STRENGTHENING BRAINS

IAS/IFoS

3 D ANALYTICAL GEOMETRY

Systematically designed mathematics optional book

Concepts, Examples & PYQs Analysis

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WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFOS :

01 Conceptual Development

02 Problem Solving Techniques

03 Assignments

04 Chapter wise PYQs Analysis

05 Test

MINDSET MAKERS

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Prepare in Right Way: Book perfectly aligned with YouTube Live Lectures

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📌□□□□□□ □□□□□□□□, □□□□□□□□□ □□□□□: Do you know what the real challenge is in such exam preps? Students attend a lecture from some teacher, read books from another professor, solve PYQs from booklets done by content development teams, practice questions from assignments prepared by another person, and struggle to align their information for the exam. So…

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Upendra Singh Sir has done this with his experiences of all three stages of the exam as a UPSC aspirant, teaching journey of over a decade, and UPSC Mentoring learnings. He actually feels this process and puts in his best efforts.

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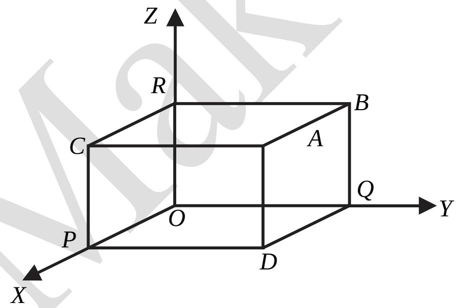
Three Dimensional Coordinate Geometry

Some basics from 10+2 level of coordinate geometry Revision Through Examples

Example:-1 The point A (1,2,3) is a vertex of the rectangular parallelepiped formed by the co-ordinate planes and the planes passing through A parallel to the co-ordinate planes. Find the coordinates of the other vertices of the parallelepiped.

Solution: - Form the figure it is clear that the co-ordinates of the other vertices of the rectangular parallelepiped are:

$$O(0,0,0),\ \ P(1,0,0)$$
$$Q(0,2,0),\ \ R(0,0,3)$$
$$B(0,2,3),\ \ C(1,0,3)$$
$$D(1,2,0)$$



Example: 1. Find the distance between the points $(2,-5,4)$ and $(8,-2,6)$

Solution: - We know that the distance between the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ us given by

$$PQ=\sqrt{\left\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right\}}$$

Putting $x_1=2,\ y_1=-5,\ z_1=4$

And $x_2=8,\ y_2=-2,\ z_2=6$

The required distance $=\sqrt{\left\{(8-2)^2+(-2+5)^2+(6-4)^2\right\}}$

$$=\sqrt{\left\{\left(6^2+3^2+2^2\right)\right\}}=\sqrt{(36+9+4)}=\sqrt{(49)}=7$$

Example: 2 Find the distance between points $(2,0,-5)$ and $(0,7,-3)$

Solution: - We know that the distance between the points (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\sqrt{\left\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right\}}$$

Hence the distance between the given points

$$=\sqrt{\left\{(0-2)^2+(7-0)^2+(-3+5)^2\right\}}$$

$$=\sqrt{(4+49+4)}=\sqrt{(57)}$$

Example: -3 Prove that the points $(-2,4,-3),(4,-3,-2)$ and $(-3,-2,4)$ from and equilateral triangle.

Solution: - Denoting the given points by A, B and C respectively. We find that

$$AB=\sqrt{\left\{(4+2)^2+(-3-4)^2+(-2+3)^2\right\}}$$
$$=\sqrt{(36+49+1)}=\sqrt{(86)}$$
$$BC=\sqrt{\left\{(-3-4)^2+(-2+3)^2+(4+2)^2\right\}}$$
$$=\sqrt{(49+1+36)}=\sqrt{(86)}$$
and $$CA=\sqrt{\left\{(-2+3)^2+(4+2)^2+(-3-4)^2\right\}}$$
$$=\sqrt{(1+36+49)}=\sqrt{(86)}$$

Clearly $AB=BC=CA$. Hence the given points form an equilateral triangle.

Example:5 - Find the co-ordinates of the point which divides the line joining the points $(2,-4,3)$ and $(-4,5,-6)$ in the ration 2: 1

Solution: - Here $x_1=2$, $y_1=-4$, $z_1=3$;
$x_2=-4$, $y_2=5$, $z_2=-6$;

And $m_1=2,\ m_2=1$

If (x,y,z) be the required co-ordinates, we have

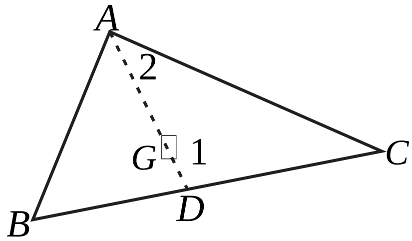
$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2}=\frac{2(-4)+1(2)}{2+1}=\frac{-8+2}{3}=-2$$
$$y=\frac{m_1y_2+m_2y_1}{m_1+m_2}=\frac{2(5)+1(-4)}{2+1}=\frac{10-4}{3}=2$$
And $$z=\frac{m_1z_2+m_2z_1}{m_1+m_2}=\frac{2(-6)+1(3)}{2+1}=\frac{-12+3}{3}=-3$$

Hence the co-ordinates of the point which divides the line joining the given points in the ration 2:1 are $(-2,2,-3)$

Example:-6 Show that the centroid of the triangle whose vertices are $(x_1,y_1,z_1)(x_2,y_2,z_2)$ and (x_3,y_3,z_3) is $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$.

Solution:- Let the given vertices of the triangle be denoted by A,B and C respectively. Let D be the mid point of BC. Then we known that the co-ordinates of D are

$$\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_2+z_3}{2}\right)$$



Also, from geometry we know that the point G dividing AD in the ration 2:1 is the centroid of ΔABC. If (x, y, z) be the co-ordinates of G. we have

$$x = \frac{2\left(\frac{x_2 + x_3}{2}\right) + 1.\ x_1}{2+1} = \frac{(x_2 + x_3) + x_1}{3} = \frac{x_1 + x_2 + x_3}{3}$$

Similarly, $y = \frac{y_1 + y_2 + y_3}{3}$ and $z = \frac{z_1 + z_2 + z_3}{3}$

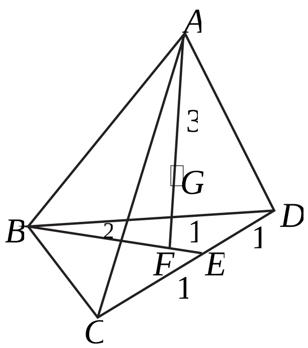
Thus we obtain the required co-ordinates of the centroid.

Remark:- The result of this example may be regarded as a standard result for further use.

Example:-7 Find the centroid of the tetrahedron having the points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) for its vertices.

Solution:- Let the given points be denoted by A,B,C and D respectively. BE is the median from B on CD so that the co-ordinates of E are

$$\left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2}, \frac{z_3 + z_4}{2}\right)$$



Clearly, the point F which divides BE in the ratio 2: 1 is the centroid of the triangular base BCD. The co-ordinates of F are

$$\left(\frac{x_2 + 2\left(\frac{x_3 + x_4}{2}\right)}{1+2}, \frac{y_2 + 2\left(\frac{y_3 + y_4}{2}\right)}{1+2}, \frac{z_2 + 2\left(\frac{z_3 + z_4}{2}\right)}{1+2}\right)$$

i.e. $$\left(\frac{x_2 + x_3 + x_4}{3}, \frac{y_2 + y_3 + y_4}{3}, \frac{z_2 + z_3 + z_4}{3}\right)$$

Further, the centroid G of the tetrahedron ABCD divides AF in the ratio 3:1. Hence the co-ordinates of G are

$$\left(\frac{x_1+3\left(\frac{x_2+x_3+x_4}{3}\right)}{1+3},\frac{y_1+3\left(\frac{y_2+y_3+y_4}{3}\right)}{1+3},\frac{z_1+3\left(\frac{z_2+z_3+z_4}{3}\right)}{1+3}\right)$$

i.e. $\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$

Example:-8 Find the ratio in which the $yz-$plane divides the line joining the points $(3,5,-7)$ and $(-2,1,8)$. Find also the co-ordinates of the point of division.

Solution:- Let $m_1 : m_2$ be the ratio in which the $yz-$plane divide the line joining the points $(3,5,-7)$ and $(-2,1,8)$. Then the co-ordinates of the point of division are

$$\left(\frac{-2m_1+3m_2}{m_1+m_2},\frac{m_1+5m_2}{m_1+m_2},\frac{8m_1-7m_2}{m_1+m_2}\right) \qquad (1)$$

Since the point lies on the $yz-$plane, its x co-ordinate should be zero.
Thus we must have

$-2m_1+3m_2=0$, which gives $m_1 : m_2 = 3:2$

Now the other co-ordinates of the point of division are

$$y=\frac{m_1+5m_2}{m_1+m_2}=\frac{3(1)+5(2)}{3+2}=\frac{13}{5}$$

And $z=\frac{8m_1-7m_2}{m_1+m_2}=\frac{8(3)-7(2)}{3+2}=\frac{10}{5}=2$.

Hence the co-ordinates of the point of division are $\left(0,\frac{13}{5},2\right)$

Example:-9 Prove that the three points P,Q,R whose co-ordinates are respectively $(2,5,-4),(1,4,-3),(4,7,-6)$ are collinear, and find the ratio in which the point Q divides PR.

Solution:- The co-ordinates of the point which divides the point $P(2,5,-4)$ and $R(4,7,-6)$ in the ratio $\lambda:1$ are

$$\left(\frac{4\lambda+2}{\lambda+1},\frac{7\lambda+5}{\lambda+1},\frac{-6\lambda-4}{\lambda+1}\right)$$

If the point P,Q,R have to be collinear, then the given co-ordinates of Q must coincide with these co-ordinates, respectively, for a certain value of λ. That is the equations.

$$\frac{4\lambda+2}{\lambda+1}=1,\quad \frac{7\lambda+5}{\lambda+1}=4,\quad \frac{-6\lambda-4}{\lambda+1}=-3$$

Must be satisfied simultaneously for the same value of λ.
Solving the first of these we obtain

$4\lambda+2=\lambda+1$, i.e. $3\lambda=-1$, i.e. $\lambda=-\frac{1}{3}$

Now we see that the remaining two equations in (1) are also satisfied for $\lambda = -\frac{1}{3}$.

Hence the points P, Q, R are collinear. Also Q divides PR in the ratio $-\frac{1}{3}:1$, i.e. $-1, 3$

Remark:-10 The mere collinearity of P,Q,R can be proved by showing that $PQ + PR = QR$ which means that the point P lies on the segment of the line QR.

Example:- Find the equation of the locus of the points which are equidistant from the points $A(1,2,3)$ and $B(3,2,1)$.

Solution:- Let (x, y, z) be the co-ordinates of the point P which is equidistant from A and B. Thus

$PA = PB$

i.e. $\sqrt{\left\{(x-1)^2 + (y-2)^2 + (z-3)^2\right\}} = \sqrt{\left\{(x-3)^2 + (y-2)^2 + (z-1)^2\right\}}$

Squaring both the sides and then simplifying, this equation reduces to

$$x - z = 0$$

Which is the required equation of the locus.

Example:-11 If A, B be the points $(1,2,3)$ and $(4,5,6)$ respectively, find the locus of a variable point P such that:

$$4PA^2 - 3PB^2 = 10$$

Solution:- Let the co-ordinates of P be (x, y, z). Then the relation

$$4PA^2 - 3PB^2 = 10$$

Gives $4\left\{(x-1)^2 + (y-2)^2 + (z-3)^2\right\} = 3\left\{(x-4^2) + (y-5)^2 + (z-6)^2\right\} = 10$

i.e. $4\left\{\left(x^2 - 2x + 1\right) + \left(y^2 - 4y + 4\right) + \left(z^2 - 6z + 9\right)\right\}$

$-3\left\{\left(x^2 - 8x + 16\right) + \left(y^2 - 10y + 25\right) + \left(z^2 - 12z + 36\right)\right\} = 10$

i.e. $x^2 + y^2 + z^2 + 16x + 14y + 12z - 175 = 0$

This is the required locus of the point P.

Previous years questions IAS/IFoS(2008-2023)
CHAPTER 1. 2D GEOMETRY
Note: less probability in CSE exam.

Q1. If the straight lines, joining the origin to the points of intersection of the curve $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$ and the straight line $2x + 3y + k = 0$, are at right angles, then show that $6k^2 + 5k + 52 = 0$. **[1e 2020 IFoS]**

Q2. If the coordinates of the points A and B are respectively $(b\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ and if the line joining A and B is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then show that $x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0$. **[1e 2019 IFoS]**

Q3. If the point $(2,3)$ is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord. **[1d 2016 IFoS]**

Q4. A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

[(2b) 2016 IFoS]

Q5. Find the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$.

[(4c) 2015 IFoS]

Q6. The tangent at $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.

[(1e) 2015 IFoS]

Q7. Prove that two of the straight lines represented by the equation
$x^3 + bx^2y + cxy^2 + y^3 = 0$ will be at right angles, if $b + c = -2$. **[(1e) UPSC CSE 2012].**

Hint: We can attempt this question by properties of orthogonal trajectories which we learnt in the topic ODE.

differentiatingw.r.tx

$$3x^2 + b(2xy + x^2dy/dx) + c(y^2 + 2xydy/dx) + 3y^2dy/dx = 0$$

$replacing : dy/dx >> -1/(dy/dx)$

Q8. Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord. **[(4c) 2011 IFoS]**

2.Projections and Direction Cosines

2.1 Projection:- In this section we employ the familiar concept of 'projection' in different useful situations in three dimensional co-ordinates geometry.

Projection of a point on a Line:- The foot of the perpendicular from a point P on a given line AB is called the projection of the point P on the line AB. In figure 2.1 (a) below, the point Q is the projection of P on AB.

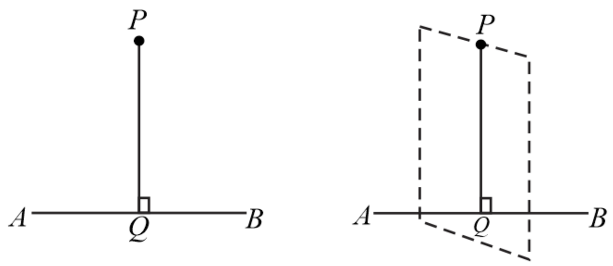


FIGURE 2.1 (a) FIGURE 2.1 (b)

Evidently, Q is the point where the line AB meets a plane through P and perpendicular to AB (see figure 2.1 (b)).

Projection of a Line Segment on a Given Line:- The projection of a line segment on another line is the length intercepted between the projections of its extremities on the line. In Figure 2.2 (a) below, the projection of the line segment AB on the line CD is $A'B'$.

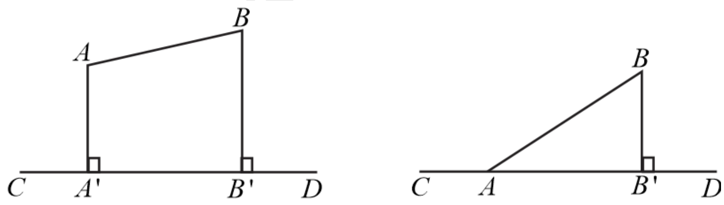


FIGURE 2.2 (a) FIGURE 2. (b)

In particular, if A lies on the line CD, then the projection of AB on CD is equal to AB' (see figure 2.2 (b))

We now obtain two useful results concerning projections.

Result:- To show that $A'B' = AB\cos\theta$, where θ is the angle between AB and CD

Proof:- Since $A'$and B' are the projection of A and B respectively on the line, CD, we can say that A' and B' are the points in which the planes through A and B perpendicular to the line CD meet it.

Through A draw a line AP parallel to CD to meet the plane through B at P. Then AP is perpendicular on the plane drawn through B, and $AP = A'B'$. Join B with P. since BP lies on the plane through B, AP is perpendicular to BP'.

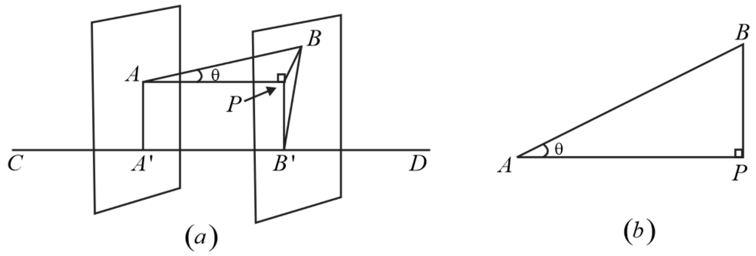
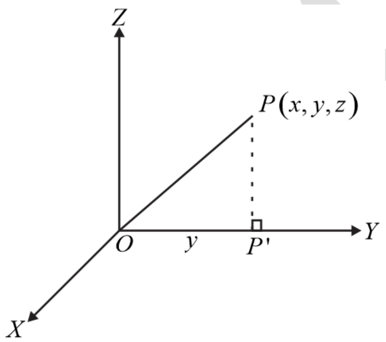


FIGURE 2.3

Let θ be the angle between AB and CD. Since AP is parallel to CD, the angle between AB and AP is also equal to θ, i.e. $\angle BAP = \theta$.
Now from the right-angled triangle APB, we find that $AP = AB\cos\theta$ i.e. $A'B' = AB\cos\theta$, since $AP = A'B'$.

Projection of OP on the Co-ordinates Axes:- Let $P(x, y, z)$ be any point .Then

Projection of OP on the y-axis $= OP' = y$
Similarly,



projection of OP on the x-axis $= x$, and projection of OP on the z-axis $= z$,
Thus the co-ordinates of any point P are the projections of OP on the respective co-ordinates axes.

2.2 Direction Cosines:- The direction of a straight line in space can be determined when the angles made by it with the co-ordinates axes are known. These angles are called the direction angles of the line. A direction angle may have any value from 0° to 180° inclusive. In case the line is oppositely directed, its direction angles are the respective supplementary angles.
The cosines of direction angles play significant role in the subject. We define:

Direction Cosines:- If α, β, γ be the angles which any line makes with the positive directions of the axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line and are abbreviated as $d.c.'s$
In general direction cosines of a line are denoted by l, m, n. Thus $l = \cos\theta$, $m = \cos\beta$, $n = \cos\gamma$.
The following facts are obvious to understand:
(1) The direction cosines of a given line are three uniquely determined numbers.
(2) Each of the direction cosines of line lies between -1 and 1.

(3) Two or more parallel lines have the same three direction cosines.

Direction Cosines of the Co-ordinates Axes:- We know that the angles made by the x-axis with the co-ordinates axes are $0^\circ, 90^\circ, 90^\circ$. Therefore the direction cosines of the x-axis are $\cos 0^\circ, \cos 90^\circ, \cos 90^\circ$, i.e. $1, 0, 0$
Similarly, the direction cosines of the y-axis are $0,1,0$ and those of the z-axis $0,0,1$.

2.3 Relation Between the Direction Cosines:- To show that if l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

Another Statements:- If a line makes angles α, β, γ with the co-ordinates axes, then $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
Let AB be the given lines which makes angles a, β, γ with the co-ordinates axes. Draw a line OP parallel to AB. Let the co-ordinates of the point P be (x, y, z) and the measure of OP be r.
Draw PN perpendicular to the $xy-$plane and NM perpendicular to the x-axis. Then $OM = x$, $MN = y$, $NP = z$.
Since OM is the projection of OP on OX, we have $OM = OP \cos \alpha$, i.e. $x = r \cos \alpha$.

Thus $\cos \alpha = \frac{x}{r}$.

Similarly, $\cos \beta = \frac{y}{r}$ and $\cos \gamma = \frac{z}{r}$.

So $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{x^2 + y^2 + z^2}{r^2} = 1$, since $r = \sqrt{(x^2 + y^2 + z^2)}$

Hence $l^2 + m^2 + n^2 = 1$.

Corollary:- If a line makes angles α, β, γ with the co-ordinates axes, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$
Proof:- we have $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = (1 - \cos^2 \alpha) + (1 - \cos^2 \beta) + (1 - \cos^2 \gamma)$
$= 3 - (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$
$= 3 - 1$, since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
$= 2$.

2.4 Direction Ratios (OR direction Numbers)
If l, m, n are the direction cosines of a line, then any three numbers which are in the same proportion as l, m, n are called direction ratios (or direction numbers) of that line. These are abbreviated as $d.r.'s$.
In general, direction ration of a line are denoted by a, b, c.

It is clear that a given line will have infinitely many different triples of direction rations. For example, if $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0$ are the direction cosines of a line, then its direction ratios are:
$\sqrt{2},\sqrt{2},0$: or $1,1,0$: or $2,2,0$ etc.

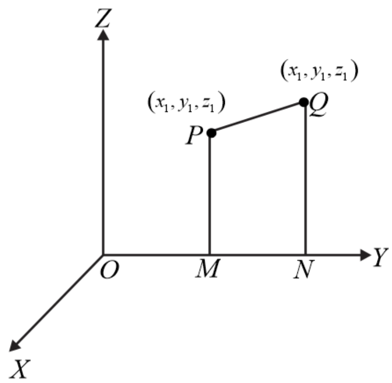
To find direction cosines when direction ratios are known:- Let a,b,c be direction ratios of a line. If l,m,n denote the direction cosines of that line, we must have $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$

$=\pm\frac{\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{\left(a^2+b^2+c^2\right)}}$, using a result of algebra $=\pm\frac{1}{\sqrt{\left(a^2+b^2+c^2\right)}}$ since $l^2+m^2+n^2=1$ whence $l=\pm\frac{a}{\sqrt{\left(a^2+b^2+c^2\right)}}$, $m=\pm\frac{b}{\sqrt{\left(a^2+b^2+c^2\right)}}$, $n=\pm\frac{c}{\sqrt{\left(a^2+b^2+c^2\right)}}$.

These relations can be used to evaluated $d.c.'s$ of a lines when it's $d.c.'s$ are known.

2.5 Direction cosines of the join of two given points:- To find the direction cosines of the line joining the two points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$.

Let l,m,n be the direction cosines of then join PQ and α,β,γ be the angles made by it with the co-ordinates axes. Let $PQ=r$.



Draw perpendicular PM and QN on the y-axis. Then

$MN=ON-OM=y_2-y_1$ (1)

Also, $MN=$ projection of PQ on the y-axis

$=PQ\cos\beta=r\cos\beta$

$=rm$, since $=r\cos\beta$ (2)

Equating the value of MN obtained in(1) and (2), we have $y_2-y_1=rm$, which given $m=\frac{y_2-y_1}{r}$.

Similarly, we can show t hat $l = \frac{x_2 - x_1}{r}$ and $n = \frac{z_2 - z_1}{r}$.

Hence the direction cosines of the join PQ are $\frac{x_2 - x_1}{r}, \frac{y_2 - y_2}{r}, \frac{z_2 - z_1}{r}$ where $r = \sqrt{\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 + z_1)^2\right\}}$

Corollary:- Direction ratios of the line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

This corollary follows at once on multiplying each of the $d.c.'s$ of PQ obtained above by r .

2.6 Projection of the join of two points on a line:- To show that the projection of the join of the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line having l, m, n as it $d.c.'s$ us $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.

Complete the parallelepiped taking PQ as its one diagonal and its faces parallel to the co-ordinates planes. Let CD be the given line whose $d.c.'s$ are l, m, n .

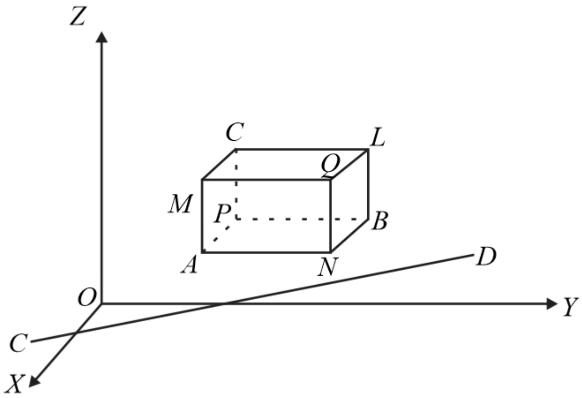
We consider the broken line $PANQ$.

From the figure, it is clear that

$PA = x_2 - x_1$

$AN = PB = y_2 - y_1$

And $NQ = z_2 - z_1$



Clearly, the projection of the join PQ on the given line CD should be equal to the sum of the perpendicular of PA, AN and NQ on CD. (by Result 2 of section 2.1)

Let α, β, γ be the angles made by CD with the co-ordinates axes. Since PA, NA and NQ are parallel to the $x-, y-$ and $z-$axes, the line CD will makes the same angles α, β, γ with PA, AN and NQ, respectively. So

Projection of PA on $CD = PA \cos\alpha = (x_2 - x_1)l$,

Projection of AN on $CD = AB \cos\beta = (y_2 - y_1)m$,

And Projection of NQ on $CD = NQ \cos\gamma = (z_2 - z_1)n$.

Adding these equations we find that the projection of the join PQ on the line CD is equal to $(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$.

Corollary:- The projection of OP, where P is (x',y',z') on a line having l,m,n as its direction cosines is $lx'+my'+nz'$.

2.7 Lagrange's Identity:- If l,m,n and l',m',n' are any real numbers, then we can prove the following by simple multiplication.

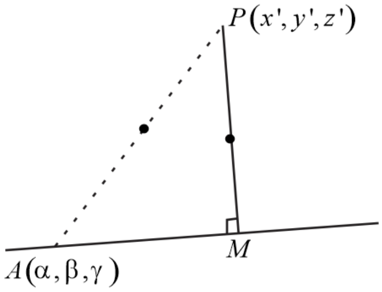
$$(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$$
$$=(mn'-nm')^2+(nl'-\mathrm{ln}')^2+(lm'-ml')^2.$$

It is known as Lagrange's identity and may be written in sigma notation $(\Sigma l^2)(\Sigma l'^2)-(\Sigma ll')^2=\Sigma(mn'-nm')^2$.

We shall occasionally use Lagrange's identity in our further work in this book.

2.8 Distance of a point from a line:- To find the distance of a point $P(x',y'z')$ from a line passing through the point $A(\alpha,\beta,\gamma)$ and having the direction cosines l,m,n.

Let M be the foot of perpendicular from P on the given line. Join A with P. Then $AM=$projection of AP on the given line $(x'-\alpha)l+(y'-\beta)m+(z'-\gamma)n$.



Now in the right-angled triangle AMP, we have $PM^2=AP^2-AM^2$

$$=\{(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2\}-\{(x'-\alpha)l+(y'-\beta)m+(z'-\gamma)n\}^2$$
$$=\{(x'-\alpha)^2+(y'-\beta)^2+(z'-\gamma)^2\}(l^2+m^2+n^2)$$
$$-\{(x'-\alpha)l+(y'-\beta)m+(z'-\gamma)n\}^2 \text{ since } l^2+m^2+n^2=1$$
$$=\{(y'-\beta)n+(z'-\gamma)m\}^2+\{(z'-y)l+(x'-\alpha)n\}^2+\{(x'-\alpha)m+(y'-\beta)l\}^2$$

Using Lagrange's identity.

$$=\begin{vmatrix} y'-\beta & z'-\gamma \\ m & n \end{vmatrix}^2+\begin{vmatrix} z'-\gamma & x'-\alpha \\ n & m \end{vmatrix}^2+\begin{vmatrix} x'-\alpha & y'-\beta \\ l & m \end{vmatrix}^2$$

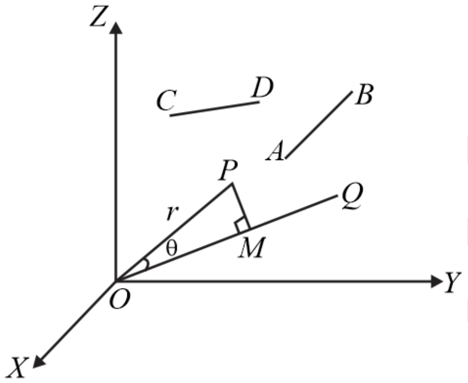
, writing the terms in determinant form.

Taking square root we obtain the desired distance PM.

Corollary:- The distance of a point $P(x', y', z')$ from a line passing through the origin and having the direction cosines l, m, n is $\sqrt{\{(ny'-mz')^2+(lz'-nx')^2+(mx'-ly')^2\}}$.

2.9 Angle between two lines:- To find the angle between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2.

Let AB and CD be the given liens whose direction cosines are, respectively, l_1, m_1, n_1 and l_2, m_2, n_2. Let θ be the angle between these lines.



Draw the lines OP and OQ parallel to AB and CD respectively. Let the co-ordinates of P be (x, y, z) and put $OP = r$. Draw the perpendicular PM from P to the line OQ.

Now in the right-angled triangle OMP, we have $\cos\theta = \dfrac{OM}{OP} = \dfrac{projection\, of\; OP\, on\, OQ}{OP}$.

$= \dfrac{l_2 x + m_2 y + n_2 z}{r}$ (using the Corollary of Section 2.7)

$= \left(\dfrac{x}{r}\right) l_2 + \left(\dfrac{y}{r}\right) m_2 + \left(\dfrac{z}{r}\right) z_2$. i.e. $\boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}$ since $x/r = l_1, y/r = m_1, z/r = n_1$.

This formula gives the required value of θ.

Corollary 1:- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines, then the angle θ between them is given by

$$\sin\theta = \sqrt{\{(m_1 n_1 - n_1 m_2)^2 + (n_2 l_2 - l_1 n_2)^2 + (l_1 m_2 - m_1 l_2)^2\}}$$

i.e. $\tan\theta = \dfrac{\sqrt{\{(m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - m_1 l_2)^2\}}}{l_1 l_2 + m_1 m_2 + n_1 n_2}$

Proof:- We have $\sin\theta = \sqrt{(1-\cos^2\theta)} = \sqrt{(1.1 - \cos^2\theta)}$ **(Note)**

$$=\sqrt{\left\{\left(l_1^2+m_1^2+n_1^2\right)\left(l_2^2+m_2^2+n_2^2\right)-\left(l_1l_2+m_1m_2+n_1n_2\right)^2\right\}}$$

$$=\sqrt{\left\{\left(m_1n_2-n_1m_2\right)^2+\left(n_1l_2-l_1n_2\right)^2+\left(l_1m_2-m_1l_2\right)^2\right\}}$$

Using Lagrange's identity.

Further $\tan=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sqrt{\left\{\left(m_1n_2-n_1m_2\right)^2+\left(n_1l_2-l_1n_2\right)^2+\left(l_1m_2-m_1l_2\right)^2\right\}}}{l_1l_2+m_1m_2+n_1n_2}$

Corollary 2:- The two lines whose direction cosines are l_1,m_1,n_1 and l_2,m_2,n_2 are perpendicular if and only if $l_1l_2+m_1m_2+n_1n_2=0$.

Proof:- We know that the angle θ between the given lines its expressed by $\cos\theta=l_1l_2+m_1m_2+n_1n_2$.

If the liens are perpendicular, we have $\theta=90^\circ$. Putting this value of θ, we obtain the desired result.

$\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$.

However, this test seems to the slightly unrefined in view of the following obvious fact.

'The two lines having $d.c.'s$ l_1,m_1,n_1 and l_2,m_2,n_2 are parallel if $l_1=l_2,\ m_1=m_2,n_1=n_2$.

Corollary 3:- If a_1,b_1,c_1 and a_2,b_2,c_2 be direction ratios of two lines, then the angle θ between them is given by $\cos\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}.\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$.

This result follows by putting $l_1=\pm\dfrac{a_1}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}}$ etc. and $l_2=\pm\dfrac{a_2}{\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$ in the formula $\cos\theta=l_1l_2+m_1m_2+n_1n_2$.

Further, it can be easily shown that the two liens are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$ and parallel if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$.

Since direction rations of al lines may be regarded as its direction ratios, we have the following result.

Corollary 4:- If a,b,c are direction cosines of a line and a,b,c direction ratios of another, then the two liens are perpendicular if $al+bm+cn=0$.

2.11 Relation Between the Direction Cosines of Three Mutually Perpendicular Lines:-

Let $l_1,m_1,n_1;l_2,m_2,n_2:l_3,m_3,n_3$ be the direction consider of three mutually perpendicular lies referred to OX,OY,OZ as the axes. Then

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_3^2+m_3^2+n_3^2=1 \end{array}\right\} \qquad \text{(a)}$$

And
$$\left.\begin{array}{l} l_1l_2+m_1m_2+n_1n_2=0 \\ l_2l_3+m_2m_3+n_2n_3=0 \\ l_3l_1+m_3m_1+n_3n_1=0 \end{array}\right\} \qquad \text{(b)}$$

Now l_1,l_2,l_3; m_1,m_2,m_3; n_1,n_2,n_3 will be the direction cosines of OX,OY,OZ referred to three mutually perpendicular given lines as the axes. Therefore

$$\left.\begin{array}{r} l_1^2+l_2^2+l_3^2=1 \\ m_1^2+m_2^2+m_3^2=1 \\ n_1^2+n_2^2+n_3^2=1 \end{array}\right\} \qquad \text{(c)}$$

And
$$\left.\begin{array}{r} l_1m_1+l_2m_2+l_3m_3=0 \\ m_1n_1+m_2n_2+m_3n_3=0 \\ n_1l_1+n_2l_2+n_3l_3=0 \end{array}\right\} \qquad \text{(d)}$$

Again if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=\delta$ then $\delta^2=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\times\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

$$=\begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_3l_1+m_3m_1+n_3n_1 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

i.e. $\delta^2=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=1$

Therefore, $\delta=\pm 1$ i.e. $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=\pm 1$

Example1:- A line which makes angles. $45^\circ, 60^\circ, 60^\circ$ with the positive direction of x-axis, y-axis respectively, then find the direction cosines of the line.

Solution:- Since the line makes angles $45^\circ, 60^\circ$ and 60° with the positive directions of the $x-, y-$ and $z-$axes, its direction cosines are

$$\cos 45^\circ,\ \cos 60^\circ, \cos 60^\circ$$

i.e. $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$

Example2:- Find the direction cosines of the line which is equally inclined to the positive direction of the axes.

Solution:- Let l, m, n be the direction cosines of the line. Since the line is equally inclined to the positive direction of the axes, we have

$l = m = n$ and $l, m, n > 0$

But $l^2 + m^2 + n^2 = 1$ is always true. Putting $m = l$ and $n = l$, this gives,

$l^2 + l^2 + l^2 = 1$, i.e. $3l^2 = 1$, i.e. $l = 1/\sqrt{3}$, neglecting the negative sign.

Hence the required direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Example3:- Find the direction cosines of a line whose direction ratios are given to be $1, 2, 3$

Solution:- Let l, m, n be the $d.c.'s$ of the line under consideration. Here $a = 1$, $b = 2$, and $c = 3$. Therefore

$$l = \pm \frac{a}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{1}{(1^2 + 2^2 + 3^2)} = \pm \frac{1}{\sqrt{(14)}}$$

$$m = \pm \frac{b}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{2}{\sqrt{(14)}}$$

And $n = \pm \frac{c}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{3}{\sqrt{(14)}}$

Hence the required $d.c.'s$ of the line are $\pm \frac{1}{\sqrt{(14)}}, \pm \frac{2}{\sqrt{(14)}}, \pm \frac{3}{\sqrt{(14)}}$

Example4:- Find $d.c.'s$ l, m, n of two lines which are connected by the relations

$l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$

Solution:- Given: $l - 5m + 3n = 0$

And $7l^2 + 5m^2 - 3n^2 = 0$

Substituting the value of l from (1) in (2), we get

$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$ i.e. $180m^2 - 210mn + 60n^2 = 0$

i.e. $6m^2 - 7mn + 2n^2 = 0$, i.e. $(3m - 2n)(2m - n) = 0$

This gives $3m - 2n = 0,\ 2m - n = 0$

i.e. $\frac{m}{2}=\frac{n}{3}, \quad \frac{m}{1}=\frac{n}{2}$

Case I:- When $\frac{m}{2}=\frac{n}{3}$ using (1) we have

$$\frac{m}{2}=\frac{n}{3}=\frac{5m-3n}{5.2-3.3}=\frac{l}{1}$$

i.e. $\frac{l}{1}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{\left(1^2+2^2+3^3\right)}}=\frac{1}{\sqrt{(14)}}$

This gives $l=1/\sqrt{14}$, $m=2/\sqrt{14}$, $n=3/\sqrt{14}$

Case II:- When $\frac{m}{1}=\frac{n}{2}$, again using(1) we have

$$\frac{m}{1}=\frac{n}{2}=\frac{5m-3n}{5.1-3.2}=\frac{l}{-1}$$

i.e. $\frac{l}{-1}=\frac{m}{1}=\frac{n}{2}=\frac{\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{\left\{(-1)^2+1^2+2^2\right\}}}=\frac{1}{\sqrt{6}}$

This gives $l=-1/\sqrt{6}$, $m=1/\sqrt{6}$, $n=2/\sqrt{6}$

Example5:- Find the direction cosines of the line joining the points $(-2,1,-8)$ and $(4,3,-5)$

Solution:- Let r be the distance between the given points. Then

$$r=\sqrt{\left[\left\{4-(-2)^2\right\}+(3-1)^2+\left\{-5-(8)\right\}^2\right]}=\sqrt{(36+4+9)}=\sqrt{(49)}=7$$

Hence the direction cosines of the line joining the given points are

$$\frac{4-(-2)}{7},\frac{3-1}{7},\frac{-5-(-8)}{7} \text{ i.e. } \frac{6}{7},\frac{2}{7},\frac{3}{7}$$

Example6:- Show that the equation to the right circular cone whose vertex is the origin, whose axis has direction cosines l,m,n and whose semi-vertical angle is α is

$$(ny-mz)^2(lz-nx)^2+(mx-ly)^2=\left(x^2+y^2+z^2\right)\sin^2\alpha$$

Example7:- If the direction cosines of two lines are relation by equations $l+2m+3n=0$ and $mn-4nl+3lm=0$, then find the direction cosines of these. Also find angle between these two lines.

Solution:- We are given that

$l+2m+3n=0$ (1)

And $mn-4nl+3lm=0$ (2)

i.e. $mn-l(4n-3m)=0$ (2')

By putting $l=-(2m+3n)$ from (1) into (2'), we get

$mn+(2m+3n)(4n-3m)=0$, i.e. $-6m^2+12n^2=0$

i.e. $m^2-2n^2=0$, i.e. $\left(m+\sqrt{2}n\right)\left(m-\sqrt{2}\,n\right)=0$

This gives $m=\sqrt{2}\,n=0,\ m-\sqrt{2}\,n=0$,

I.e. $\dfrac{m}{\sqrt{2}}=\dfrac{n}{-1},\ \dfrac{m}{\sqrt{2}}=\dfrac{n}{1}$

Case I. When $\dfrac{m}{\sqrt{2}}=\dfrac{n}{-1}$, using (1) we have

$$\frac{m}{\sqrt{2}}=\frac{n}{-1}=\frac{-2m-3n}{-2\sqrt{2}+3}=\frac{l}{3-2\sqrt{2}}$$

i.e. $\dfrac{l}{3-2\sqrt{2}}=\dfrac{m}{\sqrt{2}}=\dfrac{n}{-1}$

$$=\frac{\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{3\left\{\left(3-2\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2+\left(-1\right)^2\right\}}}=\frac{1}{\sqrt{20-12\sqrt{2}}}$$

This gives $l=\dfrac{3-2\sqrt{2}}{20-12\sqrt{2}},\ m=\dfrac{\sqrt{2}}{20-12\sqrt{2}},\ n=\dfrac{-1}{20-12\sqrt{2}}$ (3)

Case II. When $\dfrac{m}{\sqrt{2}}=\dfrac{n}{1}$, using (1) we have

$$\frac{m}{\sqrt{2}}=\frac{n}{1}=\frac{-2m-3n}{-2\sqrt{2}-3}=\frac{l}{-\left(3+2\sqrt{2}\right)}$$

i.e. $\dfrac{l}{-\left(3+2\sqrt{2}\right)}=\dfrac{m}{\sqrt{2}}=\dfrac{n}{1}$

$$=\frac{\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{\left\{\left(3+2\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2+1^2\right\}}}=\frac{1}{\sqrt{20+12\sqrt{2}}}$$

This gives $l=\dfrac{-\left(3+2\sqrt{2}\right)}{\sqrt{20+12\sqrt{2}}},m=\dfrac{\sqrt{2}}{\sqrt{20+12\sqrt{2}}},n=\dfrac{1}{\sqrt{20+12\sqrt{2}}}$ (4)

Thus the direction cosines of the two lines are given by (3) and (4)

Further, if θ be the angle between these lines, then

$\cos\theta=l_1l_2+m_1m_2+n_1n_2$

$$=\frac{\left(3-2\sqrt{2}\right)\left\{-\left(3+2\sqrt{2}\right)\right\}+\sqrt{2}.\sqrt{2}+(-1).1}{\left(20-12\sqrt{2}\right)\left(20+12\sqrt{2}\right)}=\frac{-1+2-1}{400-288}=0$$

This gives $\theta=\dfrac{1}{2}\pi$, showing that the two lines are perpendicular.

Example8:- A variable line in two adjacent positions has direction cosines l,m,n and $l+\delta l,m+\delta n,n+\delta n$. Show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2+\delta l^2+\delta m^2+\delta n^2$$

Solution:- We have

$\cos\theta = l(l+\delta l) + m(m+\delta m) + n(n+\delta n)$

$= (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$

$= 1 + (l\,\delta l + m\,\delta m + n\,\delta n)$, since $l^2 + m^2 + n^2 = 1$

i.e. $1 - \cos\delta\theta = -(l\,\delta l + m\,\delta m + n\,\delta n)$,

i.e. $2\sin^2\frac{1}{2}\delta\theta = -(l\,\delta l + m\,\delta m + n\,\delta n)$,

i.e. $2\left(\frac{1}{2}\delta\theta\right)^2 = -(l\,\delta l + m\,\delta m + n\,\delta n)$, since $\sin\alpha = \alpha$ when α is very small

i.e. $\delta\theta^2 = -2(l\,\delta l + m\,\delta m + n\,\delta n)$ (1)

Further, $l^2 + m^2 + n^2 = 1$ (2)

And $(l+\delta l)^2 + (m+\delta m)^2 + (n+\delta n)^2 = 1$ (3)

Subtracting (2) from (3), we obtain

$(\delta l + 2l\,\delta l) + (\delta m^2 + 2m\,\delta m) + (\delta n^2 + 2n\,\delta n) = 0$

i.e. $\delta l^2 + \delta m^2 + \delta n^2 = -2(l\,\delta l + m\,\delta m + n\,\delta n)$

Since the right hand sides of (1) and (4) are identical, their left hand sides must also be equal. Thus

$$\delta\theta^2 + \delta l^2 + \delta m^2 + \delta n^2$$

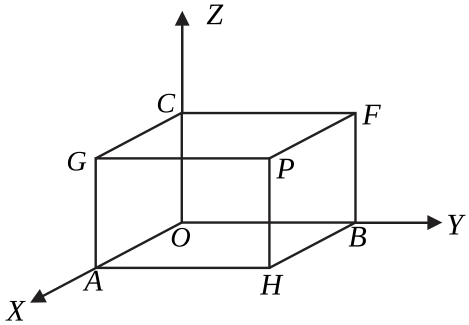
Example9:- Show that the angles between the four diagonals of a rectangular parallelopiped whose edges are a,b,c are given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$$

OR

If the edges of a rectangular parallelopiped are a,b,c, find the angle between any two diagonals of the rectangular parallelepiped.

Solution:- In a rectangular parallelopiped as shown in the adjacent figure, let



$OA = a,\ OB = b,\ OC = c$

Then the co-ordinates of the vertices are given by

$O(0,0,0)$

$A(a,0,0), B(0,b,0), C(0,0,c)$

$F(0,b,c), G(a,0,c), H(a,b,0)$

$P(a,b,c)$

The diagonals are OP, AF, BG and CH. Clearly

Direction rations of OP are a,b,c;

Direction rations of AF are $-a,b,c$;

Direction rations of BG are $a,-b,c$;

Direction rations of CH are $a,b,-c$

Here 2 diagonals out 4 can be selected in ${}^4C_2$, i.e. 6 ways

Now the angle between OP and AF

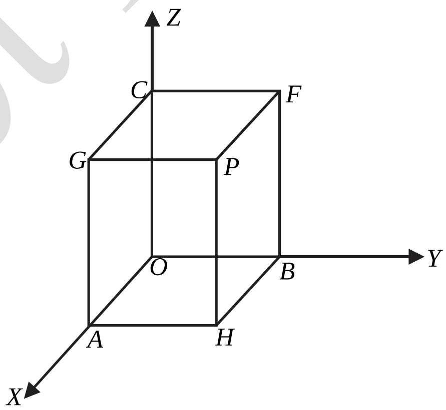
$$= \cos^{-1}\left\{\frac{a(-a)+b.b+c.c}{\sqrt{(a^2+b^2+c^2)}.\sqrt{\{(-a)^2+b^2+c^2\}}}\right\} = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right)$$

Similarly, we can find out the angles between other pairs of diagonals.
These angles can be combined to get the required form.

Example10:- A line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution:- Let the side of the cube be a. Take one vertex of the cube as the origin and three edges through this vertex as the co-ordinate axes (see adjacent figure). Then the co-ordinates of the vertices are.



$O(0,0,0)$

$A(a,0,0), B(0,a,0), C(0,0,a)$

$F(0,a,a), G(a,0,a), H(a,a,0)$

$P(a,b,c)$

Whence direction ratios of the diagonals OP, AF, BG, CH are

$a,a,a;\ -a,a,a;\ a,-a,a;\ ,a,a,-a$ respectively.

Since the sum of squares of direction ratios in each of these four cases is $3a^2$, dividing the above direction ratios by the square root of $3a^2$, i.e. $a\sqrt{3}$, the direction cosines of OP, AF, BG, CH are

$\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};\ -\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};\ \frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};\ \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}$ respectively.

If l,m,n be the $d.c.'s$ of the line which makes angles $\alpha,\beta,\gamma,\delta$ with the diagonals OP, AF, BG, CH, then we find that

$$\cos\alpha = l,\ \frac{1}{\sqrt{3}} + m\ .\frac{1}{\sqrt{3}} + n.\frac{1}{\sqrt{3}} \text{ i.e. } \cos\alpha\frac{1+m+n}{\sqrt{3}}$$

Similarly,

$$\cos\beta = \frac{-l+m+n}{\sqrt{3}},\ \cos\gamma\frac{l-m+n}{\sqrt{3}}, \cos\delta = \frac{l+m-n}{\sqrt{3}}$$

Squaring and adding these equations we find that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$

$$= \frac{(l+m+n)^2+(-l+m+n)^2+(l-m+n)^2+(l+m-n)^2}{3}$$

$$= \frac{4(l^2+m^2+n^2)}{3}, \text{ on simplifying the numerator}$$

$$= \frac{4}{3}, \text{ since } l^2+m^2+n^2 = 1.$$

Example11:- Prove that the three lines drawn from O with direction numbers $2,1,5;\ 2,-1,1;\ 6,-4,1$ are coplanar.

Solution:- Let l,m,n be the direction cosines of the normal to the plane containing the first two lines. Then using the condition of perpendicularity, we have

$$2l+m+5n=0 \qquad (1)$$

And $\quad 2l-m-n=0 \qquad (2)$

Solving these equations, we obtain

$$\frac{l}{1.1-5(-1)} = \frac{m}{5.2-2.1} = \frac{n}{2(-1)-1.2}, \text{ i.e. } \frac{l}{6}=\frac{m}{8}=\frac{n}{-4}$$

i.e. $\quad \frac{l}{3}=\frac{m}{4}=\frac{n}{-2}$

This gives $3,4,-2$ as direction number of the normal.

Since all the three lines pass through a common point (here O), the third line will lie on the plane containing the first two if it satisfies the condition of perpendicularity with the normal considered above. For this using direction numbers of the third line with those of the above normal, we have.

$$6.3+(-4).4+1(-2)=18-16-2=0$$

Hence the given three lines are coplanar.

Example12:- Find the point in which the perpendicular from the origin on the line joining the points $(-9,4,5)$ and $(11,0,-1)$ meets it.

Solution:- Any point P on the line joining the points $A(-9,4,5)$ and $B(11,0,-1)$ is

$$\left(\frac{11\lambda-9}{\lambda+1},\frac{4}{\lambda+1},\frac{-\lambda+5}{\lambda+1}\right)$$

If this the foot of the perpendicular from the origin on the line AB, the lines OP and AB are at right angles. Now the direction ratios of the line OP are
$11\lambda-9,\ 4,\ -\lambda+5$

And those of AB are $11-(-9),\ 0-4,\ -1-5,$ i.e. $20,-4,-6$

Therefore we have

$20(11\lambda-9)-4,\ 4-6(-\lambda+5)=0$, i.e. $226\lambda=226$

Which gives.

Putting this value of λ in (1), the co-ordinates of P are $(1,2,2)$

Example13:- Prove that if two pairs of opposite edges of a tetrahedron are perpendicular, then the third pair are also perpendicular.

Solution:- Let one vertex of the tetrahedron be the origin O and the other three vertices be the points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$. Then the direction ratios of various edges of the tetrahedron are shown in the following table

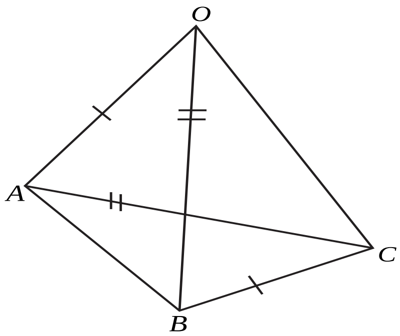
Edge	Direction Ratio
OA	x_1, y_1, z_1
OB	x_2, y_2, z_2
OC	x_3, y_3, z_3
AB	$x_2-x_1, y_2-y_1, z_2-z_1$
BC	$x_3-x_2, y_3-y_2, z_3-z_2$
CA	$x_1-x_3, y_1-y_3, z_1-z_3$

Let the edges OA and OB be perpendicular to their opposite edges BC and CA respectively. Then by using the condition of perpendicularity, we have

$x_1(x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)=0$ (1)

And $x_2(x_1-x_3)+y_3(y_1-y_2)+z_2(z_1-z_3)=0$ (2)

On adding (1) and (2) column wise, we get



$x_3(x_1-x_2)+y_3(y_1-y_2)+z_3(z_1-z_2)=0$

i.e. $x_3(x_2-x_1)+y_3(y_2-y_1)+z_3(z_2-z_1)=0$

This shows that the opposite edges OC and AB are also perpendicular to each other. Hence the result.

Example14:- If l_1,m_1,n_1 ; l_2,m_2,n_2 ; l_3,m_3,n_3 are the direction cosines of three mutually perpendicular lines, show that the line whose direction cosines and proportional to $l_1+l_2+l_3$; $m_1+m_2+m_3$; $n_1+n_2+n_3$, makes equal angle with them.

Solution:- Let $\theta_1,\theta_2,\theta_3$ be the angles which the three mutually perpendicular lines, whose $d.c.'s$ are l_1,m_1,n_1; l_2,m_2,n_2; l_3,m_3,n_3, make with the line whose direction ratios are

$$l_1+l_2+l_3,\ m_1+m_2+m_3,\ n_1+n_2+n_3.$$

Further, let

$$\sqrt{\left\{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2\right\}}=k$$

So that the $d.c.'s$ of the fourth line are

$$\frac{l_1+l_2+l_3}{k},\ \frac{m_1+m_2+m_3}{k},\ \frac{n_1+n_2+n_3}{k}$$

Then we have

$$\cos\theta_1=l_1\left(\frac{l_1+l_2+l_3}{k}\right)+m_1\left(\frac{m_1+m_2+m_3}{k}\right)+n_1\left(\frac{n_1+n_2+n_3}{k}\right)$$

$$=\frac{l_1^2+m_1^2+n_1^2}{k},\frac{l_1l_2+m_1m_2+n_1n_2}{k}+\frac{l_1l_3+m_1m_3+n_1n_3}{k} \qquad (1)$$

But $l_1^2+m_1^2+n_1^2=1$ as l_1,m_1,n_1 are $d.c.'s$ of a line. Further.

$l_1l_2+m_1m_2+n_1n_2=0$ and $l_1l_3+m_1m_3+n_1n_3=0$ as the three lines are mutually perpendicular. Therefore (1) reduces to

$$\cos\theta_1=1/k$$

Similarly, we can show that

$$\cos\theta_2=1/k \text{ and } \cos\theta_3=1/k$$

From (2) and (3) it follows at once that

$$\theta_1=\theta_2=\theta_3$$

Hence the given three mutually perpendicular lines make equal angle with the further one.

Example15:- Prove that the lines whose direction cosines are given by the equations.

$$al+bm+cn=0. \qquad ul^2+vm^2+wn^2=0$$

Are perpendicular if $a^2(v+w)+b^2(w+u)+c^2(u+v)=0$

And parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$

Solution:- Form the equation: $al+bm+cn=0$, we have

$$n=-\frac{al+bm}{c}$$

Substituting this value of n in the equation: $ul^2+vm^2+wn^2=0$, we obtain.

$$ul^2+vm^2+w\left(-\frac{al+bm}{c}\right)^2=0$$

i.e. $c^2ul^2+c^2vm^2+a^2wl^2+b^2wn^2+2abwlm=0$

i.e. $\left(a^2w+c^2u\right)\left(\frac{1}{m}\right)^2+2abw\left(\frac{1}{m}\right)+\left(b^2w+c^2v\right)=0$

Let l_1,m_1,n_1 and l_2,m_2,n_2 be the direction cosines of the lines. Then l_1/m_1 and l_2/m_2 are the roots of the equation (1). So, the product of roots:

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{b^2w+c^2v}{a^2w+c^2u}$$

Which gives $\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}$ (2)

If in place of n we eliminate l from the given equations in the same manner as above, then by symmetry we shall have

$$\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u} \quad (3)$$

Combining (2) and (3), we obtain

$$\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u}=\lambda_1\text{, say} \quad (4)$$

Now the lines under consideration will be perpendicular if

$$l_1l_2+m_1m_2+n_1n_2=0$$

i.e. $\lambda\left(b_2w+c_2v\right)+\lambda\left(c^2u+a^2w\right)+\lambda\left(a^2v+b^2u\right)=0$ using (4)

i.e. $a^2\left(v+w\right)+b^2\left(w+u\right)+c^2\left(u+v\right)=0$ dividing by λ and rearranging the terms.

Further, the lines will be parallel if

$$\frac{l_1}{l_2}=\frac{m_1}{m_1}(=1),\text{ i.e. }\frac{l_1}{m_1}=\frac{l_2}{m_2}$$

For this the roots of (1) should be identical. This is possible if the discriminate of (1) is zero i.e.

$$4a^2b^2w^2-4\left(a^2w+c^2u\right)\left(b^2w+c^2v\right)=0$$

i.e. $a^2c^2vw+b^2c^2wu+c^4uv=0$,

i.e. $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$, dividing by c^2uvw.

Example16:- Show that the straight lines whose direction cosines are given by the equations.

$$ul+vm+wn=0. \qquad fmn+gnl+hlm=0$$

Are perpendicular if $\frac{f}{u}+\frac{g}{v}+\frac{h}{w}=0$

And parallel if $\pm\sqrt{uf}\pm\sqrt{vg}\pm\sqrt{wh}=0$

Solution:- Given: $ul+vm+wn=0$ (1)

And $fmn+gnl+hlm=0$. (2)

Substituting the value of n from (1) in (2), we get

$$fm\left(-\frac{ul+vm}{w}\right)+gl\left(-\frac{ul+vm}{w}\right)+hlm=0,$$

i.e. $fm\left(ul+vm\right)+gl\left(ul+vm\right)-hlmw=0$, i.e. $gul^2 + (fu + gv - hw)lm + fvm^2 = 0$,

i.e. $gu\left(\frac{l}{m}\right)^2 + (fu + gv - hw)\left(\frac{1}{m}\right) + fv = 0$,

This is a quadratic equation in l/m. If l_1, m_1, n_1 and l_2, m_2, n_2 be the $d.c.'s$ of the lines under consideration, then l_1/m_1 and l_2/m_2 will be the roots of equation (3). So

The product of roots $= \frac{consant\ terms}{coefficient\ of\ (l/m)^2}$

i.e. $\frac{l_1}{m_1}, \frac{l_2}{m_2}, \frac{fv}{gu}$, i.e. $\frac{l_1 l_2}{fvw} = \frac{m_1 m_2}{gwu}$.

By symmetry we now have

$\frac{l_1 l_2}{fvw} = \frac{m_1 m_2}{gwu} = \frac{n_1 n_2}{huv} = k$, say

Hence using the condition of perpendicularity, viz

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

It follows from (4) that the two lines will be perpendicular if

$k(fvw + gwu + huv) = 0$

i.e. $\frac{f}{u} + \frac{g}{v} + \frac{h}{w} = 0$, on dividing by $kuvw$.

Further, the two lines will be parallel if the roots of quadratic equation (3) are equal. This is possible if the discriminate of (3) is zero i.e.

$(fu + gv - hw)^2 - 4gu.fv = 0$ i.e. $(fu + gv - hw)^2 = 4\,fu.\,gv$,

i.e. $fu + gv - hw = \pm 2\sqrt{(fu.\,gv)}$,

i.e. $hw = fu + gv \pm \sqrt{(fu.\,gv)}$,

$= \left(\sqrt{fu} \pm \sqrt{gv}\right)^2$

Taking square root: $\pm\sqrt{hw} = \pm\sqrt{fu} \pm \sqrt{gv}$

i.e. $\pm\sqrt{fu} \pm \sqrt{gv} \pm \sqrt{hw} = 0$

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in the last of this book.

CHAPTER 2. DIRECTION COSINES, DIRECTION RATIOS

Q3(c) Show that the straight lines whose direction cosines are given by the equations $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ (where a,b,c,u,v,w are constants) are parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$ and perpendicular if $a^2(v+w)+b^2(w+u)+c^2(u+v)=0$. **IFoS 2021**

Q1. Prove that the angle between two straight lines whose direction cosines are given by $l+m+n=0$ and $fmn+gnl+hlm=0$ is $\frac{\pi}{3}$, if $\frac{1}{f}+\frac{1}{g}+\frac{1}{h}=0$. **[2c 2020 IFoS]**

Q2. A line makes a angle $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube. Show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$. **[(2c) 2019 IFoS]**

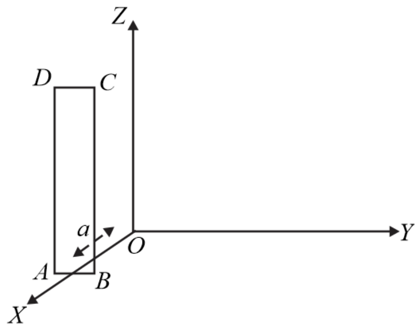
Q3. Find the angle between the lines whose direction cosines are given by the relations $l+m+n=0$ and $2lm+2\ln-mn=0$. **[(3d) 2017 IFoS]**

3.The Plane

3.1 **The plane:-** A plane is surface such that the line joining any two points on its lies completely in the surface.
We know that the equation of yz, zx and $xy-$planes are, respectively,

$$x=0\ , y=0,\quad z=0$$

It can be easily seen that the equation of the plane parallel to and at a distance a from that $yz-$ plane is $x=a$



Similarly, then equation of the plane parallel to and at a distance b from the $zx-$ plane is $y=b$, and the equation of the plane to and at a distance c from the $xy-$plane is $z=c$.
All these equations of planes are of the first degree. In the next section we shall prove that general equation of the first degree always represents a plane.

3.2 **The General Equation of The First Degree:-** To show that the general equation of the first degree in x, y, z always represents a plane.

$$Ax+By+Cz+D=0 \qquad (1)$$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the locus represented by this equations. Then the co-ordinates of these point must satisfy (1). Thus we have

$$Ax_1+By_1+Cz_1+D=0 \qquad (2)$$

And $Ax_2+By_2+Cz_2+D=0 \qquad (3)$

Multiplying (3) by $\lambda(\lambda>0)$ and adding the resulting equation to (2), we get $A(\lambda x_2+x_1)+B(\lambda y_2+y_1)+C(\lambda z_2+z_1)+D(\lambda+1)=0$.

Dividing by $\lambda+1$, it gives $A\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)+B\left(\frac{\lambda y_2+y_1}{\lambda+1}\right)+C\left(\frac{\lambda z_2+z_1}{\lambda+1}\right)+D=0$

This relation, on comparing with (1), shows that the point $\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right)$, which divides the line PQ in the ratio $\lambda:1$, also lies on the locus represented by (1).
As λ can have any positive value, it follows that the co-ordinates of all the points on the lines segment PQ satisfy (1). Thus we have shown that the line joining two points of the locus lies completely on the locus. It follows that the locus of (1) is a plane.

Some Particular Cases:- The following particular cases of (1) can be easily verified.

(i) If $D=0$, the plane passes through the origin

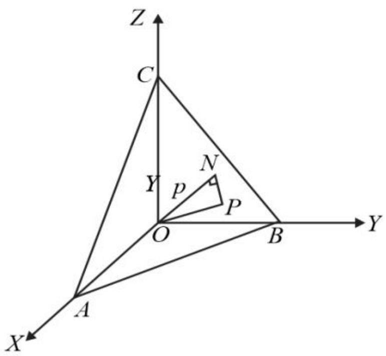
(ii) If $A=0$, the plane is parallel to the x-axis.
If $B=0$, the plane is parallel to the y-axis
If $C=0$, the plane is parallel to the z-axis

(iii) If $A=0=B$, the plane is parallel to both the $x-$ and $y-$ axes, i..e. parallel to the xy- plane.
If $B=0=C$, the plane is parallel to the $yz-$ plane
If $C=0=A$, the plane is parallel to the $xz-$ plane

(iv) If $A=0=B=C$, while D remains finite, the plane is at an infinite distance.

Number of Arbitrary Constants in the Equation of a Plane:- The equation $Ax+Bx+Cz+D=0$ of the plane involves four constants A,B,C and D. But only three of them are arbitrary. To prove it suppose $D\neq 0$. Then dividing by D, it gives $A'x+B'y+C'z+1=0$, where $A'=A/D,\ B'=B/D, C'=C/D$. This equation involves only three arbitrary constants.
In case $D=0$, then we can divide by A,B or C whichever is different from zero. (We note that all the four constants cannot be zero.) Thus from either case we find that there are only three arbitrary constants in the equation of a plane. As such to specify completely the equation of a plane, we need three conditions only.

33. Normal Form of The Equation of A Plane:- To find the equation of the plane in terms of the length of the normal drawn from the origin to the plane and its direction cosines.
Let ON be the normal from O to plane ABC, the positive direction being in the sense from O to N. Let the length $ON=p$, and the direction cosines of ON be l,m,n.
Let $P(x,y,z)$ be any point on the plane. Then PN is perpendicular to the lines ON. Also. $ON=$ the projection of OP on ON. i.e. $p=lx+my+nz$ (using corollary of section 2.6)
i.e. $lx+my+nz=p$.

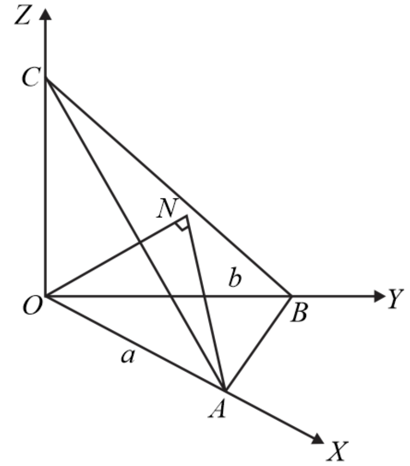


Since this relation is true every point $P(x,y,z)$ on the plane, it is the equation of the plane. The above equation is known as the normal form of the equation of a plane.
It may be noted that the co-ordinates of N are (lp,mp,np).

Aliter:- As before, the line PN is perpendicular to ON. The co-ordinates of the point N are (lp, mp, np). Therefore direction rations of the line PN are $x-lp, y-mp, z-np$. Using the condition of perpendicular between ON and PN, we obtain $l(x-lp)+m(y-mp)+n(z-np)=0$. i.e. $lx+my+nz=(l^2+m^2+n^2)p$, i.e. $lx+my+mz=p$ since $l^2+m^2+n^2=1$.

Note:- The sign of p is always taken as positive.

3.4 Intercept From of The Equation of A Plane:- To find the equation of a plane in terms of the intercepts which it makes on the axes.



Let the plane ABC cut the co-ordinate axes in the points A, B and C, where $OA=a$, $OB=b$, $OC=c$.

Let ON be the normal from O to the plane ABC with direction cosines l, m, n. Then $ON=$ the projection of OA on the line ON i.e. $p=al$.

Similarly, $p=bm$ and $p=cn$.

Therefore, $l=\frac{p}{a}, m=\frac{p}{b}, n=\frac{p}{c}$. Using these equations in the normal form $lx+my+nz=p$, we get $\frac{p}{a}x+\frac{p}{b}y+\frac{p}{c}z=p$, i.e. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

This is known as the intercept form of the equation of the plane.

3.6 Equation of The Plane through Given Three Points:- To find the equation of a plane passes through three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3).

Let the equation of the plane be $Ax+By+Cz+D=0$ (1)

It will pass through the points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) if

$Ax_1+By_1+Cz_1+D=0$ (2)

$Ax_2+By_2+Cz_2+D=0$ (3)

And $Ax_3+By_3+Cz_3+D=0$ (4)

Eliminating A,B,C and D from the equation (1), (2), (3) and (4), we get

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

This is the required equation of the plane.
The following corollary provides a necessary and sufficient **condition for co planarity of four points.**

Corollary:- The four points $(x_r, y_r, z_r), r = 1,2,3,4$, are coplanar if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

3.7 Equation of Any Plane Passing Through A Given Point:- To find the equations of a plane passing through a given point (α, β, γ).

Let the equation of the plane be $Ax + By + Cz + D = 0$ (1)

Since it passes through the point (α, β, γ), we have $A\alpha + B\beta + C\gamma + D = 0$ (2)

Subtracting (2) from (1), we have $A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0$ i.e. $Ax + By + Cz - (A\alpha + B\beta + C\gamma) = 0$.

This equation represents a system of planes passing through the point (α, β, γ).

It is clear that infinitely many planes can be made pass through a given point for different choices of A, B and C.

3.8 Angle Between Two Planes:- The angle between two planes is defined as the angle between the positive direction of the normal's to the planes.
To find the angle between any two given planes.

Let the equations of the given planes be $a_1x + b_1y + c_1z = d_1 = 0$ and $a_2x + b_2y + c_2z = d_2 = 0$

where both d_1 and d_2 are either positive or negative.

Clearly, direction ratios of the normal's to these planes are a_1, b_1, c_1 and a_2, b_2, c_2.

Hence the angle θ between these normal's is given by

$$\left| \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}} \right|$$

Thus the angle between the given planes is $\cos^{-1}\left\{ \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}} \right\}$

Corollary 1:- If θ be the angle between the planes $l_1x+m_1y+n_1z=p_1$ and $l_2x+m_2y+n_2z=p_2$ then $\cos\theta=l_1l_2+m_1m_2+n_1n_2$, i.e.

$$\tan\theta=\frac{\sqrt{\left\{(m_1-n_2-n_1m_2)^2+(n_1l_2-l_1n_2)^2+(l_1m_2-m_1l_2)^2\right\}}}{l_1l_2+m_1m_2+n_1n_2}$$

The following corollary provides **conditions of perpendicularity and parallelism** for two planes.

Corollary 2:- The planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$ and parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Plane Parallel to a Given Plane:- Let the given plane be $ax+by+cz+d=0$. If another plane parallel to this plane is $a'x+b'y+c'z+d'=0$, then we have $\frac{a'}{a}=\frac{b}{b}=\frac{c'}{c}=k$, so that $a'=ka, b'=kb, c'=kc$.

Therefore, the equation of later plane can be written as $kax+kby+kcz+d'=0$, i.e. $ax+by+cz+\lambda=0$, where $\lambda=d'/k$.

This show that the equation of two parallel planes differ only in constant term.

3.9 Position of Points Relative To Two Sides of A Planes:- To find the condition that the two given points are on opposite sides or on the same side of a given plane.

Let the equation of the plane be $Ax+By+Cz+D=0$. (1)

Suppose that $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are any two points neither of which lies on this plane. Let the straight line PQ meet the plane at a point R which is such that $PR:RQ=\lambda:1$. Then the co-ordinates of R are $\left(\frac{\lambda x_2+x_1}{\lambda+1},\frac{\lambda y_2+y_1}{\lambda+1},\frac{\lambda z_2+z_1}{\lambda+1}\right)$.

Clearly λ is positive or negative according as R divides PQ internally or externally i.e. according as P and Q are on opposite sides or on the same side of the plane (1).

Since R lies on the plane (1), its co-ordinates must satisfy (1). Thus we have $A\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)+B\left(\frac{\lambda y_2+y_1}{\lambda+1}\right)+C\left(\frac{\lambda z_2+z_1}{\lambda+1}\right)+D=0$.

On multiplying by $\lambda+1$, this gives $A(\lambda x_2+x_1)+B(\lambda y_2+y_1)+C(\lambda z_2+z_1)+D(\lambda+1)=0$ i.e.

$\lambda(Ax_2+By_2+Cz_2+D)=-(Ax_1+By_1+Cz_1+D)$ i.e. $\lambda=-\left(\frac{Ax_1+By_1+Cz_1+D}{Ax_2+By_2+Cz_2+D}\right)$.

Now if P and Q are **on the same side** of the plane λ is negative it follows from (2) that the expression $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ have the same sign. Further, if P and Q are on the opposite sides of the plane λ is positive. Therefore the above expression have different signs. Thus we obtain the following result l condition.

The two point (x_1, y_1, z_1) and (x_2, y_2, z_2) are on the same side or on the opposite sides of the plane $Ax + By + Cz + D = 0$ according as the expressions $Ax_1 + By_1 + Cz_1 + D$ and $Ax_2 + By_2 + Cz_2 + D$ have the same sign or opposite signs.

Corollary 1:- The expression $Ax + By + Cz + D$ has t he same sign for all the points lying on one side of the plane $Ax + By + Cz + D = 0$.

Corollary 2:- The point (x_1, y_1, z_1) and the origin then same or opposite sides of the plane $Ax + By + Cz + D = 0$ according as the expression $Ax_1 + By_1 + Cz_1 + D$ has the same or opposite signs as that of D.

Corollary 1:- The distance of the point $P(x_1, y_1, z_1)$ from the plane $Ax + By + Cz + D = 0$ i.e. $\pm \dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{(A^2 + B^2 + C^2)}}$.

This result follows by converting the given general equation of the plane into normal form. Then positive or negative sign is to be taken according as D is positive or negative provided P lies on the same side of plane as the origin.

Corollary 2:- The distance of the origin from the plane $Ax + By + Cz + D = 0$ i.e. $\pm \dfrac{D}{\sqrt{(A^2 + B^2 + C^2)}}$.

Corollary 3:- The distance between the parallel planes $lx + my + nz = p$ and $lx + my + nz = p'$ is $|p - p'|$.

3.10 Planes Bisecting the Angle Between Two Planes:-

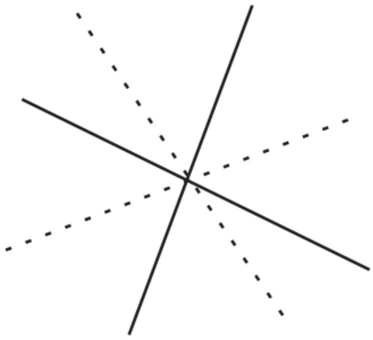
Bisectors of Angles Between Two Planes:- The locus of a point moves in such a way that its distances from the two planes are equal in magnitude, is called the bisector of angles between the planes.

We shall observe below that any two intersecting planes have two bisectors of the angles between them, and these bisectors are indeed, planes.

Let the two intersecting plane be given by $a_1x + b_1y + c_1z + d_1 = 0$ (1)

and $a_2x + b_2y + c_2z + d_2 = 0$ (2)

Let $P(x', y', z')$ be any point on the bisector. Then the perpendicular distance of this point from the planes (1) and (2) must be equal in magnitude. Thus we have



$$\frac{a_1x'+b_1y'+c_1z'+d_1}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}}=\pm\frac{a_2x'+b_2y'+c_2z'+d_2}{\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$$

Hence the locus of P is $\dfrac{a_1x+b_1y+c_1z+d_1}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}}=\pm\dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$ (3)

Evidently, this equation represents two planes, which are the bisectors of the angles between the given planes.

How to Distinguish Between The Two Bisector Planes:- We now distinguish between the two bisector planes given by (3) by finding that plane which bisects the angle containing the origin.

Without loss of generality, we assume that both d_1 and d_2 are positive. If any of these constants happens to be negative, we multiplying the concerning equation by -1 to make the constant positive.

Then the expression

And $\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1 \\ a_2x+b_2y+c_2z+d_2 \end{array}\right\}$ (4)

Are both positive at the origin.

Using corollary 1 or Section 3.9, it follows that both the expressions in (4) are positive for all the points lying on the bisector of the angle containing the origin. Hence the bisector is obtained by taking the positive sign on the right in (3). Thus its equation is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}} \quad (5)$$

This bisector is known as the **internal bisector.**

Further, both the expression in (4) have opposite signs for all the points lying on the bisector of the angle not containing the origin. Hence this bisector is obtained by taking the negative sign on the right in (3). Thus its equation is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}} \quad (6)$$

This bisector is known as the **external bisector.**

Remark:- From equation (5) and (6) we can easily prove that these bisector plane are perpendicular to each other.

Caution:- While solving numerical problems to obtain and identify internal and external bisector, the equation of given planes should be so written that the constant terms in both the equations be positive.

3.11 Plane Through The Line of Intersection of Two Planes:- To find the equation of a plane passing through the line of intersection of two given planes.

Let the equations of the given planes be $S_1 = a_1x + b_1y + c_1z + d_1 = 0$ (1)

And $S_2 = a_2x + b_2y + c_2z + d_2 = 0$ (2)

For λ to be any constant, consider the equation $S_1 + \lambda S_2 = 0$ i.e.

$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2)$ (3)

i.e. $(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0$ (3')

Since this is a first degree equation in x, y and z, it represents a plane. Again if any point (x', y', z') satisfies both the equation (1) and (2), it will also satisfy (3) and hence (3'), for all values of λ. Therefore, the plane (3') passes through all the points common to the planes (1) and (2).

Hence (3') is the general equation of a plane passing through the intersection of the given planes. But the planes (1) and (2) intersect in a line unless they are parallel. It follows that (3) is the equation of the plane through the line intersection of the planes (1) and (2). For distance values of λ, we get different such planes. A particular values of λ can be found from the . If the given planes happen to be parallel, (3') is also a plane parallel to them.

3.12 Homogeneous Equation of The Second Degree:- The equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (1)$$

Is called the homogenous equation of the second degree in three variables x, y and z.

(1) To find the condition that the homogenous equation of the second degree represents a pair of plane.

Method 1:- Suppose the equations of two planes, passing through the origin, represented by (1) are

$$\text{And} \quad \left.\begin{array}{l} l_1x + m_1y + n_1z = 0 \\ l_2x + m_2y + n_2z = 0 \end{array}\right\} \quad (2)$$

Then the joint equation of these planes is $(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z) = 0$ i.e.

$$l_1l_2x^2 + m_1m_2y^2 + n_1n_2z^2 + (m_1n_2 + n_1m_2)yz + (n_1l_2 + l_1n_2)zx$$
$$+(l_1m_2 + m_1l_2)xy = 0 \quad (3)$$

Since (1) and (3) represents the same surface, comparing the coefficients, we have

$$\frac{l_1l_2}{a} = \frac{m_1m_2}{b} = \frac{n_1n_2}{c} - \frac{m_1n_2 + n_1m_2}{2f} = \frac{n_1l_2 + l_1n_2}{2g} = \frac{l_1m_2 + m_1l_2}{2h} = k \text{ say}$$

Thus $l_1l_2 = ka$, $m_1m_2 = kb$, $n_1n_2 = kc$ (4)

$m_1n_2 + n_1m_2 = 2kf$, $n_1l_2 + l_1n_2 = 2kg$, $l_1m_2 + m_1l_2 = 2kh$

To find the required condition, we need to eliminate l_2, m_2, n_2 and k from (4). For this, consider the two determinants.

$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix}$ and $\begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix}$ each of which is equal to zero. Then their product:

$$\begin{vmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ m_1l_2 + m_2l_1 & 2m_1m_2 & m_1n_1 + m_2n_1 \\ n_1l_2 + n_2l_1 & n_1m_2 + n_2m_1 & 2n_1n_2 \end{vmatrix} = 0$$

i.e. $\begin{vmatrix} 2ka & 2kh & 2kg \\ 2kh & 2kb & 2kf \\ 2kg & 2kf & 2kc \end{vmatrix} = 0$ using (4)

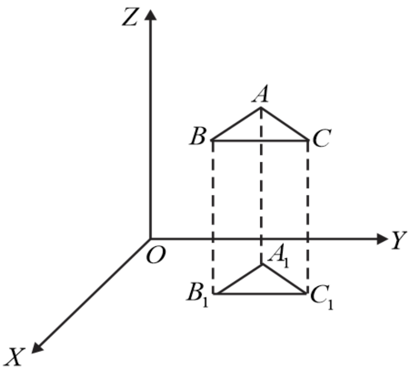
i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$

3.13 The Volume of A Tetrahedron :- To find the volume of the tetrahedron whose vertices are the point $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$.

Let $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ be the vertices of a tetrahedron.

Let p be the length of perpendicular from A to the plane BCD. Then we known (from geometry) that the volume V of the tetrahedron is given by $V = \frac{1}{3} p \times \Delta BCD$.

(1)



Now equation to the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

i.e. $\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} x - \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} y + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} z - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0$

therefore, $\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} x_1 - \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} y_1 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} z_1 - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 2\sqrt{\left(\Delta_x^2 + \Delta_y^2 + \Delta_z^2\right)}$$

Where $\Delta_x, \Delta_y, \Delta_z$ are the projection of the triangle BCD on $yz-, zx-$ and $xy-$ plane respectively.

We have $\Delta BCD = \sqrt{\left(\Delta_x^2 + \Delta_y^2 + \Delta_z^2\right)}$.

So $p = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div 2\Delta BCD$

Hence from (1), we find that

$$V = \left[\frac{1}{3} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div 2\Delta BCD \right] \times \Delta BCD = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Example1:- A plane meets the co-ordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a,b,c). Show that the equation to the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$$

Solution:- Let the equation of the plane be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \qquad (1)$$

If the plane meets the co-ordinate axes at A, B and C then the co-ordinates of these points are, respectively.

So the co-ordinates of the centroid of the triangle ABC are

$$\left(\frac{\alpha+0+0}{3},\frac{0+\beta+0}{3},\frac{0+0+\gamma}{3}\right), \text{ i.e. } \left(\frac{1}{3}\alpha,\frac{1}{3}\beta,\frac{1}{3}\gamma\right)$$

But the centroid is given to be (a,b,c). Therefore

$$\frac{1}{3}\alpha=a\,,\ \frac{1}{3}\beta=b\,,\ \frac{1}{3}\gamma=c$$

Using these values of α, β and γ in (1), the required equation of the plane is

$$\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1, \qquad \text{i.e. } \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$$

Example2:- A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the triangle ABC is:

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2};$$

And the locus of the centroid of the tetrahedron $OABC$ is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}.$$

Solution:- Let the equation of the variable plane be

$$lx+my+nz=p$$

Where $l^2+m^2+n^2=1$ and p is fixed.

Then the co-ordinates of A, B and C are $(p/l,0,0),(0,p/m,0)$ and $(0,0,p/n)$ respectively

If (x,y,z) be the co-ordinates of the centroid of the triangle ABC, then we have

$$x=\frac{p/l+0+0}{3},\ y=\frac{0+p/m+0}{3}, z=\frac{0+0+p/n}{3}$$

i.e. $\quad x=p/3l\,, \qquad y=p/3m$ m $\qquad z=p/3n$

The locus of the centroid can be obtained by eliminating l, m and n from these equations. For this, from (2), we have

$$l=p/3x\,, \qquad m=p/3y\,, \qquad n=p/3z\,.$$

Squaring and adding these equations we obtain

$$l^2+m^2+n^2=\frac{p^2}{9x^2}+\frac{p^2}{9y^2}+\frac{p^2}{9z^2},$$

i.e. $$1=\frac{p^2}{9}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right), \text{ since } l^2+m^2+n^2=1$$

i.e. $$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2}$$

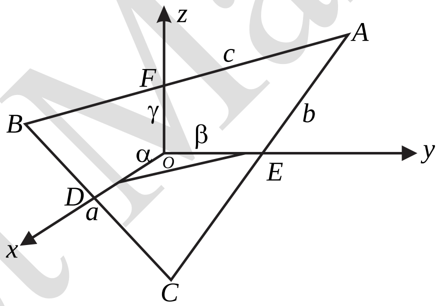
This is the required locus.
Proceeding in the same manner we can prove the second result.

Example3:- A triangle, the lengths of whose sides are a,b,c is placed so that the middle points of the sides are on the co-ordinates axes. Show that the equation to its plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$$

Where $8\alpha^2=b^2+c^2-a^2$, $8\beta^2=c^2-a^2-b^2$, $8\gamma^2=a^2+b^2-c^2$

Solution:- Let ABC be the given triangle which has intercepts α,β,γ respectively on the axes. Let D,E,F be the middle points of the sides of the triangle. Clearly, the co-ordinates of D,E,F are $(\alpha,0,0)$, $(0,\beta,0)$, $(0,0,\gamma)$ respectively.



From geometry, we known that $EF \parallel BC$

And $$EF=\frac{1}{2}\ BC=\frac{1}{2}a$$

Similarly, $FD=\frac{1}{2}b$ and $DE=\frac{1}{2}c$

Therefore, in right-angled triangle FOE, we have

$$\beta^2+\gamma^2+\frac{1}{4}a^2 \qquad (1)$$

Similarly, $$\gamma^2+\alpha^2=\frac{1}{4}b^2 \qquad (2)$$

And $$\alpha^2+\beta^2=\frac{1}{4}c^2 \qquad (3)$$

Adding (1), (2) and (3) we get

$$2\left(\alpha^2+\beta^2+\gamma^2\right)=\frac{1}{4}\left(a^2+b^2+c^2\right)$$

i.e. $$\alpha^2+\beta^2+\gamma^2=\frac{1}{8}\left(a^2+b^2+c^2\right) \qquad (4)$$

Subtracting (1) from (4), we get

$$\alpha^2=\frac{1}{8}\left(b^2+c^2-a^2\right)$$

i.e. $8\alpha^2 = b^2 + c^2 - a^2$

Similarly, subtracting (2) from (4), and also (3) from (4), we get

$$8\beta^2 = c^2 + a^2 - b^2$$

And $8\gamma^2 = c^2 + b^2 - c^2$.

Example4:- Find the intercepts made on the co-ordinates axes by the plane $5x - 3y + 2z = 8$.

Solution:- The given equation of plane is

$$5x - 3y + 2z = 8$$

Dividing by 8 (the constant term), it assumes the form

$$\frac{5x}{8} - \frac{3y}{8} + \frac{2z}{8} = 1, \qquad \text{i.e. } \frac{x}{8/5} + \frac{y}{-8/3} + \frac{z}{4} = 1$$

This is the intercept form of the given plane. Whence the intercepts made on the co-ordinates axes by the plane are $8/5, -8/3, 4$.

Example5:- Find the equation of the plane passing through the point $(1,-1,2)$ and parallel to the plane $3x - 2y + 4z = 7$

Solution:- Any plane parallel to the given plane is

$$3x - 2y + 4z = d$$

If it passes through the point $(1,-1,2)$, we must have

$$3.1 - 2(-1) + 4.2 = d \quad \text{i.e. } d = 13$$

Using the value of d in (1), the required equation of the plane is

$$3x - 2y + 4z = 13.$$

Example6:- If P is the point $(7,8,9)$, find the equation to the plane through P at right angle to OP, where O is the origin.

Solution:- Any plane through the given point $P(7,8,9)$ is

$$A(x-7) + B(y-8) + C(z-9) = 0 \qquad (1)$$

If it has to make a right angle with OP, the coefficients A, B, C must be in the same proportion as direction ratios of OP.

But direction ratios of OP are

$$7-0,\ 8-0,\ 9-0, \qquad \text{i.e. } 7,8,9$$

Replacing A, B, C in (1) by these numbers, we obtain the required equation of plane as

$$7(x-7) + 8(y-8) + 9(z-9) = 0$$

i.e. $7x + 8y + 9z = 194$.

Example7:- Find the equation of the plane passing through the point $(1,3,2)$ which is perpendicular to OP, where O is the origin and the co-ordinates of P are $(1,1,1)$.

Solution:- Direction ratios of OP are

$1-0,\ 1-0,\ 1-0$, i.e. $1,1,1$

Hence equation of the plane passing through the point $(1,3,2)$ and perpendicular to OP is

$$1.(x-1)+1.(y-3)+1.(z-2)=0$$
$$x+y+z=6.$$

Example8:- Show that the four points $(0,-1,0),(2,1,-1),(1,1,1),(3,3,0)$ are coplanar.

Solution:- Any plane through the first point $(0,-1,0)$ is

$$A(x-0)+B(y+1)+C(z-0)=0,$$

i.e. $$Ax+By+Cz+B=0 \qquad (1)$$

it will pass through the second and third points, viz. $(2,1,-1)$ and $(1,1,1)$ if

$$2A+2B-C=0$$

and $$A+2B+C=0$$

Solving these two equations, we have

$$\frac{A}{2-(-2)}=\frac{B}{-1-2}=\frac{C}{4-2}, \quad \text{i.e. } \frac{A}{4}=\frac{B}{-.}=\frac{C}{2}=k, \text{ say.}$$

So, $A=4k,\ B=-3k,\ C=2k$

Putting these values in (1), we have

$$4kx-3ky+2kz-3k=0$$

i.e. $$4x-3y+2z-3=0$$

Clearly, the fourth point $(3,3,0)$ satisfies this equation, since

$$4\times3-3\times3+2\times0-3=0$$

Hence the given four points are coplanar.

Aliter:- Equation of the plane passing through the first three given points is.

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0,$$

i.e. $$\begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} x-\begin{vmatrix} 0 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} y+\begin{vmatrix} 0 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} z-\begin{vmatrix} 0 & -1 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}=0$$

i.e. $4x-3y+2z-3=0$, on simplifying

Since the co-ordinates of the fourth given point satisfy this equation as

$$4\times3-3\times3+2\times0-3=0$$

We find that the given four points are coplanar.

Example9:- A point P moves on the fixed plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, and a plane through P, perpendicular to OP meets the axes in A,B,C. If the planes through A,B,C parallel to the co-ordinate planes meet a Q, show that the locus of Q is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$

Solution:- Let the co-ordinates of P be (α,β,γ) so that

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \qquad (1)$$

Also, direction ratios of OP are α,β,γ. Therefore, the equation of plane passing through $P(\alpha,\beta,\gamma)$ and perpendicular to OP is.

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0,$$

i.e. $\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2$

i.e. $\frac{x}{r^2/\alpha}+\frac{y}{r^2/\beta}+\frac{z}{r^2/\gamma}=1$, where $r^2=\alpha^2+\beta^2+\gamma^2$.

This plane meets the axes in A,B,C. Clearly, the co-ordinates of A,B,C are $(r^2/\alpha,0,0),(0,r^2/\beta,0),(0,0,r^2/\gamma)$ respectively.

Since Q is the point of intersection of the planes through A,B,C parallel to the co-ordinates planes, we have

x co-ordinates of $Q=x$ co-ordinates of $A=r^2/\alpha$

y co-ordinates of $Q=y$ co-ordinates of $B=r^2/\beta$

And z co-ordinates of $Q=z$ co-ordinates of $C=r^2/\gamma$

Thus the co-ordinates of Q are $(r^2/\alpha,r^2/\beta,r^3/\gamma)$. Using these co-ordinates, we have

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2}{r^4}+\frac{\beta^2}{r^4}+\frac{\gamma^2}{r^4}=\frac{\alpha^2+\beta^2+\gamma^2}{r^4}=\frac{r^2}{r^4}=\frac{1}{r^2}$$

And $\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{\alpha}{ar^2}+\frac{\beta}{br^2}+\frac{\gamma}{cr^2}=\frac{1}{r^2}\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}\right)=\frac{1}{r^2}$, using (1)

Thus the co-ordinates of Q satisfy the equation

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$$

Hence this is the locus of Q.

Example10:- Find the angle between the planes $3x+5y-2z+1=0$ and $2x+4y+9z+7=0$

Solution:- The angle between the given planes

$$=\cos^{-1}\left\{\frac{aa'+bb'+cc'}{\sqrt{(a^2+b^2+c^2)}\sqrt{(a'^2+b'^2+c'^2)}}\right\}$$

$$= \cos^{-1}\left[\frac{3\times 2+5\times 4+(-2)\times 9}{\sqrt{\left(3^2+5^2+(-2)^2\right)}\sqrt{\left(2^2+4^2+9^2\right)}}\right]$$

$$= \cos^{-1}\left\{\frac{1}{\sqrt{(38)}\sqrt{(101)}}\right\} = \cos^{-1}\left\{\frac{4\sqrt{2}}{\sqrt{(1919)}}\right\}$$

Example11:- Show that the planes $2x-5y+z+2=0$ and $x+y+3z-1=0$ are perpendicular to each other.

Solution:- We know that the planes

$$ax+by+cz+d=0$$

And $$a'x+b'y+c'z+d'=0$$

Are perpendicular if $$aa'+bb'+cc'=0$$

Here $a=2, b=-5, c=1$; $a'=1, b'=1, c'=3$. So

$$aa'+bb'+cc'=2.1+(-5).1+1.3=2-5+3=0$$

Hence the given planes are perpendicular to each other.

Example12:- Find the equation of a planes passing through the point $(1,2,3)$ and parallel to the plane $x-y+z=1$

Solution:- Any planes parallel to the given plane $x-y+z=1$ is

$x-y+z=c$, c is any constant.

If it has pass through the given points $(1,2,3)$, we must have.

$1-2+3=c$, i.e. $c=2$

Using this value of c in (1), the required equation of plane is

$$x-y+z=0.$$

Example13:- Find the equation to the plane through the point $(-1,3,2)$ and perpendicular to the planes.

$x+2y+2z=5$ and $3x+3y+2z=8$

Solution:- Equation of any plane through the point $(-1,3,2)$ is

$$A(x+1)+B(y-3)+C(z-2)=0 \quad (1)$$

If will be perpendicular to the given planes if

$$A+2B+2C=0$$

And $$3A+3B+2C=0$$

Solving these two equations, we obtain

$$\frac{A}{2\times 2-2\times 3}=\frac{B}{2\times 3-1\times 2}=\frac{C}{1\times 3-2\times 3},$$

i.e. $\frac{A}{-2}=\frac{B}{4}=\frac{C}{-3}$, i.e. $\frac{A}{2}=\frac{B}{-4}=\frac{C}{3}$

Using these proportionate values, namely $2,-4,3$ for A,B,C in (1), we obtain

$$2(x+1)-4(y-3)+3(z-2)=0$$

i.e. $2x-4y+3z+8=0$

This is the required equation of the plane.

Example14:- Find the equation to the plane through the point $(2,2,1)$ and $(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$

Solution:- Any plane through the point $(2,2,1)$ can be given by

$$A(x-2)+B(y-2)+C(z-1)=0 \qquad (1)$$

If it passes through the point $(9,3,6)$ also, then we have

$$A(9-2)+B(3-2)+C(z-1)=0$$

i.e. $7A+B+5C=0$ (2)

Further, if plane (1) is perpendicular to the given plane, from the condition of perpendicularity, we must have

$$2A+6B+6C=0$$

i.e. $A+3B+3C=0$ (3)

From (2) and (3), we have

$$\frac{A}{1.3-5.3}=\frac{B}{5.1-7.3}=\frac{C}{7.3-1.1}$$

i.e. $\frac{A}{-12}=\frac{B}{-16}=\frac{C}{20}$, i.e. $\frac{A}{3}=\frac{B}{4}=\frac{C}{-5}$

Using these proportionate values of A,B,C in (1), we get

$$3(x-2)+4(y-2)-5(z-1)=0$$

Simplifying it, the desired equation of plane is

$$3x+4y-5z=9.$$

Example15:- Prove that the points $(1,2,3)$, $(0,5,1)$ are on the same side of the plane $y+z-4=0$

Solution:- The given plane is

$$y+z-4=0$$

Using the co-ordinates $(1,2,3)$ of the first given point, we have

$$y+z-4=2+3-4=1>0$$

Similarly, using the co-ordinates $y+z-4=2+3-4=1>0$ of the second point, we have

$$y+z-4=5+1-4=2>0.$$

Since the expression $y+z-4$ has the same sign at both the points, the given points lie on the same side of the plane.

Example16:- Find the distance of the point $(1,1,-2)$ from the plane $3x+4y-12z+8=0$.

Solution:- The equation of the given plane in normal form is $\frac{3x+4y-12z+8}{-\sqrt{\{3^2+4^2+(-12)^2\}}}=0$ i.e. $\frac{3x+4y-12z+8}{-13}=0$.

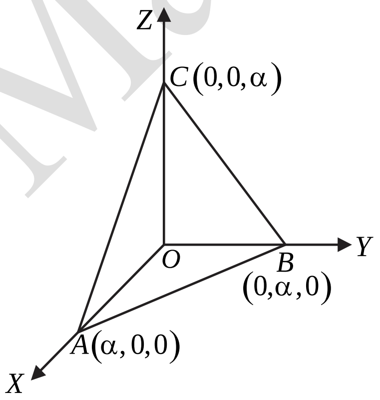
i.e. $$-\frac{3}{13}x-\frac{4}{13}y+\frac{12}{13}z=\frac{8}{13}$$

Hence the distance of the point $(1,1,-2)$ from this plane is

$$\left|-\frac{3}{13}\times1-\frac{4}{13}\times1+\frac{12}{13}\times(-2)-\frac{8}{13}\right| \text{ i.e. , } \frac{39}{13} \text{ i.e. } 3.$$

Example17:- Find the incentre of the tetrahedron formed by the planes $x=0, y=0, z=0$, $x+y+z=a$.

Solution:- The tetrahedron along with the co-ordinates of its vertices is shown in the adjacent figure.



We known that the incentre of a tetrahedron is the point equidistant from the four faces of the tetrahedron.

By symmetric of faces of given tetrahedron, it is clear that the three co-ordinates of the incentre should be equal So, let $P(\alpha,\alpha,\alpha)$ be the incentre. Then

Distance of P from the plane OAB
$=$ distance of P from $xy-$ plane.

i.e. $\frac{a-(\alpha+\alpha+\alpha)}{\sqrt{3}}=\alpha$, i.e. $a-3\alpha=\alpha\sqrt{3}$

i.e. $\alpha(3+\sqrt{3})=a$ i.e. $\alpha=a/(3+\sqrt{3})$

Hence the co-ordinates of the incentre are
$\left((a/3+\sqrt{3}),a/(3+\sqrt{3}),a/(3+\sqrt{3})\right)$.

Example18:- Two systems of the rectangular axes have the same origin. If a plane cuts them at distance a,b,c and a',b',c' respectively from the origin, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2+b'^2+c'^2}$$

Solution:- The equations to the plane in two systems of rectangular axes with the same origin will be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \text{ and } \frac{X}{a'}+\frac{Y}{b'}+\frac{Z}{c'}=1$$

Since the distance of the given plane from the origin is constant, we have

$$\frac{0+0+0-1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=\pm\frac{0+0+0-1}{\sqrt{\left(\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}\right)}}$$

Squaring both sides and taking reciprocals, this equation given

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2+b'^2+c'^2}$$

Example19:- Show that the origin lies in the acute angle between the planes $x+2y+2z=9$ and $4x-3y+12z+13=0$. Find the planes bisecting the angles between then and point out which bisects the acute angle.

Solution:- Let us first make the constant term of the first equation positive, and write the given equations as below:

$$-x-2y-2z+9=0 \qquad (1)$$

And $$4x-3y+12z+13=0 \qquad (2)$$

Now $a_1a_2+b_1b_2+c_1c_2=(-1).4+(-2)(-3)+(-2).12=-22<0$

Hence the origin lies in the acute angle between the given planes.

Further, the equation of planes bisecting the angles between the planes (1) and (2) are

$$\frac{-x-2y-2z+9}{\sqrt{\left\{(-1)^2+(-2)^2+(-2)^2\right\}}}=\pm\frac{4x-3y+12z+13}{\sqrt{\left\{4^2+(-3)^2+12^2\right\}}}$$

i.e. $$\frac{-x+2y-2z+9}{3}=\pm\frac{4x-3y+12z+13}{13} \qquad (3)$$

Since the origin lies in the acute angle, the equation of plane bisecting the acute angle can be obtained by choosing positive sign in (3). Hence the bisector plane of acute angle is.

$$\frac{-x-2y-2z+9}{3}=\frac{4x-3y+12z+13}{13}$$

i.e. $$25x+17y+62z-78=0$$

Further, taking negative sign in (3), the bisector of obtuse angle is

$$\frac{-x-2y-2z+9}{3}=-\frac{4x-3y+12z+13}{13}$$

i.e. $$x+35y-10z-156=0$$

Example20:- Find the equation of the plane through the line of intersection of the planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$ and perpendicular to the $xy-$ plane.

Solution:- Equation of any plane through the line of intersection of the given planes is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0 \quad (1)$$

i.e. $$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \quad (1')$$

Further, the equation of the $xy-$plane is $z=0$, which can be written as

$$0x+0y+1.z=0 \quad (2)$$

Now using the condition of perpendicularity, we find that the plane (1') will be perpendicular to the plane (2) if

$$(a+\lambda a').0+(b+\lambda b').0+(c+\lambda c').1=0$$

i.e. $$\lambda=-c/c'$$

Substituting this value of λ in (1), we find the equation of the required plane as

$$(a+by+cz+d)-\frac{c}{c'}(a'x+b'y+c'z+d')=0$$

i.e. $$c'(ax+by+cz+d)-c(a'x+b'y+c'z+d')=0$$

i.e. $$(ac'-a'c)x+(bc'-b'c)y+(dc'-d'c)=0$$

Example21:- The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z-5=0$. Find the equation of the plane in new form.

Solution:- Any plane through the line in intersection of the given planes is

$$(x-2y+3z)+\lambda(2x+3y-4z-5)=0$$

i.e. $$(1+2\lambda)+(3\lambda-2)y+(3-4\lambda)z-5\lambda=0 \quad (1)$$

This plane will be at right angles to the plane $x-2y+3z=0$ if.

$$(1+2\lambda).1+(3\lambda-2)(-2)+(3-4\lambda).3=0,$$

i.e. $-16\lambda+14=0$, i.e. $\lambda=\frac{7}{8}$

Putting this value of λ in (1), we find the equation of the required plane as

$$\left(1+\frac{7}{4}\right)x+\left(\frac{21}{8}-2\right)y+\left(3-\frac{7}{2}\right)z-\frac{35}{8}=0$$

i.e. $$22x+5y-4z-35=0$$

Example22:- The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation to the plane in its now position is

$$lx+my \pm z\sqrt{(l^2+m^2)}\tan\alpha=0$$

Solution:- Any plane passing through the line of intersection of the planes $lx+my=0$ and $z=0$ is

$$lx+my+\lambda z=0 \quad (1)$$

This plane makes an angle α with the plane $lx+my=0$, so we have

$$\cos\alpha=\frac{l.l+m.m+\lambda.0}{\sqrt{(l^2+m^2+\lambda^2)}\sqrt{(l^2+m^2)}}=\frac{l^2+m^2}{\sqrt{(l^2+m^2+\lambda^2)}\sqrt{(l^2+m^2)}}$$

This gives $\cos^2\alpha = \dfrac{l^2+m^2}{l^2+m^2+\lambda^2}$, so that

$$\tan^2\alpha = \sec^2\alpha - 1 = \frac{l^2+m^2+\lambda^2}{l^2+m^2} - 1 = \frac{\lambda^2}{l^2+m^2}$$

Whence $\lambda = \pm\sqrt{(l^2+m^2)}\tan\alpha$

Substituting this value of λ in (1), the equation of the required plane is

$$lx + my \pm z\sqrt{(l^2+m^2)}\tan\alpha = 0.$$

Example23:- Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes.

Solution:- The given equation is $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$.

Comparing it with the standard form:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

We get $a = 2$, $b = -6$, $c = -12$, $f = 9$, $g = 1$, $h = \dfrac{1}{2}$

Now $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 2.(-6),(-12) + 2.9.1.\frac{1}{2} - 2.9^2 - (-6).1^2 - (-12).\left(\frac{1}{2}\right)^2$$

$$= 144 + 9 - 162 + 6 + 3 = 162 - 162 = 0$$

Hence the given equation represents a pair of planes.

Example24:- Find the area of the triangle included between the plane $2x - 3y + 4z = 12$ and the co-ordinates planes.

Solution:- The equation of given plane can be written in the following intercept form:

$$\frac{x}{6} + \frac{y}{-4} + \frac{z}{3} = 1$$

Clearly this plane intersects x, y and z axes at the points

$A(6,0,0)$, $B(0,-4,0)$ and $C(0,0,3)$ respectively

Now $A_x = \Delta BOC = \dfrac{1}{2}\, OB \times OC = \dfrac{1}{2} \times 4 \times 3 = 6$

$$A_y = \Delta COA = \frac{1}{2}\, OC \times OA = \frac{1}{2} \times 3 \times 6 = 9$$

$$A_z = \Delta AOB = \frac{1}{2}\, OA \times OB = \frac{1}{2} \times 6 \times 4 = 15$$

Hence $\Delta ABC = \sqrt{(A_x^2 + A_y^2 + A_z^2)} = \sqrt{(6^2 + 9^2 + 12^2)} = 3\sqrt{29}$

Example25:- Through a point $P(\alpha, \beta, \gamma)$ a plane s drawn at right-angle to OP to meet the axes in A, B, C. Prove that the area of the triangle ABC is $\dfrac{r^5}{2\alpha\beta\gamma}$, where $OP = r$

Solution:-Direction ratios if OP are α, β, γ. Therefore, the equation of the plane passing through $P(\alpha, \beta, \gamma)$ and perpendicular to OP is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$$

i.e. $\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2$

i.e. $\alpha x+\beta y+\gamma z=r^2$, since $r=OP\sqrt{\left(\alpha^2+\beta^2+\gamma^2\right)}$

clearly this plane intersects the axes at the points

$$A\left(\frac{r^2}{\alpha},0,0\right),\quad B\left(0,\frac{r^2}{\beta},0\right),\quad C\left(0,0,\frac{r^2}{\gamma}\right)$$

Now $A_x=\Delta BOC=\frac{1}{2}OB\times OC\frac{1}{2}\frac{r^2}{\beta}\frac{r^2}{\gamma}=\frac{r^4}{2\beta\gamma}$

$$A_y=\Delta COA=\frac{1}{2}OC\times OA\frac{1}{2}\frac{r^2}{\gamma}\frac{r^2}{\alpha}=\frac{r^4}{2\gamma\alpha}$$

And $A_z=\Delta AOB=\frac{1}{2}OA\times OB\frac{1}{2}\frac{r^2}{\alpha}\frac{r^2}{\beta}=\frac{r^4}{2\alpha\beta}$

Hence $\Delta ABC=\sqrt{\left(A_x^2+A_y^2+A_z^2\right)}=\sqrt{\left(\frac{r^8}{4\beta^2\gamma^2}+\frac{r^8}{4\gamma^2\alpha^2}+\frac{r^8}{4\alpha^2\beta^2}\right)}$

$$=\frac{r^4}{2}\sqrt{\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2}\right)}=\frac{r^4}{2}\frac{r}{\alpha\beta\gamma}=\frac{r^5}{2\alpha\beta\gamma}$$

Example26:- Show that the volume of the tetrahedron formed by the planes $lm+my=0$, $my+nz=0$, $nz+lx=0$, $lx+my+nz=p$ is $\frac{2}{3}p^3/(lmn)$

Solution:- The equations of given planes are

$$lm+my=0, \qquad (1)$$
$$my+nz=0, \qquad (2)$$
$$lx+nz=0, \qquad (3)$$
$$lx+my+nz=p \qquad (4)$$

Solving (1), (2) and (3), we have

$$x=0,\qquad y=0,\qquad z=0$$

Further solving (1), (3) and (4), we have

$$x=-p/l,\qquad y=p/m,\qquad z=p/n$$

Similarly, solving (1), (2) and (4), we have

$$x=p/l,\qquad y=-p/m,\qquad z=p/n$$

And solving (2) , (3) and (4), we have

$$x=p/l\qquad y=p/m,\qquad z=-p/n$$

Thus the co-ordinate of four vertices of the given tetrahedron are

$(0,0,0),(-p/l,p/m,p/n),\ (p/l,-p/m,p/n),(p/l,p/m,-p/n)$

Hence the volume of the tetrahedron

$$=\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -p/l & p/m & p/n & 1 \\ p/l & -p/m & p/n & 1 \\ p/l & p/m & -p/n & 1 \end{vmatrix} = \frac{1}{6}\times\frac{p^3}{lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}, = \frac{p^3}{6lmn}\times 4$$, on evaluating the determinant.

$$\frac{2}{3}p^3/(lmn)$$

Example27:- Find the locus of the centroid of the tetrahedron of constant volume $64k^3$, formed by the three co-ordinate planes and a variable plane.

OR

A variable plane makes with the co-ordinate planes a tetrahedron of constant volume $64k^3$. Show that the locus of the centroid of the tetrahedron is $xyz = 6k^3$

Solution:- Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ be the given variable plane which makes with the co-ordinate planes a tetrahedron of constant volume $64k^3$. Clearly the vertices of this tetrahedron are

$$(0,0,0), \quad (a,0,0,), \quad (0,b,0), \quad (0,0,c) \tag{1}$$

Therefore,

$$64k^3 = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$. Neglecting the negative sigh.

i.e. $6k^3 = \frac{1}{64}abc$, i.e. $\left(\frac{1}{4}a\right)\left(\frac{1}{4}b\right),\left(\frac{1}{4}c\right)=6k^3$ (2)

Since the co-ordinates of the centroid of tetrahedron whose vertices are given in (1) are $\left(\frac{1}{4}a,\frac{1}{4}b,\frac{1}{4}c\right)$ it follows from (2) that the locus of the centroid is.

$$xyz = 6k^3$$

Example28:- Find the volume of the tetrahedron $OABC$ where O is the origin, the lengths OA, OB, OC are a,b,c and the angles BOC, COA, AOB are, respectively, λ, μ, ν

OR

Find the volume of a tetrahedron in terms of the edges which meet in a point and of the angles which they make each other.

Solution:- Let OA, OB, OC be the edges of tetrahedron meeting at the origin O such that

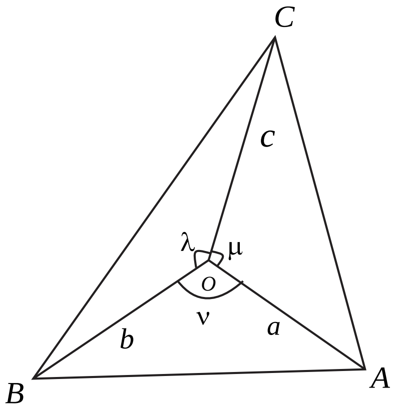
$$OA = a,\ OB = b, OC = c$$

Also, let $\angle BOC = \lambda$, $\angle COA = \mu$, $\angle AOB = \nu$

We need to find the volume of tetrahedron in terms of a,b,c and λ,μ,ν.

Let $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3;$ be the direction cosines of OA, OB, OC respectively.

Since $OA = a$, the co-ordinates of A are $(l_1 a, m_1 a, n_1 a)$. Similarly, the co-ordinates of B and C are $(l_2 b, m_2 b, n_2 b)$ and $(l_3 c, m_3 c, n_3 c)$ respectively.



Therefore, the volume V of the tetrahedron $OABC$ is given by

$$V = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 \\ l_1 a & m_1 a & n_1 a \\ l_2 b & m_2 b & n_2 b \\ l_3 c & m_3 c & n_3 c \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} l_1 a & m_1 a & n_1 a \\ l_2 b & m_2 b & n_2 b \\ l_3 c & m_3 c & n_3 c \end{vmatrix} = -\frac{1}{6} abc \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Now to evaluate the last determinant, we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1 l_2 + m_1 m_2 + n_1 n_2 & l_1 l_3 + m_1 m_3 + n_1 n_3 \\ l_1 l_2 + m_1 m_2 + n_1 n_2 & l_2^2 + m_2^2 + n_2^2 & l_2 l_3 + m_2 m_3 + n_2 n_3 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 & l_2 l_3 + m_2 m_3 + n_2 n_3 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix},$$

Since $l_1^2 + m_1^2 + n_1^2 = 1$ etc.

and $l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos\nu$ etc.

Hence $V = \dfrac{1}{6} abc \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in last of this book.

CHAPTER 3. PLANE

Q1(e) A variable plane which is at a constant distance $3p$ from the origin O cuts the axes in the points A, B, C respectively. Show that the locus of the centroid of the tetrahedron OABC is $9\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\frac{16}{p^2}$ **UPSC CSE 2023**

Q1(e) A variable plane is at a constant distance of 6 units from the origin and meets the axes in $A:(a,0,0), B:(0,b,0)$ and $C:(0,0,c)$. Find the locus of the centroid of the triangle ABC. **IFoS 2022**

Q1(e) Find the equation of the plane passing through the points $(1,-1,1)$ and $(-2,1,-1)$ and perpendicular to the plane $2x+y+z+5=0$. **IFoS 2021**

Q1. A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, which is fixed. The plane through P and perpendicular to OP meets the axes in A, B, C respectively. The planes through A, B, C parallel yz, zx and xy planes respectively intersect at Q. Prove that the locus of Q is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$. **[3c 2020 IFoS]**

Q2. Find the equation of the plane parallel to $3x-y+3z=8$ and passing through the point $(1,1,1)$.

[(4d) UPSC CSE 2018]

Q3. Show that the angles between the planes given by the equation $2x^2-y^2+3z^2-xy+7zx+2yz=0$ is $\tan^{-1}\frac{\sqrt{50}}{4}$. **[(2d) 2017 IFoS]**

Q4. Find the equations of the planes parallel to the plane $3x-2y+6z+8=0$ and at a distance 2 from it. **[(1e) 2017 IFoS]**

Q5. Obtain the equation of the plane passing through the points $(2,3,1)$ and $(4,-5,3)$ parallel to x-axis. **[(3c(i) UPSC CSE 2015]**

Q6. Find the equation of the plane containing the straight line $y+z=1, x=0$ and parallel to the straight line $x-z=1, y=0$. **[(3d) 2015 IFoS]**

Q7. Find the equation of the plane which passes through the points $(0,1,1)$ and $(2,0,-1)$, and is parallel to the line joining the points $(-1,1,-2),(3,-2,4)$. Find also the distance between the line and the plane. **[(1d) UPSC CSE 2013]**

Q8. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Prove that the locus of the centroid of the tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$.

[(1e) 2011 IFoS]

Q9. If a plane cuts the axes in A, B, C and (a, b, c) are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$. **[(1e) 2010 IFoS]**

4.THE STRAIGHT LINE

4.1 Equation To A Straight Line:- Unsymmetrical From

A straight line is obtained by the intersection of two non-parallel planes, and is completely determined when the equations of the planes are known. If

$S_1 = a_1x + b_1y + c_1z + d_1 = 0$

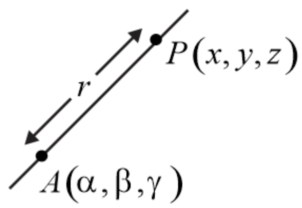
And $S_2 = a_2x + b_2y + c_2z + d_2 = 0$ (1)

Be the equation of two planes passing through a straight line AB, then these two equations taken together, represent the equations of the line AB. This representation is called the unsymmetrical from of the equations of a straight line.

4.2 Symmetrical From of The Equations:- The straight line is also determined by the co-ordinates of a fixed point on it and direction cosines of the line.

Let $A(\alpha, \beta, \gamma)$ be a fixed point of the straight line and let its direction cosines be l, m, n. If $P(x, y, z)$ is any point on the line at a distance r from A, then by Section 2.5, we have $\frac{x-\alpha}{r} = l, \frac{y-\beta}{r} = m, \frac{z-r}{r} = n$ (1)

These equations can be written as $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-r}{n} (= r)$ (1')



and are satisfied by the co-ordinates of any point on the straight line. Hence they are the required equations of the straight line.

The equation in (1') are called the symmetrical from of the equations of a straight line.

From (1) or (1'), we find that $x = \alpha + lr, \ \ y = \beta + mr, \ \ z = \gamma + nr$ (2)

Thus the co-ordinates (x, y, z) of any point on a straight line can be expressed in terms of the distance of that point from a fixed point (α, β, γ) on the straight line. Here r is a parameter.

The following corollary shows that direction ratios can be used instead of actual direction cosines in the symmetrical from introduced above.

Corollary:- If (α, β, γ) is a fixed point on a straight line having a, b, c as its direction ratios. Then the equations of the straight line are $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ (1")

Since $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ say, so that $l = ka, m = kb, n = kc$, this result follow at once from equation (1'). In what follows we shall notice that equation (1") are more convenient to use then (1').

Note:- As usual the co-ordinates of any point on (1”) may be given by $x=\alpha+ar,\ y=\beta+br,\ z=\gamma+cr$ (2’)
But in this r does not represent actual distance of (x,y,z) from (α,β,γ)

Equations of The Co-ordinates Axe in Symmetrical Form:- We known that each of the co-ordinate axes passes through the origin i.e. $(0,0,0)$. Since direction cosines of the x-axis are $1,0,0$, its equations in symmetrical form are $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$.
Similarly, the equations of the $y-$ and $z-$ axes are, respectively $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$ and $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$.

4.3 Straight Line Passing Through Two Given Points:- To find the equation of straight line passing through two given points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$.
It is clear that direction ratios of the line AB are $x_2-x_1,\ y_2-y_1,\ z_2-z_1$.
Now taking A as a fixed point, the equations of the line AB are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.
Further taking B as a fixed point and using direction ratios as $x_1-x_2, y_1-y_2-z_1-z_2$, the equations of the line AB can be written in the form $\frac{x-x_2}{x_1-x_2}=\frac{y-y_2}{y_1-y_2}=\frac{z-z_2}{z_1-z_2}$.

4.4 Transformation of The Unsymmetrical Form Into The Symmetrical One:- To transform the equations

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1&=0\\ a_2x+b_2y+c_2z+d_2&=0\end{aligned}\right\} \quad (1)$$

Of a straight line into symmetrical form.
In order to write the equations in symmetrical form we need:
(i) Direction ration (or direction cosines) of the line
(ii) The Co-ordinates of any one point on it.
We note that the equations.

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=0\\ a_2x+b_2y+c_2z&=0\end{aligned}\right\} \quad (2)$$

Together represent the straight line through the origin parallel to the given line (1), solving (2), we find that $\frac{x}{b_1c_2-c_1b_2}=\frac{y}{c_1a_2-a_1c_2}=\frac{z}{a_1b_2-b_1a_2}$.
Therefore direction ratios of the given line are $b_1c_2-c_1b_2,\ c_1a_2-a_1c_2,\ a_1b_2-b_1a_2$.
Next we have to find the co-ordinates of any one point out of infinitely, many points on the line. For the sake of convenience we choose the point of intersection of the line with plane $z=0$. This point is given by the equations.

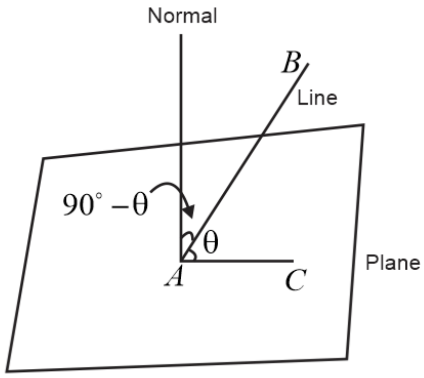
$$a_1x+b_1y+d_1=0,\ a_2x+b_2y+d_2=0,\ z=0 \qquad (3)$$

And is easily found to be $\left(\dfrac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2},\dfrac{d_1a_2-a_1d_2}{a_1b_2-b_1a_2},0\right)$.

Hence the equations of the straight line in the symmetrical form are $\dfrac{x-\left(\dfrac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2}\right)}{b_1c_2-c_1b_2}=\dfrac{y\left(\dfrac{d_1a_2-a_1d_2}{a_1b_2-b_1a_2}\right)}{c_1a_2-a_1c_2}=\dfrac{z-0}{a_1b_2-b_1a_2}$.

Note:- (i) It may not always be convenient to make z equal to zero. Depending upon the values of the coefficient is the given equations we may find it more convenient to assign some other particular value of z. (ii) In case the first two equations in (3) happen to be inconsistent, the line under consideration will be parallel to the plane $z=0$. Then we may choose the point of intersection of the line with the plane $y=0$ or $x=0$.

4.5. Angle Between A Straight Line And A Plane:- The acute angle between a straight line and its projection on a plane is called the angle between the straight line and the plane.



Evidently, the angle between a straight line and a plane is the complement of the angle between the straight line and the normal to the plane.

To find the angle between a straight line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ (1)

And to plane $ax+by+cz+d=0$ (2)

Let θ be the angle between the line (1) and the plane (2). Then the angle between the line (1) and the normal to the plane (2) is $90^\circ-\theta$. Also direction cosines of the line are l,m,n and direction ratios of the normal to the plane are a,b,c. Hence

$$\cos\left(90^\circ-0\right)=\frac{al+bm+cn}{\sqrt{\left(a^2+b^2+c^2\right)}\sqrt{\left(l^2+m^2+n^2\right)}},$$ i.e.

$$\sin\theta=\frac{al+bm+cn}{\sqrt{\left(a^2+b^2+c^2\right)}\sqrt{\left(l^2+m^2+n^2\right)}}$$

Here the factor $\sqrt{\left(l^2+m^2+n^2\right)}$ is equal to 1. However, if l,m,n are used for direction ratios of the line, we shall retain this factor in the denominator.

Corollary:- The straight line (1) is parallel to the plane (2) if $al+bm+cn=0$ and perpendicular if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

4.6 Conditions For A Line To Lie In A Plane:- To find the conditions that the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad (1)$$

May lie in the plane $ax+by+cz+d=0$ (2)

Any point on the line is $(lr+\alpha, mr+\beta, nr+\gamma)$.This point will lie on the plane if $a(lr+\alpha)+b(mr+\beta)+c(nr+\gamma)+d=0$, i.e. $(al+bm+cn)r+(a\alpha+b\beta+c\gamma+d)=0$.

But this equation should be true for every value of r. Therefore it must be an identity. Hence and

$$\left.\begin{array}{l} al+bm+cn=0 \\ a\alpha+b\beta+c\gamma+d=0 \end{array}\right\} \quad (3)$$

These are the conditions for the line (1) to the lie in the plane (2).

Remark:- The first condition in (3) show that the line (1) is parallel to the plane (2) whereas the second conditions shows that the point (α,β,γ) lies on the plane (2). Hence all the points of the line lie on the plane.

4.7 Plane Containing A Line:- To find the equation of a plane which contains the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$.

The equation of any plane passing through the point (α,β,γ) lying on the given line is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$ (1)

It will contain the given lien if it is parallel to that line i.e. if $al+bm+cn=0$

Hence the plane (1) contains the given line provided a,b,c involved in it satisfy (2).

4.8 Coplanar Lines:- Two or more lines are called coplanar if they lie on the same plane

(I) To find the condition that the lines whose equations are given in symmetrical from:

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$ may lie on the same plane.

Also to find the equation of the plane containing these lines.

Any plane containing the first line is $A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0$ (1)

Where $Al+Bm+Cn=0$ (2)

This plane will contain the second line if $A(\alpha'-\alpha)+B(\beta'-\beta)+C(\gamma'-\gamma)=0$ (3)

And $Al'+Bm'+Cn'=0$ (4)

Eliminating A,B,C from the equations (3), (2) and (4), we get

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0 \qquad (5)$$

This is the required condition.
The equation to the plane containing the first line and parallel to the second one can be obtained by eliminating A, B, C from equation (1), (2) and (4).
Hence the plane.
$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$ will contain both the given lines if condition (5) is satisfied.

(II) To find the condition that the lines whose equations are given in unsymmetrical from:

$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ and

$a_3x+b_3y+c_3z+d_3=0=a_4x+b_4y+c_4z+d_4$ may be coplanar.

Any plane containing the first line is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$
i.e. $(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0$ (1)
Similarly any plane containing the second line is
$(a_3x+b_3y+c_3z+d_3)+\mu(a_4x+b_4y+c_4z+d_4)=0$, i.e.
$(a_3+\mu a_4)x+(b_3+\mu b_4)y+(c_3+\mu c_4)z+(d_3+\mu d_4)=0$ (2)
Now the given lines will be coplanar if we can find λ and μ such that (1) and (2) represent the same plane. Hence on comparing the coefficients in equation (1) and (2), we get $\dfrac{a_1+\lambda a_2}{a_3+\mu a_4}=\dfrac{b_1+\lambda b_2}{b_3+\mu b_4}=\dfrac{c_1+\lambda c_2}{c_3+\mu c_4}=\dfrac{d_1+\lambda d_2}{d_3+\mu d_4}=k$ say.

Whence $\dfrac{a_1+\lambda a_2}{a_3+\mu a_4}=k$ i.e. $a_1+\lambda a_2=ka_3+k\mu a_4$, i.e. similarly and

$$\left.\begin{aligned} a_1+\lambda a_2-ka_3-k\mu a_4&=0\\ b_1+\lambda b_2-kb_3-k\mu b_4&=0\\ c_1+\lambda c_2-kc_3-k\mu c_4&=0\\ d_1+\lambda d_2-kd_3-k\mu d_4&=0 \end{aligned}\right\} \qquad (3)$$

The given lines will be coplanar if the four equations in (3) have a non-trivial solution for λ, μ and k, the condition for this can be obtained by eliminating $\lambda, -k$ and $-k\mu$ from equations in (3). Thus he required condition is $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0$

Note:- In case the given lien satisfy the above condition of co planarity and are not found to be parallel, they will intersect. The co-ordinates of the point of intersection can be determined then by solving any three of the four given equations of plane.

(III) To find the condition that the lines $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ may be coplanar.

Any plane containing the second line is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0 (1)$

i.e. $(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0$ (1')

This plane will contain the first line also, if (i) it passes through the point (α,β,γ)

i.e. $(a_1\alpha+b_1\beta+c_1\gamma+d_1)+\lambda(a_2\alpha+b_2\beta+c_2\gamma+d_2)=0$, using (1) (2)

And (ii) it is parallel to the first line i.e. $(a_1+\lambda a_2)l+(b_1+\lambda b_2)m+(c_1+\lambda c_2)n=0$ using (1')

$(a_1l+b_1m+c_1n)+\lambda(a_2l+b_1m+c_2n)=0$ (3)

Now eliminating λ (2) and (3) by equating the values of λ obtained from them, we obtain.

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_2\alpha+b_2\beta+c_2\gamma+d_2}=\frac{a_1l+b_1m+c_1n}{a_2l+b_2m+c_2n} \text{ i.e.}$$

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}=\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{a_2l+b_2m+c_2n}$$

This is required conditions.

4.9 Plane containing A Line And Parallel To Other Line:- To find the equation of the plane containing the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ (1)

and parallel to the line $\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$ (2)

Equation of any plane containing the line (1) is $A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0$ (3)

where $Al+Bm+Cn=0$ (4)

Now the plane (3) will be parallel to the line (2) if $Al'+Bm'+Cn'=0$ (5)

Eliminating A,B,C from equations (3), (4) and (5), we get

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0$$

This is the required equation of the plane.

4.10 Lines Intersecting Two Given Lines:- To find the equations of a line intersecting tow given liens $u_1=0=v_1$ and $u_2=0=v_2$ where the equation $u_1=0, v_1=0, u_2=0, v_2=0$ represent planes.

We know (from Section 3.13) that the equations $u_1 + \lambda_1 v_1 = 0$ (1)

Represents a plane through the line of intersection of planes $u_1 = 0$ and $v_1 = 0$ i.e. through the first given line.

Similarly, the equation $u_2 + \lambda_2 v_2 = 0$ (2)

Represents a plane through the second given line.

Now we consider the line of intersection of the plane (1) and (2). Since it lies in the plane (1), it is coplanar with the first given line. Therefore, it intersects the first given line, provided it is not parallel to the first line. Similarly, this line intersects the second given line, provided it is not parallel to the second line.

Hence the equation (1) and (2) together determine the required line.

The value of λ_1 and λ_2 are calculated from the additional condition given in the problem. In general, we can produce in finitely many lines intersecting the two given lines.

4.11 Shortest Distance (S.D):- Two non-intersecting and non-parallel line in space are called **skew lines.** Evidently, any two skew lines are always non-coplanar. Further, two skew lines can be connected by infinitely many line segments between them, as a variable point on one can be joined to a variable point on the other. Out of such line segments we shall be intersected here in that which is of the shortest distance.

In fact, we shall prove that the line segment which is perpendicular to both the skew lines has the shortest distance. But first we show below that such a line segment exists indeed.

Let the equation of two skew lines be $\dfrac{x-\alpha_1}{l_1} = \dfrac{y-\beta_1}{m_1} = \dfrac{z-\gamma_1}{n_1}$ (1)

and $\dfrac{x-\alpha_2}{l_2} = \dfrac{y-\beta_2}{m_2} = \dfrac{z-\gamma_2}{n_2}$ (2)

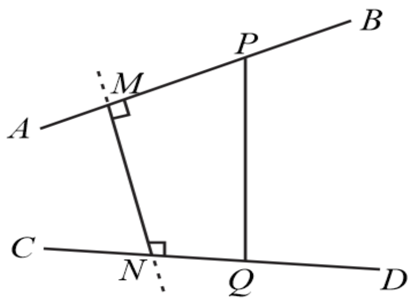
The equations of the planes passing through the origin and perpendicular to these lines are $l_1 x + m_1 y + n_1 z = 0$ (3)

And $l_2 x + m_2 y + n_2 z = 0$ (4)

As the lines (1) and (2) are non-parallel, so will be the planes (3) and (4). Let $x = a,\ y = b, z = c$ be a point (other than the origin) on the line of intersection of the planes (3) and (4). Then this point will obviously satisfy (3) and (4). Thus $l_1 a + m_1 b + n_1 c = 0$ and $l_2 a + m_2 b + n_2 c = 0$.

These two relations imply that a line with direction ratios a, b, c is perpendicular to both the given lines (1) and (2).

Let AB and CD be any two skew lines. Let MN be the line segment between the lines such that it is perpendicular to both of them. Let PQ be any line segment between the lines AB and CD.



Then both PM and QN are perpendicular to MN. Therefore MN is the projection of PQ on extended MN. If θ be the angle between MN and PQ. Then (by Section 2.1 result 1) we have $MN = PQ\cos\theta$.

But $|\cos\theta| \le 1$. Therefore $MN \le PQ$. Hence the length of MN is shortest one.

4.12 Length And Equations of The Shortest Distance:- To find the length and the equations of the shortest distance between the lines $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$

First Method:- Let λ, μ, ν be the direction cosines of the line of the shortest distance (S.D) between the given skew lines. Since the line of S.D is perpendicular to the two skew lines, we have $\lambda l + \mu m + \nu n = 0$ and $\lambda l' + \mu m' + \nu n' = 0$.

Solving these equations, we get $\frac{\lambda}{mn' - nm'} = \frac{\mu}{nl' - \ln'} = \frac{\nu}{lm' - ml'}$

$$= \pm \frac{\sqrt{\left(\lambda^2 + \mu^2 + \nu^2\right)}}{\sqrt{\left\{\Sigma\left(mn' - nm'\right)^2\right\}}} = \pm \frac{1}{\sqrt{\left\{\Sigma\left(mn' - nm'\right)^2\right\}}}.$$

Choosing positive sign, these equations given $\lambda = \frac{mn' - nm'}{\sqrt{\left\{\Sigma\left(mn' - nm'\right)^2\right\}}}$,

$$\mu = \frac{nl' - ln'}{\sqrt{\left\{\Sigma\left(mn' - nm'\right)^2\right\}}},\ \nu = \frac{lm' - ml'}{\sqrt{\left\{\Sigma\left(mn' - nm'\right)^2\right\}}}.$$

We know that the S.D. is the projection of the line joining the points (α, β, γ) and $(\alpha', \beta', \gamma')$ along the line whose direction cosines are λ, μ, ν as obtained above. Hence

S.D. $= (\alpha - \alpha')\lambda + (\beta - \beta')\mu + (\gamma - \gamma')\nu$

$$= \frac{(\alpha - \alpha')(mn' - mn') + (\beta - \beta')(nl' - \ln') + (\gamma - \gamma')(lm' - ml')}{\sqrt{\left\{\Sigma\left(mn' - nm'\right)^2\right\}}}$$

i.e. S.D. $=\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} + \sqrt{\left\{\Sigma\left(mn'-nm'\right)^2\right\}}$

further, we know that the S.D. is along the line of intersection of two planes containing the two given lines and parallel to the line whose direction cosines are λ,μ,ν as determined above. The equations to such planes are $\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}=0$ and

$$\begin{vmatrix} x-\alpha' & y-\beta' & z-\gamma' \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix}=0$$

Hence these are the equations of the line of shortest distance.

Remark:- In place of actual $d.c.'s$ l,m,n; l',m',n'; λ,μ,ν the corresponding direction ratios can also be used in these equations.

Second Method:- Refer to the figure of Section 4.11. Let $MP=r$ and $NQ=r'$. Then the co-ordinates of M and N are respectively.

$(\alpha+lr,\beta+mr,\gamma+nr)$ and $(\alpha'+l'r',\beta'+m'r',\gamma'+n'r')$.

Therefore direction ratios of MN are $\alpha+lr-\alpha'-l'r',\beta+mr-\beta'-m'r'$, $\gamma+nr-\gamma'-n'r'$.

The line segment MN will be the shortest distance between AB and CD if it is perpendicular to both of them. For this, we must have

$l(\alpha+lr-\alpha'-l'r')+m(\beta+mr-\beta'-m'r')+n(\gamma+nr-\gamma'-n'r')=0$

And $l'(\alpha+lr-\alpha'-l'r')+m'(\beta+mr-\beta'-m'r')+n'(\gamma+nr-\gamma'-n'r')=0$

Solving these two equations simultaneously, we get value of r and r'. After substituting these values of r and r' in the co-ordinates of M and N, the distance between the points M and N will give the shortest distance. Further, we can write the equations of the line of the shortest distance in symmetrical form.

Third Method:- The equation of the plane containing the first line and parallel to the second one is (by Section 4.9)

$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0$ i.e. $(x-\alpha)(mn'-nm')+(y-\beta)(nl'-\text{ln}')+(z-\gamma)(lm'-lm')=0$

Now the shortest distance between the given skew lines is the distance of any point on the second line from the above plane. But (α',β',γ') is a point on the second line. Hence

S.D. $=\dfrac{(\alpha'-\alpha)(mn'-nm')+(\beta'-\beta)(nl'-\text{ln}')(\gamma'-\gamma)(lm'-ml')}{\sqrt{\left\{\Sigma\left(mn'-nm'\right)^2\right\}}}$.

This method does not provided the equations of the shortest distance.

4.13 Shortest Distance Between The Lines in Unsymmetrical Form:- To find the length and the equations of the shortest distance between the lines

$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ and

$a_3x+b_3y+c_3z+d_3=0=a_4x+b_4y+c_4z+d_4$

Any plane containing the first line is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$ (1)

And any plane containing the second line is

$(a_3x+b_3y+c_3z+d_3)+\mu(a_4x+b_4y+c_4z+d_4)=0$ (2)

Where λ and μ are constants.

We may choose λ and μ such that the plane (1) and (2) are parallel. Then the shortest between the given lines is the distance between the planes (1) and (2).

Further, the equations of the line of the shortest distance are given by (i) the plane containing the first line and perpendicular to (1) or (2), and (ii) the plane containing the second line and perpendicular to (1) to (2). These two lines together represent the line of the shortest distance.

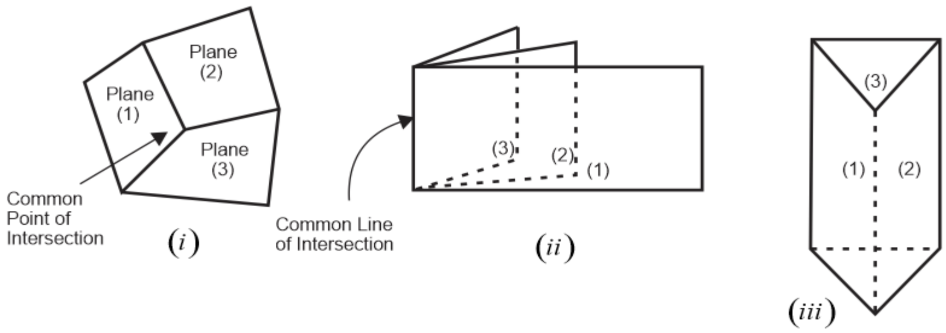
Note:- After transformation the given unsymmetrical equations into symmetrical form, we can use the method of section 4.12 conveniently.

4.14 Intersection of Three Planes:- Consider the three planes $a_1x+b_1y+c_1z+d_1=0$ (1)

$a_2x+b_2y+c_2z+d_2=0$ (2)

$a_3x+b_3y+c_3z+d_3=0$ (3)

No two of which are parallel. Then taking them in pairs, we shall obtain three lines of intersection of planes. These three lines may have (i) only one point in common, or (ii) all the points in common, or (iii) no point in common. According the three planes will intersect in a common point, or pass through a common line or, from a triangular prism, as shown in the following figure.



We know discuss these cases with the purpose to derive conditions of detect them.

(i) **Intersection of Three Planes in a Point:-** To find the condition that the three planes may intersect in a point.

Solving the equation (1), (2) and(3) by the method of determinants, we obtain $\frac{x}{\Delta_1}=\frac{y}{\Delta_2}=\frac{z}{\Delta_3}=\frac{-1}{\Delta_4}$

Where $\Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} c_1 & a_1 & d_1 \\ c_2 & a_2 & d_2 \\ c_3 & a_3 & d_3 \end{vmatrix}$

$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ $\Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Hence we see that the planes intersect at a (finite) point if $\Delta_4 \neq 0$ i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

(ii) **Intersection of Three Planes in a Common Line:-** To find the condition that the three planes may intersect in a common line.

Equation of any plane through the line of intersection of (1) and (2) is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$ i.e.

$$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0 \qquad (4)$$

Where λ is an arbitrary constant.

If the plane (1), (2) and (3) intersect in a common line, we can make (4) represent the same plane as (3) by property choosing the value of λ. For this comparing the corresponding coefficients in (4) and (3), we have $\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}=\frac{d_1+\lambda d_2}{d_3}=k$ say

Now equating each of the first three fractions to k and simplifying the resulting equations, we obtain.

$$a_1+\lambda a_2-ka_3=0$$
$$b_1+\lambda b_2-kb_3=0$$
$$c_1+\lambda c_2-kc_3=0$$
$$d_1+\lambda d_2-kd_3=0$$

Eliminating λ and $-k$ from these four equations taking three at a time, we get

$\Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} = 0$ $\Delta_2 = \begin{vmatrix} c_1 & a_1 & d_1 \\ c_2 & a_2 & d_2 \\ c_3 & a_3 & d_3 \end{vmatrix} = 0$

$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$ $\Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

These are four conditions but only two of them are independent. Thus if two of the determinants $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are zero, the remaining two will also be zero. Hence we may take $\Delta_4 = 0$ and $\Delta_3 = 0$ (5)

As the conditions for the planes to intersect in a common line.

We now show that Δ_1 and Δ_2 will also be zero under the conditions (5).

The equations in (5) can be written as $c_1A_1+c_2A_2+c_3A_3=0$ and $d_1A_1+d_2A_2+d_3A_3=0$, where A_1,A_2,A_3 are the cofactors of c_1,c_2,c_3 in Δ_4 and the cofactors of d_1,d_2,d_3 in Δ_3

Solving these relations, we obtain

$$\frac{A_1}{c_2d_3-c_3d_2}=\frac{A_2}{c_3d_1-c_1d_3}=\frac{A_3}{c_1d_2-c_2d_1}=\frac{1}{r}\text{ say}$$

i.e. $c_2d_3-c_3d_2=rA_1$, $c_3d_1-c_1d_3=rA_2$, $c_1d_2-c_2d_1=rA_3$ (6)

multiplying these equations by b_1,b_2,b_3, respectively, and adding the resulting equations, we get

$b_1(c_2d_3-c_3d_2)+b_2(c_3d_1-c_1d_3)+b_3(c_1d_2-c_2d_1)=r(b_1A_1+b_2A_2+b_3A_3)$ i.e.

$\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}=0$ since $b_1A_1+b_2A_2+b_3A_3=0$ from Δ_3 and Δ_4.

Thus $\Delta_1=0$. Similarly, multiplying the equations in (6) by a_1,a_2,a_3 respectively, and adding the resulting equations, we can show that $\Delta_2=0$.

(iii) **Triangular Prism:-** To find the condition that the three planes may form a triangular prism.

The three planes will from a triangular prism if the line of intersection of any two of them is parallel to the third one, but does not lie in that plane. As shown in section 4.4 the line of intersection of the planes (1) and (2) can represented by the equations.

$$\frac{x-\left(\dfrac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2}\right)}{b_1c_2-c_1b_2}=\frac{y-\left(\dfrac{d_1a_2-a_1d_2}{a_1b_2-b_1a_2}\right)}{c_1a_2-a_1c_2}=\frac{z=0}{a_1b_2-b_1a_2} \quad (7)$$

The condition that the plane (3) be parallel to this line is $a_3(b_1c_2-c_1b_2)+b_3(c_1a_2-a_2c_1)+c_3(a_1b_2-b_1a_2)=0$

i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$ i.e. $\Delta_4=0$ (8)

further, the condition that the line (7) does not lie on the plane (3) is that the fixed point used in (7) does not satisfy the equation (3). Thus

$$a_3\left(\frac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2}\right)+b_3\left(\frac{d_1a_2-a_1d_2}{a_1b_2-b_1a_2}\right)+c_3\ .\ 0+d_3\neq 0$$

i.e. $a_3(b_1d_2-d_1b_2)+b_3(d_1a_2-a_1d_2)+d_3(a_1b_2+b_1a_2)\neq 0$

i.e. $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \neq 0$ i.e. $\Delta_3 \neq 0$

(9)

combining (8) and (9) the condition that the three planes may from a triangular prism is that $\Delta_4 = 0$ and $\Delta_3 \neq 0$

An Intersecting Remark:- The equations of the three planes constitute a system of simultaneous linear equations. The three cases discussed above on the basis of geometrical configuration are equivalent to the following corresponding cases based on analytical ground:

(i) The system of equations has a unique solution

(ii) The system of equations has infinitely many solutions

(iii) The system of equations has no solution

Interestingly the same condition are obtained for these cases in matrix theory as above.

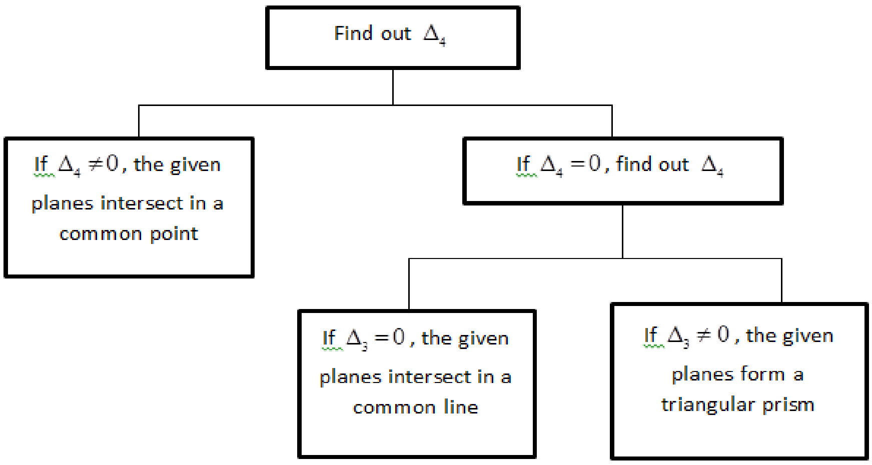
Working Rule:-

(1) Find out the value of Δ_4. If is non-zero the given planes intersect in a common point.

(2) If $\Delta_4 = 0$, find out Δ_3. If Δ_3 is also zero, the given planes intersect in a common line.

(3) If $\Delta_4 = 0$ but $\Delta_3 \neq 0$, the given planes from a triangular prism.

The above sets can be displayed in the form of a flow chart given below.



4.15 Line Intersecting Three Given Lines:- To find the locus which intersects the three lines $u_1 = 0 = v_1$, $u_2 = 0 = v_2$, $u_3 = 0 = v_3$ where $u_r = a_r x + b_r y + c_r z + d_r$ and $v_r = a_r' x + b_r' y + c_r' z + d_r'$, $r = 1, 2, 3$

As described in section 4.10 any line intersecting the first two given lines can be represented by the equations and $\left.\begin{array}{l} u_1 + \lambda v_1 = 0 \\ u_2 + \mu v_2 = 0 \end{array}\right\}$ (1)

In order that this line may intersect the third given line, it must be coplanar with the third line. Using section 4.8 (ii) the condition for this is:

$$\begin{vmatrix} a_3 & a_3^{'} & a_1 + \lambda a_1^{'} & a_2 + \lambda a_2^{'} \\ b_3 & b_3^{'} & b_1 + \lambda b_1^{'} & b_2 + \lambda b_2^{'} \\ c_3 & c_3^{'} & c_1 + \lambda c_1^{'} & c_2 + \lambda c_2^{'} \\ d_3 & d_3^{'} & d_1 + \lambda d_1^{'} & d_2 + \lambda d_2^{'} \end{vmatrix} = 0 \qquad (2)$$

This condition is a relation between λ and μ only. Let it be written as $f(\lambda, \mu) = 0$.

This equation is satisfied by an infinite number of values of λ and μ. Therefore an infinite number of lines can be found to intersect the given lines. The locus of all such lines is clearly a surface generated by them.

Now the required locus is obtained by putting the values of λ and μ received from (1) into (2).

Note:- While solving problems, the relation between λ and μ can be obtained by using the formula (2) or by some other manipulation.

4.16 Intersection of A Straight Line and A Curve:- If a variable line intersects a given curve, the co-ordinates of the plane of intersection will satisfy the equations of the straight line and that of the curve simultaneously. The condition for intersection is obtained by eliminating x, y, z from the equations of the straight line and curve. Then the locus of the line is found by eliminating the parameters involved in the condition obtained in this way.

Example1:- Find the equations of the line passing through the point $(2,1,3)$ and parallel to the line joining the points.

$(-2,3,-4)$ and $(2,5,1)$

Solution:- Direction rations of the line joining the points $(-2,3,-4)$ and $(2,5,1)$ are

$2-(-2), 5-3, 1-4(-4)$, i.e. $4,2,5$

Hence the equations of the line parallel to this line and passing through the point $(2,1,3)$ are

$$\frac{x-2}{4}=\frac{y-1}{2}=\frac{z-3}{5}.$$

Example2:- Find the equations of a line passing through the point $(2,1,3)$ and parallel to the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z+1}{-1}$

Solution:- Direction ratios of any line parallel to the line

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z+1}{-1}$$

Will be $2,3,-1$

Hence the equations of the line passing through the point $(2,1,3)$ and parallel to the line (1) are given by

$$\frac{x-2}{2}=\frac{y-1}{3}=\frac{z-3}{-1}$$

Example3:- (i) Show that the equation of the plane passing through the point (a,b,c) and perpendicular to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is:

$$(x-a)l+(y-b)m(z-c)n=0$$

(ii) Find the equation of the plane passing through the point $(2,3,4)$ and perpendicular to the line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{5}$.

Solution:- (i) Equation of any plane perpendicular to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is

$$lx+my+nz=k \qquad (1)$$

If, in addition, it passes through the point (a,b,c), its equation must be satisfied by this point. So, we have

$$la+mb+nc=k$$

Putting this value of k is (1), we have

$$lx+my+nz=la+mb+nc$$

I.e. $(x-a)l(y-b)m+(z-c)n=0$

This is the required equation of plane.

(ii) Proceeding as in part (i) above, the required equation of plane is

$$-1.(x-2)+2.(y-3)+5.(z-4)=0,$$

i.e. $-x+2y+5z-24=0$

Example4:- Find the point at which the line $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-2}{3}$ meets the plane $2x-3y+z+5=0$.

Solution:- Co-ordinates of any point on the line

$$\frac{x-1}{2}=\frac{y-1}{3}=\frac{z-2}{3}(=r)$$

And $(1+2r,1+3r,2+3r)$. If this point lies on the plane

$$2x-3y+z+5=0$$

We must have

$$2(1+2r)-3(1+3r)+(2+3r)+5=0$$

i.e. $(2-3+2+5)+(4-9+3)r=0,$

i.e. $6-2r=0$, i.e. $r=3$

Hence the co-ordinates of the required point are

$(1+2\times3,1+3\times3,2+3\times3)$, i.e. $(7,10,11)$

Example5: - Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-12}{2}$ and the plane $x-y+z=5$.

Solution: - Any point on the given line is

$$(3r+2, 4r-1, 12r+2) \qquad (1)$$

If it lies on the given plane, it must satisfy the equation of the plane.

Thus $(3r+2)-(4r-1)+(12r+2)=5$, i.e. $11r+5=5$

It gives $r=0$. Using this value of r in (1), we find $(2,-1,2)$ as the point of intersection of the given line and plane.

Now the required distance of the given point $(-1,-5,-10)$ from this point will be

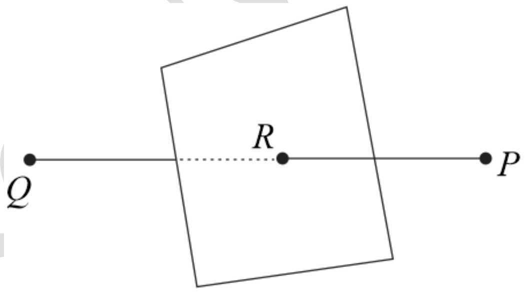
$$\sqrt{\left\{(2+1)^2+(-1+5)^2+(2+10)^2\right\}}$$, i.e. 13 (on simplifying).

Example6:- Find the image of the point $(3,5,7)$ in the plane

$$2x+y+z=6$$

Solution:- Let P denote the given point and Q be its image in the given plane. Then PQ is perpendicular to the given plane and is bisected by it at R, say, on the given plane.

Clearly, $2,1,1$ are direction ratios of PQ



Now equation of PQ are

$$\frac{x-3}{2}=\frac{y-5}{1}=\frac{z-7}{1}$$

Any point on this line is $(3+2r, 5+r, 7+r)$. Let it be Q. Since R is the midpoint of PQ, its co-ordinates will be

$$\left(\frac{3+(3+2r)}{2}, \frac{5+(2+r)}{2}, \frac{7+(7+r)}{2}\right), \text{ i.e. } \left(3+r, 5+\frac{1}{2}r, 7+\frac{1}{2}r\right)$$

These co-ordinates of R must satisfy the equation of given plane. So, we have

$$2(3+r)+\left(5+\frac{1}{2}r\right)+\left(7+\frac{1}{2}r\right)=6, \text{ which gives } r=-4$$

Hence the co-ordinates of Q are

$$(3-8, 5-4, 7-4), \text{ i.e. } (-5,1,3)$$

Example7:- Find the distance of the point $(1,1,1)$ from the plane

$$2x-y+2z+3=0$$

Solution:- Equation of the line passing through the point $(1,1,1)$ and perpendicular to the plane $2x-y+2z+3=0$ are

$$\frac{x-1}{2}=\frac{y-1}{-1}=\frac{z-1}{2}$$

Any point on this line is $(2r+1,-r+1,2r+1)$. If this point lies on the given plane, we must have

$$2(2r+1)-(-r+1)+2(2r+1)+3=0$$

i.e. $9r=-6$, which gives $r=-\frac{2}{3}$

Putting this value of r, the above point comes out to be

$$\left(-\frac{4}{3}+1,\frac{2}{3}+1,-\frac{4}{3}+1\right), \text{ i.e. } \left(-\frac{1}{3},\frac{5}{3},-\frac{1}{3}\right)$$

Clearly, this point is the foot of perpendicular from the point $(1,1,1)$ on the given plane.

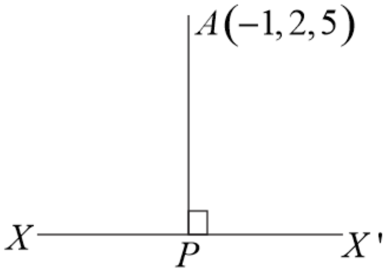
Hence the required distance.

$$=\sqrt{\left\{\left(1+\frac{1}{3}\right)^2+\left(1-\frac{5}{3}\right)^2+\left(1+\frac{1}{3}\right)^2\right\}}=\sqrt{\left(\frac{16}{9}+\frac{4}{9}+\frac{16}{9}\right)}=\sqrt{\left(\frac{36}{9}\right)}=\sqrt{4}=2$$

Example8:- Prove that the distance of the point $(-1,2,5)$ from the line which passes through $(3,4,5)$ and whose direction cosines are proportional to $2,-3,6$ is $4\sqrt{61}/7$.

Solution:- The equations of the given line, say XX', are

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-5}{6}$$



Any point on this line is

$$P(3+2r,4-3r,5+6r)$$

If the given point $(-1,2,5)$ is denoted by A, then direction ratios of AP are

$$(3+2r)-(-1),(4-3r)-2,(5+6r)-5,$$

i.e. $4+2r,\ 2-3r,6r$

If P is foot of the perpendicular from A on XX', then from the condition of perpendicularity, we must have

$$2(4+2r)-3(2-3r)+6(6r)=0, \text{ i.e. } 8+4r-6+9r+36r=0$$

i.e. $49r+2=0$, which given $r=-2/49$

So, the co-ordinates of P (as the foot of perpendicular from A on XX')

Are $\left(3+2\left(-\frac{2}{49}\right),4-3\left(-\frac{2}{49}\right),5+6\left(-\frac{2}{49}\right)\right)$, i.e. $\left(\frac{143}{49},\frac{202}{49},\frac{233}{49}\right)$

Hence the required distance AP

$$=\sqrt{\left\{\left(\frac{143}{49}+1\right)^2+\left(\frac{202}{49}-2\right)^2+\left(\frac{233}{49}-5\right)^2\right\}}$$

$$=\sqrt{\left\{\left(\frac{192}{49}\right)^2+\left(\frac{104}{49}\right)^2+\left(-\frac{12}{49}\right)^2\right\}}=\frac{1}{49}\sqrt{\left\{(192)^2+(104)^2+144\right\}}$$

$$=\frac{1}{49}\sqrt{(36864+10816+144)}=\frac{1}{49}\sqrt{(47824)}$$

$$=\frac{1}{49}\sqrt{(16\times 49\times 61)}=\frac{1}{49}\times 4\times 7\sqrt{(61)}=\frac{4}{7}\sqrt{(61)}.$$

Example9:- From the point $P(1,2,3)$, PN is drawn perpendicular to the line

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

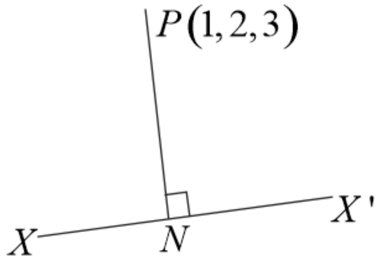
Find the distance PN, equations of PN and the co-ordinates of N

Solution:- Any point on the given line is

$$(3r+2,4r+3,5r+4) \qquad (1)$$

Direction ratios of the line joining the given point $P(1,2,3)$ and this point are $(3r+2)-1,(4r+3)-2,(5r+4)-3$,

i.e. $3r+1,\ 4r+1,\ 5r+1$ (2)



If the point given by (1) is N, the foot of perpendicular on P on the given line, then the numbers given by (2) would be direction ratios of PN. Then by the condition of perpendicularity between PN and the given line, we have

$3(3r+1)+4(4r+1)+5(5r+1)=0$, i.e. $50r+12=0$

Which gives $r=-\frac{6}{25}$

Using this value of r in (1) and (2), the co-ordinates of N are

$$\left(-\frac{18}{25}+2,-\frac{24}{25}+3,-\frac{6}{5}+4\right),\text{ i.e. }\left(\frac{32}{25},\frac{51}{25},\frac{14}{5}\right), \qquad (A)$$

Now PN $=\sqrt{\left\{\left(\frac{32}{25}-1\right)^2+\left(\frac{51}{25}-2\right)^2+\left(\frac{14}{5}-3\right)^2\right\}}=\sqrt{\left\{\left(\frac{7}{25}\right)^2+\left(\frac{1}{25}\right)^2+\left(-\frac{1}{5}\right)^2\right\}}$

$$=\sqrt{\left(\frac{49}{625}+\frac{1}{625}+\frac{25}{625}\right)}=\sqrt{\left(\frac{75}{652}\right)}=\sqrt{\left(\frac{3}{25}\right)}=\frac{1}{5}\sqrt{3} \qquad \text{(B)}$$

Lastly, direction ratios of PN are

$$\frac{32}{25}-1,\frac{51}{25}-2,\frac{14}{5}-3, \text{ i.e. } \frac{7}{25},\frac{1}{25},-\frac{1}{25}, \text{ i.e. } 7,1,-5$$

Hence the equations of PN are (using the fixed point P)

$$\frac{x-1}{7}=\frac{y-2}{1}=\frac{z-3}{-5} \qquad \text{(C)}$$

The required results are obtained in (B), (C) and (A)

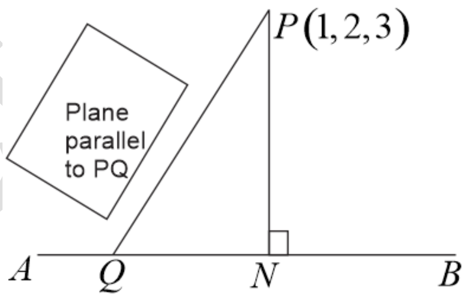
Example10: - The equations to AB are $\frac{x}{2}=\frac{y}{-3}=\frac{z}{5}$. Through a point $P(1,2,3)$, PN is drawn perpendicular to AB and PQ is drawn parallel to the plane $3x+4y+5z=0$ to meet AB in Q. Find the equations of PN and PQ.

Solution:- The points N and Q are on the line AB given by

$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{5} \qquad (1)$$

Such that PN is perpendicular to AB and PQ is parallel to the plane.

$$3x+4y+5z=0 \qquad (2)$$



From (1), the co-ordinates of N and Q may be taken as

$$N(2r,-3r,5r),Q(2r',-3r',5r')$$

Therefore, we have

Direction ratios of PN : $\quad 2r-1,-3r-2,5r-3 \qquad (3)$

And direction ratios of PQ: $\quad 2r'-1,-3r'-2,5r'-3 \qquad (4)$

Since PN is perpendicular to the line (1), using the condition of perpendicularity between the two lines, we have

$$2(2r-1)-3(-3r-2)+5(5r-3)=0, \text{ which gives } r=\frac{11}{38}$$

Using this value of r in (3), direction ratios of PN are $-\frac{16}{38},-\frac{109}{38},-\frac{59}{38}$, or more simply, $16,109,59$. Hence the equations of PN are

$$\frac{x-1}{16}=\frac{y-2}{109}-\frac{z-3}{59}$$

Further, since PQ is parallel to the plane (2), using the condition of parallelism between a plane and a line, we have

$$3(2r'-1)+4(-3r'-2)+5(5r'-3)=0, \text{ which gives } r'=\frac{26}{19}$$

Using this value of r' in (4), direction ratios of PQ are $\frac{33}{19},-\frac{116}{19},\frac{73}{19}$ or more simply, $33,-116,73$. Hence the equations of PQ are

$$\frac{x-1}{33}=\frac{y-2}{-116}=\frac{z-3}{73}.$$

Example11:- Find the equation of the line passing through the points $(1,2,3)$ and $(2,3,5)$

Solution:- Direction ratios of the line passing through the points $(1,2,3)$ and $(2,3,5)$ are

$2-1, 3-2, 5-3$, i.e. $1,1,2$

Hence the equations of the lines are

$$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{2}, \text{ (using the point } (1,2,3))$$

Or $$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-5}{2}, \text{ (using the point } (2,3,5))$$

Example12:- Find the point where the perpendicular from the origin on the line joining the points $(-9,4,5)$ and $(11,0,-1)$ meets it.

Solution:- The equations to the line joining the points $(-9,4,5)$ and $(11,0,-1)$ are

$$\frac{x-(-9)}{11-(-9)}=\frac{y-4}{0-4}=\frac{z-5}{-1-5}, \text{ i.e. } \frac{x+9}{20}=\frac{y-4}{-4}=\frac{z-5}{-6}$$

i.e. $$\frac{x+9}{-10}=\frac{y-4}{2}=\frac{z-5}{3}. \quad (1)$$

Any point P on this line is

$(-10r-9, 2r+4, 3r+5)$ (2)

Evidently, direction ratios of OP are

$-10r-9,\ 2r+4, 3r+5$

If OP is perpendicular on the line (1), using the condition of perpendicularity, we have

$$-10(-10r-9)+2(2r+4)+3(3r+5)=0, \text{ which gives } r=-1$$

Using this value of r in (2), the co-ordinates of P (the foot of perpendicular from O on the line (1)) are

$(10-9,-2+4,-3+5)$, i.e. $(1,2,2)$

Example13:- Find the direction ratios of the line determined by the planes $2x-y-z=2$ and $x+2y-3z=11$.

Solution:- Equations of the planes determining the lines are given to be

$$2x-y-z=2$$

And $x+2y-3z=11$

If l,m,n be the $d.c.'s$ of the line, we have

$$\frac{l}{(-1)(-3)-(-1).2}=\frac{m}{(-1).1-2.(-3)}=\frac{n}{2.2-(-1).1}$$

i.e. $\frac{l}{5}=\frac{m}{5}=\frac{n}{5}$ i.e. $\frac{l}{1}=\frac{m}{1}=\frac{n}{1}$

Hence the required direction ratios of the given line are $1,1,1$

Example14:- Find the symmetrical from of equation of the line determined by the planes $x+y-z=2$ and $2x+y+z=1$.

Solution:- Equations of the planes determining the required lines are given to be

$$x+y-z=2 \qquad (1)$$

And $2x+y+z+=1$ (2)

If l,m,n be the $d.c.'s$ of the line then we have

$$\frac{l}{1.1-(-1).1}=\frac{m}{(-1).2-1.1}=\frac{n}{1.1-1.2}, \text{ i.e. } \frac{l}{2}=\frac{m}{-3}=\frac{n}{-1}.$$

Further, to obtain the co-ordinates of one point on the line, let us put $z=0$ in (1) and (2). This gives

$$x+y=z, \qquad 2x+y=1$$

Solving these, we have $x=-1, y=3$

Thus $(-1,3,0)$ is a point on the line

Hence the equations of the line are

$$\frac{x+1}{2}=\frac{y-3}{-3}=\frac{z}{-1}$$

Example15:- Show that the angle between the lines

$$x-2y+z=0=x+y-z$$

And $x+2y+z=0=8x+12y+5z$ is $\cos^{-1}\left(\frac{8}{\sqrt{(406)}}\right)/$

Solution:- Clearly, both the lines pass through the origin.

The equations of the first line in symmetrical form can be written as

$$\frac{x}{(-2)(-1)-1.1}=\frac{y}{1.1-1(-1)}=\frac{z}{1.1-(-2).1}$$

i.e. $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ (1)

Similarly, the equations of the second line in symmetrical form are

$$\frac{x}{2.5-1.12}=\frac{y}{1.8-1.5}=\frac{z}{1.12-2.8}$$

i.e. $\frac{x}{2}=\frac{y}{-3}=\frac{z}{4}$ (2)

If θ be the angle between (1) and (2), we have

$$\cos\theta=\frac{1.2+2(-3)+3.4}{\sqrt{(1^2+2^2+3^2)}\sqrt{\{2^2++(-3)^2+4^2)\}}}=\frac{8}{\sqrt{(14)}\sqrt{(29)}}=\frac{8}{\sqrt{(406)}}.$$

Hence the required angle is $\cos^{-1}\left(\frac{8}{\sqrt{(406)}}\right)$

Example16:- Prove that the lines

$x=ay+b, z=cy+d$ and $x=a'y+b', z=c'y+d'$

Are perpendicular if and only if $aa'+cc'+1=0$

Solution:- The equations of the first line can be written as

$$\frac{x-b}{a}=y, \frac{z-d}{c}=y$$

These equations can be written as

$$\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c} \quad (1)$$

Which is symmetrical form of the first line.

Similarly, the symmetrical form of the second line is

$$\frac{x-b'}{a'}=\frac{y}{1}=\frac{z-d'}{c'} \quad (2)$$

Now using the condition of perpendicularity, we find that the lines (1) and (2) are perpendicular if and only if

$aa'+1.1+cc'=0$, i.e. $aa'+cc'+1=0$

Example17:- Find the angle between the plane $2x-3y+6x=8$ and the line $\frac{x-1}{2}=\frac{y-2}{5}=\frac{z+4}{4}$.

Solution:- We known that the angle between a plane and a straight line is the complement of the angle between the normal to that plane and the straight line.

Hence if θ is the required angle, then

$$\cos(90^\circ-\theta)=\frac{2.2-3.5+6.4}{\sqrt{\{2^2+(-3)^2+6^2\}}\sqrt{(2^2+5^2+4^2)}}$$

i.e. $\sin\theta=\frac{4-15+24}{\sqrt{(4+9+36)}\sqrt{(4+25+16)}}=\frac{13}{\sqrt{(49)}\sqrt{(45)}}=\frac{13}{7.3\sqrt{5}}=\frac{13}{21\sqrt{5}}$

Hence $0=\sin^{-1}(13/21\sqrt{5})$.

Example18:- Find the angle between the line given by

$y+z-5=0=x+y+z$ and $x-y+7=0$

Solution:- The equations of given line can be written as

$0.x+y+z=5$

And $x+y+z=0$

If l,m,n be the $d.c.'s$ of this line, then we have

$$\frac{l}{1.1-1.1}=\frac{m}{1.1-0.1}=\frac{n}{0.1-1.1} \text{ i.e. } \frac{l}{0},\frac{m}{1}=\frac{n}{-1}$$

Thus direction ratios of the given line are $0,1,-1$. Also, direction ratios of normal to the given plane are $1,-1,0$

If θ denote the angle between the given line and plane, then the angle between the given line and normal to the given plane will be $90^\circ-0$. So, we have

$$\cos\left(90^\circ-0\right)=\frac{0.1+1(-1)+(-1).0}{\sqrt{\left\{0^2+1^2+(-1)^2\right\}}\sqrt{\left\{1^2+(-1)^2+0^2\right\}}} \text{ i.e. } \sin\theta=-\frac{1}{2}$$

Hence $\theta=-30^\circ,-150^\circ$.

Example19:- Find the equation to the plane which contains the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ and is perpendicular to the plane $x+2y+z=12$.

Hence or otherwise find the direction cosines of the projection of the given line on the given plane.

Solution:- Equation of any plane containing the given line is

$$A(x-1)+B(y+1)+C(z-3)=0 \qquad (1)$$

Where $2A-B+4C=0$ (2)

This plane will be perpendicular to the given plane if

$$A+2B+C=0$$

Solving (2) and (3) to get proportionate value of $A,B,C,$ we obtain

$$\frac{A}{(-1)1-4.2}=\frac{B}{4.1-2.1}=\frac{C}{2.2-(-1).1}, \text{ i.e. } \frac{A}{-9}=\frac{B}{2}=\frac{C}{5}.$$

Using these proportional values of A,B,C in (1), we have

$$-9(x-1)+2(y+1)+5(z-3)=0$$

i.e. $9x-2y-5z+4=0$ (First result)

Now the projection of the given line on the given plane is the line of intersection of the given plane and the plane determined above. Hence this projection is given by the equations.

$$x+2y+z-12=0 \qquad (4)$$

And $9x-2y-5z+4=0$ (5)

If l,m,n be the direction cosines of the projection, then since this projection is perpendicular to both (4) and (5), we have

$$\frac{l}{2(-5)-1(-2)}=\frac{m}{1.9-1(-5)}=\frac{n}{1(-2)-2.9}, \text{ i.e. } \frac{l}{-8}=\frac{m}{14}=\frac{n}{-20}$$

i.e. $\dfrac{l}{4}=\dfrac{m}{-7}=\dfrac{n}{10}=\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{4^2+(-7)^2+10^2\}}}=\pm\dfrac{1}{\sqrt{(165)}}$

Hence $l=\pm\dfrac{4}{\sqrt{(165)}},\ m=\mp\dfrac{7}{\sqrt{(165)}},n=\pm\dfrac{10}{\sqrt{(165)}}$ (second result)

Example20:- Find the projection of the line $x=3-6t,\ y=2t, z=3+2t$ in the plane $3x+4y-5z+26=0$.

Solution:- The equations of the given line can be written as

$$\frac{x-3}{-6}=\frac{y-0}{2}=\frac{z-3}{2}(=t) \qquad (1)$$

Also, the given plane is

$$3x+4y-5z+26=0 \qquad (2)$$

Any plane through the line (1) is

$$a(x-3)+b(y-0)+c(z-3)=0 \qquad (3)$$

Where $-6a+2b+2c=0$ (4)

If the plane (3) be orthogonal to (2), we must have

$$3a+4b-5c=0 \qquad (5)$$

Solving (4) and (5), we have

$$\frac{a}{2(-5)-2.4}=\frac{b}{2.3-(-6)(-5)}=\frac{c}{(-6).4-2.3},$$

i.e. $\dfrac{a}{-18}=\dfrac{b}{-24}=\dfrac{c}{-30}$, i.e. $\dfrac{a}{3}=\dfrac{b}{4}=\dfrac{c}{5}$

Using these proportionate values of a,b,c in (3), we have

$$3(x-3)-4(y-0)+5(z-3)=0$$

i.e. $3x+4y+5z-24=0$ (6)

Since the plane (6) contains the given line (1) and is perpendicular to the given plane (2), it follows that the line of intersection of planes (2) and (6) is the projection of (1) on (2).

Hence the required projection is given by

$$3x+4y-26=0=3x+4y+5z-24$$

Example21:- Show that the line $\dfrac{x-2}{2}=\dfrac{y}{1}=\dfrac{z+2}{2}$ and $\dfrac{x-6}{3}=\dfrac{y-2}{2}=\dfrac{z-2}{4}$ are coplanar.

Solution:- Here $\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ a & b & c \\ a' & b' & c'\end{vmatrix}=\begin{vmatrix}2-6 & 0-2 & -2-2 \\ 2 & 1 & 2 \\ 3 & 2 & 4\end{vmatrix}$

$$=\begin{vmatrix}-4 & -2 & -4 \\ 2 & 1 & 2 \\ 3 & 2 & 4\end{vmatrix}=\begin{vmatrix}0 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 4\end{vmatrix}=0 \text{ by } R_{12}(2)$$

Hence the given lines are coplanar.

Example22:- Show that the line $\frac{x-4}{1}=\frac{y+1}{-2}=\frac{z}{1}$ and $4x-y+5z-7=0=2x-5y-z-3$ are coplanar.

Solution:- Any plane through the second line is

$$(4x-y+5z-7)+\lambda(2x-5y-z-3)=0$$

i.e. $(4+2\lambda)x-(1+5\lambda)y+(5-\lambda)z-(7+3\lambda)=0$

This plane contains the first line if

$$(4.4-(-1)+5.0-7)+\lambda(2.4-5y-z-3)=0$$

and $1(4+2\lambda)-(-2)(1+5\lambda)+1(5-\lambda)=0$

i.e. $(16+1-7)+\lambda(8+5-3)=0$ and $(4+2+5)=\lambda(2+10-1)=0$

i.e. $10+10\lambda=0$ and $11+11\lambda=0$

Since either of these equations gives the same value of λ. Viz. -1, the two liens are coplanar.

Example23:- Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ intersect. Find the co-ordinates of the points of intersection and the equation to the plane containing term.

Solution:- Any point on the first line is $(-1,-3r,3+2r,-2+r)$, and any point on the second line is $(r',7-3r',-7+2r')$. The two lines will intersect if these two point are the same for some values of r and r'. For this, equating the corresponding co-ordinates, we have

$-1-3r=r'$ i.e. $3r+r'=-1$ (1)

$3+2r=7-3r'$ i.e. $2r+3r=4$ (2)

$-2+r=-7+2r'$ i.e. $r-2r'=-5$ (3)

Solving (1) and (2), we have $r=-1$ and $r'=2$. These values satisfy equation (3) also. Therefore, the point on the first lien for $r=-1$ is the same as that on the second line for $r'=2$. Clearly, the co-ordinates of this point are $(2,1,-3)$

Hence the two lines intersect and the co-ordinates of their point of intersection are $(2,1,-3)$

Further, the equation to the plane containing the given lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0 \text{ i.e. } x+y+z=0$$

Example24:- Show that the equation to the plane through $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and perpendicular to the plane containing $\frac{x}{m}=\frac{y}{n}=\frac{z}{l}$ and $\frac{x}{n}=\frac{y}{l}=\frac{z}{m}$ is $(m-n)x+(n-l)y+(l-m)z=0$

Solution:- Clearly, all the three lines pass through the origin. Any plane through the first line is

$$Ax+By+Cz=0 \tag{1}$$

Where $Al+Bm+Cn=0$ (2)

Also, the equation of plane containing second and third lines is

$$\begin{vmatrix} x & y & z \\ m & n & l \\ n & l & m \end{vmatrix} = 0$$

i.e. $\quad (nm-l)^2 x + \left(l n + m^2\right) y + \left(ml + n^2\right) z = 0$

If plane (1) is perpendicular to this plane, we have

$$A\left(nm - l^2\right) + B\left(l n - m^2\right) + C\left(ml - n^2\right) z = 0 \qquad (3)$$

Solving (2) and (3), we have

$$\frac{A}{m\left(ml - n^2\right) - n\left(l n - m^2\right)} = \frac{B}{n\left(nm - l^2\right) - l\left(ml - n^2\right)} = \frac{C}{l\left(l n - m^2\right) - m\left(nm - l^2\right)}$$

i.e. $\dfrac{A}{m^2 l - nm^2 - l n^2 + nm^2} = \dfrac{B}{n^2 m - nl^2 - ml^2 + l n^2} = \dfrac{C}{l^2 n - lm^2 - nm^2 + ml^2}$

i.e. $\dfrac{A}{\left(m^2 - n^2\right) l + mn(m-n)} = \dfrac{B}{\left(n^2 - l^2\right) m + l n(n-1)} = \dfrac{C}{\left(l^2 - m^2\right) n + lm(l-m)}$

i.e. $\dfrac{A}{(m-n)(m+n) l + mn\}} = \dfrac{B}{(n-1)(n+1) m + l n\}} = \dfrac{C}{(l-m)(l+m) n + lm\}}$

i.e. $\dfrac{A}{m-n} = \dfrac{B}{n-l} = \dfrac{C}{l-m}$

Using these proportionate values of A, B, C in (1), the required equation of plane is

$$(m-n)x + (n-l)y + (l-m)z = 0$$

Example25:- $A, A'; B, B'; C, C'$ are points on the axes. Show that the lines of intersection of the panes $A'BC, AB'C; B'CA : C'AB, CA'B$ are coplanar.

Solution:- If the points $A, A'; B, B'; C, C'$ be taken as $(a,0,0), (a',0,0); (0,b,0); (0,0,c), (0,0,c')$ respectively, then equations to the planes $A'BC, AB'C$ are $\dfrac{x}{a'} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, $\dfrac{x}{a} + \dfrac{y}{b'} + \dfrac{z}{c'} = 1 \qquad (1)$

These equation together represent the line of intersection of the planes. $A'BC, AB'C$.

Similarly, the lines of intersection of the planes $B'CA, BC'A$ and $C'AB$, $CA'B$ are given by

$$\frac{x}{a} + \frac{y}{b'} + \frac{z}{c} = 1, \ \frac{x}{a'} + \frac{y}{b} + \frac{z}{c'} = 1 \qquad (2)$$

And $\quad \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c'} = 1, \qquad \dfrac{x}{a'} + \dfrac{y}{b'} + \dfrac{z}{c} = 1 \qquad (3)$

Any planes through the line given by (1), (2) and (3) are

$$\left(\frac{x}{a'} + \frac{y}{b} + \frac{z}{c} - 1\right) + \lambda\left(\frac{x}{a} + \frac{y}{b'} + \frac{z}{c'} - 1\right) = 0 \qquad (4)$$

$$\left(\frac{x}{a} + \frac{y}{b'} + \frac{z}{c} - 1\right) + \mu\left(\frac{x}{a'} + \frac{y}{b} + \frac{z}{c'} - 1\right) = 0 \qquad (5)$$

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c'}-1\right)+\nu\left(\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c}-1\right)=0 \qquad (6)$$

By inspection, we find that for $\lambda=\mu=\nu=1$ each of the planes given by (4), (5) and (6) reduces to

$$\left(\frac{1}{a}+\frac{1}{a'}\right)x+\left(\frac{1}{b}+\frac{1}{b'}\right)y+\left(\frac{1}{c}+\frac{1}{c'}\right)z=2 \qquad (7)$$

Hence the three given lines are coplanar and each lies on the plane (7).

Example26:- Find the equation to the plane which passes through the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ and is parallel to the lines $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Solution:- Equation of any plane containing the first given line is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0 \qquad (1)$$

i.e. $(a+\lambda a')x+(b+\lambda b')+y(c+\lambda c')z+(d+\lambda d')=0$

if this is parallel to the second given line, then we must have

$$l(a+\lambda a')+m(b+\lambda b')+n(c+\lambda c')=0$$

i.e. $\quad (al+bm+cn)+\lambda(a'l+b'm+c'n)=0$

i.e. $\quad \lambda=-\dfrac{al+bm+cn}{a'l+b'm+c'n}$

Using this value of λ in (1), the required equation of the plane is

$$(ax+by+cz+d)-\left(\frac{al+bm+cn}{a'l+b'm+c'n}\right)(a'x+b'y+c'z+d')=0$$

i.e. $\quad (a'l+b'm+c'n)(ax+by+cz+d)=(al+bm+cn)(a'x+b'y+c'z+d')=0$

Example27:- Find the equation to the line which intersection the lines

$x+y+z=1,\quad 2x-y-z=2$;

$x-y-z=3,\quad 2x+4y-z=4$, and passes through the point $(1,1,1)$.

Solution:- Any line intersection the given lines can be represented by

$$\left.\begin{aligned}(x+y+z-1)+\lambda(2x-y-z-2)&=0\\(x-y-z-3)+\mu(2x+4y-z-4)&=0\end{aligned}\right\} \qquad (1)$$

If it passes through the point $(1,1,1)$ these co-ordinates must satisfy the equations (1). Thus we have

$(1+1+1-1)+\lambda(2-1-1-2)=0$, which gives $\lambda=1$;

And $\quad (1-1-1-3)+\mu(2+4-1-4)=0$, which gives $\mu=4$

Substituting these values of λ and μ in (1), the required equations of the line are

$(x+y+z-1)+(2x-y-z-2)=0$

And $\quad (x-y-z-3)+4(2x+4y-z-4)=0$

i.e. $\quad 3x-3=0$ and $9x+15y-5z-9=0$

i.e. $\quad x-1=0$ and $9x+15y-5z-19=0$

Using the first of these equations to the second one, the equations of the same line can be written as

$$x-1=0,\ 15y-5z-10=0$$

i.e. $$x-1=0=3y-z-2$$

Example28:- Find the equations to the line which intersects the lines $2x+y-1=0=x-2y+3z$; $3x-y+z+2=0=4x+5y-2z-3$ and is parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

Solution:- Any line intersecting the first two given lines is

$$(2x+y-1)+\lambda(x-2y+3z)=0$$

And $$(3x-y+z+2)+\mu(4x+5y-2z-3)=0$$

i.e. $$\left.\begin{array}{l}(2+\lambda)x+(1-2\lambda)y+3\lambda z-1=0\\(3+4\mu)x+(-1+5\mu)y+(1-2\mu)z+(2-3\mu)=0\end{array}\right\} \quad (1)$$

This line will be parallel to the third given line if each of the planes in (1) is parallel to the third line. Using the condition of parallelism between a plane and a line, it follows that

$$1.(2+\lambda)+2.(1-2\lambda)+3.(3\lambda)=0 \text{ which gives } \lambda=-\frac{2}{3}$$

And $$1.(3+4\mu)x+(-1+5\mu)+3.(1-2\mu)=0,\text{ which gives } \mu=-\frac{1}{2}$$

Substituting these values of λ and μ in (1), the required equation are

$$\left(2-\frac{2}{3}\right)x+\left(1+\frac{4}{3}\right)y-3.\frac{2}{3}z-1=0$$

And $$\left(3-4.\frac{1}{2}\right)x+\left(-1-5.\frac{1}{2}\right)y+\left(1+2.\frac{1}{2}\right)z+\left(2+3.\frac{1}{2}\right)=0$$

Which, on simplifying, can be written as

$$4x+7y-6z-3=0=2x-7y+4z+7$$

Example29:- Find the shortest distance, between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$.

Solution:- Let l,m,n be the direction cosines of the line of S.D. between the two given lines. Since the line of S.D is perpendicular to both the given lines, we have

$$2l+3m+4n=0$$

And $$3l+4m+5n=0$$

Solving these equation, we get

$$\frac{l}{3.5-4.4}=\frac{m}{4.3-2.5}=\frac{n}{2.4-3.3},\text{ i.e. } \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1},$$

i.e. $$\frac{l}{1}=\frac{m}{-2}=\frac{n}{1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{1^2+(-2)^2+1^2\}}}=\frac{1}{\sqrt{6}}$$

Hence $1=\frac{l}{\sqrt{6}},\ m=\frac{-2}{\sqrt{6}},\ n=\frac{1}{\sqrt{6}}$ (1)

Now the length of S.D. is the projection of the join of the points $(1,2,3)$ and $(2,3,4)$ along the line whose $d.c.'s$ are given by (1). Hence

$$S.D.=(1-2).\frac{1}{\sqrt{6}}+(2-4)\left(\frac{-2}{\sqrt{6}}\right)+(3-5).\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6}}.$$

Example30:- Find the length and the equations to the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

Solution:- Let l,m,n be the direction cosines of the lie of S.D. between the two given lines. Since the line of S.D. is perpendicular to both the given lines, we have

$$3l-m+n=0$$

And $-3l+2m+4n=0$

Solving these equation, we get

$$\frac{l}{(-1).4-1.2}=\frac{m}{1.(-3)-3.4}=\frac{n}{3.2-(-1)(-3)},\ \text{i.e. } \frac{l}{-6}=\frac{m}{-15}=\frac{n}{3}$$

i.e. $$\frac{l}{2}=\frac{m}{5}=\frac{n}{-1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{2^2+5^2+(-1)^2\}}}=\frac{1}{\sqrt{30}}$$

Hence $l=\frac{2}{\sqrt{30}},m=\frac{5}{\sqrt{30}},n=-\frac{1}{\sqrt{30}}$ (1)

Now the length of S.D. is the projection of the join of the point $(3,8,3)$ and $(-3,-7,6)$ along the line whose $d.c.'s$ are given by (1). Hence

$$S.D.=\{3-(-3)\}\frac{2}{\sqrt{30}}+\{8-(-7)\}\frac{5}{\sqrt{30}}+(3-6)\left(-\frac{1}{\sqrt{30}}\right)$$

$$=\frac{90}{\sqrt{30}}=3\sqrt{30}$$

Further, using direction ratios $2,5,-1$ of S.D. line, its equation are

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 5 & -1 \end{vmatrix}=0=\begin{vmatrix} x+3 & x+7 & z-6 \\ -3 & 2 & 4 \\ 2 & 5 & -1 \end{vmatrix}$$

Simplifying these equations, we have

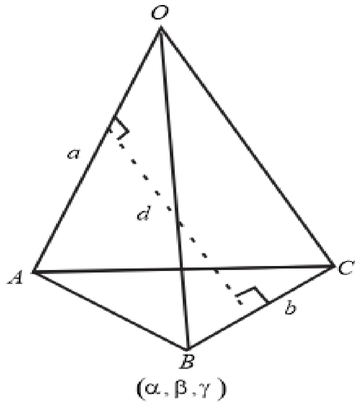
These are the equation to the S.D. line

Example31:- The length of two edges of a tetrahedron are a,b, their shortest distance is equal to d, where the angle between them is θ, prove that the volume is $\frac{1}{6}abd\ \sin\theta$

Solution :- Let $OABC$ be a tetrahedron, and let the direction cosines of the two opposite edges OA and BC be l,m,n and l',m',n', respectively.

Let B be the point (α, β, γ). Then the co-ordinates of A are (al, am, an) and the equation of BC are

$$\frac{x-\alpha}{l'} = \frac{y-\beta}{m'} = \frac{z-\gamma}{n'}$$



Therefore, the co-ordinates of C are $(bl'+\alpha, bm'+\beta, bn'+\gamma)$. Hence the volume of the tetrahedron $OABC$ is given by

$$V = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ al & am & an & 1 \\ bl'+\alpha & bm'+\beta & bn'+\gamma & 1 \end{vmatrix}$$

$$= -\frac{1}{6}\begin{vmatrix} al & am & an \\ \alpha & \beta & \gamma \\ bl'+\alpha & bm'+\beta & bn'+\gamma \end{vmatrix}$$

$$= -\frac{1}{6}a\begin{vmatrix} l & m & n \\ \alpha & \beta & \gamma \\ bl' & bm' & bn' \end{vmatrix}$$, taking a common from R_1 and then subtracting R_2 from R_3,

i.e. $V = \frac{1}{6}ab\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}$, interchanging R_1 and R_2 and taking b common from R_3 (1)

Now the shortest distance between OA and BC is given by

$$d = \begin{vmatrix} \alpha-0 & \beta-0 & \gamma-0 \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\left\{\Sigma\left(mn'-nm'\right)^2\right\}}$$

i.e. $d = \begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sin\theta$ (2)

Using the value of the determinant from (2) in (1), the required volume is:

$$V = \frac{1}{6} abd \ \sin\theta$$

Example32:- Show that the shortest distance between any two opposite edges of the tetrahedron formed by the plane $y+z=0,\ z+x=0, x+y=0,\ x+y+z=1$ is $2a/\sqrt{6}$, and that the three lies of shortest distance intersect at the point $x=y=z-1$

Solution:- Edge of intersection of the planes $y+z=0=z+x$ is

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1} \qquad (1)$$

Also, edge of intersection of the planes $x+y=0=x+y+z-a$ is

$$\frac{x}{1}=\frac{y}{-1}=\frac{z-a}{0} \qquad (2)$$

Since the first three given planes pass through $(0,0,0)$ while the fourth does not pass through it, the (1) and (2) will not intersect each other.

Let λ, μ, ν be the direction cosines of the line of S.D. since it is perpendicular to both (1) and (2), we have

$\lambda+\mu-\nu=0$ and $\lambda-\mu+0.\nu=0$

Solving these, we get

$$\frac{\lambda}{1}=\frac{\mu}{1}=\frac{\nu}{2}=\frac{\sqrt{\left(\lambda^2+\mu^2+\nu^2\right)}}{\sqrt{\left(1^2+1^2+2^2\right)}}=\frac{1}{\sqrt{6}}$$

Therefore, $\lambda=1/\sqrt{6}$, $\mu=1/\sqrt{6}$, $\nu=2/\sqrt{6}$

Hence $S.D.=\lambda(0-0)+\mu(0-0)+\nu(a-0)$

$\nu a=2a/\sqrt{6}$

Further, the equations to S.D. line are

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{vmatrix} = 0 = \begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{vmatrix}$$

Which on simplifying can be written as

$$x-y=0=x+y-z+a$$

Clearly, this lien passes through the point $x=y=z=-a$

Because of the symmetry of this result, other two S.D. lines will also pass through this point

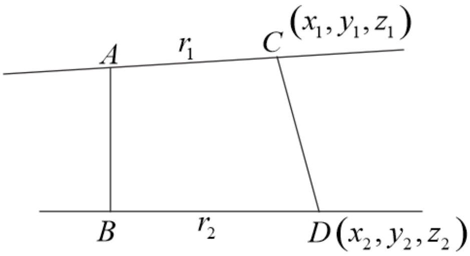
Hence the three lines of S.D. intersect at the point

$x=y=z=-a$.

Example33:- Show that the line of shortest distance between the lines $\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1}$, $\frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_2}{\cos\gamma_2}$ meets the first line at a point whose distance from (x_1, y_1, z_1) is $\dfrac{\sum(x_1-x_2)(\cos\alpha_1-\cos\theta\cos\alpha_2)}{\sin^2\theta}$, where θ is the angle between the lines.

Solution:- Let $C(x_1, y_1, z_1)$ and $D(x_2, y_2, z_2)$ be the points on the given lines.

Also, let AB be the line of shortest distance between the given lines.



Let $CA = r_1$ and $DB = r_2$. We need to find out the value of r_1.

The co-ordinates of A are $(x_1 + r_1\cos\alpha_1, y_1 + r_1\cos\beta_1, z_1 + r_1\cos\gamma_1)$ and those of B are $(x_2 + r_2\cos\alpha_2, y_2 + r_2\cos\beta_2, z_2 + r_2\cos\gamma_2)$

Then the direction ratios of AB are

$$x_1 - x_2 + r_1\cos\alpha_1 - r_2\cos\alpha_2$$
$$y_1 - y_2 + r_1\cos\beta_1 - r_2\cos\beta_2$$
$$z_1 - z_2 + r_1\cos\gamma_1 - r_2\cos\gamma_2$$

Since AB is the line of shortest distance. It is perpendicular to the both the given lines. Hence by the condition of perpendicularity with the first line we have

$$\sum(x_1 - x_2 + r_1\cos\alpha_1 - r_2\cos\alpha_2)\cos\alpha_1 = 0$$

i.e. $\sum(x_1 - x_2)\cos\alpha_1 + r_1\sum\cos^2\alpha_1 - r_2\sum\cos\alpha_2\cos\alpha_1 = 0$

i.e. $\sum(x_1 - x_2)\cos\alpha_2 + r_1 - r_2 - \cos\theta = 0$ (1)

Where $\cos\theta = \sum\cos\alpha_2\cos\alpha_1$ and $\sum\cos^2\alpha_1 = 1$

Similarly, with the second line, we have

$$\sum(x_1 - x_2 + r_1\cos\alpha_1 - r_2\cos\alpha_2)\cos\alpha_2 = 0$$

i.e. $\sum(x_1 - x_2)\cos\alpha_2 + r_1\sum\cos\alpha_1 - r_2\sum\cos^2\alpha_2 = 0$

i.e. $\sum(x_1 - x_2)\cos\alpha_2 + r_1\cos\theta - r_2 = 0$

i.e. $r_2 = \sum(x_1 - x_2)\cos\alpha_2 + r_1\cos\theta$

Putting this value of r_2 in (1), we obtain

$$\sum(x_1 - x_2)\cos\alpha_1 + r_1 - \cos\theta\sum(x_1 - x_2)\cos\alpha_2 - r_1\cos^2\theta = 0$$

i.e. $r_1\sin^2\theta + \sum(x_1 - x_2)(\cos\alpha_1 - \cos\theta\cos\alpha_2) = 0$

i.e. $r_1 = \dfrac{\sum(x_1 - x_2)(\cos\alpha_1 - \cos\theta\cos\alpha_2)}{\sin^2\theta}$, neglecting the minus sign

Example34:- Find the length and the equations of the line of the shortest distance between the lines $3x - 9y + 5z = 0 = x + y - z$ and $6x + 8y + 3z - 13 = 0 = x + 2y + z - 3$

Solution:- Since the constant term is zero in both the equations of the first line, this line passes through the origin.

Now any plane through the first line is

$$(3x-9y+5z)+\lambda(x+y+z)=0,$$

i.e. $$(3+\lambda)x+(-9+\lambda)y+(5-\lambda)z=0 \quad (1)$$

and any plane through the second line is

$$(6x+8y+3z-12)+\mu(x+2y+z-3)=0$$

i.e. $$(6+\mu)x+(8+2\mu)y+(3+\mu)z-(13+3\mu)-0 \quad (2)$$

We choose λ and μ such that the plane (1) and (2) may be parallel. For this, comparing to corresponding coefficient in the two equations, we find that $\frac{3+\lambda}{6+\mu}=\frac{9+\lambda}{8+2\mu}=\frac{5-\lambda}{3+\mu}=\nu$ say

Then we have

$$3+\lambda-6\nu-\mu\nu=0 \quad (3)$$

$$-9+\lambda-8\nu-2\mu\nu=0 \quad (4)$$

$$5-\lambda-3\nu-\mu\nu=0 \quad (5)$$

Subtracting twice the equation (3) from (4), we get

$$-15-\lambda+4\nu=0, \text{ i.e. } -\lambda+4\nu=15 \quad (6)$$

Also. Subtracting the equation (3) from (5), we get

$$2-2\lambda+3\nu=0 \text{ i.e. } \lambda-\frac{3}{2}\nu=1 \quad (7)$$

Adding (6) and (7), we have $\frac{5}{2}\nu=16$, so that $\nu=\frac{32}{8}$. Putting this value of ν in (7), we obtain

$$\lambda-\frac{48}{5}=1, \text{ i.e. } \lambda=\frac{53}{5}$$

Now, substituting the values of λ and ν in (3) we get

$$3+\frac{53}{5}-\frac{168}{5}-\frac{32}{5}\mu=0 \text{ which gives } \mu=-\frac{31}{8}$$

Finally using these values of λ and μ, the plane (1) and (2) become

$$\left(3+\frac{53}{5}\right)x+\left(-9+\frac{53}{5}\right)y+\left(5-\frac{53}{5}\right)z=0 \text{ i.e. } 17x+2y-7z=0 \quad (8)$$

And $$\left(6-\frac{31}{8}\right)x+\left(8-\frac{62}{8}\right)y+\left(3-\frac{31}{8}\right)z-\left(13-\frac{93}{8}\right)=0 \text{ i.e. } 17x+2y-7z=11 \quad (9)$$

Now S.D. between the given lines

$=$ Perpendicular distance between the plane (8) and (9)

$=$ perpendicular distance of $(0,0,0)$ from the plane (9), since $(0,0,0)$ lies on (8)

$$=\frac{11}{\sqrt{\{17^2+2^2+(-7)^2\}}}=\frac{11}{\sqrt{(342)}}$$

Further, the plane (1) will be perpendicular to the plane (8) or (9) if $17(3+\lambda)+2(-9+\lambda)-7(5-\lambda)=0$ which gives $\lambda=\frac{1}{13}$

Similarly, the plane (2) will be perpendicular to the plane (8) or (9) if $17(6+\mu)+2(8+2\mu)-7(3+\mu)=0$ which gives $\mu=-\frac{97}{14}$

Using these values of λ and μ, the planes (1) and (2) become

$$\left(3+\frac{1}{13}\right)x+\left(-9+\frac{1}{13}\right)y+\left(5-\frac{1}{13}\right)z=0 \text{ i.e. } 10x-29y+16z=0$$

(10)

And $$\left(6-\frac{97}{14}\right)x+\left(8-\frac{194}{14}\right)y+\left(3-\frac{97}{14}\right)z-\left(13-\frac{291}{14}\right)=0$$

i.e. $$13x+82y+55z-109=0$$

(11)

Equations (10) and (11) together represent the equations of the line of the shortest distance.

Example35:- Show that the shortest distance between the z-axis and the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ is $\frac{dc'-cd'}{\sqrt{\left\{(ac'-a'c)^2+(bc'+cb')^2\right\}}}$

Solution:- Any plane containing the given line is $(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$

$$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \qquad (1)$$

This plane will be parallel to the z-axis whose $d.c.'s$ are $0,0,1$

$$0(a+\lambda a')+0(b+\lambda b')+1(c+\lambda c')=0 \text{ which gives } \lambda=-c/c'$$

Putting this value of λ in (1), the equation to the plane through the given line and parallel to the z-axis is

$$\left(a-\frac{c}{c'}a'\right)x+\left(b-\frac{c}{c'}b'\right)y+\left(c-\frac{c}{c'}c'\right)z+\left(d+\frac{c}{c'}d'\right)=0$$

i.e. $$(ac'-ca')x+(bc'-cb')y+(dc'-cd')=0 \qquad (2)$$

The shortest distance between the given line and the z-axis is the distance of any point of the z-axis from the above plane. Hence

S.D. = perpendicular from $(0,0,0)$ to the plane (2)

$$=\pm\frac{dc'-cd'}{\sqrt{\left\{(ac'-ca')^2+(bc'-cb')^2\right\}}}$$

Example36:- Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1,\ x=0$ and parallel to the line $\frac{x}{a}-\frac{z}{c}=1, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$. If $2d$ is the shortest distance between the given lines, show that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{d^2}$

Solution:- Any plane containing the line $\frac{y}{b}+\frac{z}{c}=1,\ x=0$ is

$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0 \qquad (1)$$

The line $\frac{x}{a}-\frac{z}{a}=1,\ y=0$ in symmetrical form is

$$\frac{x-a}{a}=\frac{y}{0}=\frac{z}{c'} \qquad (2)$$

If the plane (1) is parallel to the line (2), then

$$\lambda.\ a+\frac{1}{b}.0+\frac{1}{x}.c=0 \text{ which gives } \lambda=-\frac{1}{a}$$

Putting this value of λ in (1), we have

$$\frac{y}{b}+\frac{z}{c}-1-\frac{x}{a}=0 \text{ i.e. } \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$$

This is equation to the required plane.

Further, a known point on the line (2) is $(a,0,0)$. Therefore

S.D. $=$ perpendicular from $(a,0,0)$ to the plane (3)

i.e. $$2d=\frac{\frac{a}{a}-\frac{0}{b}-\frac{0}{c}=1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=\frac{2}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}$$

Which at once gives $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$

Example37:- Find the nature of intersection of the following planes $2x-y+z=4$, $5x+7y+2z=0$, $3x+4y-2z+3=0$.

Solution:- Here

$$\Delta_4=\begin{vmatrix}2 & -1 & 1\\ 5 & 7 & 2\\ 3 & 4 & -2\end{vmatrix}=2(-14-8)-1(6+10)+1(20-21)$$

$$=-44-16-1=-51\neq 0$$

Hence the given planes intersect in a common point

Example38:- Prove that the planes $2x+4y+2z=7$, $5x+y-z=9$, $x-y-z=6$ from a triangular prism.

Solution:- Here

$$\Delta_4 = \begin{vmatrix} 2 & 4 & 2 \\ 5 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = 2(-1-1)+4(-1+5)+2(-5-1)=0$$

Also, $\Delta_3 = \begin{vmatrix} 2 & 4 & -7 \\ 5 & 1 & -9 \\ 1 & -1 & -6 \end{vmatrix} = 2(-6-9)+4(-9+30)-7(-5-1)=96 \neq 0$

Hence the given planes form a triangular prism.

Example39:- Find the condition that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ may intersect in a line, and show that the line of intersection is $\dfrac{x}{\sqrt{(1-a^2)}} = \dfrac{y}{\sqrt{(1-b^2)}} = \dfrac{z}{\sqrt{(1-c^2)}}$

Solution:- The equations of given planes can be written as

$$x - cy - bz = 0 \qquad (1)$$
$$cx - y + az = 0 \qquad (2)$$
$$bx + ay = z = 0 \qquad (3)$$

Clearly, all these planes pass through the point $(0,0,0)$. Therefore, they will intersect in a line if.

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

i.e. $a^2 + b^2 + c^2 + 2abc = 1$

The line o intersection of (1) and (2) is

$$\frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2} \qquad (4)$$

The line of intersection of (2) and (3) is

$$\frac{x}{1-a^2} = \frac{y}{ab+c} = \frac{z}{ac+b} \qquad (5)$$

The line of intersection of (3) and (1) is

$$\frac{x}{ab+c} = \frac{y}{1-b^2} = \frac{z}{a+bc} \qquad (6)$$

Now from (4) and (5), we have

$$\frac{x^2}{(ac+b)(1-a^2)} = \frac{z^2}{(1-c^2)(ac+b)}$$

Which gives $\dfrac{x}{\sqrt{(1-a^2)}} = \dfrac{z}{\sqrt{(1-c^2)}}$ (7)

Similarly, from (5) and (6), we have

$$\frac{x^2}{(1-a^2)(ab+c)} = \frac{y^2}{(ab+c)(1-b^2)}$$

Which gives $\dfrac{x}{\sqrt{(1-a^2)}}=\dfrac{y}{\sqrt{(1-b^2)}}$ (8)

From (7) and (8) equations of the common line of intersection are $\dfrac{x}{\sqrt{(1-a^2)}}=\dfrac{y}{\sqrt{(1-b^2)}}=\dfrac{z}{\sqrt{(1-c^2)}}$

Example40:- Prove that the planes $ny-mz=\lambda$, $lz-nx=\mu$, $mx=ly=\nu$ have a common line if and only if $l\lambda+m\mu+n\nu=0$.

Show also that the distance of the line from the origin is then $\left\{\left(\lambda^2+\mu^2+\nu^2\right)/\left(l^2+m^2+n^2\right)\right\}^{1/2}$

Solution:- The given equations of planes can be written as

$$ny-mz-\lambda=0, \quad (1)$$
$$-nx \quad +lz-\mu=0 \quad (2)$$
$$mx-ly \quad -\nu=0 \quad (3)$$

Now these there planes will pass through a common line if any two determinants formed by the coefficients x,y,z and the constant terms in the equations to the plane are equal to zero. For this, we must have

$$\Delta_4=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix}=0 \text{ and } \Delta_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_2 & d_3 \end{vmatrix}=0$$

i.e. $\begin{vmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -l & 0 \end{vmatrix}=0$ and $\begin{vmatrix} 0 & n & -\lambda \\ -n & 0 & -\mu \\ m & -l & -\nu \end{vmatrix}=0$

i.e. $0=0$ and $l\lambda+m\mu+n\nu=0$

Hence the given planes have a common line if

$$l\lambda+m\mu+n\nu=0 \quad (4)$$

If a,b,c be direction ratios of the common line, then it will be perpendicular to (1) and (2) (and to (3) also). So, we have

$$0.\ n.b-m.c=c \text{ and } -n.a+0.b+l.c=0$$

Solving these, we have

$\dfrac{a}{nl}=\dfrac{b}{mn}=\dfrac{c}{n^2}$, i.e. $\dfrac{a}{l}=\dfrac{b}{m}=\dfrac{c}{n}$.

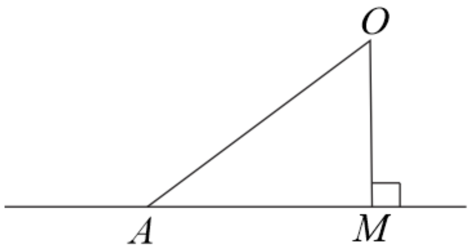
Therefore $d.c.'s$ of the common line are

$$\frac{l}{\sqrt{(l^2+m^2+n^2)}},\frac{m}{\sqrt{(l^2+m^2+n^2)}},\frac{n}{\sqrt{(l^2+m^2+n^2)}}$$

[We note that l,m,n are not given to be $d.c.'s$ of any lines so that we cannot put $l^2+m^2+n^2=1$]

Also, putting $z=0$ in (1) and (2), we have $y=\lambda/n$ and $x=-\mu/n$

Thus $A(-\mu/n,\lambda/m,0)$ is a point on the common line.



If OM be the perpendicular from the origin O on the common line, then (as shown in the figure above)

$$OM^2 = OA^2 - AM^2$$

$$= OA^2 - (\text{projection of } OA \text{ on the common line})^2$$

$$= \frac{\mu^2+\lambda^2}{n^2} - \frac{(\lambda m - \mu l)^2}{n^2\left(l^2+m^2+n^2\right)}$$

$$= \frac{\left(\mu^2+\lambda^2\right)\left(l^2+m^2+n^2\right)-(\lambda m-\mu l)^2}{n^2\left(l^2+m^2+n^2\right)}$$

$$= \frac{\left(\mu^2 l^2+\lambda^2 l^2+\mu^2 m^2+\lambda^2 m^2+\mu^2 n^2+\lambda^2 n^2\right)-\left(\lambda^2 m^2+\mu^2 l^2-2\lambda m.\mu l\right)}{n^2\left(l^2+m^2+n^2\right)}$$

$$= \frac{\lambda^2 l^2+\mu^2 m^2+\left(\mu^2+\lambda^2\right)n^2+2\lambda l.\mu m}{n^2\left(l^2+m^2+n^2\right)}$$

$$= \frac{(\lambda l+\mu m)^2+\left(\mu^2+\lambda^2\right)n^2}{n^2\left(l^2+m^2+n^2\right)} = \frac{n^2\nu^2+\left(\mu^2+\lambda^2\right)n^2}{n^2\left(l^2+m^2+n^2\right)}, \quad \text{Using (4)}$$

$$= \frac{\lambda^2+\mu^2+\nu^2}{l^2+m^2+n^2}$$

Hence $OM = \left\{\left(\lambda^2+\mu^2+\gamma^2\right)/\left(l^2+m^2+n^2\right)\right\}^{1/2}$

Example41:- Prove that the locus of the line which intersects the three lines $y-z, x=0$, $z-x=1, y=0$; $x-y=1, z=0$ is $x^2+y^2+z^2-2yz-2zx-2xy=1$

Solution:- The straight line which intersect the first two of the given lines is represented by

$$y-z+1+\lambda x = 0 = z-x-1+\mu y \qquad (1)$$

This line will intersect the third line given by the equations

$$x-y=1 \qquad (2)$$

$$z=0 \qquad (3)$$

If $y+\lambda x=1$ and $\mu y-x=1$ (using (3) in (1))

Solving these two equations we get

$$x = \frac{-1+\mu}{1+\lambda\mu} \text{ and } y = \frac{1+\lambda}{1+\lambda\mu}$$

These values must satisfy the equations (2). Thus we have

$$\frac{-1+\mu}{1+\lambda\mu}-\frac{1+\lambda}{1+\lambda\mu}=1,\text{ i.e. } -2+\mu-\lambda=1+\lambda\mu$$

i.e. $\lambda-\mu+\lambda\mu+3=0$

The required locus can be obtained now by eliminating λ and μ from (1), and (4). For this, from (1) we have

$$\lambda=\frac{1-y+z}{x} \text{ and } \mu=\frac{1+x-z}{y}$$

Using these values of λ and μ in (4), we obtain

$$\frac{1-y+z}{x}-\frac{1+x-z}{y}+\left(\frac{1-y+z}{x}\right)\left(\frac{1+x-z}{y}\right)+3=0$$

i.e. $y(1-y+z)-x(1+x-z)+(1-y+z)(1+x-z)+3xy=0$

i.e. $(y-y^2+yz)-(x+x^2-xz)+(1+x-z-y-xy+yz+z+zx-z^2)+3xy=0$

i.e. $x^2+y^2+z^2-2yz-2zx-2xy=1$

This is the required locus.

Example42:- Find the equation to the surface traced out by the lines which pass through a fixed point (α,β,γ) and intersect the curve $ax^2+by^2=1,\ z=0$

Solution:- Any line through the fixed point (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad (1)$$

Where l,m,n are parameters.

If this line intersects the curve given by the equations

$$ax^2+by^2=1 \qquad (2)$$

$$z=0 \qquad (3)$$

Then using (3) in (1), we have

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=-\frac{\gamma}{n}$$

Whence $x=\alpha-\gamma,\frac{1}{n}$ and $y=\beta-\gamma,\frac{m}{n}$. Using these values in (2), we get

$$a\left(\alpha-\gamma.\frac{1}{n}\right)^2+b\left(\beta-\gamma.\frac{y-\beta}{z-\gamma}\right)^2=1$$

i.e. $a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=1$

This is the required locus.

Example43:- Show that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ will intersect the locus $ax^2+by^2, z=0$ if

$$a(\alpha n-l\gamma)^2+b(\beta n-m\gamma)^2=n^2$$

Solution:- Proceed as in Example 1 and get the required condition from equation (4)

Example44:- Prove that the locus of the straight line which meets the lines $y=\pm mx, z=\pm c$ and the circle $x^2+y^2=a^2,\ z=0$ is

$$m^2c^2\left(cy-mzx\right)^2+c^2\left(yz-mcx\right)^2=a^2m^2\left(z^2-c^2\right)^2$$

Solution:- The straight line

$y-mx+\lambda\left(z-c\right)=0=y+mx+\mu\left(z+c\right)$ meets the given lines.

The equations of given straight lines can be written as

$$y-mx=0,\quad z-c=0 \tag{1}$$

And $$y+mx=0,\quad z+c=0 \tag{2}$$

Also, the equations of given circle are

$$x^2+y^2=c^2 \tag{3}$$

And $$z=0 \tag{4}$$

Equations of any straight line intersecting the lines (1) and (2) are

$$y-mx+\lambda\left(z-c\right)=0 \tag{5}$$

And $$y+mx+\mu\left(z+c\right)=0 \tag{6}$$

If this line has to intersect the circle given by (3) and (4), we must be able to find out a relation between λ and μ by eliminating x,y,z from (3), (4), (5) and (6). For this, using (4) in (5) and (6), we have

$$y-mx-\lambda c=0$$

And $$y+mx-\mu c=0$$

Solving these equations, we get

$$x=-\frac{1}{2}\left(\lambda+\mu\right)c/m \text{ and } y=\frac{1}{2}\left(\lambda-\mu\right)c$$

Putting these values of x and y in (3), we have

$$\left\{-\frac{1}{2}\left(\lambda-\mu\right)c/m\right\}^2+\left\{\frac{1}{2}\left(\lambda-\mu\right)c\right\}^2=a^2$$

i.e. $$c^2\left(\lambda+\mu\right)^2+m^2c^2\left(\lambda-\mu\right)^2=4m^2a^2 \tag{7}$$

This is a relation between λ and μ

Putting in (7) the values of λ and μ from (5) and (6) ,we get

$$c^2\left(-\frac{y-mx}{z-c}-\frac{y+mx}{z+c}\right)^2..m^2c^2\left(-\frac{y-mx}{z-c}+\frac{y+mx}{z+c}\right)^2=4m^2a^2$$

On simplifying, this equation reduces to

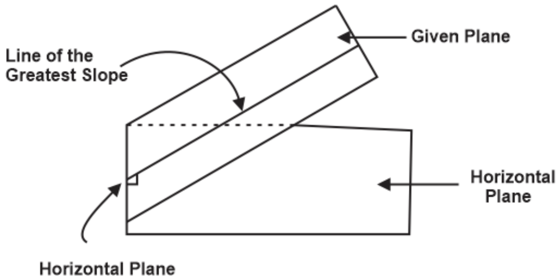
$$m^2c^2\left(cy-mzx\right)^2+c^2\left(yz-mcx\right)=a^2m^2\left(z^2-c^2\right)^2$$

This is the required locus.

Example45:- With the given rectangular axes the line $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ is vertical. Find the direction cosines of the line of greatest slop in the plane $3x-2y+z=5$

Solution:- Since the line $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ is vertical, it is normal to a horizontal plane. Hence the equations of the horizontal plane is

$$2x-3y+z=0 \qquad (1)$$



The horizontal line in the given plane is the line of intersection of this plane with the given plane

$$3x-2y+z=5 \qquad (2)$$

If l,m,n be the $d.c.'s$ of this horizontal line, then using (1) and (2), we have

$$2l-3m+n=0$$

And $\quad 3l-2m+n=0$

Solving these we obtain

$$\frac{l}{(-3)1-1.(-2)}=\frac{m}{1.3-2.1}=\frac{n}{2.(-2)-(-3).3}\text{, i.e. }\frac{l}{-1}=\frac{m}{1}=\frac{n}{5}$$

Now, let l',m',n' be the $d.c.'s$ of the line of the greatest slope, since this line is perpendicular to the horizontal line, we have $ll'+mm'+nn'=0$ i.e.

$$-l'+m'+5n'=0 \qquad (3)$$

And since it lies on given plane (2), we have

$$3l'-2m'+n'=0 \qquad (4)$$

Solving (3) an (4), we get

$$\frac{l'}{1.1-5.(-2)}=\frac{m'}{5.3-(-1).1}=\frac{n'}{(-1)(-2)-1.3}$$

i.e. $$\frac{l'}{11}=\frac{m'}{16}=\frac{n'}{-1}=\frac{\sqrt{\left(l'^2+m'^2+n'^2\right)}}{\sqrt{\left\{11^2+16^2+(-1)^2\right\}}}=\frac{1}{\sqrt{(378)}}$$

Hence $l'=\frac{11}{\sqrt{(378)}},m'=\frac{16}{\sqrt{(378)}},n'=-\frac{1}{\sqrt{(378)}}$ are the required $d.c.'s$ of the line of the greatest slope.

Example46:- Show that the volume of the tetrahedron of which a pair of opposite edges are of the lengths. r,r' on the lines whose equations are $\frac{x-a}{l}-\frac{y-b}{m}=\frac{z-c}{n}$ and $\frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n}$ is $\frac{1}{6}rr'\begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$

Solution:- Let the two vertices of the tetrahedron, lying on the first given line, be

$$A\left(a+lr_1,b+mr_1,c+nr_1\right)$$

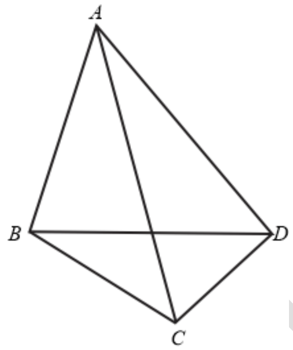
And $\quad B\left(a+lr_2,b+mr_2,c+nr_2\right)$

Where $r_2 - r_1 = r$ (given) (1)

Similarly, let the other two vertices of the tetrahedron, lying on the second given line, be $C(a'+l'r_3, b'+m'r_3, c'+n'r_3)$

And $D(a'+l'r_4, b'+m'r_4, c'+n'r_4)$

Where $r_4 - r_3 = r'$ (given) (2)



Then the volume of the tetrahedron is:

$$V = \frac{1}{6}\begin{vmatrix} a+lr_1 & b+mr_1 & c+nr_1 & 1 \\ a+lr_2 & b+mr_2 & c+nr_2 & 1 \\ a'+l'r_3 & b'+m'r_3 & c'+n'r_3 & 1 \\ a'+l'r_4 & b'+m'r_4 & c'+n'r_4 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} a+lr_1 & b+mr_1 & c+nr_1 & 1 \\ l(r_2-r_1) & m(r_2-r_1) & n(r_2-r_1) & 0 \\ a'+l'r_3 & b'+m'r_3 & c'+n'r_3 & 1 \\ l'(r_4-r_3) & m'(r_4-r_3) & n'(r_4-r_3) & 0 \end{vmatrix}$$, subtracting R_1 from R_2 and R_3 from R_4

$$= \frac{1}{6}rr'\begin{vmatrix} a+lr_1+ & b+mr_1 & c+nr_1 & 1 \\ l & m & n & 0 \\ a'+l'r_3 & b'+m'r_3 & c'+n'r_3 & 1 \\ l' & m' & n' & 0 \end{vmatrix}$$, using (1) and (2)

$$= \frac{1}{6}rr'\begin{vmatrix} a & b & c & 1 \\ l & m & n & 0 \\ a' & b' & c' & 1 \\ l' & m' & n' & 0 \end{vmatrix}$$, subtracting r_1 times R_2 from R_1 and r_3 times R_4 from R_3

$$= \frac{1}{6}rr'\begin{vmatrix} a-a' & b-b' & c-c' & 0 \\ l & m & n & 1 \\ a' & b' & c' & 1 \\ l' & m' & n' & 0 \end{vmatrix}$$, subtracting R_3 from R_1

$$=\frac{1}{6}rr'\begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$, expanding along C_4 and neglecting the negative sign.

Example47:- Two straight lines $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$ are cut by a third line whose direction cosines are λ, μ, ν. Show that the length intercepted on the third line is

$$\begin{vmatrix} \alpha-a' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix}$$

Hence obtain the shortest distance between the given liens.

Solution:- The co-ordinates of any points A and B on the given lines are $(lr+\alpha,\ mr+\beta, nr+\gamma)$ and $(l'r'+\alpha',\ m'r'+\alpha, m'r'+\beta', n'r'+\gamma')$, respectively. Then direction ratios of the line AB are

$$(lr+\alpha)-(l'r'+\alpha'),\quad (mr+\beta)-(m'r'+\beta'),\qquad (nr-\gamma)-(n'r'-\gamma')$$

i.e. $\quad lr-l'r'+\alpha-\alpha',\ mr-m'r'+\beta-\beta',\ nr-n'r'+\gamma+\gamma'$

If the third line, whose direction cosines are λ, μ, ν intersects the given lines at the points A and B, respectively, we must have

$$\frac{lr+l'r'+\alpha-\alpha'}{\lambda}=\frac{mr-m'r'+\beta-\beta'}{\mu}=\frac{nr'-n'r'+\gamma+\gamma'}{\nu}=k \text{ say} \quad (1)$$

The equations may be written as

$$lr-l'r'+(\alpha-\alpha'-k\lambda)=0$$

$$mr-m'r'+(\beta-\beta'-k\mu)=0$$

and $\quad nr-n'r'+(\gamma-\gamma'-k\nu)=0$

Eliminating r and $-r'$ from these equations, we obtain

$$\begin{vmatrix} l & l' & \alpha-\alpha'-k\lambda \\ m & m' & \beta-\beta' & k\mu \\ n & n' & \gamma-\gamma' & k\nu \end{vmatrix}=0 \text{ i.e. } \begin{vmatrix} l & l' & \alpha-\alpha' \\ m & m' & \beta-\beta' \\ n & n' & \gamma-\gamma' \end{vmatrix}-k\begin{vmatrix} l & l' & \lambda \\ m & m' & \mu \\ n & n' & \nu \end{vmatrix}=0$$

Which gives $k=\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix}$, interchanging rows and columns in the two determinants.

Now the length AB intercepted on the third line

$$=\sqrt{\left\{(lr-l'r'+\alpha-\alpha')^2(mr-m'r'+\beta+\beta')^2+(nr-n'r'+\gamma-\gamma')^2\right\}}$$

$$=\sqrt{(k^2\lambda^2+k^2\mu^2+k^2\nu^2)}\text{, using (1)}$$

$$=k\sqrt{(\lambda^2+\mu^2+\nu^2)}=k$$

$$=\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n'\end{vmatrix}\div\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu\end{vmatrix}$$

This length AB will be the shortest distance between the two given lines if the third line is perpendicular to them. For this, we must have

$$l\lambda+m\mu+n\nu=0$$

And $\quad l'\lambda+m'\mu+n'\nu=0$

Solving these, we have

$$\frac{\lambda}{mn'-nm'}=\frac{\mu}{nl'-ln'}=\frac{\nu}{lm'-ml'}$$

$$=\frac{\sqrt{\left(\lambda^2+\mu^2+\nu^2\right)}}{\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}}=\frac{1}{\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}}$$

So, $\lambda=\dfrac{mn'-nm'}{\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}},\mu=\dfrac{nl'-ln'}{\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}},\nu=\dfrac{lm'-ml'}{\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}}$

Substituting these values of λ,μ,ν in the determinant written after the sign of division in the first result, we have

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \lambda & \mu & \nu\end{vmatrix}=\begin{vmatrix} \lambda & \mu & \nu \\ l & m & n \\ l' & m' & n'\end{vmatrix}$$

$$=\frac{1}{\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}}\begin{vmatrix} mn'-nm' & nl'-ln' & lm'-ml' \\ l & m & n \\ l' & m' & n'\end{vmatrix}$$

$$=\frac{1}{\sqrt{\left\{\sum\left(nm'-nm'\right)^2\right\}}},\ \sum\left(mn'-nm'\right)^2=\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}$$

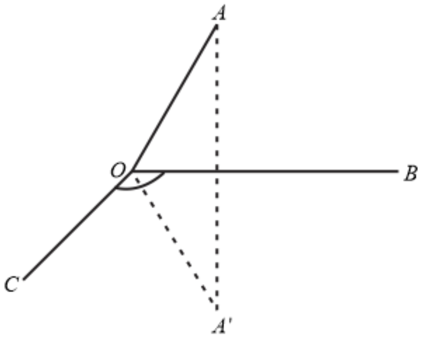
Hence the required shortest distance is

$$\begin{vmatrix}\alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n'\end{vmatrix}\div\sqrt{\left\{\sum\left(mn'-nm'\right)^2\right\}}$$

Example48:- OA,OB,OC have direction ratios $l_r,m_r,n_r,r=1,2,3$ and OA',OB',OC' bisect the angles BOC,COA,AOB respectively, Prove that the planes AOA',BOB',COC' pass through the line $\dfrac{x}{l_1+l_2+l_3}=\dfrac{y}{m_1+m_2+m_3}=\dfrac{z}{n_1+n_2+n_3}$

Solution:- Direction ratios of OB and OC are l_2,m_2,n_2 and l_3,m_3,n_3, respectively. Therefore, direction ratios of OA', the internal bisector of the angle BOC, are

$$l_2+l_3,m_2+m_3,n_2+n_3$$



Let the equation of the plane AOA' be

$$ax+by+cz=0 \qquad (1)$$

Since the lines OA and OA' lie on this plane, their direction ratios must satisfy the following conditions of perpendicularity with direction ratios a,b,c of the normal to the plane (1)

$$al_1 \quad +bm_1 \quad +cn_1 \quad =0 \qquad (2)$$

And $$a(l_2+l_3)+b(m_2+m_3)+c(n_2+n_3)=0 \qquad (3)$$

Eliminating a,b,c from equations (1), (2) and (3), the equations of plane AOA' comes out to be

$$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_2+l_1 & m_2+m_2 & n_2+n_3 \end{vmatrix}=0$$

i.e. $$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_1+l_2+l_3 & m_1+m_2+m_3 & n_1+n_2+n_3 \end{vmatrix}=0 \qquad (4)$$

adding R_2 to R_3

Now the co-ordinates of any point on the given line are

$$\left(r(l_1+l_2+l_3), r(m_1+m_2+m_3), r(n_1+n_2+n_3)\right)$$

Evidently, these co-ordinates satisfy (4). Hence the plane AOA' passes through the given line

By symmetry of notations etc., it follows that the planes BOB' and COC' also pass through the given line.

Example49:- Show that the planes $x=y\sin\psi+z\sin\phi$, $y=z\sin\theta+x\sin\psi$, $z=x\sin\phi+y\sin\theta$ intersect in the line $\dfrac{x}{\cos\theta}=\dfrac{y}{\cos\phi}=\dfrac{z}{\cos\psi}$, if $\theta+\phi+\psi=\dfrac{1}{2}\pi$

Solution:- The given equations of planes can be written as

$$-x \quad +y\sin\psi+z\sin\phi=0$$
$$x\sin\psi-y \quad +z\sin\theta=0$$
$$x\sin\phi+y\sin\theta-z \quad =0$$

Since the constant term in each of these equations is zero, we have $\Delta_3=0$. Also

$$\Delta_4 = \begin{vmatrix} 1 & \sin\psi & \sin\phi \\ \sin\psi & -1 & \sin\theta \\ \sin\phi & \sin\theta & -1 \end{vmatrix}$$

$$= -1\left(1-\sin^2\theta\right)+\sin\psi\left(\sin\theta\sin\phi+\sin\psi\right)+\sin\phi\left(\sin\psi\sin\theta+\sin\phi\right)$$

$$= \sin^2\theta\left(2\sin\phi\sin\psi\right)\sin\theta+\left(\sin^2\phi+\sin^2\psi+1\right) \qquad \textbf{(note)}$$

Thus Δ_4 has been expressed as a quadratic expression in $\sin\theta$. Its roots are given by

$$\sin\theta \frac{-2\sin\theta\sin\psi \pm \sqrt{\left\{4\sin^2\phi\sin^2\psi-4\left(\sin^2\phi+\sin^2\psi+1\right)\right\}}}{2}$$

$$= -\sin\phi\sin\psi \pm \sqrt{\left\{\left(1-\cos^2\phi\right)\left(1-\cos^2\psi\right)-\left(\sin^2\phi+\sin^2\psi+1\right)\right\}}$$

$= -\sin\phi\sin\psi \pm \cos\phi\cos\psi$, on simplifying

Taking positive sign. We have

$\sin\theta = -\sin\phi\sin\psi+\cos\phi\cos\psi$, i.e. $\sin\theta=\cos\left(\phi+\psi\right)$

i.e. $\sin\theta=\sin\left(\frac{1}{2}\pi-\phi-\psi\right)$, which gives $\theta=\frac{1}{2}\pi-\phi-\psi$

i.e. $\theta+\phi+\psi=\frac{1}{2}\pi$

Hence $\Delta_4=0$ and therefore the given planes intersect in a line, if $\theta+\phi+\psi=\frac{1}{2}\pi$

To obtain the equations of the line of intersection the given planes, we note that this line will pass through the origin as all the three planes pass through the origin. Further, let l,m,n be the $d.c.'s$ of this line. Since it lies on the first plane, we have

$$-l+m\sin\psi+n\sin\phi=0 \qquad (1)$$

Similarly, with respect to the second planes, we have

$$l+\sin\psi-m+n\sin\theta=0 \qquad (2)$$

Solving equations (1) and (2), we obtain

$$\frac{l}{\sin\psi\sin\theta+\sin\phi}=\frac{m}{\sin\phi\sin\psi+\sin\theta}=\frac{n}{1-\sin^2\psi} \qquad (3)$$

But $\sin\psi\sin\theta+\sin\phi=\sin\psi\sin\theta+\sin\left(\frac{1}{2}\pi-\theta-\psi\right)$

$$=\sin\psi\sin\theta+\cos\left(\theta+\psi\right)=\cos\theta\cos\psi$$

$$\sin\phi\sin\psi+\sin\theta=\sin\phi\sin\psi+\sin\left(\frac{1}{2}\pi-\phi-\psi\right)$$

$$=\sin\phi\sin\psi+\cos\left(\theta+\psi\right)=\cos\phi\cos\psi$$

And $1-\sin^2\psi=\cos^2\psi$

Using these relations in (3), we have

$$\frac{l}{\cos\theta\cos\psi}=\frac{m}{\cos\phi\cos\psi}=\frac{n}{\cos^2\psi}\text{, i.e. }\frac{l}{\cos\theta}=\frac{m}{\cos\phi}=\frac{n}{\cos\psi}$$

Hence the required equations of the line of intersection of the given planes are

$$\frac{x}{\cos\theta}=\frac{y}{\cos\phi}=\frac{z}{\cos\psi}.$$

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in last of this book.

CHAPTER 4. STRAIGHT LINE

Q4c Prove that the locus of a line which meets the lines $y=mx,\ z=c;\ y=-mx, z=-c$ and the circle $x^2+y^2=a^2, z=0$ is $c^2m^2(cy-mzx)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2$. **UPSC CSE 2023 (15)**

Q4(c) Find equation of the plane containing the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7},$$

$$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}.$$

Also find the point of intersection of the given lines. **UPSC CSE 2021**

Q1. Show that the lines

$$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1} \text{ and } \frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$$

intersect. Find the coordinates of the point of intersection and the equation of the plane containing them. **[(1e) UPSC CSE 2019]**

Q2. Find the projection of the straight line $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z+1}{-1}$ on the plane $x+y+2z=6$.

[(1e) UPSC CSE 2018]

Q3. Verify if the lines:

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta} \text{ and } \frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar. If yes, then find the equation of the plane in which they lie.

[(3c(ii) UPSC CSE 2015]

Q4. Find the equations of the straight line through the point $(3,1,2)$ to intersect the straight line $x+4=y+1=2(z-2)$ and parallel to the plane $4x+y+5z=0$. **[(1e) UPSC CSE 2011]**

SHORTEST DISTANCE

Q2(c) Obtain the coordinates of the points where the shortest distance line between the straight lines $\frac{x-3}{-1}=\frac{y-2}{2}=\frac{z-2}{-1}$; $\frac{x-2}{2}=\frac{y+3}{3}=\frac{z+2}{2}$ meets them. Also find the magnitude of the shortest distance and the equation of the shortest distance line between the straight lines mentioned above.

IFoS 2022

Q1. Show that the shortest distance between the straight lines

$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

is $3\sqrt{30}$. Find also the equation of the line of shortest distance. **[(3c) 2019 IFoS]**

Q2. Find the shortest distance between the lines

$a_1x+b_1y+c_1z+d_1=0$

$a_2x+b_2y+c_2z+d_2=0$

and the z-axis. **[(2d) UPSC CSE 2018]**

Q3. Find the shortest distance between the skew lines:

$\frac{x-3}{3}=\frac{8-y}{1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. **[(1e) UPSC CSE 2017]**

Q4. Find the shortest distance and the equation of the line of the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. **[(4d) 2017 IFoS]**

Q5. Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{4}=z-3$ and $y-mx=z=0$. For what value of m will the two lines intersect? **[(1e) UPSC CSE 2016]**

Q6. Find the magnitude and the equations of the line of shortest distance between the lines

$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$

$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **[(4c) 2013 IFoS]**

Q7. If 2C is the shortest distance between the lines

$\frac{x}{l}-\frac{z}{n}=1,\ y=0$ and $\frac{y}{m}+\frac{z}{n}=1,\ x=0$ then show that $\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}=\frac{1}{c^2}$.

[(3c) 2012 IFoS]

SKEW LINES

Q1. Find the surface generated by a line which intersects the lines $y=a=z$, $x+3z=a=y+z$ and parallel to the plane $x+y=0$. **[(4a) UPSC CSE 2016]**

Q2. Find the locus of the variable straight line that always intersects $x=1, y=0;\ y=1, z=0;\ z=1, x=0$. **[(4b) 2015 IFoS]**

Q3. Prove that the locus of a variable line which intersects the three lines:

$y=mx,\ z=c;\ y=-mx, z=-c;\ y=z, mx=-c$ is the surface $y^2-m^2x^2=z^2-c^2$.

[(1e) 2014 IFoS]

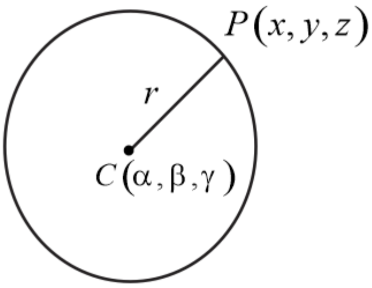
Q4. Find the surface generated by the straight line which intersects the lines $y=z=a$ and $x+3z=a=y+z$ and is parallel to the plane $x+y=0$. **[(1d) 2013 IFoS]**

5.1 The Sphere And Its Equations:-

The Sphere:- A sphere is the locus of a point which moves such that its distance from a fixed point in space remains constant.

Here the fixed point is called the centre of the sphere and the constant distance the radius of the sphere.

The Equation of a Sphere:- To find the equation of a sphere whose centre is the point (α, β, γ) and radius r.



Let $C(\alpha, \beta, \gamma)$ be the centre and r be the radius of the sphere. If $P(x, y, z)$ is any point on the sphere, we have $CP^2 = r^2$. i.e. $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$ (1)

Since this equation is satisfied by every point on the sphere it is the required equations of the sphere.

Corollary:- The equation of the sphere centred at the origin and having radius r, is $x^2 + y^2 + z^2 = r^2$.

We note that the equation of a sphere described above is a generalization of the equation of a circle in two dimensional geometry. Further, equation (1) on expanding can be written as $x^2 + y^2 + z^2 - 2\alpha x - 2\beta y + 2\gamma z + (\alpha^2 + \beta^2 + \gamma^2 - r^2) = 0$

We observe the following facts about this equations:

(i) It is a second degree equation in x, y and z.

(ii) Each of the coefficients of x^2, y^2 and z^2 is equal to 1

(iii) There are no terms containing the products yz, zx and xy.

These observation will help to discuss the general equation of a sphere in the next section.

5.2 General Equation of A Sphere:-

General Equation of A Sphere:- To show that the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (1)$$

Represents a sphere. [It is called the general equation of a sphere]

Equation (1) can be written as $(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) + d = 0$

i.e. $\{(x+u)^2 - u^2\} + \{(y-v)^2 - v^2\} + \{(z+w)^2 - w^2\} + d = 0$

i.e. $(x+u)^2 + (y+v)^2 + (z+w)^2 + u^2 + v^2 + w^2 - d$

comparing this equation with equation (1) of Section 5.1 we find that it represents a sphere with centre at the point $(-u,-v,-w)$ and radius $\sqrt{(u^2+v^2+w^2-d)}$.

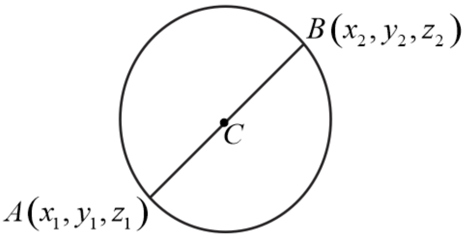
Sometimes the equation $ax^2+ay^2+az^2+2Ax+2By+2Cz+D=0$, $a\neq 0$ is taken as the general equation of the sphere. In fact, it reduces to (1) on dividing by a.

Note:- (i) In order that the sphere may be real, we must have $u^2+v^2+w^2 \geq d$.

(ii) If $u^2+v^2+w^2=d$, the general equation will represent a sphere with centre at $(-u,-v,-w)$ and radius zero. Thus the sphere will coincide with its centre. Such a sphere is called a point sphere. (iii) It $u^2+v^2+w^2<d$ the general equations represents a sphere with imaginary radius. In this case there are no points on the sphere and we call it a **virtual sphere.**

Number of Arbitrary Constants:- It is clear that in equation (1) there are four arbitrary constants.

5.3 Equation of A Sphere On The Join of Two Points As Diameter:- To find the equation of a sphere described on the line joining the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ as diameters.



The centre C of the sphere is the middle point of AB, so that its co-ordinates aer $\left\{\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2),\frac{1}{2}(z_1+z_2)\right\}$.

The radius of the sphere is $CA=\frac{1}{2}AB=\frac{1}{2}\sqrt{\left\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right\}}$

Therefore the equation of the sphere is

$$\left\{x-\frac{1}{2}(x_1+x_2)^2\right\}+\left\{y-\frac{1}{2}(y_1-y_2)^2\right\}+\left\{z-\frac{1}{2}(z_1+z_2)^2\right\}$$

$$=\frac{1}{4}\left\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right\} \text{ i.e. } \sum\left[\left\{x-\frac{1}{2}(x_1+x_2)\right\}^2-\frac{1}{4}(x_2-x_1)^2\right]=0$$

i.e. $\sum\left[\left\{x^2-x(x_1+x_2)+\frac{1}{4}(x_2-x_1)^2\right\}-\frac{1}{4}(x_2-x_1)^2\right]=0$

i.e. $\sum\left[x^2-x(x_1+x_2)+\frac{1}{4}\left\{(x_2-x_1)^2-(x_2-x_1)^2\right\}\right]=0$

i.e. $\sum\left\{x^2-x(x_1+x_2)+x_1x_2\right\}=0$ i.e. $\sum(x-x_1)(x-x_2)=0$

i.e. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

This is the required equation of the sphere.

Remark:- An alternatively. Simpler method to obtain the above equation shall be produced in section 5.5

5.4 Sphere Through Four Given Non-Coplanar Points:- To find the equation of the sphere passing through four given non-coplanar point $(x_r, y_r, z_r), r = 1,2,3,4$

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)

If it passes through the four points $(x_1, y_1, z_1), (x_2, y_2, z_2)(x_3, y_3, z_3)$ and (x_4, y_4, z_4), these points must satisfy the above equations. Thus

$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$ (2)

$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0$ (3)

$x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0$ (4)

And $x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0$ (5)

Eliminating $2u, 2v, 2w$ and d from the equations (1), (2), (3), (4) and (5) treating $x^2 + y^2 + z^2$ etc. as single terms, we get

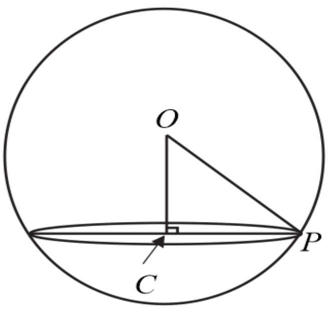
$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

This is the required equation of the sphere.

Note:- Since the given four points are non-coplanar, the cofactor of the term $x^2 + y^2 + z^2$ does not vanish. Hence the coefficient of $x^2 + y^2 + z^2$ (which is the said cofactor) in the above equation is not zero.

5.5 Plane Sections of A Sphere:- To show that the section of a sphere by a plane is a circle.

Let O be the centre of the sphere and P any point on the section of the sphere by the plane.



Let C be the foot of perpendicular from O on this plane section. Join C and O with P. Since CP lies on the plane, OC is at right angles to CP.

Now in the right angled triangle OCP, we have $OP^2 = OC^2 + CP^2$ which gives $CP + \sqrt{(OP^2 - OC^2)}$ = constant. Since OP and OC are constant.

It follows that P remains at a constant distance from C. Hence P lies on the circle centred at C and having radius CP.
Hence the section of a sphere by a plane is a circle.
If r be the radius of the sphere and h be the distance of the plane from the centre of the sphere, it is clear that $CP = \sqrt{(r^2 - h^2)}$ is the radius of the circle obtained above.
Here $h \le r$ and $\sqrt{(r^2 - h^2)} \le r$.

Small Circle:- If the radius of the circle is less than the radius of the sphere i.e. $\sqrt{(r^2 - h^2)} < r$, the circle is called a small circle.

Great Circle:- If the radius of the circle is equal to the radius of the sphere so that $h = 0$,the circle is called a great circle. Evidently, great circles are obtained as sections of a sphere by the planes which pass through the centre of the sphere.

5.6 Equations of A Circle:- In the preceding section we have seen that a circle is a plane section of a sphere. Therefore, the co-ordinates of any point on the circle satisfy simultaneously the equation of the sphere and the plane. Hence the two equation of the from $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $lx + my + nz = p$ taken together represent a circle.

5.7 Intersection of Two Spheres:- To show that the curve of intersection of two spheres is a circle.
Let the two spheres be represented by the equations

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \qquad (1)$$

And $S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ (2)

The co-ordinates of points common to these two spheres satisfy both the equations and therefore also the equation $S_1 - S_2 = 0$

i.e. $2(u_1 - u_2)x + 2(v_1 - v_2)y + (2w_1 + w_2)z + (d_1 - d_2) = 0$ (3)

since this is a first degree equations, it represents a plane. Thus the curve of intersection of the sphere (1) and(2) is the same as the curve of intersection of any one of them plane (3). Hence it is a circle.

5.8 Sphere Through A Given Circle:- To find the equation of the sphere passing through a circle.

Let the equation of a circle be $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)

And $U \equiv lx + my + nz - p = 0$ (2)

Consider the equation $S + \lambda U = 0$ i.e.

$$(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) + \lambda(lx + my + nz - p) = 0 \qquad (3)$$

Where the terms can be rearranged to given

$$x^2 + y^2 + z^2 + 2\left(u + \frac{1}{2}\lambda l\right)x + 2\left(v + \frac{1}{2}\lambda m\right)y + 2\left(v + \frac{1}{2}\lambda n\right)z + (d - \lambda p) = 0 \qquad (3')$$

Evidently, equation (3) is satisfied by all the points lying on the circle given by (1) and (2). Also the from (3') shows that (3) represents a sphere.
Hence for each value of λ equation (3) represent a sphere passing through the circle given by (1) and (2). Thus (3) represents a system of sphere passing through the given circle.
In the same way, with the notations of Section 5.7, the equations $S_1 + \mu S_2 = 0$ also represents a sphere through the circle given by sphere $S_1 = 0$ and $S_2 = 0$.

5.9 Intersection of A Sphere And A Straight Line:- Let the equations of a sphere and a straight line be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)

and $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = r$ (say) (2)

Any point on the line (2) is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point also lies on the sphere (1), we have $(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2$ $+2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr)^2 + d = 0$ i.e.

$$r^2(l^2 + m^2 + n^2) + 2r\left[l(\alpha + u) + m(\beta + v) + n(r + w)\right]$$
$$+(\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad (3)$$

But this is a quadratic in r given two values of r corresponding to which there are two points common to the sphere and the straight line. These point may be real. Coincident or imaginary, depending upon the discriminate of equation (3).
If $l, m, n,$ are the actual direction cosines of the line (2), then equation (3) reduces to

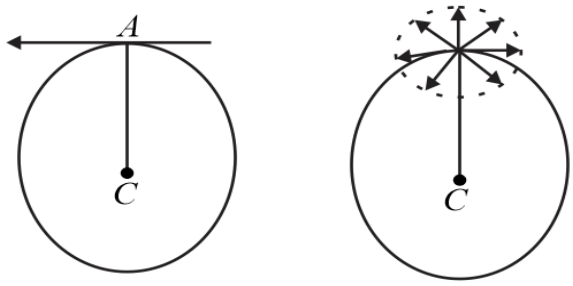
$$r^2 + 2r\left[l(\alpha + u) + m(\beta + v) + n(\gamma + w)\right] + (\alpha + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0 \quad (4)$$

This equation given the distances of the points of intersection from the point (α, β, γ).
If r_1 and r_2 be the roots of equations (4), then the product of roots:

$r_1 r_2 = \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d$.

We see that the value of $r_1 r_2$ is independent of l, m, n and as such that product $r_1 r_2$ is a constant quantity. Thus $r_1 r_2$ is independent of the line whatsover we may take through the point (α, β, γ).
Power of a Point:- The magnitude of the product $r_1 r_2$ is called the power of the point (α, β, γ) with respect to the sphere. Evidently, the power of the point (α, β, γ) with respect to the sphere $f(x, y, z) = 0$ is $f(\alpha, \beta, \gamma)$.

5.10 Tangent Line and Tangent Plane:-
Tangent Line:- A straight line which meets a sphere in two coincident points is called a tangent line of the sphere.
Tangent Plane:- Two locus of tangent line at a point on a sphere is called the tangent plane at that point.



Tangent Line Tangent Plane

To find the equation of tangent plane at the point (α,β,γ) on the sphere

$x^2+y^2+z^2+2ux+2vy+2wz++d=0$ (1)

Equations of any line through the point (α,β,γ) are $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}=r$ say (2)

Any point on this line is $(\alpha+lr,\beta+mr,\gamma+nr)$. If this point on the sphere (1), we must have

$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0$

i.e.

$r^2\left(l^2+m^2+n^2\right)+2r\left[l(\alpha+u)+m(\beta+v)+n(\gamma+w)\right]$

$+\left(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right)=0$ (3)

Since the point (α,β,γ) line on the sphere (1), we have

$\alpha+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0$ (4)

Using this equation (3) reduces to

(5)

Clearly one roots of this equation is zero showing that one of the points of intersection coincides with (α,β,γ). Then it further reduces to

$r^2\left(l^2+m^2+n^2\right)+2r\left\{l(\alpha+u)+m(\beta+v)+n(\gamma+w)\right\}=0$ (6)

In order that the line (2) should be a tangent line of the sphere at (α,β,γ) the other root of (5), which is the only root of (6), should also vanish. This requires

$l(\alpha+u)+m(\beta+v)+n(\gamma+w)=0$ (7)

Thus the line (2) is a tangent line to the sphere (1) if l,m,n satisfy the condition (7).

To obtain the equation of the tangent plane at (α,β,γ) on the sphere, we eliminate l,m,n between (2) and (7). Thus

$(x-\alpha)(\alpha+u)+(y-\beta)(\beta+v)+(z-\gamma)(\gamma+w)=0$ i.e.

$x(\alpha+u)+y(\beta+v)+z(\gamma+w)-\left(\alpha^2+\beta^2+\gamma^2+u\alpha+v\beta+w\gamma\right)=0$

i.e. $x(\alpha+u)+y(\beta+v)+z(\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0$ (8)

Using (4)

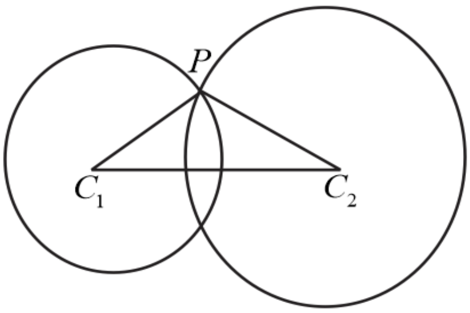
i.e. $\alpha x+\beta y+\gamma z+u(x-\alpha)+v(y-\beta)+w(z+\gamma)+d=0$

This is the required equation of the tangent plane.

Note:- (i) since direction ratios of the radius of the sphere passing through the point (α,β,γ) are $\alpha+u,\beta+v,\gamma+w$, it is clear from the equation s (7) and (8) that every tangent line through (α,β,γ), as also the tangent plane at (α,β,γ), are perpendicular to the radius of the sphere through this point.
(ii) The equation of the tangent plane derived above is such that the equations of sphere were obtained by replacing α,β,γ in it by x,y,z respectively.

5.11 Angle Of Intersection of Two Sphere:- The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point.
This angle is also equal to the angle between the radii of the sphere to the common point, since the radii are normal to the tangent places.
If the angle of intersection is a right angle, the sphere are said to be orthogonal.

5.12 Condition for Orthogonal Intersection:- To find the condition that the sphere $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ and $x^2+y^2+z^2+2u_2x+2v_1y+2w_2z+d_2=0$ intersect orthogonally.
Let C_1 and C_2 be the centres of the given spheres and P be a common point intersection. Join C_1,C_2 and P to form a triangle.



Evidently, the spheres intersect orthogonally if the angle C_1PC_2 is a right angle i.e. if $C_1C_2^2=C_1P^2+C_2P^2$, i.e.
$(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2=(u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2)$ which reduces to $2(u_1u_2+v_1v_2+w_1w_2)=d_1+d_2$. This is the required condition.

5.13 Plane of Contact:- To prove that the points of contact of the tangent planes drawn through a point exterior to the sphere, lie on a plane.
Let (α,β,γ) be a point exterior to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ (1)
And suppose that (x',y',z') is the point of contact of a tangent plane to the sphere which passes through the point (α,β,γ). Then the equation of the tangent plane at (x',y',z') is $x'x+y'y+z'z+u(x+x')+v(y+y')+w(z+z')+d=0$.
Since this plane passes through the point (α,β,γ) we have

$$x'\alpha+y'\beta+z'\gamma+u(\alpha+x')+v(\beta+y')+w(\gamma+\gamma')+d=0$$

This relation shows that the point (x',y',z') satisfies the equation

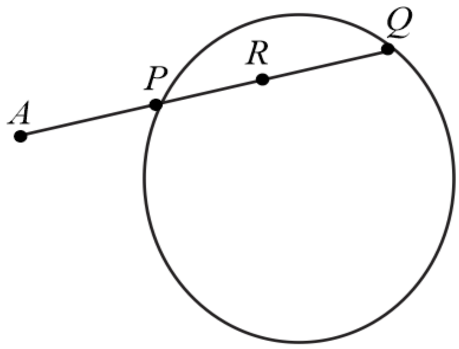
$$\alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0 \qquad (2)$$

Which represents a plane.
Thus the point of contact of the tangent planes to the sphere (1) drawn through an exterior point (α,β,γ) lie on the plane given by (2).

The plane (2) obtained above is called the **plane of contact** of the point (α,β,γ). The locus of the points of contact is the circle in which this plane cuts the sphere.

5.14 Polar Plane:-

Polar Plane And Pole:- If any line through a fixed point A meets a sphere in two point P and Q and if R is the harmonic conjugate of A with respect to P and Q (i.e. the length AR is the harmonic mean between AP and AQ), then the locus of R is a plane and is called the polar plane of A with respect to the sphere. The point A is then called the pole of the polar plane.



To find the equation of the polar plane of the point (α,β,γ) with respect to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ (1)

Let A be the point (α,β,γ) and the equations of the line through the point A meeting the sphere (1) in P and Q be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$, where l,m,n are the actual direction cosines of the line. Any point on this line at distance r from A is $(\alpha+lr,\beta+mr,\gamma+nr)$.

If this point lies on the sphere (1), then we have

$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0$ i.e.

$$r^2+2r\{l(\alpha+u)+m(\beta+v)+n(\gamma+w)\}$$
$$+\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0 \qquad (2)$$

Since $l^2+m^2+n^2=1$.

This equations being quadratic in r has two roots, say r_1 and r_2. Let $AP=r_1$ and $AQ=r_2$.

If the point $R(\xi,\eta,\zeta)$ is the harmonic conjugate of A with respect to P and Q such that $AR=\rho$, then $\frac{2}{\rho}=\frac{1}{r_1}+\frac{1}{r_2}$ i.e. $\frac{2}{\rho}=\frac{r_2+r_1}{r_1r_2}$, i.e.

$\rho=\frac{2r_1r_2}{r_1+r_2}=-\frac{\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d}{l(\alpha+u)+m(\beta+v)+n(\gamma+w)}$. Where the values of r_1r_2 (the product of roots) and r_1+r_2 (the sum of roots) have been substituted from (2). This relation can be written as $l\rho(\alpha+u)+m\rho(\beta+v)+n\rho(\gamma+w)$

$+\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0$ (3)

Also, we have $\xi-\alpha=l\rho$, $\eta-\beta=m\rho$, $\zeta-\gamma=n\rho$.

Using these values of $l\rho, m\rho$ and $n\rho$ in (3), we get

$(\xi-\alpha)(\alpha+u)+(\eta-\beta)(\beta+v)+(\zeta-\gamma)(\gamma+w)$

$+\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0$ i.e.

$\alpha\xi+\beta\eta+\gamma\zeta+u(\xi+\alpha)+v(\eta+\beta)+w(\zeta+\gamma)+d=0$

Hence the equation of the lacus of the point $R(\xi,\eta,\zeta)$ is

$\alpha x+\beta y+\gamma z+u(x-\alpha)+v(y-\beta)+w(z-\gamma)+d=0$

This is the required equation of the polar plane.

Remark:- On comparing the equation of polar plane with that of the tangent plane and the plane of contact, we find that. (i) the polar plane of a point lying **on the sphere** is the tangent plane at that point and (ii) the polar plane of a point lying **outside the sphere** is the plane of contact of that point.

We now define the following:

Conjugate Point:- Two points such that the polar plane of one passes through the other, are known as conjugate points.

Conjugate Planes:- Two planes such that the pole of one lies on the other, are known as conjugate plane.

Conjugate Lines:- Two lines such that the polar plane of any point on one passes through the other, are known as polar lines.

5.15 Radical Plane:-

Radical Plane:- The locus of a point whose r_1r_2 with respect to two spheres are equal is a plane. It is known as the radical plane of the two spheres.

To find the equation of the radical plane of two sphere

$S_1\equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ (1)

And $S_2\equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ (2)

If P is the point (α,β,γ) such that its power with respect to the spheres (1) and (2) are equal, then we have

$\alpha^2+\beta^2+\gamma^2+2u_1\alpha+2v_1\beta+2w_1\gamma+d_1=\alpha^2+\beta^2+\gamma^2+2u_2\alpha+2v_2\beta+2w_2\gamma+d_2$ i.e.
$2(u_1-u_2)\alpha+2(v_1-v_2)\beta+2(w_1-w_2)z+(d_1-d_2)=0$

Hence the locus of P is $2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+(d_1-d_2)=0$ (3)

Which represents a plane. This is the required equations of the radical plane of the given spheres. It can be written as $S_1-S_2=0$.

Note:- (1) Since direction ratios of the line joining the centres of the given spheres are $u_1-u_2, v_1-v_2, w_1-w_2$ it follows from the equations of the radical plane that it is perpendicular to this line.

(2) Since the equation of the radical plane can be written as $S_1-S_2=0$, it follows from section 5.7 that if the spheres intersect. Their circle of intersection lies in the radical plane. In particular, if the spheres touch, their redical plane is the common tangent plane at the point contact.

(3) The radical plane of the two spheres may also be defined as the locus of points, the tangents from which to the two spheres are equal.

5.16 Radical Line:- The radical plane of three spheres taken in pairs pass through a line. This line is called the radical line or the radical axis of the three spheres.

To find the equations of the radical line of the three spheres $S_1=0$, $S_2=0$ and $S_3=0$ where $S_r=x^2+y^2+z^2+2u_rx+2v_ry+2w_rz+d_r$, $r=1,2,3$.

The equations of radical plane of these spheres taken pairs are $S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_1=0$, i.e. $S_1=S_2$, $S_2=S_3$, $S_3=S_1$.

These equations show that the radical planes represented by them pass through the line $S_1=S_2=S_3$.

These are the required equations of the radical line of the three spheres.

5.17 Radical Centre:- The radical planes of four spheres taken in pairs pass through one point which is known as the radical centre of the spheres.

Alternatively, it may be defined as the point of intersection of four radical liens of the four spheres taken three at a time.

If the equations of four spheres are given by $S_r=0,\ r=1,2,3,4$, where $S_r\equiv x^2+y^2+z^2+2u_rx+2v_ry+2w_rz+d_r$, then the point common to the radical planes of these spheres is given by the equations.

$$S_1=S_2=S_3=S_4$$

This shows that it is also common to all the radical lines of the spheres.

5.18 Coaxal System of Spheres:- A system of spheres such that any two of them have the same radical plane is called the co-axal system of spheres.

To find a Co-axal System of Spheres:- Let $S_1\equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ and $S_1\equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ be the equations of any two spheres.

Consider the equation $S_1+\lambda S_2=0$ (1)

Where λ can assume any real value. The value $\lambda = -1$ corresponds to the radical planes $S_1 - S_2 = 0$ of the spheres $S_1 = 0$, and $S_2 = 0$, where as any other value of λ yield a sphere. Thus equation (1) represents a system of spheres **except when** $\lambda = -1$.
Now consider two distinct values λ_1 and $\lambda_2 (\neq -1)$ of λ. Then $S_1 + \lambda_1 S_2 = 0$ and $S_1 + \lambda_2 S_2 = 0$ are the equations of the two spheres of the family (1). Making the coefficients of the second degree terms unity, we get equations of these spheres as $\frac{S_1 + \lambda_1 S_2}{1+\lambda_1} = 0$ and $\frac{S_1 + \lambda_2 S_2}{1+\lambda_2} = 0$. Then radical plane of these two spheres is given by $\frac{S_1 + \lambda_1 S_2}{1+\lambda_1} - \frac{S_1 + \lambda_2 S_2}{1+\lambda_2} = 0$, i.e. $(1+\lambda_2)(S_1 + \lambda_1 S_2) - (1-\lambda_1)(S_1 + \lambda S_2) = 0$ i.e. $(\lambda_2 - \lambda_1)(S_1 - S_2) = 0$ on simplifying i.e. $S_1 - S_2 = 0$ since $\lambda_2 - \lambda_1 \neq 0$.
This equation is clearly independent of the value(s) of λ.
Thus any two distinct members of the family (1) have the same radical plane. Hence the equation $S_1 + \lambda S_2 = 0$ represents a system of coaxal sphere.

Note:- It might similarly be shown that the equation $S + \lambda P = 0$ also gives a co-axal system of spheres, where $S = 0$ is a sphere, $P = 0$ is a plane and λ can taken any real value. The common radical plane here will be $P = 0$.

The centres of all spheres of a co-axal system are collinear:- Since a coaxal system of spheres has a fixed radical plane, and this is at right angles to the line joining the centres of any two spheres of the system, it follow that the centres of all spheres of the system are collinear.

Example1:- Find the equation of the sphere whose centre is $(1,2,1)$ and radius $\sqrt{6}$.

Solution:- The equation to the sphere centred at $(1,2,1)$ and having radius $\sqrt{6}$ is

$$(x-1)^2+(y-2)^2+(z+1)^2=(\sqrt{6})^2$$

i.e. $$(x^2-2x+1)+(y^2-4y+4)+(z^2-2z+1)=6$$

i.e. $$x^2+y^2+z^2-2x-4y-2z=0$$

Example2:- Find the centre and radius of the sphere:

$$2x^2+2y^2+2z^2-6x+8y-8z=1$$

Solution:- The given equation of the sphere is:

$$2x^2+2y^2+2z^2-6x+8y-8z=1$$

i.e. $$x^2+y^2+z^2-3x+4y-4z=\frac{1}{2}$$

Rearranging the terms, it can be written as

$$(x^2-3x)+(y^2-4y)+(z^2-4z)=\frac{1}{2}$$

i.e. $$\left(x-\frac{3}{2}\right)^2+(y+2)^2+(z-2)^2=\frac{1}{2}+\left\{\left(\frac{3}{2}\right)^2+(-2)^2+2^2\right\}$$

i.e. $$\left(x-\frac{3}{2}\right)^2+(y+2)^2+(z-2)^2=\left(\frac{\sqrt{43}}{2}\right)^2$$

Hence the centre of the sphere is the point $\left(\frac{3}{2},-2,2\right)$ and its radius is $\sqrt{43}/2$.

Example3:- Find the equation to the sphere which passes through the points $(1,-3,4),(1,-5,2),(1,-3,0)$ and the centre lies on the plane $z+y+z=0$.

Solution:- The general equation to the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

If the passes through the given points, we must have

$$2u-6v+8w+d+26=0 \qquad (2)$$

$$2u-10v+4w+d+30=0 \qquad (3)$$

And $$2u-6v \quad +d+10=0 \qquad (4)$$

Further, if the centre lies on the plane $x+y+z=0$, then

$$u+v+w \quad =0 \qquad (5)$$

Solving (2) through (5), we get

$$u=-1, v=3, \quad w=-2 \text{ and } d=10$$

Putting these values in (1), the required equation to the sphere is

$$x^2+y^2+z^2-2x+6y-4z+10=0$$

Example4:- Find the equation of the sphere whose diameter is the line joining the points $(1,-2,3)$ and $(3,-4,-5)$

Solution:- We know that the equation of the sphere having its diameter as the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

Hence the required equation of the sphere is

$$(x-1)(x-3)+(y+2)(y+4)+(z-3)(z+5)=0$$

i.e. $(x^2-4x+3)+(y^2+6y+8)+(z^2+2z-15)=0$

i.e. $x^2+y^2+z^2-4x+6y+2z-4=0$

Example5:- Find the equation of the sphere passing through the origin and the points $(1,0,0),(0,2,0)$ and $(0,0,3)$

Solution:- We know that the equation of the sphere passing through the origin and the points $(x_r, y_r, z_r), r=1,2,3$, is:

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 \end{vmatrix} = 0$$

Using the given co-ordinates of three points, this gives

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z \\ 1^2 & 1 & 0 & 0 \\ 2^2 & 0 & 2 & 0 \\ 3^2 & 0 & 0 & 3 \end{vmatrix} = 0$$

By $C_{12}(-1), C_{13}(-2)$ and $C_4(-3)$,this becomes

$$\begin{vmatrix} x^2+y^2+z^2-x-2y-3z & x & y & z \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Expanding along C_1, this reduces to

$$(x^2+y^2+z^2-x-2y-3z)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}=0$$

i.e. $(x^2+y^2+z^2-x-2y-3z)\times 6=0$

Dividing by 6, the required equation of the sphere is

$$x^2+y^2+z^2-x-2y-3z=0$$

Example6:- Show that the equation of the sphere passing through the origin and the points $(a,0,0),(0,b,0)$ and $(0,0,c)$ is $x^2+y^2+z^2-ax-by-cz=0$

Solution:- The general equation of the sphere is

$$x^2+y^2+z^2-2ux+2vy+2wz+d=0 \qquad (1)$$

If it passes through the points $(0,0,0),(a,0,0),(0,b,0)$ and $(0,0,c)$ these co-ordinates must satisfy the above equation. Therefore, putting the co-ordinates of these point one by one, we get

$$d=0,\ a^2+2ua+d=0,\ b^2+2vb+d=0,\ c^2+2wc+d=0$$

Using the first of these in the remaining three, we obtain

$$a^2+2ua=0 \quad b^2+2vb=0 \quad c^2+2wc=0$$

i.e. $a+2u=0 \quad b+2v=0 \quad c+2w=0$

i.e. $2u=-a \quad 2v=-b \quad 2w=-c$

Using these values in (1), the required equation of the sphere is

$$x^2+y^2+z^2-ax-by-cz=0$$

Example7:- A plane passes through the fixed point (a,b,c) and cuts the axes at A,B,C. Show that the locus of the centre of the sphere $OABC$ is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$$

Solution:- Let the equation to the plane ABC be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{c}{\gamma}=1 \qquad (1)$$

Then the co-ordinates of A,B,C are $(\alpha,0,0),(0,\beta,0),(0,0,\gamma)$ respectively.

If this plane (1) passes through the fixed point (a,b,c) then

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1 \qquad (2)$$

Also, any sphere through the origin can be given by

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \qquad (3)$$

If it passes through the points A,B,C then substituting the co-ordinates of these points one by one, we obtain

$$u=-\frac{1}{2}\alpha \qquad v=-\frac{1}{2}\beta \qquad w=\frac{1}{2}\gamma$$

Using these values in (3), the equation to the sphere $OABC$ is

$$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$$

Evidently, the centre of this sphere is given by

$$x=\frac{1}{2}\alpha \qquad y=\frac{1}{2}\beta \qquad z=\frac{1}{2}\gamma$$

Substituting the values of α,β,γ from these equation in (2), we have

$$\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=1 \text{ i.e. } \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$$

This is the locus of the centre of the sphere $OABC$

Example8:- A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9\left(x^2+y^2+z^2\right)=4k^2$

Solution:- Let the co-ordinates of A, B, C be $(a,0,0),(0,b,0),(0,0,c)$ respectively.

Any sphere passing through the origin is given by

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \qquad (1)$$

Since it passes through $A(a,0,0), B(0,b,0)$ and $C(0,0,c)$ respectively, we have

$$a^2+2ua=0\,,\quad b^2+2vy=0 \quad c^2+2wc=0$$

Which given $u=-\frac{1}{2}a\,,\qquad v=-\frac{1}{2}b \qquad w=-\frac{1}{2}c$

Putting these values in (1), the equation to the sphere $OABC$ is

$$x^2+y^2+z^2-ax-by-cz=0$$

The radius of this sphere:

$$\sqrt{\left(\frac{1}{4}a^2+\frac{1}{4}b^2+\frac{1}{4}c^2\right)}=k \text{ given}$$

Which gives $a^2+b^2+c^2+4k^2$ (2)

Now the centroid of the triangle ABC is given by

$$x=\frac{1}{3}a\,,\qquad y=\frac{1}{3}b\,,\qquad z=\frac{1}{3}c$$

Substituting the values of a,b,c from these equations in (2), we obtain

$$9\left(x^2+y^2+z^2\right)=4k^2$$

This is the locus of the centroid of triangle ABC

Example9:- Find the equation of the sphere which passes through the points $(1,0,0),(0,1,0),(0,0,1)$ and has its radius as small as possible

Solution:- Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

If it passes through the points $(1,0,0),(0,1,0)$ and $(0,0,1)$, we have

$$1+2u+d=0 \quad 1+2v+d=0 \quad 1+2w+d=0$$

Whence $\qquad u=v=w=-\frac{1}{2}(1+d)$ (2)

Therefore, radius $=\sqrt{\left(u^2+v^2+w^2-d\right)}=\sqrt{\left\{\frac{3}{4}(1+d)^2-d\right\}}$

$$=\frac{1}{2}\sqrt{\left(3d^2+2d+3\right)}=\frac{\sqrt{3}}{8}\sqrt{\left(d^2+\frac{2}{3}d+1\right)}$$

$$=\frac{\sqrt{3}}{2}\sqrt{\left\{\left(d+\frac{1}{3}\right)^2+\frac{8}{9}\right\}}$$

Evidently, the radius will be as small as possible if

$$d+\frac{1}{2}=0 \text{ i.e. } d=-\frac{1}{3}$$

Then from (2), we have $u = v = w = -\dfrac{1}{3}$

Using the values of u, v, w and d in (1), the required equation of sphere is

$$x^2 + y^2 + z^2 - \frac{2}{3}x - \frac{2}{5}y - \frac{2}{3}z - \frac{1}{3} = 0$$

i.e. $3x^2 + 3y^2 + 3z^2 - 2x - 2y - 2z - 1 = 0$

Example10:- Find the centre and the radius of the circle $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$, $x - 2y + 2z = 3$

Solution:- The centre of the sphere represented by the first of the given equations is $C(4, -2, -4)$. Also, the radius CQ of this sphere is given by

$$CQ = \sqrt{\left\{(-4)^2 + 2^2 + 4^2 + 45\right\}} = \sqrt{(81)} = 9$$

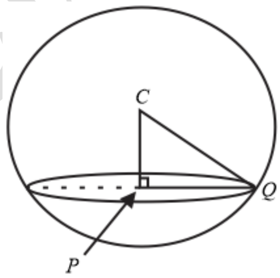
Next the given equation of the plane can be written as

$$-x + 2y - 2z + 3 = 0$$

So, the distance of $C(4, -2, -4)$ from this plane is:

$$CP = \frac{-4 + 2(-2) - 2(-4) + 3}{\sqrt{\left\{(-1)^2 + 2^2 + (-2)^2\right\}}}$$

$$= \frac{-4 - 4 + 8 + 3}{\sqrt{(9)}} = \frac{3}{3} = 1$$



Therefore, the radius of the circle is given by

$$PQ = \sqrt{\left(CQ^2 - CP^2\right)} = \sqrt{(81 - 1)} = \sqrt{(80)} = \sqrt{(16 \times 5)} = 4\sqrt{5}$$

Further, the centre P of the circle is the foot of perpendicular drawn from C on the given plane. Clearly, the equations of this perpendicular are $\dfrac{x-4}{1} = \dfrac{y+2}{-2} = \dfrac{z+4}{2} = r$, say

Any point on this line is $(r + 4, -2r - 2, 2r - 4)$. This point will be the centre of the circle if it lies on the given plane. For this, we must have

$$(r + 4) - 2(-2r - 2) + 2(2r - 4) = 3 \text{, i.e. } r = \frac{1}{3}$$

Using this value, the required co-ordinates of the centre of the circle are

$$\left(\frac{1}{3} + 4, -\frac{2}{3} - 2, \frac{2}{3} - 4\right) \text{, i.e. } \left(\frac{13}{3}, -\frac{8}{3}, -\frac{10}{3}\right)$$

Example11:- A circle of radius 2 and the centre $(2,3,0)$ lies in the plane $z=0$. Find the equation of the sphere containing this circle and passing through the point $(1,1,1)$

Solution:- Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad (1)$$

The section of this sphere by the plane $z=0$ is the circle

$$x^2+y^2+2ux+2vy+d=0, \quad z=0$$

The centre of this circle is given to be $(2,3,0)$. Therefore, we must have

$$-u=2, -v=3, \text{ i.e. } u=-2, v=-3 \quad (2)$$

Further, the radius of the circle is 2. Therefore

$$\sqrt{(u^2+v^2-d)}=2, \text{ i.e. } \sqrt{(4+9-d)}=2$$

Which gives $13-d=4$ i.e. $d=9$ (3)

Since the sphere passes through the point $(1,1,1)$, from (1) we lastly have

$$1+1+1+2u+2v+2w+d=0$$

i.e. $3+2(-2)+2(-3)+2w+9=0$, using (2) and (3)

i.e. $w=-1$ (4)

Using (2), (3) and (4), the equation of the required sphere is

$$x^2+y^2+z^2-4x-6y-2z+9=0$$

Example12:- The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes at the points A, B, C. Find the equations to circumcircle of the triangle ABC, and also find its centre.

Solution:- Evidently, the given plane intersects the axes at the points

$$A(a,0,0) \qquad B(0,b,0) \qquad c(0,0,c)$$

Any sphere through the origin is

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \quad (1)$$

If this sphere passes through the points A, B and C, using the co-ordinates of these points in (1), we obtain

$$2u=-a \qquad 2v=-b \qquad 2w=-c$$

i.e. $u=-\frac{1}{2}a \qquad v=-\frac{1}{2}b \qquad w=-\frac{1}{2}c$

Using these values in (1), the equation to the sphere O, ABC is

$$x^2+y^2+z^2-ax-by-cz=0 \quad (2)$$

Clearly, the circumcircle of the triangle ABC is the section of this sphere with the given plane. Hence the required equations of the circumcircle are

$$x^2+y^2+z^2-ax-by-cz=0, \ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Further, the centre of the sphere (2) is $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right)$

Now the equations of the straight line through the centre of the sphere and perpendicular to the given plane are $\dfrac{x-\frac{1}{2}a}{1/a}=\dfrac{y-\frac{1}{2}b}{1/b}=\dfrac{z-\frac{1}{c}2}{1/c}=r$, say

Any point on this line is $\left(\dfrac{r}{a}+\dfrac{a}{2},\dfrac{r}{b}+\dfrac{b}{2},\dfrac{r}{c}+\dfrac{c}{2}\right)$ (3)

This point will be the centre of the circle ABC, if it lies on the given plane. For this, we must have

$$\frac{\frac{r}{a}+\frac{a}{2}}{a}+\frac{\frac{r}{b}+\frac{b}{2}}{b}+\frac{\frac{r}{c}+\frac{c}{2}}{c}=1 \text{ i.e. } r\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{3}{2}=1$$

i.e. $$r=-\frac{1}{2\left(a^{-2}+b^{-2}+c^{-2}\right)}$$

Using this value of r to evaluate the co-ordinates in (3), we have

$$\frac{r}{a}+\frac{a}{2}=-\frac{1}{2a\left(a^{-2}+b^{-2}+c^{-2}\right)}+\frac{a}{2}=\frac{-1+a^2\left(a^{-2}+b^{-2}+c^{-2}\right)}{2a\left(a^{-2}+b^{-2}+c^{-2}\right)}$$

$$=\frac{-1+1+a^2\left(b^{-2}+c^{-2}\right)}{2a\left(a^{-2}+b^{-2}+c^{-2}\right)}=\frac{a\left(b^{-2}+c^{-2}\right)}{2\left(a^{-2}+b^{-2}+c^{-2}\right)}$$

Similarly, $\dfrac{r}{b}+\dfrac{b}{2}=\dfrac{b\left(c^{-2}+a^{-2}\right)}{2\left(a^{-2}+b^{-2}+c^{-2}\right)}$ and $\dfrac{r}{c}+\dfrac{c}{2}=\dfrac{c\left(a^{-2}+b^{-2}\right)}{2\left(a^{-2}+b^{-2}+c^{-2}\right)}$

Hence the required co-ordinates of the centre of the circumcircle of the triangle ABC are

$$\left(\frac{a\left(b^{-2}+c^{-2}\right)}{2\left(a^{-2}+b^{-2}+c^{-2}\right)},\frac{b\left(c^{-2}+a^{-2}\right)}{2\left(a^{-2}+b^{-2}+c^{-2}\right)},\frac{c\left(a^{-2}+b^{-2}\right)}{2\left(a^{-2}+b^{-2}+c^{-2}\right)}\right)$$

Example13:- Prove that the circles $x^2+y^2+z^2-2x+3y+4z-5=0$, $5y+6z+1=0$; and $x^2+y^2+z^2-3x-4y+5z-6=0$, $x+2y-7z=0$ lie on the same sphere and find its equation.

Solution:- Any sphere through the first circle is

$$x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0$$

i.e. $$x^2+y^2+z^2-2x+(3+5\lambda)y+(4+6\lambda)z+(\lambda-5)=0 \quad (1)$$

Similarly, any sphere through the second circle is

$$x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2y-7z)=0$$

i.e. $$x^2+y^2+z^2+(\mu-3)x+(2\mu-4)y+(5-7\mu)z-6=0 \quad (2)$$

Now the given circle will lie on the same sphere if equations (1) and (2) are identical for some values of λ and μ. For checking this, comparing the coefficients in (1) and (2), we have

$$\mu-3=-2 \qquad 2\mu-4=3+5\lambda,\ 5-7\mu=4+6\lambda,\ \lambda-5=-6$$

The first and last of these equations given

$\mu = 1$ and $\lambda = -1$

These values satisfy also the second and third equations

Thus (1) and (2) ARE identical for $\lambda = -1, \mu = 1$

Hence the given circles lie on the same sphere

The equations of this sphere can be obtained by putting $\lambda = -1$ in (1) or $\mu = 1$ in (2), it is

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

Example14:- Show that the equation to the sphere which passes through the point (α, β, γ) and the circle $z = 0, x^2 + y^2 + z^2 = a^2$ is $\gamma(x^2 + y^2 - a^2) = z(\alpha^2 + \beta^2 + \gamma^2 + a^2)$

Solution:- Equation to any sphere through the given circle $x^2 + y2 + z^2 - a^2 = 0$, $z = 0$ is

$$(x^2 + y^2 + z^2 - a^2) + \lambda z = 0 \qquad (1)$$

It will pass through the point (α, β, γ) if

$$\alpha^2 + \beta^2 + \gamma^2 - a^2 + \lambda\gamma = 0$$

Which gives $\lambda = -\dfrac{\alpha^2 + \beta^2 + \gamma^2 - a^2}{\gamma}$

Substituting this value of λ in (1), we get

$$(x^2 + y^2 + z^2 - a^2) = \frac{z(\alpha^2 + \beta^2 + \gamma^2 - a^2)}{\gamma} = 0$$

i.e. $\gamma(x^2 + y^2 + z^2 - a^2) = z(\alpha^2 + \beta^2 + \gamma^2 - a^2)$

Example15:- Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$ at the point $(-1, 0, -1)$

Solution:- We know that the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point (α, β, γ) on it is

$$\alpha x + \beta y + \gamma z + u(x - \alpha) + v(y - \beta) + w(z + \gamma) + d = 0$$

Here $u = -1$, $v = -1$, $w = -1$, $d = -6$ and $\alpha = -1, \beta = 0, \gamma = -1$

Hence the required equation of the tangent plane is

$$(-1)x + 0.y + (-1)z + (-1)(x - 1) + (-1)(y + 0) + (-1)(z - 1) - 6 = 0$$

i.e. $-x - z - (x - 1) - y - z(z - 1) - 6 = 0$

i.e. $-2x - y - 2z - 4 = 0$

i.e. $2x + y + 2z + 4 = 0$

Example16:- Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$, and find the point of contact

Solution:- The centre of the given sphere is $(1, 2, -1)$ and its

$$\text{Radius} = \sqrt{\{(-1^2) + (-2)^3 + 1^2 + 3\}} = \sqrt{(9)} = 3$$

Also, the length of the perpendicular from the centre $(1,2,-1)$ of the sphere to the given plane is

$$\frac{2\times1-2\times2+1(-1)+12}{\sqrt{\left\{2^2+(-2)^2+1^2\right\}}} \text{ i.e. } 3$$

Which is equal to the radius of the sphere.
Hence the given plane touches the given sphere.
Further, the point of contact is the foot of the perpendicular from the centre of the sphere to the tangent plane. Now the equations of the perpendicular from the centre to the given plane are $\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r$, say

Any point on this line is $(2r+1,-2r+2,r-1)$. If this is the point of contact, it must lie on the tangent plane. Thus we have $2(2r+1)-2(-2r+2)+(r-1)+12=0$ which gives $r=-1$.

Hence the point of contact is $(-1,4,-2)$

Example17:- Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$

Solution:- The general equation of sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad (1)$$

The equation of its tangent plane at the point $(1,-2,1)$ is

$$x-2y+z+u(x-1)+v(y-2)+w(z+1)+d=0$$

i.e. $(1+u)x+(v-2)y+(1+w)z+(u-2v+w+d)=0$ (2)

But we are given that the plane

$$3x+2y-z+2=0 \qquad (3)$$

Touches the sphere at the same point. Therefore, equation (2) and (3) must be equivalent. So, comparing the coefficients in (2) and (3), we have

$$\frac{1+u}{3}=\frac{v-2}{2}=\frac{1+w}{-1}=\frac{u-2v+w+d}{2}=k \text{ Say}$$

Whence $u=3k-1,\ v=2k+2,\ w=-(k+1)$ (4)

And $u+2v+w+d=2k$ (5)

Using the values of u,v,w from (4) in (5), we get

$$(3k-1)-2(2k+2)-(k+1)+d=2k$$

i.e. $d=4k+6$ (6)

Further, since the given sphere

$$x^2+y^2+z^2+4x+6y+4=0$$

Intersects sphere (1) orthogonally, we have

$$-4u+6v=d+4$$

Putting the values of u,v and d from (4) and (6), this given

$$-4(3k-1)+6(2k+2)=(4k+6)+4 \text{ i.e. } k=\frac{3}{2}$$

Putting this value of k in (4) and (6), we get

$$u=\frac{7}{2},\ v=5,\ w=-\frac{5}{2},\ d=12$$

Finally, using these values in (1), the required equation of sphere is

$$x^2+y^2+z^2+7x+10y-5z+12=0$$

Example18:- Find the equations to the sphere which pass through the circle

$$x^2+y^2+z^2=5,\ x+2y+3z=3$$

And touch the plane $4x+3y-15=0$

Solution:- Equation of any sphere passing through the given circle is

$$x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0 \qquad (1)$$

Centre of this sphere is $\left(-\frac{1}{2}\lambda,\lambda-\frac{3}{2}\lambda\right)$

Also, radius $=\sqrt{\left\{\left(-\frac{1}{2}\lambda\right)^2(-\lambda)^2+\left(-\frac{3}{2}\lambda\right)^2-(-5-3\lambda)\right\}}$

$$=\frac{1}{2}\sqrt{\left(14\lambda^2+12\lambda+20\right)}$$

Now the sphere (1) will touch the given plane

$$4x-3y-15=0$$

If the perpendicular distance of this plane from the centre of the sphere is equal to its radius i.e. if

$$\frac{4\left(-\frac{1}{2}\lambda\right)+3(-\lambda)-15}{\sqrt{\left(4^2+3^2\right)}}=\frac{1}{2}\sqrt{\left(14\lambda^2+12\lambda+20\right)}$$

i.e. $\quad 2(-5\lambda-15)=5\sqrt{\left(14\lambda^2+12\lambda+20\right)}$

On squaring, this gives

$$4\left(25\lambda^2+150\lambda+225\right)=25\left(14\lambda^2+12\lambda+20\right)$$

i.e. $\quad 250\lambda^2-300\lambda-400=0$ i.e. $5\lambda^2-6\lambda-8=0$

i.e. $\quad 5\lambda^2-10\lambda+4\lambda-8=0$ i.e. $5\lambda(\lambda-2)+4(\lambda-2)=0$

i.e. $\quad (\lambda-2)(5\lambda+4)=0$

This gives $\lambda=2,-\frac{4}{5}$

Substituting these values of λ one by one in (1), the equations of required shperes are

$$x^2+y^2+z^2+2x+4y+6z-11=0$$

And $x^2+y^2+z^2-\frac{4}{5}x-\frac{8}{5}y-\frac{12}{5}z-\frac{13}{5}=0$

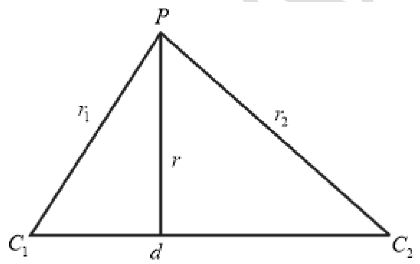
Example19:- Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\frac{r_1r_2}{\sqrt{(r_1^2+r_2^2)}}$

Solution:- If d be the distance between the centres C_1 and C_2 of the two given spheres intersecting at a point P, then $d^2=(C_1,C_2)^2=r_1^2+r_2^2$

Also, if r be the radius of the common circle then

$\frac{1}{2}r_1r_2=\frac{1}{3}rd$ (area of ΔC_1C_2P)

Which gives $d=r_1r_2/r$



Putting the value of d in (1), we get

$\frac{r_1^2r_2^2}{r^2}=r_1^2+r_2^2$, i.e. $r^2=\frac{r_1^2r_2^2}{r_1^2+r_2^2}$

This at once gives $r=\frac{r_1r_2}{\sqrt{(r_1^2+r_2^2)}}$

Example20:- If a tangent plane of the sphere $x^2+y^2+z^2=r^2$ makes intercepts a,b and c on x,y and z axes respectively, then show that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{r^2}$

Solution:- We known that the equation of the tangent plane at a point (x_1,y_1,z_1) on the sphere

$$x^2 + y^2 + 2ux + 2vy + 2wz + d = 0 \text{ is}$$

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

Hence the equation of the tangent plane at any point (x_1, y_1, z_1) on the sphere

$$x^2 + y^2 + z^2 = r^2 \text{ is}$$

$$xx_1 + yy_1 + zz_1 + r^2$$

In intercept form this equation assumes the form

$$\frac{x}{r^2 / x_1} + \frac{y}{r^2 / y_1} + \frac{z}{r^2 / z_1} = 1$$

It is given that $\dfrac{r^2}{x_1} = a,\ \dfrac{r^2}{y_1} = b,\ \dfrac{r^2}{z_1} = c$

i.e. $\dfrac{1}{a} = \dfrac{x_1}{r^2},\ \dfrac{1}{b} = \dfrac{y_1}{r^2},\ \dfrac{1}{c} = \dfrac{z_1}{r^2}$

Therefore, $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} = \dfrac{x_1^2 + y_1^2 + z_1^2}{r^4}$, i.e. $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} = \dfrac{1}{r^2}$

Since $x_1^2 + y_1^2 + z_1^2 = r^2$ as the point (x_1, y_1, z_1) lies on $x^2 + y^2 + z^2 = r^2$.

Example21:- Two points P and Q are conjugate with respect to a sphere S. Prove that sphere on PQ as diameter cuts S orthogonally.

Solution:- Let the equation of the sphere be

$$x^2 + y^2 + z^2 = a^2 \qquad (1)$$

Also, let P and Q be the points (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. Then the equation to the polar of P is

$$xx_1 + yy_1 + zz_1 = a^2 \qquad (2)$$

It passes through the point Q. therefore

$$x_1x_2 + y_1y_2 + z_1z_2 = a^2$$

Now the equation to the sphere having PQ as the diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

i.e. $x^2+y^2+z^2-(x_1+x\,)x-(y_1+y_2)y-(z_1+z_2)z+x_1x_2+y_1y_2+z_1z_2=0$

Clearly, this sphere will cut the sphere (1) orthogonally if

$$-(x_1+x_2).0-(y_1-y_2).0-(z_1+z_2).0=x_1x_2+y_1y_2+z_1z_2-a^2$$

i.e. $\quad x_1x_2+y_1y_2+z_1z_2=a^2$

But this is the same as the condition (2) for which P and Q are conjugate to each other. Hence the result

Example22:- Show that the sphere $x^2+y^2+z^2=64$ and $x^2+y^2+z^2=12x+4y-6z+48=0$ touch internally, and find their point of contact.

Solution:- The given sphere are

$$S_1 \equiv x^2+y^2+z^2-64=0 \qquad (1)$$

and $$S_2 \equiv x^2+y^2+z^2+12x+4y-6z+48=0 \qquad (2)$$

The equation of their radical plane is $S_2-S_1=0$ i.e. $12x+4y-6z+112=0$

i.e. $$6x+2y-3z+56=0 \qquad (3)$$

The given sphere will touch each other if the radical plane is their common tangent plane.

The centre of sphere (1) is $(0,0,0)$ and its radius is 8. The distance of the centre $(0,0,0)$ from the radical plane (3) is

$$\frac{56}{\sqrt{\left\{6^2+2^2+(-3)^2\right\}}}, \text{ i.e. } 8$$

Which is equal to its radius.

Therefore, the radical plane (3) touches the sphere (1). But we know that if the radical plane touches one sphere, then it will touch the other sphere also. Hence both the spheres touch each other.

Further, the centre of sphere (2) is $(-6,-2,3)$, and its

$$\text{Radius } = \sqrt{\left\{6^2+2^2+(-3)^2-48\right\}} = 1$$

Also, the distance between the centres of the given spheres

$$= \sqrt{\left\{(0+6)^2+(0+2)^2+(0-3)^2\right\}}$$

$$= \sqrt{(36+4+9)} = \sqrt{(49)} = 7$$

$$= 8-1$$

$$= \text{Difference of their radii}$$

Therefore, the given spheres touch each other internally.

Let (α, β, γ) be the point of contact of the given spheres. Then the equation of the tangent plane to the sphere (1) at this point is

$$\alpha x + \beta y + \gamma z - 64 = 0$$

Comparing in with the radical plane (3), we get

$$\frac{\alpha}{6} = \frac{\beta}{2} = \frac{\gamma}{-3} = \frac{-64}{56} = \frac{-8}{7}$$

Whence $\alpha = -48/7$, $\beta = -16/7$, $\gamma = 24/7$.

Therefore, the required point of contact is:

$$(-48/7, -16/7, 24/7)$$

Example23:- Find the equation of the sphere which is coaxal with the spheres $x^2+y^2+z^2+3x-3y+2z=0$, $x^2+y^2+z^2+2x-y-z+10=0$, and passes through the point $(0,1,2)$

Solution:- The equation of any sphere which is coaxal with the given spheres is

$$\left(x^2+y^2+z^2+3x+3y+2z\right)+\lambda\left(x^2+y^2+z^2+2x-y-z+10\right)=0 \qquad (1)$$

If it passes through the point $(0,1,2)$, we must have

$$\left(0^2+1^2+2^2+3.0-3.1+2.2\right)+\lambda\left(0^2+1^2+2^2+2.0-1-2+10\right)=0$$

Which gives

Using this value of λ in (1), the required equation of coaxal sphere is

$$\left(x^2+y^2+z^2+3x-3y+2z\right)-\frac{1}{2}\left(x^2+y^2+z^2+2x-y-z+10\right)=0$$

i.e. $\quad x^2+y^2+z^2+4x-5y+5z-10=0$

Example24:- Find the limiting points of the co-axal system of spheres determined by $x^2+y^2+z^2+3x+3y+6=0$, and $x^2+y^2+z^2-6y-6y+6=0$

Solution:- Subtracting the second equation from the first one, the radical plane of the given spheres is $3x+3y+6z=0$ i.e. $x+y+2z=0$

Therefore, the equation of the co-axal system is

$$\left(x^2+y^2+z^2+3x-3y+6\right)+\lambda\left(x+y+2z\right)=0$$

i.e. $\quad x^2+y^2+z^2\left(3+\lambda\right)x-\left(3-\lambda\right)y+2\lambda z+6=0$

From this, the centre of the co-axal system is

$$\left(-\frac{1}{2}(3+\lambda),\frac{1}{2}(3-\lambda),-\lambda\right)$$

And radius $=\sqrt{\left\{\frac{1}{4}(3+\lambda)^2+\frac{1}{4}(3-\lambda)^2+(-\lambda)^2-6\right\}}$

For limiting points of the co-axal system, the radius should be zero.

Thus we have

$$\frac{1}{4}(3+\lambda)^2+\frac{1}{4}(3-\lambda)^2+\lambda^2-6=0$$

i.e. $\quad (3+\lambda)^2+(3-\lambda)^2+4\lambda^2-24=0$

i.e. $\quad 6\lambda^2-6=0$, which gives $\lambda=\pm 1$

Substituting these values of in the co-ordinates of the centre given by (1), we get

$$(-1,2,1) \text{ and } (-2,1,-1)$$

These are the required limiting points.

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in last of this book.

CHAPTER 5. SPHERE

Q3c Find the equation of the sphere through the circle $x^2+y^2+z^2-4x-6y+2z-16=0;$ $3x+y+3z-4=0$ in the following two cases.

(i) the point (1, 0, -3) lies on the sphere.

(ii) the given circle is a great circle of the sphere. **UPSC CSE 2023** **(15)**

Q(c) Find the equation of the sphere of smallest possible radius which touches the straight lines: $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. **UPSC CSE 2022**

Q3.(b) A sphere of constant radius r passes through the origin O and cuts the axes at the points A, B and C. Find the locus of the foot of the perpendicular drawn from O to the plane ABC.

UPSC CSE 2021

Q4.(a) Find the equation of the sphere passing through the points $(1,1,2)$, $(1,-1,2)$ and having centre on the line $x+y-z-1=0=2x+y-z-2$. **IFoS 2021**

Q1(e) A variable plane passes through a fixed point (a,b,c) and meets the axes at points A, B and C respectively. Find the locus of the centre of the sphere passing through the points O, A, B and C, O being the origin. **UPSC CSE 2022**

Q1. The plane $x+2y+3z=12$ cuts the axes of coordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC. **[(2c)(i) UPSC CSE 2019]**

Q2. Find the equation of the sphere in *xyz*-plane passing through the points $(0,0,0),(0,1,-1),(-1,2,0)$ and $(1,2,3)$. **[(3d) UPSC CSE 2018]**

Q3. Find the equation of the tangent plane that can be drawn to the sphere $x^2+y^2+z^2-2x+6y+2z+8=0$, through the straight line $3x-4y-8=0=y-3z+2$.

[(3a) 2018 IFoS]

Q4. A plane passes through a fixed point (a,b,c) and cuts the axes at the points A, B, C respectively. Find the locus of the centre of the sphere which passes through the origin O and A, B, C.

[(2b) UPSC CSE 2017]

Q5. Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$. Find the point of contact. **[(2c) UPSC CSE 2017]**

Q6. Find the equation of the sphere which passes through the circle $x^2+y^2=4$; $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3. **[(1d) UPSC CSE 2016]**

Q7. Obtain the equation of the sphere on which the intersection of the plane $x+2y+2z=0$ with the sphere which has $(0,1,0)$ and $(3,-5,2)$ as the end points of its diameter is a great circle.

[(3d) 2016 IFoS]

Q8. For what positive value of a, the plane $ax-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and hence find the point of contact. **[(1e) UPSE CSE 2015]**

Q9. Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point $(2,1,3)$?

[(3b) UPSE CSE 2015]

Q10. Find the co-ordinates of the points on the sphere $x^2+y^2+z^2-4x+2y=4$, the tangent planes at which are parallel to the plane $2x-y+2z=1$. **[(4a(i) UPSE CSE 2014]**

Q11. Prove that every sphere passing through the circle $x^2+y^2-2ax+r^2=0, z=0$ cut orthogonally every sphere through the circle $x^2+z^2=r^2, y=0$. **[(2c) 2014 IFoS]**

Q12. A moving plane passes through a fixed point $(2,2,2)$ and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C. **[(3b) 2014 IFoS]**

Q13. A sphere S has points $(0,1,0),(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle. **[(1e) UPSC CSE 2013]**

Q14. Show that the three mutually perpendicular tangent lines can be drawn to the sphere $x^2+y^2+z^2=r^2$ from any point on the sphere $2(x^2+y^2+z^2)=3r^2$. **[(4a) UPSC CSE 2013]**

Q15. Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere $x^2+y^2+2fx+2gy+2hz=0$ if $af+bg+ch=0$.

[(4b) 2012 IFoS]

Q16. Show that the equation of the sphere which touches the sphere $4(x^2+y^2+z^2)+10x-25y-2z=0$ at the point $(1,2,-2)$ and passes through the point $(-1,0,0)$ is $x^2+y^2+z^2+2x-6y+1=0$. **[(1f) UPSC CSE 2011]**

Q17. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$. **[(3b) UPSC CSE 2011]**

Q18. Show that the plane $x+y-2z=3$ cuts the sphere $x^2+y^2+z^2-x+y=2$ in a circle of radius 1 and find the equation of the sphere which has this circle as a great circle.

[(1e) UPSC CSE 2010]

Q19. Show that every sphere through the circle $x^2 + y^2 - 2ax + r^2 = 0,\ z = 0$ cuts orthogonally every sphere through the circle $x^2 + z^2 = r^2, y = 0$. **[(3c) UPSC CSE 2010]**

Q20. Find the equations of the spheres passing through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, y = 0$ and touching the plane $3y + 4z + 5 = 0$.

[(1f) 2010 IFoS]

6.The Cone & Cylinder

6.1 The Cone:-Definition:- A cone is a surface generated by a variable straight line which passes through a fixed point and intersects a given curve (or touches a given surface).

The fixed point is called the vertex and the given curve (or surface) the **guiding curve** (or **guiding surface)** of the cone. The variable straight line is called the **generator** of the cone.

A cone whose equation is of second degree, is called a quadratic (or quadric) cone. Here we shall confine ourselves to the study of quadratic cones.

6.2 Cone With Vertex At The Origin:-To find the equations of the cone whose vertex is at the origin.

Let the general equation of the second degree in x, y, z , viz.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \qquad (1)$$

Represent a cone whose vertex is at the origin O. If $P(x', y', z')$ is any point on the cone, the line OP being a generator of the cone, lies completely on it. The equations of OP are $\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}=r$ say.

Clearly the co-ordinates of any point on OP are $(rx', ry' rz')$ and the equations (1) should be satisfied by these co-ordinates for all values of r. Thus we have $r'\left(ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'\right)+2r\left(ux'+vy'+wz'\right)+d=0$, which must be an identity. Therefore $ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'=0$ (2)

$$ux'+vy'+wz'=0 \qquad (3)$$

and $$d=0 \qquad (4)$$

We now observe the following.

(i) The equation (4) is obviously satisfied as the cone passes through the origin.

(ii) From equation (3) it is clear that $u=v=w=0$ for otherwise the point $P(x', y', z')$ will lie on the plane $ux+vy+wz=0$, which is not the case here.

In view of these observations, we are left with (2) only. Hence the equations of a cone with its vertex at the origin is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ (5)

Which is a homogeneous equation of the second degree in x, y and z.

Note:- (1) Conversely, every homogenous equation of the second degree represents a cone with its vertex at the origin

It is obviously that if $P(x', y', z')$ satisfies equation (5), then for all values of r, the point $(rx', ry' rz')$ also satisfies (5). Thus shows that every point on OP, and hence the whole lines OP, lies on the surface represented by (5).

Note:- (2) In section 3.14 we have seen that equation (5) represents a pair of plane under a certain condition satisfied by the coefficients. Thus such a pair of planes of course intersecting may be regarded as a cone with vertex to be any point on the line intersection of the planes.

Corollary:-If l,m,n are the direction cosines of the generator of a cone $f(x,y,z)=0$ then $f(l,m,n)=0$

OR

If the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is a generator of the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hky=0$ then $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$

OR

The direction cosines of the generators of cone represented by the homogenous equation of the second degree satisfy the equation to the cone .

Proof:- Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ (1)

Be a generator of the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ (2) whose vertex is a the origin.

Any point on (1) can be given by $P(rl,rm,rn)$. Since the line (1) is a generator of the cone (2), this point must lie on the cone for each value of r .

Therefore, $a(rl)^2+b(rm)^2+c(rn)^2+2f(rm)(rn)+2g(rn)(rl)+2h(rl)(rm)=0$.

Dividing by r^2, this reduces to $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$.

Thus $f(l,m,n)=0$ when the equation of the cone is $f(x,y,z)=0$.

6.3 Condition For The General Equation of The Second Degree To Represent A Cone:-To find the condition that the general equations of the second degree, *viz* .

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ (1)

May represent a cone.

Let the equation (1) represent a cone with its vertex at (α,β,γ). Shifting the origin to (α,β,γ) the equation (1) transforms to

$$a(x+\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2$$
$$+2f(y+\beta)(z+\gamma)+2g(z+\gamma)(x+\alpha)+2g(x+\alpha)(y+\beta)$$
$$+2u(x-\alpha)+2v(y+\beta)+2w(z+\gamma)+d=0$$

i.e.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2(a\alpha+h\beta+g\gamma+u)x+2(h\alpha+b\beta+f\gamma+v)y+2(g\alpha+f\beta+c\gamma+w)z$$
$$+a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma a+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d=0.$$

Since this equation represents a cone whose vertex is the origin, it should be a homogenous equation. Therefore, we have $a\alpha+h\beta+g\gamma+u=0$ (2)

$$h\alpha+b\beta+f\gamma+v=0 \quad (3)$$
$$g\alpha+f\beta+c\gamma+w=0 \quad (4)$$

And $a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d=0$ i.e.
$\alpha(a\alpha+h\beta+g\gamma+u)+\beta(h\alpha+b\beta+f\gamma+v)$
$+\gamma(g\alpha+f\beta+c\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0$ **(note)**
i.e. $u\alpha+v\beta+w\gamma+d=0$ (5)
Using (2), (3) and (4)
Eliminating α,β,γ between the equations (2), (3), (4) and (5), the required condition is obtained as

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}=0 \qquad (6)$$

Note:- (1) When the above condition holds good, the co-ordinates (α,β,γ) of the vertex can be determined by solving any three of the equations (2) to (5)
Note:- (2) Let us denote the equation (1) by the function notation $\phi(x,y,z)=0$. Making the equation homogenous introducing a new variable t, we obtain
$\phi(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2vxt+2vyt+2wzt+dt^2=0$.
Differentiating partially, we get

$$\frac{\partial\phi}{\partial x}=2(ax+hy+gz+ut), \qquad \frac{\partial\phi}{\partial y}=2(hx+by+fz+vt)$$

$$\frac{\partial\phi}{\partial z}=2(gx+fy+cz+wt), \qquad \frac{\partial\phi}{\partial t}=2(ux+vy+wz+dt).$$

Taking $t=1$, the equations
$\frac{\partial\phi}{\partial x}=0$, $\frac{\partial\phi}{\partial y}=0$, $\frac{\partial\phi}{\partial z}=0$, $\frac{\partial\phi}{\partial t}=0$ are found to be the same as the equations (2), (3), (4) and (5) respectively.

6.4 Equation of A Cone Whose Vertex And The Guiding Curve Are Given:-To find the equation of a cone whose vertex is the point (α,β,γ) and the guiding curve is the conic. $ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0$ (1)

Any straight line through (α,β,γ) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$.

This meets the plane $z=0$ in the point given by $\frac{x-\alpha}{l}=\frac{y-\beta}{m}-\frac{\gamma}{n}$, i.e. the point $\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$.

This point lie on the conic given by (1). Therefore

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta+\frac{m\gamma}{n}\right)^2$$

$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0 \qquad (3)$$

This is the condition for the line (2) to intersect the given conic. Therefore to obtain the locus of this line, we need to eliminate l, m, n between (2) and(3).

Hence the equation of the required cone is

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+b\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2$$

$$+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+c=0$$ which on multiplication by $(z-\gamma)^2$ can be written as $a(\alpha z-\gamma x)^2+2h(\alpha z-\gamma x)(\beta z-y\gamma)+b(\beta z-y\gamma)^2$

$$+2g(\alpha z-\gamma x)(z-\gamma)+2f(\beta z-y\gamma)(z-\gamma)+c(z-\gamma)^2=0 \quad (4)$$

This is the required equation of the cone.

6.5 Intersection of A Cone With A Plane:-To prove that a cone is cut a plane through the vertex of the cone in two generating lines.

Without any loss of generally, let us take the cone with vertex at the origin whose equation is given by $\phi(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ (1)

Let the plane through the vertex of the cone be given by $ux+y+wz=0$ (2)

Now let one of the liens in which this plane curs the cone be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Since this line lies on the cone (1) as-well-as on the plane (2), we have $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$ (3)

(using Corollary of Section 6.2)

And $ul+vm+wn=0$ (4)

(using the condition of perpendicularity).

Putting the value of n from (4) and (3), we obtain

$$al^2+bm^2+c\left(\frac{ul+vm}{m}\right)^2-2f\left(\frac{ul+vm}{w}\right)^2 m-2g\left(\frac{ul+vm}{w}\right)l+2hlm=0 \text{ i.e.}$$

$$\left(cu^2+aw^2-2gwu\right)^2+2\left(hw^2+cuv-fwu-gvw\right)lm+\left(bw^2+cv^2-2fvw\right)m^2=0.$$

Dividing by m^2 this equation becomes

$$\left(cu^2+aw^2-2gwu\right)\left(\frac{l}{m}\right)^2+2\left(hw^2+cuv-fwn-gvw\right)\frac{l}{m}+\left(bw^2+cv^2-2fvw\right)=0 \quad (5)$$

This is a quadratic equation in l/m, giving two values of l/m (real and distinct, coincident or imaginary) corresponding to which there are two lines on intersection of the cone and the plane.

Note:- Hence forth by a plane cuts a cone we shall always mean the plane through the vertex of cone unless specified otherwise. The section of a cone with an arbitrary plane (not passing through the vertex of the cone) would be some conic rather than the generating lines.

6.9 Angle Between The Two Generating Line:-To find the angle between the two generating line in which a plane cuts a cone.

Let the equations of the cone and the plane through its vertex be equation (1) and(2) of the preceding section. We have just observed via equation (5) of the same section that the plane cuts the cone in two generating lines. If l_1, m_1, n_1 and l_2, m_2, n_2 be direction ratios of these lines, then l_1/m_1 and l_2/m_2 will be the roots of the equation (5).

Using the formula for the sum and product of roots of a quadratic, we have $\frac{l_1}{m_1}+\frac{l_2}{m_2}+\frac{l_1m_2+m_1l_2}{m_1m_2}=-\frac{2\left(hw^2+cuv-fwv-gvw\right)}{cu^2+aw^2-2gwu}$ and $\frac{l_1}{m_1}\frac{l_2}{m_2}=\frac{bw^2+cv^2-2fvw}{cu^2+aw^2-2gwu}$.

These relations can also be written as $\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}=\frac{l_1m_2+m_1l_2}{-2\left(hw^2+cuv-fwu-gvw\right)}=\lambda_1$ say.

Now $\left(l_1m_2-m_1l_2\right)^2=\left(l_1m_2+m_1l_2\right)^2-4l_1l_2, m_1m_2$

$$=4\lambda^2\left(hw^2+cuv-fwu-gvw\right)^2$$

$$-4\lambda\left(bw^2+cv^2-2fvw\right).\,\lambda\left(cu^2+aw^2-2gwu\right)$$

$$=4\lambda^2\left\{h^2w^4-2hw^3\left(fu+gv\right)+w^2\left(fu+gv\right)^2+2chuvw^2-2chvw\left(fu+gv\right)+c^2u^2v^2\right\}$$

$$-4\lambda^2\left\{abw^2-2w^3\left(afv+bgu\right)+w^2\left(bcu^2+cuv^2+4guw\right)-2cuvw\left(fu+gv\right)+c^2u^2v^2\right\}$$

$$=4\lambda^2w^2\left\{-\left(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv\right)\right\}$$

$=4\lambda^2w^2P^2$ on simplification, where the capital letter A, B, C, F, G, H are the cofactors of the corresponding small letter a, b, c, f, g, h in the determinant.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & g & c \end{vmatrix} \text{ and } P^2=\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}.$$

The value of n_1n_2 and $m_1n_2-n_1m_2$, $n_1l_2-l_1n_2$ can be written by symmetry. Thus we obtain.

$$l_1l_2+m_1m_2+n_1n_2=\lambda\left\{\left(bw^2+cv^2-2fvw\right)+\left(cu^2+aw^2-2gwu\right)+\left(av^2+bu^2-2huv\right)\right\}$$

$$=\lambda\left\{a\left(v^2+w^2\right)+b\left(w^2+u^2\right)+c\left(u^2+v^2\right)-2fvw-2gwu-2huv\right\}$$

$=\lambda\left\{\left(a+b+c\right)\left(u^2+v^2+w^2\right)-\phi\left(u,v,w\right)\right\}$ and also

$$\sum\left(m_1n_2-n_1m_2\right)^2=4\lambda^2P^2\left(u^2+v^2+w^2\right).$$

Hence if θ be the angle between the lines of intersection, then

$$\tan\theta=\pm\frac{\sqrt{\left\{\sum\left(m_1n_2-n_1m_2\right)^2\right\}}}{l_1l_2+m_1m_2+n_1n_2}$$

$=\pm\frac{2\lambda P\sqrt{\left(u^2+v^2+w^2\right)}}{\lambda\left\{\left(a+b+c\right)\left(u^2+v^2+w^2\right)-\phi\left(u,v,w\right)\right\}}$, i.e.

$$\tan\theta = \pm \frac{2P^2\sqrt{\left(u^2+v^2+w^2\right)}}{(a+b+c)\left(u^2+v^2+w^2\right)-\phi(u,v,w)}.$$

Corollary 1:- The plane $ux+vy+wz=0$ cuts the cone $\phi(x,y,z)=0$ in two coincident generators if $\theta=0^\circ$ i.e. if $P=0$ i.e. $Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0$.

Corollary 2:- The plane $ux+vy+wz=0$ cuts the cone $\phi(x,y,z)=0$ in perpendicular generators if $\theta=90^\circ$ i.e. if $(a+b+c)\left(u^2+v^2+w^2\right)-\phi(u,v,w)=0$.

Three Mutually Perpendicular Generators:-To find the condition when a cone has three distinct mutually perpendicular generators.

Let $\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}$ (1)

Be a generator of the cone $\phi(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ (2)

Then $\phi(u,v,w)\equiv au^2+bv^2+cw^2+2fvw+2gwu+2huv=0$ (3)

The equation of the plane passing through the origin and perpendicular to the line (1) is $ux+vy+wz=0$ (4)

Now (using Corollary (2) of the preceding section) this plane will cut the cone in two perpendicular generators (of course, other then (1)) if $(a+b+c)\left(u^2+v^2+w^2\right)-\phi(u,v,w)=0$.

Using (3), this relation reduces to $(a+b+c)\left(u^2+v^2+w^2\right)=0$.

Since $u^2+v^2+w^2$ cannot be zero, it follow that $a+b+c=0$. (5)

This is the condition that the cone (2) has there distinct mutually exclusive generators namely the generator (1) and the two lines of intersection of the plane (4) and the cone.

Remark:-The condition $a+b+c=0$ obtained above is independent of u,v,w. Therefore, we may start with any of the infinitely many generators of the cone, and the plane passing through the origin and normal to it will be found to cut the cone in two other perpendicular generator, if $a+b+c=0$. Thus a set of there mutually perpendicular generator is obtained for each given generator of the cone, provided $a+b+c=0$. Hence we observe the following:

If $a+b+c=0$, the cone given by (2) has an infinite number of sets of three mutually perpendicular generators.

In other words, either a cone has an infinite number of sets of three mutually perpendicular generators or none.

We now define:

Rectangular Cone:-A cone having three distinct mutually perpendicular generators is called a rectangular cone..

Evidently, a cone given by (2) is rectangular if $a+b+c=0$, i.e. the sum of coefficient of x^2, y^2 and $z^2=0$.

It is interesting to note that this condition is analogous to that for a rectangular hyperbola in two dimensional geometry.

6.8 Tangent Lines And Tangent Plane:-

Intersection of a Cone a Straight Line:-Let

$$\phi(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad (1)$$

Be the equation of cone and $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}=r$, say (2)

Be the equations of a straight line passing through the point $P(\alpha,\beta,\gamma)$.

Any point on the line (2) is $(\alpha+lr, \beta+mr, \gamma+nr)$. This point will lie on the cone (1) if its co-ordinates satisfy (1), i.e.

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2+2f(\beta+mr)(\gamma+nr)$$

$$+2g(\gamma+nr)(\alpha+lr)+2h(\alpha+lr)(\beta+mr)=0, \text{ i.e.}$$

$$\phi(l,m,n)r^2+2\{(a\alpha+h\beta+g\gamma)l+(h\alpha+b\beta+f\gamma)m$$

$$+(g\alpha+f\beta+c\gamma)n\}r+\phi(\alpha,\beta,\gamma)=0 \quad (3)$$

This equation, being quadratic in r, given two values of r' which may be real and distinct, coincident of imaginary. Accordingly there are two points of intersection of the cone and the straight line which may be real and distinct, coincident or imaginary.

Tangent Line:-A line which meets a cone in two coincident points is called a tangent line to the cone.

If the point $P(\alpha,\beta,\gamma)$ lies on the cone (1), we have $\phi(\alpha,\beta,\gamma)=0$. If follows from (3) that one root of equation (3) is zero. Further if

$$(a\alpha+h\beta+g\gamma)l+(h\alpha+b\beta+f\gamma)m+(g\alpha+f\beta+c\gamma)n=0 \quad (4)$$

Equation (3) reduces to $\phi(l,m,n)r=0$, showing that the other roots also zero. Consequently, the two point of intersection coincide at P. In this case the line (2) is a tangent line to the cone (1) at P.

Tangent Plane:-The locus of all the tangent liens at a point on the cone is called a tangent plane to the cone at that point.

The point P is called the point of contact.

Evidently, the equation of the tangent plane at a point $P(\alpha,\beta,\gamma)$ on the cone (1) is obtained by eliminating l,m,n between (2) and (4). Thus we have

$$(a\alpha+h\beta+g\gamma)(x-\alpha)+(h\alpha+b\beta+f\gamma)(y-\beta)+(g\alpha+f\beta+c\gamma)(z-\gamma)=0$$

i.e. $(a\alpha+h\beta+g\gamma)x+(h\alpha+b\beta+f\gamma)y+(g\alpha+f\beta+c\gamma)z=\phi(\alpha,\beta,\gamma)0$.

But $\phi(\alpha,\beta,\gamma)=0$. Hence the equation of the tangent plane to the cone (1) at the point $P(\alpha,\beta,\gamma)$ is

$$(a\alpha+h\beta+g\gamma)x+(h\alpha+b\beta+f\gamma)y+(g\alpha+f\beta+c\gamma)z=0$$

Note:- (1) The tangent plane at any point P of a cone passes through is vertex O, and thus contains the generator OP, such that generator is called the generator of contact.

(2) There is a unique tangent plane at every point of the surface of cone, except the vertex, through which all the tangent planes pass. This point (the vertex) is called a singular of the surface.

Working Rule:-The equation of the tangent plane at any point (α,β,γ) on the cone be obtained from the equation of the cone on replacing $x^2+y^2+z^2$ by $\alpha x,\beta y,c\gamma$ and $2yz,2zx,2xy$ by $\gamma y+\beta z,\alpha z+\gamma x,\beta x+\alpha y$, respectively.

6.9 Condition of Tangency:-To find the condition when the plane $ux+vy+wz=0$ (1) becomes a tangent plane of the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

Let $P(\alpha,\beta,\gamma)$ be the point of contact of the plane with the cone. Then the tangent plane at P to be cone is $(a\alpha+h\beta+g\gamma)x+(h\alpha+b\beta+f\gamma)y+(g\alpha+f\beta+c\gamma)z=0$ (2)

Now the equation (1) will be the tangent plane when it is equivalent to (2) i.e. $\frac{a\alpha+h\beta+g\gamma}{u}=\frac{h\alpha+b\beta+f\gamma}{v}=\frac{g\alpha+f\beta+c\gamma}{w}=\lambda$, say i.e.

$$a\alpha+h\beta+g\gamma-\lambda u=0 \qquad (3)$$
$$h\alpha+b\beta+f\gamma-\lambda v=0 \qquad (4)$$
$$g\alpha+f\beta+c\gamma-\lambda w=0 \qquad (5)$$

Further $\quad u\alpha+v\beta+c\gamma=0 \qquad (6)$

Since the point (α,β,γ) lies on (1) also.

Eliminating α,β,γ and $-\lambda$ between (3), (4) (5) and (6), we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}=0 \text{ i.e. } \quad Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Hvu=0$$

where $A=bc-f^2,\ B=ca-g^2,\ c=ab-h^2,$
$F=gh-af,\ G=hf-bg,\ H=fg-ch,$

Then the is condition when the plane (1) touches the given cone.

Remarks:-The condition of tangency is similar to the condition that a plane cuts a cone in two coincident generators (see Corollary 1, section 6.6), which ensure the tangency of the plane.

6.10 Reciprocal Cone:- To find the locus of the lines drawn through the vertex of a cone perpendicular to the tangent planes.

Let the equation of a cone be $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ (1)

Any tangent plane to this cone is $ux+vy+wz=0$ (2)

Provided $Au^2+bv^2+cw^2+2Fvw+2Gwu+2Huv=0$ (3)

Where A,B,C,F,G,H are the same as mentioned in the preceding section.

The normal to the plane (2) passing through the vertex $(0,0,0)$ of the cone (1) is

$$\frac{x}{u}=\frac{y}{v}=\frac{z}{w} \tag{4}$$

Eliminating u,v,w from (3) and (4), we get

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \tag{5}$$

This is the equation of the required locus. Evidently it represents a cone, and is called the reciprocal cone of the cone (1), we thus define:

Reciprocal Cone:-The locus of the lines drawn through the vertex of a cone perpendicular to the tangent plane is a cone and is called the reciprocal cone of the give cone.

Further, to find the reciprocal cone of (5)

Any tangent plane to the cone (5) is $ux+vy+wz=0$ (6)

Provided $A'u^2+B'v^2+C'w^2+2F'vw+2G'wu+2H'uv=0$ (7)

Where $A'=BC-F^2aD$, $B'=CA-G^2=bB$, $C'=AB-H^2=cD$

$F'=GH-AF=fD$, $G'=HF-BG=gD$, $H'=FG-CH-hD$, with

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Thus the reciprocal cone of (5) is $aDx^2+bDy^2+cDz^2+2fDyz+2gDzx+2hDxy=0$

i.e. $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

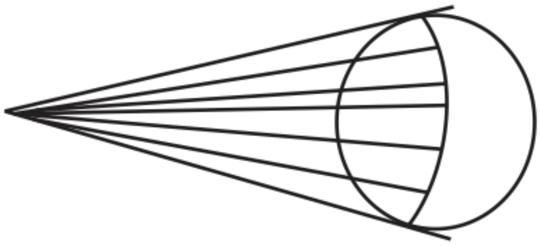
This is, in fact the origin cone (1).

Hence the reciprocal of the reciprocal of a cone is the cone itself. Thus in the language of Algebra, the relation of being reciprocal cone of a cone is reflexive.

Corollary:-The condition for the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ to have three mutually perpendicular tangent planes, is that $bc+ca+ab=f^2+g^2+h^2$.

Proof:- The given plane will have three mutually perpendicular tangent plane if its reciprocal cone has tree mutually perpendicular generator. The condition for the later situation is $A+B+C=0$ i.e. $(bc-f^2)+(ca-g^2)+0$ i.e. $ba+ca+ab=f^2+g^2+h^2$.

6.11 Enveloping Cone:-We know from analytical geometry of two dimensions that two tangents can be drawn from a point to a cone (circle, parabola, ellipse or hyperbola). In analogy with that, an infinite number of tangent lines can be drawn from a point to a coincide in space. It will observed that all such tangent lines generate a cone with a given point as the vertex (see the adjacent figure also). Such a cone is called an enveloping cone (or tangent cone) since it envelops all the tangent lines drawn from a point. We now define:



Enveloping Cone:-The locus of the tangent lies drawn from a given point to a given surface is called the enveloping cone of that surface with the given point as vertex.

To find the enveloping cone of the sphere x $x^2+y^2+z^2=u^2$ with vertex at the point (α,β,γ)

Equation of any line through the point (α,β,γ) are $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$, say (1)

Any point on this line is $(\alpha+lr,\beta+mr,\gamma+nr)$. This point will lie on the given sphere if $(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2=a^2$ i.e.

$$\left(l^2+m^2+n^2\right)r^2+2(\alpha l+\beta m+\gamma n)r+\left(\alpha^2+\beta^2+\gamma^2-a^2\right)=0 \qquad (2)$$

This is a quadratic equation in r, giving tow values of r corresponding to which three are two points common to the sphere and the line (1). Now the line (1) will be a tangent to the given sphere if both the roots of (2) are equal, which requires the discriminant of (2) to be zero i.e. $4(\alpha l+\beta m+\gamma n)^2-4\left(l^2+m^2+n^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right)=0$.

$$(\alpha l+\beta m+\gamma n)^2=\left(l^2+m^2+n^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right) \qquad (3)$$

Therefore the locus of the tangent lies i.e. the equation of the enveloping cone, is

$$\{\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)\}^2$$
$$=\left\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right\}\left(\alpha^2+\beta^2+\gamma^2-a^2\right) \qquad (4)$$

The equation is conveniently simplified by using the following notation. Let $S=x^2+y^2+z^2-a^2$, $S_1=\alpha^2+\beta^2+\gamma^2-a^2$, $T=\alpha x+\beta y+\gamma z-a^2$

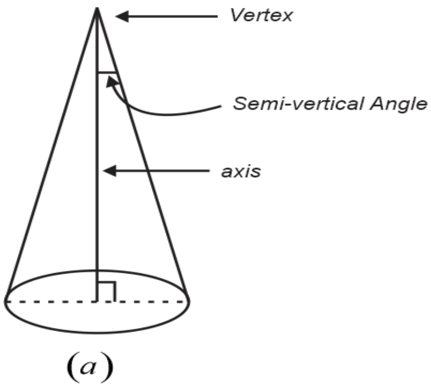
Then (4) can be written as $(T-S_1)^2=(S-2T+S_1)S_1$, which immediately reduces to $SS_1=T^2$ i.e. $\left(x^2+y^2+z^2-a^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right)=\left(\alpha x+\beta y+\gamma z-a^2\right)^2$.

This can be regarded as the simplest form of the equation of enveloping cone.

6.12 Right Circular Cone:-The surface generated by a line which passes through a fixed point and make a constant angle with a fixed line through the fixed point is called a right circular cone.

The fixed point is called the vertex, the fixed line the axis and the constant angle the semi-vertical angle of the cone.

It can be easily seen that the section of a right circular cone by a plane perpendicular to its axis is a circle (see figure (a) below)



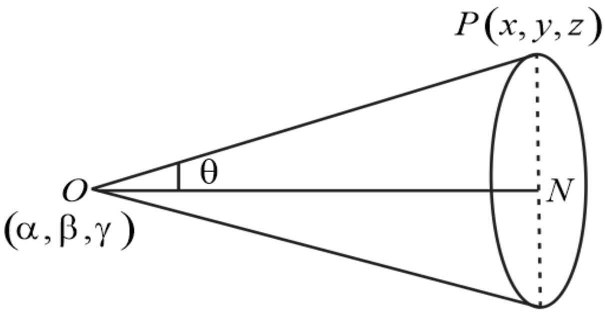
Further, a cone with base curve as a circle is not necessarily a right circular cone (see figure (b) above)

Equation of a Right Circular Cone:-To find the equation of a right circular cone whose vertex is $O(\alpha,\beta,\gamma)$, the axis a line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$ and the semi-vertical angle θ.

Let $P(x,y,z)$be any point on the cone. Then the direction cosines of OP are proportional to

$$x-\alpha, y-\beta, z-\gamma.$$



Then the angle θ between the axis ON and generator OP is given by

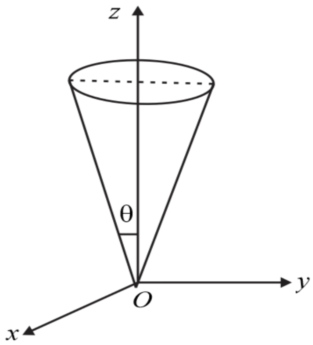
$$\cos\theta=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Taking square on both the sides, and clearing the denominators, we get

$$\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}^2=(l^2+m^2+n^2)\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}\cos\theta$$

This is the required equation of the cone.

Corollary 1:- The equation of a right circular cone with vertex at the origin the origin is



$\left(lx+my+nz\right)^2=\left(l^2+m^2+n^2\right)\left(x^2+y^2+z^2\right)\cos^2\theta$.

If the axis of the cone is the z-axis we have $l=0=m$ so that the above equation reduces to $z^2=\left(x^2+y^2+z^2\right)\cos^2\theta$ i.e. $x^2+y^2=z^2\tan^2\theta$.

Thus we obtain the following result.

Corollary 2:- The equation of a right circular cone with vertex at the origin and axis as the z-axis is $x^2+y^2=z^2\tan^2\theta$.

6.13 Equation of A Cylinder Through A Given Conic:-To find the equation of the cylinder whose axis and guiding curve are given.

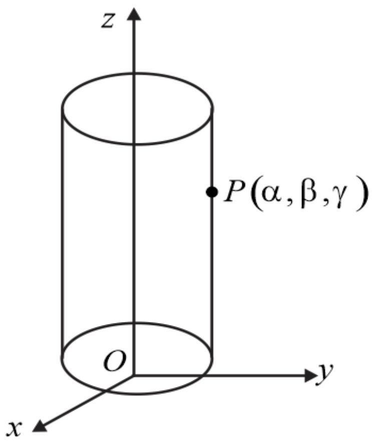
Let the guiding be the conic $ax^2+2hxy+b^2+2gx+2fy+c=0,\ z=0$ (1)

and the axis be the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \tag{2}$$

If $P\left(\alpha,\beta,\gamma\right)$ is any point on the cylinder, the equations of a generator will be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \tag{3}$$



This line meets the plane $z=0$(as given in (1)) at the point given by

$$\frac{x-\alpha}{m}=\frac{y-\beta}{m}=\frac{-\gamma}{m} \quad \text{i.e. the point } \left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right).$$

Therefore the generator (3) will intersect the conic (1) if this point lies on (1) i.e. its co-ordinates satisfy the first equation in (5). Thus we have

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+\left(\beta-\frac{m\gamma}{n}\right)^2$$

$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)=0. \text{ i.e.}$$

$$a(n\alpha-l\gamma)^2+2h(n\alpha-l\gamma)(n\beta+m\gamma)+b(n\beta+m\gamma)^2$$
$$+2gn(n\alpha-l\gamma)+2fn(n\beta-m\gamma)+cn^2=0$$

Hence the locus of the point $P(\alpha,\beta,\gamma)$ is

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2gn(ny-mz)+cn^2=0 \qquad (4)$$

This is the required equation of the cylinder.

Note:- We can write the above equation in a convenient from by collecting the coefficients of z^2, nz and n^2. For this, if the equation of the given conic are written as $\phi(x,y)=0$ and $\psi(x,y)=ax^2+2hxy=by^2$, the equation (4) assumes the form $z^2\psi(l,m)-nz\left(l\frac{\partial\phi}{\partial x}-m\frac{\partial\phi}{\partial y}\right)+n^2\phi(x,y)=0$.

Corollary:-The equation of the cylinder which passes through the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, $z=0$ and whose generators are parallel the z-axis is $ax^2+2hxy+by^2+2gx+2fy+c=0$.
(5)

This result follows by putting $l=0=m$ and $n=1$ (**the** $d.c's$ of the z-axis) in equation (4) obtained above.

It is worth noting that equation (5) in two dimensions represents a curve, i.e. the given conic on the xy-plane, while in three dimension it represents a cylinder whose generators are parallel to the z-axis. Alternatively, equation (5) can be obtained by putting $z=0$ in (4).

6.14 Enveloping Cylinder:-The locus of the lines drawn in a given direction so as to touch a given surface is called enveloping cylinder to the surface.

To find the enveloping cylinder of the sphere $x^2+y^2+z^2=a^2$ (1)

Whose generators are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad (2)$$

Let $P(\alpha,\beta,\gamma)$ be any point on the enveloping cylinder. Then the line through P and parallel to (2) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ say
(3)

Any point on this line $(\alpha+lr,\beta+mr,\gamma-nr)$. This point will lie on the given sphere (1) if its co-ordinates satisfy (1) i.e. $(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2=a^2$ i.e.

$$r^2\left(l^2+m^2+n^2\right)+2r\left(l\alpha+m\beta+n\gamma\right)+\alpha^2+\beta^2+\gamma^2-a^2=0$$

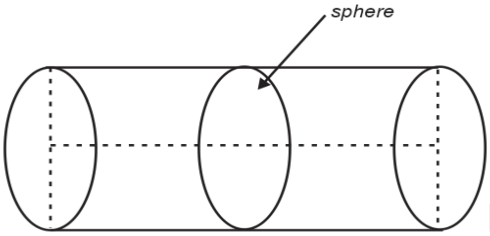
Now the generator (3) will touch the sphere (1) if the roots of this quadratic equation in r are equal i.e., its discriminate is zero. Thus we have

$$\left(l\alpha+m\beta+n\gamma\right)^2=\left(l^2+m^2+n^2\right)\left(\alpha^2+\beta^2+\gamma^2-a^2\right).$$

Hence the locus of the point $P(\alpha,\beta,\gamma)$ is

$$\left(lx+my+nz\right)^2=\left(l^2+m^2+n^2\right)\left(x^2+y^2+z^2-a^2\right)$$

This is the required equation of the enveloping cylinder

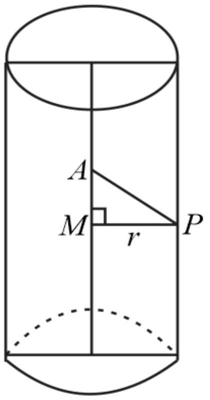


6.15 Equation of A Right Circular Cylinder:- We recall that the surface generated by a line which intersects a fixed circle and is perpendicular to plane of the circle is called the right circular cylinder.

Evidently, the normal to the plane of the circle through its centre is the axis of the cylinder and the radius of the circle the radius of the cylinder.

To find the equation of the right circular cylinder whose radius is r and axis the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$.

Let $P(x',y',z')$ be any point on the cylinder $A(\alpha,\beta,\gamma)$ be the fixed on the axis and r then radius. Let PM be the perpendicular from P on the axis. Then from the right-angled triangle AMP, we have $AP^2-AM^2=PM^2$.



i.e. $\left\{\left(x'-\alpha\right)^2+\left(y'-\beta\right)^2+\left(z'-\gamma\right)^2\right\}-\left\{\dfrac{l\left(x'-\alpha\right)+m\left(y'-\beta\right)+n\left(z'-\gamma\right)}{\sqrt{\left(l^2+m^2+n^2\right)}}\right\}^2=r^2$

i.e. $\left(l^2+m^2+n^2\right)\left\{\left(x'-\alpha\right)^2+\left(y'-\beta\right)^2+\left(z'-\gamma\right)^2\right\}$

$$-\left\{l\left(x'-\alpha\right)+m\left(y'-\beta\right)+n\left(z'-\gamma\right)^2\right\}=r^2\left(l^2+m^2+n^2\right)/$$

Hence the locus $P(x',y',z')$ is $(l^2+m^2+n^2)\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}$

$$-\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}^2=r^2(l^2+m^2+n^2).$$

This is the required equation of the right circular cylinder.

Corollary 1:- The equation of the right circular cylinder whose axis is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and the radius r, is $(l^2+m^2+n^2)(x^2+y^2+z^2-r^2)=(lx+my+nz)^2$.

Corollary 2:- The equation of the right circular cylinder whose axis is the z-axis and the radius r, is $x^2+y^2=r^2$

Example1:- Prove that the general equation of the cone of second degree passing through the co-ordinate axes is $fyz+gzx+hxy=0$

Solution:- The cone passing through the axes will have the vertex at the origin. Let its general equation be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad (1)$$

But it passes through the axes. Therefore, the direction cosines of the axes viz.

$1,0,0$; $0,1,0$; $0,0,1$

Must satisfy (1). Thus we get

$a=0$ $b=0$ $c=0$

Hence the equation (1) of the cone reduces to

$$fyz-gzx+hxy=0$$

Example2:- Prove that a cone of second degree can be found to pass through two sets of rectangular axes through the same origin.

Solution:- The general equation of a cone of the second degree having its vertex at the origin is

$$ax^2+by^2+cz^2+2gzx+2fyz+2hxy=0 \qquad (1)$$

Let the co-ordinate axes be one set of rectangular axes through which the cone passes. Then the equation of the cone passing through the co-ordinate axes is (give details as in Example 1 above)

$$fyz+gzx+hxy=0 \qquad (2)$$

Since the direction cosines $1,0,0;0,1,0;0,0,1$ of the axes satisfy (1).

Let the second set of rectangular axes through the same origin be OP,OQ,QR having direction cosines $l_1,m_1,n_1;\ l_2,m_2,n_2;\ l_3,m_3,n_3;$respectively.

Let the cone (2) pass through OP and OQ. Then

$$fm_1n_1+gm_1l_1+hl_1m_1=0 \qquad (3)$$

And $$fm_2n_2+gn_2l_2+hl_2m_2=0 \qquad (4)$$

On adding (3) and (4), we have

$$f(m_1n_1+m_2n_2)+g(n_1l_1+n_2l_2)+h(l_1m_1+l_2m_2)=0 \qquad (5)$$

But we know from the properties of rectangular axes (see section 2.11) that

$m_1n_1+m_2n_2+m_3n_3=0$ etc. i.e. $m_1n_1+m_2n_2=-m_3n_3$ etc.

Using these relations, (5) reduces to $-fm_3n_3 - gn_3l_3 - hl_3m_3 = 0$, i.e.
$fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$
This shows that the cone (2) also passes through OR. Hence the result.

Example3:- Show that the equation of the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + 2z$ is

$$p\left(ax^2 + by^2\right) = 2z\left(lx + my + nz\right)$$

Solution:- The equation of the cone will be obtained by making the equation

$$ax^2 + by^2 = 2z \qquad (1)$$

Of the given plane.

For this (2) can be put as $\dfrac{lx + my + nz}{p} = 1$

Using it in (1), we obtain $\quad ax^2 + by^2 + 2z\left(\dfrac{lx + my + nz}{p}\right)$

$$p\left(ax^2 + by^2\right) = 2z\left(lx + my + nz\right)$$

This is the required equation of the cone under question.

Example4:- The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the co-ordinate axes in A, B, C. Prove that the equation to the cone generated by the lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

Solution:- The circle ABC is determined by the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (1)$$

And a sphere passing through the points A, B, C.

Evidently, the co-ordinates of A, B, C are $(a,0,0),(0,b,0)(0,0,c)$ respectively. Since one more point is required to find the equation of the sphere, we taken the origin as the fourth convenient point. Then the equation of the sphere O, ABC is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \qquad (2)$$

Now the circle ABC is given by the equations (2) in (1).

To obtain the equation of the cone whose vertex is the origin and which passes through the circle ABC, it suffices to make the equation (2) homogeneous with the help of (1). For, such an equation will represent a surface through the intersection of the surfaces given by (1) and (2). Moreover, being a homogeneous equation it will also represent a cone.

Hence the equation of the required cone is

$$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0,$$

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

Example5:- Show that $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$

Solution:- Making the given equation homogenous by introducing a variable t, we have

$$F(x,y,z,t)=ax^2+by^2+cz^2+2uxt, 2vyt, 2wzt+dt^2=0$$

Now $\frac{\partial F}{\partial x}=0$

i.e. $2ax+2ut+0$ for $t=1$gives $x=-\frac{u}{a}$ (1)

Similarly, $\frac{\partial F}{\partial y}=0$ for $t=1$ gives $y=-\frac{v}{b}$ (2)

$\frac{\partial F}{\partial z}=0$ for $t=1$ gives $z=-\frac{w}{c}$ (3)

And $\frac{\partial F}{\partial t}=0$ for $t=1$ gives $ux+vy+wz+d=0$ (4)

Substituting the values of x,y,z from (1), (2), (3) in (4), we have

$$u\left(-\frac{u}{a}\right)+v\left(-\frac{v}{b}\right)+w\left(-\frac{w}{c}\right)+d=0$$

i.e. $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$

Example6:- Show that the equation $2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$ represents a cone vertex is $(2,2,1)$

Solution:- Making the given equation homogeneous by introducing a new variable t, we get

$$F(x,y,z,t)=2x^2+2y^2+7z^2-10yz+10zx+2xt+2yt+26zt-17t^2=0$$

Now equating to zero, the first order partial derivatives of F, we obtain

$$\frac{\partial F}{\partial x}=4x-10z+2t=0, \qquad \frac{\partial F}{\partial y}=4y-10z+2t=0$$

$$\frac{\partial F}{\partial z}=14z-10y-10x+26t=0 \qquad \frac{\partial F}{\partial t}=2x+2y+26z-34t=0$$

Taking $t=1$ and simplifying these equations, we have

$2x \quad -5z+1=0$ (1)

$2x-5z+1=0$ (2)

$5x+5y-7z-13=0$ (3)

$x+y+13z-17=0$ (4)

Solving the equations (1), (2) and (3), we get

$x=2,\ \ y=2,\ \ z=1$

These values of x,y and z also satisfy equation (4). Therefore, the equation (1) to (4) are consistent.

Hence the given equation represents a cone whose vertex is at the point $(2,2,1)$

Example7:- The section of a cone whose guiding curve is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ by the plane $x=0$, is a rectangular hyperbola. Show that the locus of the vertex is the surface.

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

Solution:- Let the vertex of the cone be (α,β,γ). Then the equations of a generator of the cone can be written as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad (1)$$

This line meets the plane $z=0$ in the point $\left(\alpha-\gamma,\frac{1}{n},\beta-\gamma\frac{m}{n},0\right)$

This point lies on the given ellipse if

$$\frac{1}{a^2}\left(\alpha-\gamma.\frac{1}{n}\right)^2+\frac{1}{b^2}\left(\beta-\gamma.\frac{m}{n}\right)^2=1 \qquad (2)$$

Eliminating l,m,n from (1) and (2) by putting the values of l/n and m/n from (1) into (2), we get

$$\frac{1}{a^2}\left(\alpha-\gamma.\frac{x-\alpha}{z-\gamma}\right)^2+\frac{1}{b^2}\left(\beta-\gamma.\frac{y-\beta}{z-\gamma}\right)^2=1$$

i.e. $\frac{1}{a^2}\{\alpha(z-\gamma)-\gamma(x-\alpha)\}^2+\frac{1}{b^2}\{\beta(z-\gamma)-\gamma(y-\beta)\}^2=(z-\gamma)^2$

i.e. $\frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$

This is the equation of the cone with vertex at (α,β,γ). The section of this cone by the plane $x=0$ is given by $\frac{1}{a^2}\alpha^2z^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$

i.e. $\left(\frac{\gamma^2}{b^2}\right)^2 y^2+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)z^2-\left(\frac{2\beta\gamma}{b^2}\right)yz+2\gamma z-\gamma^2=0$

This section represents a rectangular hyperbola if coefficient of y^2+ coefficient of $z^2=0$

i.e. $\frac{\gamma^2}{b^2}+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)=0$, i.e. $\frac{\alpha^2}{a^2}+\frac{\beta^2+\gamma^2}{b^2}=1$

Hence the locus of the vertex (α,β,γ) is the surface $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$

Example8:- Find the equations of the lines of intersection of the plane $6x-10y-7z=0$ and the cone $108x^2-20y^2-7z^2=0$. Find also the angle between them.

Solution:- Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the lines of intersection of the given plane and cone. Since it lies on both the plane and the cone. We have

$$6l-10m-7n=0 \quad (1)$$

And $$108l^2-20m^2-7n^2=0 \quad (2)$$

Putting the value of n from (1) in (2), we get

$$108l^2-20m^2-7\left(\frac{6l-10m}{7}\right)^2=0$$

i.e. $$756l^2-140m^2-\left(36l^2+100m^2-120lm\right)=0$$

i.e. $720l^2-120lm-240m^2=0$ i.e. $6l^2+lm-2m^2=0$

i.e. $$(3l+2m)(2l-m)=0$$

This gives $3l+2m=0$ or $2l-m=0$, i.e.

$$\frac{l}{2}=\frac{m}{-3} \text{ or } \frac{l}{1}=\frac{m}{2}$$

(i) When $\frac{l}{2}=\frac{m}{-3}=\lambda$ say. We have $l=2\lambda,\ m=-3\lambda$. Substituting these values in (1), we get $12\lambda+30\lambda-7n=0$, which gives $n=6\lambda$

Therefore, $$\frac{l}{2}=\frac{m}{-3}=\frac{n}{6} \quad (3)$$

(ii) When $\frac{l}{2}=\frac{m}{2}=\mu$ say. We have $l=\mu, m=2\mu$. Substituting these values in (1), we get $6\mu-20\mu-7n=0$, which gives $n=-2\mu$

Therefore, $$\frac{l}{1}=\frac{m}{2}=\frac{n}{-2} \quad (4)$$

From (3) and (4) the equations of the lines of intersection of the given plane and the cone, are $\frac{x}{2}=\frac{y}{-3}=\frac{n}{6}$ and $\frac{x}{1}=\frac{y}{2}=\frac{n}{-2}$

Further, if θ be the angle between these lines, then

$$\cos\theta=\frac{2\times1+(-3)\times2+6\times(-2)}{\sqrt{\left\{2^2+(-3)^2+6^2\right\}}\sqrt{\left\{1^2+2^2+(-2)^2\right\}}}=\frac{-16}{\sqrt{(49)}\sqrt{(9)}}=-\frac{16}{21}$$

Hence $\theta=\cos^{-1}(16/21)$

Example9:- Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in two perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Solution:- Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the lines of intersection of the given plane and cone. Since it lies on both the plane and the cone, we have

$$al+bm+cn=0 \quad (1)$$

And $$mn+nl+lm=0 \quad (2)$$

Putting the value of n from (1) in (2), we obtain

$$m\left(-\frac{al+bm}{c}\right)+\left(-\frac{al+bm}{c}\right)l+lm=0$$

i.e. $al^2+(a+b-c)lm+bm^2=0$, $a\left(\frac{l}{m}\right)^2+(a+b-c)\frac{l}{m}+b=0$

If l_1/m_1 and l_2/m_2 be the roots of this equation, then the product of roots.

$$\frac{l_1}{m_1}.\frac{l_2}{m_2}=\frac{b}{a}, \text{ i.e. } \frac{l_1l_2}{1/a}=\frac{m_1m_2}{1/b} \qquad (3)$$

By symmetry we obtain

$$\frac{m_1}{n_1}.\frac{m_2}{n_2}=\frac{c}{b}, \text{ i.e. } \frac{m_1m_2}{1/b}=\frac{n_1n_2}{1/c} \qquad (4)$$

From (3) and (4) we have

$$\frac{l_1l_2}{1/a}=\frac{m_1m_2}{1/b}=\frac{n_1n_2}{1/c}$$

Thus the lines with direction ratios l_1,m_1,n_1 and l_2,m_2,n_2 will be perpendicular if

$$l_1l_2+m_1m_2+n_1n_2=0$$

i.e. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Example10:- Show that the equation $\sqrt{fx}+\sqrt{gy}+\sqrt{gz}=0$ represents a cone which touches the co-ordinates planes, and the equation to its reciprocal cone $fyz+gzx+hxy=0$

Solution:- The given equation can be written as $\sqrt{fx}+\sqrt{gy}=-\sqrt{hz}$

Squaring this, we get

$$fx+gy+2\sqrt{fy}+\sqrt{gy}=hz$$

i.e. $fx+gy-hz=-2\sqrt{fx\,gy}$

Again squaring, this gives

$$f^2x^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0 \qquad (1)$$

This is a homogeneous equation of the second degree. Hence it represent a cone whose vertex is at the origin.

This cone meets the plane $z=0$, where

$$f^2x^2+g^2y^2-2fgxy=0, \text{ i.e. } (fx-gy)^2=0.$$

Which gives $fx-gy=0$

This shows that the cone (1) intersects the plane $z=0$ at two coincident straight lines.

Hence the cone touches the plane $z=0$.

Similarly, we can show that the cone touches the plane $x=0$ and $y=0$ also.

The reciprocal cone of cone (1) is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \qquad (2)$$

Where A, B, C, F, G, H are the cofactors of $f^2, g^2, h^2, gh, hf, fg$ respectively in the determinant $\begin{vmatrix} f^2 & -fg & -hf \\ -fg & g^2 & -gh \\ -hf & -gh & h^2 \end{vmatrix}$

Thus $A = g^2h^2 - (-gh)^2 = 0$ similarly, $B = 0 = C$

$$F = (-fg)(-hf) - f^2(-gh) = 2f^2gh$$

Similarly, $G = 2fg^2h$ and $H = 2fgh^2$

Hence from (2) the equation to the reciprocal cone is

$2(2f^2gh)yz + 2(2fg^2h)zx + 2(2fgh^2)xy = 0$

i.e. $\quad fyz + gzx + hxy = 0$

Example11:- Show that $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal to each other

Solution:- Let the reciprocal cone of $ax^2 + by^2 + cz^2 = 0$ be

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \qquad (1)$$

Where $A, B, C....$ are cofactors of the corresponding letters $a, b, c, ...$ in the determinant

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

Thus $A = bc, B = ca, C = ab$ and $F = 0 = G = H$

Using these values in (1) the reciprocal cone is $bcx^2 + cay^2 + abz^2 = 0$, i.e. $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$.

Evidently, $ax^2 + by^2 + cz^2 = 0$ will be the reciprocal cone of $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

Hence the given cones are reciprocal to each other.

Example12:- Find the equation of the enveloping cone of the sphere given below, whose vertex is at the origin $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

OR

Prove that the lines drawn from the origin so as to touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ lie on the cone

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Solution:- Equation of the enveloping cone is given by $SS_1 = T^2$

Here $\quad S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$S_1 = d$

$T = ux + vy + wz + d$ and the vertex is the point $(0, 0, 0)$

Hence the required enveloping cone is

$$\left(x^2+y^2+z^2+2ux+2vy+2wz+d\right)d=\left(ux+vy+wz+d\right)^2$$

i.e. $$\left(x^2+y^2+z^2\right)d+\left(2ux+2vy+2wz\right)d+d^2$$
$$=\left(ux+vy+wz\right)^2+2\left(ux+vy+wz\right)d+d^2$$

i.e. $\left(x^2+y^2+z^2\right)d=\left(ux+vy+wz\right)^2$, on simplifying

Example13:- Find the equation to the cone formed by rotating the line $2x+3y+6, z=0$ about the y-axis.

Solution:- Equations of the given line can by written in the following symmetrical form

$$\frac{x}{3}=\frac{y-2}{-2}=\frac{z}{0} \qquad (1)$$

Where $(0,2,0)$ is its point of intersection with the y-axis, and hence the vertex of the cone.

Since the $d.c.'s$ of the y-axis are $0,1,0$, the angle θ made by the line (1) with the y-axis is given by

$$\cos\theta=\frac{3.0+(-2).1+0.0}{\sqrt{\left\{3^2+(-2)^2+0^2\right\}}}=-\frac{2}{\sqrt{(13)}} \qquad (2)$$

Further, if the equations of the generator in any other position be

$$\frac{x}{l}=\frac{y-2}{m}=\frac{z}{n} \qquad (3)$$

The angle which it makes with the y-axis is given by

$$\cos\theta=\frac{l.0+m.1+n.0}{\sqrt{\left(l^2+m^2+n^2\right)}}=\frac{m}{\sqrt{\left(l^2+m^2+n^2\right)}} \qquad (4)$$

Equating the two values of $\cos\theta$ obtained in (2) and (4), we obtain

$$-\frac{2}{\sqrt{(13)}}=\frac{m}{\sqrt{\left(l^2+m^2+n^2\right)}}\text{, i.e. } 4\left(l^2+m^2+n^2\right)=13m^2$$

i.e. $4l^2-9m^2+4n^2=0 \qquad (5)$

now eliminating l,m,n between (3) and (5), we have

$$4x^2-9\left(y-2\right)^2+4z^2=0,$$

i.e. $4x^2-9y^2+4z^2+36y-36=0$

This is the required equation of the cone.

Example14:- If a right circular cone has three mutually perpendicular generators, prove that the semi-vertical angle is $\tan^{-1}\sqrt{2}$.

Solution:- Without loss of generality, we may take the origin as the vertex of the cone and the z-axis as its axis. Then we know the equation of the cone is

$$x^2+y^2-z^2\tan^2\theta=0 \qquad (1)$$

Where θ is the semi-vertical angle of the cone

Also we know that a cone has three mutually perpendicular generators if

Coefficient of $x^2 +$ coefficient of $y^2 +$ coefficient of $z^2 = 0$

Hence from (1), we have

$1+1-\tan^2\theta = 0$ i.e. $\tan^2\theta = 2$, i.e. $\theta = \tan^{-1}\sqrt{2}$

Example15:- Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$

Solution:- Let (α, β, γ) be any point on the cylinder. Then the equations to the generator of the cylinder through this point are $\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3}$

This generator meets the plane $z = 3$ at the point

$$\left(\alpha + 1 - \frac{1}{3}\gamma,\ \beta - 2 + \frac{2}{3}\gamma,\ 3\right)$$

If this point lies on the given ellipse, we have

$$\left(\alpha + 1 - \frac{1}{3}\gamma\right)^2 + 2\left(\beta - 2 + \frac{2}{3}\gamma\right)^2 = 1$$

i.e. $\{3(\alpha+1) - \gamma\}^2 + 2\{3(\beta-2) + 2\gamma\}^2 = 9$

i.e. $9\alpha^2 + 18\beta^2 + 9\gamma^2 + 24\beta\gamma - 6\alpha\gamma + 18\alpha - 72\beta + 54\gamma + 72 = 0$

i.e. $3(\alpha^2 + 2\beta^2 + \gamma^2) + 8\beta\gamma - 2\gamma\alpha + 6\alpha - 24\beta - 18\gamma + 24 = 0$

Hence the locus of (α, β, γ) is

$$3(x^2 + 2y^2 + z^2) + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$$

This is the required equation to the cylinder.

Example16:- Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and pass through the curve $x^2 + y^2 = 16,\ z = 0$

Solution:- Let (α, β, γ) be any point on the cylinder. Then the equations of the generator passing through this point are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{2} = \frac{z-\gamma}{3} = r \text{ (say)} \qquad (1)$$

This generator meets the plane $z = 0$ in the point given by

$$\left(\alpha - \frac{1}{3}\gamma, \beta - \frac{2}{3}\gamma, 0\right)$$

So, the generator (1) passes through the curve

$$x^2 + y^2 = 16,\ z = 0$$

If $\left(\alpha - \frac{1}{3}\gamma\right)^2 + \left(\beta - \frac{2}{3}\gamma\right)^2 = 16$

i.e. $9\alpha^2 + 9\beta^2 + 5\gamma^2 - 6\alpha\gamma - 12\beta\gamma = 144$

Hence the locus of (α, β, γ) is

$$9x^2 + 9y^2 + 5z^2 - 6xz + 12yz = 144$$

This is the equation of the required cylinder.

Example17:- Find the equation to the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generator parallel to the line $x = y = z$

Solution:- Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder. Then the equations to the generator through this point are

$$x - \alpha = y - \beta - z - \gamma = r \text{, say} \qquad (1)$$

Any point on this generator is $(\alpha + r, \beta + r, \gamma + r)$. If this point lies on the given sphere, its co-ordinates must satisfy the equation of the sphere. Thus we have

$$(\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 - 2(\alpha + r) + 4(\beta - r) = 1$$

i.e. $$3r^2 + 2r(\alpha + \beta + \gamma + 1) + (\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1) = 0 \qquad (2)$$

The generator (1) will touch the given sphere if the roots of the quadratic equation (2) are equal. For this, the discriminant of (2) should be zero, i.e.

$$(\alpha + \beta + \gamma + 1)^2 - 3(\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1) = 0$$

i.e. $$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta - 4\alpha + 5\beta - \gamma - 2 = 0$$

Hence the locus of the point $P(\alpha, \beta, \gamma)$ is

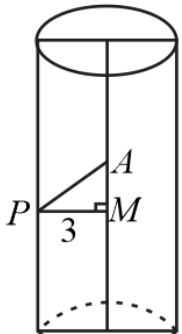
$$x^2 + y^2 + z^2 - yz - zx - xy - 4x + 5y - z - 2 = 0$$

This is the required equation to the enveloping cylinder.

Example18:- Find the equation of the right circular cylinder whose axis is the z-axis and the radius r, is $\dfrac{x-1}{2} = \dfrac{y-3}{2} = \dfrac{5-z}{7}$

Solution:- The given equations of the axis of cylinder can be written as

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-7} \qquad (1)$$



Let A denote the point $(1,3,5)$ on this axis. Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder and PM the perpendicular from P on the axis of cylinder. As given, $PM = 3$ (radius)

Now in the right-angled triangle PMA, we have $AP^2 - AM^2 = PM^2$

Since AM = projection of AP on the axis of cylinder, this gives

$$(\alpha-1)^2+(\beta-3^2)+(\gamma-5)^2=\left\{\frac{2(\alpha-1)+2(\beta-3)-7(\gamma-5)}{\sqrt{\left\{2^2+2^2+(-7)^2\right\}}}\right\}^2=3^2,$$

i.e. $$57\left\{(\alpha-1)^2+(\beta-3)^2+(\gamma-5)^2\right\}-\left\{2(\alpha-1)+2(\beta-3)-7(\gamma-5)\right\}^2=9\times 57$$

i.e. $$57\left\{(\alpha-1)^2+(\beta-3)^2+(\gamma-5)^2\right\}-\Big[4(\alpha-1)^2+4(\beta-3)^2-49(\gamma-5)^2$$
$$+2\left\{4(\alpha-1)(\beta-3)-14(\beta-3)(\gamma-5)-14(\gamma-5)(\alpha-1)\right\}\Big]=513$$

i.e. $$53(\alpha^2-1)^2+53(\beta-3)^2+8(\gamma-5)^2-8(\alpha-1)(\beta-3)+28(\beta-3)(\gamma-5)$$
$$+28(\gamma-5)(\alpha-1)=513$$

i.e. $$53(\alpha^2-2\alpha+1)+53(\beta^2-6\beta+9)+8(\gamma^2-10\gamma+25)-8(\alpha\beta-3\alpha-\beta+3)$$
$$+28(\beta\gamma-5\beta-3\gamma+15)+28(\gamma\alpha-\gamma-5\alpha+5)=513$$

i.e. $$53\alpha^2+53\beta^2+8\gamma^2+28\beta\gamma+28\gamma\alpha-8\alpha\beta-222\alpha-450\beta-192\gamma+753=0$$

This is the equation satisfied by any point (α,β,γ) on the cylinder.

Hence the equation of cylinder is

$$53x^2+53y^2+8z^2+28yz+28zx-8xy-222x-450y-192z+753=0$$

Example19:- Find the equation to the right circular cylinder whose guiding curve is the circle $x^2+y^2+z^2=9$, $x-y+z=3$

OR

Find the equation of the right circular cylinder whose guiding circle is $x^2+y^2+z^2=9$, $x-y+z=3$ and the axis is the line $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$

Solution: - Since the axis of the cylinder is perpendicular to the plane $x-y+z=3$,

Its direction ratios are $1,-1,1$. Let (α,β,γ) be any point on the cylinder. Then the equation of the generator through this point are $\frac{x-\alpha}{1}=\frac{y-\beta}{-1}=\frac{z-\gamma}{1}=r$ (say)

Any point on this generator is $(\alpha+r,\beta-r,\gamma+r)$. If this point lies on the guiding curve, then we must have

$$\begin{cases}(\alpha+r)^2+(\beta-r)^2+(\gamma+r)^2=9\\(\alpha+r)-(\beta-r)+(\gamma+r)=3\end{cases}$$

i.e. $$\begin{cases}\alpha^2+\beta^2+\gamma^2+2r(\alpha-\beta+\gamma)+3r^2=9 & (1)\\ \alpha-\beta+\gamma+3r=3 & (2)\end{cases}$$

Putting the value of r from (2) in (1), we get

$$\alpha^2+\beta^2+\gamma^2-\frac{2}{3}(\alpha-\beta+\gamma-3)(\alpha-\beta+\gamma)+\frac{1}{3}(\alpha-\beta+\gamma-3)^2=9$$

i.e. $\alpha^2+\beta^2+\gamma^2+\beta\gamma-\gamma\alpha+\alpha\beta-9=0$, on simplifying.

Hence the locus of (α,β,γ) is

$$x^2+y^2+z^2+yz-zx-xy-9=0$$

This is the required equation of right circular cylinder.

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the in the last of this book.

CHAPTER 6. CONE & CYLINDER

Q4(c) If the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2=0$ in perpendicular generators, then prove that $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$. **UPSC CSE 2022**

2.(a) Show that the planes, which cut the cone $ax^2+by^2+cz^2=0$ in perpendicular generators, touch the cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$. **UPSC CSE 2021**

(c) Find the equation of the cone whose vertex is $(1,2,1)$ and which passes through the circle $x^2+y^2+z^2=5, x+y-z=1$. **IFoS 2021**

Q1. A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes at the points A, B and C. Prove that the circle ABC lies on the cone

$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$. **[(4c) 2019 IFoS]**

Q2. Find the equation of the cone with $(0,0,1)$ as the vertex and $2x^2-y^2=4,\ z=0$ as the guiding curve. **[(4c) UPSC CSE 2018]**

Q3. Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios $(1,-2,2)$. **[(4c) 2017 IFoS]**

Q4. A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C.

[(4d) 2016 IFoS]

Q5. Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$, represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$. **[(4a(ii) UPSC CSE 2014]**

Q6. Prove that the equation $4x^2-y^2+z^2-3yz+2xy+12x-11y+6z+4=0$ represents a cone with vertex at $(-1,-2,-3)$. **[(3d) 2014 IFoS]**

Q7. A cone has for its guiding curve the circle $x^2+y^2+2ax+2by=0, z=0$ and passes through a fixed point $(0,0,c)$. If the section of the cone by the plane $y=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle $x^2+y^2+z^2+2ax+2by=0$, $2ax+2by+cz=0$.

[(4b) UPSC CSE 2013]

Q8. A variable plane is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$. **[(4b) UPSC CSE 2012]**

Q9. Prove that the second degree equation

$x^2-2y^2+3z^2+5yz-6zx-4xy+8x-19y-2z-20=0$ represents a cone whose vertex is $(1,-2,3)$. **[(4a) 2010 IFoS]**

Intersection of a plane and cone

Q2C(ii)The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C respectively. Prove that the equation of the cone generated by the lines drawn from the origin O to meet the circle ABC is $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0$ **UPSC CSE 2023** **(10)**

Q1.Find the equations of the straight lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. Find the angle between the two straight lines. **[(4a) 2018 IFoS]**

Q2. Examine whether the plane $x+y+z=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines.

[(1e) UPSC CSE 2014]

Q3. Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. **[(4b) 2014 IFoS]**

Q4. Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. **[(1e) 2012 IFoS]**

Q5. Show that the cone $yz+zx+xy=0$ cuts the sphere $x^2+y^2+z^2=a^2$ in two equal circles, and find their area. **[(4b) UPSC CSE 2011]**

Three mutually perpendicular generators

Q1. If the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, then find the equations of the other two generators.

[(3c) UPSC CSE 2020]

Q2. Show that the cone $3yz-2zx-2xy=0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ is a generator belonging to one such set, find the other two.

[(4b) UPSC CSE 2016]

Q3. If $6x=3y=2z$ represents one of the three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$ then obtain the equations of the other two generators.

[(2d) UPSC CSE 2015]

Q4. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two. **[(4c) 2010 IFoS]**

Enveloping cone

Q1. Prove that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has the vertex at $(2,4,1)$ in a rectangular hyperbola. **[(2c) (ii) UPSC CSE 2019]**

CYLINDER

Q3(c) Find the equation of the cylinder whose generators intersect the curve, $2x^2+3y^2=4z, x-y+2z=3$ and are parallel to the line $3x=-2y=4z$. **IFoS 2022**

Q1(e) Find the equation of the cylinder whose generators are parallel to the line $x=-\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is $x^2+2y^2=1, z=0$. **UPSC CSE 2021**

Q1. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is $x^2+y^2=4, z=2$. **[(2c) UPSC CSE 2020]**

Q2. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is $x^2+y^2=4, z=2$. **[(2a) 2018 IFoS]**

Q3. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$. **[(4a) 2011 IFoS]**

7. CENTRAL CONICOIDS

7.1 Central Conicoids:-The following surfaces are called central conicoids, since their sections by the plane parallel to the co-ordinate planes are conics:

i. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$ (Ellipsoid)

ii. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;$ (Hyperboloid of One Sheet)

iii. $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$ (Hyperboloid of Two Sheet)

iv. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z;$ (Elliptic Paraboloid)

v. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z;$ (Hyperboloid Paraboloid)

In this chapter we shall deal with the first three of the above. The remaining two shall be considered in the next chapter.

7.2 Ellipsoid:-Consider the equation of the ellipsoid:-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

The following observations are immediately drawn from it.

(i) It is symmetrical with respect to all the three co-ordinate axes. In fact if (x,y,z) lies on the surface, then so do the point $(-x,y,z),(x,-y,z)$ and $(x,y,-z)$.

(ii) The surface is bounded and lies inside the box whose boundaries are the planes $x=\pm a$, $y=\pm b$ and $z=\pm c$. In fact, for any point (x,y,z) on the surface, we have $|x|\le a$, $|y|\le b$ and $|z|\le c$. Moreover, the surface touches the faces of the box at the points $(\pm a,0,0),(0,\pm b,0)$ and $(0,0,\pm c)$.

(iii) The plane section of the surface in the xy-plane is the ellipse, given by

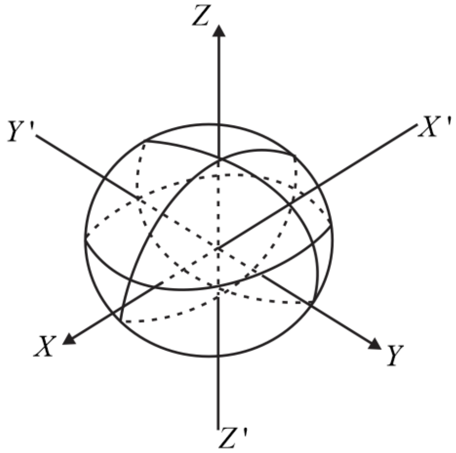
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad z=0.$$

Similarly, the plane section of the surface in yz- and zx- planes are the ellipses.

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,,x=0 \text{ and } \frac{x^2}{a^2}+\frac{z^2}{c^2}=1\,,y=0$$

(iv) The plane sections by the plane $z=k$ (k being an arbitrary but fixed constant) are the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{c^2-k^2}{k^2}$, $z=k$, provided $|k|<c$. Also the size of the ellipses decreases as $|k|$ increases. If $|k|=c$, the corresponding planes $z=\pm c$ are tangent to the surface, whereas if $|k|>c$, there are no real points of intersection.

Similar conclusions can be drawn for the sections by the planes $x=k$ and $y=k$.
We are now in a position to draw the sketch of the ellipsoid as shown below:



The following concepts regarding the ellipsoid are analogous to those for an ellipse in two dimensional geometry.

Vertices:-The six points $(\pm a,0,0),(0,\pm b,0)$ and $(0,0,\pm c)$ in which the co-ordinates axes meet the ellipsoid, are called the vertices of the ellipsoid.

Axes:-The three line segments joining the pairs of vertices on each co-ordinate axes called the axes of the ellipsoid.

Centre:-The three axes of the ellipsoid intersect at the point $(0,0,0)$ called the centre of the ellipsoid.

Semi-axes:- The line segments joining the centre to the vertices are of lengths a,b and c. These lengths are called the semi-axes of the ellipsoid.

Major Mean and Minor Axes:-The largest number of $2a,2b$ and $2c$ is called the major axis, the next to the largest the mean axis and the smallest the minor axis of the ellipsoid.

Note:- If $a=b=c,$ the ellipsoid reduces to a sphere.

7.3 Hyperbolaoid of One Sheet:-Consider the equation of the huperboloid of one sheet:-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

We observe the following:-

(i) It is symmetrical with respect to all the three co-ordinate axes.

(ii) The surface is unbounded since each of the co-ordinates x,y and z can take any values provided only that $\frac{x^2}{a^2}+\frac{y^2}{b^2}\geq 1.$

(iii) The points of intersections. Of the surfaces with the x- and y- axes are $(\pm a, 0, 0)$ and $(0, \pm b, 0)$, respectively, whereas with the z-axis imaginary.

(iv) The plane sections by the plane $z = k$ are the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^2 + k^2}{c^2},\ z = k.$$

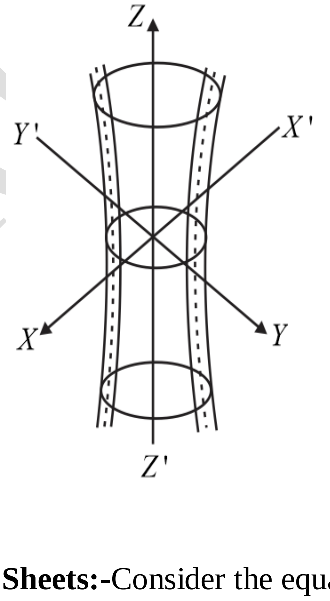
These ellipse increase in size as $|k|$ increase, so much so that they become of infinite size when $|k| \to \infty$. However, the smallest ellipse is that which corresponds to $k = 0$. Thus it follows that the surface is generated by a variable ellipse lying in the plane parallel to the xy-plane which increase in size as it move farther away on either side of the xy-plane.

(v) Unlike in the case of ellipsoid, the sections by the planes parallel to the other co-ordinate planes are not ellipse. For instance, the sections by the planes $x = k$ are the hyperbolas.

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{a^2 - k^2}{a^2},\ x = k \quad \text{(provided } |k| < a\text{)}$$ or the conjugate hyperbolas.

$$-\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2 - a^2}{a^2},\ x = k \quad \text{(provided } |k| > a\text{)}$$

The surface, therefore, may be regarded as generated by a variable hyperbola lying in the plane parallel to the yz-plane (or zx-plane).
Consequently, it is called a hyperboloid of one sheet.
We are now in a position to draw the sketch of the hyperboloid of one sheet as shown below:-



7.4 Hyperboloid of Two Sheets:-Consider the equation of the hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We observe the following:

(i) The surface is symmetrical with respect to all the three co-ordinate axes.

(ii) The surface is unbounded since each of the co-ordinates x, y and z can take any values, provided only that $|z| \geq c$.

(iii) The intersection of the surface with the x- and y-axes are imaginary, whereas with the z-axis is real, meeting at the point $(0, 0 \pm c)$.

(iv) Neither the xy-plane nor any plane parallel to it $(z = k)$ meets the surface, provided $|k| < c$. However, if $|k| > c$, its section by the plane $z = k$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2 - c^2}{c^2}$, $z = k$.

These ellipse increase in size as $|k|$ increases and become of infinite size when $|k| \to \infty$. Thus it follows that the surface is generated in two parts by a variable ellipse lying in the plane parallel to the xy-plane starting from the planes $z = \pm c$ and extending to infinity on both sides.

(v) The plane sections by the planes $x = k$, parallel to the yz-plane, are the hyperbolas

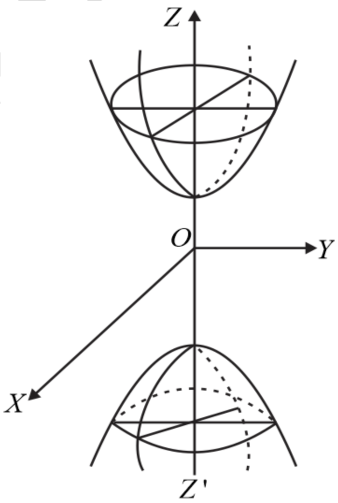
$$-\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 + k^2}{a^2}, \quad x = k$$

Similarly those by the plane $y = k$, parallel to the zx-plane, are the hyperbolas.

$$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{b^2 + k^2}{b^2}, \quad y = k.$$

Both these families of hyperbolas extend to infinity as $|k|$ varies from 0 to ∞. The surface of two sheets, therefore, may be regarded as generated by a variable hyperbolas lying in the planes parallel to the yz-plane (or zx-plane). Consequently, the surface is called the hyperbolas of two sheets.

We are now in a position to draw the sketch of the surface as shown below:



7.5 **Standard Equation of The Conicoid:-**The equation $ax^2 + by^2 + cz^2 = 1$ is called the standard equation of the conicoid. It represents (i) an ellipsoid when a, b, c are all positive, (ii) a hyperboloid of one sheet when two of a, b, c are positive and one negative, (iii) a hyperboloid of two sheets when two of a, b, c are negative and one positive, and (iv) an imaginary ellipsoid when all of a, b, c are negative.

It is woth noting that in each of these cases, the origin bisects the chords passing through it. Hence the surface represented is central. The origin is the centre of the surface.

The surface is also symmetrical about the co-ordinate plane. These co-ordinate planes are called the **principal planes,** and the co-ordinate axes as the **principal axes,** of the conicoid.

7.6 Tangent Plane:-To find the equation of tangent plane at the point (α, β, γ) the conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad (1)$$

Since (α, β, γ) lies on the conicoid (1), these co-ordinates must satisfy (1), i.e.

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \quad (2)$$

Now any straight line through the point (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$. Say (3)

Any point on this line is $(lr+\alpha, mr+\beta, nr+\gamma)$. This point lies on the conicoid (1) if

$a(lr+\alpha)^2 + b(mr+\beta)^2 + c(nr+\gamma)^2 = 1$ i.e.

$r^2(al^2+bm^2+cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2) = 1.$ i.e.

$r^2(al^2+bm^2+cn^2) + 2r(al\alpha + bm\beta + cn\gamma) = 0$ (using (2))

The line (3) will touch the conicoid (1) if both the values of r are zero. i.e.

$$al\alpha + bm\beta + cn\gamma = 0 \quad (4)$$

The locus of such lines will be obtained by eliminating l, m, n between (3) and (4). Thus we have $a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) = 0$

i.e. $a\alpha x + b\beta y + c\gamma z - (a\alpha^2 + b\beta^2 + c\gamma^2) = 0$ i.e. $a\alpha x + b\beta y + c\gamma z = 1$

- (using (2),

This is the required equation of the tangent plane at the point (α, β, γ).

7.7 Condition of Tangency:-To find the condition that the plane

$$lx + my + nz = p \quad (1)$$

May touch the conicoid $ax^2 + by^2 + cz^2 = 1$ (2)

Let the plane (1) touch the conicoid (2) at the point (α, β, γ). Then the equations

$a\alpha x + b\beta y + c\gamma z = 1$will represent the same plane as (1). This gives

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p} \text{ i.e. } \alpha = \frac{1}{ap},\ \beta = \frac{m}{bp},\ \gamma = \frac{n}{cp} \quad (3)$$

Since (α, β, γ) lies on the coincoid (2), we have $a\alpha^2 + b\beta^2 + c\gamma^2 = 1$

Using (3), it gives $a\left(\frac{l^2}{a^2p^2}\right) + b\left(\frac{m^2}{b^2p^2}\right) + c\left(\frac{n^2}{c^2p^2}\right) = 1$ i.e. $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

This is the required condition.

Corollary 1:- The plane $lx+my+nz=p$ will touch the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ if $a^2l^2+b^2m^2+c^2n^2=p^2$.

Corollary 2:- The plane $lx+my+nz=\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}$ always touches the conicoid $ax^2+by^2+cz^2=1$.

7.8 Director Sphere:-The locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid is a sphere, which is called the director sphere.

Let $l_1x+m_1y+n_1z=\sqrt{\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)}$ (1)

$l_2x+m_2y+n_2z=\sqrt{\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)}$ (2)

And $l_3x+m_3y+n_3z=\sqrt{\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)}$ (3)

Be three mutually perpendicular tangent planes to the conicoid

$$ax^2+by^2+cz^2=1$$

Then we have $\left.\begin{aligned}&\sum l_1m_1=\sum m_1n_1=\sum n_1l_1=0\\ and\ &\sum l_1^2=\sum m_1^2=\sum n_1^2=1\end{aligned}\right\}$ (4)

The co-ordinates of the point of intersection of these three planes will satisfy these three equations, and its locus can be obtained by eliminating $l_1,m_1,n_1\,;l_2,m_2,n_2\,;\,l_3,m_3,n_3$. For this squaring and adding the equation (1), (2), (3) and using (4), we obtain after a moderate calculation that $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

This is the required equations of the director sphere.

7.9 Normal:-To find the equation of the normal to the conicoid $ax^2+by^2+cz^2=1$ at any point (α,β,γ) on it.

The equation to the tangent plane at (α,β,γ) to the given conicoid is

$$a\alpha x+b\beta y+c\gamma z=1 \quad (1)$$

Evidently, the direction cosines of the normal to this plane are proportional to $a\alpha,b\beta,c\gamma$.

Hence the equation of the normal to the given conicoid at the point (α,β,γ) on it are $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$ (2)

Equation of the normal in terms of the actual direction cosines:-If p is the length of the perpendicular from the origin to the tangent plane (1), then we have $p = \dfrac{1}{\sqrt{\left\{(a\alpha)^2 + (b\beta)^2 + (c\gamma)^2\right\}}}$.

This gives $(a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1$. It follows that $a\alpha p, b\beta p, c\gamma p$ are the actual direction cosines of the normal. Hence from (2) the equations of the normal at (α, β, γ) in terms of actual $d.c's$ are $\dfrac{x-\alpha}{a\alpha p} = \dfrac{y-\beta}{b\beta p} = \dfrac{z-\gamma}{c\gamma p}$.

Normal of normals from any point to a conicoid:-If the normal at the point (α, β, γ) passes through the point (f, g, h), then from (2) we have $\dfrac{f-\alpha}{a\alpha} = \dfrac{g-\beta}{b\beta} = \dfrac{h-\gamma}{c\gamma} = r$. Say

So, $\alpha = \dfrac{f}{1+ar}$, $\quad \beta = \dfrac{g}{1+br}$, $\quad \gamma = \dfrac{h}{1+cr}$. (3)

Since the point (α, β, γ) lies on the conicoid $ax^2 + by^2 + cz^2 = 1$, these values of α, β, γ must satisfy the equation of the conicoid. Thus $\dfrac{af^2}{(1+ar)^2} + \dfrac{bg^2}{(1+br)^2} + \dfrac{ch^2}{(1+cr)^2} = 1$.

Upon clearing the denominators, this becomes a sixth degree equation in r, giving there by six values of r

It follows from (3) that six normal can be drawn from any point to a conicoid.

7.10 Cubic Curve Through The Feet of The Normals:-To prove that the feet of the six normal from any given point to a conicoid are the points of intersection of the conicoidanda certain cubic curve.

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$ (1)

The feet of the six normal drawn through the point (f, g, h) to this conicoid lie on it.

Also, we know (from equation (3) of section 7.10) that the feet of the normals lie on the curve whose parametric equation are $x = \dfrac{f}{1+ar}$, $y = \dfrac{g}{1+br}$, $z = \dfrac{h}{1+cr}$ (2)

Where r is a parameter.

Thus the feet of six normals drawn through a point (f, g, h) to the conicoid (1) are the points of intersection of the conicoid (1) and the curve (2).

We finally show that the curve (2) is a cubic in r. For this, we note that the curve (2) meets the plane.

$$Ax + By + Cz + D = 0$$

Where $$A\left(\frac{f}{1+ar}\right) + B\left(\frac{g}{1+br}\right) + C\left(\frac{h}{1+cr}\right) + D = 0$$

This equation is a cubic in r, and therefore given three values of r.

Hence the curve (2), is cubic curve through the feet of the normals.

7.11 Cone Through The Six Concurrent Normals:- To prove that the six normals drawn from any given point to a conicoid generate a cone of the second degree.

Let $ax^2+by^2+cz^2=1$ be the equation of the conicoid and (f,g,h) be the given point. The equations of the cubic curve through the feet of the normals are given by

$$x=\frac{f}{1+ar},\ y=\frac{g}{1+br},z=\frac{h}{1+cr} \quad (1)$$

Any line through the point (f,g,h) is $\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$ (2)

This line will intersect the curve (1) if the equations in (1) satisfy (2), i.e.

$$\frac{\frac{f}{1+ar}-f}{l}=\frac{\frac{g}{1+br}-g}{m}=\frac{\frac{h}{1+cr}-h}{n},\ \text{i.e.}\ \frac{af}{l(1+ar)}=\frac{bg}{m(1+br)}=\frac{ch}{n(1+cr)}.$$

To eliminate r from these equations, let each one of these fractions be equal to k, so that $\frac{af}{l}=k(1+ar)$, $\frac{bg}{m}=k(1+br)$, $\frac{ch}{n}=k(1+cr)$

So, $\frac{af}{1}(b-c)+\frac{bg}{m}(c-a)+\frac{ch}{n}(a-b)$ **(note)**

$$=k\{(b-c)(1+ar)+(c-a)(1+br)+(a-b)(1+cr)\}=0$$

i.e. $\frac{af(b-c)}{l}+\frac{bg(c-a)}{m}+\frac{ch(a-b)}{n}=0$ (3)

The locus of the line (2) is obtained by eliminating l,m,n from (2) and (3). Thus we have $\frac{af(b-c)}{x+f}+\frac{bg(c-a)}{y-g}+\frac{ch(a-b)}{z-h}=0$ (4)

This equation represents a cone of the second degree generated by the lines (2) and with the guiding curve (1). These include, in particular, all the six normals drawn from (f,g,h) to the conicoid.

Hence (4) is the required cone.

7.12 Plane of Contact Polar Plane And Pole:-

Plane of Contact:-To prove that the points of contact of the tangent planes drawn through a point to a conicoid lie on a plane.

Let (α,β,γ) be a given point and $ax^2+by^2+cz^2=1$ (1)

A given conicoid.

Let (x',y',z') be the point of contact of the tangent plane drawn through the point (α,β,γ) to the coincoid (1)

Then the equation of the tangent plane at (x',y',z') is $axx'+byy'+czz'=1$.

But this plane passes through the point (α,β,γ). Therefore $a\alpha x'+b\beta y'+c\gamma z'=1$.

From this it is clearly that the point (x',y',z') satisfies the equation $a\alpha x+b\beta y+c\gamma z=1$, which represent a plane. Hence the result.
This plane is known as the plane of a contact.

Polar Plane of a Point :- To find the equation of the polar plane of the point (α,β,γ) with respect in the conicoid $ax^2+by^2+cz^2=1$ (1)

Let A be the point (α,β,γ).Then the equation of the line through $A(\alpha,\beta,\gamma)$ meeting the given conicoid in P and Q are presented by $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ say (2)

Any point on (2) is $(\alpha+lr,\beta+mr,\gamma+nr)$. If this point lies on the conicoid (1),we must have $a(l+\alpha r)^2+b(m+\beta r)^2+c(n+\gamma r)^2=1$. i.e.

$$\left(al^2+bm^2+cn^2\right)+2(a\alpha l+b\beta m+c\gamma n)r+a\alpha^2+b\beta^2+c\gamma^2-1=0 \quad (3)$$

It is a quadratic equation in r, giving two values, say r_1 and r_2. Let $AP=r_1$ and $AQ=r_2$. If $R(\xi,\eta,\zeta)$be the point on APQ such that $AR=\rho$, then the co-ordinates of R are $(\alpha+l\rho,\beta+m\rho,\gamma+n\rho)$ and $\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{\rho}$ i.e. $\rho=\frac{2r_1r_2}{r_1+r_2}$ (4)

Now from (3) we have $r_1+r_2=-\frac{2(a\alpha l+b\beta m+c\gamma n)}{al^2+bm^2+cn^2}$ and $r_1r_2=\frac{a\alpha^2+b\beta^2+c\gamma^2-1}{al^2+bm^2+cn^2}$.

Using these relations in (4), we obtain $\rho=-\frac{a\alpha^2+b\beta^2+c\gamma^2-1}{a\alpha l+b\beta m+c\gamma n}$, i.e.

$$a\alpha l\rho+b\beta m\rho+c\gamma n\rho+a\alpha^2+b\beta^2+c\gamma^2-1=0 \quad (5)$$

Also, using (2) we have $\xi-\alpha=l\rho$, $\eta-\beta=m\rho$, $\zeta-\gamma=n\rho$.
Using these relation in (5) we get $a\alpha(\xi-\alpha)+b\beta(\eta-\beta)+c\gamma(\zeta-\gamma)+a\alpha^2+b\beta^2+c\gamma^2-1=0$ i.e. $a\alpha\xi+b\beta\eta+c\gamma\zeta=1$.

Hence the locus of $R(\xi,\eta,\zeta)$ is $a\alpha x+b\beta y+c\gamma z=1$ which is the required equation of the polar plane.

Conjugate Points:- We can easily prove the following.
If the polar plane of a point P passes thorough the point Q, then the polar plane of Q passes through P.
Two point such that each lies on the polar plane of the other are called conjugate points.

Pole of a Plane:- To find the pole of the plane $lx+my+nz=p$ (1)
with respect to the conicoid $ax^2+by^2+cz^2=1$ (2)

Let (x_1, y_1, z_1) be the pole of the plane (1) with respect to the conicoid (2). Then this plane must be identical with the polar plane.

$$ax_1x + by_1y + cz_1z = 1 \tag{3}$$

Of (x_1, y_1, z_1) with respect to the coincoid (2).

Comparing (1) and (3) we have $\frac{l}{ax_1} = \frac{m}{by_1} = \frac{n}{cz_1} = \frac{p}{1}$, which gives $x_1 = \frac{l}{ap}, y_1 = \frac{m}{bp}, z_1 = \frac{n}{cp}$.

Hence the required pole is $(l/ap, m/bp, n/cp)$.

Conjugate Planes:- We can easily prove the following:
If the pole of a plane P_1 lies on another plane P_2 then the pole of P_2 lies of P_1.
Two planes such that each contains the pole of the other, are called conjugate planes.

7.13 Polar Lines:- Consider the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{, say} \tag{1}$$

Any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$. The polar plane of this point with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is $a(\alpha + lr)x + b(\beta + mr)y + c(\gamma + nr)z = 1$, i.e. $a\alpha x + b\beta y + c\gamma z - 1 + r(alx + bmy + cnz) = 0$.

Evidently, this plane passes through the line of intersection of the planes

$$\text{and} \quad \left.\begin{aligned} a\alpha x + b\beta y + c\gamma z &= 0 \\ alx + bmy + cnz &= 0 \end{aligned}\right\} \tag{2}$$

Hence the polar plane of any point on the line (1) passes through the line (2). Similarly, we can show that the polar plane of any point on the line (2) passes through the line (1). Two such lines are called the polar lines with respect to the conicoid. We define:

Polar Lines:- Two lines such that the polar plane of each point one line passes through the other line are called polar lines.

7.14 Enveloping Cone of A Conicoid:- The locus of the tangent lines drawn from a given point to a conicoid is a cone and is called the enveloping cone or tangent cone. The given point is called the vertex of the cone.

To find the equation of the enveloping cone of the conicoid $ax^2 + by^2 + cz^2 = 1$

(1) whose vertex is the point (α, β, γ).

The equation of any line through (α, β, γ) are $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$, say

(2)

Any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point be on the conicoid (1), we must have $a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^{2=1}$, i.e.

$$(al^2 + bm^2 + cn^2)r^2 + 2(a\alpha l + b\beta m + c\gamma n)r + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0 \quad (3)$$

But this is a quadratic equation in r, giving two values of r corresponding to which there are two points common to the conicoid and the line (2).
If the line (2) be a tangent to the conicoid (1), both the values of r given by (3) must be identical This is possible if the discriminant of (3) is zero, i.e.

$$4(a\alpha l + b\beta m + c\gamma n)^2 - 4(al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \text{ i.e}$$

$$(a\alpha l + b\beta m + c\gamma n)^2 = (al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) \quad (4)$$

The locus of the tangent lines is tangents by eliminating l, m, n from (2) and (4). Thus the required locus is given by $la\alpha(x - \alpha) + b\beta(y - \beta) + c\gamma(z - \gamma)^2$

$$= \left\{a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2\right\}(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) \quad (5)$$

If we use the notations $S \equiv ax^2 + by^2 + cz^2 - 1$, $S_1 \equiv a\alpha^2 + b\beta^2 + c\gamma^2 - 1$ and $T \equiv a\alpha x + b\beta y + c\gamma z - 1$, then, after a moderate manipulation, equation (5) assume the form $(T - S_1)^2 = (S - 2T + S_1)S_1$ i.e. $T^2 = SS_1$, i.e.

$$(a\alpha x + b\beta y + c\gamma z - 1)^2 = (ax^2 + by^2 + cz^2 - 1)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

This is more useful form of the equation of the enveloping cone.

7.15 Enveloping Cylinder of A Conicoid:- The locus of the tangent lines drawn to a conicoid and parallel to a given line is a cylinder and is called the enveloping cylinder of the conicoid.

To find the equation of the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ (1) whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Let (x', y', z') be any point on the enveloping cylinder. Then the equations of the generator through this point are given by $\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n}$.

This line will touch conicoid (1) if

$(alx' + bmy' + cnz')^2 = (al^2 + bm^2 + cn^2)(ax'^2 + by'^2 + cz'^2 - 1)$ as shown the preceding section.

Hence the locus of the point (x', y', z') is

$$(alx + bmy + cnz)^2 = (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1)$$

This is the required equation of the enveloping cylinder.

Note:- The plane of contact of the enveloping cylinder and the conicoid is $alx + bmy + cnz = 0$.

7.16 Section With A Given Centre:-
(Locus of the chords bisected at a Given Point)

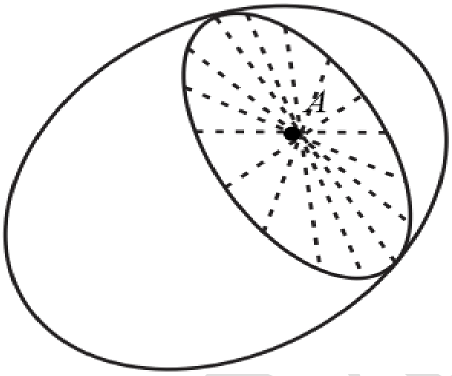
To find the locus of the chords of the conicoid $ax^2+by^2+cz^2=1$ which are bisected at the given point (α,β,γ).

The equations of any chord through the point (α,β,γ) are $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$, say (2)

Any point on this chord is $(\alpha+lr,\beta+mr,\gamma+nr)$. This point lie on the conicoid (1) if

$$\left(al^2+bm^2+cn^2\right)+2\left(a\alpha l+b\beta m+c\gamma n\right)r+a\alpha^2+b\beta^2+c\gamma^2-1=0 \quad (3)$$

as shown in section 7.14.



Since the point (α,β,γ) is the middle point of the chord (2), the roots of the equation (3) must be equal in magnitude but opposite in signs. Thus the sum of roots must be zero. This requires the coefficient of r to be zero.

i.e. $a\alpha l+b\beta m+c\gamma n=0$ (4)

the required locus is obtained by eliminating l,m,n between (2) and (4). Hence we have $a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)=0$ i.e. $a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2$ (5) which represents a plane.

This is the equation of the locus of the chords of the conicoid which are bisected at the point (α,β,γ).

Note:- The equation (5) may also be written as $T=S_1$, where T and S_1 have the same meaning as in section 7.14.

7.17 Diametral Plane:-

Diameter:- Any chord of a central conicoid through its centre is called a diameter.

Diametral Plane:- It is the locus of the middle points of a system of parallel chords of a conicoid.

Let $(\alpha,\beta\gamma)$ be the middle point of the chord $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ of the conicoid $ax^2+by^2+cz^2=1$.

Then by section 7.16, we have $al\alpha+bm\beta+cn\gamma=0$.

Therefore, the locus of the middle point of all such chords is the plane $alx+bmy+cnz=0$.

This plane passes through the centre of the conicoid, and is called the diametral plane conjugate to the given line or diameter.

Example1:- Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through the point (α,β,γ). Prove that perpendicular to them from the origin generator the cone.

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=(\alpha x+\beta y+\gamma z)^2$$

Solution:- Any plane though $(\alpha+\beta+\gamma)$ is

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$

Or $$lx+my+nz=l\alpha+m\beta+n\gamma \qquad (1)$$

If it touches $ax^2+by^2+cz^2=1$ we have

$$\frac{t^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=(l\alpha+m\beta+n\gamma)^2 \qquad (2)$$

The equations to the normal to the plane(1) passing though the origin

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad (3)$$

The surface generated by these lines can be obtained by eliminating l,m,n between (1) and (3) thus we obtain the cone.

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=(\alpha x+\beta y+\gamma z)^2$$

Example2:- The tangent plane to the conicoid $x^2+12y^2+4z^2=8$ at the point $\left(1,\frac{1}{2},1\right)$ meets the co-ordinates axes at the points A,B,C. Find the centroid and the area of the triangle ABC.

Solution:- The equation of the tangent plane at the point $\left(1,\frac{1}{2},1\right)$ of the given conicoid is $x.1+12y.\frac{1}{2}+4z.1=8$ i.e. $x+6y+4z=8$m, $\frac{x}{8}+\frac{y}{4/3}+\frac{z}{2}=1$

This plane meets the co-ordinates axes at the point $A(8,0,0),B\left(0,\frac{4}{3},0\right)$ and $C(0,0,2)$. Therefore, the centroid of the triangle ABC is

$$\left(\frac{8+0+0}{3},\frac{0+\frac{4}{3}+0}{3},\frac{0+0+2}{3}\right) \text{ Or } \left(\frac{8}{3},\frac{4}{9},\frac{2}{3}\right)$$

The areas of the triangles OAB, OBC and OAC are $\frac{1}{2},8,\frac{4}{3}$, $\frac{1}{2},\frac{4}{3},2$, $\frac{1}{2}2,8$ or $\frac{16}{3},\frac{4}{3},8$ respectively.

Therefore, area of the triangle ABC

$$=\sqrt{\left\{\left(\frac{16}{3}\right)^2+\left(\frac{4}{3}\right)^2+8^2\right\}}=\frac{4}{2}\sqrt{53}$$

Example3:- Obtain the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to the plane $lx+my+nz=0$. If $2r$ is the distance between two parallel tangent plane to the ellipsoid, prove that the line through the origin and perpendicular to the plane lies on the cone $x^2\left(a^2-r^2\right)+y^2\left(b^2-r^2\right)+z^2\left(c^2-r^2\right)=0$

Solution:- The tangent plane parallel to the planes $lx+my+nz=0$ are $lx+my+nz=+\sqrt{\left(a^2l^2+b^2m^2+c^2n^2\right)}$

The distance between these parallel plane is (which is twice the distance of either from the origin)

$$2r=r\sqrt{\left(\frac{a^2l^2+b^2m^2+c^2n^2}{l^2+m^2+n^2}\right)}, \text{ or } r^2=\frac{a^2l^2+b^2m^2+c^2n^2}{l^2+m^2+n^2}$$

Or $\quad \left(a^2-r^2\right)l^2+\left(b^2-r^2\right)+m^2\left(c^2-r^2\right)n^2=0 \qquad (1)$

Also, the equation of the line through the origin and perpendicular to the plane

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad (2)$$

Eliminating l,m,n between (1) and (2), the locus of the line is the cone

$$x^2\left(a^2-r^2\right)+y^2\left(b^2-r^2\right)+z^2\left(c^2-r^2\right)=0$$

Example4:- The normal at any point P of a central conicoid meets the three principal at G_1,G_2,G_3; show that PG_1,PG_2,PG_3 are in constant ratio.

Solution:- The equations of the normal at (α,β,γ) to the central conicoid $ax^2+by^2+cz^2=1$ are $\frac{x-\alpha}{a\alpha p}=\frac{y-\beta}{b\beta p}=\frac{z-\gamma}{c\gamma p}$.

Now since $a\alpha p$, $b\beta p$, $c\gamma p$ are the actual direction cosines. Each of these fractions represents the distance between the points (α,β,γ) and (x,y,z).

Thus the distance PG_1 of the point $P(\alpha,\beta,\gamma)$ from the point G_1 the normal meets the co-ordinates plane $x=0$, is equal to $-1/ap$

Similarly, $PG_2=-\frac{1}{bp}$, $PG_3=-\frac{1}{cp}$

Hence $PG_1 : PG_2 : PG_3 :: a^{-1} : b^{-1} : c^{-1}$

Consequently, PG_1,PG_2,PG_3, are in constant ratio.

Example5:- Prove that the line drawn the origin parallel to the normal to $ax^2+by^2+cz^2=1$ at the points of intersection with the plane $lx+my+nz=p$, generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

Solution:- The equation to the normal at the point (α,β,γ) of the conicoid are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma} \qquad (1)$$

Any straight line through the origin and parallel to the normal (1)

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma} \qquad (2)$$

Now the point (α,β,γ) lies one the conicoid, and on the given plane. Therefore, we have $a\alpha^2+b\beta^2+c\gamma^2=1$ and $l\alpha+m\beta+n\gamma=p$

These equations may be combined as

$$a\alpha^2+b\beta^2+c\gamma^2=\left(\frac{l\alpha+m\beta+n\gamma}{p}\right)^2 \qquad (3)$$

Now eliminating α,β,γ between (2) and (3), we get

$$a\left(\frac{x}{a}\right)^2+b\left(\frac{y}{b}\right)^2+c\left(\frac{z}{c}\right)^2=\left(\frac{lx/a+my/b+nz/c}{p}\right)^2$$

i.e. $$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

Which is the required equation.

Example6:- Prove that the feet of the six normal drawn to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ from any point (α,β,γ) lie on the curve of intersection of the ellipsoid and the cone $\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$

Solution:- The co-ordinates of any foot of the normal from (α,β,γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are $x=\frac{a^2\alpha}{a^2+\lambda}$, $y=\frac{b^2\beta}{b^2+\lambda}$, $z=\frac{c^2\gamma}{c^2+\lambda}$

Or $\lambda=a^2\left(\frac{\alpha}{x}\right)-a^2$, $\lambda=b^2\left(\frac{\beta}{y}\right)-b^2$, $\lambda=c^2\left(\frac{\gamma}{z}\right)-c^2$

Multiplying them by $(b^2-c^2),(c^2-a^2),(a^2-b^2)$ respectively and them adding the resulting equation we have $0=a^2\alpha\frac{(b^2-c^2)}{x}+b^2\beta\frac{(c^2-a^2)}{y}+c^2\gamma\frac{(a^2-b^2)}{z}$

Hence the feet lie on the curve of intersection of the ellipsoid and cone

$$\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$$

Example7:- Prove that the locus of the poles of the tangent plane of the conicoid $ax^2+by^2+cz^2=1$ with respect to the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$ is the conicoid $\frac{\alpha^2x^2}{a}+\frac{\beta^2y^2}{b}+\frac{\gamma^2z^2}{c}=1$

Solution:- Let (x',y',z') be the pole. Then the equation of the polar plane with respect to the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$ is $\alpha x'x+\beta y'y+\gamma z'z=1$

This plane with touch the conicoid $ax^2+by^2+cz^2=1$ if $\frac{(\alpha x')^2}{a}+\frac{(\beta y')^2}{b}+\frac{(\gamma z')^2}{c}=1$

Hence the locus of the pole $(x',y'z')$ is the conicoid $\frac{\alpha^2x^2}{a}+\frac{\beta^2y^2}{b}+\frac{\gamma^2z^2}{c}=1$

Example8:- Find the conditions that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$ are polar lines with respect to the conicoid $ax^2+by^2+cz^2=1$

Solution:- The polar line of the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is $a\alpha x+b\beta y+c\gamma z-1=0$, $alx,bmy+cnz=0$

This line will be identical with the line

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

If
$$\left.\begin{aligned} a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1&=0\\ a\alpha l'+b\beta m'+c\gamma n' &=0\\ al\alpha'+bm\beta'+cn\gamma' &=0\\ all'+bmm'+cnn' &=0\end{aligned}\right\}$$

These four equation are the required conditions.

Example9:- Prove that the line joining a point P to the centre of a conicoid passes through the centre of the section of the conicoid by the polar plane of P.

Solution:- Let the equation of the conicoid be

$$ax^2+by^2+cz^2=1 \qquad (1)$$

And the co-ordinates of P be (α,β,γ)

Then the equations of the line joining the point P to the centre of (1) are

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma} \qquad (2)$$

Also, the polar plane of $P(\alpha,\beta,\gamma)$ with respect to the conicoid (1) is

$$a\alpha x+b\beta y+c\gamma z=1 \qquad (3)$$

Suppose (f,g,h) is the centre of the section of the conicoid (1) by the plane (3). Then the equation of the plane section is $af(x-f)+bg(y-g)+ch(z-h)=0$ i.e. $afx+bgy+chz=af^2+bg^2+ch^2$

Comparing this equation with (3), we get $\frac{af}{a\alpha}=\frac{bg}{b\beta}=\frac{ch}{c\gamma}=\frac{af^2+bg^2+ch^2}{1}$

Which gives $\frac{f}{\alpha}=\frac{g}{\beta}=\frac{h}{\gamma}$

This shows that the centre (f,g,h) lies on the line (2)

Example10:- If the section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Whose vertex is P, by the plane $z=0$, is a rectangular hyperbola, prove that the locus of P is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:- Let (α,β,γ) be the co-ordinates of the vertex P of the enveloping cone. Then its equation is $\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$

The section of this cone by the plane $z=0$ is given by $\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right), z=0$

This section will represent a rectangular hyperbola if

coefficient of x^2+ coefficient of $y^2=0$

i.e. $\frac{1}{a^2}\left(\frac{\beta^2}{b^2}+\frac{\gamma}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)=0$

i.e. $b^2\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)+a^2\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)=0$

i.e. $\beta^2+\alpha^2+\left(\frac{b^2+a^2}{c^2}\right)\gamma^2=b^2+a^2$

i.e. $\frac{\alpha^2+\beta^2}{a^2+b^2}+\frac{\gamma^2}{c^2}=1$

Hence the locus of $P(\alpha,\beta,\gamma)$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Example11:- Find the locus of a luminous point if the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ casts a circular shadow on the plane $z=0$,

Solution:- Let (α,β,γ) be the co-ordinates of the luminous point. Then the equation of the enveloping cone of the ellipsoid with vertex at (α,β,γ) is

$$\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$$

The section of this coin by the plane $z=0$ is given by

$$\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right),\ z=0$$

This section will represent a circle if

(i) Coefficient of x^2 = coefficient of y^2

i.e. $\dfrac{1}{a^2}\left(\dfrac{\beta^2}{b^2}+\dfrac{\gamma^2}{c^2}-1\right)=\dfrac{1}{b^2}\left(\dfrac{\alpha^2}{a^2}+\dfrac{\gamma^2}{c^2}-1\right)$ (1)

and (ii) coefficient of $xy=0$

i.e. $\alpha\beta=0$ (2)

The following two cases do arise in view of (2).

Case I:- If $\alpha=0$, equation (1) reduces to $\dfrac{1}{a^2}\left(\dfrac{\beta^2}{b^2}+\dfrac{\gamma^2}{c^2}-1\right)=\dfrac{1}{b^2}\left(\dfrac{\gamma^2}{c^2}-1\right)$, i.e.

$$b^2\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=a^2\left(\frac{\gamma^2}{c^2}-1\right).$$

i.e. $\beta^2+\left(\dfrac{b^2-a^2}{c^2}\right)\gamma^2=b^2-a^2$, i.e. $\dfrac{\beta^2}{b^2-a^2}+\dfrac{\gamma^2}{c^2}=1$

Hence the locus if (α,β,γ) is

$$x=0,\qquad \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1$$

Case II:- If $\beta=0$, equation (1) reduces to

$$\frac{1}{a^2}\left(\frac{\gamma^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right),\text{i.e. } b^2\left(\frac{\gamma^2}{c^2}-1\right)=a^2\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)$$

i.e. $\alpha^2+\left(\dfrac{a^2+b^2}{c^2}\right)\gamma^2=a^2-b^2$, i.e. $\dfrac{\alpha^2}{a^2-b^2}=\dfrac{\gamma^2}{c^2}=1$

Hence the locus of (α,β,γ) is

$$y=0,\ \frac{x^2}{a^2-b^2}=\frac{z^2}{c^2}=1$$

Example12:- Prove that the enveloping cylinder of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, whose generators are parallel to the lines $\dfrac{x}{0}=\dfrac{y}{\pm\sqrt{(a^2-b^2)}}=\dfrac{z}{c}$ meets the plane $z=0$ in circles.

Solution:- Since the generators are parallel to the given lines, the equation of the enveloping cylinder is $\left(\frac{0.x}{a^2}+\frac{\pm\sqrt{(a^2-b^2)}}{b^2}+\frac{cz}{c^2}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{a^2-b^2}{b^2}+\frac{c^2}{c^2}\right)$

This will meet the plane $z=0$ in the conic

$$z=0,\quad \frac{(a^2-b^2)y^2}{b^2}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2}{b^2}\right)$$

i.e. $\quad x^2+y^2=a^2,\quad z=0$

Which is a circle.

Example13:- Find the locus of the centres of the sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Which are at a constant distance p from the centre of the ellipsoid

Solution:- Let (α,β,γ) be the centre of the section of the given ellipsoid. Then the equation of the section is $T=S_1$, i.e.

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}$$

Its distance from the centre $(0,0,0)$ of the ellipsoid is p. Therefore

$$\frac{\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}}{\sqrt{\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}\right)}}=p\text{, i.e. } p^2\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}\right)=\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2$$

Hence the locus of the centre (α,β,γ) is

$$p^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2$$

Example14:- Find the equation to the plane through the extremities of three conjugate semi-diameters of an ellipsoid

Solution:- Let the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be

$$P(x_1,y_1,z_1),\quad Q(x_2,y_2,z_2),\quad R(x_3,y_3,z_3)$$

And let the equation of the plane PQR be

$$lx+my+nz=p \qquad (1)$$

Therefore, $\quad l_1x_1+m_1y_1+n_1z_1=p \qquad (2)$

$$l_2x_2+m_2y_2+n_2z_2=p \qquad (3)$$

$$l_3x_3+m_3y_3+n_3z_3=p \qquad (4)$$

Multiplying (2), (3), (4) by x_1, x_2, x_3 respectively, and adding the resulting equations column wise we get.

$$l\sum x_1^2 + m\sum x_1 y_1 + n\sum z_1 y_1 = p\sum x_1$$

i.e. $\quad l.a^2 \quad +m.0 \quad +n.0 \quad = p\left(x_1 + x_2 + x_3\right)$

i.e. $\quad l = \dfrac{p\left(x_1 + x_2 + x_3\right)}{a^2}$.

Similarly, $m = \dfrac{p\left(y_1 + y_2 + y_3\right)}{b^2}$ and $n = \dfrac{p\left(z_1 + z_2 + z_3\right)}{c^2}$

Using these values of l, m, n in (1), the required equations of the plane PQR is

$$\left(\frac{x_1 + x_2 + x_3}{a^2}\right)x + \left(\frac{y_1 + y_2 + y_3}{b^2}\right)y + \left(\frac{z_1 + z_2 + z_3}{c^2}\right)z = 1$$

Example15:- If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Solution:- Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), be the co-ordinates of the extremities of equal conjugate semi-diameters. And r be their length. Then we have

$$3r^2 = a^2 + b^2 + c^2 \qquad (1)$$

If l, m, n are the direction cosines of one of the diameters, its equations will be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad (2)$$

The co-ordinates of the extremity of this diameters will be (lr, mr, nr)

This point lies on the ellipsoid, so we have

$$\frac{l^2 r^2}{a^2} + \frac{m^2 r^2}{b^2} + \frac{n^2 r^2}{c^2} = 1 = l^2 + m^2 + n^2$$

Or $\quad \dfrac{l^2}{a^2} + \dfrac{m^2}{b^2} + \dfrac{n^2}{c^2} = \dfrac{l^2 + m^2 + n^2}{r^2}$

$= \dfrac{3\left(l^2 + m^2 + n^2\right)}{a^2 + b^2 + c^2}$, using (1)

Therefore, using (2) the locus of the diameter is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{3\left(x^2 + y^2 + z^2\right)}{a^2 + b^2 + c^2}$$

Example16:- Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Which are parallel of the conjugate diametral plane of the of the ellipsoid $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} + \dfrac{z^2}{\gamma^2} = 1$ is $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} + \dfrac{z^2}{\gamma^2} = \dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2} + \dfrac{c^2}{\gamma^2}$

Solution:- Let (x_r, y_r, z_r) be the extremities of the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$$

Then the equation of the conjugate diametral planes are given by

$$\frac{xx_r}{\alpha^2}+\frac{yy_r}{\beta^2}+\frac{zz_r}{\gamma^2}=0, \quad r=1,2,3 \qquad (1)$$

The tangent planes to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Parallel to the conjugate diametral planes (1), are given by

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}=\sqrt{\left(\frac{a^2x_1^2}{\alpha^4}+\frac{b^2y_1^2}{\beta^4}+\frac{c^2z_1^2}{\gamma^4}\right)} \qquad (2)$$

$$\frac{xx_2}{\alpha^2}+\frac{yy_2}{\beta^2}+\frac{zz_2}{\gamma^2}=\sqrt{\left(\frac{a^2x_2^2}{\alpha^4}+\frac{b^2y_2^2}{\beta^4}+\frac{c^2z_2^2}{\gamma^4}\right)} \qquad (3)$$

$$\frac{xx_3}{\alpha^2}+\frac{yy_3}{\beta^2}+\frac{zz_3}{\gamma^2}=\sqrt{\left(\frac{a^2x_3^2}{\alpha^4}+\frac{b^2y_3^2}{\beta^4}+\frac{c^2z_3^2}{\gamma^4}\right)} \qquad (4)$$

The required locus is obtained by eliminating $x_r, y_r, z_r\,(r=1,2,3)$ from (2), (3) and (4). For this, squaring these equation and adding the resulting equations, we get

$$\frac{x^2}{\alpha^4}\sum x_1^2+\frac{y^2}{\beta^4}\sum y_1^2+\frac{z^2}{\gamma^4}\sum z_1^2+\frac{2xy}{\alpha^2\beta^2}\sum y_1z_1+\frac{2zx}{\gamma^2\alpha^2}\sum z_1x_1$$
$$=\frac{a^2}{\alpha^4}\sum x_1^2+\frac{b^2}{\beta^4}\sum y_1^2+\frac{c^2}{\gamma^4}\sum z_1^2.$$

i.e. $$\frac{x^2}{\alpha^4}.\,\alpha^2+\frac{y^2}{\beta^4}.\beta^2+\frac{z^2}{\gamma^4}.\gamma^2+0+0+0=\frac{a^2}{\alpha^4}.\alpha^2+\frac{b^2}{\beta^4}.\beta^2+\frac{c^2}{\gamma^4}.\gamma^2$$

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$$

8. PARABOLOID

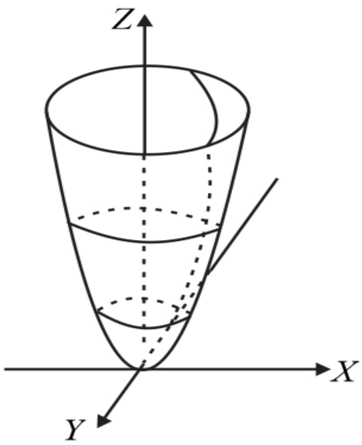
8.1 **The Paraboloid:-** The surface represented by the equation $ax^2 + by^2 = 2cz$ (1) is called a paraboloid as it can be generated by a variable parabola in two different ways. It can also be generated by a variable ellipse or a variable hyperbola, giving rise to two different surfaces.

There are two different types of paraboloids represented by (1), depending upon whether the coefficients a be b are of the same or different signs. Accordingly, they are known as the elliptic paraboloid or hyperbolic paraboloid. Whenever we intend to distinguish them, their standard equations are written in the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c} \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c} \quad (2)$$

respectively. However equation (1) is used when it is not necessary to distinguish between the two surfaces.

8.2 **The Elliptic Paraboloid:-** The locus of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$ is called an elliptic paraboloid.



We have the following properties about the nature and shape of this surface.

(i) No point bisects all chords through it and therefore there is no centre of the surface.

(ii) The co-ordinate plane $x = 0$ and $y = 0$ bisect all chords perpendicular to them and are, therefore, its two plane of symmetry or two principal planes.

(iii) z cannot be negative (c being positive). Hence there is no part of the surface below the xy-plane, i.e. the surface is above the xy-plane.

(iv) The section by the plane $z = k\left(k > 0\right)$ is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$ which represents an ellipse whose semi-axes are $a\sqrt{\left(\frac{2k}{c}\right)}$, $b\sqrt{\left(\frac{2k}{c}\right)}$ and whose centre lies on the z-axis. It increase in size as

k increases, there being no limit to the increase. For $k=0$ it is a point ellipse. The surface is therefore generated by a variable ellipse parallel to the xy-plane and is consequently called the elliptic paraboloid. Therefore the surface is entirely on the positive side of the xy-plane and extends to infinity.

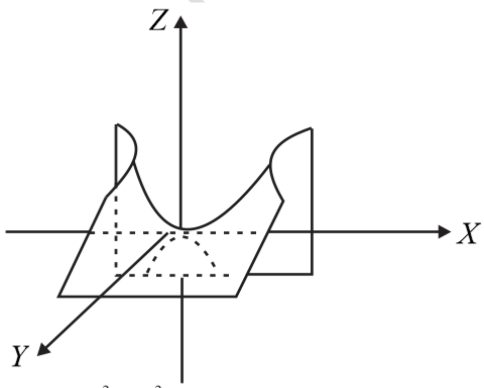
The section of the surface by a plane parallel to xz-plane is the parabola given by the equations $y=k$, $x^2=\frac{2a^2}{c}\left(z-\frac{k^2c}{2b^2}\right)$

In the same manner the section of the surface by the plane $x=k$ is the parabola whose equation are $x-k$, $y^2=\frac{2b^2}{c}\left(z-\frac{k^2c}{2a^2}\right)$.

Thus the paraboloid is also generated by a variable parabola in two different ways.

When $a=b$, the surface becomes a becomes a paraboloid of revolution, formed by revolving the parabola $y=0$, $x^2=\frac{2a^2}{c}z$ about the z-axis.

8.3 The Hyperbolic Paraboloid:- The locus of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$ is called a hyperbolic paraboloid.



$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}\ c>0$

We have the following properties about the nature and shape of this surface.

(i) The surface is symmetric with respect to the yz- and xz-planes. Thus to co-ordinate plane $x=0$ and $y=0$ are the two principal planes.

(ii) The section of the surface by the plane $z=k(k\neq 0)$ is given by the equations

$$z=k,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$$

It represent a hyperbola with its centre on the z-axis. The transverse axis of this hyperbola is parallel to the x-axis or the y-axis according as k is positive or negative.

Further, the section of the surface by the plane $z=0$ is the pair of lines $\frac{x}{a}+\frac{y}{b}=0$, $z=0$ and $\frac{x}{a}-\frac{y}{b}=0$, $z=0$.

These lines are parallel to the asymptotes of all hyperbolic sections. It follows that the surface is generated by a variable hyperbola parallel to the xy-plane. That is why the surface is called a hyperbolic paraboloid.

The sections by the planes parallel to the yz- and xy-planes are parabolas whose equations are given by $x=k,\ y^2=-\frac{2b^2}{c}\left(z-\frac{k^2c}{2a^2}\right)$ and $y=k\ ,\ x^2=\frac{2a^2}{c}\left(z+\frac{k^2c}{2b^2}\right)$ respectively.

8.4 Intersection of A Line With A Paraboloid:- To find the points of intersection of the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$,say (1)

with the paraboloid ax^2+by^2+2cz (2)

To co-ordinates of any point on (1) are $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$.

If this point lies on (2), then we must have $a(\alpha+lr)^2+b(\beta+mr)+2c(\gamma+nr)^2$ i.e.

$r^2(al^2+bm^2)+2r(a\alpha l+b\beta m-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0$ (3)

This equation is quadratic in r and therefore gives two values of r showing that every line meets a paraboloid in two points. It follows that the plane section of a paraboloid are conics.

If $l=m=0$, then one root of this equation is infinity. Therefore, one point of intersection is at infinity, and the other point is at a finite distance.

Thus a line parallel to the z-axis meets the paraboloid in one point at an infinite distance from the (α,β,γ) and therefore meets it in one finite point only. Its distance is given by $r=\frac{a\alpha^2+b\beta^2-2c\gamma}{2cn}$.

In particular, the axis meets the surface at the origin only.

8.5 Tangent Lines And Tangent Plane:- To find the equation of tangent lines and tangent plane to the paraboloid $ax^2+by^2=2cz$.

We know that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ say

(1) intersection the paraboloid $ax^2+by^2=2cz$ in two points for which r is given by $r^2(al^2+bm^2)+2r(a\alpha l+b\beta m+cn)+(a\alpha^2+b\beta^2-2c\gamma)=0$

(2)

If the point (α,β,γ) lies on the surface $ax^2+by^2=2cz$, we have $a\alpha^2+b\beta^2-2c\gamma=0$

(3)

This shows that one root of equation (2) is zero. The other roots is also zero if in addition to it we have $a\alpha l+b\beta m-cn=0$

(4)

Eliminating l, m, n between (1) and (4) we get the locus of all the tangent lines through the point (α, β, γ), which is given by $a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma)=0$

i.e. $a\alpha x+b\beta y-cz=a\alpha^2+b\beta^2-c\gamma$ i.e. $a\alpha x+b\beta y=c(z+\gamma)$ (using (3)

This is the equation of the tangent plane at (α, β, γ) to the paraboloid.

8.6 Condition of Tangency:- To find the condition that the plane $lx+my+nz=p$

(1) touches the paraboloid ax^2+by^2+2cz

(2) at the point $(\alpha, \beta\gamma)$.

If the plane (1) touches the conicoid (2) at the point (α, β, γ), this plane must be identical with the tangent plane $a\alpha x+b\beta y=c(z+\gamma)$, i.e. $a\alpha x+b\beta y-cz=c\gamma$

(3)

Comparing the equations (1) and (3), we have

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=-\frac{c}{n}=\frac{c\gamma}{p}$$

This given $\alpha=-\frac{cl}{an},\ \beta=-\frac{cn}{bn},\ \gamma=-\frac{p}{n}$

Since the point (α, β, γ) lies on the paraboloid these values of α, β, γ must satisfy the equation (2). Thus $a\left(-\frac{cl}{cn}\right)^2+b\left(-\frac{cm}{bn}\right)^2=2c\left(-\frac{p}{n}\right)$, i.e. $c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)+2pn=0$

This is the required condition of tangency.

Corollary:- This plane $2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$ always touches the paraboloid $ax^2+by^2=2cz$.

8.7 Normal:- To find the equation of the normal at any point (α, β, γ) on the

$$ax^2+by^2=2cz \qquad (1)$$

We have that the equation of the tangent plane at the point (α, β, γ) on the paraboloid (1) is given by $a\alpha x+b\beta y=c(z-\gamma)$, i.e. $a\alpha x+b\beta y-cz=c\gamma$.

Clearly, the direction cosines of the normal to this plane are proportional to $a\alpha, b\beta, -c$.

Hence equations of the normal of (α, β, γ) on (1) are $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c}$.

8.8 Enveloping Cone of A Paraboloid:- The locus of the tangent lines from a given point (α, β, γ) to the paraboloid $ax^2+by^2=2cz$ is called the enveloping cone of the paraboloid. The given point (α, β, γ) is called the vertex of this enveloping cone.

To find the equation of the enveloping cone of the paraboloid

$$ax^2+by^2=2cz \qquad (1)$$

whose vertex is the point (α,β,γ).

The equations of any line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ say} \qquad (2)$$

Any point on this line is $(\alpha+lr,\beta+mr,\gamma+nr)$. If this point be on the paraboloid (1), we must have $a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$, i.e.

$(al^2+bm^2)r^2+2(a\alpha l+b\beta m-cn)r+(a\alpha^2+b\beta^2-2c\gamma)=0$.

Clearly, the line (2) will be a tangent line to the paraboloid (1) if both the roots of the quadratic in r are equal. For this, its discriminant must be zero i.e. $4(a\alpha l+b\beta m-cn)^2-4(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma)=0$ i.e.

$$(a\alpha l+b\beta m-cn)^2=(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma) \qquad (3)$$

The locus of the tangent lines is obtained by eliminating l,m,n from (2) and(3), Thus the required locus is given by

$$\{a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma)\}^2=\{a(x-\alpha)^2+b(y-\beta)^2\}(a\alpha^2+b\beta^2-2c\gamma) \quad (4)$$

If we use the notations

$$S\equiv ax^2+by^2-2cz$$
$$S_1\equiv a\alpha^2+b\beta^2-2c\gamma$$

And $T\equiv a\alpha x+b\beta y-c(z+\gamma)$, then after a moderate manipulation, equation (4) assume the from

$(T-S_1)^2=(S+S_1-2T)S_1$, i.e. $T^2=SS_1$, i.e.

$$\{a\alpha x+b\beta y-c(z+\gamma)\}^2=(ax^2+by^2-2cz)(a\alpha^2+b\beta^2-2c\gamma)$$

This is a more useful form of the equation of the enveloping cone.

8.9 Enveloping Cylinder of A Paraboloid:- The locus of the tangent lines drawn to a paraboloid and parallel to a given line is called the enveloping cylinder of the paraboloid.

To find the equation of the enveloping cylinder of the paraboloid $ax^2+by^2=2cz$ (1) whose generators are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Example1:- Show that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2+2y^2=3z$. Find also the co-ordinates of the point of contact.

Solution:- We know that the equation of tangent plane at any point (α,β,γ) of paraboloid ax^2+by^2+2cz is

$$a\alpha x+b\beta y=c(z+\gamma)$$

Comparing the given equation of paraboloid with the equation ax^2+by^2+2cz, we find that

$$a=1,\quad b=-2 \text{ and } 2c=3$$

Therefore, tangent plane at the point (α,β,γ) to the given paraboloid is

$$\alpha x-2\beta y-\frac{3}{2}(z+y),$$

i.e. $\quad 2\alpha x-4\beta y-3z-3\gamma=0$

Now this plane and the given plane should be the same. So, comparing the coefficient, we have $\frac{2\alpha}{2}=\frac{-4\beta}{4}=\frac{-3}{-1}=\frac{-3\gamma}{3}$

Whence $\quad \alpha=\beta=-\gamma=3$

Clearly, the point $(3,3,-3)$ satisfies the equation of the given paraboloid.

Hence the given plane touches the paraboloid, and its point of contact is $(3,3,-3)$

Example2:- Find the condition that the paraboloids $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1}$, $\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2}$, $\frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3}$, may have a common tangent plane.

Solution:- Let the common tangent plane be

$$lx+my+nz=p$$

It will be a tangent plane to the given paraboloids if

$$l^2a_1^2+m^2b_1^2+2npc_1=0$$

$$l^2a_2^2+m^2b_2^2+2npc_2=0$$

And $\quad l^2a_3^2+m^2b_3^2+2npc_3=0$

Eliminating the unknowns l^2, m^2 and $2np$, we get the required condition as

$$\begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix}=0$$

Example3:- Find the locus of the point of intersection of three mutually perpendicular tangent planes to the paraboloid ax^2+by^2+2cz

Solution:- We know that the plane $lm+my+nz=p$ touches the paraboloid $ax^2+by^2=2cz$ when $\frac{l^2}{a}+\frac{m^2}{b}+\frac{2np}{0}$ i.e. $p=-\frac{c}{2n}\left(\frac{l^2}{a}+\frac{m^2}{b}\right)$

By putting this value p, the equation of plane assumes the form $lx+my+nz+\frac{c}{2n}\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$

i.e. $2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$

This plane always touches the given paraboloid

Now consider the three mutually perpendicular tangent planes given by

$$2n_r(l_r x+m_r y+n_r z)+c\left(\frac{l_r^2}{a}+\frac{m_r^2}{b}\right)=0, r=1,2,3 \qquad (1)$$

Since the three planes and hence their normal are mutually perpendicular, we have $l_1^2+m_1^2+n_1^2=1$. $\quad l_2^2+m_2^2+n_2^2=1$ etc.

And $l_1l_2+m_1m_2+n_1n_2=0$, $l_1m_1+l_2m_2+l_3n_3=0$

The locus to the point of intersection can be obtained by eliminating $l,m,n,$ from the equations of these planes (1)

For this, we add column wise the equations of these there planes obtained by putting $r=1,2,3$ and use the above relations. Thus, we get

$$2(n_1^2+n_2^2+n_3^2)z+c\left(\frac{l_1^2+l_2^2+l_3^2}{a}+\frac{m_1^2+m_2^2+m_3^2}{b}\right)=0$$

i.e. $2z+c\left(\frac{1}{a}+\frac{1}{b}\right)=0$

This is the required plane.

Example4:- Find out the number of normal that can be drawn from a given point to a paraboloid.

OR

Show that in tangent, five normal can be drawn to a paraboloid through a fixed point.

Solution:- We known that the equations of normal through any point (α,β,γ) of the paraboloid

$$ax^2+by^2+2cz \qquad (1)$$

Are $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}$

If this normal passes through a given point (x',y',z'), say, then we have $\frac{x'-\alpha}{a\alpha}=\frac{y'-\beta}{b\beta}=\frac{z'-\gamma}{-c}=\lambda$, say,

Whence $\alpha=\frac{x'}{1+a\lambda},\beta=\frac{y'}{1+b\lambda}, \gamma=z'+c\lambda$

But (α,β,γ) lies on the paraboloid. Therefore, using these co-ordinates in (1), we have

$$a=\frac{x'^2}{(1+a\lambda)^2}+b=\frac{y'^2}{(1+b\lambda)^2}=2c(z'-c\lambda).$$

This is an equation of fifth degree in λ. So, it will give five values of λ. It follows that there are five points on the paraboloid, the normal at which pass through the given point (x', y', z')

Thus, in general, five normal can be drawn through a given point to a paraboloid.

Example5:- Show that the normal from the point (α, β, γ) on the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$, lie on the cone $\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$

Solution:- We know that the normal at the point (x', y', z') on the given paraboloid is

$$\frac{x-x'}{x'/a^2} = \frac{y-y'}{y'/b^2} = \frac{z-z'}{-1} \qquad (1)$$

Now the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (2)$$

Passing through the point (α, β, γ) is the normal at the point (x', y', z') if (1) and (2) are the same. This required

$$\frac{x'/a^2}{l} = \frac{y'/b^2}{m} = \frac{-1}{n} \qquad (3)$$

Since the normal (1) passes through the point (α, β, γ), we have

$$\frac{\alpha - x'}{x'/a^2} = \frac{\beta - y'}{y'/b^2} = \frac{\gamma = z'}{-1} = r, \text{ say,}$$

This gives $\frac{x'}{a^2} = \frac{\alpha}{a^2+r}$ and $\frac{y'}{b^2} = \frac{\beta}{b^2+r}$

Using these relation in (3), we have

$$\frac{\alpha/\left(a^2+r\right)}{l} = \frac{\beta/\left(b^2+r\right)}{m} = \frac{-1}{n}, \text{ i.e. } \frac{\alpha}{l\left(a^2+r\right)} = \frac{\beta}{m\left(b^2+r\right)} = -\frac{1}{n}$$

Which gives $\frac{\alpha}{l} = -\frac{a^2+r}{n}$ and $\frac{\beta}{m} = -\frac{b^2+r}{n}$

On subtraction these relations give

$$\frac{\alpha}{l} - \frac{\beta}{m} = -\frac{a^2-b^2}{n} \text{ i.e. } \frac{\alpha}{l} - \frac{\beta}{m} + \frac{a^2-b^2}{n} \qquad (4)$$

Now, eliminating l, m, n between (2) and (4), we obtain

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$$

This equation of the cone is the required locus of the normal

Example6:- Show that the feet of the normal from the point (α, β, γ) to the paraboloid $x^2 + y^2 + 2az$ lie on the sphere $x^2 + y^2 + z^2 = y\left(\frac{\alpha^2+\beta^2}{2\beta}\right) - z(a+\gamma) = 0$

Solution:- We known that the normal at any point (x',y',z') to the given paraboloid has the equations $\frac{x-x'}{x'}=\frac{y-y'}{y'}=\frac{z-z'}{-a}=\lambda$, say.

If it passes through the point (α,β,γ), then we have

$$\frac{\alpha-x'}{x'}=\frac{\beta-y'}{y'}=\frac{\gamma-z'}{-a}=\lambda$$

Whence $x'=\frac{\alpha}{1+\lambda}, y'=\frac{\beta}{1+\lambda}, z'=\gamma+a\lambda$

i.e. the feet to the normal are given by

$$\left(\frac{\alpha}{1+\lambda},\frac{\beta}{1+\lambda},\gamma+a\lambda\right)$$

Since it lies on the given paraboloid, therefore,

$$\left(\frac{\alpha}{1+\lambda}\right)^2+\left(\frac{\beta}{1+\lambda}\right)^2=2a(\gamma+a\lambda)$$

$$\alpha^2+\beta^2=2a(\gamma+a\lambda)(1+\lambda)^2 \qquad (1)$$

If it lies on the sphere, then

$$\left(\frac{\alpha}{1+\lambda}\right)^2+\left(\frac{\beta}{1+\lambda}\right)^2+(\gamma+a\lambda)^2-\frac{\beta}{1+\lambda}\left(\frac{\alpha^2+\beta^2}{2\beta}\right)-(\gamma+a\lambda)(a+\gamma)=0$$

i.e. $$\frac{\alpha^2+\beta^2}{(1+\lambda)^2}-\frac{\alpha^2+\beta^2}{2(1+\lambda)}+(\gamma+a\lambda)(\gamma+a\lambda-a-\gamma)$$

i.e. $$\frac{\alpha^2+\beta^2}{2(1+\lambda)^2}(2-1-\lambda)-a(\gamma+a\lambda)(1-\lambda)=0$$

i.e. $$\frac{1-\lambda}{2(1+\lambda)^2}\left\{(\alpha^2+\beta^2)-2a(\gamma+a\lambda)(1+\lambda)^2\right\}=0$$

Therefore,

$$\alpha^2+\beta^2-2a(\gamma+a\lambda)(1+\lambda)^2=0, \text{ since } \lambda\neq 1$$

Which is the same as (1)

Example7:- Prove that the planes $lx+my+p=0$ and $l'x+m'y+p'=0$ are the conjugate diametral planes of the paraboloid $ax^2+by^2=2cz$, if $\frac{ll'}{a}+\frac{mm'}{b}=0$

Solution:- The given planes are the conjugate diametral planes of the given paraboloid, if one bisects the chords parallel to the other.

But the first diametral plane i.e. $lx+my+p=0$ bisects the chords parallel to the line

$$\frac{x}{l/a}=\frac{y}{l/b}=\frac{z}{-p/c} \qquad (1)$$

Thus the diametral plane $lx+my+p=0$ will bisect the chords parallel to the plane $l'x+m'y+p'=0$. If this second plane is parallel to the line (1) which required

$$\frac{l}{a}.l'+\frac{m}{b}.m'-\frac{p}{c}.0=0 \text{ i.e. } \frac{ll'}{a}+\frac{mm'}{b}=0$$

Example8:- Prove that the section of the conicoid $ax^2+by^2+cz^2=1$ by a tangent plane to the cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$ is a rectangular hyperbola

Solution:- The plane

$$lx+my+nz=0 \qquad (1)$$

Will touch the given cone if $\frac{l^2}{1/(b+c)}+\frac{m^2}{1/(c+a)}=\frac{n^2}{1/(a+b)}=0$

i.e. $$(b+c)l^2+(c+a)m^2+(a+b)n^2=0 \qquad (2)$$

the lengths of the semi-axes of the section of $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ are given by

$$r^4\left(bcl^2+cam^2+abn^2\right)-r^2\left\{(b+c)^2l^2+(c+a)m^2+(a+b)n^2\right\}+\left(l^2+m^2+n^2\right)=0$$

The section will be a rectangular hyperbola if the sum of the squares of its semi-axes is zero, i.e. $r_1^2+r_2^2=0$. i.e. $(b+c)l^2+(c+a)m^2+(a+b)n^2=0$

This is the same as the condition (2). Hence the result.

Example9:- Find the condition that the two lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$, $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$, be the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by a plane through them.

Solution:- Since the given lines are the axes of the central section, each line must bisect chords of the section to the other, i.e. each of them lies in the diametral plne conjugate of the other.

Now the equation of the diametral plane conjugate to $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ is

$$al_1x+bm_1y+cn_1z=0 \qquad (1)$$

This plane will contain the second line if we have

$$al_1l_2+bm_1m_2+cn_1n_2=0 \qquad (2)$$

Further, the axes being rectangular, the two given lines must be at right-angles to each other, i.e.

$$l_1l_2+m_1m_2+n_1n_2=0 \qquad (3)$$

Equations (2) and (3) are the required conditions.

Example10:- Prove that the axes of the section of t' conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ lie on the cone $\frac{(b-c)l}{x}+\frac{(c-a)m}{y}+\frac{(a-b)n}{z}=0$

Solution:- We know that the direction cosines λ,μ,ν of the axes of length $2r$ of the section are given by the equations

$$\frac{(ar^2-1)\lambda}{l}=\frac{(br^2-1)\mu}{m}=\frac{(cr^2-1)\nu}{n}=k\text{, say,}$$

So, $\frac{kl}{\lambda}=ar^2-1$, $\frac{km}{\mu}=br^2-1$, $\frac{kn}{\nu}=cr^2-1$. Therefore,

$$\frac{kl}{\lambda}(b-c)+\frac{km}{\mu}(c-a)+\frac{kn}{\nu}(a-b) \quad \textbf{(Note)}$$

$$=(ar^2-1)(b-c)+(br^2-1)(c-a)+(cr^2-1)(a-b)$$

$=0$, on simplification.

Eliminating λ,μ,ν between (1) and (2), we obtain the required equation of the cone as

$$\frac{kl}{x}(b-a)+\frac{km}{y}(c-a)+\frac{kn}{z}(a-b)=0$$

i.e. $$\frac{(b-c)l}{x}+\frac{(c-a)m}{y}+\frac{(a-b)n}{z}=0$$

Example11:- Find the equation of the central plane section of the conicoid $ax^2+by^2+cz^2=1$ which has one of its axes along the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Solution:- Let

$$Lx+My+Nz=0 \tag{1}$$

Be any plane through the centre of the given conicoid. Let the two axes of the section of this conicoid by the plane (1) be

$$\frac{x}{l_1}=\frac{y}{m_2}=\frac{z}{n_1} \tag{2}$$

And $$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \tag{3}$$

Since (2) and (3) are the axes of the section, we have

$$all_1+bmm_1+cnn_1=0 \tag{4}$$

And $$ll_1+mm_1+nn_1=0 \tag{5}$$

Solving (4) and (5), we obtain

$$\frac{l}{mn(b-c)}=\frac{m}{ml(c-a)}=\frac{n}{lm(a-b)}=\lambda\text{, say.} \tag{6}$$

Further, since the plane (1) contains both the axes given by (2) and (3) the normal to the plane must be perpendicular to each of the lines given by (2) and (3). Therefore, we have

$$\frac{L}{mn_1 - m_1 n} = \frac{M}{nl_1 - n_1 l} = \frac{N}{lm_1 - l_1 m} = \mu \text{, say.} \qquad (7)$$

Whence $L = \mu(mn_1 - m_1 n)$

$= \mu\{m\lambda lm(a-b) - n\lambda nl(c-a)\}$. Using (6)

$= \lambda\mu l\{(a-b)m^2 - n^2(c-a)\}$

Hence the equation of required plane will be

$$Lx + My + Nz = 0$$

i.e. $\quad \sum \lambda\mu l\{m^2(a-b) - n^2(c-a)x\} = 0$

i.e. $\quad \sum\{m^2(a-b) + n^2(c-a)\}lx = 0$

Example12:- Prove that tangent planes to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$ which cut $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$ in ellipse of constant area πk^2, have their points of contact on the surface $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{k^4}{4a^2b^2c^2}$

Solution:- Equation of the tangent plane to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$ at any point (x_1, y_1, z_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - \frac{zz_1}{c^2} = -1 \qquad (1)$$

Where $\quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = -1 \qquad (2)$

Now, if A_0 be the area of the corresponding central section $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - \frac{zz_1}{c^2} = 0$ of the second conicoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, then we have

$$A = \left(1 - \frac{p^2}{p_0^2}\right)A_0 \qquad (3)$$

Now $A = \pi k^2$ (given), $p^2 = (-1)^2 = 1$; $p_0^2 = a^2\frac{x_1^2}{a^4} + b^2\frac{y_1^2}{b^4} - c^2\frac{z_1^2}{c^4} = -1$ from (2)

And $\quad A_0 = \frac{\pi b\sqrt{(-c^2)}\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2l^2 + b^2m^2 - c^2n^2)}}$

Putting $-c^2$ in place of c^2 in the usual formula $\frac{\pi b\sqrt{\{(-c^2)\}}\sqrt{\left\{\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^2}\right)\right\}}}{\sqrt{(-1)}}$ from (2)

Squaring relation (3) and putting the values, we get

$$\pi^2 k^{3\backslash 4} = \left(1-\frac{1}{-1}\right)^2 \frac{\pi^2 a^2 b^2 \left(-c^2\right)}{1}\left(\frac{x_1^2+y_1^2+z_1^2}{a^4+b^4+c^4}\right)$$

i.e. $$\frac{k^4}{4a^2b^2c^2} = \frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}$$

Hence the locus of (x_1, y_1, z_1) is given by

$$\frac{x^2}{a^4}+\frac{y^2}{b^2}+\frac{z^2}{c^4} = \frac{k^4}{4a^2b^2c^2}$$

Example13:- Prove that if $l_1, m_1, n_1;\ l_2, m_2, n_2$ are the direction cosines of the axes of any plane section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ then $\frac{l_1 l_2}{a^2\left(b^2-c^2\right)} = \frac{m_1 m_2}{b^2\left(c^2-a^2\right)} = \frac{n_1 n_2}{c^2\left(a^{-2}-b^2\right)}$

Solution:- Let l, m, n be the $d.c.'s$ of either axis of the section of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$

By the plane $Lx+My+Nz=p$ (2)

Then we have $\frac{r^2-a^2k^2}{La^2k^2/l} = \frac{r^2-b^2k^2}{Mb^2k^2/m} = \frac{r^2-c^2k^2}{Nc^2k^2/n} = \lambda$, say.

So, $$\frac{La^2k^2}{l}\left(b^2-c^2\right)+\frac{Mb^2k^2}{m}\left(c^2-a^2\right)+\frac{Nc^2k^2}{n}\left(a^2-b^2\right)$$ **(note)**

$$=\left(\frac{1}{\lambda}\right)\left\{\left(r^2-a^2k^2\right)\left(b^2-c^2\right)+\left(r^2-b^2k^2\right)\left(c^2-a^2\right)+\left(r^2+c^2k^2\right)\left(a^2-b^2\right)\right\}=0$$

i.e. $$\sum \frac{La^2}{l}\left(b^2-c^2\right)=0 \tag{3}$$

Also, since the axes lie in plane (2), we have

$$lL+mM+nN=0 \tag{4}$$

Eliminating n between (3) and (4), we get

$$\frac{La^2\left(b^2-c^2\right)}{l}+\frac{Mb^2\left(c^2-a^2\right)}{m}-\frac{Nc^2\left(a^2-b^2\right)N}{Ll+Mn}=0$$

i.e. $$LMb^2l^2\left(c^2-a^2\right)+lm\left\{L^2a^2\left(b^2-c^2\right)+M^2b^2\left(c^2-a^2\right)\right.$$
$$\left.-N^2c^2\left(a^2-b^2\right)\right\}+LMa^2m^2\left(b^2-c^2\right)=0 \tag{5}$$

Now, the $d.c.'s$ of axes satisfy (3) and (4), and hence satisfy (5).

Thus if l_1, m_1, n_1 and l_2, m_2, n_2 are the $d.c.'s$ of the axes, from (5) we have

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{LMa^2\left(b^2-c^2\right)}{LMb^2\left(c^2-a^2\right)}$$

i.e. $$\frac{l_1 l_2}{LMa^2\left(b^2-c^2\right)} = \frac{m_1 m_2}{LMb^2\left(c^2-a^2\right)}$$

Hence by symmetric we have

$$\frac{l_1 l_2}{a^2\left(b^2-c^2\right)}=\frac{m_1 m_2}{b^2\left(c^2-a^2\right)}=\frac{n_1 n_2}{c^2\left(a^2-b^2\right)}$$

Example14:- If OP, OQ, OR are conjugate semi-diameters of an ellipsoid prove that the area of the section of the ellipsoid by the plane PQR is two thirds of the area of the parallel central section.

Solution:- If $\left(x_r, y_r, z_r\right)$, $r=1,2,3$ be the co-ordinates of the extremities P, Q, R of semi-conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (1)$$

Then the equation of the plane PQR is

$$\frac{x}{a^2}\left(x_1+x_2+x_3\right)+\frac{y}{b^2}\left(y_1+y_2+y_3\right)+\frac{z}{c^2}\left(z_1+z_2+z_3\right)=0 \qquad (2)$$

If A and A_0 be the areas of the sections of (1) by (2) and a central plane parallel to (2), respectively, then we have

$A=A_0\left(1-\frac{p^2}{p_0^2}\right)=A_0\left(1-\frac{1}{\sum a^2 l^2}\right)$. Where $l=\frac{x_1+x_2+x_3}{a^2}$ etc. which is obvious from (2)

$$=A_0\left\{1-\frac{1}{\sum a^2\left(\frac{x_1+x_2+x_3}{a^2}\right)}\right\}=A_0\left\{1-\frac{1}{\sum a^2\left(\frac{a^2}{a^4}\right)}\right\}$$

Since $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$, etc. and $\frac{x_2 x_3}{a^2}+\frac{y_2 y_3}{b^2}+\frac{z_2 z_2}{c^2}=0$ etc.

$=A_0\left(1-\frac{1}{3}\right)=\frac{2}{3}A_0$.

Hence the result.

Example15:- Find the locus of the centres of sections of the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ which are of constant area πk^2.

Solution:- Let (α, β, γ) be the centre of the section of the paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \qquad (1)$$

By the plane $lx+my+nz=p$ say, (2)

The equation of the plane section of (1) through (α, β, γ) is

$$\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2}=z-\gamma$$

i.e. $\dfrac{\alpha}{a^2}x+\dfrac{\beta}{b^2}y-z=\dfrac{\alpha^2}{a^2}+\dfrac{\beta^2}{b^2}-\gamma$

Comparing this equations with (2), we have

$$\frac{\alpha/a^2}{l}=\frac{\beta/b^2}{m}=\frac{-1}{n}=\frac{\alpha^2/a^2+\beta^2/b^2-\gamma}{p}$$

Therefore, $\dfrac{l}{n}=-\dfrac{\alpha}{a^2},\ \dfrac{m}{n}=-\dfrac{\beta}{b^2},\ \dfrac{p}{n}=-\left(\dfrac{\alpha^2}{a^2}+\dfrac{\beta^2}{b^2}-\gamma\right)$ (3)

The area of the plane section is given to be πk^2. Hence

$$\frac{\pi}{n^3}\left(a^2l^2+b^2m^2+2np\right)ab\sqrt{\left(l^2+m^2+n^2\right)}=\pi k^2$$

i.e. $ab\left\{a^2\left(\dfrac{l}{n}\right)^2+b^2\left(\dfrac{m}{n}\right)^2+\dfrac{2p}{n}\right\}\sqrt{\left\{\left(\dfrac{l}{n}\right)^2+\left(\dfrac{m}{n}\right)^2+1\right\}}=k^2$

squaring both sides and using and values from (3), the gives

i.e. $a^2b^2\left\{a^2\left(-\dfrac{\alpha}{a^2}\right)^2+b^2\left(-\dfrac{\beta}{b^2}\right)^2-2\left(\dfrac{\alpha^2}{a^2}+\dfrac{\beta^2}{b}-\gamma\right)\right\}^2$

$$\left\{\left(-\frac{\alpha}{a^2}\right)^2+\left(-\frac{\beta}{b^2}\right)^2+1\right\}=k^4$$

i.e. $a^2b^2\left(\dfrac{\alpha^2}{a^2}+\dfrac{\beta^2}{b^2}-2\gamma\right)^2\left(\dfrac{\alpha^2}{a^4}+\dfrac{\beta^2}{b^4}+1\right)=k^2$

Hence the locus of the centre (α,β,γ) is

$$a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2z\right)^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+1\right)=k^4$$

9. PLANE SECTION OF CONICOIDS

9.1 Parallel Plane Sections:- To show that the parallel plane sections of the surface represented by the general equation of the second degree are similar and similarly situated conics.

First we will see that the plane sections of the surface represented by the general equation of the second degree are conics. We take the co-ordinate axes in such a way that the plane section becomes one of the co-ordinate planes, say $z=0$.

If the equation of the surface after the transformation is of the form $ax^2+by^2+cz^2+2fyz+2gzx+2hku+2ux+2vy+2wz+d=0$, the equations of the section are $ax^2+2hxy+by^2+2ux+2vy+d=0$, $z=0$, which clearly represent a conic.

Now we will see that the parallel plane sections are similar and similarly situated conics. For this, we take the co-ordinate plane $z=0$ parallel to the system of parallel plane sections. Then the equations of the section by the plane $z=k$ are

$z=k,\ ax^2+2hky+2by^2+2(gk+u)x+2(fk+v)y+(ck^2+2wk+d)=0$

For different values of k, the sections are therefore similar conics, with their axes in the same directions.

9.2 Nature of The Plane Sections of A Central Conicoid:- To determine the nature of the sections of the central conicoid $ax^2+by^2+cz^2=1$

(1) by the plane $lx+my+nz=p$.

It is sufficient to examine the sections of (1) by the parallel plane $lx+my+nz=0$ (2) as we know that parallel plane sections are similar.

The equation of the projection of the section on the plane $z=0$ is obtained by eliminating z between their equations. Thus eliminating between (1) and (2), we have

$$ax^2+by^2+c\left(-\frac{lx+my}{n}\right)^2=1,$$

i.e. $(an^2+cl^2)x^2+2clmxy\,.\,(bn^2+cm^2)y^2=n^2$.

Hence the projection and so also the given section is an ellipse parabola or hyperbola according as $c^2l^2m^2M <,=,$ or $>(an^2+cl^2)(bn^2+cm^2)$, i.e. $0<,=,$ or $>n^2(bcl^2+cam^2+abn^2)$, i.e. $bcl^2+cam^2+abn^2 <,=,$ or >0.

9.3 Nature of The Plane Sections of A Paraboloid:- To determine the nature of the sections of the paraboloid $ax^2+by^2=2cz$

(1) by the plane $lx+my+nz=p$

(2)

As in the preceding section, it is sufficient to consider the parallel plane $lx+my+nz=0$

It is assumed here that $n\neq 0$ so that the plane of the section is not perpendicular to the $xy-$ plane .

Eliminating z between (1) and (2), the equation of the projection of the section on the plane $z=0$, is given by the equation $ax^2+by^2=2c\left(-\frac{lx+my}{n}\right)$, i.e. $n\left(ax^2+by^2\right)+2c\left(lx+my\right)=0$.

This is an ellipse, parabola or hyperbola according as $na.nb>,=,\ or<0$ i.e. $abn^2>,=,or<0$.

Since none of the numbers a,b,n is zero, the section cannot be a parabola. It is an ellipse or hyperbola, according as a and b are of the same or opposite signs.

Thus the section of the paraboloid by a plane which is not perpendicular to the xy-plane (i.e. which is not parallel to the axis of the paraboloid) is an ellipse if the paraboloid is elliptic, and a hyperbola if the paraboloid is hyperbolic.

Lastly, to determine the nature of the section by a plane perpendicular to the xy-plane i.e. the plane given by $lx+my=0$

(3)

We consider the projection of the section on the yz- or zx-plane.

Eliminating x between (1) and (3), the equation of the projection of the section on the $yz-$ plane is $a\left(-\frac{m}{l}y\right)^2+by^2=2cz$, i.e. $\left(am^2+bl^2\right)y^2=2cl^2z$, which is clearly a parabola.

Hence the section of the paraboloid by a plane parallel to its axis is a parabola.

9.4 Axes of A Central Section of The Central Conicoid:- The discuss the lengths and direction cosines of the axes and the area of the central section of the central conicoid

$$ax^2+by^2+cz^2=1 \qquad (1)$$

by the plane $lx+my+nz=0$ (2)

Clearly, the centre of the conicoid (i.e. the origin) is also the centre of the section.

If λ,μ,ν be the direction cosines of a semi-dimater of length r of the conicoid, then the extremity of that semi-diameter is the point $\left(\lambda r,\mu r,\nu r\right)$.

Since this point lies on the conicoid (1), we have $a\left(\lambda r\right)^2+b\left(\mu r\right)^2+c\left(\nu r\right)^2=1$ i.e. $a\lambda^2+r^2+b\,\mu^2r^2+c\nu^2r^2=\lambda^2+\mu^2+\nu^2$, i.e. $\left(ar^2-1\right)\lambda^2+\left(br^2-1\right)\mu^2+\left(cr^2-1\right)z^2=0$.

Therefore, the semi-diameter $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ of length r will generate the cone

$$\left(ar^2-1\right)x^2+\left(br^2-1\right)y^2+\left(cr^2-1\right)z^2=0 \qquad (3)$$

The plane (2) will intersect the cone (3) along two lines which are the semi-diameters of length r of the conicoid. They are also the semi-diameters of the plane section. But we know that if two equal semi-diameters of a conic conicoid, they will conicoid on either axes of the conic. Thus the plane (2) will intersect the cone (3) along two coincident generators, i.e. it will touch the cone.

The condition of tangency gives $\frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0$

(4) which can be written as

$$\left(bcl^2+cam^2+abn^2\right)r^4-\left\{(b+c)l^2+(c+a)m^2+(a+b)n^2\right\}r^2+\left(l^2+m^2+n^2\right)=0$$

The roots of this quadratic in r^2 are the squares of the semi-axes of the section. The plane (2) touches the cone (3) along the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$.

But the equation of the tangent plane of the cone (3) along this line is $\left(ar^2-1\right)\lambda x+\left(br^2-1\right)\mu y+\left(cr^2-1\right)\nu z=0$ (5)

Therefore, this plane is identical with the plane (2). Hence

$$\frac{\left(ar^2-1\right)\lambda}{l}=\frac{\left(br^2-1\right)\mu}{m}=\frac{\left(cr^2-1\right)\nu}{n} \qquad (5)$$

These equations determine the direction cosines of the axes of length $2r$.

If r_1 and r_2 be the lengths of the semi-axes of the conic section r_1^2 and r_2^2 will be the roots of (4). So the product of roots.

$$r_1^2r_2^2=\frac{l^2+m^2+n^2}{bcl^2+cam^2+abn^2}.$$

Denoting the area of the section by A_0, we thus have $A_0=\pi r_1r_2$, i.e.

$$A_0=\frac{\pi\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{\left(bcl^2+cam^2+abn^2\right)}}.$$

Further, if p is the length of perpendicular from the origin on the tangent plane parallel to the given section, whose equation is $lx+my+nz=\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}$, we have

$$p=\frac{\sqrt{\left(l^2/a+m^2/b+n^2/c\right)}}{\sqrt{\left(l^2+m^2+n^2\right)}}=\frac{\sqrt{\left(bcl^2+cam^2+abn^2\right)}}{\sqrt{(abc)}\sqrt{\left(l^2+m^2+n^2\right)}}$$

Therefore, $A_0=\frac{\pi}{p\sqrt{(abc)}}$.

Corollary 1:- The section is a rectangular hyperbola if $r_1^2+r_2^2=0$ i.e. $(b+c)l^2+(c+a)m^2+(a+b)n^2=0$.

Corollary 2:- The axes of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $lx+my+nz=0$ are given by the equation $\frac{a^2l^2}{r^2-a^2}+\frac{b^2m^2}{r^2-b^2}+\frac{c^2n^2}{r^2-c^2}=0$ the direction cosines of the axes by the equations $\frac{\lambda\left(r^2-a^2\right)}{a^2l}=\frac{\mu\left(r^2-b^2\right)}{b^2m}=\frac{\nu\left(r^2-c^2\right)}{c^2n}$ and the area of the section is $A_0 = \dfrac{\pi abc\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}} = \dfrac{\pi abc}{p}$, where p is the length of the perpendicular from the centre on the parallel tangent plane.

Corollary 3:- The axes of the section of the conicoid $ax^2+by^2+cz^2=k$, by the plane $lx+my+nz=0$, are given by the equation $\dfrac{l^2}{ar^2-k}+\dfrac{m^2}{br^2-k}+\dfrac{n^2}{cr^2-k}=0$ and the direction cosines of the axes, by the equations $\dfrac{\lambda(ar^2-k)}{l}=\dfrac{\mu(br^2-k)}{m}=\dfrac{\nu(cr^2-k)}{n}$.

9.5 Axes of Any Section of The Central Conicoid:- To discuss the lengths and direction cosines of the axes, and the area of the section of the central conicoid
$ax^2+by^2+cz^2=1$ (1)
by the plane $lx+my+nz=p$ (2)
Here the centre of the section is not the centre of the conicoid, *viz*, the origin. In fact we have to find the centre of the section first.
If (α,β,γ) is this centre, the equation (2) is identical with the equation of the plane section:

$$a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)=0 \text{ i.e.}$$
$$a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2 \quad (3)$$

Comparing (2) and (3), we get $\dfrac{a\alpha}{l}=\dfrac{b\beta}{m}=\dfrac{c\gamma}{n}=\dfrac{a\alpha^2+b\beta^2+c\gamma^2}{p}$ (4)

Squaring each of the fractions, we have $\dfrac{a\alpha^2}{l^2/a}=\dfrac{b\beta^2}{m^2/b}=\dfrac{c\gamma^2}{n^2/c}=\left(\dfrac{a\alpha^2+b\beta^2+c\gamma^2}{p}\right)^2$,

so that

$$\frac{a\alpha^2+b\beta^2+c\gamma^2}{l^2/a+m^2/b+c^2/n}=\frac{(a\alpha^2+b\beta^2+c\gamma^2)^2}{p^2} \quad \textbf{(note)}$$

This given $a\alpha^2+b\beta^2+c\gamma^2=\dfrac{p^2}{p_0^2}$,where $p_0^2=\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}$
(5)

Thus (4) assumes the form $\dfrac{a\alpha}{l}=\dfrac{b\beta}{m}=\dfrac{c\gamma}{n}=\dfrac{p}{p_0^2}$ (6)

i.e. $ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$ (7)

But, from (6), we have $\dfrac{a\alpha x+b\beta y+c\gamma z}{lx+my+nz}=\dfrac{p}{p_0^2}$
(8)
Using (5) and (8), the equation (7) becomes

$$ax^2+by^2+cz^2+2(p/p_0^2)(lx+my+nz)-k^2=0 \quad (9)$$

Where $k^2 = 1 - p^2 / p_0^2$.

Also the equation of the plane reduces to $lx + my + nz = 0$ (10)

Now the section of the conicoid (9) by the plane (10) is the same as the section of the surface $ax^2 + by^2 + cz^2 = k^2$, i.e. $(a/k)^2 x^2 + (b/k)^2 y^2 + (c/k^2) z^2 = 1$ by the same. Thus the problem reduces to that of the preceding article.

The semi-diameters of the section of length r lie on cone $(ar^2 - k^2)x^2 + (br^2 - k^2)y^2 + (cr^2 - k^2)z^2 = 0$, and the lengths of the semi-axes are given by $\frac{l^2}{ar^2 - k^2} + \frac{m^2}{br^2 - k^2} + \frac{n^2}{cr^2 - k^2} = 0$.

Also, the direction cosines of the axes are given by $\frac{(ar^2 - k^2)\lambda}{l} = \frac{(br^2 - k^2)\mu}{m} = \frac{(cr^2 - k^2)\nu}{n}$.

It is clear from the above result that if r_1 and r_2 are the lengths of the semi-axes of the section of the conicoid (1) by the plane (10), then the semi-axes of the section by the plane (2) are kr_1 and kr_2, where $k = \sqrt{(1 - p^2 / p_0^2)}$. Also the corresponding axes are parallel.

Then area A of the section is given by $A = \pi k^2 r_1 r_2 = A_0 k^2$, where A_0 is the area of the parallel plane section through the centre.

9.6 Axes of A Section of The Paraboloid:- The discuss the lengths and direction cosines of the axes, and the area of the section of the paraboloid $ax^2 + by^2 = 2z$ (1) by the plane $lx + my + nz = p$ (2)

If (α, β, γ) is the centre of the section, the plane is also represented by the equation

$$a\alpha(x - \alpha) + b\beta(y - \beta) = z - \gamma,$$

i.e. $$a\alpha x + b\beta y - z = a\alpha^2 + b\beta^2 - \gamma$$

comparing this equation with (2), we get $\frac{a\alpha}{l} + \frac{b\beta}{m} = \frac{-1}{n} = \frac{a\alpha^2 + b\beta^2 - \gamma}{p} = \frac{a\alpha^2 + b\beta^2 - 2\gamma}{2p - (l\alpha + m\beta)}$

(note)

Therefore $\alpha = -\frac{l}{an}$, $\beta = -\frac{m}{bn}$ and so $\gamma = a\alpha^2 + b\beta^2 + \frac{p}{n} = a \cdot \frac{l^2}{a^2 n^2} + b \cdot \frac{m^2}{b^2 n^2} + \frac{p}{n}$,

$= \frac{1}{n^2}\left(\frac{l^2}{a} + \frac{m^2}{b} + np\right) = \frac{k}{n^2}$, where $k = \frac{l^2}{a} + \frac{m^2}{b} + np$.

Transferring the origin at the centre (α, β, γ), the equation to the given paraboloid becomes $a(x + \alpha)^2 + b(y - \beta)^2 = 2(z + \gamma)$, i.e.

$ax^2 + by^2 + 2(a\alpha x + b\beta y - z) + (a\alpha^2 + b\beta^2 - 2\gamma) = 0$, i.e.

$$ax^2+by^2-\frac{2}{n}(lx+my+nz)-\frac{p_0^2}{n^2}=0 \qquad (3)$$

Where $p_0^2=np+k=\frac{l^2}{a}+\frac{m^2}{b}+2np$.

Also, the equation of the plane section reduces to $lx+my+nz=0$ (4)

Now the section of the paraboloid (3) by the plane (4) is the same as the section of the surface $ax^2+by-p_0^2/n^2=0$

(5) by the plane.

If λ,μ,ν be the direction cosines of the semi-diameter of length r of the section, the plane $(\lambda r,\mu r,\nu r)$ will lie on the surface (5). Hence $a\lambda^2r^2+b\mu^2r^2,-\frac{p_0^2}{n^2}(\lambda^2,\mu^2,n^2)=0$ **(note)**

i.e. $(an^2r^2-p_0^2)\lambda^2+(bn^2r^2-p_0^2)\mu^2-p_0^2\nu^2=0$.

Therefore, the semi-diameter of length r will lie on the cone.

$$(an^2r^2-p_0^2)x^2+(bn^2r^2-p_0^2)y^2-p_0^2z^2=0 \qquad (6)$$

Clearly, the plane (4) will intersect this cone along two lines which are the semi-diameters of length r of the section. Consequently, if r is the length of either semi-axes of the section, the plane (4) will touch the cone (6) along the semi-axes. Hence the lengths of the semi-axes are given by the equation $\frac{l^2}{an^2r^2-p_0^2}+\frac{m^2}{bn^2r^2-p_0^2}+\frac{n^2}{-p_0^2}=0$, i.e.

$$abn^6r^4-n^2p_0^2\left\{(a+b)n^2+am^2+bl^2\right\}r^2+p_0^4(l^2+m^2+n^2)=0 \qquad (7)$$

The cone represented by (6) touches the plane section along the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$.

Therefore the equation of the section will be of the form $\lambda(an^2r^2-p_0^2)x+\mu(bn^2r^2-p_0^2)y-\nu p_0^2z=0$.

Comparing this equation with (4), have $\frac{\lambda(an^2r^2-p_0^2)}{l}=\frac{\mu(bn^2r^2-p_0^2)}{m}=\frac{-\nu p_0^2}{n}$.

These equation will determine the direction cosines of the axes of the section of length $2r$.

Lastly, denoting the area of the section by A, we have $A=\pi r_1r_2=\frac{\pi p_0^2}{n^3}\sqrt{\left(\frac{l^2+m^2+n^2}{ab}\right)}$, since using (7), $r_1^2r_2^2=\frac{p_0^4(l^2+m^2+n^2)}{abn^6}$, i.e.

$$A=\frac{\pi}{n^3}\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right)\sqrt{\left(\frac{l^2+m^2+n^2}{ab}\right)}.$$

Corollary:- The section is a rectangular hyperbola if $r_1^2+r_2^2=0$ i.e. $(a+b)n^2+am^2+bl^2=0$.

9.7 Circular Sections:- Suppose the equation $F=0$ of a conicoid can be written as $S+\lambda uv=0$, where $S=0$ represents a sphere and $u=0$, $v=0$ the planes. Then the common point of $F=0$ and $u=0$ or $v=0$ lie on the sphere $S=0$ i.e. the sections of the conicoid by the planes $u=0, v=0$ are circle. Such section are known as circular sections.

To find the circular sections of the conicoid

$$\phi(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

We write this equation in the form

$$\phi(x,y,z)-\lambda\left(x^2+y^2+z^2\right)+\lambda\left(x^2+y^2+z^2-1/\lambda\right)=0$$

(note)

Hence, if the equation $\phi(x,y,z)-\lambda\left(x^2+y^2+z^2\right)=0$, i.e.

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy=0 \tag{1}$$

represents a pair of planes, these planes will give the required circular sections.

Now the condition that (1) represents a pair of planes is $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$.

This equation being a cubic in λ determines there values of λ, for which (1) represents a pair of planes. It will be shown in chapter 11 that the values of λ obtained are all real, but only the mean value of λ gives real planes.

Example1:- Plane are drawn through a fixed point (α,β,γ) so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone $\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)}{a+b}=0$

Solution:- Any plane through (α,β,γ) is

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0 \qquad (1)$$

If r is the length of either semi-axis of the section of $ax^2+by^2=2z$ by (1), then we have

$$abn^6r^4-n^2p_0^2r^2\left\{(a+b)n^2+am^2+bl^2\right\}+p_0^4\left(l^2+m^2+n^2\right)=0 \qquad (2)$$

Where $p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+2n(l\alpha+m\beta+n\gamma)$

If r_1 and r_2 are the length of the semi-axis of the section, then the section will be a rectangular hyperbola if

$$r_1^2+r_2^2=0$$

i.e. $(a+b)n^2+am^2+bl^2=0$ using (2)

Also, the normal to (1) through (α,β,γ) is given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

The normal will generate the cone

$$(a+b)(z-\gamma)^2+a(y-\beta)^2+b(x-\alpha)^2=0$$

Hence the plane (1) will touch the reciprocal cone

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0$$

Example2:- Find the real circular section of the surface $4x^2+2y^2+z^2+3yz+zx=1$

Solution:- Writing the equation of the surface in the form

$$\left\{4x^2+2y^2+z^2+3yz+zx-\lambda\left(x^2+y^2+z^2\right)\right\}+\lambda\left(x^2+y^2+z^2\right)-1=0$$

We find that the equation

$$4x^2+2y^2+z^2+3y+zx-\lambda\left(x^2+y^2+z^2\right)=0$$

i.e. $(4-\lambda)x^2+(2-\lambda)y^2+(1-\lambda)z^2+3y+zx=0$ represents of pair of planes if

$$\begin{vmatrix}4-\lambda & 0 & 1/2\\ 0 & 2-\lambda & 3/2\\ 1/2 & 3/2 & 1-\lambda\end{vmatrix}=0$$

i.e. $2\lambda^2-14\lambda^2+23\lambda+3=0$ on simplifying

i.e. $(\lambda-3)\left(2\lambda^2-8\lambda-1\right)=0$

This gives $\lambda=3,\frac{1}{2}\left(4\pm3\sqrt{2}\right)$

Since only the mean value of λ gives real planes, we have, corresponding to $\lambda = 3$.

$$4x^2 + 2y^2 + z^2 + 3yz + zx - 3\left(x^2 + y^2 + z^2\right) = x^2 - y^2 - 2z^2 + 3xy + zx$$

$$= (x+y)(x-y) + \{2z(x+y) - z(x-y)\} - 2z^2 \qquad \textbf{(note)}$$

$$= (x+y)(x-y+2z) - z(x-y+2z)$$

$$= (x+y-z)(x-y+2z)$$

Hence the real circular sections of the surface are given by the planes parallel to $x + y - z = 0$ and $x - y + 2z = 0$

Example3:- Find the equations to the sections of the circular conicoid

$$yz\left(\frac{b}{c}+\frac{c}{a}\right) + zx\left(\frac{c}{a}+\frac{a}{c}\right) + xy\left(\frac{a}{b}+\frac{b}{a}\right) + 1 = 0$$

Solution:- The equation to the given conicoid may be written as

$$yz\left(\frac{b}{c}+\frac{c}{b}\right) + zx\left(\frac{c}{a}+\frac{a}{c}\right) + xy\left(\frac{a}{b}+\frac{b}{a}\right) + \lambda\left\{\left(x^2+y^2+z^2\right) - \lambda\left(x^2+y^2+z^2-\frac{1}{\lambda}\right)\right\} = 0 \qquad (1)$$

If $yz\left(\frac{b}{c}+\frac{c}{b}\right) + zx\left(\frac{c}{a}+\frac{c}{a}\right) + xy\left(\frac{a}{b}+\frac{b}{a}\right) + \lambda\left(x^2+y^2+z^2\right) = 0$ (2)

Represents a pair of planes, then they will cut the conicoid in circles.
Now (2) will represent a pair of planes if

$$\begin{vmatrix} \lambda & \dfrac{a^2+b^2}{2ab} & \dfrac{c^2+a^2}{2ca} \\ \dfrac{a^2+b^2}{2ab} & \lambda & \dfrac{b^2+c^2}{2ca} \\ \dfrac{c^2+a^2}{2ca} & \dfrac{b^2+c^2}{2bc} & \lambda \end{vmatrix} = 0$$

Or $\lambda^2 + \dfrac{2\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}{8a^2b^2c^2} - \lambda\left\{\dfrac{\left(b^2+c^2\right)^2}{4b^2c^2} + \dfrac{\left(c^2+a^2\right)^2}{4c^2a^2} + \dfrac{\left(a^2+b^2\right)^2}{4a^2b^2}\right\} = 0$

Or $\lambda^2 - \dfrac{\lambda}{4}\left\{\dfrac{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) + 4a^2b^2c^2}{a^2b^2c^2}\right\} + \dfrac{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}{4a^2b^2c^2} = 0$

(3)

This equation is satisfied if $\lambda = 1$
Hence putting $\lambda = 1$ in (2), we hace

$$\left(x^2+y^2+z^2\right) + yz\left(\frac{b}{c}+\frac{c}{a}\right) + zx\left(\frac{c}{a}+\frac{a}{c}\right) + xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$$

Or $\left(\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\right)(ax+by+cz) = 0$

This represents a pair of circular sections.

Hence any circular section being parallel to central circular section are $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\lambda_1$ and $ax+by+cz=\lambda_2$.

Example4:- Prove that the radius of the circle in which the plane $\frac{x}{a}\sqrt{\left(a^2-b^2\right)}+\frac{z}{c}\sqrt{\left(b^2-c^2\right)}=\lambda$ cuts the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $b\sqrt{\left(1-\frac{\lambda^2}{a^2-c^2}\right)}$

Solution:- We know that the radius of the parallel central circular section $\frac{x}{a}\sqrt{\left(a^2-b^2\right)}+\frac{z}{c}\sqrt{\left(b^2-c^2\right)}=0$ is b. Therefore, (by section 9.5), the radius of the given circular section is kb, where $k^2=1-p^2/p_0^2$

Here $p=\lambda$ and $p_0^2=a^2l^2+b^2m^2+c^2n^2$

$$=a^2\left(\frac{a^2-b^2}{a^2}\right)+b^2.0+c^2\left(\frac{b^2-c^2}{c^2}\right)=a^2-c^2$$

Therefore, the required radius of the circular section

$$=b\sqrt{\left(1-\frac{p^2}{p_0^2}\right)}=b\sqrt{\left(1-\frac{\lambda^2}{a^2-c^2}\right)}$$

Example5:- Prove that the sections of hyperboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ by the plane $\frac{x}{a}\sqrt{\left(a^2-b^2\right)}+\frac{z}{c}\sqrt{\left(b^2-c^2\right)}=\lambda$ is real if $\lambda^2>a^2+c^2$

Solution;- Equation the a sphere through the section of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (1)$$

By the plane $\frac{x}{a}\sqrt{\left(a^2+b^2\right)}+\frac{z}{c}\sqrt{\left(b^2-c^2\right)}=\lambda$ (2)

Is given by $b^2\left(-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1\right)+\left[\frac{x}{a}\sqrt{\left(a^2+b^2\right)}-\frac{z}{c}\sqrt{\left(b^2-c^2\right)}\right]$

$$\times\left[\frac{x}{a}\sqrt{\left(a^2+b^2\right)}+\frac{z}{c}\sqrt{\left(b^2-c^2\right)}-\lambda\right]=0 \qquad (3)$$

The centre C of the sphere (1) is

$$\left(\frac{\lambda\sqrt{\left(a^2+b^2\right)}}{2a},0,\frac{-\lambda\sqrt{\left(b^2-c^2\right)}}{2c}\right)$$

Radius of the spere (3), is given by

$$R^2=\frac{\lambda^2\left(a^2+b^2\right)}{4a^2}+\frac{\lambda^2\left(b^2-c^2\right)}{4c^2}-b^2$$

i.e. $$R^2 = \frac{\lambda^2 b^2 \left(c^2 + a^2\right)}{4a^2c^2} - b^2$$

Again, if p be the distance of the section from the centre of the sphere, then

$p =$ Perpendicular from the centre C on (2)

$$\frac{\dfrac{\lambda\left(a^2+b^2\right)}{2a^2} - \dfrac{\lambda\left(b^2-c^2\right)}{2c^2} - \lambda}{\sqrt{\left\{\left(\dfrac{a^2+b^2}{a^2} + \dfrac{b^2-c^2}{c^2}\right)\right\}}} = \frac{\lambda b^2\left(c^2-a^2\right)}{2ca\sqrt{\left\{b^2\left(a^2+c^2\right)\right\}}}$$

Also, r the radius of the circular section is given by

$$r^2 = R^2 - p^2$$

So, the circular section will be real if r^2 is positive, i.e.

$$R^2 > p^2$$

i.e. $$\frac{\lambda^2 b^2\left(a^2+c^2\right)}{4a^2c^2} - b^2 > \frac{\lambda^2 b^2\left(c^2-a^2\right)}{4c^2a^2\left(a^2+c^2\right)}$$

i.e. $$\frac{\lambda^2 b^2}{4a^2c^2\left(a^2+c^2\right)}\left\{\left(a^2+c^2\right)^2 - \left(c^2-a^2\right)\right\} > b^2$$

i.e. $$\frac{\lambda^2 b^2}{a^2+c^2} > b^2$$

i.e. $\lambda^2 > a^2 + c^2$, since $b \neq 0$

Example6:- Show that the circular section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passing through are extremity of the x-axis are both of radius r, where $\frac{r^2}{b^2} = \frac{b^2-c^2}{a^2-c^2}$.

Solution:- We know that the real circular sections of the ellipsoid are

$$\frac{x}{a}\sqrt{\left(a^2-b^2\right)} + \frac{z}{c}\sqrt{\left(b^2-c^2\right)} = \lambda_1 \qquad (1)$$

And $$\frac{x}{a}\sqrt{\left(a^2-b^2\right)} - \frac{z}{c}\sqrt{\left(b^2-c^2\right)} = \lambda_2 \qquad (2)$$

Where $a > b > c$

The radius r of the circle in which (1) cuts the ellipsoid is given by

$$r = b\sqrt{\left\{\left(1 - \frac{\lambda_1^2}{a^2-c^2}\right)\right\}} \qquad (3)$$

Similarly, the radius of the circle in which (2) cuts the ellipsoid is $b\sqrt{\left\{\left(1-\frac{\lambda_2^2}{a^2-c^2}\right)\right\}}$

Now as the plane (1) passes through the point $(a,0,0)$, one extremity of the x-axis

$$\lambda_1 = \sqrt{\left(a^2-b^2\right)}$$

So, $r^2 = b^2\left(1-\dfrac{a^2-b^2}{a^2-c^2}\right)$

Substituting for λ_1 in (3)

$$\frac{r^2}{b^2} = \frac{b^2-c^2}{a^2-c^2}$$

Again if the plane (2) also passes through $(a,0,0)$, we have

$$\lambda_1 = \sqrt{\left(a^2-b^2\right)} \text{ and } \frac{r^2}{b^2} = \frac{b^2-c^2}{a^2-c^2} \text{ as above.}$$

Example7:- Find the circular sections of the paraboloid $13y^2+4z^2=2x$

Solution:- Equation of the paraboloid can be written as $4\left(x^2+y^2+z^2-\dfrac{x}{2}\right)+\left(9y^2-4x^2\right)=0$

Hence $9y^2-4x^2=0$ represents a pair of real planes which meet the given paraboloid in circles and the system of real circular section of the surface are parallel to $9y^2-4x^2=0$ or $2x \pm 3y = 0$.

i.e. they are given by $2x \pm 3y = \lambda$

Example8:- Prove that the umbilics of the conicoid $\dfrac{x^2}{a+b}+\dfrac{y^2}{a}+\dfrac{z^2}{a-b}=1$ are the extremities of the equal conjugate diameters of the ellipse $y=0$, $\dfrac{x^2}{a+b}+\dfrac{z^2}{a-b}=1$

Solution:- The given equation of conicoid can be written as

$$\frac{1}{a}\left(x^2+y^2+z^2-a\right)-\left\{\left(\frac{1}{a}-\frac{1}{a+b}\right)x^2+\left(\frac{1}{a}-\frac{1}{a-b}\right)z^2\right\}=0$$

i.e. $$\frac{1}{a}\left(x^2+y^2+z^2-a\right)-\frac{b}{a}\left(\frac{x^2}{a+b}-\frac{z^2}{a-b}\right)=0$$

If $a>b>0$, the real central circular sections are $\dfrac{x^2}{a+b}-\dfrac{z^2}{a-b}=0$, i.e.

$$\frac{x}{\sqrt{(a+b)}} \pm \frac{z}{\sqrt{(a-b)}} = 0 \qquad (1)$$

Let $P(\xi,\eta,\zeta)$ be an umbilic. Then the equation of tangent plane at P is

$$\frac{\xi x}{a+b}+\frac{\eta y}{a}+\frac{\zeta z}{a-b}=1$$

This plane is parallel to (1), so we have

$$\frac{\xi/(a+b)}{1/\sqrt{(a+b)}} = \frac{\eta/a}{0} = \frac{\zeta/(a-b)}{\pm 1/\sqrt{(a-b)}}$$

i.e. $$\frac{\xi}{\sqrt{(a+b)}} = \frac{\eta}{0} = \frac{\zeta}{\sqrt{(a-b)}} = r \text{, say.}$$

So that $\xi = r\sqrt{(a+b)}$, $\eta = 0$, $\zeta = \pm r\sqrt{(a-b)}$ (2)

Since the umbilic (ξ,η,ζ) lies on the given conicoid, we have

$$\frac{\left\{r\sqrt{(a+b)}\right\}^2}{a+b} + \frac{0}{a} + \frac{\left\{\pm r\sqrt{(a-b)}\right\}^2}{a-b} = 1 \text{ i.e. } 2r^2 = 1 \text{ i.e. } r = \pm\frac{1}{\sqrt{2}}$$

Hence from (2), we obtained the required umbilics:

$$\xi = \pm\sqrt{\left\{\frac{1}{2}(a+b)\right\}},\ \eta = 0,\quad \zeta = \pm\sqrt{\left\{\frac{1}{2}(a-b)\right\}}$$

Evidently, these umbilics lie on the ellipse

$$y = 0,\ \frac{x^2}{a+b} + \frac{y^2}{a-b} = 1$$

As these equation are satisfied by them.

Further, $\xi^2 + \eta^2 = \frac{1}{2}\left\{(a+b)+(a-b)\right\}$, so that the distance of the umbilic from the centre of the ellipse is equal to half the sum of the squares of its semi-axes.

Hence the umbilics are extremities of the equi-conjugate diameters of the above ellipse.

Example9:- Prove that the distance of the tangent plane at an umbilic of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ from the centre is $\frac{ac}{b}$

Solution:- Let (ξ,η,ζ) be an umbilic of the given ellipsoid. Then the equation of the tangent plane at the umbilic is $\frac{\xi x}{a^2} + \frac{\eta y}{b^2} + \frac{\zeta z}{c^2} = 1$

This tangent plane must b e parallel to the central circular section:

$$\sqrt{(a^2-b^2)}\frac{x}{a} \pm \sqrt{(b^2-c^2)}\frac{z}{c} = 0$$

Therefore, $$\frac{\xi/a^2}{\sqrt{(a^2-b^2)}/a} = \frac{\eta/b^2}{0} = \frac{\zeta/c^2}{\pm\sqrt{(b^2-c^2)}/c}$$

i.e. $$\frac{\xi/a}{\sqrt{(a^2-b^2)}} = \frac{\eta/b}{0} = \frac{\zeta/c}{\pm\sqrt{(b^2-c^2)}}$$

$$= \frac{\sqrt{(\xi^2/a^2 + \eta^2/b^2 + \zeta^2/c^2)}}{\pm\sqrt{\left\{(a^2-b^2)+(b^2-c^2)\right\}}} = \frac{1}{\pm\sqrt{(a^2-c^2)}}$$ as the umbilic (ξ,η,ζ) lies on the ellipsoid.

Hence $$\xi = \pm\frac{a\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}},\ \eta = 0,\ \zeta = \pm\frac{c\sqrt{(b^2-c^2)}}{\sqrt{(a^2-b^2)}}$$

Now the distance of the tangent plane at the umbilic from the centre of the ellipsoid

$$=\frac{1}{\sqrt{\left(\frac{\xi^2}{a^4}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\right)}}=\frac{1}{\sqrt{\left\{\frac{a^2-b^2}{a^2\left(a^2-c^2\right)}+0+\frac{b^2-c^2}{c^2\left(a^2-c^2\right)}\right\}}}$$

$$=\sqrt{\left\{\frac{a^2c^2\left(a^2-c^2\right)}{c^2\left(a^2-b^2\right)+a^2\left(b^2-c^2\right)}\right\}}=\sqrt{\left\{\frac{a^2c^2\left(a^2-c^2\right)}{b^2\left(a^2-c^2\right)}\right\}}=\frac{ac}{b}$$

Example10:- Show that if there points of a straight line lie on a conicoid, then the straight line lies wholly on the conicoid.

Solution:- Let the equation of a straight line be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,$ say.

(1)

Let the equation of a conicoid of the second degree be

$$F(x,y,z)=0$$

The co-ordinates of any point on the line (1) are $(lr+\alpha, mr+\beta, nr+\gamma)$. If this point lies on the conicoid (1), then we shall have a quadratic equation in r, let it be of the form:

$$Ar^2+Br+C=0$$

Now if the three points of the line (1) lie on the conicoid (1), then the equation (3) is satisfied for three distinct of r. Hence the relation (3) should be an identity in r. It follows that

$$A=B=C=0$$

But them the equation (3) is satisfied for all values of r, showing that every point of the line (1) lies on the conicoid (2). Hence the straight line (1) lies wholly on the conicoid.

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in the last of this book.

CHAPTER 7. CONICOID- ELLIPSOID, HYPERBOLOID

Q2(c) If P, Q, R; P', Q', R' are feet of the six normals drawn from a point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the plane PQR is represented by $lx+my+nz=p$, show that the plane P'Q'R' is given by $\frac{x}{a^2 l}+\frac{y}{b^2 m}+\frac{z}{c^2 n}+\frac{1}{p}=0$. **UPSC CSE 2022**

Q1. Find the equations of the tangent plane to the ellipsoid $2x^2+6y^2+3z^2=27$ which passes through the line $x-y-z=0=x-y+2z-9$. **[(1e) UPSC CSE 2020]**

Q2. Let P be the vertex of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. If the section of this cone made by the plane $z=0$ is a rectangular hyperbola, then find the locus of P.

[4a 2020 IFoS]

Q3. Find the length of the normal chord through a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the *xy*-plane, then P lies on the cone $\frac{x^2}{a^6}\left(2c^2-a^2\right)+\frac{y^2}{b^6}\left(2c^2-b^2\right)+\frac{z^2}{c^4}=0$.

[(4b) UPSC CSE 2019]

Q4. Find the equations of the tangent planes to the ellipsoid $2x^2+6y^2+3z^2=27$ which pass through the line $x-y-z=0=x-y+2z-9$. **[(1d) 2018 IFoS]**

Q5. Find the locus of the point of intersection of three mutually perpendicular tangent planes to $ax^2+by^2+cz^2=1$. **[(3d) UPSC CSE 2017]**

Q6. Find the locus of the point of intersection of three mutually perpendicular tangent planes to the coincoid $ax^2+by^2+cz^2=1$. **[(4d) UPSC CSE 2016]**

Q7. Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2+by^2+cz^2=1$ at, its points of intersection with the plane $lx+my+nz=p$ generate the cone

$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$. **[(4b) UPSC CSE 2014]**

Q8. Find the equations to the tangent planes to the surface $7x^2-3y^2-z^2+21=0$, which pass through the line $7x-6y+9=0, z=3$. **[(3d) 2013 IFoS]**

Q9. A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2+4y^2+9z^2=36$. Find its equation. **[(4c) 2012 IFoS]**

Q10. Find the equations of the tangent plane to the ellipsoid $2x^2+6y^2+3z^2=27$ which passes through the line $x-y-z=0=x-y+2z-9$. **[(2c) 2012 IFoS]**

Q11. Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere.

[(4a) UPSC CSE 2011]

Q12. Find the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$. **[(4b) 2011 IFoS]**

Q13. Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$. **[(4d) 2010 IFoS]**

Q14. If the feet of three normals drawn from a point P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in the lane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, prove that the feet of the other three normals lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1=0$.

[(4b) 2010 IFoS]

PARABOLOID

Q1, Prove that, in general, three normals can be drawn from a given point to the parabholoid $x^2+y^2=2az$, but if the point lies on the surface $27a\left(x^2+y^2\right)+8\left(a-z\right)^3=0$ then two of the three normals coincide. **[(3b) UPSC CSE 2019]**

Q2. Find the equation of the tangent plane at point $\left(1,1,1\right)$ to the conicoid $3x^2-y^2=2z$.

[(1d) UPSC CSE 2017]

Q3. Two perpendicular tangent planes to the paraboloid $x^2+y^2=2z$ intersect in a straight line in the plane $x=0$. Obtain the curve to which this straight line touches. **[(4c) UPSC CSE 2015]**

Q4. Show that the locus of a point from which the three mutually perpendicular tangent lines can be drawn to the paraboloid $x^2+y^2+2z=0$ is $x^2+y^2+4z=1$. **[(4c) UPSC CSE 2012]**

Q5. Tangent planes to two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid.

[(4d) 2011 IFoS]

Q6. Show that the plane $3x+4y+7z+\frac{5}{2}=0$ touches the paraboloid $3x^2+4y^2=10z$ and find the point of contact. **[(2c) UPSC CSE 2010]**

10. GENERATING LINES

10.1 Ruled Surfaces:-A surface which is generated by a variable line is called a ruled surface and the generating lines are called its generators.
Alternatively, a ruled may be defined as one through every points of which a straight line can be drawn so as to lie completely on it.
The ruled surfaces may be classified as (i) ***developable surface***, and (ii) ***Skew surfaces***. A developable surface is one on which the consecutive generators intersect, while on a skew surface, these generators do not intersect.
The cone is a developable surface as all its generators pass through a common vertex. The cylinder is also a surface of this kind as consecutive generators touch all along their length. In this chapter we shall observe that the hyperboloid of one sheet and the hyperbolic paraboloid are skew surfaces.

10.2 Conditions That A Line Be A Generator of AConicoid:- To find conditions that the straight line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ (1)

is a generator of the conicoid $ax^2+by^2+cz^2=1$ (2)

the co-ordinates of any point on the line (1) are $(lr+\alpha, mr+\beta, nr+\gamma)$. If this point lies on the conicoid (2), then we must have $a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1$,

i.e. $(al^2+bm^2+cn^2)r^2+2(al\alpha+bm\beta+cn\gamma)r+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$ (3)

Now the line (1) will be a generator of the conicoid (2) if this line lies wholly on (2). For this, equation (3) should be an identity in r, the condition for which are given by

$$a\alpha^2+b\beta^2+c\gamma^2=1 \quad (4)$$
$$al\alpha+bm\beta+cn\gamma=0 \quad (5)$$
And $$al^2+bm^2+cn^2=0 \quad (6)$$

The following facts can be observed now from these conditions.
The condition (4) states that the points (α,β,γ) on the line should also lie on the conicoid (1).
The condition (5) shows that the generating line wit h$d.c's\ l,m,n$ lies on the plane $a\alpha x+b\beta y+cyz=1$ which is the equation of the tangent plane at the point (α,β,γ) on the conicoid (2).
The condition (6) shows that the lines parallel to the generating lines, passing through the centre$(0,0,0)$ and having their $d.c's$ as l,m,n of the conicoid (2) i.e. the lines whose equations are $x/l=y/m=z/n$, generate the cone.

$$ax^2+by^2+cz^2=0$$

The cone given by this equations is called the asymptotic cone
Note:- The equations (4) and(5) give direction rates of the generating lines.

10.3 The Hyperboloid of One Sheet is A Ruled Surface:-To show that the hyperboloid of one sheet is a ruled surface, and to find out its generators.

The equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ of the hyperboloid of one sheet can be written as $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$ i.e. $\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right)$ (1)

Now tentatively consider the line whose equations as $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$,

$$\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right) \qquad (2)$$

where λ is any arbitrary non-zero constant. If we eliminate λ in these equations by multiplying them, we obtain (1). It follows that all those point which lie on the line (2) must also satisfy (1).

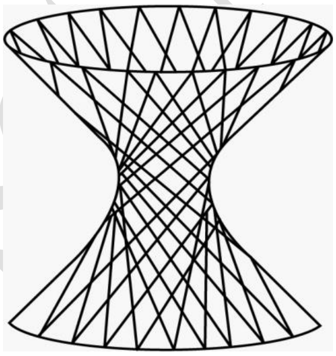
Thus the line (2) lies completely on the surface (1).

Exactly in the same manner we can show that the line whose equations are given by

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \qquad (3)$$

where μ is any arbitrary non-zero constant, also lies completely on the surface (1).

Assigning different values λ and μ we observe that the equations (2) and (3) represent to infinite families of lines lying on the hyperboloid of one sheet. These families are known as λ-system and μ-system of the lines. The two systems are shown in the following diagram.



Now if (α,β,γ) is any point on hyperboloid, then $\frac{\alpha^2}{a^2}-\frac{\gamma^2}{c^2}=1-\frac{\beta^2}{b^2}$, i.e.

$$\left(\frac{\alpha}{a}+\frac{\gamma}{c}\right)\left(\frac{\alpha}{a}-\frac{\gamma}{c}\right)-\left(1+\frac{\beta}{b}\right)\left(1-\frac{\beta}{b}\right),\ \text{i.e.}\ \frac{\frac{\alpha}{a}+\frac{\gamma}{c}}{1+\frac{\beta}{b}}=\frac{1-\frac{\beta}{b}}{\frac{\alpha}{a}-\frac{\gamma}{c}}=\lambda,\ \text{say} \qquad (4)$$

Then $\frac{\alpha}{a}+\frac{\gamma}{c}=\lambda,\left(1+\frac{\beta}{b}\right)$ and $\frac{\alpha}{a}-\frac{\gamma}{c}=\lambda,\left(1-\frac{\beta}{b}\right)$, which shows that the point (α,β,γ) lies on the line (2) for the value of λ given by (4). Thus every point of the hypeboloid lies on a line of λ-system

Similarly, we can show that every point of the hyperboloid lies on a line of μ-system.

Hence the hyperboloid of one sheet is a ruled surface and its generators are given by the system (2) and (3).

10.4 Properties of Generators of The Hyperboloid of One Sheet:-

Properties 1:- No two generators of the same system intersect.

Proof:- Let, it possible, two generators of the $\lambda-$ system given by $\frac{x}{a}+\frac{z}{c}=\lambda_1\left(1+\frac{y}{b}\right)$,

$$\left(\frac{x}{a}-\frac{z}{c}\right)=\lambda_1\left(1-\frac{y}{b}\right) \qquad (1)$$

And $\frac{x}{a}+\frac{z}{c}=\lambda_2\left(1+\frac{y}{b}\right)$, $\quad \left(\frac{x}{a}-\frac{z}{c}\right)=\lambda_2\left(1-\frac{y}{b}\right) \qquad (2)$

where $\lambda_1 \neq \lambda_2$ intersect. We shall prove that there is no solution of these two sets of equations of generators.

Comparing the first equation of each of (1) and (2), we get $\lambda_1=\left(1+\frac{y}{b}\right)=\lambda_2\left(1+\frac{y}{b}\right)$,

i.e. $(\lambda_1-\lambda_2)\left(1+\frac{y}{b}\right)=0$ and since $\lambda_1 \neq \lambda_2$, we find that $1+\frac{y}{b}=0$ i.e. $y=-b$.

Again comparing the second equation of each of (1) and (2), we get $\lambda_1\left(1-\frac{y}{b}\right)=\lambda_2\left(1-\frac{y}{b}\right)$, which implies that $y=b$.

Now from the equations $y=-b$ and $y=b$ obtained in this way, we see that $-b=b$ i.e. $b=0$. But this is absurd as b cannot be zero.

Hence the two generators given by (1) and (2) do not intersect.

Similarly, we can show that the two generators of $\mu-$ system do not intersect.

Property 2:- Through any point on the hyperboloid, passes, one and only one member of each system of generator.

Proof:-Let P be any point on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

The result that through P, at least one member of $\lambda-$ system and one member of $\mu-$ system passes follows from the fact that the lines of each one of these systems generate the hyperboloid.

Uniqueness follows from Property 1 above, for no two members of the same system intersect.

Property 3:- Any generator of $\lambda-$ system intersects a generator of $\mu-$ system, and vice-versa.

Proof:-Let $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$, $\frac{x}{a}-\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$ be any member of $\lambda-$ system. It will intersect a member, say $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$, $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$ of $\mu-$ system if there exists a point (x,y,z) which satisfies all the above four equations i.e. if we can solve them for x,y,z.

Equating the two values of $\frac{x}{a}+\frac{z}{c}$ from these equations, we get $\lambda\left(1+\frac{y}{b}\right)=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$,

i.e. $\left(\frac{1+\lambda\mu}{b}\right)=1-\lambda\mu$, which gives $y=\frac{b(1-\lambda\mu)}{1+\lambda\mu}$, provided $\lambda\mu=-1$ (1)

Further, $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)=\lambda\left(1+\frac{1-\lambda\mu}{1+\lambda\mu}\right)=\frac{2\lambda}{1+\lambda\mu}$ and $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)=\mu\left(1+\frac{1-\lambda\mu}{1+\lambda\mu}\right)=\frac{2\mu}{1+\lambda\mu}$

By addition of these two equations, we obtain $\frac{2x}{a}=\frac{2(\lambda+\mu)}{1+\lambda\mu}$, which gives $x=\frac{a(\lambda+\mu)}{1+\lambda\mu}$ (2)

Whereas by their subtraction, we get $\frac{2z}{c}=\frac{2(\lambda-\mu)}{1+\lambda\mu}$, which gives $z=\frac{c(\lambda-\mu)}{1+\lambda\mu}$ (3)

Thus the two generators do intersect, and from (2), (1) and (3), their point of intersection is $\left(\frac{a(\lambda+\mu)}{1+},\frac{b(1-\lambda\mu)}{1+\lambda\mu},\frac{c(\lambda-\mu)}{1+\lambda\mu}\right)$.

Note:- (i) The values of x,y,z obtained above without using the second equation of the first generator also satisfy that equation *viz,* $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$, for L.H.S $=\frac{\lambda+\mu}{1+\lambda\mu}-\frac{\lambda-\mu}{1+\lambda\mu}=\frac{2\mu}{1+\lambda\mu}$ and R.H.S. $=\frac{1}{\lambda}\left(1-\frac{1-\lambda\mu}{1+\lambda\mu}\right)=\frac{2\mu}{1+\lambda\mu}$.

(ii) The equation $x=\frac{a(\lambda+\mu)}{1+\lambda\mu},y=\frac{b(1-\lambda\mu)}{1+\lambda\mu}$, $z=\frac{c(\lambda-\mu)}{1+\lambda\mu}$ any be regarded as parametric equations of the paraboloid of one sheet.

Property 4:- Plane through one member of $\lambda-$system and one member of $\mu-$system is tangent plane to the hyperboloid at the point of intersection.

Proof:-Let $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$, $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$ (1)

And $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$, $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$ (2)

Be the members of two system intersecting at the point P given by $\left(\frac{a(\lambda+\mu)}{1+\lambda\mu},\frac{b(1-\lambda\mu)}{1+\lambda\mu},\frac{c(\lambda-\mu)}{1+\lambda\mu}\right)$.

Any plane through the line (1) is $\frac{x}{a}+\frac{z}{c}-\lambda\left(1+\frac{y}{b}\right)+k\left\{\frac{x}{a}-\frac{z}{c}-\frac{1}{\lambda}\left(1-\frac{y}{b}\right)\right\}=0$,

i.e. $(1+k)\frac{x}{a}+\left(-\lambda+\frac{k}{\lambda}\right)\frac{y}{b}+(1-k)\frac{z}{c}-\left(1+\frac{k}{\lambda}\right)=0$ (3)

Similarly, any plane through the line (2) is $\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)+k'\left\{\frac{x}{a}+\frac{z}{c}-\frac{1}{\mu}\left(1-\frac{y}{b}\right)\right\}=0$ i.e.

$(1+k')\frac{x}{a}+\left(-\mu+\frac{k'}{\mu}\right)\frac{y}{b}+(-1+k')\frac{z}{c}-\left(\mu+\frac{k'}{\mu}\right)=0$ (4)

Now (3) and (4) will represent the same plane if $\frac{1+k}{1+k'}=\frac{-\lambda+k/\lambda}{-\mu+k'/\mu}=\frac{1-k}{-1-k'}=\frac{\lambda+k/\lambda}{\mu+k'/\mu}$.

Whereas $\frac{1+k}{1+k'}=\frac{1-k}{-1+k'}$ implies that $kk'=1$. Also $\frac{-\lambda+k/\lambda}{-\mu+k'/\mu}=\frac{\lambda+k/\lambda}{\mu+k'/\mu}$ implies that $\left(-\lambda+\frac{k}{\lambda}\right)\left(\mu+\frac{k'}{\mu}\right)=\left(\lambda+\frac{k}{\lambda}\right)\left(-\mu+\frac{k'}{\mu}\right)$.

On simplifying, this gives $k=\pm\,\lambda/\mu$ and $k'=\pm\,\mu/\lambda$.

But $\frac{1+k}{1+k'}=\frac{-\lambda+k/\lambda}{-\mu+k'/\mu}$ is only satisfied for $k=\lambda/\mu$ and $k'=\mu/\lambda$.

Hence from (3) the plane through the given two generators is $\left(1+\frac{\lambda}{\mu}\right)\frac{x}{a}+\left(-\lambda+\frac{1}{\mu}\right)\frac{y}{b}+\left(1-\frac{\lambda}{\mu}\right)\frac{z}{c}-\left(\lambda+\frac{1}{\mu}\right)=0$ i.e.

$(\mu+\lambda)\frac{x}{a}+(-\lambda\mu+1)\frac{y}{b}+(\mu-\lambda)\frac{z}{c}-(\lambda\mu+1)=0$ i.e.

$$\frac{\lambda+\mu}{a(1+\lambda\mu)}x+\frac{1-\lambda\mu}{b(1+\lambda\mu)}y-\frac{\lambda-\mu}{c(1+\lambda\mu)}z=1$$

The same plane also can be found from (4)

Evidently, this is the equation of tangent plane at the point P on the hyperboloid. Hence the result.

10.5 Perpendicular Generators of The Hyperboloid of One Sheet:-To find out the locus of the point intersection of two perpendicular generators.

The equations of a generator of $\lambda-$system can be written in the form $\frac{1}{a}x-\frac{\lambda}{b}y+\frac{1}{c}z=\lambda\ \frac{1}{a}x+\frac{1}{\lambda b}y-\frac{1}{c}z=\frac{1}{\lambda}$.

Its direction cosines, say l_1, m_1, n_1 are given by

$$\frac{l_1}{\lambda/bc-1/\lambda bc}=\frac{m_1}{1/ac-(-1/ac)}=\frac{n_1}{1/\lambda ab-(-\lambda/ab)}$$ i.e.

$\frac{l_1}{a(\lambda^2-1)}=\frac{m_1}{2b\lambda}=\frac{n_1}{c(1+\lambda^2)}$, multiplying the denominators by λabc.

Further, the equations of generators of $\mu-$ system can be written in the form $\frac{1}{a}x-\frac{\mu}{b}y-\frac{1}{c}z=\mu$, $\frac{1}{a}x+\frac{1}{\mu b}y+\frac{1}{c}z=\frac{1}{\mu}$.

Its direction cosines, say l_2, m_2, n_2 are given by

$$\frac{l_2}{-\mu/bc-(-1/\mu bc)}=\frac{m_2}{-1/ac-1/ac}=\frac{n_2}{1/\mu ab-(-\mu/ab)} \text{ i.e.}$$

$$\frac{l_2}{a(1-\mu^2)}=\frac{m_2}{-2b\mu}=\frac{n_2}{c(1-\mu^2)} \qquad (2)$$

Multiplying the denominators by μabc.

If the two generators under consideration are perpendicular, we must have $l_1l_2+m_1m_2+n_1n_2=0$, i.e.

$a(\lambda^2-1).a(1-\mu^2)+2b\lambda.(-2b\mu)+c(1+\lambda^2).c(1+\mu^2)=0$ using (1) and (2) i.e.

$a^2(\lambda^2+\mu^2-1-\lambda^2\mu^2)-4b^2\lambda\mu+c^2(\lambda^2+\mu^2+1+\lambda^2\mu^2)=0$ i.e.

$$a^2\left\{(\lambda+\mu)^2-(1+\lambda\mu)^2\right\}+b^2\left\{(1-\lambda\mu)^2-(1+\lambda\mu)^2\right\}+c^2\left\{(\lambda-\mu)^2+(1+\lambda\mu)^2\right\}=0$$

(note)

i.e. $a^2(\lambda+\mu^2)+b^2(1-\lambda\mu)^2+c^2(\lambda-\mu)^2=(a^2+b^2-c^2)(1+\lambda\mu)^2$, i.e.

$$\left\{\frac{a(\lambda+\mu)}{1+\lambda\mu}\right\}^2+\left\{\frac{b(1-\lambda\mu)}{1+\lambda\mu}\right\}^2+\left\{\frac{c(\lambda-\mu)}{1+\lambda\mu}\right\}^2=a^2+b^2-c^2.$$

This is shows that the point of intersection of the two generators, *viz*.

$\left(\frac{a(\lambda+\mu)}{1+\lambda\mu},\frac{b(1-\lambda\mu)}{1+\lambda\mu},\frac{c(\lambda-\mu)}{1+\lambda\mu}\right)$, lies on the sphere.

$$x^2+y^2+z^2=a^2+b^2-c^2.$$

This sphere is called the ***director sphere*** of the hyperboloid of one sheet.

Since the point of intersection of the generators lies on the hyperboloid also, the required locus is the curve of intersection of the hyperboloid and its director sphere obtained above.

10.6 Projections of Generators on The Principal Planes:-To show that the projections of the generators of hyperboloid of one sheet on the principal planes are the tangents to the section of the hyperboloid by that principal plane.

Let $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$, $\qquad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$ (1)

And $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$, $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$ (2)

be the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

Then, it $\lambda=\mu$, the two systems of generators have the same projection on the plane $z=0$.

The projection of the generators of $\lambda-$ system can be arranged as $z=0$, $\lambda^2\left(1+\frac{y}{b}\right)-\frac{2\lambda}{b}x+\left(1-\frac{y}{b}\right)=0$, whose envelope is the curve.

$z=0$, $\frac{4x^2}{a^2}=4\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right)$, (from Differential Calculus) i.e. $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which represents an ellipse. It is principal elliptic section of the hyperboloid by the plane $z=0$

When $\lambda=\mu=\tan\theta$ say, the point of intersection of the $\lambda-$ and $\mu-$generators *viz*. $\left(\frac{a(\lambda+\mu)}{1+\lambda\mu},\frac{b(1-\lambda\mu)}{1+\lambda\mu},\frac{c(\lambda-\mu)}{1+\lambda\mu}\right)$ assumes the form $\left(\frac{2\tan\theta}{1+\tan^2\theta}a,\frac{1-\tan^2\theta}{1+\tan^2\theta}b,0\right)$ i.e. $(a\sin 2\theta, b\cos 2\theta, 0)$.

If $\theta=\frac{1}{4}\pi-\frac{1}{2}\alpha$, these co-ordinates becomes $(a\cos\alpha, b\sin\alpha, 0)$, which is a point on the principal elliptic section.

Hence for equal values of the parameters, the two generators intersect on the principal elliptic section at the point 'α', where $\lambda=\mu=\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$.

The projection of these generators on the xy-plane is given by the equations $z=0$ and $\frac{2x}{a}=\left(1+\frac{y}{b}\right)\tan\theta+\left(1-\frac{y}{b}\right)\cot\theta$, i.e. $\frac{2x}{a}\tan\theta+(1-\tan^2\theta)\frac{y}{b}=1+\tan^2\theta$, i.e. $\frac{x}{a}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\frac{y}{b}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=1$ i.e. $\frac{x}{a}\sin 2\theta+\frac{y}{b}\cos 2\theta=1$, i.e. $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$, which is the tangent to the principal elliptic section at the point α.

Similar results hold for projections of the generators on the other principal planes.

10.7 Generators Through A Point on The Principal Elliptic Section:-To find equations of generators through a point on the principal elliptic section of the hyperboloid of one sheet.

In the preceding we have shown that the point of intersection of the $\lambda-$and $\mu-$generators, when $\lambda=\mu=\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$ is the point $(a\cos\alpha, b\sin\alpha, 0)$ on the principal elliptic section of the hyperboloid.

Further, we know from section 10.5 that direction numbers of the generators of $\lambda-$ system are $(a\lambda^2-1), 2\lambda, c(\lambda^2+1)$, i.e. $a\frac{\lambda^2-1}{\lambda^2+1}, b\frac{2\lambda}{\lambda^2+1}, c$, i.e. $-a\sin\alpha, b\cos\alpha, c$, putting $\lambda=\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$ i.e. $a\sin\alpha, -b\cos\alpha, -c$.

Similarly direction number of the generator of $\mu-$ system are $a\sin\alpha, -b\cos\alpha, c$.

Hence the equations of the $\lambda-$ and $\mu-$ generators through the point α on the principal elliptic section of the hyperboloid, are respectively.

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\mp c}.$$

10.8 Generators Through any Point of The Hyperboloid:-The parametric co-ordinates of any point on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ can be conveniently written as $(a\cos\theta\sec\phi, b\sin\theta, \sec\phi, c\tan\phi)$, (1)

where θ and ϕ are parameters. However in view of the present context, we obtain these co-ordinates as follow.

Let P and Q be the points α and β on the principal elliptic section of the hyperboloid. Let the generator of $\lambda-$system of P intersects the generator of $\mu-$system at Q in the point R whose co-ordinates are given by $\left(\frac{\lambda+\mu}{1+\lambda\mu}a, \frac{1-\lambda\mu}{1+\lambda\mu}b, \frac{\lambda-\mu}{1+\lambda\mu}\right)$.

Then if $\lambda=\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$ and $\mu=\tan\left(\frac{1}{4}\pi-\frac{1}{2}\beta\right)$, the above co-ordinates, after a simple calculation, transform to $\left(a\frac{\cos\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}, b\frac{\sin\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}, c\tan\frac{1}{2}(\beta-\alpha)\right)$.

Therefore, writing $\theta=\frac{1}{2}(\alpha+\beta)$ and $\phi=\frac{1}{2}(\beta-\alpha)$ i.e. $\alpha=\theta-\phi$ and $\beta=\theta+\phi$, we obtain the point $'\theta,\phi'$ given by (1).

Similarly, if we take the generator of $\mu-$ system at P and that of $\lambda-$ system at Q, then their point of intersection R' is given by $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, -c\tan\phi)$, i.e. the point $'\theta,\phi'$.

Since direction ratios of the generator RP of $\lambda-$ system are $a\sin\alpha, -b\cos\alpha, -c$, i.e. $a\sin(\theta-\phi), -b\cos(\theta-\phi), -c$, and direction ratios of the generator RQ of $\mu-$ system are $a\sin\beta, -b\cos\beta, c$, i.e. $a\sin(\theta+\phi), -b\cos(\theta+\phi), c$, the equations of the two generators through R can be written as $\frac{x-a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)}=\frac{z-c\tan\phi}{\pm c}$.

Finally suppose P remains fixed while Q moves on the principal elliptic section, i.e. α is constant while β varies. As the point R moves on the generator PR, the co-ordinates of R very in such that way that α. i.e. $\theta-\phi$, remains constant. Therefore, we find that $\theta-\phi$ is constant for points on a generator of $\lambda-$ system.

Similarly, supposing Q to remain fixed while P moving, we find that $\theta+\phi$ is constant for points on a generator of $\mu-$ system.

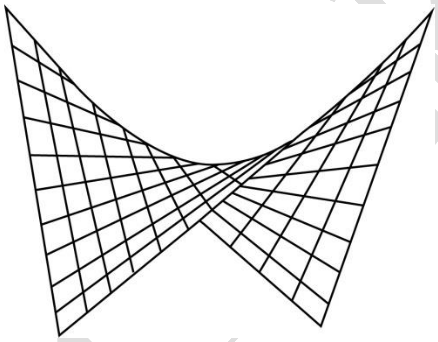
10.9 The Hyperbolic Paraboloid is A Ruled Surface:- The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ of the hyperbolic paraboloid can be written in the form $\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z$

So either $\frac{x}{a}+\frac{y}{b}=2\lambda$, $\frac{x}{a}-\frac{y}{b}=\frac{z}{\lambda}$ (1)

Or $\frac{x}{a}-\frac{y}{b}=2\mu$, $\frac{x}{a}+\frac{y}{b}=\frac{z}{\mu}$ (2)

generated the paraboloid. (The reasoning is exactly same as in the case of the hyperboloid of one sheet)

Hence the hyperbolic paraboloid is a ruled surface. it can be shown that the two system (1) and (2) of generators are distinct. Here(1) is called $\lambda-$ system and (2) is called $\mu-$ system. The two systems are shown in the following diagram.



10.10 Properties of Generators of The Hyperbolic Paraboloid:- As in the case of the hyperboloid of one sheet, we can prove the following properties for the two systems of generators of the hyperbolic paraboloid.

Properties 1:- No two generators of the same system intersect.

Properties 2:- Through any point on the hyperbolic paraboloid passes, one and only one member of each system of generators.

Properties 3:- Any generator of $\lambda-$ system intersects a generator of $\mu-$ system and vice-verse.

Properties 4:- Plane through one member of $\lambda-$ system and one member of $\mu-$ system is tangent plane to the paraboloid at the point of intersection.

Here we give details of Properties 3, and the proofs of the remaining properties are left for reader.

Let $\frac{x}{a}+\frac{y}{b}=2\lambda$, $\frac{x}{a}-\frac{y}{b}=\frac{z}{\lambda}$, and $\frac{x}{a}-\frac{y}{b}=2\mu$, $\frac{x}{a}+\frac{y}{b}=\frac{z}{\mu}$ be any generators belonging to $\lambda-$ and $\mu-$ systems respectively. They will intersect if it is possible to solve these four equations for x, y, z.

Equating the two values of $\frac{x}{a}-\frac{y}{b}$, we get $\frac{z}{\lambda}=2\mu$,which gives $z=2\lambda\mu$.

Further $\frac{x}{a}-\frac{y}{b}=\frac{z}{\lambda}=\frac{2\lambda\mu}{\lambda}=2\mu$ and $\frac{x}{a}+\frac{y}{b}=\frac{z}{\mu}=\frac{2\lambda\mu}{\mu}=2\lambda$.

By addition and subtraction of these equations, we get $\frac{2x}{a}=2(\mu+\lambda)$ and $\frac{2y}{n}=2(\lambda-\mu)$ and therefore $x=a(\mu+\lambda)$ and $y=b(\lambda-\mu)$.

Thus the two generators do intersect, and their point of intersection is $(a(\lambda+\mu),b(\lambda-\mu),2\lambda\mu)$.

These co-ordinates are also seen to satisfy the second of the four equations of generators.

Note:- The equation $x=a(\lambda+\mu), y=b(\lambda-\mu), z=2\lambda\mu$ may be regarded as parameters equations of the hyperboloid paraboloid.

10.11 Plane Through Generators of The Paraboloid:- Any plane through a generator of $\lambda-$ system is $\frac{x}{a}+\frac{y}{b}-2\lambda+k\left(\frac{x}{a}-\frac{y}{b}-\frac{z}{\lambda}\right)=0$ and any plane through a generators of $\mu-$ system is $\frac{x}{a}-\frac{y}{b}-2\mu+k'\left(\frac{x}{a}+\frac{y}{b}-\frac{z}{\mu}\right)=0$.

When is, indeed, the equation of the tangent plane to the paraboloid at the point of intersection of the generators.

It follows that the two generators through any point on the paraboloid lie on the tangent plane at that point.

Further, the first plane is the tangent plane to the paraboloid at the point of intersection of the generators of $\lambda-$ system with that generator of $\mu-$ system for which $\mu=\lambda/k$.

Hence any plane through a generator is a tangent plane to the paraboloid at some point.

10.12 Perpendicular Generators of The Paraboloid:- To find out the locus of the point of intersection of two perpendicular generators.

The equations of a generator of $\lambda-$ system can be written in the form $\frac{1}{a}x+\frac{1}{b}y+0.z=2\lambda, \frac{1}{a}x-\frac{1}{b}y-\frac{1}{\lambda}z=0$.

It is direction cosines, say l_1,m_1,n_1 are given by $\frac{l_1}{-1/b\lambda-0}=\frac{m_1}{0-(-1/a\lambda)}=\frac{n_1}{-1/ab-1/ab}$ i.e. $\frac{l_1}{a}=\frac{m_1}{-b}=\frac{n_1}{2\lambda}$,

(1) multiplying the denominators by $-ab\lambda$.

Further, the equations of a generator of $\mu-$ system can be written in the form $\frac{1}{a}x-\frac{1}{b}y+0.z=2\mu, \frac{1}{a}x+\frac{1}{b}y+\frac{1}{\mu}z=0$.

It direction cosines, say l_2,m_2,n_2, are given by

$$\frac{l_2}{1/b\mu-0}=\frac{m_2}{0-(-1/a\mu)}=\frac{n_2}{1/ab-(-1/ab)} \text{ i.e. } \frac{l_2}{a}=\frac{m_2}{b}=\frac{n_2}{2\mu}$$

(2) multiplying the denominators by $ab\mu$.

If the two generators under consideration are perpendicular, we must have $l_1l_2 + m_1m_2 + n_1n_2 = 0$ i.e. $a.a + (-b)b + 2\lambda.2\mu = 0$ using (1) and (2) i.e. $a^2 - b^2 + 4\lambda\mu = 0$.

Therefore the point of intersection of the two generators, *viz*. $\left(a(\lambda+\mu), b(\lambda-\mu), 2\lambda\mu\right)$ lies on the surface $a^2 - b^2 + 2z = 0$. We recall that this is also the locus of the point of intersection of there mutually perpendicular tangent planes to the paraboloid.

10.13 Projections of Generators on The Principal Planes:- The projections of the generators of $\lambda-$ system and that of $\mu-$ system on the principal plane $y = 0$ are given by $y = 0, \dfrac{2x}{a} = 2\lambda + \dfrac{z}{\lambda}$ and $y = 0, \dfrac{2x}{a} = 2\mu + \dfrac{z}{\mu}$.

We can easily find that the envelope of each of these is given by $y = 0, \dfrac{x^2}{a^2} = 2z$, which is the section of the paraboloid by the plane $y = 0$.

Similarly it can be found that the envelope of the projections on the principal plane $x = 0$ is the section of the paraboloid by the plane $x = 0$.

When $\lambda = \mu = t$, say the point of intersection of the generators is $\left(2at, 0, 2t^2\right)$. This is the point '$t$' on the principal paraboloid section $y = 0, x^2 = 2a^2z$ and both the generators project into the same tangent $y = 0,\ 2tx = a\left(z + 2t^2\right)$ to the above section.

10.14 Generators Through Any Point of The Paraboloid:- The parametric co-ordinates of any point on the paraboloid $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$ can be written as $\left(au\cos\theta, bu\sin\theta, \dfrac{1}{2}u^2\cos 2\theta\right)$, where u and θ are the parameters. This may be called the point 'u,θ' on the paraboloid.

Comparing this point with the point 'λ,μ' obtained in Section 10.10, we find that $\lambda + \mu = u\cos\theta$ and $\lambda - \mu = u\sin\theta$.

By addition and subtraction of these equations, we obtain $\lambda = \dfrac{1}{2}u(\cos\theta + \sin\theta)$ and $\mu = \dfrac{1}{2}u(\cos\theta - \sin\theta)$.

These values of λ and μ satisfy the relation $2\lambda\mu = \dfrac{1}{2}u^2 + \cos 2\theta$.

Now from section 10.12 we know that direction ratios of generators of $\lambda-$ and $\mu-$ system are $a, -b, 2\lambda$ and $a, b, 2\mu$ respectively.

Therefore the equations of these generators through the point $'u, \theta'$ are

$$\frac{x - au\cos\theta}{a} = \frac{y - bu\sin\theta}{n} = \frac{z - \frac{1}{2}a^2\cos 2\theta}{u\left(\cos\theta \pm \sin\theta\right)}.$$

Example1:- Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$ which pass through the point $(2, 3, -4)$

Solution:- The equation of the hyperboloid is

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1 \qquad (1)$$

Let $l, m, n,$ be the $d.c.'s$ of the generator. Then the equations of the generator through the point $(2, 3, -4)$ are

$$\frac{x-2}{l} = \frac{y-3}{m} = \frac{z+4}{n} = r \text{ , say,} \qquad (2)$$

Any point on this line is $(lr + 2, mr + 3, nr - 4)$. If this point lies on (1), we get

$$\frac{1}{4}(lr - 2)^2 + \frac{1}{9}(mr + 3)^2 - \frac{1}{10}(nr - 4)^2 = 1$$

i.e. $$r^2\left(\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16}\right) + 2r\left(\frac{l}{2} + \frac{m}{3} + \frac{n}{4}\right) = 0 \qquad (3)$$

If the line given by (2) is a generator of the hyperboloid (1), then (3) is an identity in r, the condition for which are

$$\frac{l^2}{4} + \frac{m^2}{9} + \frac{n^2}{16} = 0 \qquad (4)$$

And $$\frac{l}{2} + \frac{m}{3} + \frac{n}{4} = 0 \qquad (5)$$

Eliminating n between (4) and (5), we get

$$\frac{l^2}{4} + \frac{m^2}{9} - \left(\frac{l}{2} + \frac{m}{2}\right)^2 = 0$$

On simplifying, this given $lm = 0$. Hence

Either $l = 0$ or $m = 0$

If $l = 0$, then from (5), we have

$$\frac{m}{3} + \frac{n}{4} = 0 \text{ i.e. } \frac{m}{3} = \frac{n}{-4}$$

Hence the $d.c.'s$ of one generator are given by

$$\frac{l}{0} = \frac{m}{3} = \frac{n}{-4} \qquad (6)$$

Further, if, $m = 0$ then from (5), we obtain

$$\frac{l}{2} + \frac{n}{4} = 0\text{, i.e. } \frac{l}{1} = \frac{n}{-2}$$

Hence the $d.c.'s$ of the other generator are given by

$$\frac{l}{1}=\frac{m}{0}=\frac{n}{-2} \qquad (7)$$

Thus the equations of the two generator through the point $(2,3,-4)$ can be obtained by putting the proportionate values of l,m,n from (6) and (7) in (2) and are thus given by

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \text{ and } \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}$$

Example2:- Show that the perpendicular from the origin on the generator of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on the cone $\frac{a^2\left(b^2+c^2\right)^2}{x^2}+\frac{b^2\left(c^2+a^2\right)}{y^2}-\frac{c^2\left(a^2-b^2\right)}{z^2}=0$

Solution:- The equation of the hyperboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

(1)

We know that the equations of the generator of (1) belonging to one system and passing through any point $(a\cos,\alpha,b\sin\alpha,0)$ on the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2},z=0$ are given by

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c} \qquad (2)$$

Also, the equations to any line through the origin are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad (3)$$

If (2) and (3) are perpendicular to each other, then we have

$$al\sin\alpha-bm\cos\alpha+an=0 \qquad (4)$$

The lines (2) and (3) will intersect (i.e. they will be coplanar) if we have

$$\begin{vmatrix} a\cos\alpha & b\sin\alpha & 0 \\ a\sin\alpha & -b\cos\alpha & c \\ l & m & n \end{vmatrix}=0$$

i.e. $\quad a\cos\alpha(-bn\cos\alpha-cm)-b\sin\alpha(an\sin\alpha-cl)=0$

i.e. $\quad -abn\left(\cos^2\alpha+\sin^2\alpha\right)-acm\cos\alpha+bcl\sin\alpha=0$

i.e. $\quad lbc\sin\alpha-mac\cos\alpha-nab=0 \qquad (5)$

solving (4) and (5) for $\sin\alpha$ and $\cos\alpha$, we have

$$\frac{\sin\alpha}{mnab^2+mnac^2}=\frac{\cos\alpha}{nlbc^2+nlba^2}=\frac{1}{-\ln ca^2+\ln cb^2}$$

i.e. $$\frac{\sin\alpha}{mna\left(b^2+c^2\right)}=\frac{\cos\alpha}{nlb\left(c^2+a^2\right)}=\frac{1}{-lmc\left(a^2-b^2\right)}$$

which gives $\sin\alpha=-\frac{na\left(b^2+c^2\right)}{lc\left(a^2-b^2\right)}$ and $\cos\alpha=-\frac{nb\left(c^2+a^2\right)}{mc\left(a^2-b^2\right)}$

To eliminate α, squaring and then adding these equations, we obtain

$$\frac{n^2a^2\left(b^2+c^2\right)^2}{l^2c^2\left(a^2-b^2\right)^2}+\frac{n^2b^2\left(c^2+a^2\right)^2}{m^2c^2\left(a^2-b^2\right)^2}=1$$

i.e. $$\frac{a^2}{l^2}\left(b^2+c^2\right)^2+\frac{b^2}{m^2}\left(c^2+a^2\right)^2=\frac{c^2\left(a^2-b^2\right)^2}{n^2}$$

Hence the line (3) lies on cone

$$\frac{a^2\left(b^2+c^2\right)^2}{x^2}+\frac{b^2\left(c^2+a^2\right)^2}{y^2}-\frac{c^2\left(a^2-b^2\right)^2}{z^2}=0$$

A similar result can be obtained by taking generators of the other system.

Example3:- Find the equations of the generators of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point $(x+y+z)(2x+y-z)=6z$

Solution:- The two generators of the given surface belonging to the λ and μ systems are given by

$$x+y+z=\lambda z, \qquad 2x+y-z=6/\lambda \qquad (1)$$

And $$x+y+z=6/\mu, \qquad 2x+y-z=\mu z \qquad (2)$$

If these pass through $(1,1,1)$, we have $\lambda=3,\ \mu=2$

Hence the two generators of the opposite system are

$$x+y-2z=0, \qquad 2x+y-z=2 \qquad (3)$$

Or $$x+y+2z=3, \qquad 2x+y-3z=0 \qquad (4)$$

Direction ratios of (3) and (4) are respectively given by $-1,3,1$ and $-4,5,1$. Since the lines given by (3) and (4) both pass through the point $(1,1,1)$ their equations in symmetrical form are given by $\frac{x-1}{-1}=\frac{y-1}{3}=\frac{z-1}{1}$ and $\frac{x-1}{-4}=\frac{y-1}{5}=\frac{z-1}{1}$.

Example4:- Find the locus of the perpendiculars from the vertex of the paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ to the generators of one system.

Solution:- The equations of any generator of the λ-system are

$$\frac{x}{a}-\frac{y}{b}-\lambda z, \qquad \frac{x}{a}+\frac{y}{b}=\frac{2}{\lambda} \qquad (1)$$

The $d.c.'s$ of this line are proportional to $a\lambda,-b\lambda,2$

The vertex of the paraboloid is $(0,0,0)$. The equations of any line through the vertex are given by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r,\ \text{say,} \qquad (2)$$

If the line (2) is perpendicular to the generator (1), then from the condition of perpendicularity., we must have

$$a\lambda l-b\lambda m+2n=0 \qquad (3)$$

Suppose the line (2) meets the generator (1) in the point (lr,mr,nr).

Putting the co-ordinates of this point in the first of the equations (1), we have

$$\frac{l}{a}-\frac{m}{b}=n\lambda\text{, i.e.}\qquad \lambda=\frac{lb-ma}{abn}$$

Putting this value of λ in (3), we obtain

$$\frac{(al-bm)(lb-ma)}{abm}+2n=0$$

i.e. $\quad abl^2-a^2lm-b^2lm+abm^2+2abn^2=0$

i.e. $\quad l^2+m^2+2n^2-\dfrac{(a^2+b^2)lm}{ab}=0$

Hence for varying values of l,m,n the locus of the perpendicular is given by

$$x^2+y^2+2z^2-\frac{(a^2+b^2)xy}{ab}=0$$

In the same manner, considering the μ-system of generators, the required locus is given by

$$x^2+y^2+2z^2+\frac{(a^2+b^2)xy}{ab}=0$$

Hence the required locus is

$$x^2+y^2 2z^2\pm\frac{(a^2+b^2)xy}{ab}=0$$

Example5:- Show that the polar lines with respect to the sphere $x^2+y^2+z^2=a^2$ of the generators of the quadric $x^2-y^2=2az$ all the on the quadric $x^2-y^2=-2az$

Solution:- Any generator of the quadric $x^2-y^2=2az$ belonging to the λ-system is given by

$$x-y=\lambda z,\qquad x+y=2a/\lambda$$

The symmetrical form of these equations is

$$\frac{x-a/\lambda}{1}=\frac{y-a/\lambda}{-1}=\frac{z}{2\lambda}=r\text{, say,}\qquad (1)$$

Further, the polar plane of any point $(r+a/\lambda,-r+a/\lambda,2r/\lambda)$ on the generators (1) with respect to the given sphere $x^2+y^2+z^2=a^2$ is

$$x(r+a/\lambda)+y(-r+a/\lambda)+z(2r/\lambda)=a^2$$

i.e. $\quad r(x-y+2z/\lambda)+a(x/\lambda+y/\lambda-a)=0$

So, the polar line of the λ-generator is given by

$$x-y+2z/\lambda=0,\ x/\lambda+y/\lambda-a=0$$

i.e. $\quad x-y=-2z/\lambda,\ (x+y)/\lambda=a$

Eliminating λ between these equations, we obtain

$$(x-y)/a=-2z/(x+y)\text{, i.e. } x^2-y^2=-2az$$

This is the required quadric on which the polar lines lie.

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in the last of this book.

CHAPTER 8. GENERATING LINES

Q1. Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$. **[(4b) UPSC CSE 2020]**

Q2. Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point $(1,1,1)$. **[(3c) UPSC CSE 2018]**

Q3. Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$. **[(4c) 2018 IFoS]**

Q4. Find the equations of the two generating lines through any point $(a\cos\theta, b\sin\theta, 0)$, of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$, of the hyperboloid by the plane $z=0$.

[(4c) UPSC CSE 2014]

Q5. A variable generator meets two generators of the system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2c^2=1$ in P and P'. Prove that $BP\cdot B'P'=a^2+c^2$. **[(4c) UPSC CSE 2013]**

Q6. Show that the generators through any one of the ends of the an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2+b^2=6c^2$. Find also the condition for the generators to be perpendicular to each other.

[(4c) UPSC CSE 2011]

Q7. Find the vertices of the skew quadrilateral formed by the four generators of the hyperboloid $\frac{x^2}{4}+y^2-z^2=49$ passing through $(10,5,1)$ and $(14,2,-2)$. **[(4c) UPSC CSE 2010]**

11.REDUCTION OF GENERAL EQUATIONS OF THE SECOND DEGREE

11.1 The General Equation of The Second Degree:- The general equation of the second degree, *viz*.

$F(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ can be reduced to the two forms, by transformation as axes:

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^3=\mu \qquad (1)$$

Or $$\lambda_1 x^2+\lambda_2 y^2=2\mu z \qquad (2)$$

The nature of the surfaces represented by (1) and (2) depends upon the signs of $\lambda_1, \lambda_2, \lambda_3$ and is shown in the tabular form given below:

Table for $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2=\mu$

λ_1	λ_2	λ_3	μ	**Surface**
+	+	+	+	Elliposid
+	+	−	+	Hyperboloid of One Sheet
+	−	−	+	Hyperboloid of Two Sheets
+	+	+	0	Cone
+	+	0	+	Elliptic Cylinder
+	−	0	+	Hyperboloid Cylinder
+	−	0	0	Pair of Intersecting Planes
+	0	0	+	Pair of Parallel Planes

Table for $\lambda_1 x^2+\lambda_2 y^2=2\mu z$

λ_1	λ_2	μ	**Surface**
+	+	+	Elliptic Paraboloid
+	−	+	Hyperboloid Paraboloid
+	0	±	Parabolic Cylinder
0	+	±	Parabolic Cylinder

In what follows we shall take $f(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, i.e. the homogeneous part of $F(x,y,z)$.

11.2 Centre of The Surface $F(x,y,z)=0$

Centre:- If the origin is the centre of the surface $F(x,y,z)=0$, then for each point (x',y',z') laying on it the point $(-x',-y',-z')$ must also lie on it.

Hence we have $F(x',y',z')=f(x',y',z')+2ux'+2vy'+2wz'+d=0$ and

$F(-x',-y',-z')=f(x',y',z')-2ux'+2vy'-2wz'-d=0$.

By subtraction, these equations given

$$ux'+vy'+wz'=0$$

Hence we must have $u=0,\ v=0,\ w=0$.

It follows that if the origin is the centre of a conicoid, the coefficient of the first degree terms in its equation are all zero.

Determination of the Centre:- Let (α,β,γ) be the centre of the conicoid $F(x,y,z)=0$. Shifting the origin to the centre, this equaitons transform to

$$a(x+\alpha)^2+b(y-\beta)^2+c(z+\gamma)^2$$
$$+2f(y+\beta)(z+\gamma)+2g(z+\gamma)(x+\alpha)+2h(x+\alpha)(y+\beta)$$
$$+2u(x+\alpha)+2v(y+\beta)+2w(z+\gamma)+d=0$$

i.e. $f(x,y,z)+2x(a\alpha+h\beta+g\gamma+u)+2y(h\alpha+b\beta+f\gamma+v)+2z(g\alpha+f\beta+c\gamma+w)$

$$+\left(a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma a+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d\right)=0 \qquad (1)$$

Since the centre of this equation is the origin, the first degree terms should vanish.

Thus
$$\left.\begin{aligned} &2(a\alpha+h\beta+g\gamma+u)=0 \quad i.e.\ \frac{\partial F}{\partial \alpha}=0\\ &2(h\alpha+b\beta+f\gamma+v)=0 \quad i.e.\ \frac{\partial F}{\partial \beta}=0\\ and\quad &2(g\alpha+f\beta+c\gamma+w)=0 \quad i.e.\ \frac{\partial F}{\partial \gamma}=0\end{aligned}\right\} \qquad (A)$$

Multiplying this equations in (A) by α,β,γ respectively, and adding the resulting equations, we get $a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta+u\alpha+v\beta+w\gamma+d=0$ (2)

Using (A) and (2), equations (1) reduces to $f(x,y,z)=d'=0$ where $d'=u\alpha+v\beta+w\gamma+d$.

Remark:- The equations (A) may or may not give unique centre. There may be store than one centre, a line of centres or even a plane of centres, depending upon the nature of the solutions of the equations (A).

11.3 Transformation of $f(x,y,z)$:- To prove that by rotation of axes,

$$f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy \qquad (1)$$

transforms to $\lambda_1x^2+\lambda_2y^2+\lambda_3z^2$, (2)

Where $\lambda_1,\lambda_2,\lambda_3$ are the roots of the equation

$$\begin{vmatrix} a-\lambda & h & g\\ h & b-\lambda & f\\ g & f & c-\lambda\end{vmatrix}=0 \qquad (3)$$

We know that the expression $x^2+y^2+z^2$ remains unchanged by rotation of the axes through the same. Hence if we put $x=l_1x+m_1y+n_1z$, $y=l_2x+m_2y+n_2z$, $z=l_3x+m_3y+n_3z$ in $x^2+y^2+z^2$, then it remains unchanged.

If the axes, are transformed in such a manner that the expression (1) reduces to (2), then the expression $ax^2+by^2+cz^2+2fyz+2gzx+2hxy-\lambda\left(x^2+y^2+z^2\right)$

(4) reduces to $\lambda_1x^2+\lambda_2y^2+\lambda_3z^2-\lambda\left(x^2+y^2+z^2\right)$

(5)

Now both the expression (4) and (5) will be the product of linear factors for the same value of λ The condition for (4) i.e.

$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy$ to be the product of two linear factors is the equation (3).

Further, the condition for the expression (5) i.e.

$(\lambda_1-\lambda)x^2+(\lambda_2-\lambda)y^2+(\lambda_3-\lambda)z^2$ to be the product linear factors is

$\begin{vmatrix}\lambda_1-\lambda & 0 & 0\\ 0 & \lambda_2-\lambda & 0\\ 0 & 0 & \lambda_3-\lambda\end{vmatrix}=0$ i.e. $(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)=0$.

i.e. when λ is equal to either of $\lambda_1,\lambda_2,\lambda_3$ .Then the same should be the value of λ from the condition (3).

Hence the coefficients of x^2,y^2,z^2 in the expression into which $f(x,y,z)$ is transformed are the roots of the equation (3).

Discriminating Cubic:- The equation (3) which is a cubic in λ, is called the discriminating cubic of the homogenous expression $f(x,y,z)$. It can be written in the expended form

$\lambda^3-\lambda^2(a+b+c)+\lambda\left(bc+ca+ab-f^2-g^2-h^2\right)-\left(abc+2fgh-af^2-bg^2-ch^2\right)=0$

i.e. $\lambda^3-\lambda^2(a+b+c)+\lambda(A+B+C)-D=0$, where

$A=bc-f^2,\ B=ca-g^2, c=ab-h^2$ and $D=\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}$.

11.4 Direction Cosines of The New Axes:- The direction cosines of the new axes corresponding to the value λ_1 of λ are given by any two of the equations.

$\dfrac{\partial f/\partial l}{2l}=\lambda_1,\ \dfrac{\partial f/\partial m}{2m}=\lambda_1,\ \dfrac{\partial f/\partial n}{2n}=\lambda_1$

Similarly taking λ_2 and λ_3 or, in general, by any two of the following equations:

$$\begin{aligned}(a-\lambda)l+hm+gn&=0\\ hl+(b-\lambda)m+fn&=0\\ gl+fm+(c-\lambda)n&=0\end{aligned}$$

where λ to be put $\lambda_1,\lambda_2,\lambda_3$ to get the corresponding direction cosines of the new axes.

11.5 Nature of The Conicoid Given By $F(x,y,z)=0$ When The Second Degree Terms Do Not Form A Perfect Square:- We solve the discriminating cubic to get its roots $\lambda_1, \lambda_2, \lambda_3$.

Case I:- When none of the roots of discriminating cubic is zero

In this case, we have $D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ and the conicoid will be of any one of the forms

Ellipsoid	:	$Ax^2+By^2+Cz^2=1$
Hyperboloid of One Sheet	:	$Ax^2+By^2-Cz^2=1$
Hyperboloid of Two sheets	:	$Ax^2-By^2-Cz^2=1$
Cone	:	$Ax^2+By^2+Cz^2=0$

working Rule :- (1) Find out centre (α,β,γ) solving the equations $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$

(2) Shift the origin to the centre and get the transformed equation as $f(x,y,z)+d'=0$ where $d'=u\alpha+v\beta+w\gamma+d$

(3) Again transform this equation by rotation of the axes to assume the form $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$ which may be reduced in any of the above forms.

(4) Direction ratios of the new axes can be obtained using the method suggested in Section 11.4

Case II:- When one of the roots of the discriminating cubic is zero:-

In this case we have $D\neq 0$ and the conicoid will be of any of the following forms

Ellipsoid	:	$Ax^2+By^2+2kz=0$
Hyperboloid Paraboloid	:	$Ax^2-By^2+2kz=0$

Working Method 1:- (1) Suppose that $\lambda_3=0$. Then direction ratios of the axis corresponding to this value of λ may be obtained from any two of the following equations.

$al_3+hm_3+gn_3=0$ i.e. $\partial f/\partial l_3=0$

$hl_3+bm_3+fn_3=0$ i.e. $\partial f/\partial m_3=0$

$gl_3+fm_3+cn_3=0$ i.e. $\partial f/\partial n_3=0$

(2) Evaluate $k=ul_3+vm_3+wn_3$, where l,m,n are the actual $d.c's$. In case, $k\neq 0$ the reduced equations of the conicoid is $\lambda_1 x^2+\lambda_2 y^2+2kz=0$.

It represents an elliptic or hyperbolic paraboloid according as λ_1 and λ_2 are of the same or opposite signs.

(3) The co-ordinates of the vertex are obtained by solving any two of the following three equations:

$$\frac{\partial F/\partial x}{2l_3} = \frac{\partial F/\partial y}{2m_3} = \frac{\partial F/\partial z}{2n_3} = k$$

i.e.
$$ax + hy + gz + u - l_3 k = 0$$
$$hx + by + fz + v - m_3 k = 0$$
$$gx + fy + cz + w - m_3 k = 0 \text{ with the}$$
equation $k(l_3 x + m_3 y + n_3 z) + (ux + vy + wz + d) = 0$

Case III:- One root of the discriminating cubic zero other roots different and also $k = 0$**:-**

In this case the conicoid will be of any of the following form

Elliptic Cylinder : $Ax^2 + By^2 + C = 0$

Hyperbolic Cylinder : $Ax^2 - By^2 + C = 0$

Pair of Plane : $Ax^2 - By^2 = 0$

Working Rule:- (1) In this case there is a line of centres given by any two of the following equations.

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$$

(2) Choose any point (α, β, γ) on this line as centre and shift the origin to this point.

(3) By rotation of the axes, the given equation shall reduce to the form $\lambda_1 x^2 + \lambda_2 y^2 + d = 0$ where $d' = u\alpha + v\beta + w\gamma + d$.

Case IV:- Conicoid of Revolution:-

(i) When two roots of the discriminating cubic are equal and third root not zero:- in this case the conicoid will be any of the following forms

Ellipsoid of Revolution : $A(x^2 + y^2) + Bz^2 = 1$

Hyperboloid of Revolution : $A(x^2 - y^2) + Bz^2 = 1$

(ii) When two roots of the discriminating cubic are equal but the third roots is zero:- In this case, the conicoid will be any of the following form:

Paraboloid of Revolution : $A(x^2 + y^2) + Bz = 0$

Right Circular Cylinder : $A(x^2 + y^2) + D = 0$

If the discriminating cubic has two equal roots , then the surface represented by the equation $F(x, y, z) = 0$ is a surface of revolution. In this case we have to proceed as in the Case I and II. Direction ratios of the axis of rotations are obtained from the usual equations by taking into consideration that value λ which is different from the other two equal values.

11.6 When The Second Degree Terms In $F(x, y, z)$ Form A Perfect Square:- In this case the equation $F(x, y, z) = 0$ represents a pair of parallel planes or a parabolic cylinder.

We have $F(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

Since the second degree terms from a perfect square, we have $f^2=bc$, $g^2=ca$, $h^2=ab$.

For the sake of convenience, let us put $a=A^2,b=B^2,c=C^2$, so that $f=BC$, $g=CA$, $h=AB$.

Then $F(x,y,z)=(Ax+By+Cz)^2+(2ux+2vy+2wz)+d=0$ (1)

Case 1:- If $\frac{u}{A}=\frac{v}{B}=\frac{w}{C}=\frac{1}{2}k$, say then

$F(x,y,z)=(Ax+By+Cz)^2+k(Ax+By+Cz)+d=0$.

This is a quadratic in $Ax+By+Cz$, and therefore it represents a pair of parallel planes.

$$Ax+By+Cz=\frac{-k\pm\sqrt{(k^2-4d)}}{2}.$$

Case 2:- When the condition (2) is not satisfied, we write (1) as $(Ax+By+Cz+\lambda)^2=\lambda^2+2\lambda(Ax+By+Cz)-2ux+2vy-2wz-d$, i.e.

$$(Ax+By+Cz+\lambda)^2=2x(\lambda A-u)+2y(\lambda B-v)+2z(\lambda C-w)+\lambda^2-d \quad (3)$$

Now we choose λ in such a manner that the planes.

$Ax+By+Cz+\lambda=0$ and $2x(\lambda A-u)+2y(\lambda B-v)+2z(\lambda C-w)+\lambda^2-d=0$ be perpendicular. The condition for that is $2(\lambda A-u)A+2(\lambda B-v)B+2(\lambda C-w)C=0$

which gives $\lambda=\frac{Au+Bv+Cw}{A^2+B^2+C^2}$.

Now we write (3) in the form

$$\left\{\frac{Ax+By+Cz+\lambda}{\sqrt{(A^2+B^2+C^2)}}\right\}(A^2+B^2+C^2)=2\sqrt{\left\{\sum(\lambda A-u)^2\right\}}.\frac{\left\{\sum 2x(\lambda A-u)\right\}+\lambda^2-d}{\sqrt{\left\{\sum A(\lambda A-u)^2\right\}}}.$$

Now choose the plane $Ax+By+Cz+\lambda=0$ as the plane $y=0$ and $\left\{\sum 2x(\lambda A-u)\right\}+\lambda^2-d=0$ as the plane $x=0$. Then the above equation can be put into the form $y^2=4Kx$, which represents a parabolic cylinder where $4K=\frac{2\sqrt{\left\{\sum(\lambda A-u)^2\right\}}}{A^2+B^2+C^2}$.

The generating lines of this cylinder are parallel to the line of intersection of $x=0$ and $y=0$ on which lie the vertices of the parabolic cylinder.

The plane $x=0$ is called the tangent plane which touches the cylinder along the vertices and the plane $y=0$ is called the plane through the axis.

Example1:- Find the roots of the discriminating cubic of the following equation:

$3x^2 - y^2 - z^2 + 6yz + 6x + 6y - 2z - 2 = 0$. Reduce the above equation of the standard form.

Solution:- Here the discriminating cubic is

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -1-\lambda & 3 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0$$

i.e. $\lambda^3 - \lambda^2 - 14\lambda + 24 = 0$ (1)

i.e. $(\lambda - 2)(\lambda + 4)(\lambda - 3) = 0$

This gives

Now $\dfrac{\partial F}{\partial \alpha} = 6\alpha - 6 = 0$

$$\frac{\partial F}{\partial \beta} = -2\beta + 6\gamma + 6 = 0 \text{ and } \frac{\partial F}{\partial \gamma} = 6\beta + 2\gamma - 2 = 0$$

Solving these equations, we get

$$\alpha = 1, \quad \beta = 0, \quad \gamma = -1$$

Therefore, the centre is $(1, 0, -1)$

Now $d' = (u\alpha + v\beta + w\gamma + d) = \{-3(1) - 1(-1) - 2\} = -4$

$$3x^2 + 2y^2 - 4z^2 - 4 = 0$$

Which represents a hyperboloid of one sheet

Example2:- Reduce the equation $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12 - 4z + 1 = 0$, to the standard form and state the nature of conicoid.

Solution:- The discriminating cubic is $\begin{vmatrix} 3-\lambda & 5 & -1 \\ 5 & 7-\lambda & 5 \\ -1 & 5 & 3-\lambda \end{vmatrix} = 0$

i.e. $\lambda^2 - 13\lambda^2 + 144 = 0$

i.e. $(\lambda - 4)(\lambda - 12)(\lambda + 3) = 0$

This gives $\lambda = 4, -3, 12$

Therefore, the centre (α, β, γ) is given by

$$\frac{\partial F}{\partial \alpha} = 6\alpha + 10\beta - 2\gamma + 4 = 0$$

$$\frac{\partial F}{\partial \beta} = 10\alpha + 14\beta - 10\gamma - 12 = 0$$

and $\dfrac{\partial F}{\partial \gamma} = -2\alpha + 10\beta + 6\gamma - 4 = 0$

Solving these equations, we get

$$\alpha = \frac{1}{3}, \ \beta = -\frac{1}{3}, \ \gamma = \frac{4}{3}$$

Therefore, the centre is $\left(\frac{1}{3},-\frac{1}{3},\frac{4}{3}\right)$

Now, $d'=\left(u\alpha+v\beta+w\gamma+d\right)$

$$=\left\{2\left(\frac{1}{3}\right)+(-6)\left(-\frac{1}{3}\right)-2\left(\frac{4}{3}\right)+1\right\}=1$$

Hence the reduced equation is

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0 \text{ i.e. } -3x^2+4y^2+12z^2+1=0$$

i.e. $\quad 3x^2-4y^2-12z^2=1$

This is the equation of a hyperboloid of two sheets.

Example3:- Prove that the equation $2x^2+2y^2+z^2+2yz-2zx-4xy+x+y=0$ represents an elliptic paraboloid. Reduce it to the standard from and find the co-ordinates of the vertex and equation to its axis.

Solution:- Hence the discriminating cubic is

$$\begin{vmatrix} 2-\lambda & -2 & -1 \\ -2 & 2-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix}=0$$

i.e. $\quad \lambda^3-5\lambda^2+2\lambda=0$

This gives $\lambda=0, \dfrac{5\pm\sqrt{17}}{2}$

Direction ratios of the axis corresponding to $\lambda=0$ are given by

$$\frac{\partial f}{\partial l}=\frac{\partial f}{\partial m}=\frac{\partial f}{\partial n}=0$$

Alternately, we put $\lambda=0$ in the above discriminating cubic determinant and associate each row with l,m,n. i.e. $2l-2m-n=0,\ -2l+2m+n=0$ and $-l+m+n=0$

Solving any two of these equations, we get

$$\frac{l}{-1}=\frac{m}{-1}=\frac{n}{0}=\frac{\sqrt{\left(l^2+m^2+n^2\right)}}{\sqrt{\left\{(-1)^2+(-1)^2+0^2\right\}}}=\frac{1}{\sqrt{2}}$$

This gives $l=\dfrac{-1}{\sqrt{2}},\ m=\dfrac{-1}{\sqrt{2}},\ n=0$

Now, $k=ul+vm+wn$

$$=\frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)+0=-\frac{1}{\sqrt{2}}$$

Hence the required equations is

$$\lambda_1 x^2+\lambda_2 y^2+2kz=0$$

i.e. $\quad \dfrac{5+\sqrt{17}}{2}x^2+\dfrac{5-\sqrt{17}}{2}y^2-\sqrt{2}z=0$

Evidently, this equation represents an elliptic paraboloid as both λ_1 and λ_2 are positive.

Further, co-ordinates of the vertex can be obtained by solving any two the following equations.

$$2x-2y-z+\frac{1}{2}-\frac{1}{2}=0$$

$$-2x+2y+z+\frac{1}{2}-\frac{1}{2}=0$$

And $\quad -x+y+z+0+0=0$

With $-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y+0z\right)+\left(\frac{1}{2}x+\frac{1}{2}y+0\right)+0=0$

i.e. $\quad 2x-2y-z=0$, $x+y+z=0$ and $x+y=0$

Solving we get the co-ordinates of the vertex as $(0,0,0)$

Hence the equations to the axis are given by

$$\frac{x-0}{-1}=\frac{y-0}{-1}=\frac{z-0}{0}, \text{ i.e. } \frac{x}{1}=\frac{y}{1}=\frac{z}{0}$$

i.e. $\quad x=y,\ z=0$

Example4:- Reduce to standard form $2y^2-2yz+2zx-2xy+x-2y+3z-2=0$ and state the nature of surface represented by the equations.

Solution:- Here discriminating cubic is given by

$$\begin{vmatrix} 0-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 0-\lambda \end{vmatrix}=0$$

i.e. $\quad \lambda^3-4\lambda^2-12\lambda=0$, i.e. $\lambda(\lambda-6)(\lambda+2)$

This gives $\lambda=0,6,-2$

Direction ratios of the axis corresponding to $\lambda=0$ are given by

$$0.l-2m+2n=0$$

$$-2l+4m-2n=0$$

And $\quad 2l-2m \quad =0$

Solving the first two of these equations, we get

$$\frac{l}{(-2)(-2)-2.4}=\frac{m}{2(-2)-0(-2)}=\frac{n}{0.4-(-2)(-2)}$$

$$\frac{l}{-4}=\frac{m}{-4}=\frac{n}{-4}$$

i.e. $\quad \frac{l}{1}=\frac{m}{1}=\frac{n}{1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1^2+1^2+1^2)}}=\frac{1}{\sqrt{13}}$

so, $k=ul+vm+wn=(1/\sqrt{3})\left(-\frac{1}{2}-1+\frac{3}{2}\right)=0$

In this case there is a line of centres given by any two of the following equations:

$$-y+z-\frac{1}{2}=0 \qquad \text{i.e. } \frac{\partial F}{\partial x}=0$$

$$-2x+2y-z-1=0 \qquad \text{i.e. } \frac{\partial F}{\partial y}=0$$

And $$x-y+0+\frac{3}{2}=0 \qquad \text{i.e. } \frac{\partial F}{\partial z}=0$$

Putting $z=0$ and solving these equations, we get

$$x=-2,\ y=-\frac{1}{2}$$

Therefore, a centre is $\left(-2,-\frac{1}{2},0\right)$, i.e. (α,β,γ)

Whence $d'=u\alpha+v\beta+w\gamma+d=-\frac{1}{2}(-2)-1\left(-\frac{1}{2}\right)+0-2=-\frac{1}{2}$

Hence the standard form is

$$\lambda_1 x^2+\lambda_2 y^2+d'=0 \text{ i.e. } 6x^2-2y^2=\frac{1}{2}, \text{ i.e. } 12x^2-4y^2=1$$

Evidently, this equation represents a hyperbolic cylinder.
Further, the equation of its axis are

$$\frac{x+2}{1}=\frac{y+\frac{1}{2}}{1}=\frac{z-0}{1}.$$

Example5:- Show that the surface represented by the equation $x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$ represents a paraboloid of revolution, and find the co-ordinates of the focus and the equation to the axis

Solution:- The discriminating cubic is

$$\begin{vmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{vmatrix}=0, \text{ which on solving gives } \lambda=0,\frac{3}{2},\frac{3}{2}$$

Since one roots is zero and the other two roots are equal, the given equation represents a paraboloid of revolution.

Direction ratios of the axis are given by

$$l-\frac{1}{2}m-\frac{1}{2}n=0 \qquad \text{i.e.} \qquad 2l-m-n=0$$

And $$-\frac{1}{2}l+m-\frac{1}{2}n=0 \qquad \text{i.e.} \qquad -l+2m-n=0$$

Solving these, we have

$$\frac{l}{1+2}=\frac{m}{1+2}=\frac{n}{4-1}, \qquad \text{i.e.} \qquad \frac{l}{1}=\frac{m}{1}=\frac{n}{1}=\frac{1}{\sqrt{3}}$$

Therefore, $k = ul + vm + wn = \frac{1}{\sqrt{3}}\left(-\frac{3}{2} - 3 - \frac{2}{3}\right) = -3\sqrt{3}$

Hence the reduced equation will be $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$, i.e. $\frac{3}{2}x^2 + \frac{3}{2}y^2 - 6\sqrt{3}\,z = 0$,

i.e. $x^2 + y^2 - 4\sqrt{3}\ z = 0$

Further, the co-ordinates of the vertex are given by

$$x - \frac{1}{2}y - \frac{1}{2}z - \frac{3}{2} + 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right) = 0, \text{ i.e. } 2x - y - z + 3 = 0$$

$$-\frac{1}{2}x + y - \frac{1}{2}z - 3 + 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right) = 0 \text{ i.e. } -x + 2y - z = 0$$

And $-3\sqrt{3}\left(\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right) - \frac{3}{2}x - 3y - \frac{9}{2}z - 21 = 0$

i.e. $-9x - 12y - 15z + 42 = 0$

Solving these, we get $x = 0,\ y = 1,\ z = 2$. Hence the vertex is $(0,1,2)$

Therefore, the equation of the axis are $\frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{1}$

Now focus will be a point on the axis whose actual direction cosines are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$, and will be at a distance $\frac{1}{4}, 4\sqrt{3}$ (where $4\sqrt{3}$ is the latus rectum) i.e. $\sqrt{3}$ from the vertex $(0,1,2)$. The co-ordinates of the focus are given by $\frac{x-0}{1/\sqrt{3}} = \frac{y-1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}} = \sqrt{3}$

Which gives $x = 1,\ y = 2,\ z = 3$

Hence the co-ordinates of the focus are $(1,2,3)$

Example6:- What is the surface represented by the equation

$$x^2 + 4y^2 + z^2 + 2zx - 4yz - 4xy - 2x + 4y - 3 = 0$$

Solution:- Here the second degree terms form a perfect square. Hence the equation can be put in the form $(x - 2y + z)^2 - 2(x - 2y + z) - 3 = 0$,

i.e. $$\left\{\frac{x-2y+z}{\sqrt{(1+4+1)}}\right\}^2,\ 6 - 2\sqrt{6}\left\{\frac{x-2y+z}{\sqrt{(1+4+1)}}\right\} - 3 = 0$$

Now choose the plane $x - 2y + z = 0$ as $X = 0$ and if (x, y, z) be the co-ordinates of any point, then we have

$$X = \frac{x-2y+z}{\sqrt{(1+4+1)}}$$

Hence the above equation reduces to the form $6X^2 - 2\sqrt{6}X - 3 = 0$ and represents a pair of parallel plane.

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in the last of this book.

CHAPTER 9. REDUCTION OF GENERAL EQUATION OF SECOND DEGREE

Q2c(i) Show that the equation $2x^2+3y^2-8x+6y-12z+11=0$ represents an elliptic paraboloid. Also find its principal axis and principal planes**. UPSC CSE 2023 (10)**

Q4(c) Reduce the equation,

$3x^2+6yz-y^2-z^2-6x+6y+2z+2=0$

to a canonical form and mention the name of the surface it represents**. IFoS 2022**

Q1. Reduce the following equation to the standard form and hence determine the nature of the conicoid: $x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$. **[(4a) UPSC CSE 2017]**

Q2. Reduce the following equation to its canonical form and determine the nature of the conic $4x^2+4xy+y^2-12x-6y+5=0$. **[(3b) 2013 IFoS]**

HINTS FOR SELECTED PYQs

3. ANALYTIC GEOMETRY
1. Straight Lines

1.1 A line is drawn through a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z = 0$ to met fixed lines y = mx, $z = c$ and $y = -mx$, $z = -c$. Find the locus of the line

Solution:

Use general equation of line intersecting two lines given in planar form. Given fixed lines are

$$y - mx = 0, z - c = 0 \quad \ldots(i)$$
$$y + mx = 0, z + c = 0 \quad \ldots(ii)$$

General equation of line intersecting both

$$(y - mx) + k_1 (z - c) = 0 = (y + mx) + k_2(z + c) \quad \ldots(iii)$$

If it meets ellipse we eliminate k, and k_2

Putting $z = 0$ in (iii)

$$y - mx - k_1 c = 0; \; y + mx + k_2 c = 0$$

$$\Rightarrow \quad \frac{y}{-k_2m+k_1m}=\frac{x}{-(k_1+k_2)}=\frac{c}{2m}$$

$$\Rightarrow \quad x=\frac{-(k_1+k_2)c}{2m}; y\frac{(k_1-k_2)c}{2}$$

Putting this in equation of ellipse

$$\frac{(k_1+k_2)^2c^2}{4m^2a^2}+\frac{(k_1-k_2)^2c^2}{4b^2}=1$$

$$(k_1+k_2)^2c^2b^2+(k_1-k_2)^2c^2a^2m^2=4a^2b^2m^2$$

Substituting k_1 and k_2 from (iii)

$$\left\{\left(\frac{mx-y}{z-c}\right)+\left(-\frac{mx+y}{z+c}\right)\right\}^2c^2b^2+\left\{\left(\frac{mx-y}{z-c}\right)+\left(\frac{mx+y}{z+c}\right)\right\}^2\times c^2a^2=4a^2b^2m^2$$

$\Rightarrow$ $[(mx - y)(z + c) - (mx + y)(z-c)]^2c^2b^2 + [(mx - y)(z + c) + (mx + y)(z - c)]^2 m^2 c^2a^2$

$= 4a^2b^2 m^2(z^2 - c^2)^2$

$\Rightarrow$ $[cmx - yz]^2 c^2b^2 + [mxz-cy]^2m^2c^2a^2 = a^2b^2m^2(z^2 - c^2)^2$

which is required locus.

1.2 Prove that two of the straight lines represented by the equation

$$x^3 + bx^2y + cxy^2 + y^3 = 0$$

will be at right angles, if $b + c = -2$.

Solution:

The given equation is a homogeneous equation of third degree and hence it represents three straight lines through the origin.

Let $y = mx$ be any of these lines.

Replacing $\frac{y}{x}$ by m in $x^3 + bx^2y + cxy^2 + y^3 = 0$ or $1 + b\frac{y}{x}+c\frac{y^2}{x^2}+\frac{y^3}{x^3}=0$, we get

$$m^3 + cm^2 + bm + 1 = 0 \quad \ldots(i)$$

Let m_1, m_2, m_3 be its roots, then

$$m_1.m_2.m_3 = -1$$

But, two of these lines, say with slopes, m_1 and m_2, are at right angles,

then, $m_1.m_2 = -1$

Thus, $(-m_3) = 1$ or $m_3 = 1$

But m_3 is a root of (i)

$\therefore$ $1 + c + b + 1 = 0$

Or $b + c = -2$

1.3 Verify if the lines $\dfrac{x-a+d}{\alpha-\delta}=\dfrac{y-a}{\alpha}=\dfrac{z-a-d}{\alpha-\delta}$ and $\dfrac{x-b+c}{\beta-\gamma}=\dfrac{y-b}{\beta}=\dfrac{z-b-c}{\beta+\gamma}$ are coplanar. If yes, then find the equation of the plane in which they lie?

Solution:

Two straight lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

are coplanar it

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

And equation of plane containing them, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

Here, in our case,

$$\begin{vmatrix} (b-c)-(a-d) & b-a & b+c-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\delta \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{matrix}$$

$$=\begin{vmatrix} d-c & b-a & c-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix}=0 \text{ as } C_1=-C_3$$

Hence, the given lines are coplanar.

The equation of the plane containing them, is

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0. \text{ Applying } \begin{matrix} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{matrix}$$

$$\begin{vmatrix} x-y+d & y-a & z-y-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x-2y+z & y-a & z-y-d \\ 0 & \alpha & \delta \\ 0 & \beta & \gamma \end{vmatrix}=0 \text{ as } C_1 \to C_1 + C_3$$

$x - 2y + z = 0$

2. Shortest Distance between Two Skew Lines

2.1 Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{4}=z-3$ and y – mx = z = 0. For what value of m will the two lines intersect?

Solution:

Lines are:

$$L_1: \frac{k-1}{2}=\frac{y-2}{4}=\frac{z-3}{1}$$

$$L_2: \frac{x}{1}=\frac{y}{m}=\frac{z}{0}\begin{bmatrix} y-mx=0 \\ z=0 \end{bmatrix}$$

$P_1(1, 2, 3)$ on L_1, $P_2(0, 0, 0)$ on L_2

Shortest distance (SD) is the projection of P_1P_2 on AB which is perpendicular to both lines. Direction ratio's of AB:

$$\begin{vmatrix} i & j & k \\ 2 & 4 & 1 \\ 1 & m & 0 \end{vmatrix} = i(0-m)-j(0-1)+k(2m-4)$$

$$= -mi + j + (2m - 4)k$$

$$\text{SD} = \frac{1}{\sqrt{m^2+1+(2m-4)^2}}\left[-m(1-0)+1(2-0)+(2m-4)(3-0)\right)$$

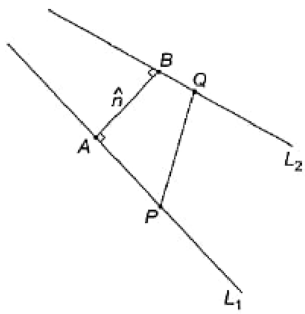
$$= \left|\frac{5m-10}{\sqrt{5m^2-16m+17}}\right|$$

The lines will intersect if, $SD = 0$, ie., $5m-10 = 0 \Rightarrow m = 2$.

2.2 Find the shortest distance between the skew lines, $= \frac{x-3}{3}=\frac{8-y}{1}=\frac{z-3}{1}$ and $\frac{x-3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

Solution:

Shortest distance lies along a direction which is perpendicular to both lines and given by the cross-product of vectors along given two lines, L_1, L_2,



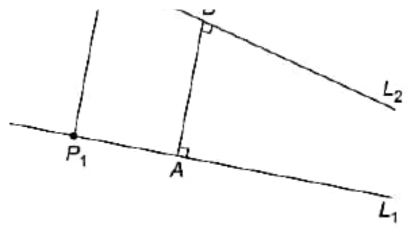
$$\vec{n}\begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$= i(-4-2)-j(12-3)+k(6-3)$$

$$= -6i - 15j + 3k$$

$$= -3(2i + 5j - k)$$

$$n = \frac{1}{\sqrt{30}}(2i + 5j - k)$$



S.D. is the projection of PQ along n

$$\text{SD} = \overrightarrow{AB}\,.\,n$$

$= \frac{1}{\sqrt{30}}$ [(3 - (-3)) 2 + (8 - (-7)) 5 - (3 - 6)·1]

$= \frac{1}{\sqrt{30}}$ (12 + 75 + 3) = $\frac{90}{\sqrt{30}}$

= 3√30

2.3 Find the shortest distance between the lines

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$

and the z-axis.

Solution:

The equation of z-axis is $x = y = 0$

$\therefore$ Any plane, P, through z-axis can be written as

$$x + \mu y = 0 \quad \ldots(i)$$

Further, any plane P_2, through given set of planes is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

i.e., $$(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + d_1 + \lambda d_2 = 0 \quad \ldots(ii)$$

For shortest distance P, and P_2 should be parallel.

$\therefore$ $$\frac{a_1 + \lambda a_2}{1} = \frac{b_1 + \lambda b_2}{\mu} = \frac{c_1 + \lambda c_2}{0}$$

i.e., $$c_1 \lambda c_2 = 0$$

$\Rightarrow$ $$\lambda = \frac{-c_1}{c_2}$$

$\therefore$ equation of P_2 is

$$\left(a_1 - \frac{c_1}{c_2}a_2\right)x + \left(b_1 - \frac{c_1}{c_2}b_2\right)y + \left(d_1 - \frac{c_1}{c_2}\right)d_2 = 0$$

Shortest distance, $$d = \frac{|d_1 + \lambda d_2 - 0|}{\sqrt{(a_1 + \lambda a_2)^2 + (b_1 + \lambda b_2)^2 + 0^2}}$$

$$d = \frac{|c_2d_1 + c_1d_2|}{\sqrt{(c_2a_1 - c_1a_2)^2 + (c_2b_1 - c_1b_2)^2}}$$

2.4 Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ intersect. Find the coordinates of the point of intersection and the equation of the plane containing them.

Solution:

Any point on the line

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$$

Is (-1 – 3r, 3 + 2r, -2 + r)

…(i)

Similarly, any part on the line

$$\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$$

Is (r^2, 7 – 3r^1, -7 + 2r^1) …(ii)

If the two given lines intersect then for some value of r and r^2 the two above points (i) and (ii) must coincide. i.e., $-1-3r = r^1$;

$$3 + 2r = 7 - 3r^1;$$
$$-2 + r = -7 + 2r^1$$

Solving the first two of these equations, we get

$$r = -1, r^1 = 2$$

These values of r and r^1 satisfy the third equation also. Hence, the given lines intersect. Substituting these values r and r^1 in (1) or (2) we get the required coordinates of the point of intersection as (2, 1, -3).

…(i)

Also, the equation of the plane containing the given lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

$\Rightarrow$ $(x + 1)(4 + 3) - (y - 3)(-6 - 1) + (z + 2)(9 - 2)$

$$x + y + z = 0$$

which is the required equation. …(ii)

3. Plane and its Properties

3.1 Find the equations of the straight line through the point (3, 1, 2) to intersect the straight line

$$x + 4 = y + 1 = 2(z - 2)$$

and parallel to the plane $4x + y + 5z = 0$.

Solution:

Let the required line intersects the given line

$$x + y = y + 1 = 2(z - 2) \quad \text{…(i)}$$

at (x_1, y_1, z_1)

$\therefore$ The equations of the required line passing through (3, 1, 2) and (x_1, y_1, z_1) are

$$\frac{x-3}{x_1-3} = \frac{y-1}{y_1-1} = \frac{z-2}{z_1-1} \quad \text{…(ii)}$$

The direction ratios of the line (ii) are $x_1 - 3, y_1 - 1, z_1 - 1$.

Because the line (ii) is parallel to the plane $4x + y + 5z = 0$, therefore the normal to the plane with direction ratios 4, 1, 5 is perpendicular to the line (ii).

$\therefore$ $4(x_1 - 3) + 1(y_1 - 1) + 5(z_1 - 1) = 0$

Or $4x_1 + y_2 + 5z_2 - 23 = 0$ …(iii)

Because the point (x_1, y_1, z_1) lies on (i).

$\therefore$ $x_1 + 4 = y_1 + 1 = 2(z_1 - 2)$

$\Rightarrow$ $x_1 = 2z_1 - 8, y_1 = 2z_1 - 5$ …(iv)

$\therefore$ from (iii), we get

$4(2z_1 - 8) + 2z_1 - 5 + 5z_1 - 23 = 0$

$\Rightarrow$ $z_1 = 4$

$\therefore$ from (iv), $x_1 = 0, y_1 = 3$

$\therefore$ from (ii), the equations of the required line are:

$$\frac{x-3}{-3} = \frac{y-1}{2} = \frac{z_2-2}{2}$$

3.2 Find the equation of the plane which passes through the points (0, 1, 1) and (2, 0, -1) and is parallel to the line joining the points (-1, 1, -2), (3, -2, 4). Find also the distance between the line and the plane.

Solution:

The general equation of plane through (0, 1, 1)

$$l(x - 0) + m(y - 1) + n(z - 1) = 0$$

$$lx + m(y - 1) + n(z - 1) = 0 \qquad \ldots(i)$$

(2, 0, -1) lies on this plane.

$$\therefore \quad 2l - m - 2n = 0 \qquad \ldots(ii)$$

Direction ratios of line passing through (-1, 1, -2), (3, -2, 4).

$$a_1 = 3 - 1 = 4,\ b_1 = (-2 - 1) = -3,\ c_1 = (4 - (-2)) = 6$$

$\therefore$ Direction ratios are 4 :: -3 :: 6

This is parallel to plane.

$$\therefore \quad 4l - 3m + 6n = 0 \qquad \ldots(iii)$$

From (ii) and (iii)

$$\frac{l}{-12} = \frac{m}{-20} = \frac{n}{-2} \Rightarrow l : m : n :: 6 : 10 : 1$$

$\therefore$ Equation of plane is

$$6l + 10(y - 1) + (z - 1) = 0$$

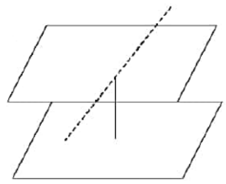
$$\Rightarrow \quad 6x + 10y + z - 11 = 0$$

Equation of plane parallel to this plane and passing through line

$$6(x + 1) + 10(y - 1) + 1(z + 2) = 0$$

$$6x + 10y + z - 2 = 0$$

$\therefore$ distance between line and plane



$$= \text{distance between two plane}$$

$$= \frac{|d_1 - d_2|}{\sqrt{l^2 + m^2 + n^2}} = \frac{9}{\sqrt{6^2 + 10^2 + 1^2}}$$

This must be on the plane for the intersection of this sphere to be a great circle.

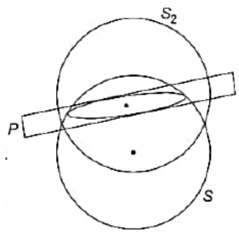
$$\therefore \quad 5\left(\frac{5\lambda - 3}{2}\right) - (4 - 2\lambda) + 2(4\lambda - 2) + 7 = 0$$

$$\Rightarrow \quad 45\lambda = 17 \Rightarrow \lambda = \frac{17}{45}$$

$\therefore$ The given sphere is

$$x^2 + y^2 + z^2 - \frac{10}{9}x + \frac{146}{45}y - \frac{22}{45}z - \frac{106}{45} = 0$$

$$\Rightarrow \quad 45(x^2 + y^2 + z^2) - 50x + 146y - 22z - 106 = 0$$



3.3 Obtain the equation of the plane passing through the points (2, 3, 1) and (4, -5, 3) parallel to x-axis.

Solution:

The equation of any plane through (2, 3, 1) is

$$a(x - 2) + b(y - 3) + c(z - 1) = 0 \qquad \ldots(i)$$

It passes through (4, -5, 3)

$$\therefore \quad a(4 - 2) + (-5 - 3) + c(3 - 1) = 0 \qquad \ldots(ii)$$

i.e.. $\quad a - 4b + c = 0$

If the plane (i) is parallel to x-axis, then it is perpendicular to yz-plane, i.e., $x = 0$, i.e.,

$$1x + 0y + 0z = 0$$

$\therefore$ $1a + 0b + 0c = 0 \Rightarrow a=0$

$\therefore$ from (ii), $-4b + c = 0$, i.e., $c = 4b$

$\therefore$ $$\frac{a}{0}=\frac{b}{1}=\frac{c}{4}$$

Hence, (i) becomes $0 + 1(y - 3) + 4(z - 1) = 0$

$$Y + 4z - 7 = 0$$

3.4 Find the surface generated by a line which intersects the lines $y = a = z$, $x + 3z = a = y + z$ and parallel to the plane $x + y = 0$.

Solution:

Topic: Equation of a straight line intersecting two given lines.

Given lines are:

$$y - a = 0 = z - a \qquad \text{...(i)}$$

$$x + 3z - a = 0 = y + z - a \qquad \text{...(ii)}$$

Hence, the equation of a line intersecting the given lines (i) and (ii) will be

$$(y - a) + \lambda(z - a) = 0 \qquad \text{...(i)*}$$

and $(x + 3z - a) + \mu(y + z - a) = 0$ ...(ii)*

$\Rightarrow$ $y + \lambda z - (a + \lambda a) = 0$

and $x + \mu y + (3 + \mu) - (a + \mu a) = 0$ ...(iii)

Line (iii) is parallel to the plane $x + y = 0$ ...(iv)

If direction ratio's of line (iii) are l, m, n, then

$$\frac{l}{3+\mu-\mu\lambda}=\frac{m}{\lambda-0}=\frac{n}{0-1}$$

$\therefore$ (iv) $\Rightarrow$ $1.(3 + \mu - \mu\lambda) + 1.\lambda + 0.(-1) = 0$

$$3 + \mu + \lambda - \mu\lambda = 0 \qquad \text{...(v)}$$

The required locus of the line is obtained by eliminating λ and μ between (i)*, (ii)* and (v).

$$3-\frac{y-a}{z-a}-\frac{x+3z-a}{y+z-a}-\frac{y-a}{z-a}.\frac{x-3z-a}{y+z-a}=0$$

Solving and simplifying: $(y + z)(x + y) = 2a(x + z)$

3.5 Find the projection of straight line $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z+1}{-1}$ on the plane $x + y + 2z = 6$.

Solution:

Given line is $$\frac{x-1}{2}=\frac{y-1}{3}=\frac{z+1}{-1}=r$$

Let this line meets given plane at $(2r + 1, 3r + 1, -r - 1)$.

The point lies on given plane, i.e.,

$$2r + r + 3r + 1 - 2r - z = 6$$

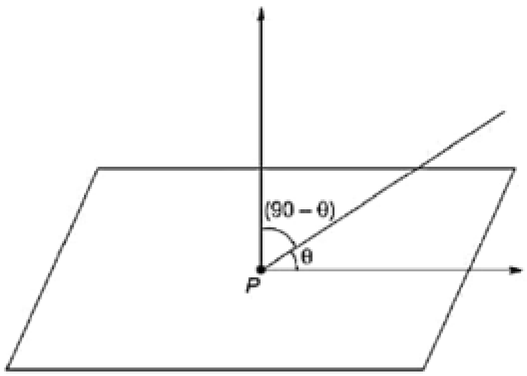
$\Rightarrow$ $r = 2$

$\therefore$ The point is $(5, 7, -3)$.

Let equation of line of projection is

$$\frac{x-5}{l}=\frac{y-7}{m}=\frac{n+3}{1}$$

Angle between given line and normal of plane is $(90 - \theta)$.



$\therefore$ $\cos(90 - \theta)\ \sin\theta = \dfrac{2\times1+3\times1-1\times2}{\sqrt{14}\sqrt{6}} = \dfrac{3}{\sqrt{84}}$

$\therefore$ Angle between given line and its projection on given plane is θ.

$\therefore$ $$cos\,\theta = \frac{\sqrt{75}}{\sqrt{84}} = \frac{2l+3m-1}{\sqrt{l^2+m^2+1}\sqrt{14}} \quad \ldots(i)$$

Also, since projected line lies on given plane

$\therefore$ $l \times 1 + m \times 1 + 1 \times 2 = 0 \Rightarrow$ m = $-l - 2$ …(ii)

Putting this value of min eqn. (i), we get

$$\frac{25}{2} = \frac{(-l-7)^2}{2l^2+4l+5}$$

$\Rightarrow$ $$l = \frac{-3}{7}, m = \frac{-5}{4}$$

$\therefore$ equation of projected line is

$$\frac{x-5}{\left(\frac{-3}{4}\right)} = \frac{y-7}{\left(\frac{-5}{4}\right)} = \frac{z+3}{1}$$

Or $$\frac{x-5}{3} = \frac{y-7}{5} = \frac{z+3}{-4}$$

3.6 Find the equation of plane parallel to 3x-y+ 3z-8 and passing through the point (1, 1, 1).

Solution:

Given, equation of plane is $3x - y + 3z = 8$

Any plane parallel to given plane has equation

$$3x - y + 3z = P \quad \ldots(i)$$

Now (i) passing through (1, 1, 1)

$\therefore$ $3 \times 1 - 1 + 3 \times 1 = P \Rightarrow P = 5$

$\therefore$ Equation of plane is $3x - y + 3z = 5$.

4.1 Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle

$$x^2 + y^2 + z^2 - 12x - 3y + 4z + 8 = 0$$
$$3x + 4y - 5z + 3 = 0$$

Solution. General equation of sphere through any circle is used. The parameter can be found by the centre satisfying equation of plane.

General equation of a sphere passing through the circle is

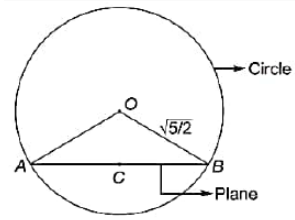
$$S + \lambda P = 0$$

or $(x^2 + y^2 + z^2 - 12x - 3y + 4z + 8) + \lambda(3x + 4y - 5z + 3) = 0$

i.e., $x^2 + y^2 + z^2 + (3\lambda - 12)x + (4\lambda - 3)y + (4 - 5\lambda)z + 3\lambda + 8 = 0$

The centre of the sphere is $\left(\frac{3\lambda-12}{2}\right),\left(\frac{4\lambda-3}{2}\right),\left(\frac{4-5\lambda}{2}\right)$

This lies on the given plane if



$$-\left[4\left(\frac{3\lambda-12}{2}\right)-5\left(\frac{4\lambda-3}{2}\right)-\left(\frac{4-5\lambda}{2}\right)\right]=3$$

$\Rightarrow$ $$3\lambda + 37 = 6$$

$\Rightarrow$ $$\lambda = \frac{-31}{3}$$

$\therefore$ Required sphere is

$$x^2 + y^2 + z^2 - 43x - \frac{133}{3}y + \frac{167}{3}z - 23 = 0$$

4.2 Show that the plane $x + y - 2z = 3$ cuts the sphere $x^2 + y^2 + z^2 - x + y = 2$ in a circle of radius 1 and find the equation of the sphere which has this circle as great cirlce.

Solution:

Given: Equation of circle is $x^2 + y^2 + z^2 - x + y = 2$, Plane = $x + y - 2z = 3$

$$\text{Centre of given circle} = \left(\frac{1}{2},-\frac{1}{2},0\right)$$

$$\text{Radius} = \sqrt{\frac{1}{4}+\frac{1}{4}+2} = \sqrt{2+\frac{1}{2}} = \sqrt{\frac{5}{2}}$$

Let O be the centre of this circle.

Distance of plane from centre

$$\frac{\left|\frac{1}{2}-\frac{1}{2}-3\right|}{\sqrt{1^2+1^2+2^2}} = \frac{3}{\sqrt{6}}$$

$\therefore$ Radius of circle with BC as radius

$$\sqrt{\frac{5}{2}-\frac{9}{6}} = \sqrt{\frac{2}{2}} = 1$$

Equation of sphere with circle as great circle.

$$x^2 + y^2 + z^2 - x + y - 2 + \lambda(x + y - 2z - 3) = 0$$

$\Rightarrow$ $$x^2 + y^2 + z^2 - x + y - 2 + \lambda x + \lambda y - 2\lambda z - \lambda 3 = 0$$

$\Rightarrow$ $$x^2 + y^2 + z^2 + (1-\lambda)x + (1+\lambda)y - 2\lambda z - 3\lambda + 2 = 0$$

Radius of this sphere = 1

$\Rightarrow$ $$\left(\frac{1-\lambda}{2}\right)^2 + \left(\frac{1+\lambda}{2}\right)^2 + \lambda^2 + 3\lambda + 2 = 1^2$$

$\Rightarrow$ $$\frac{1+\lambda^2+2\lambda}{4} + \frac{1+\lambda^2-2\lambda}{4} + \lambda^2 + 3\lambda + 2 = 1$$

$\Rightarrow$ $$\frac{3\lambda^2}{2} + \frac{1}{2} + 3\lambda = -1$$

$\Rightarrow$ $$\frac{3\lambda^2}{2} + 3\lambda + \frac{3}{2} = 0$$

$\Rightarrow$ $$\lambda^2 + 2\lambda + 1 = 0$$

$\Rightarrow$ $$\lambda = -1$$

Using this value of λ, the equation of sphere is

$$x^2 + y^2 + z^2 – x + y – 2 – 1(x + y - 2z - 3) = 0$$
$$\Rightarrow \quad x^2 + y^2 + z^2 – x + y – z - x - y - 2z + 3 = 0$$
$$\Rightarrow \quad x^2 + y^2 + z^2 + 2x + 2z + 1 = 0$$

4.3 Show that the equation of the sphere which touches the sphere
$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$$
at the point (1,2,-2) and passes through the point (-1, 0, 0) is
$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$$

Solution:

The equation of the given sphere is
$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0 \qquad \text{...(i)}$$
The equation of the tangent plane to the sphere (i) at the point (1, 2, -2) is
$$4(x – 1 + y.2 + z(-2)) + 5.(x + 1) - \frac{25}{2}(y + 2) - (z - 2) = 0$$
or $$18x - 9y - 18z + 14 = 0 \qquad \text{...(ii)}$$
$\therefore$ The equation of the sphere which touches the sphere (i) at (1,2,-2) is
$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z + \lambda(18x - 9y - 18z + 14) = 0$$
If (iii) passes through (-1, 0, 1) then,
$$4(1 + 0 + 0) + 10(-1) – 0 – 0 + \lambda(-18 – 0 – 0 + 14) = 0$$
$$4 - 10 + \lambda(-14) = 0$$
$$-6 = 4\lambda \Rightarrow \lambda = -\frac{3}{2}$$
Put $\lambda = -\frac{3}{2}$ in (iii),
$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z - \frac{3}{2}(18x - 9y - 18z + 14) = 0$$
$$\Rightarrow \quad 8(x^2 + y^2 + z^2) + 20x - 50y - 4z - 54x + 27y + 54z - 42 = 0$$
$\Rightarrow$ $8(x^2 + y^2 + z^2) -34x - 23y + 50z – 42 = 0$, which is the required equation of the sphere.

4.4 Show that three mutually perpendicular tangent lines can be drawn to the sphere $x^2 + y^2 + z^2 = r^2$ from any point on the sphere $2(x^2 + y^2 + z^2) = 3r^2$.

Solution:

Let (α, β, γ) be any point. Equation of enveloping cone from this point to sphere
$$x^2 + y^2 + z^2 = r^2 \qquad \text{...(i)}$$
is $$SS_1 = T_1^2$$
$$\Rightarrow \quad (x^2 + y^2 + z^2 - r^2)(\alpha^2 + \beta^2 + \gamma^2 - r^2) = (\alpha x + \beta y + \gamma z - r^2)^2$$
This cone will have three mutually perpendicular generators if coefficient of x^2 + coefficient of y^2 + coefficient of $z^2 = 0$.

i.e.. $$a + b + c = 0$$
$$\Rightarrow \quad (\beta^2 + \gamma^2 - r^2) + (\alpha^2 + \gamma^2 + r^2) + (\alpha^2 + \beta^2 - r^2) = 0$$
$$\Rightarrow \quad 2(\alpha^2 + \beta^2 + \gamma^2) = 3r^2$$
Since this is also the condition that three tangent lines from (α, β, γ) to sphere are mutually perpendicular, so locus of (α, β, γ) is
$$2(x^2 + y^2 + z^2) = 3r^2$$

4.5 Find the co-ordinates of the points on the sphere $x^2 + y^2 + z^2 - 4x + 2y = 4$, the tangent planes at which are paralle to the plane $2x - y + 2z = 1$.

Solution:

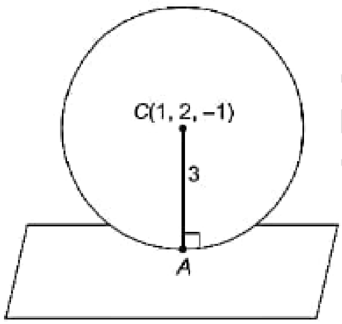
Let, the equation of planes P, and P2 parallel to

$$2x - y + 2z = 1$$
$$2x - y + 2z + \lambda = 0$$

Now, of $\qquad 2x - y + 2z + \lambda = 0$

be tangent to sphere length perpendicular to P_1 and P_2 = radius of sphere.

...(II)



$$\left|\frac{2(2)-1(-1)+2(0)+\lambda}{\sqrt{4+1+4}}\right| = 3$$

So, $\qquad \lambda = 14, -4$

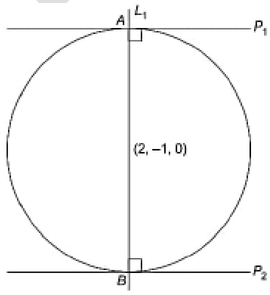
Now, to find points of constant of tangent plane, P_1, P_2 line and sphere (point A and B in diagram) equation of 'L_1' normal to tangent plane and passing through centre (2,-1, 0) is

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-0}{2} = r$$

For A and B $\Rightarrow$ r = $\pm$ B

So, $$\frac{x-2}{2/3} = \frac{y+1}{-1/3} = \frac{z-0}{2/3} = \pm 3$$

Or $\qquad (x, y, z) = (4, -2, 2), (0, 0, -2)$



4.6 For what positive value of a, the plane $ax - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and hence find the point of contact.

Solution:

Plane: $\qquad ax - 2y + z + 12 = 0$ ...(i)

$$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$$

Centre (1, 2, -1)

$$\text{Radius} = \sqrt{1+(2)^2+(-1)^2-(-3)} = 3$$

Since plane is a tangent plane.

$$CA = \text{radius} = \left|\frac{a.1 - 2.2 + (-1) + 12}{\sqrt{a^2+4+1}}\right| = 3$$

$\Rightarrow \qquad (a + 7)^2 = 9(a^2 + 5)$

i.e., $\qquad a^2 + 14a + 49 = 9a^2 + 45$

$\Rightarrow \qquad 4a^2 - 7a - 2 = 0$

$\Rightarrow \qquad 4a^2 - 8a + a - 2 = 0$

$\Rightarrow \qquad 4a(a - 2) + (a - 2) = 0$

$\Rightarrow \qquad (a - 2)(4a + 1) = 0$

$\Rightarrow \qquad a = 2 \text{ or } a = -\frac{1}{4}$

Now, equation of straight line CA is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1}$$ (perpendicular to given plane and taking $a = 2$)

Any point on this line $(2t + 1, -2t + 2, t - 1)$ ...(*)

It satisfies the equation of plane

$\therefore\ 2(2t + 1) - 2(-2t + 2) + (t - 1) + 12 = 0$

$(4t + 4t + 1) + 2 - 4 - 1 + 12 = 0$

$\Rightarrow \quad t = -1$

$\therefore$ Point of contact: (-1, 4, -2) ...(from (*))

4.7 Find the equation of the sphere which passes through the circle $x^2 + y^2 = 4$; $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

Solution:

Let the equation of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

...(i)

It passes through $x^2 + y^2 = 4$; $z = 0$

$z = 0 \Rightarrow x^2 + y^2 + 2ux + 2vy + d = 0$

$\Rightarrow \quad u = 0, v = 0, d = -4$

$\therefore$ (i) becomes

$$x^2 + y^2 + z^2 + 2wz - 4 = 0 \quad \text{...(ii)}$$

Plane, $x + 2y + 2z = 0$ cut the above sphere in the radius of 3.

$$OA^2 + AB^2 = OB^2$$

$$\left[\frac{0+2(0)+2(-W)}{\sqrt{1+4+4}}\right]^2 + (3)^2 = \left(\sqrt{0+0+w^2-(-4)^2}\right)$$

$$\frac{4W^2}{9} + 9 = W^2 + 4$$

$$\Rightarrow \quad \frac{5W^2}{9} = 5 \Rightarrow W = \pm 3$$

Hence, the required equation of sphere

$$x^2 + y^2 + z^2 \pm 6z - 4 = 0$$

4.8 A plane passes through a fixed point (a, b, c) and cuts the axis at the points A, B, C respectively. Find the locus of the centre of the sphere which passes through the origin O and A, B, C.

Solution:

Let points A(x_1, 0, 0), B(0, y_1, 0), C(0, 0, z_1).

Eqn. of sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

As it passes through O(0, 0, 0), A, B and C.

$\therefore \quad d = 0, u = -\frac{x_1}{2}, v = -\frac{y_1}{2}, w = -\frac{z_1}{2}$

Centre of sphere, C($-u$, $-v$, $-w$) = C(α, B, y)

i.e., $C\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2}\right)$ locus to be found.

Equation of plane through A, B, C is

$$\frac{z}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

As it passes through fixed point (a, b, c)

$$\therefore \quad \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1$$

i.e., $$\frac{a}{2\alpha} + \frac{b}{2\beta} + \frac{c}{2\gamma} = 1$$

Hence, the required locus of C(α, B. y) is $\left[\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2\right]$.

4.9 Show that the plane 2x-2y+z+12=0 touches the sphere x2++2-2x-4y+2z-3 = 0. Find the point of contact.

Solution:

Centre of sphere, O(1,2,-1)

Radius = $\sqrt{1+4+1-(-3)}=3$

Perpendicular distance of point O from plane

$$d = \frac{2(1)-2(2)+(-1)+12}{\sqrt{4+4+1}}$$

$$= \frac{9}{3}=3$$

which is equal to radius, hence plane touches the sphere, at A. A line through O(1,2,-1) and perpendicular to given plane is

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}$$

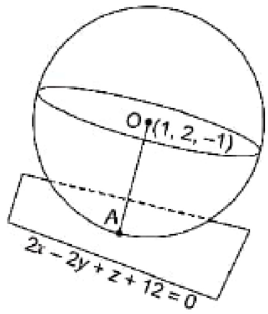
A general point on it, (2r + 1, -2r + 2, r - 1). If this lies on the plane, then

$$2(2r + 1) – 2(-2r + 2) + (r - 1) + 12 = 0$$

$$9r + 9 = 0 \Rightarrow r = -1$$

∴ Point of contact is (-1, 4, -2).

4.10 Find the equation of sphere in xyz plane passing thrugh the points (0, 0, 0), (0, 1, -1), (-1, 2, 0) and (1, 2, 3).



Solution:

Let the equation of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Since, it passes through (0, 0, 0), d = 0

Also, (0, 1, -1), (1, 2, 0) and (1, 2, 3) lie on this sphere. So, they satisfy equation of sphere.

i.e., v - w = -1 …(i)

$2u$ - $4v$ = 5 …(ii)

Solving (i), (ii) and (iii), we get

$$v=\frac{-25}{14}, w=\frac{-11}{14}, u=\frac{-15}{14}$$

∴ Equation of given sphere is

$$x^2 + y^2 + z^2 - \frac{15}{7}x-\frac{25}{7}y-\frac{11}{7}z=0$$

or $7(x^2 + y^2 + z^2) -15x - 25y - 11z = 0$

4.11 The plane $x + 2y + 3z = 12$ cuts the axes of coordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC.

Solution:

The given plane $x + 2y + 32 = 12$ ...(1)

meets the x-axis, i.e., $y = 0$, $z = 0$ in the point A whose coordinates are (12, 0, 0).

Similarly, the co-ordinates of B and C where the given plane meets y - axis are (0, 6, 0) and z-axis are (0, 0, 4).

Thus, the point A, B, C are (12, 0, 0), (0, 6, 0) and (0, 0, 4) respectively.

Let the equation of the circle circumscribing the triangle ABC be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

If it passes through A, B, C then we have

...(2)

$$(12)^2 + 2u(12) + d = 0 \Rightarrow 24u + d + 144 = 0 \quad ...(3)$$

$$(6)^2 + 2v(6) + d = 0 \Rightarrow 12v + d + 36 = 0 \quad ...(4)$$

and $(4)^2 + 2w(4) + d = 0 \Rightarrow \quad 8w + d + 116 = 0 \quad ...(5)$

From (3), (4) and (5), we get

$$2u = -\left[12 - \left(\frac{d}{12}\right)\right]$$

$$2v = -\left[6 - \left(\frac{d}{6}\right)\right] \text{ and}$$

$$2w = -\left[4 - \left(\frac{d}{4}\right)\right]$$

Substituting these values of $2u$, $2v$, $2w$ in (2), we get the required equation as

$$x^2 + y^2 + z^2 - \left[12 - \left(\frac{d}{12}\right)\right]x - \left[6 - \left(\frac{d}{6}\right)\right]y$$

$$-\left[4 - \left(\frac{d}{4}\right)\right]z + d = 0$$

where i can take vany value. Hence the result.

5. Cone and its Properties

5.1 Prove that the normals from the point (α, β, γ) to the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ be on the cone

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$$

(Note: There is an error in the question of (+) sign instead of (-) before second term.

Solution: From the general equation of normal passing through a point (α, β, γ) eliminate the direction cosines.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z \quad \text{...(i)}$$

is given equation of paraboloid equation of tangent plane to paraboloid at (f, g, h) is

$$\frac{fx}{a^2} + \frac{gy}{b^2} = (z+h)$$

$\therefore$ Normal to paraboloid at (f, g, h) has direction cosines $\left(\frac{f}{a^2}, \frac{g}{b^2} - 1\right)$ = -1 and the equation of normal is

$$\frac{a^2(x-f)}{f} = \frac{b^2(y-g)}{g} = \frac{z-h}{-1}$$

It passes through a point (α, β, γ) if

$$\frac{a^2(\alpha-f)}{f} = \frac{b^2(\beta-g)}{g} = \frac{\gamma-h}{-1} = r \text{ (let)}$$

$\Rightarrow$ $$f=\frac{a^2\alpha}{a^2+r};g=\frac{b^2\beta}{b^2+r};h=\gamma+r$$

Now let any normal through (α , β , γ) be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-y}{n} \qquad \text{...(ii)}$$

Then, $$\frac{l}{f/a^2}=\frac{m}{g/b^2}=\frac{n}{-1}$$

$\Rightarrow$ $$\frac{l\left(a^2+r\right)}{\alpha}=\frac{m\left(b^2+r\right)}{\beta}=\frac{n}{-1}$$

$\Rightarrow$ $$\frac{n}{-1}=\frac{a^2-b^2}{\frac{\alpha}{l}-\frac{\beta}{m}}$$

$\Rightarrow$ $$n\left(\frac{\alpha}{l}-\frac{\beta}{m}\right)=b^2-a^2$$

$\therefore$ Replacing *I*, *m*, *n* from (ii)

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0$$

5.2 Show that the cone yz + zx + xy=0 cuts the sphere x + y + z = a2 in two equal circles, and find

Solution:

The given equations are

$$x^2 + y^2 + z^2 = a^2 \qquad \text{...(i)}$$

and $$yz + zx + xy = 0 \qquad \text{...(ii)}$$

Multiply (ii) by 2 and add it to (i), we get

$$x^2 + y^2 + z^2 + 2(yz + 2x + xy) = a^2$$

or $$(x + y + z)^2 = x^2 \Rightarrow x + y + z = \pm a$$

$\therefore$ The equations of the required circles are

$$x^2 + y^2 + z^2 = a^2, x + y + z = a \qquad \text{...(iii)}$$

and $$x^2 + y^2 + z^2 = a^2, x + y + z = -a$$

Area of Circle (i)

Centre of the sphere $x^2 + y^2 + z^2 = a^2$ is (0,0, 0).

If l_1, is the length of perpendicular from the centre (0, 0, 0) of the sphere $x^2 + y^2 + z^2 = a^2$ to the plane $x + y + z = a$, then

$$l_1=\left|\frac{0+0+0-a}{\sqrt{1+1+1}}\right|=\frac{a}{\sqrt{3}}$$

$\therefore$ radius of the circle (iii) is

$$R_1 = \sqrt{a^2-l_1^2}=\sqrt{a^2-\frac{a^2}{3}}=\sqrt{\frac{2}{3}}a$$

Area of circle (iii) = πR_1^2

$$= \pi.\frac{2}{3}a^2=\frac{2\pi}{3}a^2$$

Similarly area of circle (iv) is $\frac{2\pi}{3}a^2$.

5.3 A variable plane is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Solution:

The equation of any plane parallel to the given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k \qquad \ldots(i)$$

It is given that the plane (i) meets the co-ordinate axes in A, B and C.

$\therefore$ A, B and C are (ak, 0, 0), (0, bk, 0) and (0, 0, ck) respectively.

Equation of any sphere passing through the points O, A, B, C is

$$x^2+y^2+z^2-kax-kby-kcz=0$$

or $\quad x^2+y^2+z^2-k(ax+by+cz)=0 \qquad \ldots(ii)$

The equation (i) and (ii) together represents the circle ABC.

Eliminating k from (i) and (ii), the required cone is:

$$x^2+y^2+z^2-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$$

or $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

5.4 A cone has for its guiding curve the circle $x^2+y^2+2ax+2by=0$, $z=0$ and passes through a fixed point (0, 0, C). If the section of the cone by the plane $y=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle.

$$x^2+y^2+z^2+2ax+2by+=0$$
$$2ax+2by+cz=0$$

Solution:

Let P(α, β, γ) be the vertex.

Any line through P

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

It passes through $z=0$

$$\Rightarrow \quad \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$$

$$\Rightarrow \quad x=\frac{-ly}{n}+ay=\left(\frac{-my}{n}+\beta\right)$$

$\therefore$ Point of intersection with $z=0$

$$\left(\alpha-\frac{ly}{n},\beta-\frac{my}{n},0\right)$$

It lies on $x^2+y^2+2ax+2by+=0$

$$\Rightarrow \quad \left(\alpha-\frac{ly}{n}\right)^2+\left(\beta-\frac{my}{n}\right)^2+2a\left(\alpha-\frac{ly}{n}\right)+2b\left(\beta-\frac{my}{n}\right)=0$$

$$\Rightarrow \quad (an-ly)^2+(na-my)^2+2an(an-ly)+2bn(\beta n-my)=0$$

Eliminating l, m, n from (i)

$[\alpha(z-\gamma)-\gamma(x-\alpha)]^2+[\beta(z-\gamma)-\gamma(\gamma-\beta)^2+2a(z-\gamma)[\alpha(z-\gamma)-(x-\alpha)\gamma]+2b(z-\gamma)[\beta(z-y)-(y-\beta)\gamma]=0$ …(ii)

$\Rightarrow \quad (\alpha z-\gamma x)^2+(\beta z-\gamma y)^2+2a(z-\gamma)(\alpha z-x\gamma)+2b(z-\gamma)(\beta z-y\gamma)=0$ …(ii)

Intersection with $y = 0$ of (ii)

$$(\alpha z-\gamma x)^2+(\beta z)^2+2a(z-\gamma)(\alpha z-x\gamma)+2b(z-\gamma)(\beta z)=0$$

This is rectangular hyperbola iff

Coefficient of x^2+Coefficient of z^2 = 0

$\Rightarrow \quad \gamma^2+\alpha^2+\beta^2+2a\alpha+2b\beta=0$ …(iii)

(ii) passes through fixed point (0, 0, c)

$$\therefore (\alpha c)^2+(\beta c)^2+2a(c-\gamma)\gamma c+2b(c-\gamma)\beta c=0$$

$$\Rightarrow (\alpha^2+\beta^2+2a\alpha+2b\beta)c^2-2a\alpha\gamma c-2b\beta\gamma c=0$$ …(iv)

Using (ii), (iii) & (iv) are equivalent to

$$-\gamma^2c^2-2(a\alpha+b\beta)\gamma c=0$$

$$\gamma c(2a\alpha+2b\beta+c\gamma)=0$$

$\Rightarrow \quad (2a\alpha+2b\beta+c\gamma)=0$ …(v)

as $c\gamma$ is not identically zero.

$\therefore$ (iii) and (iv) are required conditions.

Locus of P(α, β, γ) is

$$x^2+y^2+z^2+2ax+2by=0$$
$$2ax+2by+cz=0$$

5.5 Examine whether the plane $x + y + z = 0$ cuts the cone $xy + zx + yz = 0$ in perpendicular lines.

Solution:

From the equation of plane and the cone it is clear that the lines of intensities passes through origin. Let equation of lines be

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$ …(i)

(i) must satisfy equation of plane and cone.

So, $\quad a+b+c=0$ …(i)

And $\quad ab+bc+ca=0$ …(ii)

From (ii) and (iii)

$$ab+(a+b)\times(a+b)=0$$

$\Rightarrow \quad (a+b)^2-ab=0$

$\Rightarrow \quad a^2+b^2+ab=0$

$\Rightarrow \quad \left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)+1=0$ …(iv)

$\therefore \quad \frac{a_1a_2}{b_1b_2}=1;$ similarly $\frac{a_1a_2}{c_1c_2}=1$

$\therefore \quad \sum a_1a_2=3a_1a_2=3b_1b_2=3c_1c_2$

So, only those lines of intersection which are in yz, xz or xy planes will be perpendicular.

5.6 Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d.$$

Solution:

Let $F(x,y,z,t)=ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt=0$

$\therefore$ $\frac{\partial F}{\partial x}=0$ for t = 1 gives

$2ax + 2u = 0 \Rightarrow x = -\frac{u}{a}$ ...(i)

Similarly $\frac{\partial F}{\partial y}=0$ for t = 1 gives $y = -\frac{v}{b}$ ...(ii)

$\frac{\partial F}{\partial z}=0$ for t = 1 gives z = $-\frac{w}{c}$ ...(iii)

and $\frac{\partial F}{\partial t}=0$ for t = 1 gives $ur + vy + wz + d = 0$ ...(iv)

Substituting the values x, y, z from (i), (ii), (iii) in (iv), we get the required condition as

$\Rightarrow$ $$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$$

which is the required result.

5.7 Show that the lines drawn from the origin parallel to the normals to the central coincoid ax^2 + by^2 + cz^2 = 1 at its points of intersection with the plane $lx + my + nz = p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

Solution:

Let (α,β,γ) be the point of intersection of the given conicoid and the given plane, then we have

...(i)

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1$$

and $l\alpha + m\beta + n\gamma = p$ ...(ii)

Also, the equations of the normals to the given conicoid at (α, β, γ) are

$\therefore$ The equations of the line thrugh the origin parallel to this line are

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$$ ...(iii)

From (i) and (iii), we have

$$a\alpha^2+b\beta^2+c\gamma^2=\left(\frac{l\alpha+m\beta+n\gamma}{P}\right)^2$$

$\Rightarrow$ $P^2(a\,\alpha^2 + b\,\beta^2 + c\,\gamma^2) = (l\,\alpha + m\,\beta + n\,\gamma)2$

$\Rightarrow$ $$P^2\left(\frac{(a\alpha)^2}{a}+\frac{(b\beta)^2}{b}+\frac{(c\gamma)^2}{c}\right)=\left[\frac{l(a\alpha)}{a}+\frac{m(b\beta)}{b}+\frac{n(c\gamma)}{c}\right]^2$$

$\Rightarrow$ $$P^2\left[\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right]=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

from (iii), eliminating α, β, γ

Hence, the line (ii) generates the above cone. Henve Proved.

5.8 If $6x - 3y = 2z$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$ then obtain the equations of the other two generators.

Solution:

If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is one of the three mutually perpendicular generators, then it is normal to the plane thrugh the vertex cutting the cone in two perpendicular generators and therefore the equation of the plane is

$$x + 2y + 3z = 0 \qquad \text{…(i)}$$

Now, we are to find the lines of intersection of this plane and the given cone.

Let one of the line be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

$\therefore$ $\quad l+2m+3n=0$

and $\quad 5mn - 8nl\ 3lm = 0$

Eliminating / between these, …(ii)

$$5mn - (8n + 3n)[-(2m + 3n)] =$$

$\Rightarrow \quad 24n^2 + 30mn + 6m^2 = 0$

$\Rightarrow \quad m^2 + 5mn + 4n^2 = 0$ or $(m + n)(m + 4n) = 0$

If $m = -n$, from (ii), $l = -n$

$$\therefore \quad \frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$$

If $m = -4n$, from (ii), $l = 5n$

$$\therefore \quad \frac{l}{5}=\frac{m}{-4}=\frac{n}{1}$$

Hence, other two generators are:

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1} \text{ and } \frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$$

And evidently these three form a set of mutually perpendicular generators.

5.9 Show that the cone $3yz - 2zx - 2xy = 0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ is a generator belonging to one such set, find the other two.

Solution:

Condition for a cone to have three mutually perpendicular generators:

Cone, $ax^2 + by2 + cz^2 + 2fyz + 2gzx + 2hxy= 0$ has an infinite set of three mutually perpendicular generators,

If

$$a + b + c = 0$$

Here, $a = 0$, $b = 0$, $c = 0$ $\therefore$ $a + b + c = 0$

Part 2: $\quad \frac{x}{1}=\frac{y}{1}=\frac{z}{2} \qquad$ …(i)

is one of three mutually perpendicular generators of cone.

$$3yz - 2zx - 2xy = 0 \qquad \text{…(ii)}$$

Let a line perpendicular to (i) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \text{…(iii)}$$

$\therefore \quad l + m + 2n = 0 \qquad$ …(iv)

(iii) is generator of (ii)

$\Rightarrow$ $3mn - 2nl - 2lm = 0$ …(v)

Eliminating / between (iv) and (v)

$$3mn - (n + m)[-(m + 2n)] = 0$$

or $4n^2 + 9m + 2m^2 = 0$

or $(2m + n)(m + 4n) = 0$

or $m = -4n \text{ or } m = -\frac{n}{2}$

If $m = -4n$, from (iv), $I = 2n \Rightarrow \frac{1}{2} = \frac{m}{-4} = \frac{n}{1}$

If $m = -\frac{n}{2}$, from (iv), $2l = -3n \Rightarrow \frac{l}{3} = \frac{m}{1} = \frac{n}{-2}$

Hence, the other two generators are

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{1} \text{ and } \frac{x}{3} = \frac{y}{1} = \frac{z}{-2}$$

5.10 Find the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2 + by^2 + cz = 1$.

Solution:

$$ax^2 + by^2 + cz^2 = 1 \quad \text{…(i)}$$

The three equations of tangent planes to conicoid (i) are

$$l_1x + m_1y + n_1z = \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right)^{1/2} \quad \text{…(ii)}$$

$$l_2x + m_2y + n_2z = \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right)^{1/2} \quad \text{…(iii)}$$

$$l_3x + m_3y + n_3z = \left(\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}\right)^{1/2} \quad \text{…(iv)}$$

If the above three planes are at right angles to each other, then

$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$m_1^2 + m_2^2 + m_3^2 = 1$$
$$n_1^2 + n_2^2 + n_3^2 = 1$$

And
$$l_1m_1 + l_2m_2 + l_3m_3 = 0$$
$$m_1n_1 + m_2n_2 + m_3n_3 = 0$$
$$n_1l_1 + n_2l_2 = n_3l_3 = 0 \quad \text{…(v)}$$

To find the locus of the point of intersection of three planes, we need to eliminate l_1, m_1, n_1 from (ii), (iii), (iv) with the help of (v).

Squaring each of (ii), (ii), (iv) and adding

$$x^2\sum l_1^2 + y^2\sum m_1^2 + z^2\sum n_1^2 + 2xy\sum l_1m_1 + 2yz\sum m_1n_1 + 2zx\sum n_1l_1 = \frac{1}{a}\sum l_1^2 + \frac{1}{b}\sum m_1^2 + \frac{1}{c}\sum n_1^2$$

Or $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

which is also called the directors sphere.

5.11 Find the equation of the tangent plane at point (1, 1, 1) to the coincold $3x^2 - y^2 = 2z$.

Solution:

The equation of the tangent plane to the coincoid, $ax^2 + by^2 = 2cz$ at point (α, β, γ)

$$a\alpha x + b\beta y = c(z+\gamma)$$

Here, equation of conicoid,

$$3x^2 - y^2 = 2z \qquad \ldots(i)$$

Tangent plane at (1, 1, 1)

$$3.1x - 1.1y = \frac{2}{2}(z+1)$$

i.e., $3x - y - z = 1$

Method II: Without using formula

Any line through point (1, 1, 1) is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n} = r \qquad \ldots(ii)$$

Any point on it $(Ir + 1, mr + 1, nr + 1)$.

If this line cut the paraboloid (i) at this point then it satisfies the equation of paraboloid (i).

$$3(lr+1)^2 - (mr+1)^2 = 2(nr+1)$$

$$r^2(3l^2 - m2) + 2r(3l - \text{m} - \text{n}) + 2 = 2$$

Line touches paraboloid if both values of r given by above equation are zero, for which $3l - m - n = 0$. The locus of all such lines gives the tangent plane. Eliminating l, m, n with help of eqn. (ii)

$\Rightarrow$ $3(x-1) - (y-1) - (z-1) = 0$

i.e., $3x - y - z = 1$

5.12 Find the locus of the point of intersection of three mutually perpendicular tangent planes to $ax^2 + by^2 + cz^2 = 1$.

Solution:

$$ax^2 + by^2 + cz^2 = 1 \qquad \ldots(i)$$

The three equations of tangent planes to conicoid (i) are

$$l_1x + m_1y + n_1z = \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right)^{1/2} \qquad \ldots(ii)$$

$$l_2x + m_2y + n_2z = \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right)^{1/2} \qquad \ldots(iii)$$

$$l_3x + m_3y + n_3z = \left(\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}\right)^{1/2} \qquad \ldots(iv)$$

If the above three planes are at right angles to each other, then

$$l_1^2 + l_2^2 + l_3^2 = 1$$

$$m_1^2 + m_2^2 + m_3^2 = 1$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

And

$$l_1m_1 + l_2m_2 + l_3m_3 = 0$$

$$m_1n_1 + m_2n_2 + m_3n_3 = 0$$

$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \qquad \ldots(v)$$

To find the locus of the point of intersection of three planes, we need to eliminate l, m, n, from (ii), (ii), (iv) with the help of (v).

Squaring each of (ii), (iii), (iv) and adding

$$x^2\sum l_1^2 + y^2\sum m_1^2 + z^2\sum n_1^2 + 2xy\sum l_1m_1 + 2yz\sum m_1n_1 + 2zx\sum n_1l_1 = \frac{1}{a}\sum l_1^2 + \frac{1}{b}\sum m_1^2 + \frac{1}{c}\sum n_1^2$$

Or $$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

which is also called the directors sphere.

5.13 Find the equation of the cone with (0, 0, 1) as the vertex and $2x^2 - y^2 = 4$, $z = 0$ as the guiding curve.

Solution:

Given (0, 0, 1) is the vertex of cone.

Equation of line thrugh this vertex is

$$L_1:\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-1}{n} \qquad \ldots(i)$$

Now, L_1 meets guiding curve.

$$\therefore \quad \frac{x}{l}=\frac{y}{m}=\frac{0-1}{n}\Rightarrow x=\frac{-l}{n},y=\frac{-m}{n},z=0$$

Putting these values in equation of guiding curve, we get

$$2\left(\frac{-1}{n}\right)^2-\left(\frac{-m}{n}\right)^2=4$$

$$\Rightarrow \quad \frac{2l^2}{n^2}-\frac{m^2}{n^2}=4$$

From (i), putting values of $\frac{l}{n}$ and $\frac{m}{n}$, we have

$$2\left(\frac{x}{z-1}\right)^2-\left(\frac{y}{z-1}\right)^2=4$$

$$\Rightarrow \quad 2x^2-y^2=4\left(z^2-2z+1\right)$$

$\Rightarrow \quad 2x^2-y^2-4z^2+8z=4$ is the equation of given cone.

5.14 Prove that the plane $z = 0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2 = 11$ which has the vertex at (2, 4, 1) in a rectangular hyperbola.

Solution:

The equation of the sphere is

$$x^2+y^2+z^2 - 11 = 0$$

and the vertex is (2, 4, 1).

Here, $S=x^2+y^2+z^2-11, x_2=2, y_1=4, z_1=1$

$S_1=x_1^2+y_1^2+z_1^2-11=4+16+1-11$

i.e., $S_1=x_1^2+y_1^2+z_1^2-11=10$

and $T=xx_1+yy_1+zz_1-11$

$=2x+4y+z-11$

$\therefore$ Equation of the enveloping cone is

$(x^2+y^2+z^2 - 11)(10)\ (2x+4y+z-11)^2$ using $SS_1 = T^2$

$\Rightarrow \quad 10(x^2 + y^2 - z^2 - 11) - (2x + 4y - 11)^2 = 0$

This meets the plane $z = 0$ in the curve

$$10(x^2 + y^2 - 11) - (2x + 4y - 11)^2 = 0$$

$$10(x^2 + y^2) - (4x^2 + 16y^2 +\ldots.) = 0$$

This represents a rectangular hyperbola in the XY- plane if co-efficient of x^2+ coefficient of y^2 = 0.

If $(10 - 4) + (10 - 16) = 0$ which is true. Hence the result.

6. Paraboloid and its Properties

6.1 Show that the plane $3x + 3y + 7z + \frac{5}{2} = 0$ touches the paraboloid $3x^2 + 4y^2 = 10z$ and find the point of contact.

Solution:

Given plane $$P = 3x + 3y + 7z + \frac{5}{2} = 0 \quad \text{...(i)}$$

and Paraboloid $= 3x^2 + 4y^2 = 10z$

Now, let point of contact be (x_1, y_1, z_1).

$\therefore$ If given plane touches paraboloid at this point of contact, then the plane should be tangent plane to the paraboloid.

Now, equation of tangent plane at (x_1, y_1, z_1) is

$$3xx_1 + 4yy_1 = \frac{10(z + z_1)}{2}$$

$$\Rightarrow \quad 3xx_1 + 4yy_1 - 5z - 5z_1 = 0 \quad \text{...(ii)}$$

Comparing (1) and (2), we get

$$\frac{3}{3x_1} = \frac{4}{4y_1} = \frac{7}{-5} = \frac{5/2}{-521}$$

$$\therefore \quad \frac{3}{3x_1} = \frac{-7}{5} \Rightarrow x_1 = \frac{-5}{7}$$

$$\frac{4}{4y_1} = \frac{7}{-5} \Rightarrow y_1 = \frac{-5}{7}$$

$$\frac{7}{-5} = \frac{5}{-5 \times 221} \Rightarrow z_1 = \frac{5}{14}$$

$\therefore$ Point of contact $(x_1, y_1, z_1) = \left(\frac{-5}{7}, \frac{-5}{7}, \frac{5}{14}\right)$

6.2 Show that the locus of a point from which the three mutually perpendicular tangent lines can be drawn to the paraboloid $x^2 + y^2 + 2z = 0$ is

$$\mathbf{x^2 + y^2 + 4z = 1}$$

Solution:

Let $P(x_1, y_1, z_1)$ be the point from which three mutually perpendicular lines can be drawn to the paraboloid

$$x^2 + y^2 + 2z = 0 \quad \text{....(i)}$$

Then, the enveloping cone of (i) with the vertex at $P(x_1, y_1, z_1)$ is

$$SS_1 = T^2 \quad \text{....(ii)}$$

Where
$S = x^2 + y^2 + 2z$
$S_1 = x^2 + y^2 + 2z$
$T = xx_1, yy_1, z + z + z_1$

$\therefore$ from (ii), we have

$$(x^2 + y^2 + 2z)(x_1^2 + y_1^2 + 2z_1) = (xx_1 + yy_1 + (z + z_1))^2$$

For three mutually perpendicular generators, coefficient of x^2 + coefficient of z^2+ coefficient of $z^2 = 0$

$$x_1^2 + y_1^2 + 4z_1 - 1 = 0$$

or $$x_1^2 + y_1^2 + 4z_1 = 0$$

$\therefore$ Locus of (x_1, y_1, z_1) is

$$x^2 + y^2 + 4z = 1$$

6.3 Two perpendicular tangent planes to the paraboloid $x^2 + y^2 = 2z$ intersect in a straight line in the plane $x = 0$. Obtain the curve to which this straight line touches.

Solution:

Let the line of intersection of the two planes be:

$$my + nz = \lambda,\ x = 0 \qquad \ldots(i)$$

Since this lies on the plane $x = 0$ (given).

$\therefore$ Equation of the plane through the line (i) is

$$(my + nz - \lambda) + kx = 0$$

Or $$kx + my + nz = \lambda \qquad \ldots.(ii)$$

If the plane (ii) touches the paraboloid, then

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{2pn}{c} = 0 \text{(Condition)}$$

i.e.. $$k^2 + m^2 + 2\lambda n = 0 \qquad \ldots(iii)$$

This being quadratic in k, gives two values of k, say k_1 and k_1 such that

$$k_1.k_2 = \frac{m^2 + 2\lambda n}{l} \qquad \ldots.(iv)$$

Also from (ii), the direction ratio's of the normal to the two tangent planes whose line of intersection is (ii) are k_1, m, n and k_2, m, n.

Also, as these two tangent planes are perpendicular

$\therefore$ $$k_1.k_2 + m.m + n.n = 0$$

$\Rightarrow$ $$(m^2 + 2\lambda n) + m^2 + n^2 = 0$$

{from (iv)}

$\Rightarrow$ $$2m^2 + n^2 + 2\lambda n = 0 \qquad \ldots(v)$$

Now, we are to prove that the line (i) touches a parabola (to be found). So, we are to find the envelope of (1) which satisfies the condition (v).

Eliminating λ between (i) and (v), the equations of the line of intersection of two tangent planes is:

$$2m^2 + n^2 + 2(my + nz)n = 0,\ x = 0$$

$\Rightarrow$ $$2\left(\frac{m}{n}\right)^2 + 2y\left(\frac{m}{n}\right) + (1 + 2z) = 0, x = 0$$

It is quadratic in $\frac{m}{n}$, so its envelope is given by:

$$B^2 - 4AC = 0,\ x = 0$$

$\Rightarrow$ $$(2y)^2 - 4.2(1 + 2z) = 0,\ x = 0$$

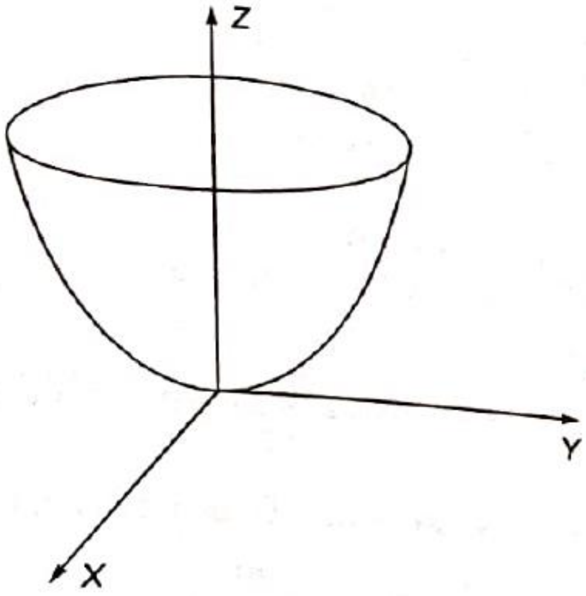
$\Rightarrow$ $$y^2 = 2(2z + 1),\ x = 0$$

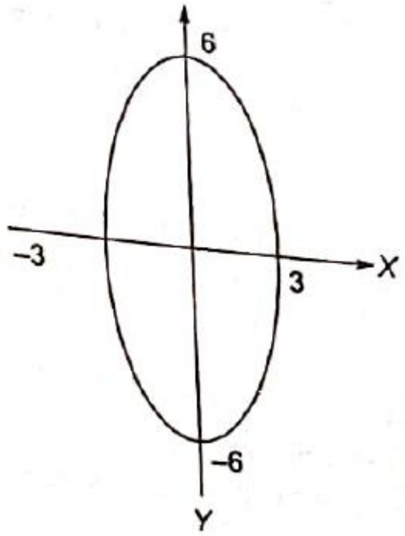
This is the required curve.

6.4 Find the volume of the solid above the xy-plane and directly below the portion of the elliptic paraboloid $x^2 + \frac{y^2}{4} = z$ which is cut off by the plane $z = 9$.

Solution:

Equation of ϕ surface, cut off cut plane





$$x^2+\frac{y^2}{4}=0; z=9$$

i.e., $$\frac{x^2}{9}+\frac{y^2}{36}=1; z=9$$

Making the transformation,

$$x = 3r\cos\theta$$

$$y = 6r\sin\theta$$

r : 0 to 1; 0:0 to 2π

$$\frac{J(x,y)}{J(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}3\cos\theta & -3r\sin\theta\\ 6\sin\theta & 6r\cos\theta\end{vmatrix}$$

$$= 18r$$

$$V=\iint_D z\,dx\,da=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(9r^2\right)18r\,dr\,d\theta \qquad \left(\because z=x^2+\frac{y^2}{4}\right)$$

$$= 9\times 18\int_0^{2\pi}d\theta\int_0^1 r^3dr$$

$$= 9\times 18\times 2\pi\times\frac{1}{4}=81\pi$$

6.5 Reduce the following equation to the standard form and hence determine the nature of the conicoid:

$$\mathbf{x^2 + y^2 + z^2 - yz - zx - xy - 3x - 6y - 9z + 21 = 0}$$

Solution:

Comparing with

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

The discriminating cubic is:

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $\quad 4\lambda^3 - 12\lambda^2 + 9\lambda = 0 \text{ or } \lambda(2\lambda - 3)^2 = 0$

$\therefore$ $\quad \lambda = \frac{3}{2}, \frac{3}{2}, 0$

As this discriminating cube has two roots equal and third root equal to zero, so it is either a paraboloid of revoltion or a right circular cylinder.

The d.r.'s of the axis are given by $al + hm + gn = 0$, $hl + bm + fn = 0$, $gl + fm + cn = 0$

i.e., $\quad l - \frac{m}{2} - \frac{n}{2} = 0, -\frac{1}{2} + m - \frac{n}{2} = 0, -\frac{1}{2} - \frac{m}{2} + n = 0$

i.e., $\quad 2l - m - n = 0, -l + 2m - n = 0, -l - m + 2n = 0$

These gives: $\quad l = m = n = \frac{1}{\sqrt{3}}$

Now, $\quad K = ul + vm + wn$

$$= \left(-\frac{3}{2} + (-3) + \left(-\frac{9}{2}\right)\right)\frac{1}{\sqrt{3}} = -3\sqrt{3} \neq 0$$

$\therefore$ Reduced equation:

$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$$

Or $\quad x^2 + y^2 = 4\sqrt{3z}$

which represents a paraboloid of revolution.

6.6 Find the equations to the generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$ which pass through (1, 1, 1).

Solution:

The given equation of paraboloid can be re-written in λ - μ form as

$$x + y + z = z\lambda,\ 2x + y - z = \frac{6}{\lambda}$$
…(i)

and $$x + y + z = \frac{6}{\mu},\ 2x + y - z = z\mu$$
…(ii)

Since these lines pass through (1, 1, 1), therefore

$$1 + 1 + 1 = 1.\lambda \Rightarrow 3\lambda = 3$$

So for (1), the equation are

$$x + y + z = 3z \Rightarrow x + y - 2z = 0$$
…(iii)

and $$2x + y - z = \frac{6}{3} \Rightarrow 2x + y - z =$$ …(iv)

Similarly, for (ii), the equation are

$$1 + 1 + 1 = \frac{6}{\mu} \Rightarrow \mu = 2$$

i.e., $x + y + z = 3$ …(v)
and $2x + y - 3z = 0$ …(vi)

From (ii) and (iv), eqn. of line is symmetrical form

$$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-1}{-1}$$

From (v) and (vi), eqn. of line in symmetrical form

$$\frac{x-1}{4} = \frac{y-1}{-5} = \frac{z-1}{1}$$

6.7 Prove that, in general, three normals can be drawn from a given point to the paraboloid $x^2 + y^2 = 2az$,

but if the point lies on the surface
$$27a(x^2 + y^2) + 8(a - z)3 = 0$$

then two of the three normals coincide.

Solution:

The equations of the normal at (x_1, y_1, z_1) to the paraboloid $x^2 + y^2 = 2az$ are

$$\frac{x-x_1}{x_1}=\frac{y-y_1}{y_1}=\frac{z-z_1}{-a}$$

This passes through a given point (α, β, γ) if

$$\frac{\alpha-x_1}{x_1}=\frac{\beta-y_1}{y_1}=\frac{\gamma-z_1}{z_1}=\lambda \text{ (say)}$$

These gives $\alpha-x_1=\lambda x_1 \Rightarrow x_1=\frac{\alpha}{(1+\lambda)}$

Similarly, $y_1=\frac{\beta}{(1+\lambda)}, z_1=\gamma+a\lambda$ …(i)

Also, (x1, y1, z1) lies on the given paraboloid, so

$$x_1^2+y_1^2=2az_1 \Rightarrow \left[\frac{\alpha}{1+\lambda}\right]^2+\left[\frac{\beta}{1+\lambda}\right]^2=2a(\gamma+a\lambda) \quad \text{[from (1)]}$$

$$\Rightarrow \quad \alpha^2+\beta^2=2a(\gamma+a\lambda)(1+\lambda)^2 \quad \text{….(ii)}$$

This being a cubic in λ gives three values of I and so from (1) there are three points on the paraboloid normal at which pass through (α, β, γ).

The equation (2) can be rewritten as

$$(\lambda)=2a(1+\lambda)^2(\gamma+a\lambda)-(\alpha^2+\beta^2)=0 \quad \text{…(iii)}$$

The condition that this equation has two equal roots is obtained by eliminating λ between λ(λ) = 0 and f(λ)=0.

From (3), f(λ) = 0 means $2a(1+\lambda)^2(a)+4a(1+\lambda)(\gamma+a\lambda)=0$

$= a(1+\lambda)+2(\gamma+a\lambda)=0 \quad (\because 1+\lambda\neq 0)$

$= (a+2\gamma)+\lambda(3a)=0$

$$\Rightarrow \quad \lambda=\frac{-(a+2\gamma)}{(3a)}$$

Substituting this value of λ in (3), we get

$$2a\left[1-\frac{a+2\gamma}{3a}\right]^2\left[\gamma-\frac{a(a+2\gamma)}{3a}\right]=\alpha^2+\beta^2$$

$$2a[2(a-\gamma)]^2[a(\gamma-a)]=27a^3(\alpha^2+\beta^2)$$

∴ Locus of the point (α, β, γ) is

$27a(x^2 + y^2) + 8(a - z)^3 = 0$. Hence, proved.

7. Ellipsoid and its Properties

7.1 Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere.

Solution:

Let the equation of the plane PQR be

$$Lx + my + nz = 1 \quad \ldots(i)$$

The equation of the given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots(ii)$$

The equation of the cone with vertex at (0, 0, 0) and the curve of intersection of (i) and the ellipsoid (ii) as the guiding curve is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(lx+my+nz)^2 \quad \ldots(iii)$$

If the cone (iii) has three mutually perpendicular generators then Coefficient of x^2 + Coefficient of y^2 + Coefficient of z^2 = 0

$$\Rightarrow \left(l^2-\frac{1}{a^2}\right)+\left(m^2-\frac{1}{b^2}\right)+\left(n^2-\frac{1}{c^2}\right)=$$

$$\Rightarrow l^2+m^2+n^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{\lambda^2}(say) \quad \ldots(iv)$$

If the plane (i) touches the sphere $x^2 + y^2 + z^2 = \lambda^2$, then the length of the perpendicular from the centre (0, 0, 0) of the sphere to (i) must be equal to the radius λ of the sphere.

i.e.,
$$\frac{1}{\sqrt{l^2+m^2+n^2}}=\lambda$$

$$\Rightarrow l^2+m^2+n^2=\frac{1}{a^2}$$ which is true by virtue of (iv).

Hence, the plane (i) touches the sphere $x^2 + y^2 + z^2 = \lambda^2$.

7.2 Find the length of the normal chord through a point P of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and prove that if it is equal to 4PG$_3$, where G$_3$ is the point where the normal chord through P meets the *xy*-plane, then P lies on the cone

$$\frac{x^2}{a^6}\left(2c^2-a^2\right)+\frac{y^2}{b^6}\left(2c^2-b^2\right)+\frac{z^2}{c^4}=1$$

Solution:

Let P be (α, β, γ), then the equations of the normal to the given ellipsoid at P(α, β, γ) are

$$\frac{x-\alpha}{\left(P\alpha/a^2\right)}=\frac{y-\beta}{\left(P\beta/b^2\right)}=\frac{z-\gamma}{\left(P\gamma/c^2\right)}=\gamma(say) \qquad \ldots(i)$$

Where $$\frac{1}{P^2}=\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^4} \qquad \ldots(ii)$$

$\therefore$ The co-ordinates of any point Q on the normal (1) are $\left(\alpha+\frac{P\alpha}{a^2},\beta+\frac{P\beta}{b^2}\gamma,\gamma+\frac{P\gamma}{c^2}\gamma\right)$

where g is the distance of Q from P.

If Q lies on the given ellipsoid i.e., PQ is the normal chord, then

$$\frac{1}{a^2}\left(\alpha+\frac{P\alpha}{a^2}\gamma\right)^2+\frac{1}{b^2}\left(\beta+\frac{P\beta}{b^2}\gamma\right)^2+\frac{1}{c^2}\left(\gamma+\frac{P\gamma}{c^2}\right)^2=1$$

$$=\gamma^2P^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2\gamma P\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^4}{c^4}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)=1$$

$$=\gamma^2P^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2\gamma P\left(\frac{1}{P^2}\right)=0$$

From (2) and $\sum\frac{\alpha^2}{a^2}=1$ as P(α, β, γ) (lies on the given coincoid.

$$\gamma=\frac{-2}{P^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)}=\text{ length of normal chord PQ} \qquad \ldots(iii)$$

Also, let the normal at P(α, β, γ) meets the coordinate planes viz. *yz*, *zx* and *xy* planes at G$_1$, G$_2$ and G$_3$, then putting x = 0, y = 0 and z = 0 in succession in the eqn. (1), we have respectively.

$$PG_1=\frac{a^2}{P},PG_2=-\frac{b^2}{P}\,and\;PG_3=\frac{c^2}{P} \qquad \ldots(iv)$$

Given, PQ = 4PG$_3$

$$PQ = 4\left(-\frac{c^2}{P}\right)$$

$$\Rightarrow \quad \frac{-2}{P^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)} = 4\left(-\frac{c^2}{P}\right)$$

$$\Rightarrow \quad 2c^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right) = \frac{1}{P^2} = \frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}$$

$$\Rightarrow \quad \frac{\alpha^2}{a^6}\left(2c^2-a^2\right)+\frac{\beta^2}{b^6}\left(2c^2-b^2\right)+\frac{\gamma^2}{c^4}\left(2c^2-c^2\right)=0$$

The locus of P(α, β, γ) is

$$\frac{x^2}{a^6}\left(2c^2-a^2\right)+\frac{y^2}{b^6}\left(2c^2-b^2\right)+\frac{z^2}{c^4}=0.$$ Hence, Proved

8. Hyperboloid of One and Two Sheets and its Properties

8.1 Find the vertices of the skew quadrilateral formed by the four generators of the hyperboloid

$$\frac{x^2}{4}+y^2-z^2=49$$

passing through (10, 5, 1) and (14, 2, -2).

Solution:

Given, the equation of hyperboloid is

$$\frac{x^2}{4}+y^2-z^2=49$$

It can be rewritten as

$$\left(\frac{x}{2}-z\right)\left(\frac{x}{2}+z\right)=(7-y)(7+y)$$

$\therefore$ The equation of two systems of generating lines are:

$$\left(\frac{x}{2}-z\right)=\lambda(7-y),\lambda\left(\frac{x}{2}+z\right)=(7+y) \quad \ldots(i)$$

$$\left(\frac{x}{2}-z\right)=\mu(7-y),\mu\left(\frac{x}{2}+z\right)=(7+y) \quad \ldots(ii)$$

(1) and (2) pass through (10, 5, 1) and (14, 2,-2) for $\lambda = 2$, $\mu = \frac{1}{3}$ and $\lambda = \frac{9}{5}$, $\mu = 1$.

So, the two systems of generating lines are

$$\left(\frac{x}{2}-z\right)=2(7-y),2\left(\frac{x}{2}+z\right)=7+y \qquad \ldots\text{(iii)}$$

And
$$\left(\frac{x}{2}-z\right)=\frac{1}{3}(7-y),\frac{1}{3}\left(\frac{x}{2}+z\right)=7-y \qquad \ldots\text{(iv)}$$

$$\left(\frac{x}{2}-z\right)=\frac{9}{5}(7-y),\frac{9}{5}\left(\frac{x}{2}+2\right)=7+y \qquad \ldots\text{(v)}$$

$$\left(\frac{x}{2}-z\right)=1(7+y),\frac{x}{2}+z=7-y \qquad \ldots\text{(vi)}$$

Solving (3) and (6), we get the vertices as $\left(14,\frac{7}{3},-\frac{7}{3}\right)$.

Solving (4) and (5), we get the vertices as $\left(\frac{21}{2},\frac{77}{16},\frac{21}{16}\right)$.

$\therefore$ Other two vertices are $\left(14,\frac{7}{3},-\frac{7}{3}\right)$ and $\left(\frac{21}{2},\frac{77}{16},\frac{21}{16}\right)$.

8.2 Show that the generators through any one of the ends of an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2 + b^2 = 6c^2$. Find also the condition for the generators to be perpendicular to each other.

Solution:

Let (a cos θ, b sin θ, 0) be the given point on the diameter. The equations of the two generators through this point are

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}$$

The direction ratios of two generators are (a sin θ, -b cos θ, c) and (a sin θ, -b cos θ, -c) respectively. Let a be the angle between two generators and let e be the parameter of the end points of conjugate *diameters.*

$$\therefore \qquad \text{COS } \alpha=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$=\frac{a^2\sin^2\theta+b^2\cos^2\theta-c^2}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta+c^2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta+c^2}}$$

$$=\frac{a^2\sin^2\theta+b^2\cos^2\theta-c^2}{a^2\sin^2\theta+b^2\cos^2\theta+c^2}$$

Putting α = 60° and θ = 45° (Q equi-conjugal diameters means equal length of conjugal diameters, i.e., $\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$ which is equal to $\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ and is possible for θ = 45°).

$\therefore$ from (i), we get

$$\cos 60° = \frac{\frac{a^2}{2}+\frac{b^2}{2}-c^2}{\frac{a^2}{2}+\frac{b^2}{2}+c^2}$$

$$\Rightarrow \quad \frac{1}{2}=\frac{a^2+b^2-2c^2}{a^2+b^2+2c^2}$$

$$\Rightarrow \quad a^2+b^2+2c^2=2a^2+2b^2-4c^2$$

$$\Rightarrow \quad a^2+b^2=6c^2$$

Again, put $\alpha = 90°$ and $\theta = 45°$ in (i), we get

$$0=\frac{\frac{a^2}{2}+\frac{b^2}{2}-c^2}{\frac{a^2}{2}+\frac{b^2}{2}+c^2}$$

$$\Rightarrow \quad a^2+b^2=2c^2$$

which is the required condition for the generators to be perpendicular to each other.

8.3 A variable generator meets two generators of the same system through the extremities B and B of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ in P and P'. Prove that BPBP = $a^2 + c^2$.

(Note: There is minor error in actual question. It must be $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ *or BP.B'P* $= a^2 + \frac{1}{C^2}$.

Solution:

The generator through any general point (a cosθ, b sinθ) on the principal elliptic section is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c} \qquad \text{...(i)}$$

(Both systems)

Taking the positive system for the extremity of minor axis $\theta=\frac{\pi}{2}$ and $\frac{3\pi}{2}$

i.e.,
$$\frac{x-a}{a}=\frac{y-b}{0}=\frac{z}{c} \qquad \text{...(ii)}$$

and
$$\frac{x}{-a}=\frac{y+b}{0}=\frac{z}{c} \qquad \text{...(iii)}$$

Any point on these lines is

$$\frac{x}{a}=\frac{y-b}{a}=\frac{z}{c}=r \text{ and } \frac{x}{-a}=\frac{y+b}{0}=\frac{z}{c}=r_2$$

i.e., $x=ar, y=b, z=cr_1$ and $x=-ar, y=-b, z=cr_2$

Note that the distance of such a point from B is $\sqrt{l^2+m^2+n^2r_1}=\sqrt{a^2+c^2r_1}$.

For, P, P this general point must be on the variable generator whose equation is given by (i) (taking the other system).

$$\frac{ar_1-a\cos\theta}{a\sin\theta}=\frac{b-b\sin\theta}{-b\cos\theta}=-r_1$$

and
$$\frac{-ar_1-a\cos\theta}{a\sin\theta}=\frac{-b-b\sin\theta}{-b\cos\theta}=-r_2$$

$\Rightarrow$
$$r_1=\frac{\cos\theta}{1+\sin\theta}.$$

$$r_2=-\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$\therefore$
$$BP=\sqrt{a^2+c^2}\,|r_1|$$

$$B'P'=\sqrt{a^2+c^2}\,|r_2|$$

$\therefore$
$$BP.B'P'=\left(a^2+c^2\right)\frac{\cos\theta}{1+\sin\theta}\times\frac{1+\sin\theta}{\cos\theta}$$

$$= a^2 + c^2$$

8.4 Find the equations of the two generating lines through any point ($a\cos\theta$, $b\sin\theta$, 0), of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ of the hyperboloid by the plane z = 0.

Solution:

Any point on the elliptic section of the hyperboloid is ($a\cos\theta$, $b\sin\theta$, 0).
$\therefore$ Equations of any line through this point is

$$\frac{x-a\cos\theta}{l}=\frac{y-b\sin\theta}{m}=\frac{z-0}{n}=r(say) \qquad \ldots(i)$$

Any point on this line is ($Ir + a\cos\theta$, $mr + b\sin\theta$, and it lies on given hyperboloid, if

$$\frac{(lr+a\cos\theta)^2}{a^2}+\frac{(mr+b\sin\theta)^2}{b^2}-\frac{n^2r^2}{c^2}=1$$

$$\text{Or } \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2+2\left(\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right)r=0 \qquad \ldots(ii)$$

If the line (i) generator of given hyperboloid, then (i) lies wholly on the hyperboloid and the condition for which from (ii) are

$$\left(\frac{l^2}{a^2}\right)+\left(\frac{m^2}{b^2}\right)-\left(\frac{n^2}{c^2}\right)=0 \qquad \ldots\text{(iii)}$$

$$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0 \qquad \ldots\text{(iv)}$$

from (iv) we get,

$$\frac{l}{a\sin\theta}=\frac{m}{-b\cos\theta} \text{ or } \frac{(l/a)}{\sin\theta}=\frac{(m/-b)}{\cos\theta}$$

$$\Rightarrow \quad \frac{(l/a)}{\sin\theta}=\frac{m/-b}{\cos\theta}=\sqrt{\frac{(l^2/a^2)+(m^2/b^2)}{\sqrt{\sin^2\theta+\cos^2\theta}}}=\frac{\sqrt{n^2/c^2}}{1} \qquad \text{from...(iii)}$$

$$\Rightarrow \quad \frac{l}{a\sin\theta}=\frac{m}{-b\cos\theta}=\frac{n}{\pm c}$$

$\therefore$ The equation to the required generated from (i) are

$$\frac{x-\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}$$