mg\left(1-\cos\theta\right)$

Or $\left[\dfrac{2}{5}.\dfrac{Mm}{M+m}+m\right]c^2\dot{\theta}^2=2mgc\left(1-\cos\theta\right)$

Or $\left(\dfrac{2}{5}.\dfrac{5m^2}{6m}+m\right)c\dot{\theta}^2=2mg\left(1-\cos\theta\right)\ \left(\because M=5m\right)$

Or $c\dot{\theta}^2=\dfrac{3}{2}g\left(1-\cos\theta\right)$ (6)

Differentiating (6) w.r.t. $'t'$ and dividing by $2\dot{\theta}$, we get $c\ddot{\theta}=\dfrac{3}{4}g\sin\theta$ (7)

From (3), we have $R=mg\cos\theta-mc\dot{\theta}^2=mg\cos\theta-\dfrac{3}{2}mg\left(1-\cos\alpha\right)$

[from (6)]

$=mg\left(\dfrac{5\cos\theta-3}{2}\right)$

From (4), we have

$F=mg\sin\theta-mc\ddot{\theta}=mg\sin\theta-\dfrac{3}{4}\sin\theta=\dfrac{1}{4}mg\sin\theta$ [from (7)]

$\therefore \quad \dfrac{F}{R}=\dfrac{1}{4}mg\sin\theta\,.\dfrac{2}{mg\left(5\cos\theta-3\right)}=\dfrac{\sin\theta}{2\left(5\cos\theta-3\right)}$

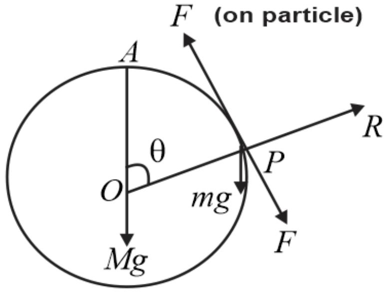
Sliding will being when $F=\mu R$ or $\dfrac{F}{R}=\mu$ i.e. when $\dfrac{\sin\theta}{2\left(5\cos\theta-3\right)}=\mu$ or

$\sin\theta=2\mu\left(5\cos\theta-3\right)$

Example:- A rough cylinder, of mass M, is capable of motion about its axis which is horizontal; a particle of mass m is placed on it vertically above the axis and the system is slightly disturbed. Show that the 'particle will slip on the cylinder when it has moved through an angle θ given by $\mu(M+6m)\cos\theta-M\sin\theta=4m\mu$, where μ is the coefficient of friction.

Solution:- Assume that F is the force of friction which keeps the particle at rest the radius OP makes an angle θ with the vertical. Referred to O as the origin, the co-ordinates of particle are $\left(a\sin\theta, a\cos\theta\right)$

Energy of the particle $\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}ma^2\dot{\theta}^2$



Energy of the cylinder $=\frac{1}{2}M\frac{a^2}{2}\dot{\theta}^2$ due to rotation

The energy equation gives $\frac{1}{2}M\frac{a^2}{2}\dot{\theta}^2+\frac{1}{2}ma^2\dot{\theta}^2=$ work done by gravity $=mga(1-\cos\theta)$

Or $a(M+2m)\dot{\theta}^2=4mg(1-\cos\theta)$ (1)

Differentiating above the dividing by $2\dot{\theta}$, we get $a(M+2m)\ddot{\theta}=2mg\sin\theta$ (2)

The particle m describes a circle about O, therefore, $ma\dot{\theta}^2=mg\cos\theta-R$ (3)

And $ma\ddot{\theta}=mg\sin\theta-F$ (4)

Hence $R=mg\cos\theta-ma\dot{\theta}^2=mg\cos\theta-\frac{4m^2g}{M+m}(1-\cos\theta)$ [From (1)]

$=\frac{mg}{M+2m}\left[(M+2m)\cos\theta-4m(1-\cos\theta)\right]$

$=\frac{mg}{M+2m}\left[(M+6m)\cos\theta-4m\right]$ and $F=mg\sin\theta-ma\ddot{\theta}=mg\sin\theta-\frac{2m^2g\sin\theta}{M+2m}$

[From (2)] $=\frac{mg\sin\theta}{M+2m}(M+2m-2m)=\frac{mMg\sin\theta}{M+2m}$ (4)

From (3), and (4) $\frac{F}{R}=\frac{M\sin\theta}{(M+6m)\cos\theta-4m}$

The particle slips from the cylinder when $F=\mu R$ i.e. when $\frac{F}{R}=\mu$

i.e. when $\frac{M\sin\theta}{(M+6m)\cos\theta-4m}=\mu$

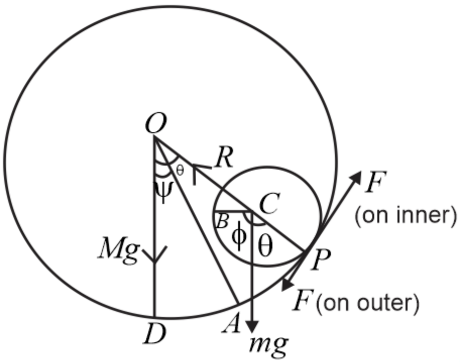
or when $\mu(M+6m)\cos\theta-4m\mu=M\sin\theta$

or when $\mu(M+6m)\cos\theta-M\sin\theta-4m\mu$, which is the required result.

Example:- A circular cylinder of radius a and of radius of gyration k rolls without slipping inside a hollow cylinder of radius b which is free to move about its axis. Show that the plane through their axis will move like a simple circular pendulum of length $(b-a)(1+n)$ where $n=\dfrac{(k^2/a^2)}{1+\dfrac{b^2}{a^2}\cdot\dfrac{mk^2}{mK^2}}$ where k and K are the radii of gyration, of the inner and outer cylinders respectively, about their axes; and m and M their masses.

Solution:- Adjoining figure is the vertical section through the centres of gravity of the two cylinders. The centre O remains fixed and the outer cylinder turns about it, let ψ be the angle turned by it when the plane of the axis makes an angle θ with the vertical. Let CB a line fixed in the inner cylinder makes an angle ϕ with the vertical a line fixed in space. Since there is no slipping so $Arc\ AP = Arc = PB$]

i.e. $b(\theta-\psi)=a(\phi+\theta)$ or $\quad b\ddot{\psi}+a\ddot{\phi}=(b-a)\ddot{\theta}$ (1)



Equations of motion are $Mk^2\ddot{\psi}=-Fb$ (2)

For outer cylinder $Mk^2\ddot{\phi}=-Fa$ (3)

And $m(b-a)\ddot{\theta}=F-mg\sin\theta$ (4) for inner cylinder

From (2) and (3), we have $-F=\dfrac{Mk^2\ddot{\psi}}{b}=\dfrac{mk^2\ddot{\phi}}{a}$ or

$$-F=\frac{b\ddot{\psi}}{(b^2/MK^2)}=\frac{a\ddot{\phi}}{(a^2/mk^2)}=\frac{b\ddot{\psi}+a\ddot{\phi}}{(b^2/MK^2)+(a/mk^2)}$$

$$=\frac{(b-a)\ddot{\theta}}{(b^2/MK^2)+(a^2/mk^2)}$$. (By virtue of (1))

Therefore, $F=-\dfrac{(b-a)\ddot{\theta}}{(b^2/MK^2)+(a^2/mk^2)}=-m(b-a).\dfrac{(k^2/a^2)}{1+\dfrac{b^2}{a^2}\cdot\dfrac{mk^2}{MK^2}}\ddot{\theta}$

$=-m(b-a)n\ddot{\theta}$,where $n=\dfrac{(k^2/a^2)}{1+\dfrac{b^2}{a^2}\cdot\dfrac{mk^2}{MK^2}}$

Putting this value of F in (4), we get $\quad m(b-a)\ddot{\theta}=-m(b-a)n\,\ddot{\theta}-mg\sin\theta$

Or $(b-a)(1+n)\ddot{\theta}=-g\sin\theta$ or $\ddot{\theta}=-\dfrac{g}{(b-a)(1+n)}\theta$

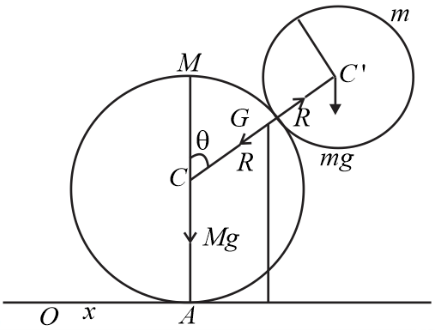
(θ is small taking θ for $\sin\theta$)

Thus length of simple equivalent pendulum is $(b-a)(1+n)$.

Example:- Two unequal smooth spheres, one placed on the top of the other are in unstable equilibrium, the lower sphere resting on a smooth table. The system is slightly disturbed, show that sphere will separate when the lines joining their centres make an angle θ with the vertical given by the equation $m\cos^3\theta = (M+m)(3\cos\theta - 2)$, where M is the mass of the lower, and m that of the upper spere.

Solution:- Let C and C' be the centres a and b the radius of the lower and upper sphere respectively and their masses are M and m respectively. Let after time t the lower sphere have moved through a distance x on the table when CC' the line joining their centre makes an angle θ with vertical.

As both the sphere are given to be smooth there are no forces acting on them to turn either sphere about its centre i.e. there is no rotation.



The co-ordinates of centres of gravity of both sphere with reference to O as origin are (x, a) (for the lower sphere) and $X = x + (a+b)\sin\theta, Y = (a+b)\cos\theta$ (for the upper sphere)

There is no horizontal force on the system, since the sphere and the planes are smooth. Thus

$$\frac{d}{dt}\left[M\dot{x} + m\left\{\dot{x} + (a+b)\cos\theta\,\dot{\theta}\right\}\right] = 0$$

Integrating above we get $(M+m)\dot{x} + m(a+b)\cos\theta\,\dot{\theta} = 0$ (2)

(Initially $\dot{x} = 0 = \dot{\theta}$, so that constant $= 0$)

Or $\dot{x} = -\dfrac{m}{M+m}(a+b)\cos\theta\,\dot{\theta}$ (3)

Energy equations gives

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right) = mg\left\{(a+b) - (a+b)\cos\theta\right\}$$

Or $\dfrac{1}{2}M\dot{x}^2 + \dfrac{1}{2}m\left[\dot{x}^2 + (a+b)^2\dot{\theta}^2 + 2(a+b)\dot{\theta}\dot{x}\cos\theta\right]$

$= mg(a+b)(1-\cos\theta)$

Putting for $\dot{x}^2$ from (3) we get $\dfrac{1}{2}\left[\dfrac{(M+m)m^2}{(M+m)^2}(a+b)^2\cos^2\theta\,\dot{\theta}^2 + m(a+b)^2\dot{\theta}^2\right.$

$\left. -2(a+b)^2\dfrac{m^2}{(M+m)}\cos^2\theta\,\dot{\theta}^2\right] = mg(a+b)(1-\cos\theta)$

Or $\left[-\frac{m(a+b)}{(M+M)}\cos^2\theta(a+b)\right]\dot{\theta}^2 = 2g(1-\cos\theta)$

or $(M+m\sin^2\theta)\dot{\theta}^2 = \frac{2g}{(a+b)}(M+m)(1-\cos\theta)$ (4)

Differentiating (4) with respect to t, we have

$(M+m\sin^2\theta)\ddot{\theta} + m\cos\theta\sin\theta\dot{\theta}^2 = \frac{M+m}{a+b}g\sin\theta$ (5)

Let R be the reaction between the two spheres.
Considering the horizontal motion of the lower sphere, we have

$$-R\sin\theta = M\ddot{x} = M\left\{-\frac{m}{M+m}(a+b)\right\}\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right)$$

Or $R\sin\theta = \frac{Mm}{M+m}(a+b)\left(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2\right)$ (6)

By (6), R vanishes i.e. sphere separate, when $\cos\theta\ddot{\theta} = \sin\theta\dot{\theta}^2$ (7)

On eliminating $\dot{\theta}^2$ from (5) and (7), we get $(M+m)\ddot{\theta} = \frac{M+m}{(a+b)}g\sin\theta$ or

$\ddot{\theta} = \frac{g}{a+b}\sin\theta$

Thus from (7) $\cos\theta\left(\frac{g}{a+b}\right)\sin\theta = \sin\theta\dot{\theta}^2$ or $\dot{\theta}^2 = \frac{g\cos\theta}{(a+b)}$

Putting this values of $\dot{\theta}^2$ in (4), we get

$$(M+m\sin^2\theta)\frac{g\cos\theta}{(a+b)} = \frac{2g}{(a+b)}(M+m)(1-\cos)$$

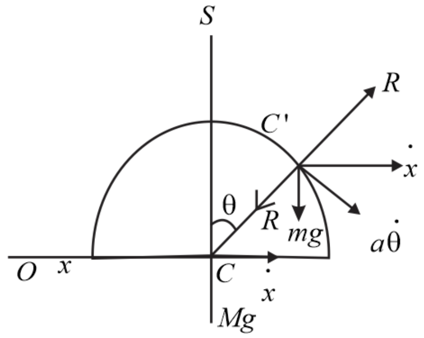
Or $\{M+m(1-\cos^2\theta)\}\cos\theta = 2(M+m)(1-\cos\theta)$

Or $m\cos^3\theta = (M+m)(3\cos\theta-2)$ which is the required result.

Example:- A hemisphere of mass M is free to slide with its base on a smooth horizontal table. A particle of mass m is placed on the hemisphere at an angular distance α from the vertex, show that the radius to the point of contact at which the particle leaves the surface, makes with the vertical an angle θ given by equation $m\cos^3\theta-(M+m)(3\cos\theta-2\cos\alpha)=0$

Solution:- Let in time t the centre of the hemisphere have moved through distance x on the placed and its velocity be $\dot{x}$ while CC' make an angle θ with the vertical where C in the centre of the hemisphere and C' is point where particle is placed. Let $\dot{x}$ and $a\dot{\theta}$ be the horizontal and tangential velocities of the particle.

With reference to O as the origin, the coordinates of centre of gravity of particle are $X = x + a + a\sin\theta,\ Y = a\cos\theta$



As there in no horizontal force on the system, so $\frac{d}{dt}\left\{M\dot{x}+m\left(\dot{x}+a\cos\theta\,\dot{\theta}\right)\right\}=0$.

Integrating above equation we get $M\dot{x}+m\left(\dot{x}+\cos\theta\,\dot{\theta}\right)=0$ (since $\dot{x}=0=\dot{\theta}$ initially, so constant $=0$)

Or $\dot{x}=-\dfrac{ma\cos\theta\,\dot{\theta}}{(M+m)}$ (1)

Kinetic energy of the hemisphere is $\frac{1}{2}M\dot{x}^2$ and that of the particle is $\frac{1}{2}m\left(\dot{X}^2+\dot{Y}^2\right)$. Since there are no-forces to turn the hemisphere, so there is no rotational energy. Hence the energy equation gives $\frac{1}{2}M\dot{x}^2+m\left(\dot{x}^2+a^2\dot{\theta}^2+2a\dot{\theta}\dot{x}\cos\theta\right)=mga(\cos\alpha-\cos\theta)$ (2)

Putting for $\dot{x}$ from (1) and (2), we get

$$\frac{m(M+m)a^2\cos^2\theta\,\dot{\theta}^2}{(M+m)^2}+a^2\dot{\theta}^2-\frac{2a^2m\cos^2\theta}{(M+m)}=\dot{\theta}^2=2ga(\cos\alpha-\cos\theta)$$

Or $\left(1-\dfrac{m}{M+m}\cos^2\theta\right)a^2\dot{\theta}^2=2ga(\cos\alpha-\cos\theta)$

Or $\left\{(M+m)-m\cos^2\theta\right\}a\dot{\theta}^2=2g(M+m)(\cos\alpha-\cos\theta)$ (3)

Considering horizontal motion of the hemisphere, we have $M\ddot{x}=-R\sin\theta$ or $M\dfrac{d}{dt}\left(\dot{x}\right)=-R\sin\theta$ or $-\dfrac{M\,ma}{(M+m)}\dfrac{d}{dt}\left(\dot{\theta}\cos\theta\right)=-R\sin\theta$.

The particle leaves the hemisphere of $R=0$ [from (1)]

i.e. if $\dfrac{d}{dt}\left(\dot{\theta}\cos\theta\right)=0$ or $\ddot{\theta}\cos\theta=\dot{\theta}^2\sin\theta$ (4)

equation (3) may be written as

$a(M+m)\sin^2\theta)\theta^2=2g(M+m)(\cos\alpha-\cos\theta)$

Differentiating it with regard to $'t'$ and dividing by 2θ, we get

$$(M+m\sin^2\theta)\ddot{\theta}+m\sin\theta\cos\theta\,\dot{\theta}^2=\frac{(M+m)g}{a}\sin\theta \qquad (5)$$

Thus from (4) and (5), we get $\ddot{\theta}=\frac{g}{a}\sin\theta$ $\qquad \therefore \dot{\theta}^2=\frac{g}{a}\cos\theta$

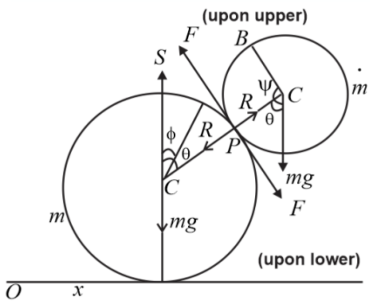
Putting this value of $\dot{\theta}^2$ in (3), we get

$$\{(M+m)-m\cos^2 x\}a\left(\frac{g}{a}\cos\theta\right)=2g(M+m)(\cos\alpha-\cos\theta)$$

Or $m\cos^3\theta-(M+m)(3\cos\theta-2\cos\alpha)=0$ which is the required result.

Example:- Two homogenous sphere of equal radii and masses m and m' rest on a smooth horizontal plane with m' on the highest point of m. If the system be disturbed show that the inclination θ of their common normal to the vertical is given by $a\dot{\theta}^2(7m+5m'\sin^2\theta)=5g(m+m')(1-\cos\theta)$

Solution:- Let C and C' be the centres of the two spheres whose masses are m and m' respectively CA and $C'B$ of are the redii (line fixed in the bodies) which were initially vertical. Let in time (t) the lower sphere have moved through distance x on the table while the line of their centres CC' make an angle θ with the vertical and the bodies have turned through angle ϕ and ψ in space. As there is no sliding, hence $Arc\ AP=Arc\ BP$ i.e. $a(\theta-\phi)=a(\psi-\theta)$ or $\theta-\phi=\psi=\theta$ or $\psi+\phi=2\theta$.



Considering the motion of the spheres and taking moments about their centre C and C', we have $m'\frac{2a^2}{5}\ddot{\psi}=Fa$ (for the upper sphere); $m'\frac{2a^2}{5}\ddot{\phi}=Fa$ (for the lower sphere)

$\therefore \qquad m'\ddot{\psi}=m\ddot{\phi} \qquad$ or $\frac{\ddot{\psi}}{m}=\frac{\ddot{\phi}}{m'}$

Integrating it, we get $\frac{\dot{\psi}}{m}=\frac{\dot{\phi}}{m'}=\frac{\dot{\psi}+\dot{\phi}}{(m+m')}=\frac{2\dot{\theta}}{(m+m')}$

(by componendo and dividend)

$\therefore \qquad \dot{\psi}=\frac{2m\dot{\theta}}{(m+m')}$ $\dot{\phi}=\frac{2m'\dot{\theta}}{(m+m')}$ (initially $\dot{\phi}=0=\dot{\psi}$)

The coordinates of C and C' with respect to O as origin, are (x,a) and $(x+2a\sin\theta,a+2a\cos\theta)$ respectively.

Since there is no horizontal force on the system, we have $\frac{d}{dt}\left\{m\dot{x}+m'\left(\dot{x}+a\cos\theta\,\dot{\theta}\right)\right\}=0$

Integrating it, we get $m\dot{x}+m'\left(\dot{x}+2a\cos\theta\dot{\theta}\right)=0$

(initially $\dot{x}=0=\dot{\theta}$, so constant = 0)

Or $(m+m')\dot{x}=-2am'\cos\theta\dot{\theta}$ or $\dot{x}=\dfrac{-2am'}{m+m'}\cos\theta\dot{\theta}$ (2)

The energy equation gives

$$\frac{1}{2}m\left(\dot{x}+\frac{2a^2}{5}\dot{\phi}^2\right)+\frac{1}{2}m'\left(\dot{x}^2+4a^2\dot{\theta}^2+4a\cos\theta\dot{\theta}\dot{x}+\frac{2a^2}{5}\dot{\psi}^2\right)$$
$$=2am'g(1-\cos\theta) \quad (3)$$

Putting for $\dot{x}$, $\dot{\phi}$ and in $\dot{\psi}$ (3), we get

$$\left[\frac{4a^2mm'^2}{(m+m')^2}\cos^2\theta\dot{\theta}+\frac{2a^2}{5}\frac{4mm'^2}{(m+m')^2}\dot{\theta}^2+\frac{4m'^3a^2}{(m+m')^2}\cos\theta\dot{\theta}^2\right.$$
$$\left.+4a^2m'\dot{\theta}^2-\frac{8a^2m'^2\cos^2\theta\dot{\theta}^2}{(m+m')}+\frac{2a^2}{5}\cdot\frac{4m^2m'\dot{\theta}^2}{(m+m')^2}\right]=4am'g(1-\cos\theta)$$

Or $\left[\dfrac{m'(m+m')}{(m+m')^2}\cos^2\theta+\dfrac{2m(m+m')}{5(m+m')^2}+1-\dfrac{2m'\cos^2\theta}{m+m'}\right]a\dot{\theta}^2=g(1-\cos\theta)$

Or $\left[\dfrac{2}{5}\dfrac{m}{(m+m')}+1-\dfrac{m'\cos^2\theta}{(m+m')}\right]a\dot{\theta}^2=g(1-\cos\theta)$

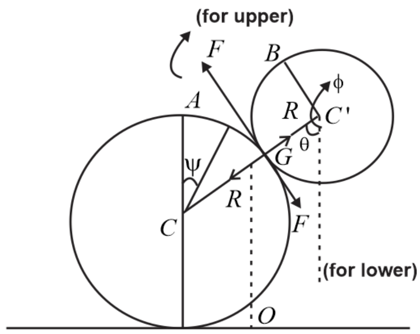
Or $(2m+5m+5m'-5m'\cos^2\theta)a\dot{\theta}^2=5(m+m')g(1-\cos\theta)$

Or $(7m+5m'\sin^2\theta)a\dot{\theta}^2=5(m+m')g(1-\cos\theta)$ which is the required result.

Example:- A uniform solid cylinder rests on a smooth horizontal plane and on it placed a second equal cylinder touching it along its highest generator, if there is no slipping between the cylinders and system moves from rest, show that the cylinder separate when the plane of either axes makes an angle θ with the vertical given by the equation $2\cos^3\theta+4\cos^2\theta-35\cos\theta+20=0$. Also show that until the cylinder separate the same generators remain in contact.

Solution:- The adjoining figure is the vertical section of the system through the centre of gravity of the cylinder. Let C and C' be the centres of two cylinders, CA and $C'B$ the lines fixed in the cylinder making angle ψ and ϕ with the vertical at time t, Initally CA and $C'B$ were vertical and B coincided with A.

Since there is no slipping, between the two cylinder, hence $arc\,AG=arc\,BG$ where G is their point of contact. The cylinder being equal (gives)



$\therefore \qquad \angle ACG = \angle BC'G$

Considering motion of the two cylinder and taking moments about C and C', we have

$m\dfrac{a^2}{2}\ddot{\psi} = Fa$ (for the lower cylinder) ; and $m\dfrac{a^2}{5}\ddot{\phi} = Fa$ (for the upper cylinder)

Integrating we get $\dot{\phi} = \dot{\psi}$

Again integrating $\phi = \psi$

The constants vanish when initially $\dot{\psi}, \dot{\phi}, \psi$ and ϕ are all zero.

Again $\angle ACG = \angle BC'G$ i.e. $\theta - \psi = \phi - \theta$ i.e. $\psi - \phi = 2\theta$ i.e. the same generators remain in contact until the cylinder separate.

Since there is no horizontal force on the two cylinder considered combined together therefore, the common centre for gravity G (which is the point of contact) will descend vertically. Let the vertical through G cut the horizontal plane in O, then 0 is a fixed point. With O as origin and horizontal and are $(a\sin\theta a + 2a\cos\theta)$, then $(-a\sin\theta, a)$ respectively.

$$\frac{1}{2}m\left[\frac{a^2}{2}\dot{\theta}^2 + a^2\cos^2\theta\,\dot{\theta}^2\right] + \frac{1}{2}m\left[\frac{a^2}{2}\dot{\theta}^2 + \left(a^2\cos^2\theta\,\dot{\theta}^2 + 4a^2\sin\theta\,\dot{\theta}^2\right)\right]$$

$$= mg\left[2a - 2a\cos\theta\right] \qquad \text{(Here we have taken } \dot{\phi} = \dot{\theta}\text{)}$$

Or $a\left(3 + 2\sin^2\theta\right)\dot{\theta}^2 = 4g\left(1 - \cos\theta\right)$

Or $a\left(5 - 2\cos^2\theta\right)\dot{\theta}^2 = 4g\left(1 - \cos\theta\right)$ (1)

Differentiating and dividing by $2\dot{\theta}$ we get

$a\left(5 - 2\cos^2\theta\right)\ddot{\theta} + 2a\sin\theta\cos\theta\,\dot{\theta}^2 = 2g\sin\theta$ (2)

Now consider the horizontal motion of the upper cylinder

$R\sin\theta - F\cos\theta = m\dfrac{d^2}{dt^2}\left(a\sin\theta\right) = ma\left(\cos\ddot{\theta} - \sin\theta\,\dot{\theta}^2\,0\right)$ (3)

And also $m\dfrac{a^2}{2}\ddot{\phi} = Fa$ (4) (taking moment about C')

Eliminating F between (3) and (4), we get

$$R\sin\theta = ma\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) + \frac{ma\ddot{\theta}}{2}\cos\theta$$

i.e. $R\sin\theta=\frac{ma}{2}\left(3\cos\theta\ddot{\theta}-2\sin\theta\dot{\theta}^2\right)$

The cylinders will separate if $R=0$, i.e. if $3\cos\theta\ddot{\theta}=2\sin\theta\dot{\theta}^2$

Now we eliminate $\ddot{\theta}$ and $\dot{\theta}^2$ between (1), (2) and (5)

Putting the value of $\ddot{\theta}$ from (5) in (2), we get

$$a\left(5-2\cos^2\theta\right)\frac{2\sin\theta}{3\cos\theta}\dot{\theta}^2+2a\sin\cos\theta\dot{\theta}^2=2g\sin\theta$$

Or $a\left(5+\cos^2\theta\right)\dot{\theta}^2=3g\sin\theta$

Putting this value of $\dot{\theta}^2$ in (1), we have

$$\left(5-2\cos^2\theta\right)\frac{3g\cos\theta}{5+\cos^2\theta}=4g\left(1-\cos\theta\right)$$

i.e. $3\left(5-2\cos^2\theta\right)\cos\theta=4\left(1-\cos\theta\right)\left(5+\cos^2\theta\right)$

i.e. $15\cos\theta-6\cos^3\theta=20-20\cos\theta+4\cos^2\theta-4\cos^3\theta$

i.e. $2\cos^3\theta+4\cos^2\theta-35\cos\theta+20=0$, which is the required result.

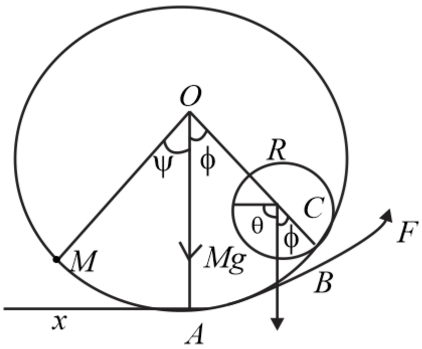
Example:- A uniform rough ball is at rest within a hollow cylindrical garden roller, and the roller is then drawn along a level-path with uniform velocity V. If $V^2>\frac{22}{7}g\left(b-a\right)$, show that the ball will not completely round the inside of the roller; a,b, being the radii of the ball and roller.

Solution:- Let O be the centre of the roller and C the centre of the spherical ball moving inside the cylindrical roller. Let CN be the radius of the ball was vertical when it was in its lowest position. When the roller has moved through a distance x, let CN have turned through an angle θ. The line joining the cetnre ϕ with the vertical and the ball has turned through an angle θ. As there is no sliding.

$arc\ BM=arc\ BN$

i.e. $b\left(\phi+\psi\right)=a\left(\theta+\phi\right)$

or $\left(b-a\right)\phi=a\theta-b\psi$ (1)



Again the velocity of the roller is constant i.e. $\dot{x}=b\dot{\psi}=V$

Then $\ddot{x}=b\ddot{\psi}=0$ (2)

Let R and F be the normal reaction and friction. As C describes a circle of radius $(b-a)$ about O so accelerations along CO and perpendicular to CO are $(b-a)\dot{\phi}^2$ and $(b-a)\ddot{\phi}$ respectively.

Thus equation of motion are $m(b-a)\dot{\phi}^2 = R\,mg\cos\phi$ (3)

$$m(b-a)\ddot{\phi} = F - mg\sin\phi \quad (4)$$

and $m\frac{2a^2}{5}\ddot{\theta} = -F\ .\ a$ (5)

Eliminating F between (4), and (5) we get

$$(b-a)\ddot{\phi} = -\frac{2a}{5}\ddot{\theta} - g\sin\phi \text{ or } (b-a)\ddot{\phi} + \frac{2}{5}(b-a)\ddot{\phi} - g\sin\phi$$

[since $(b-a)\ddot{\phi} = a\ddot{\theta} - a\ddot{\psi} = a\ddot{\theta}$ by virtue of (2)]

Or $\frac{7}{5}(b-a)\ddot{\phi} = -g\sin\phi$ (6)

Integrating it, we get $\frac{7}{5}(b-a)\dot{\phi}^2 = 2g\cos\phi + A$ (7)

Initially the velocity of the C.G. is $\dot{x} + (b-a)\dot{\phi} = 0$

i.e. $(b-a)\dot{\phi} = -\dot{x} = -V$ $\therefore A = \frac{7V^2}{5(b-a)} - 2g$

Hence the equation (7)gives

$$\frac{7}{5}(b-a)\dot{\phi}^2 = -2g(1-\cos\phi) + \frac{7V^2}{5(b-a)} \quad (8)$$

Substituting for $\dot{\phi}^2$ from (8) in (3), we get

$$\frac{R}{m} = g\cos\phi + \frac{V^2}{b-a} - \frac{10}{7}(-\cos\phi) = \frac{1}{7}\left(17g\cos\phi - 10g + \frac{7V^2}{b-a}\right)$$

The necessary condition that the ball should roll completely round the fixed cylinder is that R is positive when $\phi = \pi$, and if R is positive in this position.

Hence $\left\{\frac{7V^2}{b-a} - 10 + 17g\cos\phi\right\}_{\phi=\pi} > 0$

Or $\frac{7V^2}{b-a} > 27g$ or $V^2 > \frac{27g(b-a)}{7}$

Revision at a Glance

(i) $M\ddot{x}_G = \Sigma X$, $M\ddot{y}_G = \Sigma Y$, $Mk^2\ddot{\theta} = L$ where L is the moment of external forces about G.

(ii) K.E. of the body = K.E. due to translation $\left(\frac{1}{2}Mv_G{}^2\right)$ + K. E. due to rotation $\left(\frac{1}{2}Mk^2\dot{\theta}^2\right)$

(iii) Moment of momentum about the fixed origin $O = Mv_{GP} + Mk^2\dot{\theta}$.

PREVIOUS YEARS QUESTIONS

CHAPTER 2. EQUATION OF MOTION IN 2D / D'ALEMBERT PRINCIPLE

Q1. A rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α. Prove that the direction of reaction at the hinge makes with the vertical, an angle $\tan^{-1}\left[\frac{3}{4}\tan\alpha\right]$. **[1d IFoS 2022]**

Q2. A particle is constrained to move along a circle lying in the vertical *xy*-plane. With the help of the D'Alembert's principle, show that its equation of motion is $\ddot{x}y - \ddot{y}x - gx = 0$, where g is the acceleration due to gravity. **[5d UPSC CSE 2021]**

Q1. A uniform rod OA, of length 2*a*, free to turn about its end O, revolves with angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ; find the value of α.

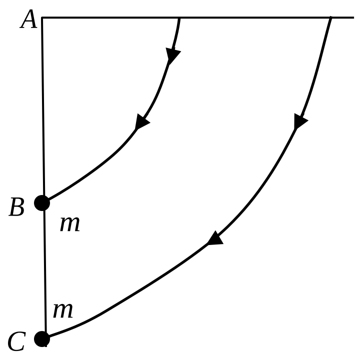
[5c UPSC CSE 2019]

Q2. A circular cylinder of radius *a* and radius of gyration *k* rolls without slipping inside a fixed hollow cylinder of radius *b*. Show that the plane through axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$. **[6c UPSC CSE 2019]**

Q3. A uniform rod OA of length 2*a* is free to turn about its end O, revolves with uniform angular velocity ω about a vertical axis OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or

$\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$. **[7c 2014 IFoS]**

Q4. A weightless rod ABC of length 2*a* is movable about the end A which is fixed and carries two particles of mass *m* each one attached to the mid-point B of the rod and the other attached to the end C of the rod. If the rod is held in the horizontal position and released from rest and allowed to move, show that the angular velocity of the rod when it is vertical is $\sqrt{\frac{6g}{5a}}$.



[8b 2012 IFoS]

Q5. The ends of a heavy rod of length $2a$ are rigidly attached to two light rings which can respectively slides on the thin smooth fixed horizontal and vertical wires O_x and O_y. The rod starts at an angle α to the horizon with an angular velocity $\sqrt{[3g(1-\sin\alpha)/2a]}$ and moves downwards. Show that it will strike the horizontal wire at the end of time

$-2\sqrt{a/(3g)}\log\left[\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right]$. **[8a UPSC CSE 2011]**

CHAPTER 3. LAGRANGE'S EQUATION OF MOTION

Q1. A particle at a distance r from the centre of force moves under the influence of the central force $F=-\frac{k}{r^2}$, where k is a constant. Obtain the Lagrangian and derive the equations of motion. **[5d UPSC CSE 2022]**

Q2. Derive the Lagrange's equation for a spherical problem. **[8a IFoS 2021]**

Q3. Obtain the Lagrangian equation for the motion of a system of two particles of unequal masses connected by an inextensible string passing over a small smooth pulley. **[6C UPSC CSE 2021]**

Q1. A particle is attached to a center by a force which varies inversely as the cube of its distance from the center. Identify the generalized coordinates and write down the Lagrangian of the system. Derive then the equations of motion and solve them for the orbits. Discuss how the nature of orbits depends on the parameters of the system. **[8a 2020 IFoS]**

Q2. For a dynamical system

$$T=\frac{1}{2}\left\{(1+2k)\dot{\theta}^2+2\dot{\theta}\dot{\varphi}+\dot{\varphi}^2\right\},$$

$$V = \frac{n^2}{2}\left\{(1+k)\theta^2 + \varphi^2\right\},$$

where θ, φ are coordinates and n, k are positive constants, write down the Lagrange's equations of motion and deduce that

$$\left(\ddot{\theta} - \ddot{\varphi}\right) + n^2\left(\frac{1+k}{k}\right)(\theta - \varphi) = 0.$$

Further show that if $\theta = \varphi, \dot{\theta} = \dot{\varphi}$ at $t = 0$, then $\theta = \varphi$ for all t. **[6c 2019 IFoS]**

Q3. Suppose the Lagrangian of a mechanical system is given by

$$L = \frac{1}{2}m\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{1}{2}k\left(ax^2 + 2bxy + cy^2\right),$$

where $a, b, c, m(>0), k(>0)$ are constants and $b^2 \neq ac$. Write down the Lagrangian equations of motion and identify the system. **[6c UPSC CSE 2018]**

Q4. A particle of mass *m* is constrained to move on the inner surface of a cone of semi-angle α under the action of gravity. Write the equation of constraint and mention the generalized coordinates. Write down the equation of motion. **[8c 2018 IFoS]**

Q5. Two uniform AB, AC, each of mass *m* and length 2*a*, are smoothly hinged together at A and move on horizontal plane. At time *t*, the mass centre of the rods is at the point (ξ, η) referred to fixed perpendicular axes O_x, O_y in the plane, and the rods make angles $\theta \pm \phi$ with O_x. Prove that the kinetic energy of the system is

$$m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right].$$

Also derive Lagrange's equations of motion for the system if an external force with components $[X, Y]$ along axes acts at A. **[6c UPSC CSE 2017]**

Q6. Consider a mass *m* on the end of a spring of natural length *l* and spring constant *k*. Let *y* be the vertical coordinate of the mass as measured from the top of the spring. Assume that the mass can only move up and down in the vertical direction. Show that

$$L = \frac{1}{2}my'^2 - \frac{1}{2}k(y - l)^2 + mgy$$

Also determine and solve the corresponding Euler-Lagrange equations of motion.

[8a 2017 IFoS]

Q7. A hoop with radius *r* is rolling, without slipping, down an inclined plane of length *l* and with angle of inclination ϕ. Assign appropriate generalized coordinates to the system. Determine the constraints, if any. Write down the Lagrangian equations for the system. Hence or otherwise determine the velocity of the hoop at the bottom of the inclined plane. **[8b UPSC CSE 2016]**

Q8. A bead slides on a wire in the shape of a cycloid described by the equations

$$x = a(\theta - \sin\theta)$$

$$y = a(1 + \cos\theta)$$

where $0 \le \theta \le 2\pi$ and the friction between the bead and the wire is negligible. Deduce Lagrange's equation of motion. **[8b 2016 IFoS]**

Q9. Two equal rods AB and BC, each of length l, smoothly joined at B, are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$ where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$. **[8a UPSC CSE 2013]**

Q10. Find the Lagrangian for a simple pendulum and obtain the equation describing its motion.

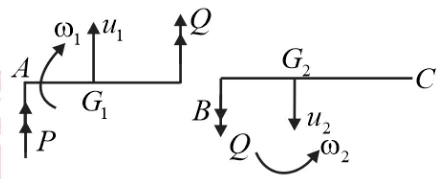
[5d 2011 IFoS]

Motion in Two Dimensions (under impulsive forces)

Example:- Two rods AB and BC of length $2a$ and $2b$ and of masses proportional to their lengths, are freely joined at B are laying in a straight line. A blow is communicated to the end A, show that the resulting kinetic energy when the system is free is to the energy when C is fixed as $(4a+3b)(3a+4b):12(a+b)^2$.

Solution:- Case I:- When the system is free. Let the mass of the unit length be m, then length of the rod $AB = 2a \Rightarrow$ mass of $AB = 2ma$; length of the rod $BC = 2b \Rightarrow$ mass of $BC = 2mb$. Further let G_1 be the C.G. of the rod AB and G_2 that of BC.

Let P be the impulse applied at A at right angles to AB then there will be an impulsive action between the two rods at B. Let the impulse be Q, in opposite directions on the two rods AB and BC respectively.



Just after the impulse, let u_1 be the velocity of the centre of gravity of AB and ω_1 its angular velocity u_2 and ω_2 similar quantities for BC. Since the rods AB, BC are started fro rest, the equations of motion of the rod AB are

$$2mau_1 = P+Q \qquad \text{(motion of } G_1\text{)} \qquad (1)$$

And $$2ma\frac{a^2}{3}\omega_1 = (P-Q)a \quad \text{(taking moment about } G_1\text{)} \qquad (2)$$

Similarly for the motion of BC, $2mbu_2 = Q$ (motion of G_1) (3)

And $$2mb\ .\ \frac{b^2}{3}\omega_2 = Qb \qquad \text{[taking moment about } G_2\text{]} \qquad (4)$$

The rods are connected at B, so the velocity of the point B of the rod AB = velocity of the point B referred to BC

i.e. $u_1 - a\omega_1 = -u_2 - b\omega_2$ (5)

substituting the values for $\omega_1, u_1; u_2, \omega_2$ in (5), we get

$$\frac{3(P-Q)}{2a} - \frac{P+Q}{2a} = \frac{Q}{2b} + \frac{3Q}{2b} \Rightarrow Q = \frac{pb}{2(a+b)}$$

$\therefore$ Velocity of the point A at which the below has been given

$$= u_1 + a\ \omega_1 = \frac{P+Q}{2am} + \frac{3(P-Q)}{2am} + \frac{P(4a+3b)}{2am(a+b)}$$

Now substituting the value of Q in (1), (2), (3) and (4), we get

$$u_1 = \frac{1}{2ma}\left\{P + \frac{bp}{2(a+b)}\right\} = \frac{(2a+3b)P}{4ma(a+b)},\ a\,\omega_1 = \frac{3}{2ma}\left\{\frac{b\,p}{2(a+b)}\right\} = \frac{3(b+2a)P}{4ma(a+b)},$$

$$u_2 = \frac{1}{2mb}\left\{\frac{b\,p}{2(a+b)}\right\} = \frac{P}{4ma(a+b)}, \text{ and } b\,\omega_2 = \frac{3}{2mb}\left\{\frac{b\,p}{2(a+b)}\right\} = \frac{3P}{4m(a+b)}$$

So in this case total kinetic energy.

= K.E. of the rod $AB + K.E.$ of the rod BC.

$$= \frac{1}{2}\,2ma\left[u_1^{\,2} + \frac{a^2\,\omega_1^2}{3}\right] + \frac{1}{2}\,.2mb\left[u^2 + \frac{b^2\,\omega_2^2}{3}\right]$$

$$= ma\left[\left\{\frac{(2a+3b)^2\,P^2}{16m^2a^2(a+b)^2}\right\} + \left\{\frac{3(b+2a)^2\,P^2}{16m^2a^2(a+b)^2}\right\}\right]$$

$$+mb\left[\left\{\frac{P^2}{16m^2(a+b)^2}\right\} + \left\{\frac{3P^2}{16m^2(a+b)^2}\right\}\right]$$

$$= ma\left[\frac{(2a+3b)^2\,P^2}{16m^2a^2(a+b)^2} + \frac{3(b+2a)^2\,P^2}{16m^2a^2(a+b)^2}\right]$$

$$+mb\left[\frac{P^2}{16m^2(a+b)^2} + \frac{3P^2}{16m^2(a+b)^2}\right]$$

$$= \frac{p^2(4a+3b)(a+b)}{4ma(a+b)^2} = \frac{(4a+3b)P^2}{4ma(a+b)} = E_1. \text{ Say}$$

Case II:- When C is kept fixed.

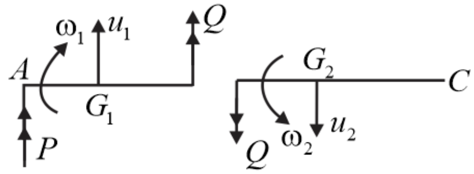
In this case, for the motion of AB, we get

$$2mau_1 = P + Q \qquad (1)$$

And $2ma\dfrac{a^2}{3}\omega_1 = (P-Q)a$

For the motion of BC, we also have $2mb\dfrac{4b^2}{3}\omega_1 = 2bQ$

[taking moments about C] (2)



Now velocity of B referred to $AB =$ velocity of B referred to BC

$\Rightarrow$ $$-u_1 + a\,\omega_1 = 2b\,\omega_2 \qquad (4)$$

From (1) and (2), we get $u_1 - a\,\omega_1 = (-2P + 4Q)\,.\,\dfrac{1}{2ma}$

Substituting from this and from (3) and (4) we readily get

$$(-2P+4Q)\frac{1}{2ma} = -3\frac{Q}{2mb} \text{ or } Q = \frac{(2bP)}{(3a+4b)}$$

Substituting this value of Q in (1), (2) and (3), we get

$$u_1 = \frac{1}{2ma}\left(P + \frac{2b\,P}{3a+4b}\right) = \frac{(3a+6b)\,P}{2ma(3a+4b)}, \text{ and}$$

$$a\,\omega_1 = \frac{3}{2ma}\left(P - \frac{2b\,P}{3a+4b}\right) = \frac{3}{2ma}\,.\,\frac{(3a+2b).P}{(3a+4b)}$$

Also, $b\,\omega_2 = \dfrac{3}{2ma}\left(\dfrac{b\,P}{3a+4b}\right)$

$\therefore$ Total K. E. of $AB + K.E.$ of BC

$$= \frac{1}{2}\,.2\,ma\left(u_1^2 + \frac{1}{3}a^2\,\omega_1^2\right) + \frac{1}{2}\,.\,2mb\left(\frac{4b_2\,\omega_2^2}{3}\right)$$

$$= ma\left[\frac{1}{4m^2a^2}\left(\frac{3a+6b}{3a+4b}\right)^2 P^2 + \frac{1}{3}\frac{9}{4m^2a^2}\left(\frac{3a+2b}{3a+4b}\right)P^2\right] + mb\left[\frac{4}{3}\,.\,\frac{9}{4m^2\,b^2}\,\frac{b^2P^2}{(3a+4b)^2}\right]$$

$$= 12b\frac{(3a^2+7ab+4b^2)}{4mab(3a+4b^2)}P^2 = \frac{3(a+b)}{ma(3a+4b)}P^2 = E_2 \text{ say}$$

$\Rightarrow$ $$\frac{E_1}{E_2} = \left\{\frac{(4a+3b)P^2}{4ma(a+b)}\right\} / \left\{\frac{3(a+b)P^2}{ma(3a+4b)}\right\}$$

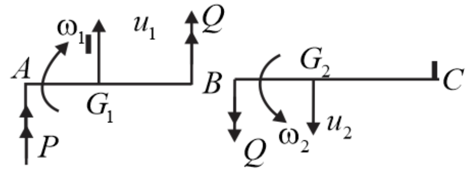
Or $\dfrac{E_1}{E_2} = \{(4a+3b)(3a+4b)\} / \{12(a+b)^2\}$

Example:- AB, BC are two equal similar rods freely hinged at B and lie in a straight line on a smooth table. The end A is struck by a below perpendicular to AB; show that resulting of A is $3\dfrac{1}{2}$ times of B

Solution:- Let P be the impulsive force applied at A. Just after the blow let u_1, ω_1 and u_2, ω_2 be the linear and angular velocities of the rods AB and BC respectively. When the blow is struck, there will be an impulsive action between the two rods at B. Let Q be the impulsive actin at the joint B in opposite directions on two rods. Also let $2a$ be the length of each rod and m be the mass, then equations of motion for the rods AB and BC are

$$\left.\begin{array}{ll} m(u_1 - 0) = P + Q & (1) \\ m, \frac{1}{3}a^2\,\omega_1 = (P - Q)a & (2) \end{array}\right\} \text{for the rod } AB$$

And
$$\left.\begin{array}{lll} & mu_2 = Q & (3) \\ and & m\frac{1}{3}a^2\,\omega_2 = Q\,a & (4) \end{array}\right\} \text{for the rod } BC$$



But the rods are connected at B so the velocity of B, as deduced from each rod must be equal,

i.e. $u_1 - a\,\omega_1 = -(u_2 + a\omega_2) \Rightarrow u_1 = \frac{5P}{4m}; a\,\omega_1 = \frac{9P}{4m}$ (5)

$$\therefore \quad \frac{Velocity\ of\ \ A}{Velocity\ of\ \ B} = \frac{u_1 + a\,\omega_1}{u_1 - a\omega_1} = \frac{\frac{5P}{4m} + \frac{9P}{4m}}{\frac{5P}{4m} - \frac{9P}{4m}} = -\frac{7}{2} = -3\frac{1}{2}$$

$\Rightarrow$ Velocity of $A = 3\frac{1}{2}$ times the velocity of B.

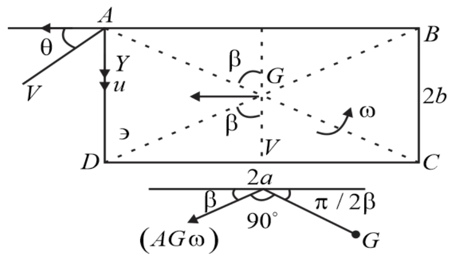
Example:- A rectangular lamina, whose sides are of lengths $2a$ and $2b$, is at rest when one corner is caught and suddenly made to move with prescribed speed V in the plane of the lamina; show that the greatest angular velocity which can thus be imparted to be lamina is $\frac{3V}{4\sqrt{(a^2+b^2)}}$

Solution:- the corner A is suddenly made to move with prescribed velocity V in plane of the lamina, such that V makes in angle θ (say) with BA, just after the impulse, let u, v be the velocities of the centre of gravity G parallel and perpendicular to AB, and let ω, be the angular velocity of the lamina.

Now $DG = \sqrt{(a^2+b^2)}$;

So $\cos\beta = \frac{b}{\sqrt{(a^2+b^2)}}$

$$\sin\beta = \frac{a}{\sqrt{(a^2+b^2)}}$$



$\therefore$ Equations of motion are $mu = X$ (1)

$mv = Y$ (2)

And $m\frac{a^2+b^2}{3}\omega = Xb + Ta$ (3)

Now velocity of A parallel $BA =$ velocity of G parallel to $BA +$ velocity of A parallel to BA relative to G

$\Rightarrow$ $V\cos\theta = u + \sqrt{(a^2+b^2)\omega\cos\beta} = u + \sqrt{(a^2+b^2)}\,\omega\frac{b}{\sqrt{(a^2+b^2)}}$

$\Rightarrow$ $V\cos\theta = u + b\,\omega$ (4)

And velocity of A perpendicular to $AB =$ velocity of G perpendicular to $AB +$ velocity of A perpendicular to AB relative to G.

$\Rightarrow$ $V\sin\theta = c + \sqrt{\left(a^1+b^2\right)}\,\omega\sin\beta = v + \sqrt{(a^2+b^2)}\,\omega\frac{b}{(a^2+b^2)}$

$\Rightarrow$ $V\sin\theta = v + a\,\omega$ (5)

Substituting the values of X and Y [using (1) and (2)] in (3), we have

$\frac{a^2+b^2}{3}\omega = ab + a\left[(V\cos\theta - b\,\omega)b + (V\sin\theta - a\omega)a\right]$ [using (4) and (5)]

$\Rightarrow$ $\frac{4(a^2+b^2)}{3}\omega = V(b\cos\theta + a\sin\theta)$ (6)

Differentiating (6), with respect to "θ", we get $\frac{4}{3}(a^2+b^2)\frac{d\omega}{d\theta} = V(-b\sin\theta + a\cos\theta)$

For ω to be maximum, we must have $\frac{d\omega}{d\theta} = 0$

i.e. $-b\sin\theta + a\cos\theta = 0$ $\Rightarrow \frac{\cos\theta}{b} = \frac{\sin\theta}{a} = \frac{1}{\sqrt{(a^2+b^2)}}$ (7)

$\therefore$ (6) and (7) give

$$\frac{4}{3}\left(a^2+b^2\right)=V\left[\frac{b^2}{\sqrt{\left(a^2+b^2\right)}}\frac{a^2}{\sqrt{\left(a^2+b^2\right)}}\right] \text{ or } \omega=\frac{3V}{4\sqrt{\left(a^2+b^2\right)}}.$$

Example:- A square lamina $ABCD$ rest on a smooth horizontal plane. If the corner A is made to move with velocity u along the $3A$ produced then determine the initial angular velocity of the lamina.

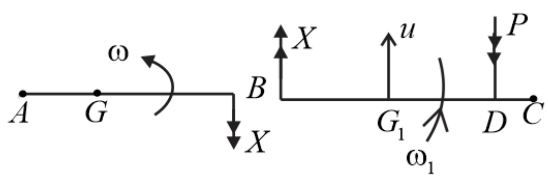
Solution:- For a square let us choose $b=a$. The corner A is made to move with velocity u along BA therefore, $V=u$ and $\theta=\pi$. Hence angular velocity u along BA is obtained by putting $V=u,\theta=\pi$ in result (6) of Ex. 3, we get.

$$\therefore \qquad \frac{4}{3}\left(a^2+a^2\right)\omega=u\left(\cos\pi+a\sin\pi\right) \text{ or } \frac{8}{3}a^2\,\omega=-au \text{ to } \omega=-\frac{3u}{8a}.$$

Above relation gives the required angular velocity.

Example:- Two equal uniform rods AB and BC are freely joined at B and turn about a smooth joint at A. When the rods, are in a straight line ω being angular velocity of AB and u the velocity of the centre of mass BC; BC impinges on a fixed inelastic obstacle at a point D; show that the rods are instantaneously brought to rest if $BD=2a\dfrac{2u-a\omega}{3u+2a\omega}$ where $2a$ is the length of the either rod.

Solution:- Let ω_1 be the angular velocity of BC before the impulse P is given at D. Here $BD=x$ say so that $G_1D=x-a$. Obviously there will be an impulsive action between the two rods at B and let it be X, in opposite directions on the two rods. Now further if the rods are instantaneously brought to rest, then the equation of motion of rod AB is $m,\dfrac{4}{3}a^2\,\omega=X.2a$ (taking moments about B) (1)



Also equations of motion of BC are $mu=P-X$ (2)

$$m\frac{1}{3}a^2\,\omega=P\left(X-a\right)+Xa \qquad \left[\because G_1C=x-a\right] \qquad (3)$$

But the rods are connected at B so the motion of B as deduced from each rod must be the same.

$\Rightarrow \qquad 2a\,\omega=$ vel. Of G_1+ vel. Rel. to $G_2=u-a\omega_1$

Putting the value of P and ω_1 from (2) and (4) in (3), we obtain

$$m,\frac{1}{3}a\left(u-2a\,\omega\right)=\left(mu+X\right)\left(x-a\right)+Xa=mu\left(x-a\right)+X\,x$$

$$\Rightarrow \qquad m\frac{1}{3}a(u-2a\omega)=mu(x-a)+m\,.\frac{2}{3}a\,\omega.x \qquad \text{[putting for } X \text{ from (1)]}$$

$$\Rightarrow \qquad \frac{1}{3}a(u-2a\,\omega+3u)-\frac{1}{3}x(3u+2\,\omega a)\Rightarrow x=BD=2a\frac{2u-a\omega}{3u+2a\omega}$$

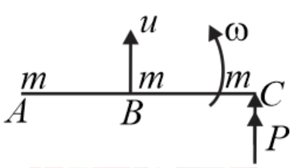
Example:- There particles of equal masses are attached to the ends A and C and the middle point B of al light rod ABC and the system is at rest on a smooth table. The particle C is struck a blow at right angles to the rod. Energy communicated when the system is free as $24\ :\ 25$

Solution:- Let P be the impulse of the blow imparted at B and mass of each particle at A, B, C be m where $AC=2a$.

1st case. Let A be no fixed. i.e. system is free.

The C.G. of the three particle is at B and the system is at rest before the action of the impulse. Let u be the velocity of the point B. When total mass $3m$ is supposed to be placed at B, let ω be the angular velocity of the rod after the action of the impulse, then the equations of motion are $3mu=P$

[motion of C.G. i.e. B] (1)



$3mk^2\,\omega=Pa$ [taking moments about B] (2)

And $3mk^2=ma^2+ma^2$

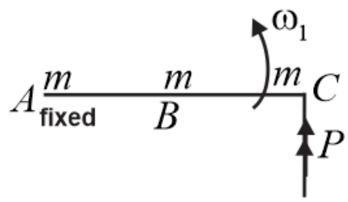
So that $k^2=\dfrac{2a}{3}$

Now K.E. $=\dfrac{1}{2}3mu^2+\dfrac{1}{2}\,.\,3mk^2\,\omega^2$

$$\Rightarrow \qquad [K.E.]_1=\frac{3m}{2}\left[\frac{P^2}{9m^2}+k^2\,\frac{P^2a^2}{9m^2\,k^2}\right]=\frac{P^2}{6m}\left(1+\frac{3}{2}\right)=\frac{5P^2}{12m}$$

2nd Case. Let A be fixed:- Here the mass m attached at A does not move, so the rod turns about A say with angular velocity ω_1. Hence the particle m and B describes a circle of radius a about the fixed point A and the particle m at C describes a circle of radius $2a$ about the point A, so that their linear velocities are $a\,\omega_1,\ 2a\omega_1$ respectively.

Then taking moments about A, we obtain



$$ma^2\,\omega_1 + m4a^2\omega_1 = P.\ 2a \Rightarrow \omega_1 = \frac{2P}{5ma}$$

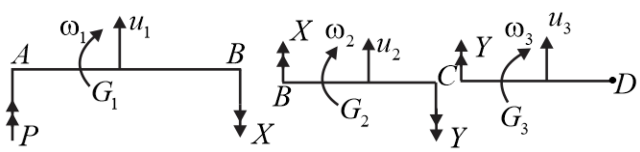
Also, $(K.E.)_2 = \frac{1}{2}ma^2\,\omega_1^2 + \frac{1}{2}m\ .\ 4a^2\omega_1^2 = \frac{5}{2}ma^2\omega_1^2$

$$= \frac{5}{2}m\ .\ \frac{4P^2}{2m5^2} = \frac{2P}{5m} = \frac{(K.E.)_2}{(K.E.)_1} = \frac{24}{5}$$

Example:- Three equal uniform rods AB, BC, CD are freely jointed and placed in a straight line on a smooth table. The rod AB is struck at its end A by a by a blow which is perpendicular to its length, find the resulting motion and show that the velocity of the centre of AB is 19 times that of CD and its angular velocity 11 times that of CD

Sol:- Let each rod be of length $2a$ and mass m and P be the impulse of the blow at A. Hence there will be impulsive action at B and C in opposite directions, which also taken as X and Y respectively.

Just before the blow, the three rods are at rest and after the blow let u_1, ω_1 ; u_2, ω_2 ; u_3 ω_3 be the velocities of C.G.'s and the angular velocities of the rods AB, BC and CA respectively. Then we have. For AB . $mu_1 = P - X$, $m\frac{a^2}{3}\ \omega_1 = Pa + Xa$ (A)



For, BC. $mu_2 = X - Y, m\frac{a^2}{3}\omega_2 = Xa + Y\,a$ (B)

For CD . $mu_3 = Y$, $m\frac{a^2}{3}\ \omega_3 = Y\,a$ (C)

Now velocity of B should be the same as deduced from motion of AB and BC and similarly velocity of C should be the same as deduced from the motion of BC and CD respectively.

For B $u_1 - a\,\omega_1 = u_2 + a\omega_2$ (1)

For C $u_2 - a\,\omega_2 = u_3 + a\omega_3$ (2)

$\Rightarrow$ $$u_1 - a\,\omega_1 = \frac{P - X}{m} - \frac{3(P - X)}{m} = \frac{-2P - 4X}{m}$$

From (B), we have $u_2 + a\,\omega_2 = \frac{X - Y}{m} + \frac{3(X - Y)}{m} = \frac{4X + 2Y}{m}$

$\therefore$ (1) $\Rightarrow$ $-2P - 4X = 4X + 2Y$ or $4X + Y = -P$ (3)

Again from (B), we obtain $u_2 - a\,\omega_2 = \frac{X - Y}{m} + \frac{3(X - Y)}{m} = \frac{-2X - 4Y}{m}$

And $u_3 + a\,\omega_3 = \frac{Y}{m} + \frac{3Y}{m} = \frac{4Y}{m}$ [using C]

$\Rightarrow$ $-2X - 4Y = 4Y$ or $X + 4Y = 0$ [using (2)] (4)

Solving (3) and (4), we get $X = -\frac{4P}{15}$ and $Y = \frac{P}{15}$

$$\Rightarrow \quad u_1 = \frac{1}{m}\left(P + \frac{4P}{15}\right) = \frac{19P}{15m},\ \omega_1 = \frac{3}{ma}\left(P - \frac{4P}{15}\right) = \frac{11P}{5ma}$$

$$u_2 = \frac{1}{m}\left(-\frac{4P}{15} - \frac{P}{15}\right) = -\frac{P}{3m},\ \omega_2 = \frac{3}{3a}\left(-\frac{4P}{15} - \frac{P}{15}\right) = \frac{-5P}{5ma}$$

$$u_3 = \frac{1}{15m},\ \omega_3 = \frac{3}{ma}\left(\frac{P}{15}\right) = \frac{P}{5ma}$$

$$\Rightarrow \quad u_1 ; u_3 = \frac{19P}{15m} ; \frac{P}{15m} = 19 ; 1 \text{ i.e. } u_1 = 19\,u_3$$

i.e. velocity of centre of AB is 19 times that of CD

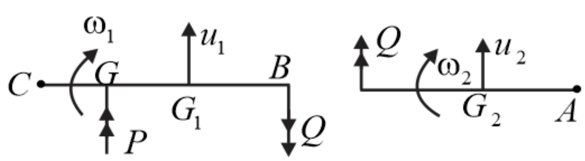
Further $\omega_1 ; \omega_2 = \frac{11P}{5ma} : \frac{P}{5ma} = 11; 1$ i.e. $\omega_1 = 11\omega_3$

i.e. the angular velocity of AB is 11 times that of CD.

Example:- The uniform rods, AB and BC, of the same material are smoothly joined to B and placed in a horizontal line; the rod BC is struck at G by a blow at right angles to it, find the position of G so that the angular velocities of AB and BC may be equal in magnitude.

Solution:- Let $AB = 2a$ and $BC = 2b$ and let m be mass of each rod per unit length. Also let G_1 be the centre of gravity of the rod BC and G_2 that of the rod AB.

Let P be impulse applied at a point G of the rod BC such that $GG_1 = x$.



After the application of the blow, let u_1 be the velocity of G_1 the centre of gravity of BC, and ω_1 it angular velocity, u_2 and ω_2 similar quantities for the rod AB, in the direction as shown in the above figure.

When the blow is struck, let the impulsive action at B between the two rods be Q, in opposite directions on the two rods.

Initially the rods BC and AB were at rest

$\therefore$ $m\,2bu_1 = P - Q$ (motion of G_1) (1)

$m\,2b\frac{b^2}{3}\omega_1 = Px + Qb$ (motion of G_2) (2)

$m\,.2au_2 = Q$; (motion of G_2) (3)

And $m\,2a\dfrac{a^2}{2}\,\omega_2 + Qa$; (motion about G_2) (4)

Further the rods are connected at B, so the velocity of B, from each rod, must be the same i.e.

$u_1 - b\,\omega_1 = u_2 + a\omega_2$ (5)

Now we have two cases:

Case (i):- If $\omega_1 = \omega_2$ (equal in magnitude and same in direction)

Equations (3) and (4) $\Rightarrow u_2 = \dfrac{a\,\omega_2}{3} = \dfrac{a\,\omega_1}{3}$ (6)

And equation (5) $\Rightarrow u_1 - b\,\omega_1 = \dfrac{a\,\omega_1}{3} + a\omega_1$, i.e. $u_1 = \left(\dfrac{4a+3b}{3}\right)\omega_1$

Also (2), $\Rightarrow$ $xP = 2m\dfrac{b^2}{3}\,\omega_1 - Q\,b = \dfrac{2mb^3}{3}\omega_1 - 2mabu_2$ [using (3)]

$$= \frac{2mb^3}{3}\,\omega_1 - \frac{2ma^2b}{3}\omega_1\left(\because u_2 - \frac{a\omega_1}{3}\right) = \frac{2mb}{3}\left(b^2 - a^2\right)\omega_1$$

$\therefore$ $P = 2mbu_1 + Q$ [using (1)] $= 2mbu_1 + 2mau_2$ [using (3)]

$= 2mb\dfrac{4a+3b}{3}\omega_1 + \dfrac{2ma^2\omega_1}{3}$ [after substituting for u_1 and u_2]

$$= \frac{2m}{3}\left(a^2 + 4ab + 3b^2\right)\omega_1 = \frac{2m}{3}(a+b)(a+3b)\omega_1$$

$\Rightarrow$ $x = \dfrac{2m\left(b^2 - a^2\right)}{3}\,.\dfrac{3}{2m(a+b)(a+3b)} = \dfrac{b-a}{a+3b}$

Case (ii):- If $\omega_1 = -\omega_2$ (magnitude equal but opposite in direction)

In this case (3) and (4) $\Rightarrow u_2 = \dfrac{a\,\omega_1}{3} = -\dfrac{a\,\omega_2}{3}$

And (5), $\Rightarrow u_1 - b\,\omega_1 = u_2 + a\omega_2 = -\dfrac{a\omega_1}{3} - o\omega_1$, $\Rightarrow u_1 = \dfrac{3b-4a}{2}\omega_1$

$\therefore$ $xP = \dfrac{2mb^3}{3}\omega_1 - Q\,b = \dfrac{2mb^3}{3}\omega_1 - 2mabu_2$ [using (2)]

$$= \frac{2mb^3}{3}\omega_1 + \frac{2ma^2b}{3}\omega_1\left(\therefore u_2 = -\frac{a\omega_1}{3}\right) = \frac{2mb}{3}\left(b^2 - a^2\right)\omega_1$$

Now $P = 2mbu_1 + Q$ [from (1)]

$= \dfrac{2mb(3b-4a)}{3}\omega_1 - \dfrac{2ma^2}{3}\omega_1$ [substituting for u_1 and u_2]

$$= \frac{2m}{3}\left(3b^2 - 4ab - a^2\right)\omega_1$$

$$\Rightarrow \qquad x = \frac{2mb\left(b^2+a^2\right)}{3}\,\frac{3}{2m\left(3b^2-4ab-a^2\right)} = \frac{b\left(b^2+a^2\right)}{3b^2-4ab-a^2}$$

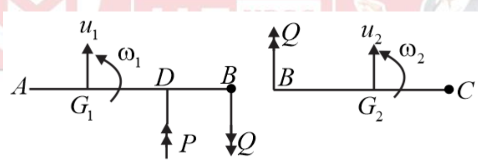
Example:- Two uniform rods AB and BC are freely joined at B and laid on a horizontal table. AB is struck by a horizontal blow of impulse P in a direction perpendicular to AB at a distance c from its centre ; the lengths of AB, BC being $2a$ and $2b$ and their masses M , find the motion immediately, after the blow.

Solution:- Let u_1, u_2 be the velocity of G_1 and G_2, the centres of gravity of the rods AB and BC respectively, and ω_1, ω_2 be the angular velocities of the rod just after blow. Let D be a point in AB where the impulse P is imparted, and $DG_1 = c$.

Before the impulse, the rods are at rest.

There will be an impulsive action at B between the two rods AB and BC when the blow is struck. Let this action Q in opposite directions on the two rods, then for the motion of AB, we have $M\left(u_1 - 0\right) = P - Q$ (1)

And $M\dfrac{a^2}{3}\omega_1 = Pc - Qa$ (2)



And for motion of BC we have $M'\left(u_2 - 0\right) = Q$ (3)

And $M'\dfrac{a^2}{3}\omega_2 = -Qb$ (4)

Now as the rods are connected at B, the velocity of the point of the rods AB is the same as that of the some point B of rod BC

$\therefore \qquad u_1 + a\,\omega_1 = u_2 - b\omega_2$ (5)

Substituting the value of $u_1, \omega_1;\, u_2, \omega_2$ from (1), (2), (3), (4), in (5), we have

$$\frac{P-Q}{M} + 3\frac{\left(Pc - Qa\right)}{aM} = \frac{Q}{M'} - \frac{3Q}{M'} \qquad \Rightarrow \qquad Q = \frac{P}{4}.\frac{M'}{M+M'}\left(1+\frac{3c}{a}\right)$$

$$\therefore \qquad u_1 = \frac{P}{M}\left[1 - \frac{1}{4}\frac{M'}{M+M'}\left(1+\frac{3c}{a}\right)\right] \text{ and } \omega_1 = \frac{3P}{Ma}\left[\frac{c}{a} - \frac{1}{4}\frac{M'}{M+M'}\left(1+\frac{3c}{a}\right)\right]$$

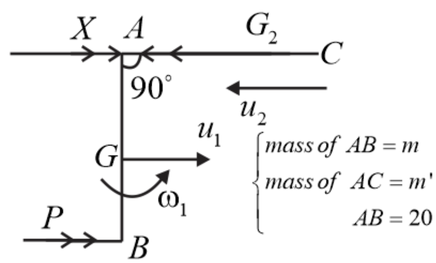
$$u_1 = \frac{1}{4}\frac{P}{M+M'}\left(1+\frac{3c}{a}\right) \text{ and } \omega_2 = -\frac{3}{4}\frac{P}{b\left(M+M'\right)}\left(1+\frac{3c}{a}\right)$$

Hence the four quantities which determine the motion have been obtained.

Example:- Two uniform rods, AB. AC are freely joined at A and laid on a smooth horizontal table so that the angle BAC is a right angle. The rod BAC is struck by a blow P at B in a direction perpendicular to AB, show that the initial velocity of A is $\frac{2P}{4m'+m}$ where m, m' are the masses of AB , AC respectively.

Solution:- The initial motion of A is to be perpendicular to AB, hence the action at A (say X) must be along AC.

Let u_1 be the linear velocity of G_1 and ω_1, the angular velocity of AB. Then since impulse on AC is along CA, the rod AC will only move in the direction CA say with linear velocity u_2. Now for equation of motion, we have



$$\left.\begin{aligned} m u_1 &= P + X \quad ....(1) \\ m\frac{1}{3}a^3\,\omega_1 &= (P - X)a, ..(2) \end{aligned}\right\} \text{motion of } AB$$

$m'\,u_2 = X$ motion of AC (3)

But the rods the connected at A, so the motion of A as deduced form the motion of AB and AC must be the same

$\Rightarrow \quad u_1 = a\,\omega_1 = -u_2 \quad (4)$

Now putting the values of u_1, ω_1 and u_2 [from (1), (2), (3) in (4) we have $\frac{(P+X)}{m} - \frac{3(P-X)}{m} = -\frac{X}{m'} \Rightarrow X = \frac{2Pm'}{4m'+m}$

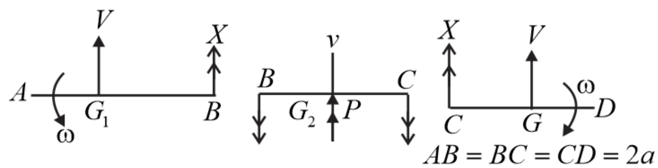
Whence substitution in (3) provides us $u_2 = \frac{X}{m'} = \frac{2P}{4m'+m}$ i.e. velocity of the rods $AC = \frac{2P}{4m'+m}$ of velocity of $A = \frac{2P}{4m'+m}$

Example:- AB, BC and CD are there equal uniform rods lying in a straight line an a smooth plane, and they are freely jointed at B and C. A blow is applied at the centre of BC in a direction perpendicular to

BC. If ω be the initial angular velocity of AB or CD, and θ the angle they make with BC any time, show that the angular velocity is $\dfrac{\omega}{\sqrt{(1+\sin^2\theta)}}$.

Solution:- Let P be the impulse of the blow applied at G_2, the centre of BC, and X the impulsive action at B or C.

Just after the impulse, let U be the velocity of the centre of BC, and V be the velocity of centre of AB and CD, the angular velocities being ω.



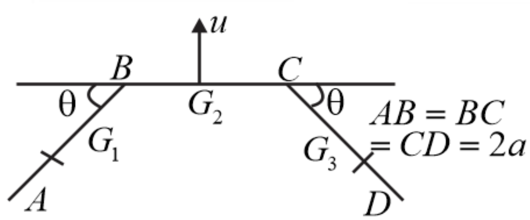
Now, we have : $mU = P = 2X$ (1) for the rod BC

$mV = X$ (2) and $m\dfrac{a^2}{3}\omega = Xa$ for the rod AB or CD

AB and BC are connected at B, so the velocity at B as deduced from BC and AB must be the same i.e. $U = V + a\,\omega$ (4)

$\therefore$ (2) and (3) $\Rightarrow V = \dfrac{a\omega}{3}$ (5) and $U = \dfrac{4a\omega}{3}$ (6)

After the action of the impulse, the rods are set in motion and move in the horizontal plane under finite forces. The rod BC retains is horizontal position where as AB, and DC turn about B and C.



At any time t, let AB or DC make an angle θ with BC and further let u be the velocity of BC. Now co-ordinates of G_1 relative to BC (with B as origin) are $x = a\cos\theta$, $y = a\sin\theta$.

$\Rightarrow$ $\dot{x} = -a\sin\theta\dot{\theta}$, $\dot{y} = a\cos\theta\dot{\theta}$

$\Rightarrow$ actual velocity of G_1 along CB and the right angles to CB is given by

$\left(-a\sin\theta\dot{\theta}\right)$ and $\left(u - a\cos\theta\dot{\theta}\right)$

Now K.E. of the system $= \dfrac{1}{2}mu^2 + \dfrac{1}{2}\,.\,2m\left[\left(u - a\cos\dot{\theta}\cos\theta\right)^2 + a^2\dot{\theta}^2\sin^2\theta + \dfrac{a^2}{3}\dot{\theta}^2\right]$

$$=\frac{1}{2}m\left[3u^2-4au\,\dot{\theta}\cos\theta+\frac{8a^2}{3}\dot{\theta}^2\right]$$

When $\theta=0,\ u=U$; and $\dot{\theta}=\omega$

$\therefore$ initial K.E. of the system $=\frac{1}{2}m\left[3U^2-4aU\ \omega+\frac{8a^2}{3}\omega^2\right]$

Now there is no displacement, in the points of application of (reactions and weights), at right angles to the horizontal plane, so no work is done by these forces, and thus implying that there should be no change in the $K.E.$ of system i.e.

$$=\frac{1}{2}m\left[3u^2-4bu\,\dot{\theta}\cos\theta+\frac{8a^2}{3}\dot{\theta}^2\right]=\frac{1}{2}m\left(3U^2-4aU\,\omega+\frac{8a^2}{5}\omega^2\right) \text{ or}$$

$$3u^2-4au\,\dot{\theta}\,\cos\theta+\frac{8a^2}{3}\dot{\theta}^2=\left(8a^2\omega^2/3\right) \qquad (7)$$

[Substituting the value of U]

Again after the action of the impulse, the motion is under finite forces hence there are no forces on the system in the horizontal plane. The horizontal momentum of the system remains constant viz.

$$mu=2m\left(u-a\,\dot{\theta}\,\cos\theta\right)=mU+2m\left(U-a\,\omega\right)$$

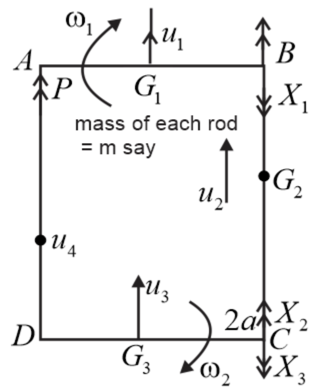
Or $3u-2a\,\dot{\theta}\cos\theta=3U-2a\,\omega$

Or $3u-2a\,\dot{\theta}\cos\theta=2a\,\omega$ [Putting the value of U] (8)

$\therefore$ (7) and (8) $\Rightarrow$ $\dot{\theta}=\frac{\omega}{\sqrt{\left(1+\sin^2\theta\right)}}$ (eliminating u)

Example:- Four equal uniform rods, AB, BC and DE are freely joined at B , C and D and lie on a smooth table in the form of a square. The rod AB is struck by a blow at A at right angles to AB from the inside of the square, show that the initial velocity of A is 79 times that of E .

Solution:- Let the impulse applied at A be P from inside, so that u_1, u_2, u_3 and u_4 are the velocities of G_1, G_2, G_3 and G_4 in the direction of blow. Further let ω_1 and ω_2 be the angular velocities of AB and CD respectively. The angular velocities of BC and DE are zero, because the impulsive reactions upon these two rods are along the rods themselves.



Then we have

$$\left.\begin{aligned} mu_1 &= P + X_1 \quad .... \quad (1) \\ \text{and} \quad m\frac{a^2}{3}\omega_1 &= Pa - X_1 a ....(2) \end{aligned}\right\} \quad \text{(for the rod AB)}$$

Where $AB = 2a$ and X_1 is the impulsive action at B,

$mu_2 = X_2 - X_1$ (for the rod BC)

Where X_2 is the impulsive action at C (3)

$$\left.\begin{aligned} mu_3 &= -(X_2 + X_3) \quad ......(4) \\ \text{and} \quad m\frac{a^2}{3}\omega_3 &= X_2 a - X_3 a \quad ....(5) \end{aligned}\right\} \text{(for the rod CD)}$$

Where X_3 is the impulsive action at D. $mu_4 = X_3$ (for the rod DE) (6)

Now the velocity of the point B as deduced from AB must be equal to the velocity as deduced from BC.

$\therefore$ $u_1 - a\omega_1 = u_2$ (7)

Similarly, for the point C $u_2 = u_3 - a\omega_3$ (8)

And for the point D. $u_3 + a\omega_3 = u_4$ (9)

Whence substituting the values of u_1, ω_1 and u_2, in (70, in readily obtain

$P + X_1 - 3(P - X_1) = X_2 - X_1$ i.e. $5X_1 - X_2 = 2P$ (10)

Also substituting the value of u_1, u_3 and ω_3 in (8), we obtain

$X_2 - X_1 = -(X_2 + X_3) - 3(X_2 - X_3)$ i.e. $5X_2 - 2X_3 = X_1$ (11)

Finally substituting the value of u_3, ω_3 and u_4 in (9), we obtain

$-(X_2 - X_3) + 3(X_2 - X_3) = X_3$ i.e. $2X_2 - 5X_3 = 0$ (12)

Soling (11) and (12), we get

$$X_2 = \frac{5}{21}X_1 \text{ and } X_3 = \frac{2}{21}X_1 \;\therefore (10) \Rightarrow 5X_1 - \frac{5}{21}X_1 = 2P \text{ i.e. } X_1 = \frac{21P}{50}$$

Hence $X_2 = \dfrac{P}{10}$ and $X_3 = \dfrac{P}{25}$

Putting these value in (1), (2) and (6), we get $mu_1 = P + \dfrac{21P}{50}$ i.e. $u_1 = \dfrac{71P}{50m}$ and

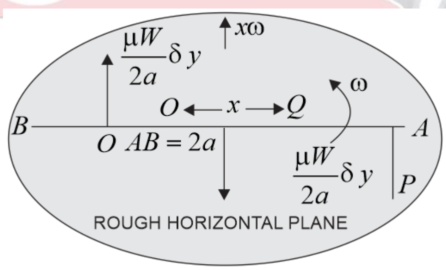
$a\omega_1 = \dfrac{3}{m}\left(P - \dfrac{21P}{50}\right)$ i.e. $a\,\omega_1 = \dfrac{87P}{50m}$ and $mu_4 = \dfrac{P}{25}$ i.e. $u_4 = \dfrac{P}{25m}$

$$\Rightarrow \qquad \frac{velocity\ of\ A}{velocity\ of E} = \frac{u_1 + a\omega_1}{u_4} = \frac{\left(\dfrac{71P}{50m}\right) + \left(\dfrac{87P}{50m}\right)}{\left(\dfrac{P}{25m}\right)} = \frac{79}{1}$$

Example:- A uniform flat rod, of length $2a$ rests on a rough horizontal plane with its weight uniformly distributed. A horizontal force P large enough to produce motion is applied suddenly at the end perpendicular to the length of the rod. Show that initially point, where x is given by the positive root of the equation $x^3 - \left(\dfrac{1}{3} - \dfrac{2P}{\mu W}\right)a^2 x - \dfrac{2}{3}\dfrac{P}{\mu W}a^3 = 0$

W being the weight of the rod and μ the coefficient of friction.

Solution:- Let the impulse P be given at the end A and let O' be the point about which the rod begins to turn where $GO' = x$. Just after the application of the impulse, let ω be the angular velocity of the rod. Hence velocity of the centre of gravity G just after the impulse is $x\omega$ as shown in the figure.



Due to the impulse P, there is an impulsive friction at each point of the rod.

Now consider an element δy of the rod to the right and also to the left of the point O' where $O'Q = y$.

Friction on each of these elements $= \mu \dfrac{W}{2a}\delta y$

Now taking moments about the point O' we have

$$M\left(\frac{a^2}{3} + x^2\right) = \omega = P(a+x) - \int_0^{a+x} \mu \frac{W}{2a} y\,dy - \int_0^{a+x} \mu \frac{W}{2a} y\ dy$$

[The moments of implusive frictions are negative because they tend to decrease ω]

$$= P(a+x) - \frac{\mu W}{2a}\left[\frac{y^2}{2}\right]_0^{a+x} - \frac{\mu W}{2a}\left[\frac{y^2}{a}\right]_0^{a-x} \text{ i.e.}$$

$$M\left(\frac{a^2}{3}+x^2\right)\omega = P(a+x)-\frac{\mu W}{2a}\left(a^2+x^2\right)$$

Also moments of centre of gravity G, gives $M\,x\,\omega = P - \int_0^{a+x}\frac{\mu W}{2a}+dy+\int_0^{a-x}\frac{\mu W}{2a}dy$

[The friction on the right hand of O is opposite to P, while on the left is in direction of P]

$$\Rightarrow \quad M\,x\,\omega = P-\frac{\mu W}{2a}(a+x)+\frac{\mu W}{2a}(a-x) \Rightarrow M\,x\,\omega = P-\frac{\mu W}{2a}x \qquad (2)$$

Whence eliminating ω between (1) and (2), we readily obtain

$$\left(\frac{a^2}{3}+x^2\right)\left(\frac{P}{x}-\frac{\mu W}{2a}\right)=P(a+x)-\frac{\mu W}{2a}\left(a^2+x^2\right)$$

$$\Rightarrow \quad x^3-\left(\frac{1}{3}-\frac{2P}{\mu W}\right)a^2\,x-\frac{2}{3}\frac{P}{\mu W}a^3=0$$

Example:- A lamina in the from of an equilateral triangle ABC lies on a smooth horizontal plane. Suddenly it receives a blow at A in a direction parallel to BC, which causes A to move the velocity V. Determine the instantaneous velocity of B and C and describe the subsequent motion of the lamina.

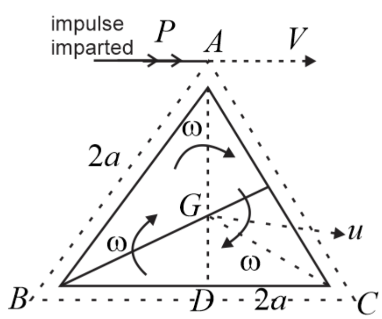
Solution:- As the impulse is parallel to BC, the velocity of G_1, the centre of gravity of the triangular lamina. ABC must also be parallel to BC. Let this velocity be u, and ω be the angular velocity of ΔABC

Equations of motion of the C.G. of the lamina are: $\quad mu = P \qquad (1)$

And $\dfrac{m}{3}\left(\dfrac{a^2}{3}+\dfrac{a^2}{3}+\dfrac{a^2}{3}\right)\omega = P\dfrac{2}{3}a\sqrt{3} \qquad (2)$

(taking moments about G)

$$\left(\because GD\,etc. = a/\sqrt{3}\right)$$



$\therefore$ Velocity of the point $A =$ (velocity of G+ velocity of A relative to G)

Or $V = u+\dfrac{2}{3}a\sqrt{3}\,\omega$. Now (1) and (2) $\Rightarrow a\,\omega = 2\sqrt{3}\,.\,u$

$\Rightarrow \quad V = u+4u = 5u$ i.e. $u=\dfrac{1}{5}V$ and $a\,\omega = \dfrac{2\sqrt{3}}{5}V$.

$\therefore$ Velocity of $B =$ velocity of G+ velocity of B relative to G

$$= u + \frac{2}{3} a\sqrt{3}\,\omega$$

$\Rightarrow$ Velocity of B parallel of BC

= (Velocity of B parallel of BC + velocity of B relative to G parallel to BC)

$= u - \frac{2}{3} a\sqrt{3}\,\omega \cos 60^\circ = \frac{1}{5}V - \frac{2}{5}V = -\frac{V}{5}$ and velocity of B at right angles to BC

= (velocity of G at right angles to BC)

+ (velocity of B, relatives to G) to BC

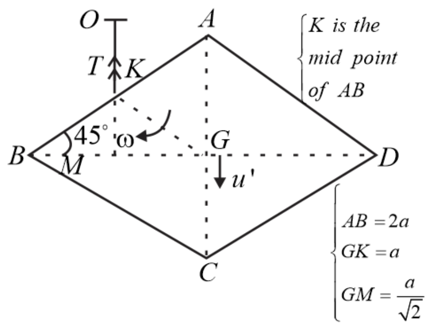
$$= -\frac{2}{3} a\sqrt{3}\,\omega \sin 60^\circ = -a\omega = -\frac{2\sqrt{3}}{5}V.$$

Finally proceeding in the same manner, the velocities of C are $-\frac{V}{5}$ and $\frac{2\sqrt{3}}{5}V$ along and perpendicular to BC

Example:- A square plate, of side $2a$, is falling with velocity u, a diagonal being vertical, when an inelastic string attached to the middle point of an upper edge becomes tight in vertical position. Show that the impulsive tension of the string is $\frac{4}{7}Mu$, where M is the mass of the plate.

Solution:- When the string becomes tight, a jerk experienced by the string resulting an impulsive tension in the string say T.

Just after the impulse let u' be the vertical velocity of G, and ω be the angular velocity of the square, while just before the jerk, the velocity of G is u and there no angular velocity. It is to note that there will be no velocity in the horizontal direction, as there is no horizontal impulse.



Equations of motion of the square plate $ABCD$ are:

$$M(u' - u) = -T \qquad (1)$$

And $M\frac{2a^2}{3}\omega = T\frac{a}{\sqrt{2}} \qquad (2)$

Also, the velocity of K relative to G is $a\omega$ at right angles to GK. Hence its resolved part is $a\omega\cos 45^\circ$ i.e. $a\omega\frac{1}{\sqrt{2}}$ in vertical upward direction.

But just after the impulse, the point K is reduced to rest.

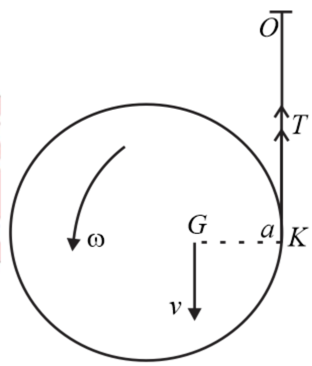
$\Rightarrow$ vertical velocity of the point $K = 0 \Rightarrow u' - \frac{a\omega}{\sqrt{2}} = 0$ (3)

$\therefore$ $\left\{a - \left(\frac{T}{W}\right)\right\} - \left(\frac{3T}{4M}\right) = 0$ i.e. $T = \frac{4}{7} M u$

Example:- A light string is wound round the circumference of a uniform reel of radius a and radius of gyration k about its axis. The free end of the string being tied to a fixed point, the reel is lifted up and let fall so that at the moment when the string becomes tight, the velocity of the centre of reel is u and the string is vertical. Find the change in the motion and show that the impulsive tension is $mu\left(\frac{k^2}{a^2+k^2}\right)$

Solution:- When the string becomes tight, a jerk is experienced by the string resulting an impulsive tension in the sting, say T.

Just after the jerk let v be the velocity of the centre of gravity G and ω the angular velocity while just before the jerk the velocity of G is u and there is no angular velocity.



Equations of motion of the reel are $m(v-u) = T$ (1)

And $mk^2\omega = Ta$ (2)

Just after the impulse, the velocity of the point contact K is zero.

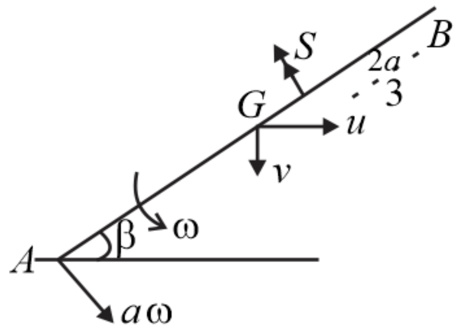
$\Rightarrow$ $v - a\omega = 0$ (3)

$\therefore$ $\left(u - \frac{1}{m}T\right) - \frac{Ta^2}{mk^2} = 0$ $\Rightarrow$ $T = mu\left(\frac{k^2}{a^2+k^2}\right)$

Example:- A uniform inelastic rod falls without rotation, being inclined at an angle β to the horizon and hits a smooth fixed peg at a distance from its upper end equal to one third of its length. Show that the lower end begins to descend vertically.

Solution:- Let AB be the rod which strikes the fixed smooth peg at C, the inclination of the rod, say β at that time. Let S be the impulse at the peg C. Perpendicular to the rod. Just before the impact, the rod was falling without rotation under gravity. Hence it must have then only the vertical velocity V. Just after impact, let u, v be the horizontal and vertical velocities of the C.G.

Now equations of motion of the rod AB are as follows:



$$\left.\begin{aligned} mu &= -S \sin\beta & .....(1) \\ m(u-V) &= -\cos\beta & .....(2) \end{aligned}\right\}$$ motion of C.G.

And $m\frac{a^2}{3}\,\omega = S\,.\frac{a}{3}$ (3) (taking moments about C.G)

Substituting the value of S from (3) in (1) and (2), we readily get $u = -a\,\omega\sin\beta$, $v = -a\,\omega\cos\beta$

Now velocity of the end A relative to $G = a\,\omega$ (perpendicular to AB)

And horizontal velocity at A= horizontal velocity of G+ horizontal velocity of A relative to G

$= u + a\,\omega\sin\beta = -a\,\omega\sin\beta + a\,\omega\sin\beta$ (Substituting value of u)

And vertical velocity of A = vertical velocity of G

+ Vertical velocity of A relative of G

$= v + a\,\omega\cos\beta = (V - a\,\omega\cos\beta) + a\omega\cos\beta = V$ (Substituting for v)

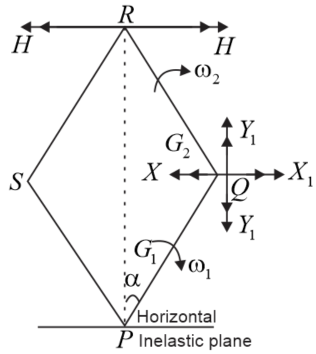
Implies that after the impact the lower end A being to descend vertically.

Example:- Four equal rods, each of mass m and length $2a$ are freely joined at their ends so as to from a rhombus. The rhombus falls with a diagonal vertical, and is moving with velocity V when it hits a fixed horizontal inelastic plane. Find the motion of the rods immediately after the impact, and show that their angular velocities are equal to $\frac{3V\sin\alpha}{2a\left(1+3\sin^2\alpha\right)}$, where α is the angle each rod makes with the vertical.

Show also that the impact destroys a fraction $\left(\frac{3\sin\alpha}{1+3\sin^2\alpha}\right)$ of the kinetic energy just before the impact.

Solution:- Let $PQRS$ be the rhombus formed of four equal rods each of length $2a$ and mass m, fall with the diagonal RP vertical. By symmetry, the motion of the rod RS is the same as the motion of RQ while the motion of the rod PS is the same that of PQ. Hence we need only to consider the motion of PQ and QR alone.

Just after impact at P with the horizontal plane, the rod PQ turns about P, say with angular velocity ω_1, while the rod QR turns about Q, say with angular velocity ω_2, with the direction as shown in the figure.



Due to symmetry, the impulsive action at R will be only horizontal, let it be equal to H. Further let X_1, Y_1 be the horizontal and vertical impulses at Q in opposite directions on the rods PQ, QR respectively. As after impact, R moves in vertical direction so horizontal velocity of the point R relative to $Q = 0$ $\Rightarrow$ horizontal velocity of $Q +$ horizontal velocity of R relative to $Q = 0$

$\Rightarrow$ $$2a\,\omega_1 \cos\alpha + 2a\,\omega_2 \cos\alpha = 0$$

$\Rightarrow$ $$\omega_2 = -\omega_1$$

Now we have horizontal velocity of $G_1 = a\,\omega_1 \cos\alpha$, and vertical velocity of $G_1 = a\,\omega_1 \sin\alpha$

$\Rightarrow$ Horizontal velocity of $G_2 =$ horizontal velocity of $Q +$ horizontal velocity of G_2 relative to Q.

$$= 2a\,\omega_1 \cos\alpha + a\,\omega_2 \cos\alpha = a\omega_1 \cos\alpha \qquad (\because \omega_2 = -\omega_1)$$

And vertical velocity of $G_2 = 2a\ \omega_1 \sin\alpha - a\omega_2 \sin\alpha = 3a\omega_1 \sin\alpha$

Now considering the combined motion of PQ and QR and taking moments about P, we get

$$\left\{m\frac{4a^2}{3}\omega_1 + m\left(\frac{a^2}{3}\omega_1 a + \omega_1 \cos\alpha . 3a\cos\alpha + 3a\omega_1 \sin\alpha . a\sin\alpha\right)\right\}$$
$$-2mVa\sin\alpha = 4a\cos\alpha\ \ H$$

$\Rightarrow$ $$2\,\omega_1 = \frac{V\sin\alpha}{a} + \frac{2H\cos\alpha}{ma} \qquad (2)$$

Again considering motion of the rod QR alone and taking moments about Q, we have

$$\left[m\left\{\frac{a^2}{3}\omega_2 + a\omega_1 \cos\alpha . a\cos\alpha - 3a\omega_1 \sin\alpha . a\sin\alpha - m(-V)a\sin\alpha\right\}\right] = H.2a\cos\alpha$$

$\Rightarrow$ $$\omega_1\left(\frac{2}{3} - 4\sin^2\alpha\right) = -\frac{V\sin\alpha}{a} + \frac{2H\cos\alpha}{ma} \qquad (3)$$

$\therefore$ (2) and (3)give

$$\omega_1\left(\frac{4}{3} + 4\sin^2\alpha\right) = \frac{2V\sin\alpha}{a} \Rightarrow \omega_1 = \frac{3V\sin\alpha}{2a\left(1 + 3\sin^2\alpha\right)}$$

Now K.E. just before the impulse $=\frac{1}{2}4mV^2=2mV^2=E_1$ (say) and K.E. just after the impulse

$=2$ [K.E. of $PQ+K.E.$ of QR] $=E_2$ (say)

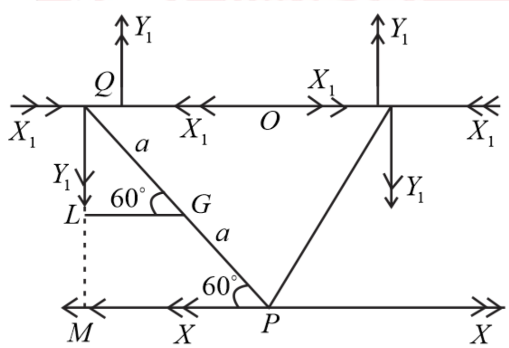
Or $E_2=2\left[\frac{1}{2}m\frac{4a^2}{3}\omega_1^2+\frac{1}{2}m\left(\frac{a^2}{3}\omega_2^2+a\omega_1^2\cos^2\alpha+9a^2\omega_1^2\sin^2\alpha\right)\right]$

$=\frac{8}{3}ma^2\left(1+3\sin^2\alpha\right)\omega_1^2 \qquad (\because \omega_2=-\omega_1)$

$=\frac{3\sin^2\alpha}{1+3\sin^2\alpha}2mV^2=\frac{3\sin^2\alpha\, E_1}{1+3\sin^2\alpha}$

Example:- An equilateral triangle, formed by inform rods freely hinged at their ends, is falling freely with one side horizontal and upper-most. If the middle point of this side be suddenly stopped, show that the impulsive actions at the upper and lower hinges are in the ration $\sqrt{(13)}:1$.

Solution:- The middle point O of the rod QR is suddenly stopped, so the impulse is imparted at O. Hence an impulsive actin between the two rods at Q is generated. A similar impulse is generated at R as there at symmetry about O. The action at P will be horizontal due to symmetry. Let it be X in opposite direction on the two rods PQ and PR.



Just before the blow there was only vertical velocity for every rod. As the system has been stopped, there are no linear velocities and no angular velocities for any rod after the blow. Now considering the motion of the rod PQ, or (PR), we get $X_1-X=0$ (1)

[$\because$ There is no horizontal velocity of G before and after the impulse]

Taking moments about G, we get

$X_1LQ-Y_1GL+X_1LM=0 \Rightarrow X_1a\sin 60^\circ-Y_1a\cos 60^\circ+Xa\sin 60^\circ=0$

$\Rightarrow \qquad X_1\frac{a\sqrt{3}}{2}+X\frac{a\sqrt{3}}{2}-Y_1\frac{a}{2}=0$ (2)

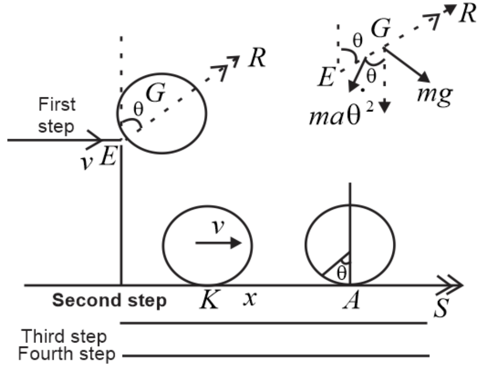
[$\because$ There is no angular velocity of PQ before and after the impulse]

$\Rightarrow \qquad \frac{reaction\ at\ Q}{reaction\ at\ P}=\frac{\sqrt{\left(X_1^2+Y_1^2\right)}}{X}=\frac{\sqrt{\left(X^2+12X^2\right)}}{X}=\frac{\sqrt{(13)}}{1}$

Example:- An inelastic sphere of radius a rolls down a flight of perfectly rough steps, show that if the velocity of the centre on the first step exceeds $\sqrt{(ga)}$ its velocity will be the same on every step, the step being such that , in its flight, the sphere never impinges on an edge.

Solution:- Let E be the edge of the first step and let v be its velocity at E. Now the sphere has a tendency to turn about the edge E and let θ be the angle through which it turns in time t.

Now equations of motion of the sphere are given



By $ma\dot{\theta}^2 = ma\cos\theta - R$ (1)

And $m\left(\frac{2a^2}{5}+a^2\right)\ddot{\theta} = mga\sin\theta$ (2)

(taking moments about E)

(2) $\Rightarrow a^2\dot{\theta}^2 = -\frac{10ga}{7}\cos\theta + c$

When $\theta = 0,\ a\dot{\theta} = v,\ \Rightarrow c = v^2 + \frac{10ga}{7}$

$\therefore$ $a^2\dot{\theta}^2 = \frac{10ga}{7}(1-\cos\theta)+v^2$ or $a\dot{\theta}^2 = \frac{10ga}{7}(1-\cos\theta)+\frac{v^2}{a}$

$\therefore$ (1) gives $\frac{10g}{7}(1-\cos\theta)+\frac{v^2}{a} = g\cos\theta - \frac{R}{m}$

Or $R = m\left[g\cos\theta - \frac{10g}{7}(1-\cos\theta)+\frac{v^2}{a}\right]$ $\therefore (R)_{\theta=0} = m\left(g-\frac{v^2}{a}\right)$

Obviously $(R)_{\theta=0}$ will not remain positive, if

$v^2 a > g$ i.e. $v > \sqrt{(ga)}$

Which implies that the sphere leaves the step at once if the velocity $v > \sqrt{(ga)}$. After traveling the distance on the first step, the sphere strikes the second step say at the point K. But the step is inelastic so the sphere will not rebound and will roll on the second step with the velocity v.

When it has rolled a distance x on the second step, let S be the force of friction, sufficient for pure rolling, then equations of motion are given by $m\ddot{x} = S$ (3)

$$m\frac{2a^2}{5}\ddot{\theta} = -Sa \qquad (4)$$

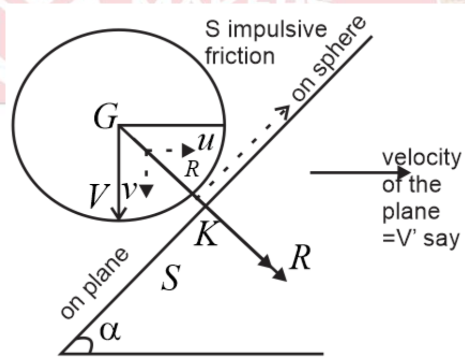
But the motion is of pure rolling, so we have $\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta}$ (5)

From above, we have $\ddot{x} = 0 \Rightarrow \dot{x} =$ constant $= v$

Which implies that the sphere rolls on the scorned step with the uniform velocity v. Hence velocity at every step is the same.

Example:- A sphere of mass m falls with velocity V on a perfectly rough inclined plane of mass M and angle α which rests on a smooth horizontal plane. Show that the vertical velocity of the centre of the sphere immediately after the impact is $\dfrac{5(M+m)V\sin^2\alpha}{7M+2m+5m\sin^2\alpha}$ the bodies being all supposed perfectly inelastic.

Solution:- Just before the impact velocity of the sphere is V in vertical direction $(\downarrow)$ there being no angular velocity then. Just after the impact let u and v be the velocities and ω the angular velocity of the sphere as shown in the figure.



just after the impact, the inclined plane also begins to move in the horizontal direction.

Now equation of motion are

$$\left.\begin{aligned} m(-u) &= R\sin\alpha - S\cos\alpha \qquad (1) \\ m(v-u) &= -R\cos\alpha - S\sin\alpha \qquad (2) \end{aligned}\right\}$$

And $m\dfrac{2a^2}{5}\omega = Sa$ (3)

Also, the motion of the inclined plane is given by $MV' = R\sin\alpha - S\cos\alpha$ (4)

Now horizontal velocity of the point K of the sphere $=$ horizontal velocity of the point K of the inclined plane.

$\therefore$ $u + a\omega\cos\alpha = V'$ and vertical velocity of the point K of the sphere = vertical velocity of the point K of the inclined plane.

$\therefore \qquad v - a\omega \sin\alpha = 0 \;\Rightarrow v = a\omega\sin\alpha$

From (1) and (4) it is easy to see that $MV' = -mu$, i.e. $V' = -\dfrac{mu}{M}$

Now multiplying (2) by $\sin\alpha$ and (1) by $\cos\alpha$, and then adding these to (3), we obtain

$$-mu\cos\alpha + m(v-V)\sin\alpha + m\frac{2a\omega}{5} = 0$$

$$\Rightarrow \quad \frac{Ma\omega\cos^2\alpha}{M+m} + (v-V)\sin\mu + \frac{2a\omega}{5} = 0 \quad \left(\because u = -\frac{Ma\omega\cos^2\alpha}{M+m}\right)$$

$$\Rightarrow \quad \left[\frac{Ma\omega\cos^2\alpha}{M+m} + \frac{2}{5}\right]\frac{v}{\sin\alpha} + (v-V)\sin\alpha = 0 \quad [\because v = a\omega\sin\alpha]$$

$$\Rightarrow \quad \left[5M\cos^2\alpha + 2(M+m) + 5(M+m)\sin^2\alpha\right]v$$

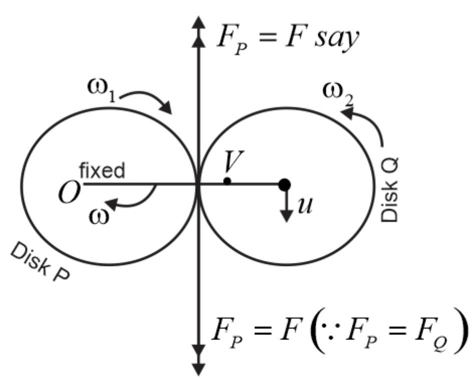
$$= (M+m)V\sin^2\alpha$$

$$\Rightarrow \quad \left[7M + 2m + 5m\sin^2\alpha\right]v = 5(M+m)V\sin^2\alpha$$

$$\Rightarrow \quad v = \frac{5(M+m)V\sin^2\alpha}{7M + 2m + 5m\sin^2\alpha}$$

Example:- Of two inelastic circular discs with milled edged each of mass m and radius a, one is rotating with angular velocity ω round its centre O, which is fixed on a smooth plane, and the order is moving with spin in the plane with velocity v directed towards O. Find the motion immediately afterwards, and show that the energy lost by the impact is $\dfrac{1}{2}m\left(v^2 + \dfrac{a^2\omega^2}{5}\right)$.

Solution:- The disc P is rotating about its centre O with angular velocity ω (say) while disc Q is moving with linear velocity v towards O as shown in the figure. Further let F be the impulsive friction at the point of contact, K. But the discs are inelastic, so they will not rebound, and hence after the impact, then velocity of the disc Q will be along O_1Q.



Further let the velocity of Q be u, along the common tangent. Also let ω_1 and ω_2 be angular velocities of P and Q after the impact, then equations of motion are:

$$\frac{ma^2}{2}(\omega_1 - \omega) = -Fa \qquad (1) \qquad \text{(for the disc P)}$$

$$mu = F \qquad (2) \qquad \text{and } \frac{ma^2}{2}\omega^2 = Fa \ \ (3) \text{ [for the disc P]}$$

But the discs P and Q touch each other at K, so the velocity of the point of contact K as deduced from disc must be equal.

i.e. $a\,\omega_1 = u + a\omega_2$

Now putting the value of u, ω_1 from (2), (1) and (3) in (4), we easily obtain $-\frac{2F}{m} + a\omega = \frac{F}{m} + \frac{2F}{m}$

i.e. $F = \frac{ma\,\omega}{5}$

$\therefore \qquad a\,\omega_1 = -\frac{2a\omega}{5} + a\omega = \frac{3a\omega}{5}$; and $u = \frac{a\omega}{5}$: Also $a\omega_2 = \frac{2a\omega}{5}$

$\therefore \qquad$ Total K.E. after the impact =K.E. of the disc $P + K.E.$ of the disc Q

$$= \frac{1}{2}m\frac{a^2}{2}\omega_1^2 + \frac{1}{2}m\left(u^2\frac{a^2}{2}\omega_2^2\right)$$

$$= 2m\left[\frac{9a^2\omega^2}{50} + \frac{a^2\omega^2}{25} + \frac{4a^2\omega^2}{50}\right] = \frac{3a^2m\omega^2}{20} = E_2 \text{ (say)}$$

But K.E. before the impact $= \frac{1}{2}m\frac{a^2}{2}\omega^2 + \frac{1}{2}mv^2 = E_1$ (say)

Hence loss in K.E. $= E_1 - E_2$

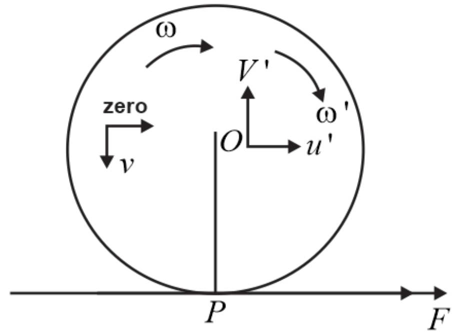
$$= \frac{1}{2}m\frac{a^2}{2}\omega^2 + \frac{1}{2}mv^2 - \frac{3ma^2\omega^2}{20} = \frac{1}{2}m\left(v^2 + \frac{a^2\omega^2}{5}\right)$$

Example:- If a hollow lawn tennis ball of elasticity e has on striking the ground supposed perfectly rough, a vertical velocity v and angular velocity ω about a horizontal axis, find its angular velocity after impact and that the range of the rebound will be $\frac{4a\,\omega}{5g}ev$

Solution:- Just before the impact v is vertical velocity and ω the angular velocity while just after impact let u', v' be the horizontal and vertical velocities and ω' the angular velocity $(\ )$, the equations of motion are

$$mu' = -F \qquad (1) \text{ and } m(v' + v) = R \qquad (2)$$

$$\text{And } m\frac{2}{3}a^2(\omega' - \omega) = Fa \qquad (3)$$



The point of contact P is reduced to rest instantaneously, there being no sliding,

$\Rightarrow$ $u' - a\omega' = 0$ also $v' = ev$

Now eliminating from (1), (3) we get

$$\frac{2}{3}a(\omega' - \omega) = -u'$$

Or $\frac{2}{3}(u' - a\omega) = -u' \;\Rightarrow \frac{5}{3}u' = \frac{2}{3}a\omega$

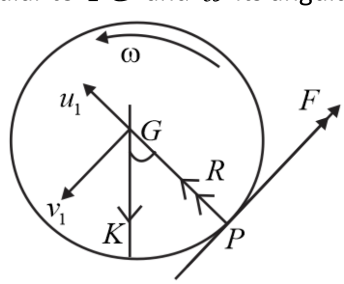
Or $u' = \frac{2}{5}a\omega$ (4)

$\therefore$ Range after rebound $= \frac{2(\text{horizonal velo.})(\text{vert.velo.})}{g}$

$$= \frac{2u'v'}{8} = \frac{2\,\frac{2}{5}\,a.\,\omega.ev}{g} = \frac{4a\omega}{5g}ev\,.$$

Example:- An imperfectly elastic sphere descending vertically comes in contact with a fixed rough point, the impact taking place taking place at a point distant α from the lowest point, and the coefficient of elasticity being e. Find the motion, and show that the sphere will start moving horizontal after the impact if $\alpha = \tan^{-1}\sqrt{\left(\frac{7e}{5}\right)}$

Solution:- Before the impact the sphere is descending vertically say with velocity V; implies that it has then no horizontal velocity and no angular velocity. Just after the impact let u_1 and v_1 be the velocities of the sphere along and perpendicular to PG and ω its angular velocity as marked in the above figure.



Then the equation of motion are

$$m(v_1 - V\sin\alpha) = -F \qquad (1)$$

$$m(v_1 + V\cos\alpha) = R \qquad (2)$$

And $m\dfrac{2a^2}{5}\,\omega = Fa \qquad (3)$

After impact then is no sliding velocity i.e. velocity of the point P along tangent at P is zero.

$\therefore \quad v_1 - a\,\omega = 0 \;\Rightarrow v_1 = a\,\omega$

Also $u_1 = e v_1 \cos\alpha$ (by Newton's Law)

From (1) and (3), we get $(v_1 - V\sin\alpha) = -\dfrac{2a}{5}\omega$

$\Rightarrow \quad v_1 - V\sin\alpha = -\dfrac{2}{5}v_1 \Rightarrow v_1 = \dfrac{5}{7}V\sin\alpha$

$\therefore \quad v_1 = a\,\omega = \dfrac{5}{7}V\sin\alpha$

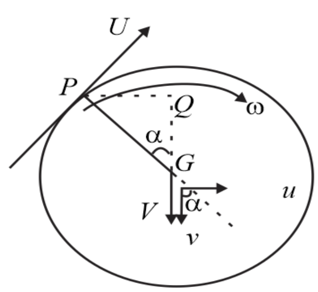
Then sphere will start moving horizontally after impact if the vertical velocity of sphere $=0$ i.e. if $u_1\cos\alpha - v_1\sin\alpha = 0$

Or $\tan\alpha = \dfrac{u_1}{v_1}$ or $\tan\alpha = \dfrac{eV\cos\alpha}{\frac{5}{7}V\sin\alpha}$

Or $\tan^2\alpha = \dfrac{7}{5}e \Rightarrow \alpha = \tan^{-1}\sqrt{\left(\dfrac{7e}{5}\right)}$

Example:- A rough imperfectly elastic ball is dropped vertically and when its velocity is V, a man suddenly moves his racket forward in its own plane with velocity U, and thus subjects the ball to pure cut in a downward direction making an angle α with the horizon. Show that, on striking the rough ground, the ball will not proceed beyond the point of impact, provided $(U - V\sin\alpha)(1-\cos\alpha) > (1+e)\left(1+\dfrac{a^2}{k^2}\right)V\sin\alpha\cos\alpha$

Solution:- Let P be a point on the ball such that it is hit by the man. Obviously the plane of the racket is tangential to the ball. Since the man moves the racket in its own plane with velocity U, the velocity of the point of contact P will also be U along the tangent at P



Just before the impact the velocity of the ball is given to be V, there being no horizontal velocity and no angular velocity.

Just after the impact, let u, v be the horizontal and vertical velocities and ω the angular velocity as marked in the figure. **Then we have for the motion of ball.**

Just before the impact, the moment of momentum about P

$= mV\,a\sin\alpha$ [$\because$ moment of momentum $= mk^2\dot{\theta} + mv\,p$]

After impact, moment of momentum about P

$= mk^2\omega - mua\cos + mva\sin\alpha$

$\therefore$ change in moment of momentum

$= \left(mk^2\omega - mua\cos\alpha + mva\sin\alpha\right) - mVa\sin\alpha$

But the impulse is applied at P, hence moment of the impulse at P is zero.

Now $mk^2\omega - mua\cos\alpha + mva\sin\alpha - mVa\sin\alpha = 0$ (1)

($\therefore$ Change in moment of momentum = moment of the impulse)

But velocity of the point P of the racket = velocity of the point of the ball.

$\Rightarrow$ $U = v\cos\alpha - v\sin\alpha + a\ \omega$

Again multiplying (1) by $\cos\alpha$ and (3) by $\sin\alpha$ and adding, we get

Also by Newton's experimental law, we have

$u\sin\alpha + v\cos\alpha = eV\cos\alpha$ (3)

Now (2) and (1) $\Rightarrow \omega = \dfrac{a\left(U - V\sin\alpha\right)}{a^2 + k^2}$

$$u = \frac{Uk^2\cos\alpha + V\sin\alpha\cos\alpha\left\{a^2 + e\left(k^2 + a^2\right)\right\}}{a^2 + k^2}$$

Now after striking the ground, let u_1, v_1, ω_1 be horizontal, vertical and angular velocities. Then taking moments about the point of contact of the ball with ground, we have

$\left(mk^2\omega_1 - mu_1a\right) - \left(mk^2\omega - mua\right) = 0$ (4)

($\therefore$ Change in moment of momentum = moment of the impulse)

Also the point of contact has no horizontal velocity so we get

$u_1 + a\,\omega_1 = 0$ (5)

(4) and (5) $\Rightarrow u_1\left(1 + \dfrac{k^2}{a^2}\right) = u - \dfrac{k^2}{a}\omega$ (6)

From (6), it is clear that u_1 will be negative if $u < \dfrac{k^2}{a}\omega$

Thus we can say that the ball will not proceed beyond the point of contact, if $u < \frac{k^2}{a}\omega$, or

$$\frac{Uk^2 \cos\alpha + V\sin\alpha\cos\alpha\left\{a^2 + e\left(k^2 + a^2\right)\right\}}{a^2 + k^2} < \frac{k^2 a\left(U - V\sin\alpha\right)}{a\left(a^2 + k^2\right)}$$ or if

$$k^2\left(U - V\sin\alpha\right)Uk^2\cos\alpha + V\sin\alpha\cos\alpha\left\{a^2 + e\left(k^2 + a^2\right)\right\}$$

$$= Uk^2\cos\alpha - Vk^2\sin\alpha\cos\alpha$$

$$+Vk^2\sin\alpha\cos\alpha + V\sin\alpha\cos\alpha\left\{a^2 + e\left(k^2 + a^2\right)\right\}$$

$$= k^2\left(U - V\sin\alpha\right)\cos\alpha + V\sin\alpha\cos\alpha\left\{\left(a^2 + k^2\right) + e\left(k^2 + a^2\right)\right\}$$

$$= k^2\left(U - V\sin\alpha\right)\cos\alpha + \left(1 + e\right)\left(k^2 + a^2\right)V\sin\alpha\cos\alpha$$

i.e. If $\left(U - V\sin\alpha\right)\left(1 - \cos\alpha\right) > \left(1 + e\right)\left(1 + \frac{a^2}{k^2}\right)V\sin\alpha\cos\alpha$

Example:- A tennis ball of hallow spherical space is given by underground cut, and hits the ground at the other side of the net at a distance c from it, if u and v its horizontal and vertical velocities and ω its angular velocity when it hits the perfect rough ground, show that the ball will return back towards the net if $2a\omega > 3u$. Further show that it will rebound over the net if $c < \frac{2ev\left(2a\omega - 3u\right)}{5g}$ and will touch the net overhead if $k < \frac{5e}{2} \cdot \frac{2ev\left(2a\omega - 3u\right) - 5gc}{\left(2a\omega - 3u\right)^2}$, where e is the coefficient of restitution and h is the height of the net.

Solution:- After the impact let u_1 and v_1 be the velocities of the centre of gravity G of the ball and ω_1 its angular velocity.

Let F be the impulsive friction and R the impulsive normal reactions of motion if the ball are

$$m\left(u_1 + u\right) = F \quad (1)$$

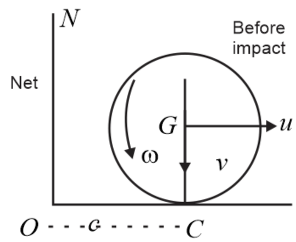
$$m\left(v_1 + v\right) = R \quad (2)$$

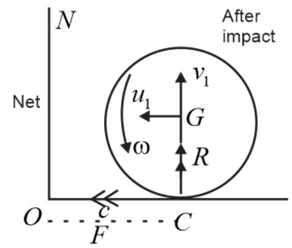
And $m\frac{2}{3}a^2\left(\omega_1 - \omega\right) = -Fa$ (3)

Where $ev = v_1$

Just after the impact there is no horizontal velocity of the point of contact, so $u_1 - a\,\omega_1 = 0$ (4)

From (1) and (3), we obtain





$u_1 = u + \dfrac{2}{3}a(\omega_1 - \omega) = 0$ i.e. $a\omega_1 = -\dfrac{2}{2}(u_1 + u) + a\omega$

$\therefore \qquad u_1 + \dfrac{3}{2}(u_1 + u) - a\,\omega = 0$

$\Rightarrow \qquad 5u_1 = 2a\,\omega - 3u \quad \Rightarrow u_1 = \dfrac{2a\omega - 3u}{5}$

Hence the ball will return back toward the net if u_1 is positive when $2a\,\omega - 3u > 0$ or when $2a\,\omega > 3u$.

Second part:- After rebounding from the ground, the ball moves in a parabolic path. Now considering the horizontal motion of the ball, we have $t = \dfrac{c}{u_1}$ where t is time taken by the ball to reach the net or

$$t = \frac{5c}{2a\,\omega - 3u}$$

Now t, must be less than the time of reaching the ground.

$\therefore \qquad \dfrac{5c}{2a\omega - 3u} < \dfrac{2ev}{g}$ or $c < \dfrac{2ev(2a\omega - 3u)}{5g}$

Third part:- Let the vertical height to which the ball rises be h then have $h' = v_1 t - \dfrac{1}{2}gt^2 = ev - \dfrac{1}{2}gt^2$

$(\because v_1 = ev)$

$$= \frac{5}{2}e\frac{2ev(2a\omega - 3u) - 5gc}{(2a\omega - 3u)^2}$$

Clearly the ball will touch the net of the height h overhead if $h < h'$ i.e. if $h < \dfrac{5c}{2}\dfrac{2ev(2a\omega - 3u) - 5gc}{(2a\omega - 3u)^2}$

Example:- Three particles of equal masses are attached to the ends, A and C and the middle point B of light rod ABC, and the system is at rest on a smooth table. The particle C is struck by a blow at right angles to the rod; show that the energy communicated to the system when A is fixed is to the energy communicated when the system is free as 24 : 25.

Solution:- Let A, B, C be the three points of a rod. Where the three particles each of mass m,be the placed. Now let the end A be free, and an impales P be applied at C, such that the rod begins to rotate about the point B. Then if the velocity of mass at is u_1, the velocities of the masses A and C are $u_1 - a\omega_1$, $u_1 + a\omega_1$ respecitvley.

Case I:- When the end A is free.

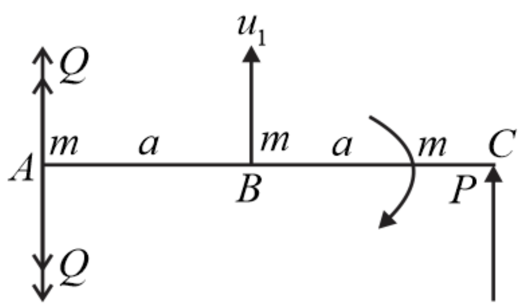
We have $mu_1 + m(u_1 - a\omega_1) + (u_1 + a\omega_1)$

$$= P \Rightarrow u_1 = \frac{P}{3m} \qquad (1)$$

Taking moments about C, we have

$$mu_1 + am(u_1 - a\omega_1)2a = 0 \Rightarrow 3u - 2a\omega_1 = 0 \qquad (2)$$

$\Rightarrow \quad a\omega_1 = \dfrac{P}{2m}$ [using (1)]



Velocity of the mass at $C = u_1 + a\omega_1 = \dfrac{P}{3m} + \dfrac{P}{2m} + \dfrac{5P}{6m}$ and the K.E. communicated to the rod

$= \dfrac{1}{2}$ impulse at $C \times$

(velocity of C)

$$= \frac{1}{2}P \cdot \frac{5P}{6m} = \frac{1}{12}\frac{5P^2}{m} = E_1 \text{ (say)}$$

Case II:- When A is fixed.

In this case the rod will begin to rotate about A, with an angular velocity ω and an impulsive thrust (=X say) will be generated at A.

Now the moment about A gives $m(2a)^2\omega + ma^2\omega = p \, . \, 2a \Rightarrow m5a\omega = 2P \Rightarrow a\omega = \dfrac{2P}{5m}$

$\therefore \quad$ Velocity communicated to $C = 2a\,\omega$

($\because$ C describes a circle, about A, of radius $2a$)

Hence its velocity perpendicular to $AC = 2a\omega = \dfrac{4P}{5m}$

$$E_2 = \frac{1}{2}P \text{ (Velocity of C) } = \frac{1}{2}P\,\frac{4P}{5m} = \frac{2P^2}{5m} = E_2 \text{ (say)}$$

$$\Rightarrow \quad \frac{E_1}{E_2} = \frac{(5P/12m)}{(2P^2/5m)} = \frac{25}{24} \Rightarrow \frac{E_2}{E_1} = \frac{24}{25}$$

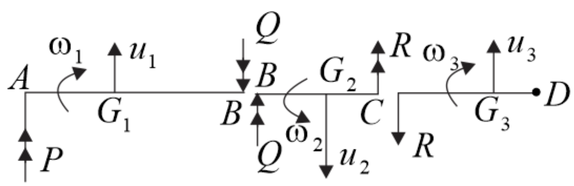
Example:- Three equal rods, AB, BC, CD are freely jointed and placed in a straight line on a smooth table. The rod AB is struck at its end A by a blow which is perpendicular to the length; find the resulting motion and show that the velocity of the centre of AB is 19 times that of CD, and its angular velocity 11 times that of CD.

Solution:- Let P be the impulse of the blow applied at A and let $(u_1, \omega_1)(u_2, \omega_2)(u_3, \omega_3)$ be the velocities angular velocities of the rods AB, BC, CD respectively just after the blow.

Again let Q and R be the impulsive reaction at B and C respectively, then we have

$mu_1 = P + Q$ (1)

$m\frac{a^2}{3}\omega_1 = (P - Q)a$ (2) **[for the rod A,B]**



$mu_1 = -R + Q$ (3)

$m\frac{a^2}{3}\omega_2 = (Q + R)a$ (4) **[for the rod BC]**

$mu_3 = R$ (5); $-m\frac{a^2}{3}\omega_3 = -Ra$ (6) **[for the rod CD]**

As the rods are connected at B and C, the velocity of B as considered from $AB =$ then velocity of A as considered from BC.

$\Rightarrow$ Velocity of $G_1 +$ velocity of B relative to G_1

$=$ Velocity of $G_2 +$ velocity of B of relative to G_2

$\Rightarrow \quad -u_1 + a\omega_1 = u_2 + a\omega_2$ (7)

Similarly for C, we have $a\omega_2 - u_2 = a\omega_3 + u_3$ (8)

Now from (7) and [(1) to (4), we easily obtain $3(P-Q)(-(P+Q)) = 3(R+Q) + (Q-R)$

$\Rightarrow \quad 2P = 8Q + 2R \Rightarrow P = 4Q + R$

Also from (8) and [(3) to (6)], we get $3(Q+R) - (-R+Q) = 3R + R$

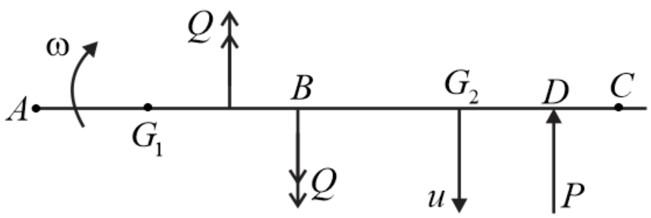
$\Rightarrow \quad Q = -4R$

From (9) to (10), we get easily $P = -15R$ and $Q = -4R$

$$\therefore \frac{\text{velocity of } G_1}{\text{velocity of } G_2} = \frac{u_1}{u_3} = \frac{15R+4R}{R} = 19 \text{ and } \frac{\text{angular velocity of} AB}{\text{angular velocity of } CD} = \frac{\omega_1}{\omega_2} = \frac{3(15R-4R)}{3R} = 11.$$

Example:- Two equal uniform rods. AB and BC, are freely jointed at B and turn about a smooth joint at A. When the rods are in a straight line, ω being the angular velocity of AB and u the velocity of the centre of mass of BC, BC impinges on a fixed inelastic obstacle at point D, show that rods instaneously brought to rest if $BC = 2a\dfrac{2u-a\omega}{3u+2a\omega}$ where $2a$ is the length of either rod.

Solution:- When the rods AB and BC are in a straight line, ω is the angular velocity of AB and u the velocity of G_2 the centre of gravity of BC, and let ω_1 be the angular velocity of BC about G_2 before impinging on an inelastic obstacle at D, such that $BC = x$.



There will be an impulsive reaction between the two rods at B denoted by Q acting on opposite directions as marked in the above figure.

Now the velocity of B as deduced from $AB =$ the velocity of B as deduced from BC.

$\Rightarrow$ $2a\omega =$ velocity of $G_2 +$ velocity of B relative of $G_2 = (u - a\,\omega_1)$.

Take moments about A to remove the unknown reaction at A, we get

$$m\,.\frac{4}{33}a = -Q^2\,\omega = -Q\,.2a \quad \Rightarrow m\frac{2a\omega}{} \qquad (1)$$

For the rod BC, we get $mu = -(P-Q)$ (2)

And $\quad m\dfrac{a^2}{3}\omega_1 = -\{P(x-a)+Qa\}$ (3)

$\therefore$ (2) and (3) $\Rightarrow m\left(u+\dfrac{a}{3}\omega_1\right) = -P - P\dfrac{(x-a)}{x} = -\dfrac{Px}{a}$ (4)

With this substitution in equation (4), we readily obtain

$$\frac{x}{a}\left(u+\frac{2a\omega}{3}\right) = \left(u+\frac{a}{3}\omega_1\right) = \left[u+\frac{(u-2a\omega)}{3}\right]$$

$$\Rightarrow \quad \frac{x}{a}\left(\frac{3u+2a\omega}{3}\right) = \left(\frac{4u-2a\omega}{3}\right) = \frac{2(2u-a\omega)}{3}$$

$$\Rightarrow \quad x = \frac{2a(2u-a\omega)}{(3u+2a\omega)}$$

Example:- A disc of any form moving in its plane without rotation with velocity V at right angles to a fixed plane strikes the plane so that the distances of its centre of gravity from the point of impact and from the plane are r and p. Assuming the plane to the be elastic and sufficiently rough to prevent sliding, show that the impulsive pressure and friction are respectively $\frac{m(1+e)V\left(p^2+k^2\right)}{r^2+k^2}$ and $\frac{m(1+e)V\,p\left(r^2-p^2\right)^{1/2}}{r^2+k^2}$ where k is the radius of gyration. Also show that loss of kinetic energy is $\frac{1}{2}\frac{m\left(1-e^2\right)\left(k^2+p^2\right)V^2}{\left(r^2+k^2\right)}$

Solution:- Let the disc strike the horizontal plane at K. Before the impact, V is the vertical velocity of G. After the impact let (u,v,w) be the velocities and angular velocity of the disc. Angular velocity of K relative to G is $r\,\omega$ perpendicular to GK where $GK=r$

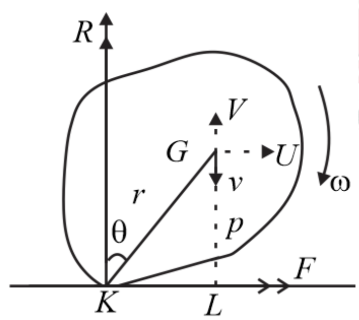
Now equations of motion are

$m\left[v-(-V)R\right]$ (1)

$mu=F$ (2)

$mk^2\omega=-Fp+Rr\sin\theta$ or $mk^2\,\omega=-Fp+R\sqrt{\left(r^2-p^2\right)}$ (3)

p being the perpendicular from G on the tangent at K.



Also since there is no sliding.

$\therefore$ tangential velocity of the point K = velocity of $G+$ Velocity of K relative to G.

$$=u-\frac{p}{r}(\omega r)=0 \Rightarrow u-p\,\omega=0 \qquad (4)$$

Also Newton's rule $(V=ev)$ $\Rightarrow v+\sqrt{\left(r^2-p^2\right)}\,\omega=eV$ (5)

Now eliminating u,ω from (4) with the help of (2) and (3), we readily obtain

$$\frac{F}{m}=p\left(-\frac{Fp}{mk^2}+\frac{R\sqrt{\left(r^2-p^2\right)}}{mk^2}\right)\Rightarrow\frac{F}{m}\left(1+\frac{p^2}{k^2}\right)=\frac{Rp\sqrt{\left(r^2-p^2\right)}}{mk^2}$$

$$\Rightarrow \quad F\left(k^2+p^2\right)=Rp\sqrt{\left(r^2-p^2\right)} \qquad (6)$$

Now $\frac{R}{m} - V + \sqrt{(r^2 - ph2)}\left\{-\frac{Fp}{mk^2} + \frac{R\sqrt{(r^2 - p^2)}}{mk^2}\right\} = eV$

[Putting the values of v, ω in (5) from (1)]

This gives, $\frac{R}{m}\left\{1 + \frac{r^2 - p^2}{k^2}\right\} - \frac{Rp\sqrt{(r^2 - p^2)}}{mk^2} = V(1+e)$

$$\Rightarrow \quad \frac{R}{mk^2}\left[k^2 + r^2 - p^2\right] - \frac{Fp^2(r^2 - p^2)}{(k^2 + p^2)mk^2} = V(1+e)$$

$$\Rightarrow \quad R = \frac{mV(1+e).(k^2 + p^2)}{k^2 + r^2}$$

Second Part:- $\therefore$ $F = \frac{Rp\sqrt{(r^2 - p^2)}}{k^2 + r^2} = \frac{mV(1+e)\sqrt{(r^2 - p^2)}}{k^2 + r^2}p$ [using (6)]

$\therefore$ Loss of K.E. $= \frac{1}{2}R$ (velocity of the point K in the direction of impulse before impact + the velocity after impact)

$= \frac{1}{2}R[V - tV] = \frac{1}{2}(1-e)RV = E$ say

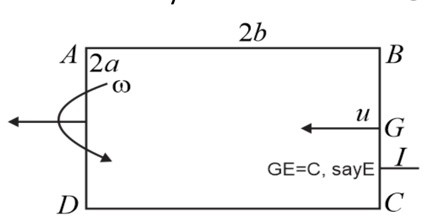
[$\because$ Velocity in the direction of F is zero, hence loss of K.E. in that direction is zero]

$\Rightarrow \quad E = \frac{1}{2}mv^2(1-e^2)\frac{k^2 + p^2}{k^2 + r^2}$ [putting the value of R]

Example:- Four freely jointed rods, of the same material and thickness, form a rectangle of sides $2a$ a $2b$ and of mass M'. When lying in this form on a horizontal plane, an inelastic particle of mass M moving with velocity V in a direction perpendicular to the rod of length $2a$ impinges on it at a distance c from its centre. Show that the K.E. lost in the impact is $\frac{1}{2}V^2 \div \left[\frac{1}{M} + \frac{1}{M'}\left(1 + \frac{3a+3b}{a+3b}\frac{c^2}{a^2}\right)\right]$

Solution:- Let AB be a rod length $2b$ and mass m_2 and BC of length $2a$ and mass m_1. Let u be the linear velocity of G_1, the C.G. of BC, and ω the angular velocity just after the action of the imulse (i.e. when the particle of mass M strikes at E such that $(G_1E = c)$.

Let I be the impulse applied. Obviously the rods AB and CD will not rotate and remain parallel.



Now velocity of AB= velocity of B= velocity of $G_1 +$ velocity of B rel. to $G_1 = u - a\omega$ velocity of CD = velocity of $G_1 +$ velocity of C rel. to $G_1 = u + a\,\omega$ and the velocity of the point E just after the blow $=$ velocity of $G_2 +$ velocity of E rel. to $G_1 = u + a\,\omega$

(Since the $a.v$ of BC is in opposite direction to the AD)

We also have $\dfrac{m_1}{2a} = \dfrac{m_2}{2b} = \dfrac{m_1 + m_2}{2(a+b)} = \dfrac{M'}{4(a-b)} \left(\because m_1 + m_2 = \dfrac{M'}{2} \right)$

Now equation of motion for the system (all the four rods) is

$$M'u = I \qquad (1)$$

And for the particle, of mass M, we have $M(V - u - c\,\omega) = I$ (2)

Taking moment of momentum about G, the centre for all the rods, we have

$$b + m_1 \frac{a^2}{3}\omega - m_1 ub + m_1 \frac{a^2}{3}\omega - m_2 (u - a\omega) a + m_2 a (u + c\omega) = Ic$$

$$\Rightarrow \quad 2\left(m_1 \frac{a^2}{3}\omega + 2m_2 a^2 \omega \right) = Ic \Rightarrow \frac{2a^2\omega}{3}\left[m_1 + 3m_2 \right] = Ic$$

$$\Rightarrow \quad \frac{a^2\omega}{3}\left[\frac{M'a + 3M'b}{(a+b)} \right] = cM(V - c\,\omega - u)$$

[Putting the value of I and m_1, m_2]

$$\Rightarrow \quad \frac{M'a^2\omega}{3} \cdot \frac{a+3b}{a+b} = cM(V - c\,\omega - u)$$

Using (1) and (2), we get $M'u = M(V - c\,\omega - u)$ (3)

$$\Rightarrow \quad (M + M')u = MV - Mc\,\omega$$

Now putting the value of in (3) from (4), we obtain

$$M'\frac{a^2}{3}\omega \cdot \frac{a+3b}{a+b} c\,M(V - c\,\omega) - \frac{c\,M^2 (V - c\,\omega)}{M + M'}$$

$$= cM(V - c\,\omega)(M + M' - M)/(M + M') = \frac{c\,M\,M'(V - c\,\omega)}{(M + M')}$$

$$\Rightarrow \quad c\,\omega\left[\frac{M}{M+M'} + \frac{a^2}{3c^2} \cdot \frac{a+3b}{a+b} \right] = \frac{VM}{M+M'}$$

$$\Rightarrow \quad c\,\omega = \frac{M\,V\,K}{M(1+K) + M'} \text{ where } K = \frac{3c^2}{a^2}\,\frac{a+b}{(a+3b)}$$

$$\therefore \quad (M + M')u = M\left[V - \frac{M\,V\,K}{M(1+K) + M'} \right] = \frac{(M + M')M\,V}{M(1+K) + M'}$$

$$\Rightarrow \quad u = \frac{M\,V}{M(1+K) + M'}$$

Now loss of K.E. = loss of the energy of the four rods + loss of energy of the particle.

$$\frac{1}{2}I \cdot \left[V+(u+c\,\omega)\right]-\frac{1}{2}\,I.(u+c\omega)=E \text{ (say)}$$

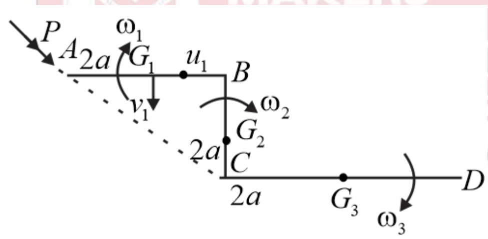
[$\therefore$ Impulse on the particle and rod is equal and opposite]

$$\therefore \qquad E=\frac{1}{2}I \cdot V=\frac{1}{2}V \cdot M\left[V-(u+c\,\omega)\right]$$

$$=\frac{1}{2}\frac{MM'V^2}{M(1+K)+M'}=\frac{V^2}{2\left[\frac{1}{M}+\frac{1}{M'}.\left\{1+\frac{3c^2}{a^2}\frac{(a+b)}{(a+3b)}\right\}\right]}$$

Example:- AB, BC, CD three equal uniform rods hinged freely at B and C are lying on a smooth horizontal table, so that ABC and BCD are at right angles on opposite sides of BC. A blow is given to A in the direction AC. Prove that D begins to move in a direction $\tan^{-1}\left(\frac{7}{4}\right)$ with CD.

Solution:- Let P be the impulse applied at A in the direction of AC. **Before impulse** the system is at rest. **After impulse,** let the velocity of G_1, the centre of gravity of AB, be (u_1, v_1) and the angular velocity of this rod AB be ω_1. Similarly the angular velocities of the other rods BC and CD are ω_2 and ω_3 respectively.



Hence, we obtain velocity of B along $AB = u_1$ (1)

Velocity if B perpendicular to $AB =$ velocity of $G_1 +$ velocity of B rel. to G_1

$= v_1 + a\,\omega_1$ (2)

Velocity of G_2 along $AB =$ velocity of $B +$ velocity of G_2 relative to $B = u_1 - a\,\omega_2$ (3)

Velocity of G_2 perpendicular to AB = the same as that of $G_1 + v_1 + a\,\omega_1$ (4)

Velocity of C along $AB =$ velocity of $B +$ velocity of C relative to $B = u_1 - 2a\,\omega_2$ (5)

Velocity of C perpendicular to $AB =$ the same as that of G_2 or $G_1 = v_1 + a\,\omega_1$ (6)

Velocity of G_3 along $AB =$ the same as that of $C = u_1 - 2a\,\omega_2$ (7)

Velocity of G_3 perpendicular to $AB =$ velocity of $G_2 +$ velocity of G_3 relative to $G_2 = v_1 + a\,\omega_1 + a\omega_3$ (8)

Velocity of D along $AB =$ the same as $C = u_1 - 2a\,\omega_2$ (9)

Velocity of D perpendicular to $AB =$ velocity of $C +$ velocity of D relative to $C = v_1 + a\,\omega_1 + 2a\omega_3$ (10)

Taking moments about C for the rod CD, we have

$$\frac{1}{3}a^2\omega_2 + a\left(v_1 + a\omega_1 + a\omega_3\right) = 0 \;\Rightarrow\; 3v_1 + 3a\omega_1 + 4a\omega_3 = 0 \qquad (11)$$

Again taking moments about B for the rods BC and CD, we get

$$\frac{1}{3}a^2\omega_3 + \frac{1}{3}a^2\omega_2 - 2a\left(u_1 - 2a\omega_2\right) + a\left(v_1 + a\omega_1 + a\omega_3\right) - a\left(u_1 - a\omega_2\right) = 0 \qquad (12)$$

$\Rightarrow$ $$\frac{a^2}{3}\omega_2 - 2a\left(u_1 - 2a\omega_2\right) - a\left(u_1 - a\omega_2\right) = 0 \qquad \text{[using (11)]}$$

$\Rightarrow$ $$9u_1 - 16a\,\omega_2 = 0 \qquad (13)$$

For all the rods, taking moments about A, we get

$$\frac{1}{3}a^2\left(\omega_1 + \omega_2 + \omega_3\right) - 2a\left(u_1 - 2a\omega_2\right) + 3a\left(v_1 + a\omega_1 + a\omega_3\right)$$
$$-\left(u_1 - a\omega_2\right) + 2a\left(v_1 + a\omega_1\right) + av_1 = 0 \qquad (14)$$

$\therefore$ $$(14) - (13) \;\Rightarrow\; \frac{a^2\omega_1}{3} + 2a\left(v_1 + a\omega_1 + a\omega_3\right) + 2a\left(v_1 + a\omega_1\right) + av_1 = 0$$

$\Rightarrow$ $$15v_1 + 13a\omega_1 + 6a\omega_3 = 0 \qquad (15)$$

Now resolving all the velocities perpendicular to AB, we have

$$\left(u_1\cos 45^\circ - v_1\cos 45^\circ\right) + \left(u_1 - a\omega_2\right)\cos 45^\circ - \left(v_1 + a\omega_1\right)\cos 45^\circ$$
$$+\left(u_1 - 2a\omega_2\right)\cos 45^\circ - \left(v_1 + a\omega_1 + a\omega_3\right)\cos 45^\circ = 0$$

Or $3\left(u_1 - v_1\right) - 2a\omega_1 - 3a\omega_2 - a\omega_2 = 0$ (16)

If the direction of motion of D makes an angle θ with DC, then obtain

$$\tan\theta = \frac{v_1 + a\omega_1 + 2a\omega_2}{u_1 - 3a\omega_2} = \frac{\dfrac{2a\omega_3}{3}}{-\dfrac{2a\omega_2}{9}} = -3\left(\frac{\omega_3}{\omega_2}\right) \qquad \text{[using (11) and(13)]}$$

From equations (11) and (15), we get $\dfrac{v_1}{-17} = \dfrac{a\omega_1}{21} = \dfrac{a\omega_2}{-3} \;\Rightarrow\; v_1 = \dfrac{17a\omega_3}{3};\; a\omega_1 = -7a\omega_3$

Putting these values in (16 and making use of (13), we have

$$\frac{16a\omega_2}{3} - 17a\omega_3 + 14a\omega_3 - 3a\omega_2 - a\omega_3 = 0$$

$\Rightarrow$ $$\frac{7a\omega_2}{3} = 4aw_3 \;\Rightarrow\; \frac{3\omega_3}{\omega_2} = \frac{7}{4} \qquad \therefore\; \tan\theta = -\frac{3\omega_3}{\omega_2} = -\frac{7}{4}$$

Example:- AB and CD are two equal similar rods connected by a string BC, AB, BC and CD form three sides of the square. The point A of the rod AB is struck by a blow in a direction perpendicular to the rod, show that the initial velocity of A is seven times that of D.

Solution:- Let u_1 be the velocity of G_1 then C.G. of AB and ω_1 the angular velocity of AB and $u_2,\ \omega_2$ those of CD.

As a matter of fact, the initial motion of B must be perpendicular to AB, so that the tension in the string at B must be along BC.

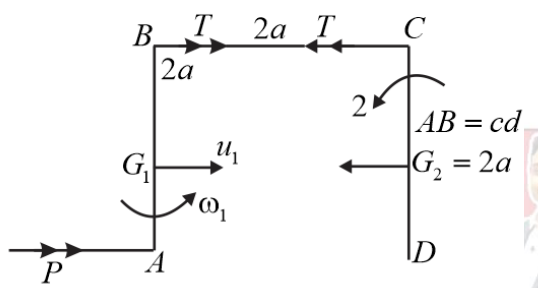
Let P be the impulse applied at A perpendicular to AB, then we have .

$$mu_1 = P + T \qquad (1)$$

$$m\frac{a^2}{3}\omega_1 = (P - T)a \qquad (2)$$

$$mu_2 = T \qquad (3)$$

$$m\frac{a^2}{3}\omega_3 = T\ .\ a \qquad (4)$$ **[motion of CD]**



As AB and CD are connected by a string, the velocity of $B =$ the velocity of $C \Rightarrow a_1\,\omega - u_1 = u_2 + a\omega_2$ (5)

Substituting the values of $u_1, \omega_1, u_2, \omega_2$ [From (1) to (4) in (5), we have

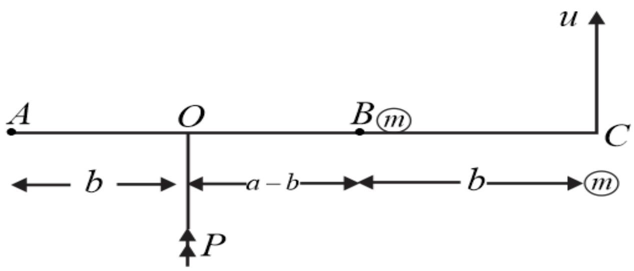
$$3(P-T)-(P+T) = T + 3T \Rightarrow T = (P/4)$$

$$\therefore \qquad \frac{Initial\ velocity\ of\ A}{Initial\ velocity\ of\ D} = \frac{u_1 + a\,\omega_1}{a\omega_2 + u_2} = \frac{\frac{5T}{m}+\frac{9T}{m}}{\frac{3T}{m}-\frac{T}{m}} = \frac{7}{1}$$

Example:- A light rod ABC has three particles each of mass m attached to it at A, B, C. The rod is struck by a blow P at right angles to it is a point distant from A equal to BC. Prove that the K.E. set up is $\frac{1}{2}\frac{P^2\left(a^2 - ab + b^2\right)}{m\left(a^2 + ab + b^2\right)}$ where $AB = a,\ BC = b$.

Solution:- Let the three particles each of mass m be placed at A, B, C of a light rod ABC, and let the impulse P be applied at O such that $AO = BC = b$, where $AB = a,\ BC = b$

Let u be the velocity of C and ω, the angular velocity of the rod just after the blow, then the velocity of B is $u+b\omega$, the velocity of A is $u+(a+b)\omega$ and the velocity of the point O at which the impulse is applied is $u+a\omega$. Since the system was at rest initially, the velocity of O just before the impact is zero.



Now equation the total momentum perpendicular to rod, the impulse, we have

$$mu+m(u+b\omega)+m\{u+(a+b)\omega\}=P \Rightarrow 3u+\omega(a+2b)=\frac{P}{m}$$

Also taking moments about O, we easily obtain

$$\{u+(a+b)\}b-m(u+b\omega)(a-b)-mua=0 \Rightarrow u(a-b)=b^2\omega$$

$$\therefore \quad \frac{u}{b^2}=\frac{\omega}{a-b} \Rightarrow \frac{3u+(a+2b)\omega}{3b^2+(a+2b)(a-b)}=\frac{(P/m)}{(a^2+ab+b^2)}$$

$$\Rightarrow \quad u=\frac{b^2}{a^2+ab+b^2}\frac{P}{m}, \text{ and } \omega=\frac{(a-b)}{(a^2+ab+b^2)}\frac{P}{m}$$

Hence the velocity of point O is given by

$$u+a\omega=\frac{P}{m}\frac{1}{m(a^2-b^2+ab)}\left[b^2+a(a-b)\right]=\frac{a^2+b^2-ab}{a^2+b^2+ab}\cdot\frac{P}{m}$$

K.E. set up $=\frac{1}{2}P$ (velocity of the point O) $=\frac{1}{2}\frac{P^2(a^2+b^2-ab)}{m(a^2+b^2+ab)}$

Alter. K.E. $=\frac{1}{2}m\{u(a+b)\omega\}^2+(u+b\omega)^2+u^2]$

$$=\frac{1}{2}\frac{P^2}{m}\left[\frac{a^2}{(a^2+ab+b^2)}+\frac{a^2b^2+b^4}{(a^2+b^2+ab)^2}\right]$$

But $u+(a+b)\ \omega=\frac{a^2}{a^2+ab+b^2}.\frac{P}{m}$ and $u+b\ \omega=\frac{ab}{a^2+ab+b^2}\cdot\frac{P}{m}$

$$\therefore \quad K.E.=\frac{1}{2}\frac{P^2}{m}\left[\frac{(a^2+b^2)^2-a^2b^2}{(a^2+b^2+ab)^2}\right]=\frac{1}{2}\frac{P^2}{m}\frac{(a^2+b^2-ab)}{(a^2+b^2+ab)}$$

Generalised Co-ordinates.

Suppose that a particle or a system of N- particles moves subject to possible constraints, as for example a particle moving along a circular wire or a rigid body moving along an inclined plane, then there will be necessarily a minimum number of independent co-ordinates then needed to specify the motion. These co-ordinates denoted by q_1, q_2, ……, q_n, are called generalized co-ordinates. These co-ordinates may be distances, angles or quantities relating to them.

Degrees of freedom.

The number of independent co-ordinates required to specify he position of a system of one or more particles is called the number of degrees of freedom of the system.

Ex. 1. A particle moving freely in space require 3 co-ordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.

Ex. 2. A system containing of N-particles moving freely in space require 3N co-ordinates to specify the position. The number of degrees of freedom is 3N.

A rigid body which can move freely in space has 6 degrees of freedom i.e., **6 co-ordinates are required to specify the position.**

Let 3 non-collinear points of a rigid body be fixed in space, then the rigid body also fixed in space. Let these points have co-ordinates (x_1, y_1, z_1) ; (x_2, y_2, z_2) ; (x_3, y_3, z_3) respectively, a total of 9. Since the body is rigid, we must have

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = \text{constant.}$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = \text{constant.}$$
$$(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = \text{constant.}$$

Hence 3 co-ordinates can be expressed in terms of the remaining six. Thus six independent co-ordiantes are needed to describe the motion i.e., there exit six degrees of freedom.

Transformation equation.

Let $r_v = x_v\, i + y_v\, j + z_v\, k$ be the position vector of v-th particle with respect to xyz co-ordinate system. The relationships of the generalized co-ordinates q_1, q_2, ……, q_n the position co-ordinates are given by the transformation equations.

$$\left.\begin{aligned} x_v &= x_v\left(q_1, q_2, \ldots\ldots, q_n; t\right) \\ y_v &= y_v\left(q_1, q_2, \ldots\ldots, q_n; t\right) \\ z_v &= z_v\left(q_1, q_2, \ldots\ldots, q_n; t\right) \end{aligned}\right\} \qquad \ldots\ldots(1)$$

Where t denotes the time. In vector (1) can be written as

$$r_v = r_v\,(q_1, q_2, \ldots\ldots q_n\,;\, t) \qquad \ldots\ldots(2)$$

Where the functions in (1) or (2) are continuous and have continuous derivatives.

Classification of Mechanical systems.

(1) Scleronomic system.

The mechanical system in which t, the time, does not enter explicity in equation (1) or (2) is called a scleronomic system.

(2) Rheonomic system.

The mechanical system in which the moving constraints are involved and the time t does enter explicitly is called a Rheonomic system.

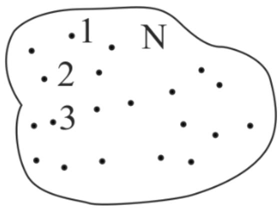
(3) Holonomic system and Non Holonomic system.

Let q_1, q_2,….., q_n, denote the generalized co-ordinates describing a system and let t denote the time. If all the constraints of the system can be expressed as equations having the form (q_1, q_2,……, q_n ; t) = 0 or their equivalent, then the system is said to be Holonomic otherwise it is be Non-Holonomic system.

(4) Conservative and non-conservative system.

If the forces acting on the system are derivable from a potential function [or potential energy] V, then the system is called conservative, otherwise it is non-conservative.

Kinetic energy and generalized velocities.



The K.E of the system is $T = \frac{1}{2}\sum_{v=1}^{n} m_v \dot{r}_v^2$.

The K.E of the system can be written as a quadratic form in the generalized co-ordinates. $q_{\propto}$.

If the system is independent of time explicitly i.e., Scleronomic then the quadratic form has only terms of the type $a_{\alpha\beta}\, q_a\, q_\beta$. In case the system is Rhenomic, linear terms in q_α are also present.

TOTAL AND PARTIAL DIFFERENTIAL COEFFICIENTS (Required further)

If $u = f(x, y)$, where x and y are function of a single variable t, we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

But $du = \frac{du}{dt}dt, dx = \frac{dx}{dt}dt$ and $dy = \frac{dy}{dt}dt$. Therefore

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

This value of du/dt is called the *total differential coefficient*.

In general, if $u = f(x, y)$, where x and y are functions of t, we can show that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}.$$

Similarly, if $u = f(x, y)$, where x and y are functions of two other variables t_1 and t_2, then we have

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t_1} \qquad ....(1)$$

and $\dfrac{\partial u}{\partial t_2} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t_2} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t_2}$. ....(2)

These results can be extended to any number of variables.

Generalised Forces.

If W is the total work done on a system of particles by forces F_v acting on the v-th particle, then

$$dW = \sum_{\alpha=1}^{N} \phi_\alpha dq_\alpha \text{ where } \phi_\alpha = \sum_{v=1}^{N} F_v \cdot \frac{\partial r_v}{\partial q_\alpha}$$

is called the **generalized force** associated with generalized co-ordinates q_x.

Suppose that a system undergoes increments $dq_1, dq_2, \ldots, dq_n$, of the generalized co-ordinates $q_1, q_2, \ldots, q_n$, then the v-th particles undergoes a displacement.

$$dr_v = \sum_{\alpha=1}^{n} \frac{\partial r_v}{\partial q_\alpha} dq_\alpha \qquad \ldots\ldots(4)$$

$\therefore$ Total work done is given by

$$dW = \sum_{v=1}^{N} F_v \cdot dr_v = \sum_{v=1}^{N} \left\{ \sum_{\alpha=1}^{n} F_v \cdot \frac{\partial r_v}{\partial q_\alpha} \right\} dq_\alpha \qquad \ldots..(5)$$

Now, let $\phi_\alpha = \sum_{v=1}^{N} F_v \cdot \dfrac{\partial r_v}{\partial q_\alpha}$

Then (5) $dW = \sum_{\alpha=1}^{N} \left(\sum_{v=1}^{N} F_v \cdot \dfrac{\partial r_v}{\partial q_\alpha} \right) d\alpha = \sum_{\alpha=1}^{n} \phi_\alpha d_\alpha$ .....(6)

We have $dW = \sum_{\alpha=1}^{N} \dfrac{\partial W}{\partial q_\alpha}$, $\therefore \dfrac{\partial W}{\partial q_\alpha} = \phi_\alpha$ .....(7)

Note. (i) α varies from (1) to n, the number of degree of freedom.

(ii) v aries from 1 to N, the number of particles in the system.

Lagrange's equations.

Let F be the net external force acting on the v-th particle of a system, then by Newton's second law

$$m_v \ddot{r} = F_v$$

$$\Rightarrow m_v \ddot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} = F_v \bullet \frac{\partial r_v}{\partial q_\alpha} \qquad \ldots\ldots(8)$$

$$\Rightarrow m_v \ddot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} = F_v \bullet \frac{\partial r_v}{\partial q_\alpha} \qquad \ldots\ldots(8)$$

$$\Rightarrow \sum_{v=1}^{N} m_v \ddot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} = \sum_{v=1}^{N} F_v \bullet \frac{\partial r_v}{\partial q_\alpha} \qquad \ldots\ldots(9)$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{v=1}^{N} m_v \dot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} \right] - \sum_{v=1}^{N} m_v \dot{r}_v \bullet \frac{d}{dt}\left(\frac{\partial r_v}{\partial q_\alpha} \right) = \sum_{v=1}^{N} F_v \bullet \frac{\partial r_v}{\partial q_\alpha}$$

But $r_v = r_v\,(q_1, q_2, \ldots, q_n\,;\, t)$ .....(10)

$$\therefore\ \dot{r}_v = \frac{\partial r_v}{\partial q_1}\dot{q}_1 + \frac{\partial r_v}{\partial q_2}\dot{q}_2 + \ldots\ldots + \frac{\partial r_v}{\partial q_2}\dot{q}_n + \frac{\partial r_v}{\partial t} \quad \ldots\ldots(11)$$

$$\Rightarrow \frac{\partial \dot{r}_v}{\partial \dot{q}_\alpha} = \frac{\partial r_v}{\partial q_\alpha} \quad \textit{[Cancellation law of the dots]} \quad \ldots\ldots(12)$$

Also, $$\frac{\partial}{\partial q_\alpha}(\dot{r}_v) = \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_v}{\partial q_1}\dot{q}_1 + \frac{\partial r_v}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial r_v}{\partial q_n}\dot{q}_n + \frac{\partial r_v}{\partial t}\right)$$

$$= \frac{\partial^2 r_v}{\partial q_\alpha \partial q_1}\dot{q}_1 + \frac{\partial^2 r_v}{\partial q_\alpha \partial q_1}\dot{q}_2 + \ldots + \frac{\partial^2 r_v}{\partial q_\alpha \partial q_n}\dot{q}_n + \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_v}{\partial t}\right)$$

$$= \frac{\partial}{\partial q_1}\left(\frac{\partial r_v}{\partial q_\alpha}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial r_v}{\partial q_\alpha}\right)\dot{q}_2 + \ldots + \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_v}{\partial q_n}\right)\dot{q}_n + \frac{\partial}{\partial t}\left(\frac{\partial r_v}{\partial q_\alpha}\right)$$

or $$\frac{\partial}{\partial q_\alpha}\left(\frac{dr_v}{dt}\right) = \frac{d}{dt}\left(\frac{\partial r_v}{\partial q_\alpha}\right) \Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial q_\alpha}\right) \equiv \frac{\partial}{\partial q_\alpha}\left(\frac{d}{dt}\right) \quad \ldots\ldots(13)$$

[interchange law of the order of operators]

Now, $$\frac{d}{dt}\left\{\sum_{v=1}^{N} mv\,\dot{r}_v \cdot \frac{\partial r_v}{\partial q_a}\right\} - \sum_{v=1}^{N} m_v\,\dot{r}_v \cdot \frac{\partial \dot{r}_v}{\partial q_\alpha} = \sum_{v=1}^{N} F_v \cdot \frac{\partial r_v}{\partial q_\alpha} \quad \ldots\ldots(14)$$

and $$T = \frac{1}{2}\sum_v m_v \dot{r}_v^2 = \frac{1}{2}\sum m_v(\dot{r}_v \cdot \dot{r}_v) \quad \ldots\ldots(15)$$

$$\Rightarrow \frac{\partial T}{\partial q_\alpha} = \sum_v m_v \dot{r}_v \cdot \frac{\partial \dot{r}_v}{\partial q_\alpha} \quad \ldots\ldots(16)$$

and $$\frac{\partial T}{\partial q_\alpha} = \sum_v m_v \dot{r}_v \cdot \frac{\partial \dot{r}_v}{\partial q_\alpha} = \sum m_v \dot{r}_v \cdot \frac{\partial r_v}{\partial q_\alpha} \ \text{[using (12)]} \quad \ldots\ldots(17)$$

$$\therefore\ (14) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \sum_v F_v \cdot \frac{\partial r_v}{\partial q_\alpha}$$

or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha = \frac{\partial W}{\partial q_\alpha} \ \text{using (7)} \quad \ldots\ldots(18)$$

Note. The quantity $P_\alpha = \dfrac{\partial T}{\partial \dot{q}_\alpha}$ is called the **generalized momentum** associated with the general co-ordinates q_α.

Lagrangian function.

If the forces are derivable from a potential function V, then

$$\phi_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$$

Since the potential, or potential energy is a function of q's only (and possibly the name t) then, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \Rightarrow \left[\frac{\partial}{\partial \dot{q}_\alpha}(T - V)\right] - \left(\frac{\partial T}{\partial q_\alpha} - \frac{\partial V}{\partial q_\alpha}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0, \text{ where } L = T - V \qquad .....(19)$$

The function L defined by $L = T - V$ is said to be Lagragian function.

Generalised momentum.

We defined $P_\alpha = \dfrac{\partial T}{\partial \dot{q}_\alpha}$ to be the generalized momentum associated with generalized co-ordinates q_α, or the conjugate momentum.

In case the system is conservative, we have

$$T = L + V \Rightarrow (\partial T / \partial \dot{q}_\alpha) = (\partial T / \partial \dot{q}_\alpha) + (\partial T / \partial \dot{q}_{\dot{\alpha}}) = (\partial T / \partial \dot{q}_\alpha)$$

Because V, he P. E. of the system does not depend upon $\dot{q}_\alpha$

$$\therefore\ p_\alpha = (\partial L / \partial q_\alpha)$$

Kinetic energy as a Quadratic function of velocities.

If at time t, the position of the v^{th} particle (mass m), of a holonomic system is defined by r_v, then K.E. is given by

$$T = \frac{1}{2}\sum_{v=1}^{N} m_v \dot{r}_v^{\,2}, \text{ where } r_v = r_v(q_1.....q_n; t) \qquad .....(1)$$

So, that $\dot{r}_v = \dot{q}_1 \dfrac{\partial r_v}{\partial q_1} + \dot{q}_2 \dfrac{\partial r_v}{\partial q_2} + ... + \dot{q}_n \dfrac{\partial r_v}{\partial q_n} + \dfrac{\partial r_v}{\partial t}$

$$\Rightarrow T = \frac{1}{2}\sum_{v=1}^{N} m_v \left(\dot{q}_1 \frac{\partial r_v}{\partial q_1} + \dot{q}_2 \frac{\partial r_v}{\partial q_2} .... + \dot{q}_n \frac{\partial r_v}{\partial q_n} + \frac{\partial r_v}{\partial t}\right)^2$$

$$= \frac{1}{2}\left[\left(a_{11}\dot{q}_1^{\,2} + a_{22}\dot{q}_2^{\,2} + ... + a_{nn}\dot{q}_n^{\,2} + 2a_{12}\dot{q}_1\dot{q}_1 + 2a_{1n}\dot{q}_1\dot{q}_n + ... + 2(a_1\dot{q}_1 + a_2\dot{q}_2 + . + a_n\dot{q}_n)\right) + a\right]$$

.....(2)

Where $a_{rs} = \sum_{v=1}^{N} m_v (\partial r_v / \partial q_r).(\partial r_v / \partial q_s)(s \geq r)$

$$a_{rr} = \sum_{v=1}^{N} m_v (\partial r_v / \partial r_r)^2, a = \sum_{v=1}^{N} m_v (\partial r_v / \partial t)^2, a_r = \sum_{v=1}^{N} m_v (\partial r_v / \partial q_r).\left(\frac{\partial r_v}{\partial t}\right)$$

From (2), we see that T is a quadratic function of the generalized velocities.

The case t is not explicitly involves, is of considerable importance. Hence,

We have $\dfrac{\partial r_v}{\partial t} = 0$ and therefore (2) implies that

$$T = \frac{1}{2}\left(a_{11}\dot{q}_1^{\,2} + a_{22}\dot{q}_2^{\,2} + ... + a_{nn}\dot{q}_n^{\,2} + 2a_{12}\dot{q}_n^{\,2} + 2a_{12}\dot{q}_1\dot{q}_2 + ...\right) \qquad .....(3)$$

$$= \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} a_{rs}\dot{q}_r\dot{q}_s \text{ where } a_{rs} = a_{sr}. \qquad .....(4)$$

Now using Euler's theorem for homogeneous functions, we get

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + \ldots + \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = 2T$$

$$\Rightarrow \; 2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = \sum_{\alpha=1}^{n} p_\alpha \, \dot{q}_\alpha$$

i.e., $2T = p_1\dot{q}_1 + p_1\dot{q}_2 + \ldots\ldots + p_n\dot{q}_n$

To deduce the principle of energy from The Lagrange's equations (Conservative field)

Lagrange's equations are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}; \qquad (a = 1, 2, \ldots. n) \qquad \ldots..(1)$$

we know that

$$T = \frac{1}{2}\left(a_{11}\dot{q}_1^{\,2} + a_{22}\dot{q}_2^{\,2} + .. + a_m\dot{q}_n^{\,2} + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots.\right)$$

That is, T can be expressed as a quadratic expression in generalized velocities. Hence applying Euler's theorem. We get

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T \qquad \ldots..(2)$$

Also, $$\frac{dT}{dt} = \sum_{\alpha=1}^{n} \frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha}\ddot{q}_\alpha \qquad \ldots..(3)$$

Now multiplying the n equations of (1) by $\dot{q}_1, \dot{q}_2, \ldots\ldots, \dot{q}_n$ respectively and then adding we get

$$\left\{\dot{q}_1 \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_1}\right] + \ldots + \dot{q}_n \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_1}\right]\right\} - \left\{\dot{q}_1 \frac{\partial T}{\partial q_1} + \ldots + q_n \frac{\partial T}{\partial q_n}\right\}$$

$$= -\left\{\dot{q}_1 \frac{\partial V}{\partial q_1} + \ldots + \dot{q}_n \frac{\partial V}{\partial q_n}\right\}$$

$$\Rightarrow \; \frac{d}{dt}\left\{\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \left\{\sum_{\alpha=1}^{n} \ddot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \left\{\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial q_\alpha}\right\} = -\left\{\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial V}{\partial q_\alpha}\right\}$$

$$\Rightarrow \; \frac{d}{dt}(2T) - \frac{dT}{dt} = -\frac{dV}{dt} \Rightarrow \frac{dT}{dt} + \frac{dV}{dt} = 0$$

$$\Rightarrow \; \frac{d}{dt}(T + V) = 0 \Rightarrow T + V = \text{ constant.}$$

Hamilton's form of the equations of Motion.

Here we shall obtain the differential equations of motion of a conservative holonomic dynamical system in a form which constitutes the basis of most of the advanced theory of dynamics.

Let (q_1, q_2 ...., q_n) be the generalised co-ordinates and let L (q_1, q_2 ,...., q_n ; $\dot{q}_1, \dot{q}_2 \ldots, \dot{q}_n$;t),

the kinetic potential of the system, so that the equations of motion in the Lagrangian form are

$$\frac{d}{dt}\left(\partial L / \partial \dot{q}_i\right) - (\partial L/\partial q_i) = 0; \; (i = 1, 2, ..., n)$$
...(*i*)

writing $p_i = \partial L / \partial \dot{q}_i$ we get $\dot{p}_i = \left(\partial L / \partial q_i\right)$ $(i = 1, 2, ..., n)$
...(*ii*)

hence from the former of these sets of equations we can regard either of the sets of quantities $(q_1, q_2 ...., a_n)$

or $(p_1, p_2, ...,p_n)$ as functions of the other set.

Now, let δ denote the increment in any function of the variables $(q_1, q_2 ...., q_n; p_1, p_2, ..., p_n)$ or $\left(q_1, q_2 ...., q_n; \dot{q}_1, \dot{q}_2, ..., \dot{q}_n\right)$ $(q_1, q_2, ..., q_n, q_1, q_2, ..., q_n)$; then we get

$$dL = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial q_i} + \frac{\partial L}{\partial \dot{q}_i} d\,\dot{q}_i\right) + \frac{\partial L}{\partial t} \quad \text{(when } L \text{ contains } t \text{ explicitly)}$$

$$= \sum_{i=1}^{n}\left(\dot{p}_1\, dq_i + p_i\, d\,\dot{p}_i\right) + \frac{\partial L}{\partial t} dt$$

$$= d\left(\sum_{i=1}^{n} p_i\, \dot{q}_i\right) + \sum_{i=1}^{n}\left(\dot{p}_i\, dq_i - \dot{p}_i\, d\, p_i\right) + \frac{\partial L}{\partial t} dt$$

$$\Rightarrow d\left[\sum_{i=1}^{n}\left(p_i\, \dot{q}_i\right) - L\right] = \sum_{i=1}^{n}\left(\dot{q}_i\, dp_i - p_i\, dq_i\right) - \left(\partial L / \partial t\right) dt.$$

Thus if the quantity $\sum_{i=1}^{n}\left(p_i\, \dot{q}_i - L\right)$ when expressed in terms of $(q_1, q_2 ...., q_n; p_1, p_2, ..., p_n; t)$ be denoted by H, we have

$$dH = \sum_{i=l}^{n}\left(\dot{q}_i\, dp_i - \dot{p}_i\, dq_i\right) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad \sum_{i=l}^{n}\frac{\partial H}{\partial q_i} dq_i + \sum_{i=l}^{n}\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt = \sum_{i=l}^{n}\left(\dot{q}_i\, dp_i - \dot{p}_i\, fq_i\right) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}, \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If H does not contain t explicity (i.e. does not contain t explicitly) we have

$$\dot{p}_i = -\left(\frac{\partial H}{\partial q_i}\right) \text{ and } \dot{q}_i = \left(\frac{\partial H}{\partial p_i}\right) \quad ....(4)$$

These equations are called as Hamilton's equation or Hamilton's canoncial equation and the function H is called Hamiltonian.

The total order of Hamilton equation is the same as the total order of Lagrange's equations, names 2n. But whereas Lagrange's equations present us with n equation each of the second order. Hamilton's present us with n equation each of the second order.

Hamilton's equations are 2n equations. Each of the first order. Hamilton's equation can also be written as $\frac{dp_i}{-\left(\frac{\partial H}{\partial q_i}\right)} = \frac{dq_i}{-\left(\frac{\partial H}{\partial p_i}\right)} = dt$.

Physical significance of the Hamiltonian.

If the Hamiltonian H is independent of t explicity prove that it is

(a) constant and (b) equal to the total energy of the system.

Proof. (a) We have $\frac{dH}{dt} = \sum_{i=l}^{n} \frac{\partial H}{\partial q_i}\frac{\partial q_i}{dt} + \sum_{i=l}^{n} \frac{\partial H}{\partial p_i}\frac{\partial p_i}{dt}$

$$= \sum_{i=l}^{n} -(\dot{p}_i)\dot{q}_i + \sum_{i=l}^{n} \dot{q}_i\,\dot{p}_i \left(\because \dot{p}_i = -\frac{\partial H}{\partial q_i} \, and \, \dot{q}_i = \frac{\partial H}{\partial p_i}\right) = 0$$

$\Rightarrow$ H = constant, say E.

(b) By Eulre's theorem on homogeneous function, we have

$\sum \dot{q}_i \frac{\partial T}{\partial q_i} = 2T$, where T is the K.E. of the system.

But $L = T - V$, $\therefore \frac{\partial L}{\partial q_i} = \frac{\partial (T-V)}{\dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$ (V does not depend on $\dot{q}_i$)

or $\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T \Rightarrow \sum \dot{q}_i\, p_i = 2T \left(\because p_i = \frac{\partial L}{\partial \dot{q}_i}\right)$

$\therefore H = \sum p_i \dot{q}_i - L = 2T - (T - V) = T + V = E$

Passage from the Hamiltonian to the Lagrangian.

Suppose that we are given a function $^{*}H(q,p,t)$ and are told that the motion of the system satisfies the canoncial equations

$$\dot{p}_i = -\left(\frac{\partial H}{\partial q_i}\right) \quad and \quad \dot{q}_i = \left(\frac{\partial H}{\partial p_i}\right) \qquad \ldots(1)$$

Then we want to find a function $L(p_1, p_2, \ldots\ldots p_n;\, q_1, q_2, \ldots\ldots q_n; t), i.e. L(p,q,t)$

Such that the motion also satisfies the equations

$$\left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \qquad \ldots(2)$$

Solve the first set of equation in (1) for the p's in terms of the q's the $\dot{q}$'s and t.

Then write $L = \sum_{i=l}^{n} \dot{q}_i\, p_i - H$ and express L as a function of the q's, the $\dot{q}$'s and t. This is the required Lagangian.

$$\therefore \left(\frac{\partial L}{\partial \dot{q}_i}\right) = p_i \left(u \sin g \; L = \sum_{i=l}^{n} p_i \dot{q}_i - H\right)$$

$$\Rightarrow \left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{p}_i \text{ and } \left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\left(\frac{\partial L}{\partial q_i}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)=\dot{p}_i+\left(\frac{\partial H}{\partial q_i}\right)=0$$

i.e. L satisfies (2) assuming (1).

Principle of Least Action.

Principle of least action states that if T is kinetic energy, at time t, of a conservative, holonomic dynamical system specified by the generalized co-ordinates, then the integral $I=\int_{t_1}^{t_2} 2T dt$ has necessary an extreme value, minimum or maximum, on actual path as compared with varied path as the systems passed from one configuration at time t_0 to another configuration at time t_1.

We know that L = T – V, i.e. Lagrangian = k.E. – P.E.

and T + V = E (const), since system is conservative.

But by Hamilton's principle, we know that

$$\int_{t_1}^{t_2} \delta L\, dt = 0 \Rightarrow \int_{t_1}^{t_2} \delta(T-V)\, dt = 0 \int_{t_1}^{t_2} \delta(2T-E)\, dt - 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta(2T)\, dt = 0 \qquad \Rightarrow \delta \int_{t_1}^{t_2} (2T)\, dt = 0. \qquad \ldots(1)$$

[$\delta E = 0$ as E, the total energy is const.]

Result (1), is know as **Principle of least action.**

Equation (1) can also be written as $\delta A = 0$, where $A = \int_{t_0}^{t_1} 2T\ dt$ and is defined by action as follws:

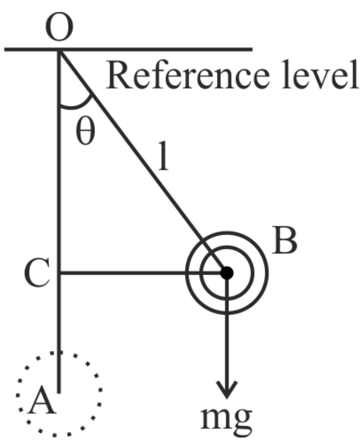
This implies that principal of least action states that the action in the actual path is minimum compared with the varied path, as the system passes from one configuration to another.

EXAMPLES TO SUBSTANTIATE.

Ex. 1. (i) Set up the Lagrangian for a simple pendulum, and

(ii) obtain an equation describing its motion.

Sol. (i)



Choose as generalized coordinates,

the angle θ made by the string *OB* of the pendulum and the vertical *OA*. Let *l* be the length of *OA*, then K.E., is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

Where m is the mass of the bob.

The potential energy of mass m is given by

$$V = mg\left(OA - OC\right) = mg\left(l - l\cos\theta\right) = mgl\left(1 - \cos\theta\right)$$

$$\therefore \; L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1 - \cos\theta\right)$$

(ii) Hence Lagrange's θ equation gives

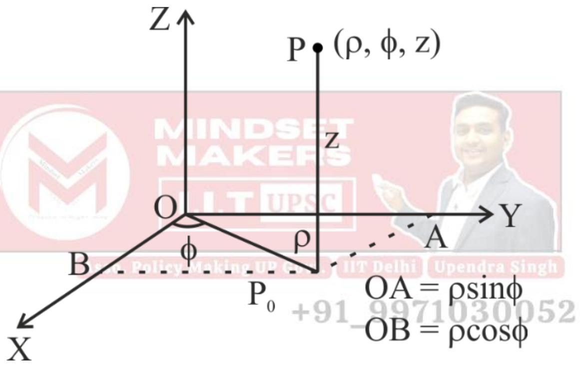
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial \dot{L}}{\partial \dot{\theta}} = 0 \Rightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\right) - \left(-mgl\sin\theta\right) = 0$$

$$\Rightarrow \; l\ddot{\theta} = -g\sin\theta \Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$$

Which is the required equation of motion.

Ex. 2. A particle of mass m moves in a conservative force field. Find

(a) the Lagrangian function, (b) the equations of motion in cylindrical co-ordinates (ρ, ϕ, z).



Sol. we have $OP = OP_0 + P_0P = OA + AP_0 + + P_0P = \vec{\rho}\left(\text{say}\right)$

$\therefore \; \vec{\rho} = \rho\sin\phi j + \rho\cos\phi i + zk$ where i, j, k

are the unit vector along OX, OY and OZ respectively.

Hence the unit vector along the direction of ρ increasing is

Given by $\vec{\rho}_1 = \dfrac{\partial\vec{\rho}}{\partial\rho} / \left|\dfrac{\partial\vec{\rho}}{\partial\rho}\right| = \sin\phi j + \cos\phi i$

Similarly $\vec{\phi}_1 = \dfrac{\partial\vec{\rho}}{\partial\phi} / \left|\dfrac{\partial\vec{\rho}}{\partial\phi}\right|$

$$= \frac{\rho\cos\phi j - \rho\sin\phi i}{\rho} = -\sin\phi i + \cos\phi j$$

Now $v = \dfrac{d\vec{\rho}}{dt} = \dfrac{d}{dt}\left(\rho\sin\phi j + \rho\cos\phi i + zk\right)$

$$= \rho\cos\phi\dot{\phi}j + \dot{\rho}\sin\phi j - \rho\sin\phi\dot{\phi}i + \dot{\rho}\cos\phi i + k\dot{z}$$

$$= \dot{\rho}\cos\phi j - \dot{\rho}\sin\phi j + \rho\dot{\phi}\left(\cos\phi j - \sin\phi i\right) + \dot{z}k$$

$= \dot{\rho}\vec{\rho}_1 + \rho\dot{\phi}\vec{\phi}_1 + \dot{z}k$

$\therefore T = \frac{1}{2}m\left\{\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right\}$ and $V = V(\rho, \phi, z)$

(a) Hence the Lagrangian function is

$$L = T - V = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right] - V(\rho, \phi, z)$$

(b) Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0 \text{ i.e., } \frac{d}{dt}\frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 \frac{\partial V}{\partial \rho}\right) = 0$$

i.e., $m\left(\ddot{\rho} - \rho\dot{\phi}^2\right) = -\frac{\partial V}{\partial \rho}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$ i.e., $\frac{d}{dt}\left(m\rho^2\dot{\phi}\right) + \frac{\partial V}{\partial \phi} = 0$

or $\frac{d}{dt}\left(\rho^2\dot{\phi}\right) = -\frac{\partial V}{\partial \phi}$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$ i.e., $\frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} = 0$ or $m\ddot{z} = -\frac{\partial V}{\partial z}$

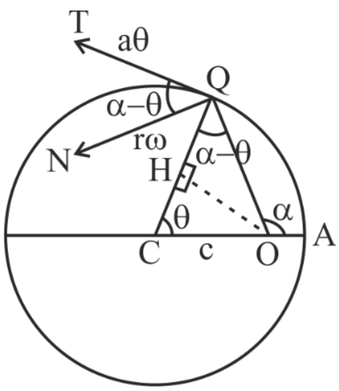
Ex. 3. A particle Q moves on a smooth horizontal circular wire of radius. A which is free to rotate about a vertical axis through a point O, distance c from the centre C. If the $\angle QCO = \theta$, show that

$a\ddot{\theta} + \dot{\omega}(a - c\cos\theta) = c\omega^2 \sin\theta.$

Where ω is the angular velocity of the wire.

Sol. Let OQ = r, and $\angle AOQ = \alpha$

$\Rightarrow r^2 = a^2 + c^2 - 2ac\cos\theta$ …..(1)



$r\cos(\alpha - \theta) = a - c\cos\theta$ …..(2)

The particle Q moves on circle of radius a, so its velocity along the tangent QT will be $\alpha\theta$ but Q revolves about O with angular velocity ω, which causes a velocity aω at the right angles to OQ.

$\Rightarrow w_Q^2$ = (velocity)^2 of the particle at Q

$$= a^2\dot{\theta}^2 + r^2\omega^2 + 2ar\dot{\theta}\cos(\alpha-\theta)\omega$$

$$\angle AOQ = \alpha, \angle CQO = \alpha - \theta$$

Now $T = \frac{1}{2}mvQ^2 = \frac{1}{2}m\left[a^2\dot{\theta}^2 + r^2\omega^2 + 2arw\dot{\theta}\cos(\alpha-\theta)\right]$

$$\left[\begin{array}{l}\angle NQT = \alpha - \theta, OQ = r \\ \\ HQ = a - c\cos\theta\end{array}\right.$$

NOTE:-

If r is the position vector of the particle at any time t. the $\partial r / \partial r$ is the vector tangent to the curve θ = constant i.e., a vector in the direction of r (increasing r), A unit vector in this direction is thus given by $r_1 = \frac{\partial r}{\partial r} / \left|\frac{\partial r}{\partial r}\right|$.

Similarly, $\partial r / \partial\theta$ is the vector tangant to the curve r = constant, A unit vector in the direction is given by $\vec{\theta}_1 = \frac{\partial r}{\partial \theta} / \left|\frac{\partial r}{\partial \theta}\right|$

$$= \frac{1}{2}m\left[a^2\dot{\theta}^2 + \left(a^2 + c^2 - 2ac\cos\theta\right)\omega^2 + 2a\omega\dot{\theta}(a - c\cos\theta)\right] \qquad = r\cos(\alpha-\theta)$$

and work function = 0 $\qquad$ ($\because$ weight does no work)

$\therefore$ Lagrange's θ equation $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = 0$

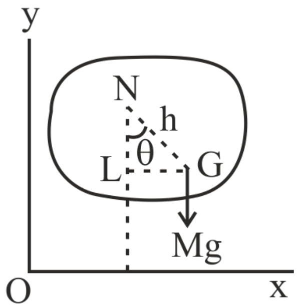
$$\Rightarrow \frac{d}{dt}\left[a^2\dot{\theta} + a\omega(a + c\cos\theta)\right] - ac\omega^2\sin\theta - ac\omega\dot{\theta}\sin\theta = 0$$

$$\Rightarrow a^2\ddot{\theta} + a\dot{\omega}(a - c\cos\theta) + a\omega c\dot{\theta}\sin\theta - ac\omega^2\sin\theta - ac\omega\dot{\theta}\sin\theta = 0$$

$$\Rightarrow a^2\ddot{\theta} + \dot{\omega}(a - c\cos\theta) = c\omega^2\sin\theta$$

Ex. 4. Use Lagrange's equations to find the differential equation for a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.

Sol.



Let the plane of oscillation be represented by xy – plane, where N is its intersection with the axis of rotation and G is the centre of gravity.

Let the mass of the pendulum be M and let its moment of interia about the axis of rotation be MK^2.

Then potential energy relative to the horizontal plane through N is $V = -Mgh\cos\theta$.

Also, $T = \frac{1}{2} Mk^2\dot{\theta}^2$

$\therefore\ L = T - V = \frac{1}{2} Mk\dot{\theta}^2 + Mgh\cos\theta$ .....(1)

$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = Mk^2\theta$ and $\frac{\partial L}{\partial \dot{\theta}} = -Mgh\sin\theta$

Now Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(Mk^2\dot{\theta}\right) + Mgh\sin\theta = 0$$

i.e., $Mk^2\ddot{\theta} + Mgh\sin\theta = 0 \Rightarrow \ddot{\theta} = -\frac{gh}{k^2}\sin\theta$

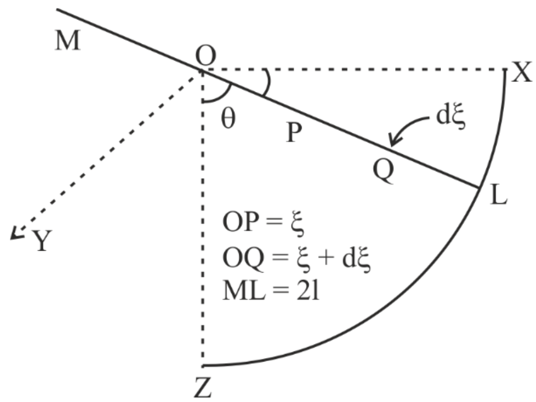
When θ is small, we have $D^2\theta = -\frac{gh}{k^2}\theta$ (sinθ = θ)

or $\left(D^2 + \frac{gh}{k^2}\right)\theta = 0$

This is the differential equation of the pendulum.

Ex.5. A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. The rod when in horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{\left(\frac{2ng}{l}\right)}$ show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}\left[\sqrt{\left(n^2+1\right)} - n\right]$ and will then rise again.

Sol. The mass *m* is attached at L. On the rod ML, take a point p such that $OP = \xi$, the element $PQ = d\xi$.



Further at any time t,

let the plane through it and the vertical have turned through an angle ϕ from its initial position and let the rod be inclined at an angle θ to the rod be inclined at an angle θ to the vertical.

Taking *O*, the mid point of the rod, as the origin and OX, OY (a line perpendicular to the plane of the paper) and OZ as axes of refrence,

then co-ordinates of the point *P* on the rod are:

$x = \xi\sin\theta\cos\phi, y = \xi\sin\theta, z = \xi\cos\theta$

$\therefore\ \dot{x} = \xi\cos\theta\cos\dot{\theta} - \xi\sin\phi\dot{\phi}\sin\theta$

$\dot{y} = \xi\cos\theta\sin\phi\dot{\theta} + \xi\sin\theta\cos\phi\dot{\phi}, \dot{z} = -\xi\sin\theta\dot{\theta}$. Thus, v_ρ^2 = (velocity)^2 of.

$P = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \xi^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$.

$\therefore\ v_L^2 = l^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$ = (velocity)^2 of mass m,

Now mass of the element $PQ = \dfrac{3m}{2l}d\xi = dm$, say.

$\therefore$ Its kinetic energy

$= \dfrac{1}{2}dm.v_\rho^2 = \dfrac{1}{2}.\dfrac{3m}{2l}d\xi\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\xi^2$

$= \dfrac{3m}{4l}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\xi^2 d\xi$

and K.E. of the rod = $\dfrac{3m}{4l}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\int_{-1}\xi^2 d\xi$

$\dfrac{1}{2}m\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)l^2$

Again, (velocity)^2 of the particle $m = l^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$.

$\therefore$ Kinetic energy of the particle of mass $m = \dfrac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$.

$\therefore$ Total K.E. = T = K.E. of the rod + K.E. of the particle

$= \dfrac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \dfrac{1}{2}ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$

$T = ml^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$

Also the work function is given by $W = mgl\cos\theta + C$

Lagrange's ϕ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\phi}}\right) - \dfrac{\partial T}{\partial\phi} = \dfrac{\partial W}{\partial\phi}$

Which gives $\dfrac{d}{dt}\left(2ml^2\dot{\phi}\sin^2\theta\right) = 0$

Integrating it, we get $\dot{\phi}\sin^2\theta = K$ (constant).

Initially, $\theta = \dfrac{\pi}{2}$ and $\dot{\phi} = \sqrt{\left(\dfrac{2ng}{l}\right)}$

$\therefore\ K = \sqrt{\left(\dfrac{2ng}{l}\right)}$

$\therefore\ \dot{\phi}\sin^2\theta = \sqrt{\left(\dfrac{2ng}{l}\right)}$ .....(1)

and Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}=\frac{\partial W}{\partial \theta}$

i.e., $\frac{d}{dt}\left(2ml^2\dot{\theta}\right)-2ml^2\dot{\phi}^2\sin\theta\cos\theta=mgl\sin\theta$

or $2l\ddot{\theta}-2l\dot{\phi}^2\sin\theta\cos\theta=-g\sin\theta$ .....(2)

Substituting value of $\dot{\phi}$ from (1) in (2), we have

$2l\ddot{\theta}-4ng\cot\theta\,\mathrm{cosec}^2\theta=-g\sin\theta$ .....(2')

Integration provides us $2l\dot{\theta}^2+4ng\cot^2\theta=2g\cos\theta+k$.

Initially $\theta=\frac{\pi}{2},\dot{\theta}=0,$ $\quad\therefore k=0$

$\therefore\ 2l\dot{\theta}^2+4ng\cot^2\theta=2g\cos\theta$ .....(3)

The red will fall till $\dot{\theta}=0$

i.e., $4ng\cot^2\theta=2g\cos\theta$ or $2n\cos^2\theta-\cos\theta\sin^2\theta=0$

$\therefore$ either $\cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$ which gives initial position.

as $2n\cos\theta-\sin^2\theta=0\Rightarrow\cos^2\theta+2n\cos\theta-1=0$.

Solving it, $\cos\theta=\frac{-2n\pm\sqrt{\left(4n^2+4\right)}}{2}\left\{-n+\sqrt{\left(n^2+1\right)}\right\}$

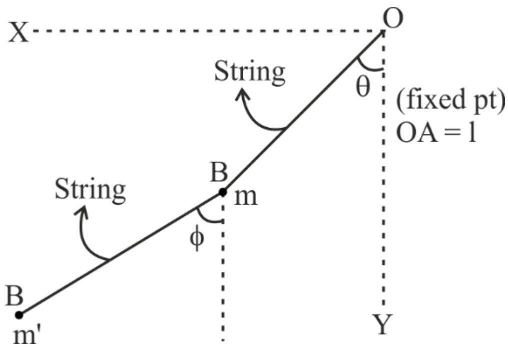
[the other value being inadmissible because θ can not be obtuse]

or $\theta=\cos^{-1}\left[-n+\sqrt{\left(n^2+1\right)}\right]$. This proves the required result. If we substitute this value of θ in equation (2'), then we find that θ comes out to be positive. Hence at that time the rod begins to rise.

Ex. 6. A mass m hangs from a fixed point by a light string of length *l* and *a* mass m' hangs from m by a second string of length *l'*. For oscillations in a vertical plane, show that the periods of the principal oscillations are the values of $\frac{2\pi}{n}$ where *n* is given by the equation

$$n^4-gn^2\frac{m+m'}{m}\left(\frac{1}{l}+\frac{1}{l'}\right)+g^2\frac{m+m'}{ml'l}=0$$

Sol. A any time *t*, let the strings be inclined at angle θ and ϕ to the vertical. Co-ordinates of *m* are $\left(l\sin\theta,l\cos\theta\right)$.



$\therefore$ (velocity)^2 of $m = l^2\dot{\theta}^2$ while co-ordinates of m' are

$x_B = l.\sin\theta + l'\sin\phi.$ $\Rightarrow$ $\dot{x}_B = l\cos\theta\dot{\theta} + l'\cos\phi\dot{\phi}$

$\dot{y}_B = l\cos\theta\theta + l'\cos\phi\phi$

$\therefore$ (velocity)^2 of

$m' = \dot{x}_B^2 + \dot{y}x_B^2 = l^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2ll\dot{\theta}\dot{\phi}$

[$\because$ θ and ϕ are small]

Now let T, be the kinetic energy and, W the work function, of the system, then we have

$W = mgl\cos\theta + m'g\left(l\cos\theta + l'\cos\phi\right)$

$= gl\left(m + m'\right)\cos\theta + m'gl'\cos\phi$

and $T = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}m'\left[l^2\dot{\theta}^2 + l'^2\theta^2 + 2ll'\dot{\theta}\dot{\phi}\right]$

$= \frac{1}{2}\left[\left(m + m'\right)l^2\dot{\theta}^2 + m'l'^2\theta^2 + 2m'\dot{\theta}\dot{\phi}\right]$

Lagrange's θ-equation is given by $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$\Rightarrow$ $\frac{d}{dt}\left[\left(m + m'\right)l^2\dot{\theta} + m'll'\dot{\phi}\right] = -gl\left(m + m'\right)\theta$ .....(1)

While Lagrange's ϕ-equation gives

$\frac{d}{dt}\left[m'l'^2\dot{\phi} + m'll'\dot{\theta}\right] = -m'gl'\phi$

$\Rightarrow$ $l'\ddot{\phi} + l\ddot{\theta} = -g\phi$ .....(2)

Equation (1) and (2) again give

$\left(m + m'\right)\left(lD^2 + g\right)\theta + m'l'D^2\phi = 0$ .....(3)

$lD^2\theta + \left(l'D^2 + g\right)\phi = 0$ .....(4)

Eliminating ϕ, we get $\left[\left(m + m'\right)\left(lD^2 + g\right)\left(l'D^2 + g\right) - m'll'D^4\right]\theta = 0$

i.e., $\left[mll'D^4 + \left(m + m'\right)\left(l + l'\right)gD^2 + \left(m + m'\right)g^2\right]\theta = 0$ .....(5)

Now let $\theta = A\cos\left(nt + B\right)$; $\therefore D\theta = -nA\sin\left(nt + B\right)$

$D^2\theta = -n^2 A\cos(nt+B) = -n^2\theta$ and $D^4\theta = n^4\theta$ .....(6)

$\therefore$ (5) and (6) give mll' $n^4 - (m+m')(l+l')gn^2 + (m+m')g^2 = 0$

or $n^4 - \dfrac{m+m'}{m}\left(\dfrac{1}{l}+\dfrac{1}{l'}\right)gn^2 + \dfrac{(m+m')}{mll'} = 0$ .....(7)

Ex. 7. (a) A mas M hangs from a fixed point at the end of a very long string whose length l is a, to M is suspended a mas m by a string whose length l is small compound with a; prove that the tiem of a small oscillation of m is $2\pi\sqrt{\left(\dfrac{M}{M+m}.\dfrac{l}{g}\right)}$

Sol. Here, we have m = M, $m' = m$, $l = a$, $l' = l$

$$\therefore\ n^4 - \frac{M+m}{M}\left(\frac{l}{a}+\frac{1}{l}\right)gn^2 + \frac{(M+m)}{Mal} = 0$$

i.e., $n^4 - \dfrac{M+m}{M}\left(\dfrac{l}{a}+1\right)\dfrac{g}{l}n^2 + \dfrac{(M+m)g^2}{Ml^2}.\dfrac{l}{a} = 0$ .....(8)

But a is larger compared to l $\therefore$ $\dfrac{l}{a}\to 0$

Hence the equation (8), gives

$$n^4 - \frac{M+m}{M}.\frac{g}{l}.n^2 = 0 \text{ i.e., } n^2 = \frac{M+m}{M}.\frac{g}{l}$$

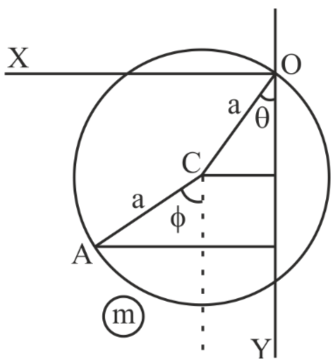
$\therefore$ Time of a small oscillation = $\dfrac{2\pi}{n} = 2\pi\sqrt{\left\{\dfrac{M}{M+m}.\dfrac{l}{g}\right\}}$

Ex.8. (b) At the lowest point of a smooth circular tube, of mass M and radius a, is placed a particle of mass M', the tube hangs in a vertical plane from its highest point, which is fixed, and can tum freely in tis own plane about this point. If the system be slightly displaced, show that the periods of the two independent oscillations of the system are

$$2\pi\sqrt{\left(\frac{2a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{Mag^{-1}}{M+M}\right)}$$

And that for one principal mode of oscillations, the particle remains at rest relative to the tube nd for the other, the centre of gravity of the particle and the tube remain at rest.

Sol. Let C be the centre of the tube and A the position of the particle M' at time t when OC and CA make angle 3θ and ϕ with the vertical



$\therefore\ x_A = a\sin\theta + a\sin\phi$,

$\therefore\ y_A = a\cos\theta + a\cos\phi$

and

(velocity)^2 of $A = \dot{x}_A^2 + \dot{y}_A^2$

$$= \left(a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi}\right)^2 + \left(-a\sin\theta\dot{\theta} - a\sin\phi\dot{\phi}\right)^2$$

[neglecting small quantities of the higher order]

Also $C \equiv \left(a\sin\theta, a\cos\theta\right)$

(velocity)^2 of $C = \left(a\cos\theta\dot{\theta}\right)^2 + \left(-a\sin\theta\dot{\theta}\right)^2 = a^2\dot{\theta}^2$

Now let T, be the kinetic energy and W the work function of the system then we readily obtain

$$W = Mga\cos\theta + M'g\left(a\cos\theta + a\cos\phi\right) + K$$

$$= \left(M + M'\right)ga\cos\theta + M'ga\cos\phi + K$$

T = K.E. of circular tube + K.E. of particle

$$= \frac{1}{2}M\left(a^2\dot{\theta}^2 + a^2\dot{\theta}^2\right) + \frac{1}{2}M\left(a^2\dot{\theta}^2 + a^2\dot{\theta}^2 + 2a^2\dot{\theta}\dot{\phi}\right)$$

$$= \frac{2M + M'}{2}a^2\dot{\theta}^2 + \frac{1}{2}M'a^2\dot{\phi}^2 + M'a^2\dot{\theta}\dot{\phi} \qquad \text{.....(2)}$$

$\therefore$ Lagrange's θ-equation gives.

$$\frac{d}{dt}\left[M'a^2\dot{\phi} + M'a^2\dot{\theta}\right] = -M'ga\ddot{\phi} \Rightarrow \phi + \ddot{\theta} = -\frac{g}{a}\phi$$

$$\Rightarrow\ \left(2M + M'\right)\ddot{\theta} + M'\ddot{\phi} = -\left(M + M'\right)\frac{g}{a}\theta. \qquad \text{.....(3)}$$

Also Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left[M'a^2\dot{\phi} + M'a^2\dot{\theta}\right] = -M'ga\ddot{\phi} \Rightarrow \phi + \ddot{\theta} = -\frac{g}{a}\phi \qquad \text{.....(4)}$$

Equations (3) and (4) can be re-written as

$$\left[\left(2M + M'\right)D^2 + \left(M + M'\right)c\right]\theta + M'D^2\phi = 0 \qquad \text{.....(5)}$$

and $D^2\theta + \left(D^2 + c\right)\phi = 0$ where $c = \frac{g}{a}$. .....(6)

Eliminating ϕ between these two equations, we get

$$\left[\left(\left(2M + M'\right)D^2 + \left(M + M'\right)c\right)\left(D^2 + c\right) - M'D^4\right]\theta = 0$$

i.e., $\left[2MD^4 + c\left(3M + 2M'\right)D^2 + c^2\left(M + M'\right)\right]\theta = 0$.

To sole (7). .....(7)

Let $\theta = A\cos\left(pt + B\right); D\theta = -pA\sin\left(pt + B\right)$

$$D^2\theta = -p^2a\cos\left(pt + B\right) = -p^2\theta \text{ and } D^4\theta = p^4\theta \qquad \text{.....(8)}$$

$\therefore$ (7) and (8) give $\left[2Mp^4 - c(3M + 2M')p^2 + c^2(M + M')\right]\theta = 0$

i.e., $2Mp^4 - c(3M + 2M')p + c^2(M + M') = 0 \qquad [\because \theta \neq 0]$

which again gives $(2p^2 - c)\left[Mp^2 - c(M + M')\right] = 0$

$$\therefore \; p_1^2 = \frac{c}{2} \text{ and } p_2^2 = \frac{c(M + M')}{M}$$

$$\text{i.e., } p_1^2 = \frac{g}{2a} \text{ and } p_2^2 = \frac{M + M'}{M}\frac{g}{a} \qquad \left(\because \; c = \frac{g}{a}\right)$$

Hence periods of oscillations are given by

$$\frac{2\pi}{p_1} \text{ and } \frac{2\pi}{p_1} \text{ i.e., by } 2\pi\sqrt{(2a/g)} \text{ and } 2\pi\sqrt{\left(\frac{M}{(M + M')}\frac{a}{g}\right)}$$

Multiplying (6) by λ and adding to (5), we have

$$D^2\left((2M + M' + \lambda)\theta + (M' + \lambda)\phi\right) = -\left\{(M + M')\right\}\theta + \lambda\phi \qquad \text{.....(9)}$$

Now choose λ such that

$$\frac{2M + M' + \lambda}{M' + \lambda} = \frac{M + M'}{\lambda} \Rightarrow \lambda = M' \text{ and } \lambda = -(M + M').$$

Taking $\lambda = M'$, equation (9) reduces to

$$D^2\left\{(M + M')\theta + M'\phi\right\} = -\frac{1}{2}c \quad \left\{M + M'\theta + M'\phi\right\}$$

and when $\lambda = -(M + M')$ equation (9) reduces to

$$D^2(\theta - \phi) = -\frac{M + M'}{M}c(\theta - \phi)$$

$\therefore$ Principal co-ordinates are $\theta - \phi$ and $\left\{(M + M')\theta + M'\phi\right\}$ 0.

For the first mode, $\theta - \phi = 0$. i.e., $\theta = \phi$. This shows that the particle is at rest relative to the tube. For the second mode we have $(M + M')\theta + M'\phi = 0$.

Further, the x-coordinates of C.G. of the particle and the tube

$$= \frac{Ma\sin\theta + M'(a\sin\theta + a\sin\phi)}{M + M'}$$

$$= \frac{a}{M + M'}\left\{M\theta + M'(\theta + \phi)\right\} \quad \text{(since } \theta \text{ and } \phi \text{ are small } \therefore \sin\theta = \theta \text{ and } \sin\phi = \phi\text{)}$$

$$= \frac{a}{M + M'}\left\{(M + M')\theta + M'\phi\right\} = 0 \qquad \text{[using above results]}$$

$\Rightarrow$ The common C.G. of the particle and the tube remains at rest.

HAMILTONIAN

Ex. 9. *A particle moves in the xy-plane under the influence of a central force depending only on its distance from the origin.*

(a) *Set up the Hamiltonian for the system.*

(b) *Write Hamilton's equations of motion.*

Sol. (a) Let the potential due the central force be V (r). Then, we have

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m \text{ [(radial velocity)]2 + (transverse velocity}^2\text{)}$$

$$= \frac{1}{2}m(r^2 + r^2\theta^2)$$

$$L = T - V = \frac{1}{2}m(r^2 + r^2\theta^2) - V(r)$$

$$\therefore \quad p_r = (\partial L / \partial \dot{r})m\dot{r},\ p_\theta = (\partial L / \partial \dot{r}) - mr^2\theta$$

$$\Rightarrow \quad \dot{r} = (p_r / m), \dot{\theta} = (p_\theta / mr^2)$$

$$\text{Thus} \quad H = \sum p_i q_i - L = p_r q_r + p_\theta q_\theta - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)V(r)$$

$$= p_r(p_r / m) + p_\theta(p_\theta + mr^2) - \frac{1}{2}m\{(p_r^2 / m^2) + r^2, (p_\theta^2 / m^2r^2)\} - V(r)\}$$

$$= (p_r^2 / 2m) + (p_\theta^2 / 2m_r^2) + V(r) = \text{total energy of the system}$$

(b) Hamilton's equations are

$$\dot{p}_i = -(\partial H / \partial qi), \dot{q}_i = (\partial H / \partial p_i)$$

$$\Rightarrow \dot{r} = (\partial H / \partial p_r) = (p_r / m), \dot{\theta} = (\partial H / \partial p_e) = (p_\theta / mr^2)$$

$$\dot{p}_i = -(\partial H / \partial r) = (p_\theta^2 / mr^3) - V(r), \dot{p}_\theta = -(\partial H / \partial \theta) = 0$$

Ex. 10. *A particle of mass m moves in a force field of potential V. Write*

(a) the Hamiltonian and

(b) Hamilton's equations in spherical polar co-ordinates.

Sol. (a) K.E. is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) \qquad ....(1)$$

$$\therefore \quad L = T - V = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V \qquad ....(2)$$

We have $p_r = (\partial L / \partial \dot{r}) = m\dot{r},\ p\theta = (\partial L / \partial \dot{\theta}) = mr^2\dot{\theta},$

$p_\phi = (\partial L / \partial \dot{\phi}) = mr^2 \sin^2\theta\dot{\phi}$

$$\Rightarrow \quad \dot{r} = (p_r / m), \dot{\theta} = (p_\theta)/mr^2), \dot{\phi} = p_\phi / (mr^2 \sin^2 \theta) \qquad ...(3)$$

Now Hamiltonian is given by

$$H = \sum p_i \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\theta^2}{2mr^2 \sin^2 \theta} + V(r, \theta, \dot{\phi})$$

= total energy of the system

(b) Hamilton's equations are given by

$$\dot{q}_i = (\partial H / \partial p_i), \dot{p}_i = -(\partial H / \partial q_i)$$

i.e. $\dot{r} = \dfrac{\partial H}{\partial p_i} = \dfrac{pr}{m}$ $\qquad$ $\dot{p}_r = -\dfrac{\partial H}{\partial r} = \dfrac{p_\theta^2}{mr^3} + \dfrac{p_\phi^2}{mr^3 \sin^2 \theta} - \dfrac{\partial V}{\partial r}$

$\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mr^2}$ $\qquad$ $\dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{p_\phi^2 \cos \theta}{mr^2 \sin^3 \theta} - \dfrac{\partial V}{\partial \theta}$

$\dot{\theta} = \dfrac{\partial H}{\partial p_\phi} = \dfrac{p_\phi}{mr^2 \sin^2 \theta}$ $\qquad$ $\dot{p}_\phi = \dfrac{\partial H}{\partial \phi} = \dfrac{\partial V}{\partial \phi}$

Ex. 11. *A particle of mass m moves in a force field of potential V.*

(a) Write the Hamiltonian and

(b) Hamilton's equations in cartesian co-ordinates.

Sol. (a) We have

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\Rightarrow \quad L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \qquad ....(1)$$

$$\because \quad p_x = (\partial L / \partial \dot{x}) = m\dot{x}, p_y = (\partial L / \partial \dot{y}) = m\dot{y}; p_z = (\partial L / \partial \dot{z}) = m\dot{z}$$

$$\Rightarrow \quad \dot{x} = (p_x / m), \dot{y} = (p_y / m), \dot{z} = (p_z / m)$$

Thus $H = \sum p_x \dot{q}x - L = \dot{p}_x x + \dot{p}_y + \dot{p}_z z - \dfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z)$

$$= p_x (p_x / m) + p_y (p_y / m) + p_z (p_z / m) \ -\frac{1}{2} m[(p_x^2 / m^2) + (p_y^2 / m^2) + (p_z^2 / m) + V(x, y, z)]$$

$$= (p_x^2 / 2m) + (p_y^2 / 2m) + (p_z^2 / 2m) + V(x, y, z)$$

= total energy of the system.

(b) Hamilton's equations are:

$\dot{p}_x = -(\partial H / \partial x); \dot{p}_y = -(\partial H / \partial y); \dot{p}_z = -(\partial H / \partial z)$ and

$\dot{x} = (\partial H / \partial p_x); \dot{y} = (\partial H / \partial p_y); z = (\partial H / \partial p_z)$

$\Rightarrow \dot{p}_x = -(\partial V / \partial x), \dot{p}_y = -(\partial V / \partial y), \dot{p}_z = -(\partial V / \partial z)$

$\dot{x} = (p_x / m), \dot{y} = (p_y / m), \dot{z} = (p_z / m)$

Ex.12. *A sphere rolls down a rough included plane; if x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration.*

Sol. We have $T = \frac{1}{2} m(\dot{x}^2 + k^2 \dot{\theta}^2) = \frac{1}{2} m\left(\dot{x}^2 + \frac{2}{5} a^2 \dot{\theta}^2\right)$ $\left(\because k^2 = \frac{2}{5} a^2\right)$

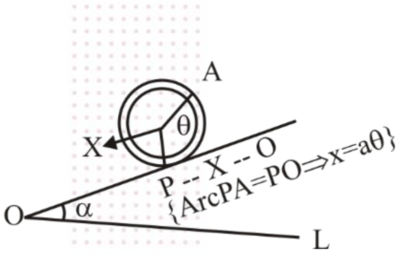
$= \frac{1}{2} m\left(\dot{x}^2 + \frac{2}{5}\dot{x}^2\right) = \frac{7}{10} m\dot{x}^2;$ ...(1) $V = m\,g\,x \sin\alpha$ ...(2)

$\therefore\ L = T - V = \frac{7}{10} m\dot{x}^2 + m\,g\,x \sin\alpha$ ...(3)

Now $p_x = (\partial L / \partial \dot{x}) = \frac{7}{5} m\dot{x} \Rightarrow \dot{x} = (5 p_x / 7m)$

Thus $H = -L + p_x \dot{x} = -\frac{7}{10} m\dot{x} - m\,g\,x \sin\alpha + p_x.(5 p_x / 7m)$

$= \frac{5}{14}(p_x^2 / m) - m\,g\,x \sin\alpha$ ...(4)



$\therefore$ One of Hamilton's equations gives

$\dot{p}_x = -(\partial H / \partial x) = mg \sin\alpha \Rightarrow \frac{7}{5} m\ddot{x} = mg \sin\alpha \Rightarrow \ddot{x} = \frac{5}{7} g \sin\alpha$

Ex. 13. *If the Hamiltonian H is independent of time explicitly, prove that it is.*

(a) *a constant, and*

(b) *equal to the total energy of the system.*

Sol. (a) $(dH / dt) = \sum_{i=1}^{n} (\partial H / \partial p_i)\dot{p}_i + \sum_{i=1}^{n} (\partial H / \partial q_i)\dot{q}_i$

$\sum_{i=1}^{n} \dot{q}_i \dot{p}_i + (-\dot{p}_i)\dot{q}_i = 0$ $[\because\ (\partial H / \partial p_i) = \dot{q}_{i,} (\partial H / \partial \dot{q}_i) = -\dot{p}_i]$

$\Rightarrow$ H = constant = E say.

(b) By Euler's theorem on homogeneous functions, we have

$\sum_{i=1}^{n} \dot{q}_i (\partial T / \partial \dot{q}_i) = 2T$ ...(2)

Put $p_i = (\partial L / \partial \dot{q}_i) = \{\partial (T - V) / \partial \dot{q}_i\} = (\partial T / \partial \dot{q}_i) - (\partial V / \partial \dot{q}_i) = (\partial T / \partial \dot{q}_i)$

$\{\because (\partial V / \partial \dot{q}_i) = 0$ as V is independent of $\partial \dot{q}_i\}$

$\therefore\ (2) \Rightarrow \sum_{i=1}^{n} \dot{q}_i\ p_i = 2T$

Thus $H = \sum_{i=1}^{n} \dot{q}_i\ p_i - L = 2T - L = 2T - (T - V) = T + V = E$

Ex. 14. *Write the Hamiltonian function and equation of motion for a compound pendulum.*

Sol. We have $L = \frac{1}{2} I\dot{\theta}^2 + mgh\cos\theta \Rightarrow p_\theta = (\partial L / \partial\theta) = I\theta$

where $I = mk^2$

$\therefore\ H = \sum p_i \dot{q}_i - L = p_\theta \dot{\theta} - L = I\,\dot{\theta}\dot{\theta} - \frac{1}{2}\dot{\theta}^2 - mgh\ \cos\dot{\theta}$

$= \frac{1}{2} I\,\dot{\theta}^2 - mgh\cos\theta$

$\Rightarrow\ \frac{1}{2} H = (p_\theta\ / I^2) - mgh\ \cos\theta = (p_\theta^2\ / 2I) - mgh\ \cos\theta\ \ \{\because \dot{\theta} = (p_\theta / I)$

$\therefore (\partial H / \partial p_\theta) = (p_\theta\ / I), (\partial H / \partial\theta) = mgh\ \sin\theta$

Thus the Hamilton's equations for $\dot{\theta}$ and $\dot{p}_\theta$ are given by

$\dot{\theta} = (\partial H / \partial p_\theta). \dot{p}_\theta = -(\partial H / \partial\theta)$

i.e. $\dot{\theta} = (p_\theta / \mathrm{I})$ and $\dot{p}_\theta = -mgh\ \sin\theta.$ But $p_\theta = I\theta \Rightarrow \dot{p}_\theta = I\,\ddot{\theta}$

$\therefore\ I\,\ddot{\theta} = -mgh\ \sin\theta \Rightarrow \ddot{\theta} + \frac{mgh}{I}\sin\theta = 0.$

This is exactly the same as obtained previously using Lagrange's equations.

Ex. 15. *Obtain Euler's equations from Hamilton's equations.*

Sol. We know that $2T = (A\omega_1^2 + B\omega_2^2 + C_3^2),$

$\Rightarrow\ L = T - V = \frac{1}{2}(A\,\omega_1^2 + B\,\omega_2^2 + C\,\omega_2^2)$ ...(1)

Also Euler's geometrical relations give

$\omega_1 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\ \cos\psi$

$\omega_2 = \dot{\theta}\cos\psi - \dot{\phi}\sin\theta\ \sin\psi;$ and

$\omega_3 = \psi + \phi\cos\theta$

Now H = T + V $= \frac{1}{2}(A\,\omega_2^1 + B\omega_2^2 + C\,\omega_3^2) + V$

Again, $P\phi = \frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\omega_1}\frac{\partial\omega_1}{\partial\dot{\theta}} + \frac{\partial L}{\partial\omega_2}\frac{\partial\omega_2}{\partial\dot{\theta}} + \frac{\partial L}{\partial\omega_3}\frac{\partial\omega_3}{\partial\dot{\theta}}$

$= A\,\omega_1\ \sin\psi + B\,\omega_2\cos\psi + C\omega_3.0$ ...(1)

$P\phi = \frac{\partial L}{\partial\dot{\theta}} = -A\omega_1\sin\theta\cos\psi + B\,\omega_2\sin\theta\ \sin\psi + C\omega_3\cos\theta$ ...(2)

and $P_\psi = (\partial L / \partial\dot{\psi}) = C\omega_3$

Solving the three equations for $\omega_1, \omega_2, \omega_3$ we have

$$\left.\begin{aligned}\omega_1 &= \frac{1}{A}\left[p_\theta \sin\psi + (p_\psi \cos\theta - p_\phi)\frac{\cos\psi}{\sin\theta}\right.\\ \omega_2 &= \frac{1}{A}\left[p_\theta \cos\psi - (p_\psi \cos\theta - p_\phi)\frac{\sin\psi}{\sin\theta}\right]\end{aligned}\right.;\text{and } \omega_3 = \frac{1}{C}.p_\psi$$

Also, Hamilton's equations are $\dot{p}_\psi = -\frac{\partial H}{\partial \psi}$ and $\dot{\psi} = \frac{\partial H}{\partial p_\psi}$

Now $\dot{p}_\psi = \frac{\partial H}{\partial \psi}$

$$\Rightarrow C\dot{\omega}_3 = -\left[\frac{\partial H}{\partial \omega_1}\frac{\partial \omega_1}{\partial \psi} + \frac{\partial H}{\partial \omega_2}\frac{\partial \omega_2}{\partial \psi} + \frac{\partial H}{\partial \omega_3}\frac{\partial \omega_3}{\partial \psi}\right] - \frac{\partial V}{\partial \psi}$$

$$= -\left[A\omega_1 . \frac{1}{A}B\omega_2 + B\omega_2\left(\frac{A}{B}\omega_1\right) + C\omega_3.0\right] - \frac{\partial V}{\partial \psi}$$

$$= (A - B)\omega_1\omega_2 - \frac{\partial V}{\partial \psi}$$

$$\Rightarrow C\frac{d\omega_1}{dt} - (A - B)\omega_1\omega_1 = N\left(\because -\frac{\partial V}{\partial \phi} = N\right)$$

This is Euler's third Familiar dynamical equation

Also, $\quad \psi = (\partial H / \partial p_\psi) = \frac{\partial H}{\partial \omega_1}\frac{\partial \omega_1}{\partial p_\psi} + \frac{\partial H}{\partial \omega_2}\frac{\partial \omega_2}{\partial p_\psi} + \frac{\partial H}{\partial \omega_3}\frac{\partial \omega_3}{\partial p_\psi}$

$$= (\omega_1 \cos\psi - \omega_2 \sin\psi)\cot\theta + \omega_3 = -\phi \sin\theta \cot\theta + \omega_3$$

i.e. $\quad \psi = -\phi\cos\theta + \omega_3 \Rightarrow \omega_3 = \psi + \phi\cos\theta$

This is Euler's third geometrical equation.

On the same lines, we can deduce Euler's other equations (dynamical and geometrical).

Ex.16 *Prove that*

$$\left(\frac{\partial H}{dt}\right) = \left(\frac{\partial H}{dt}\right) \text{ where H is the Hamilton's function.}$$

Sol. *Let q_1, q_2 ...., qn* be the generalised co-ordinates then Hamilton's equation are given by

$$\dot{p}_1 = \frac{\partial H}{\partial q_i} \text{ and } \dot{q}_1 = \frac{\partial H}{\partial p_i} \text{ (i = 1, 2, ..., n)} \qquad ...(1)$$

But Hamiltonian H is a function of q's and ps'

$$\therefore \frac{\partial H}{\partial t} = \frac{\partial H}{dt} + \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial H}{\partial t} + \sum_{i=1}^{n}(-\dot{p}_i)\dot{q}_i + \sum_{i=1}^{n}\dot{q}_i\dot{p}_i = \frac{\partial H}{\partial t} \qquad \text{[using (1)]}$$

Ex.17 *Use Hamilton's equations to find the equations of motion of a projectile in space.*

Sol. Let (x, y, z) be the co-ordinates of the projectile in space at time t, then we have

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), V = mgz$$

$$\therefore \;\; L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = mgz$$

$$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, p\dot{z} = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

But L does not involve t explicity therefore Hamiltonian H is given by $H = T + V = \frac{1}{2}$ $(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$

$$= \frac{1}{2}m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2}\right) + mgz = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

Now Hamilton's equations are given by

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \qquad \ldots(1),$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \ldots(2),$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \qquad \ldots(3),$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \qquad \ldots(4),$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg \qquad \ldots(5),$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \ldots(6)$$

Using (1) and (2), we have $\ddot{x} = 0$ …(7)

Using (3) and (4), we have $\ddot{y} = 0$ …(8)

Again making use of (6) and (5), we have

$m\ddot{z} = \dot{p}_z = -mg$ or $\ddot{z} = -g$.

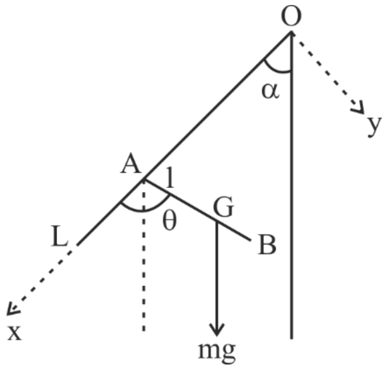
These (7, 8, 9) are the equations of motion of the projectile in space.

ASSIGNMENT TO IMPROVE

Q. 1. A bead, of mass M, slides on a smooth fixed wire, whose inclination to the vertical is α and has hinged to it a rod, of mass m and length $2l$, which can move freely in the vertical plane through the wire. IF the system starts from rest with the rod hanging vertically, show that

$\left\{4M+m\left(1+3\cos^2\theta\right)\right\};\dot{\theta}^2=6(M+m)g\sin\alpha(\sin\theta-\sin\alpha)$ where θ is the angle between the rod and the lower part of the wire.

Sol. Let OL be the fixed wire. At any time t. let the bead of mass M bet at A where OA = x, also let θ be the angle which the rod AB makes with the lower part of the fixed wire.



Take O as origin and the fixed wire OL as x axis and a lien through O and prep. To OL as y axis; the co-ordinates of G, the $C.G.$ of the rod AB, are $\{x+l\cos\theta, l\sin\theta\}$

i.e., $x_G=(x+l\cos\theta)$ and $y_G=l\sin\theta$

$\dot{x}_G=\left(\dot{x}-l\sin\theta\dot{\theta}\right)$; $\dot{y}_G=l\cos\theta\dot{\theta}$

$\therefore$ (velocity)^2 of $G=v_G^2=\dot{x}_G^2+\dot{y}_G^2=\left(\dot{x}-l\sin\theta\dot{\theta}\right)^2+\left(l\cos\theta\dot{\theta}\right)^2$.

Now let T be the kinetic energy and W the work function of the system

Then we easily get

Total energy = T = K.E. of the bead + K.E., of the rod

$$=\frac{1}{2}Mx^2+\frac{1}{2}m\left[\frac{l^2}{3}\dot{\theta}^2+\left(\dot{x}-l\sin\theta\dot{\theta}\right)^2+\left(l\cos\theta\dot{\theta}\right)^2\right]$$

$$=\frac{1}{2}(M+m)\dot{x}^2-ml\dot{x}\dot{\theta}\sin\theta+\frac{2}{3}ml^2\dot{\theta}^2$$

Also, the work function is given by

$$W=Mgx\cos\alpha+mg\left[x\cos\alpha+l\cos(\theta-\alpha)\right]$$

$$=(M+m)gx\cos\alpha+mgl\cos\cos(\theta-\alpha)$$

$\therefore$ Lagrange's x-equation gives, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)-\frac{\partial T}{\partial x}=\frac{\partial W}{\partial x}$

i.e., $\frac{d}{dt}\left[(M+m)\dot{x}-ml\dot{\theta}\sin\theta\right]=(M+m)g\cos\alpha$

or $(M+m)\ddot{x}-ml\ddot{\theta}\sin\theta-ml\dot{\theta}\sin\theta-ml\dot{\theta}^2\cos\theta=(M+m)g\cos\alpha$. .....(i)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}=\frac{\partial W}{\partial \theta}$$

i.e., $\frac{d}{dt}[ml\dot{x}\sin\theta+\frac{4}{3}ml^2\dot{\theta}]+ml\dot{x}\dot{\theta}\cos\theta$

$=-mgl\sin(\theta-\alpha)$.

or $-\ddot{x}ml\sin\theta-\dot{x}\dot{\theta}ml\cos\theta+\frac{4}{3}ml\ddot{\theta}+\dot{x}\dot{\theta}ml\cos\theta$

$=-mgl\sin(\theta-\alpha)$

or $-\ddot{x}\sin\theta+\frac{4}{3}l\ddot{\theta}-g\sin(\theta-\alpha)$ …..(ii)

Eliminating $\ddot{x}$ between (i) and (ii), we get

$\ddot{\theta}\left[-ml\sin^2\theta+\frac{4}{3}(M+m)l\right]-ml\dot{\theta}^2\sin\theta\cos\theta$

$=(M+m)g\left[\cos\alpha\sin\theta-\sin(\theta-\alpha)\right]$

or $l\ddot{\theta}\left[3M+m+3m\cos^2\theta\right]-3ml\dot{\theta}^2\sin\theta\cos\theta$

$=3(M+m)g\cos\theta\sin\alpha$.

Whence on integrating, we get

$l\dot{\theta}^2\left[4M+m+3m\cos^2\theta\right]=6(M+m)g\sin\alpha\sin\theta+C$ …..(iii)

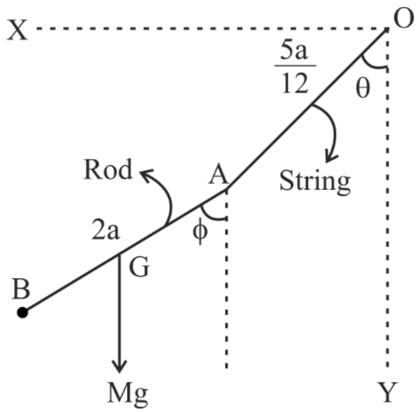
When $\theta=\alpha,\dot{\theta}=0,$ $\therefore C=-6(M+m)g\sin^2\alpha$.

Putting the value of C in (iii), we get

$l\dot{\theta}^2\left(4M+m+3m\cos^2\theta\right)=6(M+m)g\sin\alpha(\sin\theta-\sin\alpha)$

Q.2. A uniform rod, of length 2a, which has one end attached to a fixed point by a light inextensible string, of length $\frac{5}{12}a$, performing small oscillations in a vertical plane about its position of equilibrium. Find the position at any time, and show that the period of its principal oscallations are $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\frac{a}{3g}\right)}$

Sol. Figure is self explanatory. At any time t, let the string and the rod by inclined at θ and ϕ to the vertical OY.



Co-ordinates of G are given by

$$x_G = \frac{5}{12}a\sin\theta + a\sin\phi$$

$$y_G = \frac{5}{12}a\cos\theta + a\cos\phi\,.$$

$$x_G' = \frac{5a}{12}a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi}\,.$$

$$y_G' = -\left(\frac{5a}{12}a\sin\theta\dot{\theta} + a\sin\phi\dot{\phi}\right)$$

$$\therefore\ \dot{x}_G^2 + \dot{y}_G^2 = (\text{velocity})^2 \text{ of G}$$

$$= \frac{25a^2}{144}\dot{\theta}^2 + a^2\dot{\phi}^2\,\frac{5a^2}{6}\dot{\theta}\dot{\phi}\cos(\theta+\phi)$$

$$= \frac{25a^2}{144}\dot{\theta}^2 + a^2\dot{\phi}^2\,\frac{5}{6}a^2\dot{\theta}\dot{\phi}$$

[$\because$ θ and ϕ are small so $\cos(\theta+\phi)=1$]

$$T = \frac{1}{2}m\left[\frac{a^2}{3}\dot{\phi}^2 + \left(\frac{25}{144}a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{2}a^2\dot{\theta}\dot{\phi}\right)\right]$$

$$\frac{ma^2}{288}\left[25\dot{\theta}^2 + 192\dot{\phi}^2 + 120\dot{\theta}\dot{\phi}\right]$$

and $W = mg\left[\frac{5}{12}a\cos\theta + a\cos\phi\right]$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta} \Rightarrow \frac{d}{dt}\left\{\frac{ma^2}{144}\left(25\dot{\theta}^2 + 60\dot{\phi}\right)\right\} = -\frac{5mga}{12}\theta$$

{$\sin\theta = \theta$ as θ is small}

$$\Rightarrow\ 5\ddot{\theta} + 12\ddot{\phi} = -\frac{12g}{a}\theta. \qquad \text{.....(1)}$$

and Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{\frac{ma^2}{144}(192\phi + 60\dot{\theta})\right\} = -mga\phi \Rightarrow 5\ddot{\theta} + 16\ddot{\phi} = -12\frac{g}{2} \qquad \text{.....(2)}$$

Equation (1) and (2) $\Rightarrow \left(5D^2 + 12c\right)\theta + 12D^2\phi = 0$ .....(3)

and $5D^2\theta + \left(6D^2 + 12C\right)\phi = 0$ where (g/a) = c. .....(4)

Now elimination ϕ between these two equations, we get

$$\left[\left(5D^2 + 12c\right)\left(16D^2 + 12c\right) - 60D^4\right]\theta = 0$$

or $\left(5D^4 + 63cD^2 + 36c^2\right)\theta = 0$ .....(5)

Let $\theta = A\cos(pt + B)$ $\therefore D\theta = -pA\sin(pt + B)$,

$$D^2\theta = -p^2A\cos(pt + B) = 2p^2\theta \text{ and } D^2\theta = -p^4\theta.$$

Substituting these values in (5), we get

$$\left(5p^4 - 63c^2 - 36c^2\right)\theta = 0 \Rightarrow \left(5p^4 - 63cp^2 + 36c^2\right) = 0 \qquad (\because\ \theta \neq 0)$$

$$\Rightarrow \left(5p^2 - 3c\right)\left(p^2 - 12c\right) = 0 \Rightarrow \left(5p^2 - \frac{3g}{a}\right)\left(p^2 - \frac{12g}{a}\right) = 0$$

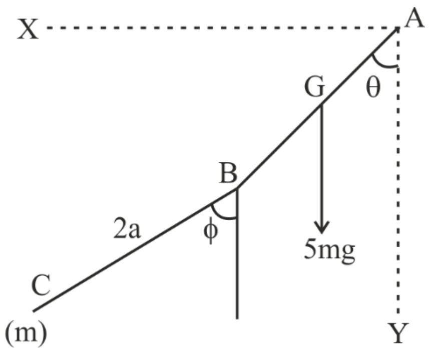
$\therefore\ p_1^2 = \frac{3g}{5a}$ and $\therefore\ p_2^2 = \frac{12g}{a}$

The periods of oscillations are $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$

i.e., $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $2\pi\sqrt{\left(\frac{a}{12g}\right)}$ i.e., $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\frac{a}{3g}\right)}$.

Q. 3. A uniform rod, of mass 5m and length 2a, turns freely about one end which is fixed, to its other extremity is attached one end of a light string of length 2a, which carries at its other end a particle of mass m, show that the periods of the small oscillations in a vertical plane are the same as those of simple pendulums of length $\frac{2a}{3}$ and $\frac{20a}{7}$

Sol. Let the string *BC* and the rod *AB* make angle ϕ a θ with the vertical at any time *t*. The particle of mass *m* is tied to the end *C* of the string.



Now $x_c = 2a\sin\theta + 2a\sin\phi$

$$\dot{x}_c = 2a\left(\cos\theta\dot{\theta} + \cos\phi\dot{\phi}\right)$$

$$y_c = 2a\cos\theta + 2a\cos\phi$$

$$\dot{y}_c = -2a\left(\sin\theta\dot{\theta} + \sin\phi\dot{\phi}\right)$$

$\therefore$ (velocity)^2 of $m = \dot{x}_c^2 + \dot{y}_c^2$

$= 4a^2\left(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right)$.

Again co-ordinates of G are $(a\sin\theta, a\cos\theta)$.

$\therefore$ (velocity)^2 of $G = a^2\dot{\theta}^2$

Now let T be the kinetic energy, and W the work function of the system, then we have

Total K.E. = K.E. of rod + K.E. of particle of mass m.

$$T = \frac{1}{2}5m\left(\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}\right) + \frac{1}{2}m.4a^2\left(\dot{\theta}^2 + \phi^2 + \dot{\theta}\dot{\phi}\right)$$

$$= ma^2\left(\frac{16}{3}\dot{\theta}^2 + 2\dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\right)$$

and $W = 5mga\cos\theta + mg.2a(\cos\theta + \cos\phi)$

$$= 7mga\cos\theta + 2mga\cos\phi = 7mag\left(1 - \frac{\theta^2}{2}\right) + 2mag\left(1 - \frac{\theta^2}{2}\right)$$

$\therefore$ Lagrange's θ equation is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{32}{3}\dot{\theta} + 4\dot{\phi}\right) = -\frac{7g}{a}\theta \Rightarrow 32\ddot{\theta} + 12\ddot{\phi} = -21\frac{g}{a}\theta \qquad \text{.....(1)}$$

Lagrange's ϕ equation is given by

$$\frac{d}{dt}\left(4\dot{\phi} + 4\dot{\theta}\right) = -\frac{2g}{a}\phi \Rightarrow i.e., 2\ddot{\theta} + 2\ddot{\phi} = -\frac{g}{a}\phi \qquad \text{.....(2)}$$

$\therefore$ (1) and (2) $\Rightarrow \left(32D^2 + 21c\right)\theta + 12D^2\phi = 0$ .....(3)

and $2D^2\theta + \left(2D^2 + c\right)\theta = 0$ where $\frac{g}{a} = c$.

Now eliminating 'ϕ' between (3) and (4), we get

$$\left[\left(32D^2 + 21c\right)\left(2D^2 + c\right) - 24D^2\right]\theta = 0$$

or $\left[40D^4 + 74cD^2 + 21c^2\right]\theta = 0$

Now let $\theta = A\cos(pt + B) \Rightarrow D\theta = -pA\sin(pt + B)$ .....(5)

$$D^2\theta = -p^2A\cos(pt + B) = -p^2\theta \text{ and } D^4\theta = p^4\theta.$$

Substituting these in (5), we get

$$\left(40p^4 - 74cp^2 + 21c^2\right)\theta = 0 \text{ i.e., } 40p^4 - 74cp^2 + 21c^3 = 0 \text{ as } \theta \neq 0$$

$$\text{or } \left(2p^2 - 2c\right)\left(20p^2 - 7c\right) = 0$$

$$\text{i.e., } \left(2p^2 - \frac{3g}{a}\right)\left(20p^2 - 7\frac{g}{a}\right) = 0$$

$$\Rightarrow p_1^2 = \frac{3g}{2a} \text{ and } p_2^2 = \frac{7g}{20a}$$

Hence length of equivalent pendulums are

$$\frac{g}{p_1^2} \text{ and } \frac{g}{p_2^2} \text{ i.e., } \frac{2a}{3} \text{ and } \frac{20}{7}a.$$

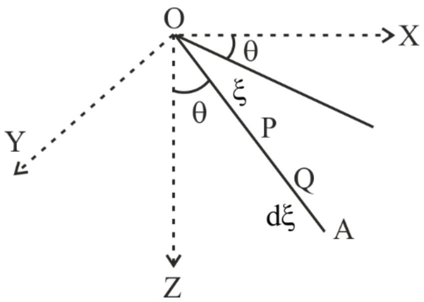
Q. 4. A uniform rod, of length $2a$, can tum freely about one end, which is fixed. Initially it is inclined at an angle α, to the down-ward drawn vertical and its is et rotating about a vertical axis through its fixed end with angular velocity ω. Show that, during the motion, the rod is always inclined to the vertical at an angle which is > or < α. According as $\omega^2 > or < \frac{3}{4a\cos\alpha}$ and that in each case its motion is inclined between the inclination α and

$$\cos^{-1}\left[-n + \sqrt{\left(1 - 2n\cos\alpha + n^2\right)}\right], \text{ when } n = \frac{a\omega^2\sin^2\alpha}{3g}$$

If it be slightly disrobed when revolving steadily at a constant angle α, show that the time of a small oscillation is

$$2\pi\left[\frac{4a\cos\alpha}{3g\left(1 + 3\cos^2\alpha\right)}\right]$$

Sol. The rod OA is turning about the end O. Take a point P on the rod such that $OP = \xi$. And the element $PQ = d\xi$.



$$\therefore \text{ mass of element } PQ = \frac{m}{2a}d\xi,$$

Where m is the mass of the rod Further at any time t, let the rod be inclined at an angle θ to the vertical and let the plane through the rod and the vertical have turned through and ϕ from its initial position OX, then co-ordinates of the point P are

$$x_p = \xi\sin\theta\cos\phi, y_p = \xi\sin\theta\sin\phi, z = \xi\cos\theta$$

$$\therefore v_P^2 = (\text{velocity})^2 \text{ of } P = \dot{x}_P^2 + \dot{y}_P^2 + \dot{z}_P^2 = \xi^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)$$

And kinetic energy of the element $PQ = \frac{1}{2}\frac{m}{2a}v_P^2$

$= \frac{1}{2}\frac{m}{2a}d\xi\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\xi^2$

Now, let T, be the K.E. of the rod OA, then we have

$$T = \frac{1}{2}\frac{m}{2a}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)\int_0^{2a} \xi^2 d\xi = \frac{2ma^2}{3}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

or $T = \frac{2ma^2}{3}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$

Also the work function, $W = mga\cos\theta + C$

Lagrange's ϕ - equation gives

$$\frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\phi}\sin^2\theta\right) = 0 \text{ i.e., } \frac{d}{dt}\left[\dot{\phi}\sin^2\theta\right] = 0 \qquad \text{.....(1)}$$

$\Rightarrow \dot{\phi}\sin^2\theta = \text{K (constant)},$ .....(2)

Initially $\theta = \alpha, \dot{\phi} = \omega,$ $\therefore K = \omega\sin^2\alpha$

Thus (2) gives $\dot{\phi}\sin^2\theta = \omega\sin^2\alpha$ .....(3)

and Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0$

When $\theta = A(pt + B)$, the period of motion is given by $T = \frac{2\pi}{p}$. If l is the length of the simple equivalent pendulum, we have

$$T = 2\pi\sqrt{(l/g)} \Rightarrow l = \frac{g}{p^2}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\theta}\right) - \frac{2ma^2}{3}\dot{\theta}^2.2\sin\theta\cos\theta = -mga\sin\theta$$

$$\Rightarrow \ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta = -\frac{g}{4a}\sin\theta \qquad \text{.....(4)}$$

Eliminating $\dot{\phi}$ between (4) and (3), we have

$$\ddot{\theta} - \frac{\omega^2\sin^4\alpha}{\sin^3\theta}\cos\dot{\theta} = -\frac{3g}{4a}\sin\theta. \qquad \text{.....(5)}$$

Initially $\theta = \alpha, \theta = 0$, $\therefore A = \omega^2\sin^2\alpha - \frac{3g}{2a}\cos\alpha$.

Substituting this value of A in (6), we get

$$\dot{\theta}^2 + \frac{\omega^2\sin^4\alpha}{\sin^2\theta} = \frac{3g}{2a}\cos\theta + \omega^2\sin^2\alpha - \frac{3g}{2a}\cos\alpha$$

or $\dot{\theta}^2 = \omega^2 \sin^2\alpha\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3g}{2a}(\cos\theta - \cos\alpha)$

$= \frac{3ng}{a}\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3g}{2a}(\cos\theta - \cos\alpha) \quad \left[\therefore n = \frac{a\omega^2\sin^2\alpha}{3g}\right]$

$= \frac{3ng}{a} \cdot \frac{\cos\alpha - \cos\theta}{\sin^2\theta}\left[2n(\cos\alpha + \cos\theta) - \sin^2\theta\right]$

i.e., $\dot{\theta}^2 = \frac{3g}{2a} \cdot \frac{\cos\alpha - \cos\theta}{\sin^2\theta}\left[\left(\cos^2\theta + 2n\cos\theta + 2n\cos\alpha - 1\right)\right]$ .....(7)

From (7), we see that $\dot{\theta} = 0$, when

$(\cos\alpha - \cos\theta)\left[\left(\cos^2\theta + 2n\cos\theta + 2n\cos\alpha - 1\right)\right] = 0$

i.e., if either $\cos\alpha - \cos\theta$ i.e., $\theta = \alpha$ (the initial position)

or $\cos^2\theta + 2n\cos\theta + 2n\cos\alpha - 1 = 0$

i.e., $\cos\theta = \frac{-2n \pm \sqrt{\left[4n^2 + 4(1 - 2\cos\alpha)\right]}}{2}$

or $\cos\theta = -n + \sqrt{\left(1 - 3n\cos\alpha + n^2\right)}$ .....(8)

(the other value being inadmissible because that gives value of $\cos\theta$ numerically greater than unity.)

Hence the motion is included between $\theta = \alpha$ and $\theta = \theta_1$ where

$\cos\theta_1\left(\sqrt{\left(1 - 2n\cos\alpha + n^2\right)} - n\right)$

The rod will move above or below its initial position, if $\theta_1 >$ or $< \alpha$ or if $\cos\theta_1 <$ or $> \cos\alpha$

i.e., if $1 - 2n\cos\alpha + n^2 <$ or $> (n + \cos\alpha)^2$

i.e., if $\frac{3ng}{a\omega^2} <$ or $> 4n\cos\alpha$ i.e., if $\omega^2 >$ or $< \frac{3g}{4ac\cos\alpha}$.

2nd part.

Small oscillations about the steady motion: The motion will be steady if the rod goes round, inclined at the same angle α with the vertically or mathematically if $\theta = \alpha$ (throughout the motion), then $\ddot{\theta} = 0$.

Making these substitutions in (5), we get,

$-\frac{\omega^2\sin^4\alpha}{\sin^3\theta}\cos\theta = -\frac{3g}{4a}\sin\theta$ i.e., $\omega^2 = \frac{3g}{4a\cos\alpha}$

When ω^2 has this value and there are small oscillation about the position

$\theta = \alpha$, then putting $\theta = \alpha + \psi$ in equation (5) we get

$$\ddot{\psi}=\frac{3g}{4a\cos\alpha}\frac{\sin^4\alpha}{\sin^3(\alpha+\psi)}\cos(\alpha+\psi)-\frac{3g}{4a}\sin(\alpha+\psi)$$

$$=\frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha\cos\psi-\sin\alpha\sin\psi)}{\cos\alpha(\sin\alpha\cos\psi+\cos\alpha\sin\psi)^3}-(\sin\alpha\cos\psi+\cos\alpha\sin\psi)\right]$$

$$=\frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha-\psi\sin\alpha)}{\cos\alpha(\sin\alpha+\psi\cos\alpha)^3}-(\sin\alpha+\psi\cos\alpha)\right],\text{ approximately}$$

$$=-\frac{3g\sin\alpha}{4a}\left[(1-\psi\tan\alpha)(1+\psi\cot\alpha)^{-3}-(1+\psi\cot\alpha)\right],\text{ approx.}$$

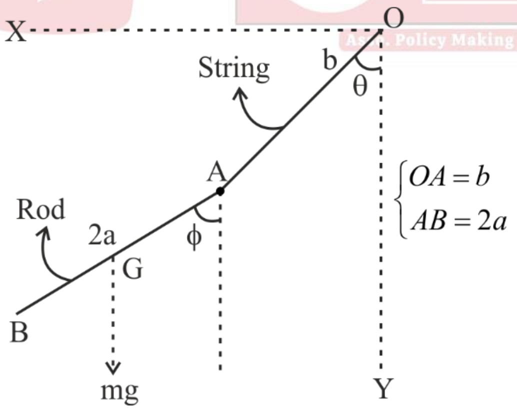
$$=-\frac{3g\sin\alpha}{4a}(4\cot\alpha+\tan\alpha)\psi\text{ , app.}=\frac{-3g\left(1+3\cos^2\alpha\right)}{4a\cos\alpha}\psi=-\mu\psi\text{ say}$$

$$\therefore\text{ time of small oscillation }=\frac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left(\frac{4a\cos\alpha}{3g\left(1+3\cos^2\alpha\right)}\right)}.$$

Q. 5. A uniform bar of length $2a$ is hung from a fixed point by a string of length b fastened to one end of the bar. Show that when the system makes small normal oscillations in a vertical plane, the length l of the equivalent simple pendulum is a root of the quadratic,

$$l^2-\left(\frac{4}{3}a+b\right)l+\frac{ab}{3}=0$$

Sol. Figure is self explanatory.



At any time t, let the string OA and the rod AB make angles θ and ϕ with the vertical.

$$x_G=b\sin\theta+a\sin\phi$$

$$y_G=b\cos\theta+a\cos\phi$$

$$\therefore\ \dot{x}_G^2+\dot{y}_G^2=(\text{velocity})^2\text{ of G}$$

$$=b^2\dot{\theta}^2+a^2\dot{\phi}^2+2ab\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$=b^2\dot{\theta}^2+a^2\dot{\phi}^2+2ab\dot{\theta}\dot{\phi}\qquad[\because\ \theta\text{ and }\phi\text{ are small}]$$

Now let T be the kinetic energy and W the work function of the system, then we easily obtain

$$W=mg[b\cos\theta+a\cos\phi]$$

And $T=\frac{1}{2}m\left[\frac{a^2}{3}\dot{\phi}^2+b^2\dot{\theta}^2+a^2\dot{\phi}^2+2ab\dot{\theta}\dot{\phi}\right]$

$=\frac{1}{2}m\left[\frac{4a^2}{3}\dot{\phi}^2+b^2\dot{\theta}^2+2ab\dot{\theta}\dot{\phi}\right]$

$\therefore$ Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}=\frac{\partial W}{\partial \theta}$

$\Rightarrow \frac{d}{dt}\left\{m\left(b^2\dot{\theta}+ab\dot{\phi}\right)\right\}=-mgb\theta$ $\qquad\{\because \ \sin\theta=0\}$

$\Rightarrow b\ddot{\theta}+a\dot{\phi}=-g\theta$ …..(1)

Lagrange's ϕ–equation is given by $\frac{d}{dt}\left\{m\frac{4}{3}a^2\dot{\phi}+ab\dot{\theta}\right\}=-mag\phi$

$\Rightarrow 4a\ddot{\phi}+3b\ddot{\theta}=-3g\phi$ …..(2)

Equations (1) and (2) again can be written as

$\left(bD^2+g\right)\theta+aD^2\phi=0$ …..(3)

and $3bD^2\theta+\left(4aD^2+3g\right)\phi=0$ …..(4)

Eliminating ϕ between these equations, we obtain

$\left[\left(bD^2+g\right)\left(4aD^2+3g\right)-3abD^4\right]\theta=0$

i.e., $\left[abD^4+(4a+3b)gD^2+3g^2\right]\theta=0$ …..(5)

Now let $\quad \theta=A\cos\left[\sqrt{\left(\frac{g}{l}\right)}t+B\right]$

Where l is the length of the simple equivalent pendulum.

Then $D\theta=-\sqrt{\left(\frac{g}{l}\right)}A\sin\left[\sqrt{\left(\frac{g}{l}\right)}t+B\right]$

$D^2\theta=-\frac{g}{l}A\cos\left[\sqrt{\left(\frac{g}{l}\right)}t+B\right]=-\frac{g}{l}\theta$ and $D^4q=\frac{g^2}{l^2}\theta,$

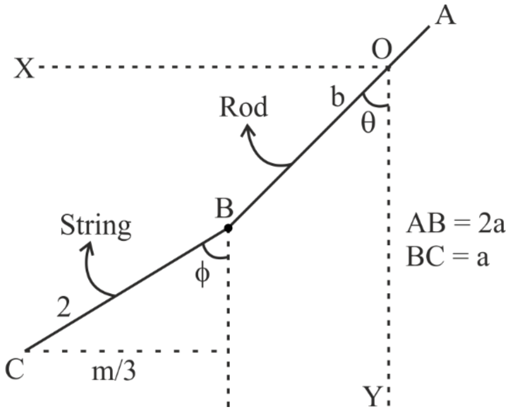
$\therefore (5)\Rightarrow\left[ab\frac{g^2}{l^2}-(4a+3b)\frac{g^2}{l}+3g^2\right]\theta=0$

$\Rightarrow 3l^2-(4a+3b)l+ab=0$ $\qquad\{\because \ \theta\neq 0\}$

$\Rightarrow l^2-\left(\frac{4}{3}a+b\right)l+\frac{ab}{3}=0$

Q. 6. A uniform straight rod of length 2a, is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string, of length a, to one end of the rod ; show that one period of principle oscillation is $\left(\sqrt{5}+1\right)\pi\sqrt{\left\{\frac{a}{g}\right\}}$

Sol. Figure is self explanatory.



At time t, let θ and ϕ be the inclinations of the rod and the string to the vertical. Co-ordinates of C are

$x_C = a\sin\theta + a\sin\phi$ and $y_C = a\cos\theta + a\cos\phi$

$\therefore\ \dot{x}_C = a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi}$ and $\dot{y}_C = -a\sin\theta + a\sin\phi\dot{\phi}$

$\Rightarrow\ \dot{x}_C^2 + \dot{y}_C^2 = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\cos(\theta-\phi)\dot{\theta}\dot{\phi}$

$= a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$

[neglecting higher powers of mall quantities]

$\therefore$ (velocity)^2 of the particle $C = v_C^2 = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$.

And velocity of the C.G. of the rod i.e., of O, is zero.

Now let T, be the kinetic energy and W, the work function of the system then we easily get

$$W = \frac{mg}{3}(a\cos\theta + a\cos\phi) + C$$

and $T = \frac{1}{2}m\frac{a^2}{3}\dot{\theta}^2 + \frac{1}{2}\left(\frac{m}{3}\right)\left[a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}\right]$

$$\frac{ma^2}{6}\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right]$$

$\therefore$ Lagrange's θ-equation is given by

$$\frac{d}{dt}\left(\frac{2ma^2}{3}\dot{\theta} + \frac{ma^2}{3}\dot{\phi}\right) = -\frac{mga}{3}\theta \Rightarrow 2\ddot{\theta} + \ddot{\phi} = -\frac{g}{a}\theta \qquad \text{.....(1)}$$

While Lagrange's ϕ-equation gives $\frac{d}{dt}\left[\frac{ma^2}{3}\dot{\phi} + \frac{ma^2}{3}\dot{\theta}\right] = -\frac{mga}{3}\phi$

i.e., $\ddot{\theta} + \ddot{\phi} = -\frac{g}{a}\phi \qquad$ .....(2)

Equation (1) and (2) again give

$$\left(2D^2+c\right)\theta+D^2\phi=0 \qquad \text{.....(3)}$$

and $$D^2\theta+\left(D^2+c\right)\phi=0 \qquad \text{.....(4)}$$

where $c=\dfrac{g}{a}$.

Eliminating ϕ in between (3) and (4), we get

$$\left\{\left(D^2+c\right)\left(2D^2+c\right)-D^4\right\}\theta=0 \text{ i.e., } \left[D^4+3cD^2+c^2\right]\theta=0 \qquad \text{.....(5)}$$

To solve (5), let $\theta=A\cos(pt+B)$ $\therefore$ $D\theta=-pA\sin(pt+B)$

$D^2\theta=-p^2A\cos(pt+B)=-p^2\theta$ and $D^4\theta=p^4\theta$.

With these substitutions, (5) gives

$$\left(p^4-3cp^2+c^2\right)\theta=0$$

$$\Rightarrow p^4-3cp^2+c^2=0 \qquad (\because \theta\neq 0)$$

$$\therefore p^2\frac{3c\pm\sqrt{\left(9c^2-4c^2\right)}}{2}=\left(\frac{3\pm\sqrt{5}}{2}\right)c=\frac{\left(3\pm\sqrt{5}\right)}{2}.\frac{g}{a}$$

$$\therefore p_1^2=\frac{3-\sqrt{5}}{2}.\frac{g}{a} \text{ and } p_2^2=\frac{3+\sqrt{5}}{2}.\frac{g}{a}.$$

$\therefore$ one period of principal oscillations corresponding to p_1, is given by

$$\frac{2\pi}{p_1}=2\pi\sqrt{\left(\frac{2}{3-\sqrt{5}}.\frac{a}{g}\right)}=2\pi\sqrt{\left(\frac{a}{g}\right)}\sqrt{\left\{\frac{2\left(3+\sqrt{5}\right)}{9-5}\right\}}$$

$$=2\pi\sqrt{\left(\frac{a}{g}\right)}\sqrt{\left(\frac{6+2\sqrt{5}}{4}\right)}=2\pi\sqrt{\left(\frac{a}{g}\right)}\sqrt{\left\{\frac{\left(\sqrt{5}+1\right)^2}{4}\right\}}$$

$$=\left(\sqrt{5}+1\right)\pi\sqrt{\left(\frac{a}{g}\right)}.$$

Q.7. A smoother circular wire, of mass 8 m and radius *a* swings in a vertical plane, being suspended by an inxtensible string of length *a* attached to one point of it, a particle of mass *m* can slide on the wire, Prove that the periods of small oscillations are

$$2\pi\sqrt{\frac{8a}{3g}},2\pi\sqrt{\frac{a}{3g}},2\pi\sqrt{\frac{8a}{9g}}.$$

Sol. At any time t, let the string OA, and the radius AC be inclined at angle θ and ϕ with the vertical and further let the radius of the particle (m) be inclined at an angle ψ with the vertical.

Now co-ordinates of C are

$(a\sin\theta + a\sin\phi, a\cos\theta + a\cos\phi)$.

$\therefore$ (velocity)^2 of $C = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}.\cos(\theta - \phi)$

$= a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$ approximately .

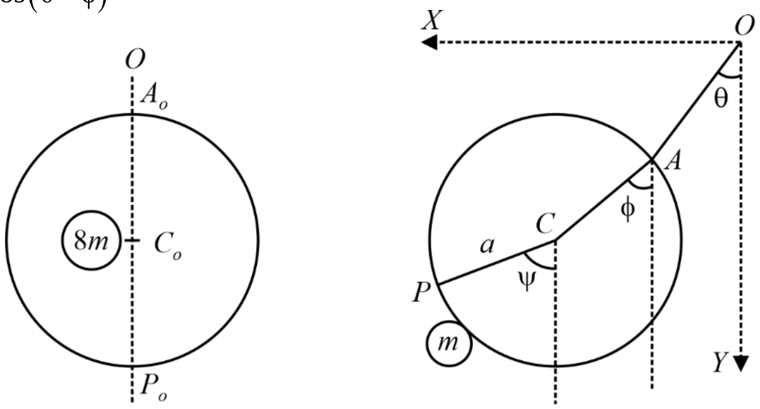
$$\left[\because \begin{array}{l} \dot{x}_C = a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi} \\ \dot{y}_C = -a\left(\sin\theta\dot{\theta}. + \sin\phi\dot{\phi}\right) \end{array}\right]$$

Also co-ordinates of the particle m (i.e. of the pt. P) are

$x_P = a(\sin\theta + \sin\phi + \sin\psi)$,

$y_P = a(\cos\theta + \cos\phi + \cos\psi)$.

$\because$



[(i)]
Rest position

[(ii)]
displaced position

$(\text{velocity})^2$ of $m = a^2\left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}\right)$app,

Let T, be the kinetic energy and W, the work function of the system, then we readily get

$W = 8mg(a\cos\theta + a\cos\phi) + mg(a\cos\theta + a\cos\phi + a\cos\psi)$

$= m\,ga[9\cos\theta + 9\cos\phi + \cos\psi]$ ...(1)

and $T = \frac{1}{2}\cdot 8m\left[a^2\dot{\phi}^2 + \left(a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}\right)\right] + \frac{1}{2}ma^2\left[\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}\right]$.

i.e. $T = \frac{1}{2}m\left[9\dot{\theta}^2 + 17\dot{\phi}^2 + \dot{\psi}^2 + 18\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}\right]$ ...(2)

Lagrange's θ, and ψ equations give

$$9\ddot{\theta} + 9\ddot{\phi} + \ddot{\psi} = -9\frac{g}{a}\theta \quad \text{...(3)}$$

$$9\ddot{\theta} + 17\ddot{\phi} + \ddot{\psi} = -9\frac{g}{a}\phi \quad \text{...(4)}$$

and $\ddot{\theta} + \ddot{\phi} + \ddot{\psi} = \frac{g}{a}\psi$, ...(5)

which can be rewritten as

$(9D^2 + 9c)\theta + 9D^2\phi + D^2\psi = 0$ ...(6)

$9D^2\theta + (17D^2 + 9c)\phi + D^2\psi = 0$ ...(7)

and $D^2\theta + D^2\phi + (D^2 + c)\psi = 0$ ...(8)

Eliminating ϕ and ψ in (6), (7) and (8), we get

$$\begin{vmatrix} 9D^2 + 9c & 9D^2 & D^2 \\ 9D^2 & 17D^2 + 9c & D^2 \\ D^2 & D^2 & D^2 + c \end{vmatrix}\theta = 0$$

i.e. $(8D^2 + 9c)\left[9c(2D^2 + c) + D^2(8D^2 + 9c)\right]\theta = 0$

i.e. $\quad (8D^2+9c)\left[8D^4+27cD^2+9c^2\right]\theta=0$

i.e. $\quad \left[(8D^2+9c)(8D^2+3c)(D^2+3c)\right]\theta=0$

Now let $\theta = A\cos(pt+B)$, then

$D\theta=-pA\sin(pt+B), D^2\theta=-p^2A\cos(pt+B)=-p^2\theta \ldots(10)$

$\therefore$ (9) gives $(8p^2-9c)(8p^2-3c)(p^2-3c)=0 \qquad [\because 0\neq\theta]$

$$\Rightarrow \quad \left(8p^2-\frac{9g}{a}\right)\left(8p^2-\frac{3g}{a}\right)\left(p^2-\frac{3g}{a}\right)=0.$$

i.e. $\quad p_1^2=\dfrac{9g}{8a}, p_2^2=\dfrac{3g}{8a}, p_3^2=\dfrac{3g}{a}$

Thus periods of small oscillations are $\dfrac{2\pi}{p_1},\dfrac{2\pi}{p_2},\dfrac{2\pi}{p_3}$

i.e. $\quad 2\pi\sqrt{\dfrac{8a}{9g}}, 2\pi\sqrt{\dfrac{8a}{3g}}, 2\pi\sqrt{\dfrac{a}{9g}}$

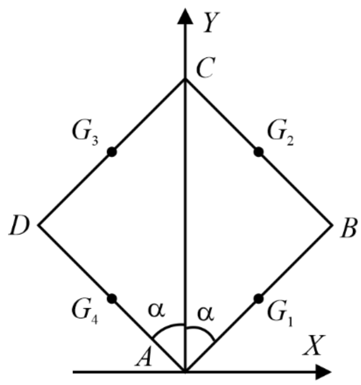
Q. 8. Four uniform rods; each of length $2a$, are hinged at their ends so as to form a rhombus *ABCD*. The angles *B* and *D* are connected by an elastic string and the lowest end *A* rests on a horizontal plane while the end *C* slides on a smooth vertical wire passing through *A*; in the position of equlibrium the string is strected to twice its natural length and the angle *BAD* is 2α. Show that the time of a small oscillation about this position is

$$2\pi\left\{\frac{2a(1+3\sin^2\alpha)}{3g\cos 2\alpha}\cos\alpha\right\}^{1/2}$$

Sol. In the position of equilibrium, rods are making angles α with the vertical.

When the system is slightly displaced from the position of equilibrium, let the rods make angle $(\alpha+\theta)$ with the vertical θ being a small displacement.

Now assuming the fixed end *A* as origin and the horizontal and vertical lines through it as co-ordinate axes, the co-ordinates of G_2 are $\{a\sin(\alpha+\theta), 3a\cos(\alpha+\theta)\}$



$\therefore (\text{velocity})^2$ of $G_2=\{a\cos(\alpha+\theta)\dot{\theta}\}^2+\{-3a\sin(\alpha+\theta)\dot{\theta}\}^2$

$=a^2[(1+8\sin^2(\alpha+\theta)]\dot{\theta}^2.$

Co-ordinates of G_1 are $\{a\sin(\alpha+\theta), a\cos(\alpha+\theta)\}$

$\therefore$ (velocity)^2 of $G_1=a^2\dot{\theta}^2$.

$\therefore$ Kinetic energy of the four rods taken together is

$$T=2.\frac{1}{2}m.\left[\frac{a^2}{3}\dot{\theta}^2+a^2\dot{\theta}^2\right]+2.\frac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2+a^2\{1+8\sin^2(\alpha+\theta)\}\dot{\theta}^2\right].$$

$$=\frac{8ma^2}{3}\left[1+\sin^2(\alpha+\theta)\right]\dot{\theta}^2\left(\because v_{G_1}=v_{G_4} \text{ and } v_{G_2}=v_{G_3}\right)$$

The work function W is given by $W = 2\left\{-mga\cos(\alpha+\theta)\right\}$

$$+2\left\{-mg\,3a\cos(\alpha+\theta)\right\}-2\int_O^{2a\sin(\theta+\alpha)}\lambda\left(\frac{y-c}{c}\right)dy$$

$$=-8\,m\,ga\cos(\alpha+\theta)-\frac{\lambda}{c}\left\{2a\sin(\alpha+\theta)-c\right\}^2$$

Lagrange's θ equation gives

$$\frac{d}{dt}\left[\frac{16ma^2}{3}\left\{1+3\sin^2(\alpha+\theta)\right\}\dot{\theta}-16ma^2\sin(\alpha+\theta)\cos(\alpha+\theta)\theta^2\right]$$

$$=8m\,ga\sin(\alpha+\theta)-\frac{4\lambda}{c}a\cos(\alpha+\theta)\left\{2a\sin(\alpha+\theta)-c\right\}$$

$$\Rightarrow\frac{16ma^2}{3}(1+3\sin^2(\alpha+\theta)\}\ddot{\theta}$$

$$=8m\,ga\sin(\alpha+\theta)-\frac{4\lambda a}{c}\cos(\alpha+\theta)\left\{2a\sin(\alpha+\theta)-c\right\}\ \ldots(1)$$

Initially when $\theta = 0$, $\dot{\theta}=0, \ddot{\theta}=0, c=a\sin\alpha$, hence (1) gives

$$\lambda=\frac{2m\,gc}{a\cos\alpha}$$

Putting this value of λ in equation (1), we get

$$\frac{16ma^2}{3}\left\{1+3\sin^2(\alpha+\theta)\right\}\ddot{\theta}$$

$$=m\,ga\,8\sin(\alpha+\theta)-\frac{8mg\cos(\alpha+\theta)}{\cos\alpha}\left\{2a\sin(\alpha+\theta)-c\right\}$$

*The force $m\omega^2 l\sin\theta$ also contributes to W. The distance of the point of application at O' of this force from the vertical OZ is equal to $l\sin\theta$, hence the contribution $m\omega^2 l^2\sin^2\theta$ to W is as given in (2).

** If W is the work function of the system, then P.E. = $C - W$.

i.e. $$\frac{16ma^2}{3}\left\{1+3\sin^2\alpha\right\}\theta=8\,m\,ga(\sin\alpha+\theta\cos\alpha)$$

$$-\frac{8mg}{\cos\alpha}(\cos\alpha-8\sin\alpha)\left\{2a(\sin\alpha+\theta\cos\alpha)-a\sin\alpha\right\}$$

$$=-\frac{8\,m\,ga\cos2\alpha}{\cos\alpha}\theta\text{ app.}\Rightarrow\ddot{\theta}=-\frac{3g\cos2\alpha}{2a\cos\alpha\left(1+3\sin^2\alpha\right)}\theta\text{ app.}$$

$\therefore$ Time of a small oscillation about the position of equilibrium is given by

$$2\pi\sqrt{\left\{\frac{2a\cos\alpha\left(1+3\sin^2\alpha\right)}{3g\cos2\alpha}\right\}}$$

HAMILTONIAN

Q.9. *Using cylindrical coordinates* $(\rho,\ \phi,\ z)$ *write the Hamiltonian and Hamilton's equations for a particle of mass m moving in a force field of potential V* $(\rho,\ \phi,\ z)$.

Sol. In cylinderical coordinates, co-ordinates of any points are

$$x = \rho \cos\phi,\ y = \rho \sin\phi,\ z = z \qquad \text{...(1)}$$

$$\therefore T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \qquad \text{...(2)}$$

$$\Rightarrow L = T - V = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - V(\rho,\ \phi,\ z) \quad \text{...(3)}$$

$$\Rightarrow p_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{p},\ p_\phi = \frac{\partial L}{\partial\dot{\phi}} m\rho^2\dot{\phi} \text{ and } p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z}$$

Evidently, L does not involve t explicity, therefore Hamiltonian ***H*** is given by

$$H = T = V = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) + V$$

$$= \frac{1}{2}m\left[\frac{p_\rho^2}{m^2} + \frac{p_\phi^2}{m^2\rho^2} + \frac{p_z^2}{m^2}\right] + V = \frac{1}{2m}\left[p_\rho^2 + \frac{p_\phi^2}{\rho} + p_z^2\right] + V$$

Hence, Hamilton's are given by:

$$\dot{p}_\rho = -\frac{\partial H}{\partial\rho} = \frac{p_\phi^2}{m\rho^2} - \frac{\partial V}{\partial\rho};\ \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial\phi} = -\frac{\partial V}{\partial\phi};\ \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z};\ \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

Q. 10. Using cylindrical coordinates, write the Hamiltonian and Hamilton's equations for a particle of mass moving on the inside of a frictionless cone $x^2 + y^2 = z^2\tan^2\alpha$

Sol.

Like previous example, we have

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + z^2\right) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \rho^2\cot^2\alpha\right)$$

$$[\because\ x = \rho\cos\phi,\ y = \rho\sin\phi, z = \rho\cot\alpha]\ \ldots(1)$$

$$= \frac{1}{2}m\left(\dot{\rho}^2\operatorname{cosec}^2\alpha + \dot{\rho}^2\dot{\phi}^2\right) \qquad \ldots(1)$$

and $\qquad V = -W = -mgz = mg\,\rho\cot\alpha,$

[$\because$ the particle is above the vertex origin)].

$$\Rightarrow\ L = T - V = \frac{1}{2}m\left(\rho^2 \text{cosec}^2\alpha + \rho^2\dot{\phi}^2\right) - mg\,\rho\cot\alpha\ \ldots(2)$$

This gives, $p_\rho = \dfrac{\partial L}{\partial \dot{p}} = m\rho\text{cosec}^2\alpha,\ p_\phi = \dfrac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi}\ \ldots(3)$

Again, L does not involve t explicitly, therefore Hamiltonian H is given by

$$H = T + V = \frac{1}{2}m\left(\dot{\rho}^2\text{cosec}^2\alpha + \rho^2\dot{\phi}^2\right) + mg\,\rho\cot\alpha$$

$$= \frac{1}{2}m\left[\frac{p_\rho^2}{m^2\text{cosec}^2\alpha} + \frac{p_\phi^2}{m^2\rho^2}\right] + mg\,\rho\cot\alpha + \frac{1}{2m}\left[\frac{p_\rho^2}{\text{cosec}^2\alpha} + \frac{p_\phi^2}{\rho^2}\right] + mg\,\rho\cot\alpha$$

Thus Hamilton's equations are given by:

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = \frac{p_\phi^2}{m\rho^2} - mg\cot\alpha;\ \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m\,\text{cosec}^2\alpha}.$$

$$\dot{\rho}_\phi = -\frac{\partial H}{\partial \phi} = 0; \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}.$$

Q.11. *Write the Hamiltonian and equation of motion for a simple pendulum.*

Sol. We have $T = \dfrac{1}{4} = ml^2\theta^2$ and $V = mgl(1-\cos\theta)$,

$$\therefore\ L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1-\cos\theta)\quad ...(1)$$

$$\Rightarrow\ H = \sum p_i\dot{q}_i - L = p_\theta\dot{\theta} - L = ml^2\dot{\theta}^2 - \left\{\frac{1}{2}ml^2\theta^2 - mgl(1-\cos\theta)\right\}$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = T + V = \text{total energy}$$

Now $p_\theta = (\partial L/\partial\dot{\theta}) = ml^2\dot{\theta}^2 = (p_\theta / ml^2)$

$$\therefore\ H = \frac{1}{2}ml^2(p_\theta / ml^2)^2 + mgl\,(1-\cos\theta) = (p_\theta^2 / 2ml^2) + mgl(1-\cos\theta)$$

$$\Rightarrow (\partial H/\partial p_\theta) = (p_\theta / ml^2), (\partial H/\partial\theta) = mgl\sin\theta$$

Now Hamilton's equation of motion of θ and p_θ are

$\dot{\theta} = (\partial H/\partial p_\theta), \dot{p}_\theta = -(\partial H/\partial\theta) \Rightarrow \dot{\theta} = (p_\theta / ml^2)$ and $\dot{p}_\theta = -m\lg\sin\theta$,

These represent Hamilton's equations for a simple pendulum.

From above, we have $p_\theta = ml^2\,\theta, i.e,\ p_\theta = ml^2\ddot{\theta}$

$\therefore\ ml^2\ddot{\theta} = -mgl\sin\theta \Rightarrow \ddot{\theta} + (g/l)\sin\theta = 0$

This gives the equation of motion of the simple pendulum.

Q.12 . *If H is the Hamiltonian, prove that if f is any function depending on position, momento 1 time then* *(df/dt) = (∂f/ .... + [H, f])*

Sol. We have

$$(df/dt) = (\partial f/\partial t) + \sum_i \{(\partial f/\partial q_i)(dq_i/dt) + (\partial f/\partial p_i)(dp_i/dt)\}$$

$$\Rightarrow (df/dt) = (\partial f/\partial t) + \sum_i \{(\partial f/\partial q_i)(\partial H/\partial p_i) - (\partial f/\partial p_i)(\partial H/)\}$$

{$\because$ By Hamilton's equation $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -(\partial H/\partial q_i)$}

$\Rightarrow (df/dt) = (\partial f/\partial t) + [H, f]$ where, $[H, f]$ is the Poisson Bracket)

PREVIOUS YEARS QUESTIONS

CHAPTER 4. HAMILTON'S EQUATION OF MOTION

Q1. By writing down the Hamiltonian, find the equations of motion of a particle of mass *m* constrained to move on the surface of a cylinder defined by $x^2 + y^2 = R^2$, R is a constant. The particle is subject to a force directed towards the origin and proportional to the distance *r* of the particle from the origin given by $\vec{F} = -k\vec{r}$, *k* is a constant. **[6c UPSC CSE 2020]**

Q2. Find the condition on *a*, *b*, *c* (real numbers) such that the dynamical system with equations $\dot{p} = aq - q^2, \dot{q} = bp + cq$ is Hamiltonian. Compute also the Hamiltonian of the system.

[5d 2020 IFoS]

Q3. Using Hamilton's equation, find the acceleration for a sphere rolling down a rough inclined plane, if *x* be the distance of the point of contact of the sphere from a fixed point on the plane.

[7a UPSC CSE 2019]

Q4. Consider a mass-spring system consisting of a mass *m* and a linear spring of stiffness *k* hanging from a fixed point. Find the equation of motion using the Hamiltonian method, assuming that the displacement *x* is measured from the unscratched position of the string.

[7b 2019 IFoS]

Q5. The Hamiltonian of a mechanical system is given by,

$H = p_1q_1 - aq_1^2 + bq_2^2 - p_2q_2$, where a, b are the constants. Solve the Hamiltonian equations and show that $\dfrac{p_2 - bq_2}{q_1}$ = constant. **[7c UPSC CSE 2018]**

Q6. For a particle having charge *q* and moving in an electromagnetic field, the potential energy is $U = q\left(\phi - \vec{v}\cdot\vec{A}\right)$, where ϕ and $\vec{A}$ are, respectively, known as the scalar and vector potentials. Derive expression for Hamiltonian for the particle in the electromagnetic field.

[6c 2018 IFoS]

Q7. Consider a single free particle of mass *m*, moving in space under no forces. If the particle starts from the origin at $t=0$ and reaches the position (x, y, z) at time τ, find the Hamilton's characteristic function S as a function of x, y, z, τ. **[5c UPSC CSE 2016]**

Q8. Solve the plane pendulum problem using the Hamiltonian approach and show that H is a constant of motion. **[6b UPSC CSE 2015]**

Q9. A Hamiltonian of a system with one degree of freedom has the form

$$H=\frac{p^2}{2\alpha}-bqpe^{-\alpha t}+\frac{b\alpha}{2}q^2e^{-\alpha t}\left(\alpha+be^{-\alpha t}\right)+\frac{k}{2}q^2$$

where α, b, k are constants, q is the generalized coordinate and p is the corresponding generalized momentum.

(i) Find a Lagrangian corresponding to this Hamiltonian.

(ii) Find an equivalent Lagrangian that is not explicitly dependent on time.

[7c UPSC CSE 2015]

Q10. Derive the Hamiltonian and equation of motion for a simple pendulum.

[5c 2015 IFoS]

Q11. Find the equation of motion of a compound pendulum using Hamilton's equations.

[5e UPSC CSE 2014]

Q12. Derive the Hamiltonian and equation of motion for a simple pendulum.

[5c 2013 IFoS]

Q13. Obtain the equations governing the motion of a spherical pendulum. **[5d UPSC CSE 2012]**

Q14. Derive the differential equation of motion for a spherical pendulum. **[6b 2012 IFoS]**

Q15. A sphere of radius *a* and mass *m* rolls down a rough plane inclined at an angle α to the horizontal. If *x* be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations. **[8a UPSC CSE 2010]**

CHAPTER 5. Work & Energy (Equilibrium/Centre of Mass)

Q1. A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

where a is the length of the plank. **[8b 2014 IFoS]**

Work & Energy (Statics)

Q2. A mass m_1, having at the end of a string, draws a mass m_2 along the surface of a smooth table. If the mass on the table be doubled, the tension of the string

is increased by one-half. Show that $m_1 : m_2 = 2:1$. **[(8a) 2010 IFoS]**