MINDSET MAKERS

UPSC
IAS / IFoS
Mathematics
Optional

Vector Calculus Book

With PYQs till year 2022

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WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFOS :

01 Conceptual Development

02 Problem Solving Techniques

03 Assignments

04 Chapter wise PYQs Analysis

05 Test

MINDSET MAKERS

I.I.T UPSC

VECTOR ANALYSIS & CALCULUS

Vector Analysis: 5% Qs

Dot Product, Cross Product, Properties of vectors and their modulus

Example PYQs-

Let $\vec{a}, \vec{b}, \vec{c}$ are some given vectors. Show that they possibly make a triangle. Also Find medians of this triangle.

Vector Calculus = 95% Qs

(I) Differential

- Gradient
- Directional derivative
- Greatest rate of increase
- Angle between two surfaces
- Divergence, Solenoid Field, Change in per unit volume per unit time (Rate)
- Curl, Rotation, Work done etc. Exactness.

(II) Integral

- Line Integral, Surface Integral, Volume Integral
- Three Important Theorems
  Green's, Stokes, Gauss Divergence Theorem.

Chapter 1: Vector Analysis

Vectors: $(\text{Vector Space}): V(\text{IF})$

- A non-empty set of vector
- A field of scalar IF
- Vector Addition
- Scalar Multiplication

e.g. A special kind of vector spaces : $\mathbf{R}^n(\mathbf{R})$ [Euclidean Space]

$$\mathbf{R}^n = \left\{ (a_1, a_2 \ldots. a_n);\ a_1, a_2 \ldots. \text{ are real numbers} \right.$$

Field is Real Numbers.

Vector Addition: $(a_1, a_2, \ldots. a_n) + (b_1, \ldots b_n)$

$= (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n)$

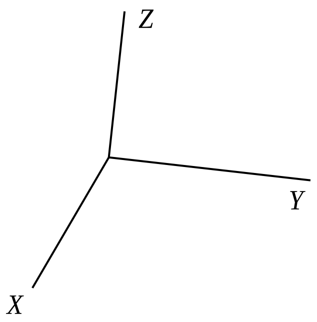
Scalar Multiplication

$\alpha(a_1, a_2 \ldots a_n) = (\alpha a_1, \alpha a_2 \ldots \alpha a_n)$

More Specifically, Here we will deal with $\mathbf{R}^3(\mathbf{R})$

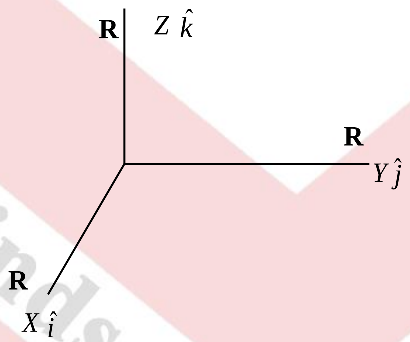
$\mathbf{R}^3 = \{(a_1, a_2, a_3); a_1, a_2, a_3 \text{ are real number}\}$

(3- Dimensional Euclidean Space)



Representation

$\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$



$\vec{a} = a_1\hat{i} + a_2\hat{j} + a\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Operation on Vectors:

(1) Scalar product:

$\vec{a} \cdot \vec{b}$ for two vectors

For three vectors-

(Scalar triple product) $\vec{a}, \vec{b}, \vec{c}$

$\vec{a} \cdot (\vec{b} \times \vec{c})$ = Volume of parallelepiped having edges $\vec{a}, \vec{b}, \vec{c}$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

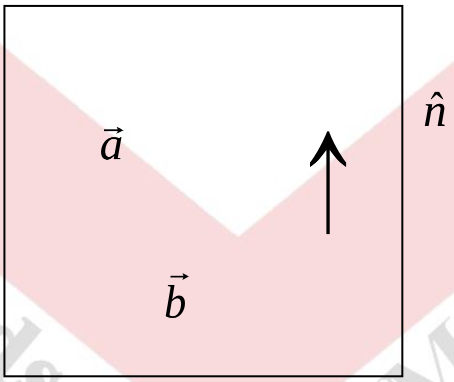
$$\vec{b}\cdot(\vec{c}\times\vec{a})=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left[\vec{b}\ \vec{c}\ \vec{a}\right]=\left[\vec{c}\ \vec{a}\ \vec{b}\right]$$

(2) <u>Vector Product</u>

$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}, 0<\theta<\pi$$

$\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.



<u>Vector Triple Product</u>

Formula

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

$$\left(\vec{a}\times\vec{b}\right)\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}$$

<u>Reciprocal Seat of Vectors</u>

$\vec{a},\vec{b},\vec{c}$ are set to form reciprocal set of vector if

$$\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$$

$$\vec{a}\cdot\vec{b}=\vec{a}'\cdot\vec{c}=\vec{b}'\vec{a}=\vec{b}'\vec{c}'=\vec{c}'\cdot\vec{a}=\vec{c}'\cdot\vec{b}=0$$

Note: $\vec{a}',\vec{b}',\vec{c}'$, $\vec{a},\vec{b},\vec{c}$ are said to be reciprocal

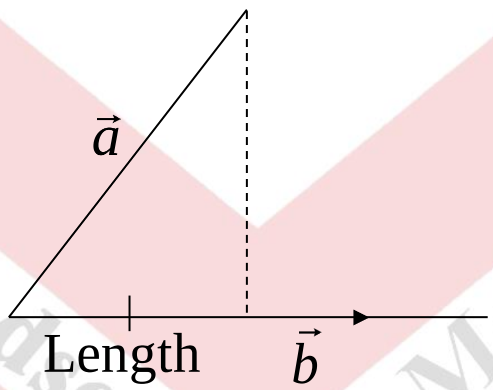
if $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)}=\dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\vec{b}\vec{c}\right]}$

$$\vec{b}'=\frac{\vec{c}\times\vec{a}}{\vec{b}\cdot(\vec{c}\times\vec{a})}=\frac{\vec{c}\times\vec{a}}{\left[\vec{b}\vec{c}\vec{a}\right]}=\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\vec{b}\vec{c}\right]}$$

Projection of $\vec{a}$ on $\vec{b}$

is given by $\vec{a}\cdot\hat{b}$

where $\hat{b}=\frac{\vec{b}}{|\vec{b}|}$



Example: Find projection of

$\vec{A}=\hat{i}-2\hat{j}+3\hat{k}$ on the vector $\hat{i}+2\hat{j}+2\hat{k}=\vec{B}$

$$\vec{A}\cdot\hat{B}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\frac{\left(\hat{i}+2\hat{j}+2\hat{k}\right)}{\left|\hat{i}+2\hat{j}+2\hat{k}\right|}=\frac{1-4+6}{\sqrt{1^2+4+4}}=\frac{3}{3}=1$$

Q1. Without making use of cross product find a vector perpendicular to the plane of

$\vec{A}=2\hat{i}-6\hat{j}-3\hat{k}$

$\vec{B}=4\hat{i}+3\hat{j}-\hat{k}$

Solution.

Let $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ is required unit vector

$\therefore\ \vec{c}\cdot\vec{A}=0$

$\Rightarrow\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)\cdot\left(2\hat{i}-6\hat{j}-3\hat{k}\right)=0$

$2c_1-6c_2-3c_3=0$ ....(i)

$\vec{c}\cdot\vec{B}=0\Rightarrow 4c_1+3c_2-c_3=0$ ....(ii)

On solving (i) and (ii) we get

$c_1=\frac{1}{2}c_3, c_2=-\frac{1}{3}c_3$

$\therefore \quad \vec{c} = \frac{1}{2}c_3\hat{i} - \frac{1}{3}c_3\hat{j} + 3c_3\hat{k}$

$$\therefore \quad \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\frac{1}{2}c_3\hat{i} - \frac{1}{3}c_3\hat{j} + c_3\hat{k}}{\left|\sqrt{\frac{1}{4}c_3^2 + \frac{1}{9}c_3^2 + c_3^2}\right|} = \frac{\frac{1}{2}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}}{\sqrt{\frac{1}{4} + \frac{1}{9} + 1}}$$

Formula

(1) Area of parallelogram with touching side as $\vec{A}, \vec{B}$

$= |\vec{A} \times \vec{B}|$

(2) Area of triangle with two adjacent sides $\vec{A}$, $\vec{B}$

$= \frac{1}{2}|\vec{A} \times \vec{B}|$

Q. Prove that the necessary and sufficient condition for $\vec{A}, \vec{B}, \vec{C}$ to be coplaner is $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

Solution.

The necessary part -

Let if $\vec{A}, \vec{B}, \vec{C}$ are coplaner then $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ must hold.

As we know that $\vec{B} \times \vec{C}$ represents a vector perpendicular to plane containing $\vec{B}$ and $\vec{C}$

$\therefore \quad \vec{A}$ must be $\perp$ to $\vec{B} \times \vec{C}$

$\therefore \quad \vec{A} \cdot (\vec{B} \times \vec{C}) = 0$

<u>Sufficient Part -</u>

Let if $\vec{A}.(\vec{B} \times \vec{C}) = 0$ then volume of parallelepiped with edges $\vec{A}, \vec{B}, \vec{C}$ must be zero.

$\Rightarrow \vec{A}, \vec{B}, \vec{C}$ must lie in same plane.

Q. Find the equation of the plane containing three vectors $P_1(2,-1,1), P_2(3,2,-1), P_3(-1,3,2)$

Solution. We know that

Equation of a plane is given as

$Ax + by + cz + d = 0$

*Two planes together represent straight line in 3D (if they intersect) represented by

$a_1x + a_2y + a_3z + a_4 = 0$

$b_1x + b_2y + b_3z + b_4 = 0$

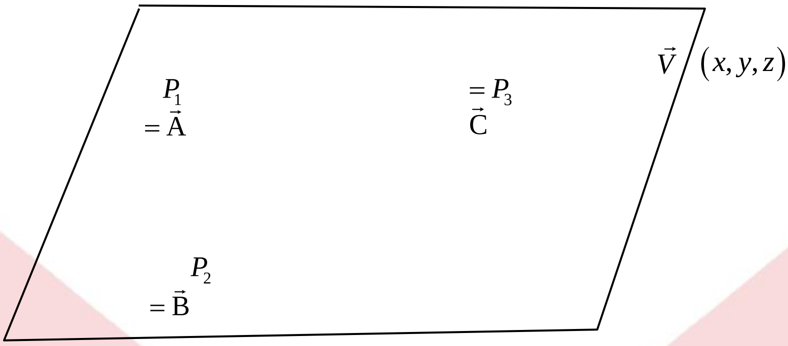
or in symmetrical form, Line passing through $(x_1, y_1, z_1), (x_2, y_2, z_2)$ is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$P_1(2,-1,1) = 2\hat{i} - \hat{j} + \hat{k} = \vec{A}$

$P_2(3,2,-1) = 3\hat{i} + 2\hat{j} - \hat{k} = \vec{B}$

$P_3(-1,3,2) = -\hat{i} + 3\hat{j} + 2\hat{k} = \vec{c}$



$(\vec{r} - \vec{A})(\vec{r} - \vec{B})(\vec{r} - \vec{C})$ are co-planer

$(\vec{r} - \vec{A}) \cdot ((\vec{r} - \vec{B}) \times (\vec{r} \times \vec{C})) = 0$

$\Rightarrow ((x-2)\hat{i} + (y+1)\hat{j} + (z-1)\hat{k}).$

$.[(x-3)\hat{i} + (y-2)\hat{j} + (z+1)\hat{k} \times (x+1)\hat{i} + (y-3)\hat{j} + (z-z)\hat{k}] = 0$

$\Rightarrow 11x + 5y + 13z = 30$

Q. Find the constant a so that the following vectors are co-planer

$2\hat{i} - \hat{j} + \hat{k}\ ,\hat{i} + 2\hat{j} - 2\hat{k},\ 3\hat{i} + a\hat{j} + 5\hat{k}$

A B C

Solution.

$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \quad (\vec{a}\ \vec{b}\ \vec{c})$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

$\Rightarrow 2(10+30) + 1(5+9) + 1(a-6) = 0$

$\Rightarrow 20 + 6a + 14 + a - 6 = 0$

$\Rightarrow 34 - 6 + 7a = 0$

$\Rightarrow 28 + 7a = 0$

$\boxed{a = -4}$

PYQ [2016]

Prove that the vector

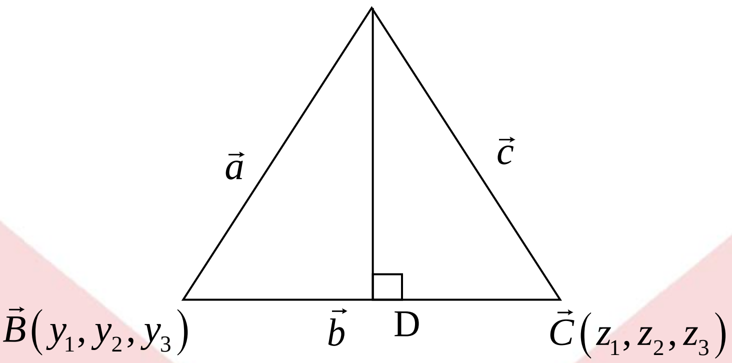
$\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$

$\vec{b} = \hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$

can form the sides of triangle. Find the median of Δ.

Hint for solution;



$\lvert\vec{a}\rvert$	$\lvert\vec{b}\rvert$	$\lvert\vec{c}\rvert$
$=\sqrt{9+1+4}$	$=\sqrt{1+9+16}$	$=\sqrt{16+4+36}$
$=\sqrt{14}$	$=\sqrt{26}$	$=\sqrt{56}$
		$=2\sqrt{14}$

Sum of the sides $\geq$ third side

$$|\vec{a}|+|\vec{b}| \geq |\vec{b}|$$

$$3\sqrt{14} \geq \sqrt{26}$$

yes possible of side of Δ

$\vec{D}$ is mid point of $\vec{B}, \vec{C}$

$$\vec{D}=\frac{1}{2}(y_1+z_1), \frac{1}{2}(y_2+z_2), \frac{1}{2}(y_3+z_3)$$

$$|\vec{D}|=\sqrt{\frac{1}{4}(y_1+z_1)^2+\frac{1}{4}(y_2+z_2)^2+\frac{1}{4}(y_3+z_3)^2}$$

$$=\frac{1}{2}\sqrt{(y_1+z_1)^2+(y_2+z_2)^2+(y_3+z_3)^2}$$

<u>Vector Differentiation</u>

Type (1) Problems

Simple Differentiation

e.g. Velocity, Acceleration, Momentum, Work done, K.E.

if position vector

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Parametric Form

e.g. if $\vec{r} = \sin t\hat{i} + e^t\hat{j} + e^{2t}\cos t + \hat{k}$ then find velocity at $t = 0$ and acceleration at $t = 0$

Note- Some added information from Vector Analysis will be needed here too.

Type II Problem

(1) Gradient

- Finding gradient at some point
- Finding normal vector to cover surface
- Angle of intersection between two level surfaces
- Gradient and greatest rate of increase/decrease

(2) Divergence

(3) Curl

Type - I

(i) $d\left(\vec{A}+\vec{B}\right) = d\vec{A} + d\vec{B}$

(ii) $d\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\cdot d\vec{B} + d\vec{A}\cdot\vec{B}$

(iii) $d\left(\vec{A}\times\vec{B}\right) = \vec{A}\times d\vec{B} + d\vec{A}\times\vec{B}$

(iv) $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\cdot\vec{B}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\vec{A}\cdot\vec{B}\right)\right)$

$= \frac{\partial}{\partial y}\left(\vec{A}\cdot\frac{\partial}{\partial x}\vec{B} + \left(\frac{\partial}{\partial x}\vec{A}\right)\vec{B}\right)$

I II I II

$= \vec{A}\cdot\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\vec{B}\right) + \left(\frac{\partial}{\partial y}\vec{A}\right)\left(\frac{\partial}{\partial z}\vec{B}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\vec{A}\right)\cdot\vec{B} + \left(\frac{\partial}{\partial y}\vec{B}\right)\left(\frac{\partial}{\partial y}\vec{A}\right)$

Similarly we can do for $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right)$

PYQ [2012]

If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

$\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, then find the value of $\frac{\partial^2}{\partial x\partial y}\left(\vec{A}\times\vec{B}\right)$ at $(1,0,-2)$

Solution. Hint:

$$\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\vec{A}\times\vec{B}\right)$$

Method-(1) First find $\vec{A}\times\vec{B}$ then derivative

Method-(2) Applying formula

$$\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right)\right)$$

$$=\frac{\partial}{\partial x}\left(\vec{A}\times\frac{\partial}{\partial y}\times\vec{B}+\vec{B}\times\frac{\partial}{\partial y}\vec{A}\right)$$

$$=\vec{A}\times\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\vec{B}\right)+\left(\frac{\partial}{\partial y}B\right)\frac{\partial}{\partial y}A+B\times\left(\frac{\partial}{\partial y}\vec{A}\right)\frac{\partial}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\vec{A}\right)\cdot\vec{B}$$

Q. For two vectors $\vec{a}$, $\vec{b}$ given by

[2017]

$$\vec{a}=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$$
$$\vec{b}=\sin t\,\hat{i}-\cos t\,\hat{j}$$

Determine $\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$, $\frac{d}{dt}\left(\vec{A}\times\vec{B}\right)$

Solution.

$$\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)=\vec{a}\frac{d\vec{b}}{dt}+\vec{b}\cdot\frac{d\vec{\theta}}{dt}$$

$$=\left(5t^2\hat{i}+t\hat{j}-t^3\hat{k}\right)\cdot\left(\cos t\,\hat{i}+\sin t\,\hat{j}\right)+\left(10t\,\hat{i}+\hat{j}-3t^2\hat{k}\right)\cdot\left(\sin t\,\hat{i}-\cos t\,\hat{i}\right)$$

$$=5t^2\cos t+t\sin t-t^3\cdot 0+10t\sin t-\cos t-3t^2\cdot 0$$

$$=5t^2\cos t+11t\sin t-\cos t$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\frac{d\vec{b}}{dt}+\vec{b}\times\frac{d\vec{a}}{dt}$$

$$=\begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix}+\begin{vmatrix} i & j & k \\ 10t & 1 & -3t \\ \sin t & -\cos t & 0 \end{vmatrix}$$

Q. The position vector of a moving particle at time t is

[2017]

$$\vec{v}=\sin t\,\hat{i}+\cos 2t\,\hat{j}+\left(t^2+2t\right)\hat{k}$$

Find the component of acceleration $\vec{a}$ in the direction parallel to the velocity vector $\vec{v}$ and perpendicular to the plane of $\vec{v}$ and $\vec{v}$ at time $t=0$.

Solution. Hint:

$$\because\quad \vec{v}=\sin t\,\hat{i}+\cos 2t\,\hat{j}+\left(t^3+2t\right)\hat{k}$$

$$\vec{v}=\frac{d\vec{r}}{dt}=\cos t\,\hat{i}-2\sin 2t\,\hat{j}+\left(2t+2\right)\hat{k}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$$

Dynamics

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{v}}{dt^2}$$
$$= -\sin t\,\hat{i} - y\cos 2t\,\hat{j} + 2\hat{k}$$

Vector Analysis required here

If $c = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Here c_1, c_2, c_3 are component of $\vec{c}$

Let's say if it is given $\vec{c}$ is parallel to $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\Rightarrow \frac{c_1}{b_1} = \frac{c_2}{b_2} = \frac{c_3}{b_3} \quad ....(1)$$

Let's say if $\vec{c}$ is perpendicular to $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

$$c_1d_1 + c_2d_2 + c_3d_3 = 0 \quad ....(2)$$

Using those condition (1) and (2)

Can we try to figure out

$c_1 =$

$c_2 =$

$c_3 =$

We need

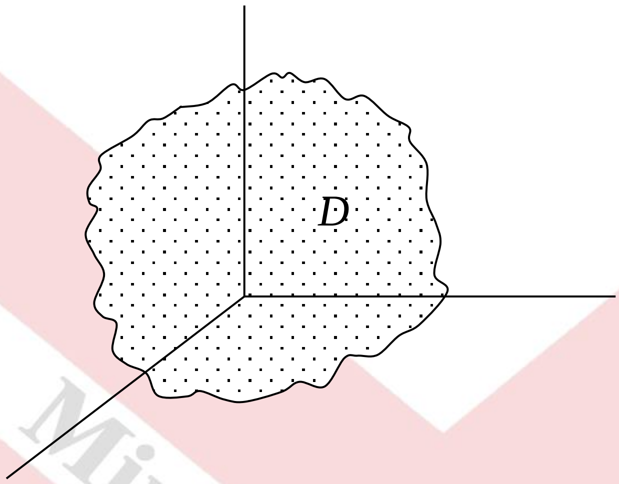
$$\frac{a_1}{v_1} = \frac{a_2}{v_2} = \frac{a_3}{v_3} \quad ....(1)$$

$$\because \quad \vec{\gamma} \times \vec{v} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$$

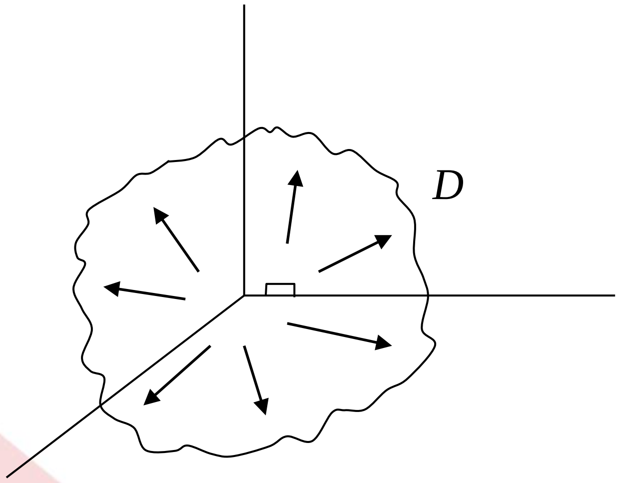
$$\therefore \quad \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0 \quad ....(2)$$

Vector Calculus

Scalar Field: If we can assign some particular scalar value to each point of a region D in some space then this scalar valued function is called scalar function of the position and we say $f(x, y, z)$ is a scalar field defined on region D.
e.g.
Temperature $T(x, y, z) = x^2 y + yz^3$ on earth's surface is a scalar field because we can assign a particular scalar value to each point on surface.
$T(1,2,3) = 1^2 - 2 + 2 \cdot 3^2 = 56$



Vector Field: Suppose to each point (x, y, z) in the region D in space there corresponds a vector $\vec{f}(x, y, z)$ then $\vec{f}$ is known as vector function of the position (x, y, z) and we say that a vector field $\vec{f}$ has been defined on D.



Level Surfaces: Let's consider a function of 3 variables $f(x, y, z)$ whose inputs are points in $\mathbf{R}^3$ and whose outputs are numbers.

e.g.

$f(x, y, z) = x^2 + y^2 + z^2$

or $f(x, y, z) = x^2 + y^3$

or $f(x, y, z) = z - (x^2 + y^2)$

A function $f(x, y, z)$ is said to be of level $\mathbf{K}$ to be the set of all points in $\mathbf{R}^3$ which are solution of $f(x, y, z) = \mathbf{K}$.

e.g.

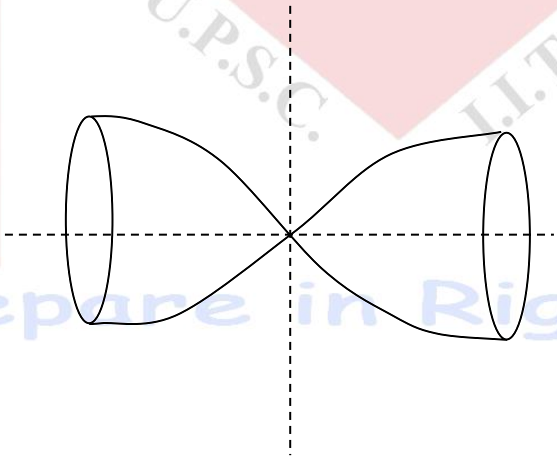
$\therefore \quad z = \phi(x^2, y^2)$

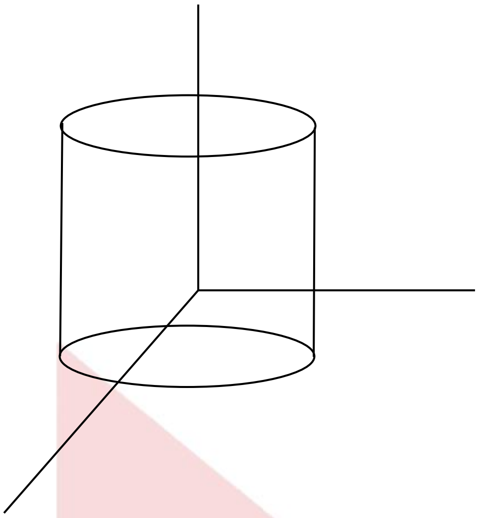
$a \le t \le b$ cylinder.

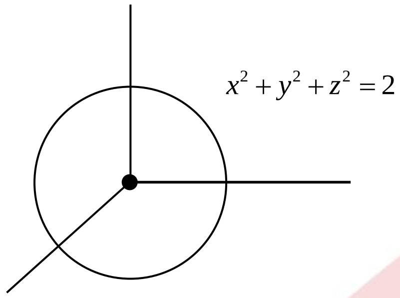
$\therefore \quad z = \phi(x^2, y^2)$

$a \le z \le b$

cylinder







(i) Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(ii) Elliptic paraboloids: $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

(iii) Hyperbolic paraboloids: $\frac{z}{c}=\frac{x^2}{a^2}-y^2/b^2$

(iv) Hyperboloid in one-shed: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(v) Hyperboloid in two-shed: $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Directional Derivative

- $\dfrac{\partial f}{\partial x}$ is the directional derivatives of f along the direction of unit normal vector $\hat{i}$.
- $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are directional derivatives of f along $\hat{j}$ and $\hat{k}$.

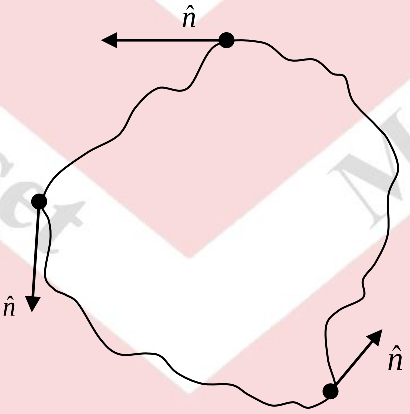
  i.e., $\dfrac{\partial f}{\partial n}$ is the directional derivative of the function f in an arbitrary direction n (along unit normal vector $\hat{n}$)

Gradient and Level Surfaces

For a scalar function f the gradient vector is defined as $\dfrac{\partial f}{\partial n}\cdot\hat{n}$ where $\hat{n}$ is the unit normal vector to the level surface f at some point in the direction of increasing f and $\dfrac{\partial f}{\partial n}$ is called the normal derivative at that point.

Grad f :

$$\vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f \;= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$



Note:

$$|\text{grad } f| = \left|\frac{\partial f}{\partial n}\right| |\hat{n}|$$

$$\therefore\; \left|\vec{\nabla} f\right| = \frac{\partial f}{\partial n}\cdot 1 = \frac{\partial f}{\partial n}$$

For a function f, the gradient vector $\vec{\nabla} f$ has the properties:

- It points in the direction in which f increases most rapidly (fastest).
- It is perpendicular to level curves or surface of f.

Divergence

"Loss" per unit volume, per unit time

Vector valued function

$$\vec{\nabla}\cdot\vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}} \qquad \ldots(1)$$

Let's consider a vector valued function

$\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$ which is defined and differentiable at each point (x,y,z) in a region of space. Then div$(\vec{f})$ is defined by equation (1).

Although $\vec{f}$ is a vector valued function but div $(\vec{f})$ is a scalar.

e.g.

Find the Divergence of $\vec{F} = \left(e^{x\log z} + \cos y\right)\hat{i} + \left(z^2 + \log x\right)\hat{j} + e^{2z}\hat{k}$ at $\left(1, e^{2\log 7}, \log 5\right)$ over a region in $\mathbf{R}^3$ in which $\vec{F}$ is defined and differentiable.

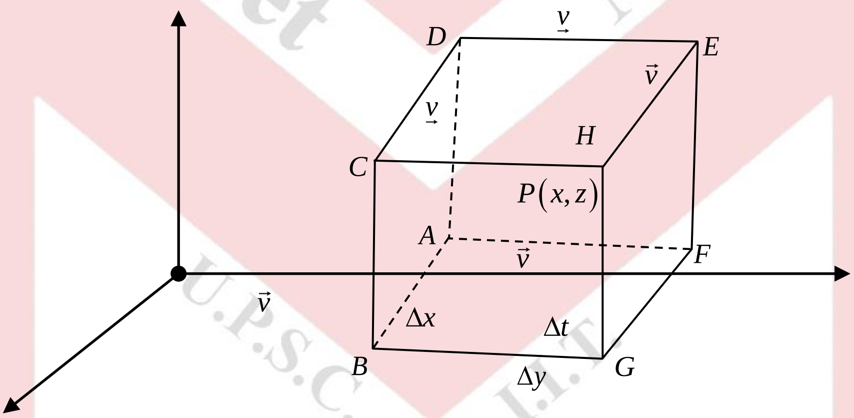
$$\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}\left(e^{x\log z} + \cos y\right) + \frac{\partial}{\partial y}\left(z^2 + \log x\right) + \frac{\partial}{\partial z}\left(e^{2z}\right)$$

$$= e^{x\log z}\log z + 0 + 2e^{2z}$$

$$= \log z\, e^{x\log z} + 2e^{2z} \text{ at } \left(1, e^{\sqrt{2}\log 7}, \log 5\right)$$

$$\vec{\nabla}\cdot\vec{f} = \log(\log 5)e^{\log(\log 5)} + 2e^{2\log 5} = \log 5\log(\log 5) + 50$$

Q. A fluid moves so that it's velocity at any point $P(x,y,z)$ is $\vec{v}(x,y,z)$. Show that the loss of fluid per unit volume per unit time in a small parallelepiped having centre at $P(x,y,z)$ and edge parallel to the coordinate axes and having magnitude $\Delta x, \Delta y$ and Δz respectively, is given approximately by div$\vec{v}$.



Let x component of velocity $\vec{v}$ at $P = v_1$

x component of $\vec{v}$ at centre of the face

AFED $= v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x$ approx.

- x component of $\vec{v}$ at centre of the face

  GHCB $= v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x$ approx.

Now it is clear volume of the fluid entering through the face GHCB per unit time

$$=\left(v_1+\frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y\Delta z$$

$$\because \quad v=\frac{d}{t}$$

$t=1, v=d$

Volume $=d\Delta y\Delta z$

Volume of the fluid existing through the face

$$\text{AFED}=\left(v_1-\frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y\Delta z$$

So loss in volume per unit time in x-direction

$$=\left(v_1+\frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y\Delta z-\left(v_1-\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y\Delta z=\frac{\partial v_1}{\partial x}\Delta x\Delta y\Delta z \qquad ....(1)$$

Similarly loss in volume of the fluid per unit time in the y-direction

$$=\frac{\partial v_2}{\partial y}\Delta x\Delta y\Delta z$$

and in z-direction

$$=\frac{\partial v_3}{\partial z}\Delta x\Delta y\Delta z$$

$\therefore$ Total loss in volume of the fluid per unit volume per unit time equal to

$$=\frac{\frac{\partial v_1}{\partial x}\Delta x\Delta y\Delta z+\frac{\partial v_2}{\partial y}\Delta x\Delta y\Delta z+\frac{\partial v_3}{\partial z}\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}=\vec{\nabla}\cdot\vec{\nabla}\ \text{div}\,\vec{v}\text{, where}$$

$$\vec{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$$

Note:

The above article is true exactly only in the limit as the parallelepiped shrinks to P i.e., $\Delta x, \Delta y, \Delta z$ approaches to '0'. If there is no loss of fluid anywhere then $\boxed{\text{div}\,\vec{v}=\vec{\nabla}\cdot\vec{v}=0}$. This is known as equation of continuity for an incompressible fluid.

i.e., neither source nor sinks such vector $\vec{v}$ is known as **Solenoidal.**

<u>Curl</u>

Let's consider a vector valued function $\vec{f}=f_1(x,y,z)\hat{i}+f_2(x,y,z)\hat{j}+f_3(x,y,z)\hat{k}$

If $\vec{F}$ is differentiable, then the curl or rotation of $\vec{F}$ is defined as

$$curl\,\vec{F}=\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}=\hat{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial t}\right)-\hat{j}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial t}\right)+\hat{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Note:

At the time of numerical solution we should take care of e.g. f_3, f_1 are functions free from y and f_2, f_3 are free from x and f_1, f_2 are from z, then the calculation is very easy.

e.g. curl $\vec{F}$, $\vec{F} = \left(e^{x^2}\cos x\right)\hat{i} + e^{2y}\hat{j} + e^{z^2}\log z\,\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial t}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) = 0 \text{ (zero vector)}$$

Note:

Suppose ϕ and $\vec{A}$ are differentiable scalar and vector functions respectively and both have continuous 2nd partial derivatives, then following laws hold.

(i) $\vec{\nabla}\cdot\left(\vec{\nabla}\phi\right) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

(ii) $\vec{\nabla}\times\left(\vec{\nabla}\phi\right) = 0$ i.e., curl grad $\phi = 0$

(iii) $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$ i.e., div curl $\vec{A} = 0$

(iv) ϕ satisfies Laplacian equation if $\nabla^2\phi = 0$

Note:

This problem indicates that the curl of a vector field has something to do with the rotational properties of the field (Because $\vec{\omega}$ is present).

- If the field $\vec{F}$ is that due to a moving fluid e.g. a paddle wheel placed at various points in the field would tend to rotate in regions where curl $\vec{F} \neq 0$, while curl $\vec{F} = 0$ in the region, there would be no rotation and in this case, the vector field $\vec{F}$ is called Irrational.
- If a field is not irrotational then sometimes it is also called as a "Vortex Field".

Vector Integration

Let's consider a vector valued function $\vec{F} = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}$ in a region D of some space.

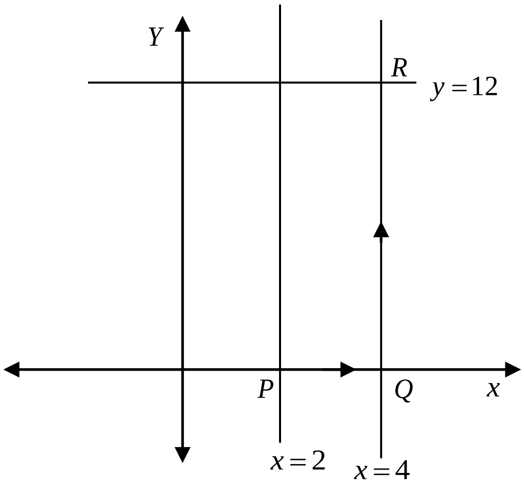
If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of some point in this region D.

Let c be a curve in this region and we want to find the value of the integral $\int_c \vec{F}\cdot d\vec{r}$ i.e., integration along the curve c.

$$\int_c \vec{F}\cdot d\vec{r} = \int_c \left\{f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k}\right\}\cdot\left\{dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right\}$$

$$\Rightarrow \boxed{\int_c \vec{F}\cdot d\vec{r} = \int_c f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz}$$

e.g. Evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = xy\,\hat{i} + \left(x^2 + y^2\right)\hat{j}$ and the curve c is the x-axis from $x = 2$ to $x = 4$ and the line $x = 4$ from $y = 0$ to $y = 12$.



$$\int_c \vec{F}\cdot d\vec{r} = \int_c xy\,dx + \left(x^2+y^2\right)dy$$

Along the line PQ; $x=2$ to $x=4$, $y=0$ and $dy=0$

$$\int_{PQ} \vec{F}\cdot d\vec{r} = \int_{x=2}^{x=4} (x\times 0\times dx) + \left(x^2+0^2\right)\cdot 0 = 0$$

Along the line QR

$x=4 \Rightarrow dx=0, y=0$ to $y=12$

$$\int_{QR} \vec{F}\cdot d\vec{r} = \int_{y=0}^{y=12} 4\times y\times 0 + \left(4^2+y^2\right)dy = \left[16y+\frac{y^3}{3}\right]_0^{12} = \left[192+\frac{(12)^3}{3}\right] = 192+576=768$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_{PQ} \vec{F}\cdot d\vec{r} + \int_{QR} \vec{F}\cdot d\vec{r} = 0+768=768$$

Conservative Fields

Suppose $\vec{F}=\vec{\nabla}\phi$ everywhere in a region $\mathbf{R}$ of the space where R is defined by $a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$ and $\phi(x,y,z)$ is a single valued function and has continuous partial derivatives in the region $\mathbf{R}$. Then

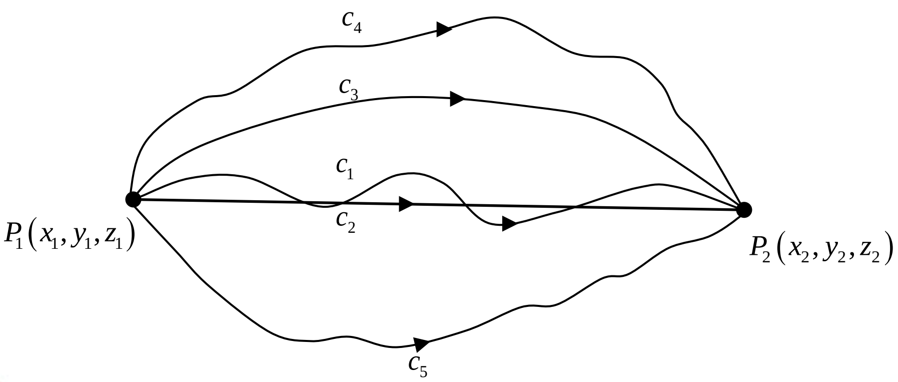
(i) $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ is independent of the path c in $\mathbf{R}$ joining the points P_1 and P_2 in $\mathbf{R}$.

(ii) $\oint_c \vec{F}\cdot d\vec{r} = 0$ around any closed curve c in $\mathbf{R}$.

In such a case $\vec{F}$ is called conservative vector field and ϕ is its scalar potential.

Q. Suppose $\vec{F}=\nabla\phi$, where ϕ is single valued and has continuous partial derivatives. Show that the work done in moving a particle from a point P_1 to $P_2(x,y,z)$ in this vector field is independent of the path joining P_1 and P_2. Conversely suppose $\int_c \vec{F}\cdot d\vec{r}$ is independent of the path c joining two points. Show that $\exists$ a function ϕ s.t. $\vec{F}=\vec{\nabla}\phi$.

Proof:



Let $\vec{F} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

Work done $= \int_{c_1} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \right)$

$= \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \right) = \int_{P_1}^{P_2} d(\phi) = \phi(P_2) - \phi(P_1)$

Since it is given that ϕ is single valued so whatever the path $c_1 / c_2 / c_3 \ldots$ joining the points P_1 and P_2 is chosen, we get

Work done $= \int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r} = \int_{c_3} \vec{F} \cdot d\vec{r} = \phi(P_2) - \phi(P_1)$

Therefore if $\vec{F} = \vec{\nabla}\phi$ the work done or the line integral $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the path.

<u>Conversely</u>

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

We have to show

If $\int_c \vec{F} \cdot d\vec{r}$ is independent of path c.

Then $\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

i.e., we have to show $F_1 = \frac{\partial \phi}{\partial x}, F_2 = \frac{\partial \phi}{\partial y}, F_3 = \frac{\partial \phi}{\partial z}$

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{F} \cdot d\vec{r} = \int_{(x_1, y_1, z_1)}^{(x, y, z)} (F_1 dx + F_2 dy + F_3 dz) \quad \text{....(1)}$$

$$\therefore \quad \phi(x + \Delta x, y, t) = \int_{(x_1, y_1, z_1)}^{(x+\Delta x, y, t)} F_1 dx + F_2 dy + F_3 dz \quad \text{....(2)}$$

Target

$$\frac{\partial \phi}{\partial x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y, z) - \phi(x, y, z)}{\Delta x}$$

On subtracting (1) from (2), we get

$\phi(x + \Delta x, y, t) - \phi(x, y, z)$

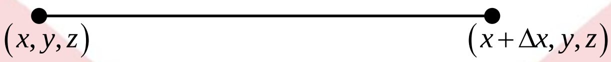
$$= \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} F_1dx + F_2dy + F_3dz - \int_{(x_1,y_1,z_1)}^{(x,y,z)} F_1dx + F_2dy + F_3dz$$

$$= \int_{(x,y,z)}^{(x_1,y_1,z_1)} F_1dx + F_2dy + F_3dz + \int_{(x_1,y_1,z_1)}^{(x+\Delta x,y,z)} F_1dx + F_2dy + F_3dz$$

$$= \int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1dx + F_2dy + F_3dz$$

$$\Rightarrow \frac{\phi(x+\Delta x, y, t) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{(x,y,z)} F_1dx + F_2dy + F_3dz \qquad ....(3)$$

Since we have taken the integral in the R.H.S. of equation (3) is independent of the path joining points (x, y, z) and $(x+\Delta x, y, z)$. So, let's choose the path as straight line



$$\therefore \ dy = 0, \ dz = 0$$

So equation (3) becomes

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\phi(x+\Delta x, y, t) - \phi(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_{(x,y,z)}^{(x+\Delta x,y,z)} F_1dx + 0 + 0}{\Delta x} = d\int F_1dx = F_1$$

$$\therefore \ \frac{\partial \phi}{\partial x} = F_1 \qquad ....(4)$$

Similarly we can find $\frac{\partial \phi}{\partial y} = F_2, \frac{\partial \phi}{\partial z} = F_3$

Therefore, we have $\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$\therefore \ \boxed{\vec{F} = \vec{\nabla}\phi}$$

<u>Theorem</u>

Suppose $\vec{F}$ is a conservative field then curl $\vec{F} = 0$ (i.e. $\vec{F}$ is irrotational) and conversely if curl $\vec{F} = 0$ then $\vec{F}$ is conservative.

Proof:

Let $\vec{F}$ is conservative field, then by definition $\vec{F} = \vec{\nabla}\phi$

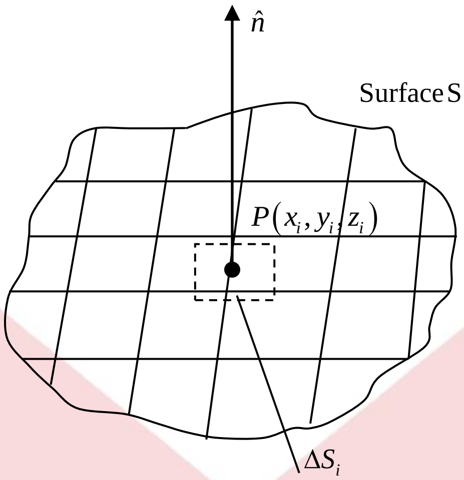
$\therefore$ curl $\vec{F} = \vec{\nabla} \times (\vec{\nabla}\phi) = 0$

i.e., $\vec{F}$ is conservative $\Rightarrow$ curl $\vec{F} = 0$.

Surface Integral

The surface integral of a function ϕ over a surface S (which need not be closed surface or phase surface). May be defined as:

Divide the surface S into m small elements $\Delta S_1, \Delta S_2, ..., \Delta S_n$ and form the expression $\phi_1 \Delta S_1 + \phi_2 \Delta S_2 + ... + \phi_m \Delta S_m$, where ϕ_i is the value of the function ϕ at point P_i. Now if $m \to \infty$ we land up with the surface integral $\int_S \phi \cdot ds$ or for vector valued functions $\int_S \vec{F} \cdot d\vec{s}$.



Let if $\hat{n}$ is the unit normal vector drawn outward to the surface S then we can define the vector elementary surface area by

$$d\vec{S} = \left| d\vec{S} \right| \cdot \hat{n}$$

$$\Rightarrow \boxed{d\vec{S} = \hat{n} \cdot ds}$$

$\downarrow$

Elementary surface area

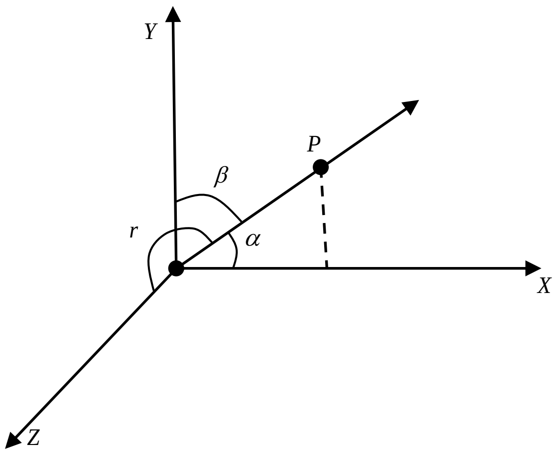
Note:

$$\therefore \int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \hat{n}\, dS$$

Here $\vec{F} \cdot \hat{n}$ is the component of $\vec{F}$ along $\hat{n}$ i.e., normal to the surface S and $\int_S \vec{F} \cdot \hat{n}\, dS$ is called the total Flux across the surface S.

How to calculate $\int_S \vec{F} \cdot d\vec{s}$?

We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = r\cos\alpha\hat{i} + r\cos\beta\hat{j} + r\cos r\hat{k}$



The same concept we apply for the elementary surface area.
If $ds\cos\alpha, ds\cos\beta, ds\cos r$ are the orthogonal projections of the elementary area ds on the YZ-plane, ZX-plane and XY-plane respectively.

Therefore, now we can write
$d\vec{S} = ds\cos\alpha\hat{i} + ds\cos\beta\hat{j} + ds\cos r\hat{k}$
Here α, β, r are direction angles of ds with x-axis, y-axis and z-axis respectively.
$\hat{n}\cdot ds = ds\cos\alpha\hat{i} + ds\cos\beta\hat{j} + ds\cos r\hat{k}$
$\hat{i}\cdot\hat{n}ds = ds\cos\alpha$

$$\hat{i}\cdot\hat{n}\,ds = dy\cdot dz \Rightarrow ds = \frac{dy\cdot dz}{\hat{i}\cdot\hat{n}} \qquad ....(1)$$

$$\hat{j}\cdot\hat{n}\,ds = ds\cos\beta \Rightarrow ds = \frac{dz\cdot dx}{\hat{j}\cdot\hat{n}} \qquad ....(2)$$

$$\hat{k}\cdot\hat{n}\,ds = ds\cos r \Rightarrow ds = \frac{dx\cdot dy}{\hat{k}\cdot\hat{n}} \qquad ....(3)$$

$$\int_S \vec{F}\cdot d\vec{s} = \int_S \vec{F}\cdot\hat{n}\,ds = \int_x\int_y \vec{F}\cdot\hat{n}\frac{dx\,dy}{\hat{k}\cdot\hat{n}} = \int_y\int_z \frac{\vec{F}\cdot\hat{n}\,dy\,dz}{\hat{i}\cdot\hat{n}} = \int_z\int_x \frac{\vec{F}\cdot\hat{n}\,dt\,dx}{\hat{j}\cdot\hat{n}} \qquad ....(4)$$

Whichever form in (4) suits you to easily integrate (According to given condition); Apply that

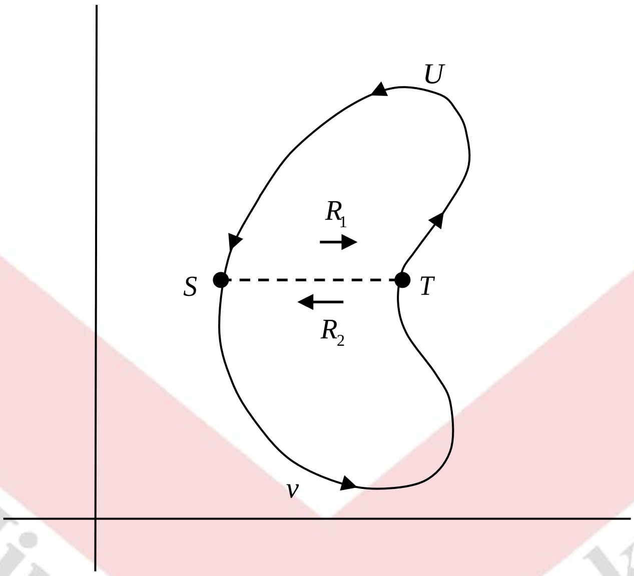
Green's Theorem

Let's consider a closed region **R** in the xy-plane bounded by a simple closed curve c and suppose $P(x,y), Q(x,y)$ are continuous function with continuous derivatives in the region R.
Then

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Note:
Unless otherwise stated, we will always consider that the line integral is described in the positive sense (i.e. the curve c is transverse in the counterclockwise direction.

Note:
We can extend the proof of Green's Theorem in the plane to the curve c for which lines parallel to the coordinate axis may cut the curve c in more than 2 points.



$$\therefore \int_{STUS} M dx + N dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad ....(1)$$

$$\int_{SVTS} M dx + N dy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad ....(2)$$

On adding (1) and (2), we get

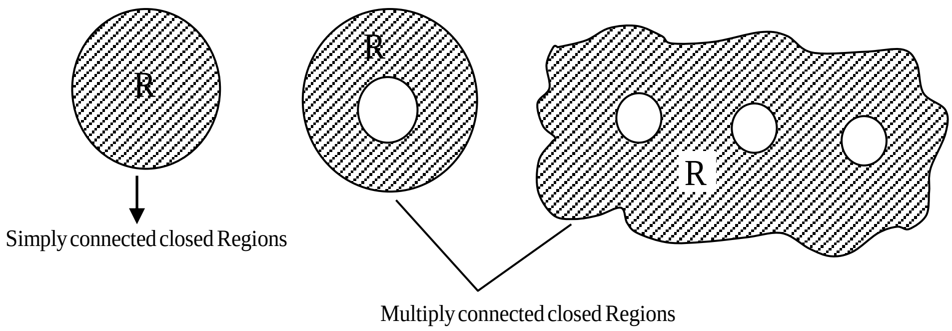
L.H.S.

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

R.H.S.

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

Note:
From the above description we just try to show that the Green's Theorem in the plane is applicable for simply connected closed Regions.



Note:
The Green's Theorem in the plane is also applicable for the multiply connected Regions. It can also be shown by the similar process as above.

Q. Express the Green's Theorem in the plane in vector notation.
Solution.
Let's consider a vector field function
$\vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$ and the position vector in the plane as $\vec{r} = x\hat{i} + y\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

Now, curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \hat{i}\left(-\frac{\partial N}{\partial z}\right) + \hat{j}\left(\frac{\partial M}{\partial y}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$\therefore$ curl $\vec{F} \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Now the Green's Theorem in the plane can be written as $\boxed{\oint \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F} \cdot \hat{k} \cdot dr}$

- A Generalization of this phenomena to the surface S in the space having a curve c as a boundary leads quite naturally to Stoke's Theorem.

Q. Show that a necessary and sufficient condition for $F_1dx + F_2dy + F_3dz$ to be an exact differential is that $\boxed{\vec{\nabla} \times \vec{F} = 0}$, where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

Q. Show that $\left(y^2z^3\cos x - 4x^3z\right)dx + 2z^3y\sin x\,dy + \left(3y^2z^2\sin x - x^4\right)dt$ is an exact differential of a function ϕ and find such ϕ.
Solution.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3\cos x - 4x^3z & 2z^3y\sin x & 3y^2z^2\sin x - x^4 \end{vmatrix}$$

Proof:
Let $F_1dx + F_2dy + F_3dz$ is an exact differential

i.e., $F_1dx + F_2dy + F_3dz = d(\phi) = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

i.e., $F_1 = \frac{\partial \phi}{\partial x}, F_2 = \frac{\partial \phi}{\partial y}, F_3 = \frac{\partial \phi}{\partial z}$

i.e., $\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \vec{\nabla}\phi$

$\therefore \quad \vec{\nabla} \times \vec{F} = \vec{\nabla} \times \left(\vec{\nabla}\phi\right) = 0$

Let $\vec{\nabla} \times \vec{F} = 0$

Then $\vec{F}$ must be of the form $\vec{\nabla}\phi$

i.e., $\vec{F} \cdot d\vec{r} = \vec{\nabla}\phi \cdot d\vec{r}$

$\Rightarrow F_1 dx + F_2 dy + F_3 dz = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$\Rightarrow F_1 dx + F_2 dy + F_3 dz = d\phi$

$\therefore \quad F_1 dx + F_2 dy + F_3 dz = d\phi$ is an exact differential equation.

Solution.

$\left(y^2 z^3 \cos x - 4x^3 z\right)dx + 2z^3 y \sin x\, dy + \left(3y^2 z^2 \sin x - x^4\right)dz$

$= F_1 dx + F_2 dy + F_3 dz$

i.e., $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\because \quad \vec{\nabla} \times \vec{F} = 0$ (on calculating)

$\Rightarrow \vec{F}$ is an exact differential i.e., $\exists$ a function ϕ s.t.

$F_1 dx + F_2 dy + F_3 dz = d\phi$

i.e., $\frac{\partial \phi}{\partial x} = F_1, \frac{\partial \phi}{\partial y} = F_2, \frac{\partial \phi}{\partial z} = F_3$

$\Rightarrow \phi_1 = y^2 z^3 \sin x - x^3 z$

$\phi_2 = z^3 y^2 \sin x$

$\phi_3 = y^2 z^3 \sin x - x^4 z$

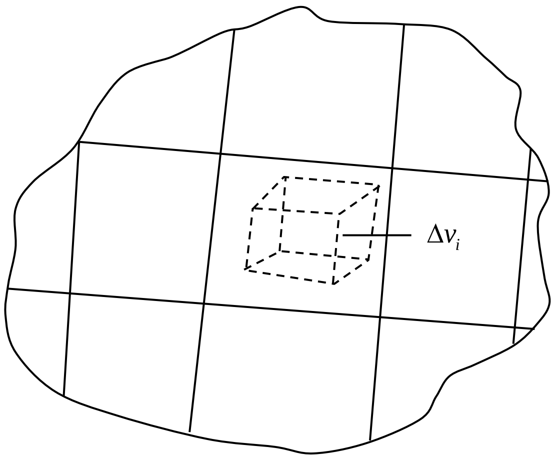
$\therefore \quad \boxed{\phi(x, y, z) = y^2 z^3 \sin x - x^4 z}$

Result

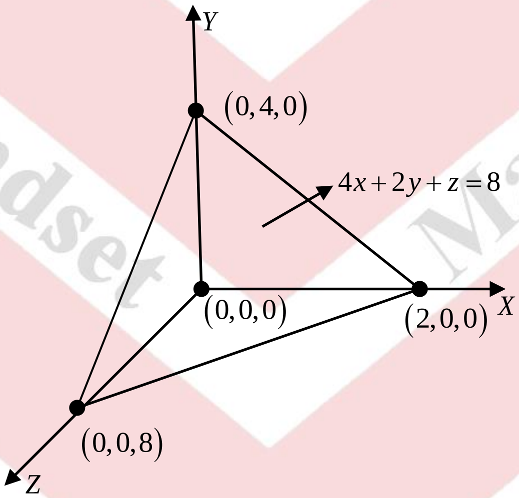
Consider a closed curve c in a simply connected region then $\oint_c M dx + N dy = 0$

iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ everywhere in the region.

Volume Integral



Q. Let's consider some scalar valued function $\phi(x, y, z) = 45x^2 y$ and let v be the closed region bounded by the planes $4x + 2y + z = 8$ and $x = 0, y = 0, z = 0$. Then evaluate the volume integral $\iiint_v \phi dv$



$$v = \int_{x=0}^{2} \int_{y=0}^{\frac{8-4x}{2}} \int_{z=0}^{8-(4x+2y)} 45x^2 y \, dxdydt$$

$$= \int_{x=0}^{2} \int_{y=0}^{4} 45x^2 y [z]_0^{8-(4x+2y)} dxdy$$

$$= \int_{x=0}^{4} \int_{y=0}^{4-2x} 45x^2 y (8 - 4x - 2y) dxdy$$

$$= \int_{x=0}^{4} \int_{y=0}^{4-2x} (360x^2 y - 180x^3 y - 90x^2 y^2) dxdy$$

$$= \int_{x=0}^{4} \left[\frac{360x^2 y^2}{2} - \frac{180x^3 y^2}{2} - \frac{90x^2 y^3}{3} \right]_0^{4-2x} dx$$

$$= \int_{x=0}^{4} \left[180x^2 (4-2x)^2 - 90x^3 (4-2x)^2 - 30x^2 (4-2x)^3 \right] dx$$

$$= \int_{x=0}^{4} \left[180x^2\left(16+4x^2-16x\right)-90x^3\left(16+4x^2-16x\right)-30x^2\left(64-96x+48x^2-8x^3\right)\right]dx$$

Gauss Divergence Theorem

Let v is the volume bounded by the closed surface S and $\vec{F}$ is a vector valued function of position with continuous derivative then

$$\boxed{\iiint_v \operatorname{div}\vec{F} = \iint_S \vec{F}\cdot\hat{n}\,dS}$$

- Applicable only for closed surface.

e.g. If the surface S is $x^2+y^2=4$, $z=5$ we cannot apply Gauss's Divergence Theorem here.
But if the surface S is $x^2+y^2=4$, $z=5$ to $z=8$; yes we can apply Gauss Divergence Theorem.

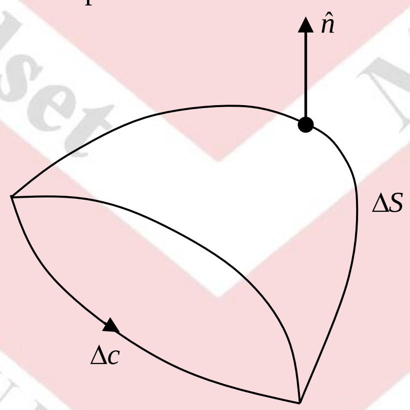
Stoke's Theorem

Alternative definition of curl:

Suppose ΔS is a surface element at a point P, the boundary of the element being the closed curve Δc and $\hat{n}$ is the unit normal vector at the point P drawn outward to the surface. Then we define a limit

$$\left(\text{curl}\,\vec{F}\right)_n = \lim_{\Delta S\to 0}\frac{\int_{\Delta c}\vec{F}\cdot d\vec{r}}{\Delta S}$$

If this limit exists independent of the shape of the curve.



Here $\left(\text{curl}\,\vec{F}\right)_n$ is the component of a certain vector curl $\vec{F}$ along the normal $\hat{n}$ to the surface.

Statement

The line integral of a vector field $\vec{F}$ around any closed curve is equal to $\iint_S \text{curl}\,\vec{F}$ (i.e., the surface integral of curl $\vec{F}$ taken over any surface of which the curve is a boundary edge.

Mathematically if $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve c, then

$$\boxed{\int_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS}$$

Here $\hat{n}$ is the unit normal vector outward to the surface S.

<u>Gradient, Divergence and Curl</u>

Solved Examples

1. If $\hat{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k},\ \vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$ **, then value of** $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)$ **at** $(1,0,-2)$ **is equal to ?**

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= \left(2x^3z^3 - xyz^2\right)\hat{i} + \left(2xz^3 + x^4yz\right)\hat{j} + \left(x^2y^2z + 4xz^4\right)\hat{k}$$

$$\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right) = -xz^2\hat{i} + x^4z\hat{j} + 2x^2yz\hat{k}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right) = -z^2\hat{i} + 4x^3z\hat{j} + 4xyz\hat{k}$$

So, at $(1,0,-2)$, $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right) = -4\hat{i} - 8\hat{j}$

2. If $f(x,y,z) = 3x^2y - y^3z^2$ **, then grad *f* and the point** $(1,-2,-1)$ **is equal to ?**

$$f = 3x^2y - y^3z^2$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = 6xy\hat{i} + \left(3x^2 - 3y^2z^2\right)\hat{j} - 2y^3z\hat{k}$$

At $(1,-2,-1)$, $\nabla f = -12\hat{i} - 9\hat{j} - 16\hat{k}$.

3. The gradient of $f(r)$ **is equal to?**

$$\nabla f(r) = \sum \hat{i}\frac{\partial}{\partial x}\left(f(r)\right) = \sum \hat{i}f'(r)\frac{\partial r}{\partial x} = \sum \hat{i}f'(r)\frac{x}{r} = \frac{f'(r)}{r}\sum \hat{i}x = \frac{f'(r)}{r}\vec{r}$$

4. $\nabla f(r)\times\vec{r}$ **is equal to?**

$\nabla f(r) = \frac{f'(r)}{r}\vec{r}$ [as solved in previous question]

$$\nabla f(r)\times\vec{r} = 0$$

5. $\nabla\left(\frac{1}{r}\right)$ **is equal to?**

$$\nabla\left(\frac{1}{r}\right) = \sum \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} = \sum \hat{i}\left(-\frac{1}{r^2}\right)\frac{x}{r} = -\frac{1}{r^3}\sum \hat{i}x = -\frac{\vec{r}}{r^3}$$

6. $\nabla \log r$ **is equal to?**

$$\nabla \log r = \sum \hat{i}\,\frac{\partial}{\partial x}\log r = \sum \hat{i}\,\frac{1}{r}\cdot\frac{\partial r}{\partial x} = \frac{1}{r^2}\sum \hat{i}x = \frac{\vec{r}}{r^2}$$

7. ∇r^n is equal to ?

$$\nabla r^n = \sum \hat{i}\,\frac{\partial}{\partial x}r^n = \sum \hat{i}nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-2}\sum \hat{i}x = nr^{n-2}\vec{r}$$

8. If $\vec{a}$ is constant vector & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then grad $(\vec{r}\cdot\vec{a})$ is equal to ?

$$\nabla(\vec{r}\cdot\vec{a}) = \sum \hat{i}\,\frac{\partial}{\partial x}(\vec{r}\cdot\vec{a}) = \sum \hat{i}\left(\frac{\partial \vec{r}}{\partial x}\cdot\vec{a}\right) = \sum \hat{i}\left(\hat{i}\cdot\vec{a}\right) = \vec{a}$$

9. Let $\vec{a}$ & $\vec{b}$ are constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ grad$\left[\vec{r}\;\vec{a}\;\vec{b}\right]$ is equal to ?

$$\text{grad}\left[\vec{r}\;\vec{a}\;\vec{b}\right] = \sum \hat{i}\,\frac{\partial}{\partial x}\left(\vec{r}\cdot\left(\vec{a}\times\vec{b}\right)\right) = \sum \hat{i}\left(\frac{\partial \vec{r}}{\partial x}\cdot\left(\vec{a}\times\vec{b}\right)\right) = \sum \hat{i}\left(\hat{i}\cdot\left(\vec{a}\times\vec{b}\right)\right) = \vec{a}\times\vec{b}$$

10. If $\vec{a}$ is a constant vector, ϕ is scalar field $(\vec{a}\cdot\nabla)\phi$ is equal to?

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$(\vec{a}\cdot\nabla)\phi = a_1\frac{\partial \phi}{\partial x} + a_2\frac{\partial \phi}{\partial y} + a_3\frac{\partial \phi}{\partial z}$$

11. If $\vec{a}$ is constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\,(\vec{a}\cdot\nabla)\vec{r}$ is equal to ?

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$(\vec{a}\cdot\nabla)\vec{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

12. The unit normal vector to the level surface $x^2 + y^2 - z = 4$ at point $(1,1,-2)$ is?

Normal vector lies in direction of ∇f , So $\hat{n} = \dfrac{\nabla f}{|\nabla f|}$

$f = x^2 + y^2 - z$, $\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$, At $(1,1,-2)$ $\nabla f = 2\hat{i} + 2\hat{j} - \hat{k}$, $|\nabla f| = \sqrt{9} = 3$

So, $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2\hat{i} + 2\hat{j} - \hat{k}}{3} = \dfrac{2}{3}\hat{i} + \dfrac{2}{3}\hat{j} - \dfrac{1}{3}\hat{k}$

13. The directional derivative of $f(x,y,z)=x^2yz+4xz^2$ at the point $(1,-2,-1)$ in the direction of vector $2\hat{i}-\hat{j}-2\hat{k}$ is?

$\nabla f=\left(2xyz+4z^2\right)\hat{i}+\left(x^2z\right)\hat{j}+\left(x^2y+8xz\right)\hat{k}$, At $(1,-2,-1)$, $\nabla f=8\hat{i}-\hat{j}-10\hat{k}$

So, directional derivative of f in direction of $2\hat{i}-\hat{j}-2\hat{k}$ is equal to

$\nabla f\cdot\hat{a}=\frac{1}{3}\left(8\hat{i}-\hat{j}-10\hat{k}\right)\cdot\left(2\hat{i}-\hat{j}-2\hat{k}\right)=\frac{37}{3}$

14. The point P closet to origin on the plane $2x+y-z-5=0$ is ?

Closest point will be foot of perpendicular from origin

$S=2x+y-z-5=0$, $\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$

Coordinate of $P=\left(\frac{2}{\sqrt{6}}r,\frac{1}{\sqrt{6}}r,\frac{-1}{\sqrt{6}}r\right)$, It lies on S. So, $r=\frac{5}{\sqrt{6}}$

Hence, $P=\left(\frac{5}{3},\frac{5}{6},-\frac{5}{6}\right)$

15. The temperature T at a surface is given by $T=x^2+y^2-z$. In which direction a mosquito at the point $(4,4,2)$ on the surface will fly so that it cools fastest?

$T=x^2+y^2-z$

Direction of fastest cooling will lie in direction opposite to the direction of gradient i.e. $-\nabla T$

$\nabla T=2x\hat{i}+2y\hat{j}-\hat{k}=8\hat{i}+8\hat{j}-\hat{k}$

16. The scalar function *f* which corresponds to $\vec{V}=\nabla f$

where $\vec{V}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ is ?

$f=\sqrt{x^2+y^2+z^2}+c$, $\nabla f=\frac{\vec{r}}{r}$

17. One of the point at which the derivative of the function $f(x,y)=x^2-xy-y+y^2$ vanishes along the direction $\frac{\hat{i}+\sqrt{3}\hat{j}}{2}$ is ?

$\nabla f=(2x-y)\hat{i}-(x+1-2y)\hat{j}$

Directional derivative in direction given by $\frac{\hat{i}+\sqrt{3}\hat{j}}{2}=\frac{1}{2}(2x-y)-\frac{\sqrt{3}}{2}(x+1-2y)$

$=\frac{2-\sqrt{3}}{2}x-\frac{\left(1-2\sqrt{3}\right)}{2}y-\frac{\sqrt{3}}{2}$. It becomes zero at $\left(-1,\frac{2}{2\sqrt{3}-1}\right)$

18. Which of the following is a unit normal vector to the surface $z = xy$ at $P(2,-1,-1)$?

The surface is $f = xy - z = 0$, $\nabla f = y\hat{i} + x\hat{j} - \hat{k} = -\hat{i} + x\hat{j} - \hat{k}$, $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{6}}$

19. Let $f(x,y) = \ln\sqrt{x+y}$ and $g(x,y) = \sqrt{x+y}$. Then the value of $\nabla^2(fg)$ at $(1,0)$?

$f = \ln(x+y)^{1/2}$, $g = \sqrt{x+y}$, $fg = \sqrt{x+y}\ln\sqrt{x+y}$, $\nabla^2 fg = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) fg$

$$\frac{\partial}{\partial x}(fg) = \frac{1}{2\sqrt{x+y}}\ln\sqrt{x+y} + \sqrt{x+y}\,\frac{1}{\sqrt{x+y}}\cdot\frac{1}{2\sqrt{x+y}}$$

$$\frac{\partial^2 fg}{\partial x^2} = -\frac{1}{4}(x+y)^{-3/2}\ln(x+y) + \frac{1}{2(x+y)}\cdot\frac{1}{2\sqrt{x+y}} - \frac{1}{4}(x+y)^{-3/2}$$

$\nabla^2 fg = 0$

20. The spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + \left(y - \sqrt{3}\right)^2 + z^2 = 4$ intersect at an angle?

$x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + z^2 - 2\sqrt{3}y = 1$. They intersect at plane $y = 0$

$(0,0,1)$ is one point of intersection which is lying on the both sphere.

Let us find normal vector at this point and find angle between them

$\hat{n}_i = x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{n}_2 = \frac{x\hat{i} + \left(y - \sqrt{3}\right)\hat{j} + z\hat{k}}{2}$$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \dfrac{1}{2}$ at point $(0,0,1)$, $\theta = \pi/3$

21. Let $\theta, 0 \le \theta \le \pi$ be the angle between the planes $x - y + z = 0$ and $2x - z = 4$
The value of θ is?

$x - y + z = 3 \Rightarrow x - y + z - 3 = 0$, $2x - z = 4 \Rightarrow 2x - z - 4 = 0$

Let us find angle between their normal

$\hat{n} = \dfrac{\nabla f}{|\nabla f|}$, $\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \dfrac{1}{\sqrt{15}}$, $\hat{n}_2 = \dfrac{2\hat{i} - \hat{k}}{\sqrt{5}}$, $\hat{n}_1 = \dfrac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$

$\Rightarrow \theta = \cos^{-1}\dfrac{1}{\sqrt{15}}$

22. $f(x,y) = xy^2 + yx^2$

Suppose the directional derivative of *f* in the direction of the unit vector (u_1, u_2) at the point $(1,-1)$ is 1. The among the following (u_1, u_2) is?

$f = xy^2 + yx^2$, $\nabla f = \left(y^2 + 2xy\right)\hat{i} + \left(x^2 + 2xy\right)\hat{j}$, At $(1,-1)$ $\nabla f = -\hat{i} - \hat{j}$

$\hat{n} = u_1\hat{i} + u_2\hat{j}$

Directional derivative of *f* in direction of unit vector (u_1, u_2) is $\nabla f \cdot \hat{n} = 1$

$\Rightarrow -u_1 - u_2 = 1$, $u_1 = -1, u_2 = 0$ satisfies above equation.

23. For what values of *a* and *b*, the directional derivative of $u(x,y,z) = ax^2yz + bxy^2z$ at $(1,1,1)$ along $\hat{i} + \hat{j} - 2\hat{k}$ is $\sqrt{6}$ and along $\hat{i} - \hat{j} + 2\hat{k}$ is $3\sqrt{6}$?

$\nabla u = (2axyz + by^2z)\hat{i} + (ax^2z + 2bxyz)\hat{j} + (ax^2y + bxy^2)\hat{k}$

The directional derivative of $u(x,y,z)$ along $(\hat{i} + \hat{j} - 2\hat{k})$ at $(1,1,1)$

$$(2a+b)\hat{i} + (a+2b)\hat{j} + (a+b)\hat{k} \cdot \frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}(2a+b+a+2b-2a-2b) = \frac{a+b}{\sqrt{6}} = \sqrt{6} \text{ (Given)}$$

So, $a + b = 6$

The directional derivative of $u(x,y,z)$ along $(\hat{i} - \hat{j} + 2\hat{k})$ at $(1,1,1)$

$$= \left((2a+b)\hat{i} + (a+2b)\hat{j} + (a+b)\hat{k}\right) \cdot \frac{\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}(3a+b) = 3\sqrt{6} \text{ (Given)}$$

$3a + b = 18$ ....(2)

Solving (1) & (2)

$a = 6, b = 0$

24. If $f(x,y,z) = x - y$ and $\nabla\left(\frac{f}{g}\right) = \frac{1}{z}(\hat{i} - \hat{j}) - \left(\frac{x-y}{z^2}\right)\hat{k}$ then $g(x,y,z)$ is ?

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} = \frac{\nabla f}{g} - \frac{f}{g^2}\nabla g$$

Given $\nabla\left(\frac{f}{g}\right) = \frac{1}{z}(\hat{i} - \hat{j}) - \left(\frac{x-y}{z^2}\right)\hat{k}$, On comparing $g(x,y,z) = z$.

25. The directional derivative of $f(x,y,z) = z^2e^{\cos xy}$ at $\left(0, \frac{\pi}{2}, 1\right)$ along $(2\hat{i} - \hat{j} + 2\hat{k})$ is?

$\nabla f = -yz^2e^{\cos xy} \cdot \sin xy\hat{i} - xz^2e^{\cos xy} \cdot \sin xy\hat{j} + 2ze^{\cos xy}\hat{k}$

Unit vector along $2\hat{i} - \hat{j} + 2\hat{k}$ is given by, $\hat{n} = \frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$

Directional derivative along $(2\hat{i} - \hat{j} + 2\hat{k})$

$$\frac{df}{ds} = \nabla f \cdot \hat{n}\left(-z^2e^{\cos xy}y\sin xy\hat{i} - z^2e^{\cos xy}x\sin xy\hat{j} + 2ze^{\cos xy}\hat{k}\right) \cdot \frac{(2\hat{i} - \hat{j} + 2\hat{k})}{3}$$

$$= \frac{1}{3}\left(-2z^2ye^{\cos xy}\sin xy - xz^2e^{\cos xy}\sin xy + 4ze^{\cos xy}\right)$$

At $\left(0,\frac{\pi}{2},1\right)$, directional derivative $=\frac{4}{3}e$.

26. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. Then $\nabla\left(\vec{r}\cdot\nabla\left(\vec{r}\cdot\vec{r}\right)\right)$ is equal to ?

$\nabla\left(\vec{r}\cdot\vec{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)=2\vec{r}$, $\vec{r}\cdot\nabla\left(\vec{r}\cdot\vec{r}\right)=2\vec{r}\cdot\vec{r}=2r^2$

$\nabla\left(\vec{r}\cdot\nabla\left(\vec{r}\cdot\vec{r}\right)\right)=2\nabla\left(r^2\right)=4\vec{r}$

27. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then $\nabla|\vec{r}|^4$ equals ?

$\nabla\left(|r|^4\right)=\nabla\left(r^4\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(r^4\right)=4r^3\frac{\sum\hat{i}x}{r}=4r^2\vec{r}$.

28. Let $T(x,y,z)=xy^2+2z-x^2z^2$ be the temperature at the point (x,y,z). The unit vector in the direction in which the temperature decreases most rapidly at $(1,0,-1)$ is ?

Temperature increases most rapidly in the direction of ∇T.

$\nabla T=\left(y^2-2xz^2\right)\hat{i}+2xy\hat{j}+\left(2-2x^2z\right)\hat{k}$, At $(1,0,-1)$, $\nabla T=-2\hat{i}+4\hat{k}$

Unit vector in the direction of $\nabla T=-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$

So, temperature decreases most rapidly in the direction of $-\nabla T$. i.e., $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$

29. The equation of a surface of revolution is $z=\pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2}$. The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}},0,1\right)$ is ?

Equation of surface is

$z^2=\frac{3}{2}\left(x^2+y^2\right)$

$F=3x^2+3y^2-2z^2=0$

Unit normal to the surface $\hat{n}=\frac{\nabla F}{|\nabla F|}=\frac{6x\hat{i}+6y\hat{j}-4z\hat{k}}{2\sqrt{9x^2+9y^2+4z^2}}=\frac{3x\hat{i}+3y\hat{j}-2z\hat{k}}{\sqrt{9x^2+9y^2+4z^2}}$

$=\frac{3\sqrt{\frac{2}{3}}\hat{i}-2\hat{k}}{\sqrt{9\times\frac{2}{3}+4}}=\frac{\sqrt{6}\hat{i}-2\hat{k}}{\sqrt{10}}=\sqrt{\frac{3}{5}}\hat{i}-\frac{2}{\sqrt{10}}\hat{k}$.

Assignment-1

1. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of line PQ where Q is the point $(5,0,4)$.

Solution.

Here, function $f = x^2 - y^2 + 2z^2$, $\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$, At $(1,2,3)$, $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

Now, vector $\overrightarrow{PQ}$ = position vector Q-position vector of P $= (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$

Unit vector in direction of $\overrightarrow{PQ}$, $\hat{a} = \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$

So, directional derivative of f in the direction of $\hat{a} = \nabla f \cdot \hat{a} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \dfrac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$

$= \dfrac{28}{\sqrt{21}} = \dfrac{4}{3}\sqrt{21}$

2. What is the greatest rate of increase of $u = xyz^2$ at the point $(1,0,3)$?

Solution.

$u = xyz^2, \nabla u = yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$, At $(1,0,3)$, $\nabla u = 9\hat{j}$

The greatest rate of increase of f lie in the direction of ∇f.

So, maximum value of directional derivative

$= \nabla u \cdot \hat{a}$ with $\hat{a}$ being unit vector parallel to $\nabla u = |\nabla u| = 9$.

3. Find the directional derivative of

(i) $4xz^3 - 3x^2y^2z^2$ at $(2,-1,2)$ along *z* axis.

(ii) $x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution.

(i) $f = 4xz^3 - 3x^2y^2z^2$, $\nabla f = (4z^3 - 6xy^2z^2)\hat{i} - 6x^2yz^2\hat{j} + (12xz^2 - 6x^2y^2z)\hat{k}$, At $(2,-1,2)$,

$\nabla f = -16\hat{i} + 96\hat{j} + 48\hat{k}$

Along *z* axis, the directional derivative along *z* axis $= \nabla f \cdot \hat{k} = 48$

(ii) $f = x^2yz + 4xz^2$, $\nabla f = (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8zx)\hat{k}$, At $(1,-2,1)$, $\nabla f = \hat{j} + 6\hat{k}$

Unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$, $\hat{a} = \dfrac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}} = \dfrac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$

Directional derivative in direction of $2\hat{i} - \hat{j} - 2\hat{k} = \nabla f \cdot \hat{a} = (\hat{j} + 6\hat{k}) \cdot \left(\dfrac{2\hat{i} - \hat{j} - 2\hat{k}}{3}\right) = -\dfrac{13}{3}$.

4. Find the directional derivative of $f(x,y) = x^2y^3 - xy$ at the point $(2,1)$ in the directional of a unit vector which makes an angle of $\pi/3$ with *x* axis?

Solution.

$f(x,y)=x^2y^3+xy$, $\nabla f=\left(2xy^3+y\right)\hat{i}+\left(3x^2y^2+x\right)\hat{j}$, At $(2,1)$, $\nabla f=5\hat{i}+14\hat{j}$

Unit vector making an angle of $\pi/3$ with x axis $\hat{a}=\cos\frac{\pi}{3}\hat{i}+\sin\frac{\pi}{3}\hat{j}=\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{3}\hat{j}$

So, directional derivative of f in the direction of unit vector making angle of $\frac{\pi}{3}$ with the x axis $=\nabla f\cdot\hat{a}$

$$=\left(5\hat{i}+14\hat{j}\right)\cdot\left(\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}\right)=\frac{5+14\sqrt{3}}{2}.$$

5. Find the constants *a* and *b* so that the surface $ax^2-byz=(a+2)x$ will be orthogonal to the surface $4x^2y+z^3=4$ at the point $(1,-1,2)$.

Solution.

The two surface are orthogonal at point P if the respective normal to the surface are perpendicular to each other.

The surface S_1 is given by

$S_1: ax^2-byz-(a+2)x=0$, Gradient of S_1, $\nabla S_1=\left(2ax-(a+2)\right)\hat{i}-bz\,\hat{j}-by\,\hat{k}$

At $(1,-1,2)$ $\nabla S_1=(a-2)\hat{i}-2b\,\hat{j}-b\hat{k}$

Unit normal vector to S_1 $\hat{n}_1=\frac{\nabla S_1}{|\nabla S_1|}$

The surface S_2 is given by

$S_2: 4x^2y+z^3=4$, Gradient of S_2, $\nabla S_2=8xy\hat{i}+4x^2\hat{j}+3z^2\hat{k}$, At $(1,-1,2)$ $\nabla S_2=8\hat{i}+4\hat{j}+12\hat{k}$

Normal to S_2, $\hat{n}_2=\frac{\nabla S_2}{|\nabla S_2|}$

Two surface S_1 & S_2 are orthogonal So, $\hat{n}_1\cdot\hat{n}_2=0$ $\frac{\nabla S_1}{|\nabla S_1|}\cdot\frac{\nabla S_2}{|\nabla S_2|}=0$ $\nabla S_1\cdot\nabla S_2=0$

$$\Rightarrow\left((a-2)\hat{i}-2b\hat{j}+b\hat{k}\right)\cdot\left(-8\hat{i}+4\hat{j}+12\hat{k}\right)=0\Rightarrow-8(a-2)-8b+12b=0\Rightarrow-8a+4b=-16$$

Point $(1,-1,2)$ lies on S_1. So, $a+2b=a+2\Rightarrow b=1$ So, $a=\frac{5}{2}$.

6. Find the values of constant *a,b* & *c* so that the directional derivative of the function. $f=axy^2+byz+cz^2x^3$ at the point $(1,2-1)$ has maximum magnitude 64 in the direction of parallel to *z*-axis

Solution.

The function

$f=axy^2+byz+cz^2x^3$, $\nabla f=\sum\hat{i}\frac{\partial}{\partial x}f=\left(ay^2+3cz^2x^2\right)\hat{i}+\left(2axy+bz\right)\hat{j}+\left(by+2czx^3\right)\hat{k}$

At $(1,2,-1)$ $\nabla f=(4a+3c)\hat{i}+(4a-b)\hat{j}+(2b-2c)\hat{k}$

The directional derivative of f is maximum along ∇f and it is given that maximum value of directional derivative is along z axis. So, ∇f is parallel to z axis. So, its x & y component should be zero.

So, $4a+3c=0 \Rightarrow c=-\dfrac{4a}{3}$, $4a-b=0 \Rightarrow b=4a$, So, $\nabla f=(2b-2c)\hat{k}=\dfrac{32}{3}a\hat{k}$

Maximum value of directional derivative is equal to $|\nabla f|=64 \Rightarrow \dfrac{32a}{3}=64$. So, $a=6$

$b=24$, $c=-8$.

7. Find the directional derivative of $f=x^2yz^3$ along $x=e^{-t}, y=1+2\sin t, z=t-\cos t$ at $t=0$.

Solution.

The function $f=x^2yz^3$, $\nabla f=2xyz^3\hat{i}+x^2z^3\hat{j}+3x^2yz^2\hat{k}$, For $t=0, x=e^{-t}=1, y=1+2\sin t=1$

$z=t-\cos t=-1$. So, at $(1,1,-1)$ $\nabla f=-2\hat{i}-\hat{j}+3\hat{k}$

The curve is described by vector $\vec{r}=e^{-t}\hat{i}+(1+2\sin t)\hat{j}+(t-\cos t)\hat{k}$

$\vec{t}=\dfrac{d\vec{r}}{dt}=-e^{-t}\hat{i}+2\cos t\vec{j}+(1+\sin t)\hat{k}$, At $t=0$ $\vec{t}=-\hat{i}+2\hat{j}+\hat{k}$

Unit vector along tangent, $\hat{t}=\dfrac{-\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}}$

Directional derivative along the curve at $t=0=\nabla f\cdot\hat{t}=\left(-2\hat{i}-\hat{j}+3\hat{k}\right)\cdot\dfrac{\left(-\hat{i}+2\hat{j}+\hat{k}\right)}{\sqrt{6}}=\dfrac{3}{\sqrt{6}}=\sqrt{\dfrac{3}{2}}$.

8. If $\vec{r}_1$ and $\vec{r}_2$ are the vector joining the fixed point. $A(x_1,y_1,z_1)$ & $B(x_2,y_2,z_2)$ respectively to a variable point $P(x,y,z)$ then find the values of grad$(\vec{r}_1\cdot\vec{r}_2)$ & $(\vec{r}_1\times\vec{r}_2)$.

Solution.

The vector

$A\vec{P}=\vec{r}_1$ = position vector of P-position vector of A $=\left(x\hat{i}+y\hat{j}+2\hat{k}\right)-\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)$

$=(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$. Similarly, $B\vec{P}=\vec{r}_1=(x-x_2)\hat{i}+(y-y_2)\hat{j}+(z-z_2)\hat{k}$

$$\vec{r}_1\times\vec{r}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x-x_1 & y-y_1 & z-z_1\\ x-x_2 & y-y_2 & z-z_2\end{vmatrix}$$

$=\left[(y-y_1)(z-z_2)-(y-y_2)(z-z_1)\right]\hat{i}+\left[(x-x_2)(z-z_1)-(x-x_1)(z-z_2)\right]\hat{j}$

$+\left[(y-y_1)(z-z_2)-(y-y_2)(z-z_1)\right]\hat{k}$

$=\left[y(z_1-z_2)+z(y_2-y_1)+(y_1z_2-y_2z_1)\right]\hat{i}+\left[z(x_1-x_2)+x(z_2-z_1)+(z_1x_2-z_2x_1)\right]\hat{j}$

$+\left[x(y_1-y_2)+y(x_2-x_1)+(x_1y_2-x_2y_1)\right]\hat{k}$

$\vec{r}_1\cdot\vec{r}_2=(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$

$$\nabla(\vec{r}_1 \cdot \vec{r}_2) = \sum \hat{i} \frac{\partial}{\partial x}(\vec{r}_1 \cdot \vec{r}_2) = \sum \hat{i}\,(2x - x_1 - x_2) = \sum \hat{i}\left[(x - x_1) + (x - x_2)\right]$$
$$= \sum \hat{i}\,(x - x_1) + \sum \hat{i}\,(x - x_2) = \vec{r}_1 + \vec{r}_2 .$$

9. Find the equation of tangent plane and normal to the surface $2xz^2 - 3xy + 4x = 1$ at the point $(1,1,2)$.

Solution.

Equation of the surface is , $f(x,y,z) = 2xz^2 - 3xy + 4x = 1$,

$\nabla f = \sum \hat{i} \frac{\partial}{\partial x} f = (2z^2 - 3y + 4)\hat{i} - 3x\hat{j} + 4xz\hat{k}$, At $(1,1,2)$ $\nabla f = 9\hat{i} - 3\hat{j} + 8\hat{k}$

Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ is a position vector of any arbitrary point (x,y,z) on the tangent plane at point P.

The position vector of point P is $\vec{r} = \hat{i} + \hat{j} + 2\hat{k}$

Equation of tangent plane at point P is

$$(\vec{R} - \vec{r}).\text{grad } f = 0 \Rightarrow (x-1)\frac{\partial f}{\partial x} + (y-1)\frac{\partial f}{\partial y} + (z-2)\frac{\partial f}{\partial z} = 0 \Rightarrow 9(x-1) - 3(y-1) + 8(z-2) = 0$$

$9x - 3y + 8z = 22$

Equation of normal to the surface at point $(1,1,2)$ is

$$\frac{x-1}{\frac{\partial f}{\partial x}} = \frac{y-1}{\frac{\partial f}{\partial y}} = \frac{z-2}{\frac{\partial f}{\partial z}},\ \frac{x-1}{9} = \frac{y-1}{-3} = \frac{z-2}{8}.$$

10. Find the equation of the tangent plane and normal to the surface $xyz = 2$ at the point $(1,2,1)$.

Solution.

The equation of surface is $f(x,y,z) = xyz - 2 = 0$

$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$, At point $(1,2,1)$ $\nabla f = 2\hat{i} + \hat{j} + 2\hat{k}$

Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of an arbitrary point (x,y,z) on the tangent plane.

Position vector of point of contact $(1,2,1)$, $\vec{r} = \hat{i} + 2\hat{j} + \hat{k}$

Equation of tangent plane is , $(\vec{R} - \vec{r}) \cdot \nabla f = 0$

$$\Rightarrow (x-1)\frac{\partial f}{\partial x} + (y-2)\frac{\partial f}{\partial y} + (z-1)\frac{\partial f}{\partial z} = 0 \Rightarrow 2(x-1) + (y-2) + 2(z-1) = 6$$

$2x + y + 2z = 6$

Equation of normal to the surface at point $(1,2,1)$

$$\frac{x-1}{\frac{\partial f}{\partial x}} = \frac{y-2}{\frac{\partial f}{\partial y}} = \frac{z-1}{\frac{\partial f}{\partial z}},\ \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{2}.$$

11. Give the curve $x^2+y^2+z^2=1, x+y+z=1$ (intersection of two surfaces) find the equation of the tangent line at the point $(1,0,0)$.

Solution.

Given is the curve of intersection of two surfaces

$S_1 \equiv x^2+y^2+z^2=0 \quad S_2 \equiv x+y+z=1$

$\nabla S_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k},\ \nabla S_2=\hat{i}+\hat{j}+\hat{k}$, At the point $(1,0,0)$ $\nabla S_1 = 2\hat{i}$, $\nabla S_2=\hat{i}+\hat{j}+\hat{k}$

The normal vector to surface S_1 & S_2 are given by, $\hat{n}_1=\frac{\nabla S_1}{|\nabla S_1|}=\hat{i}$, $\hat{n}_2=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Tangent to the curve of intersection will be perpendicular to both $\hat{n}_1$ & $\hat{n}_2$ i.e. it lies in the direction of $\hat{n}_1\times\hat{n}_2$ i.e. $\hat{i}\times\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}=-\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$

So, equation of tangent passing through $(1,0,0)$ & parallel to vector $-\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$ is

$$\frac{x-1}{0}=\frac{y-0}{-1/\sqrt{3}}=\frac{z-0}{1/\sqrt{3}}.$$

12. Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution.

Angle between two surfaces at a point will be equal to the angle between their respective normal at the point. So, let us the unit normal vector to the surfaces at given point i.e. $(2,-1,2)$

The given surfaces are , $S_1 \equiv x^2+y^2+z^2=9$, $S_2\equiv x^2+y^2-z=3$

Their gradient are , $\nabla S_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$, $\nabla S_1 = 2x\hat{i}+2y\hat{j}-\hat{k}$

At $(2,-1,2)$, $\nabla S_1 = 4\hat{i}-2\hat{j}+4\hat{k}$, $\nabla S_2 = 4\hat{i}-2\hat{j}-\hat{k}$

The unit normal vectors to the surface S_1 & S_2 at point $(2,-1,2)$ are

$$\hat{n}_1=\frac{\nabla S_1}{|\nabla S_1|}=\frac{4\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{36}}=\frac{2}{3}\hat{i}-\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$\hat{n}_2=\frac{\nabla S_2}{|\nabla S_2|}=\frac{4\hat{i}-2\hat{j}-\hat{k}}{\sqrt{21}}=\frac{4}{\sqrt{21}}\hat{i}-\frac{2}{\sqrt{21}}\hat{j}-\frac{1}{\sqrt{21}}\hat{k}$$

If θ is the angle between two vectors

then $\hat{n}_1\cdot\hat{n}_2=|\hat{n}_1||\vec{n}_2|\cos\theta \Rightarrow \cos\theta=\frac{8}{3\sqrt{21}}+\frac{2}{3\sqrt{21}}-\frac{2}{3\sqrt{21}}=\frac{8}{3\sqrt{21}}$ .So, $\theta=\cos^{-1}\frac{8}{3\sqrt{21}}$.

13. If $\vec{F}$ & *f* are two point function, show that the components of the former tangential and normal to the level surface $f=0$ are $\frac{\nabla f\times(\vec{F}\times\nabla f)}{(\nabla f)^2}$ and $\frac{(\vec{F}\cdot\nabla f)\nabla f}{(\nabla f)^2}$

Solution.

Unit normal vector to the surface $f=0$, $\hat{n}=\frac{\nabla f}{|\nabla f|}$

Magnitude of component of $\vec{F}$ normal to the surface $f=0$ is , $\vec{F}\cdot\hat{n}=\vec{F}\cdot\frac{\nabla f}{|\nabla f|}$

Component of $\vec{F}$ normal to the surface $f=0$ $\left(\vec{F}\cdot\hat{n}\right)\hat{n}=\left(\vec{F}\cdot\frac{\nabla f}{|\nabla f|}\right)\frac{\nabla f}{|\nabla f|}=\frac{\left(\vec{F}\cdot\nabla f\right)\nabla f}{\left(\nabla f\right)^2}$

Component of $\vec{F}$ tangential to surface $f=0=$ $\vec{F}$ - normal component of $\vec{F}$ $=$ $\bar{F}-\frac{\left(\vec{F}\cdot\nabla f\right)\nabla f}{\left(\nabla f\right)^2}$

$$=\frac{\left(\nabla f\cdot\nabla f\right)F-\left(\vec{F}\cdot\nabla f\right)\nabla f}{\left(\nabla f\right)^2}=\frac{\nabla f\times\left(\vec{F}\times\nabla f\right)}{\left(\nabla f\right)^2}$$

Divergence and Curl

SOLVED EXAMPLES

1. $div\,\vec{r}$ **equal to ?**

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\nabla\cdot\vec{r} = \sum \hat{i}\cdot\frac{\partial}{\partial x}\vec{r} = \sum \hat{i}\cdot\hat{i} = \sum 1 = 3$

2. curl $\vec{r}$ **is equal to?**

$$\text{curl } \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

3. The value of constant *a* for which the vector $\vec{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ **is solenoidal is?**

Vector $\vec{f}$ is solenoidal if div $\vec{f} = 0$, div $\vec{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$

$\Rightarrow 1+1+a = 0$ $a = -2$.

4. If $\vec{a}$ **is a constant vector, then** $\nabla\cdot(\vec{r}\times\vec{a})$ **is equal to ?**

div $(\vec{r}\times\vec{a}) = \sum \hat{i}\cdot\frac{\partial}{\partial x}(\vec{r}\times\vec{a}) = \sum \hat{i}\cdot\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right) = \sum \hat{i}\cdot(\hat{i}\times a) = 0$

5. If $\vec{a}$ **is a constant vector, curl** $(\vec{r}\times\vec{a})$ **is equal to?**

curl $(\vec{r}\times\vec{a}) = \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{r}\times\vec{a}) = \sum \hat{i}\times\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right) = \sum \hat{i}\times(\hat{i}\times a) = \sum\left[(\hat{i}\cdot\vec{a})\hat{i} - (\hat{i}\cdot\hat{i})\vec{a}\right]$

$= \sum(\hat{i}\cdot\vec{a})\hat{i} - \sum\vec{a} = \vec{a} - 3\vec{a} = -2\vec{a}$

6. If $\vec{f} = e^{xyz}(\hat{i} + 2\hat{j} + 3\hat{k})$ **the curl** $\vec{f}$ **at** $(1,1,1)$ **equal to ?**

$$\nabla\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & 2e^{xyz} & 3e^{xyz} \end{vmatrix} = e^{xyz}(3xz-2xy)\hat{i} + e^{xyz}(xy-yz)\hat{j} + e^{xyz}(2yz-xz)\hat{k}$$

At $(1,1,1)$, $\nabla\times\vec{f} = e(\hat{i}\times\hat{k})$

7. If $\vec{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$**, then value of div** $\vec{f}$ **at** $(1,1,1)$ **is equal to ?**

$\vec{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$, div $\vec{f} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2) = y^2 + 2x^2z - 6yz$

At $(1,1,1)$, div $\vec{f} = -3$

8. If $\vec{f} = (x^2-y^2)\hat{i} + 2xy\hat{j} + (y^2-xy)\hat{k}$**, the curl** $\vec{f}$ **at** $(1,1,1)$ **is equal to?**

$$\vec{f}=(x+y+1)\hat{i}+\hat{j}+(-x-y)\hat{k}\text{, curl }\vec{f}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+1 & 1 & -x-y\end{vmatrix}=-\hat{i}+\hat{j}-\hat{k}$$

$$\vec{f}\cdot curl\ \vec{f}=\left((x+y+1)\hat{i}+\hat{j}+(-x-y)\hat{k}\right)\cdot\left(-\hat{i}+\hat{j}-\hat{k}\right)=-x-y-1+1+x+y=0$$

<u>Assignment-2</u>

1. Prove that div $\left(r^n\vec{r}\right)=(n+3)r^n$

Solution.

div $=\left(r^n\vec{r}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(r^n\vec{r}\right)=\sum\hat{i}\cdot\left[nr^{n-1}\frac{\partial r}{\partial x}\vec{r}+r^n\frac{\partial\vec{r}}{\partial x}\right]$

$=\sum\left[nr^{n-1}\frac{x}{r}\left(\hat{i}\cdot\hat{r}\right)+r^n\hat{i}\cdot\hat{i}\right]\left(\frac{\partial r}{\partial x}=\frac{x}{r}.\frac{\partial\vec{r}}{\partial x}=\hat{i}\right)=nr^{n-2}-\Sigma x^2+r^n\Sigma1=nr^n+3r^n=(n+3)r^n$

2. Prove that $\nabla^2\left(r^n\vec{r}\right)=n(n+3)r^{n-2}\vec{r}$

Solution.

$\nabla^2\left(r^n\vec{r}\right)=\nabla\left(\nabla\cdot\left(r^n\vec{r}\right)\right)$ (from previous example), $\nabla\cdot\left(r^n\vec{r}\right)=(n+3)r^n$ So, $\nabla^2\left(r^n\vec{r}\right)=\nabla(n+3)r^n$

$=(n+3)\sum\hat{i}\frac{\partial\left(r^n\right)}{\partial x}=(n+3)\sum nr^{n-1}\hat{i}\frac{\partial r}{\partial x}=n(n+3)r^{n-2}\Sigma x\hat{i}=n(n+3)r^{n-2}\vec{r}$

3. Prove that div $\left(\frac{\vec{r}}{r^3}\right)=0$

Solution.

div $\left(\frac{\vec{r}}{r^3}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^3}\right)=\sum\hat{i}\cdot\left[\frac{1}{r^3}\frac{\partial\vec{r}}{\partial x}+\vec{r}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right]=\sum\hat{i}\cdot\left[\frac{1}{r^3}\hat{i}+\vec{r}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right]$

$=\frac{1}{r^3}\Sigma\hat{i}\cdot\hat{i}-\frac{3}{r^5}\Sigma\left(\hat{i}\cdot\vec{r}\right)x=\frac{1}{r^3}\Sigma1-\frac{3}{r^5}\Sigma x^2=\frac{3}{r^3}-\frac{3}{r^5}\cdot r^2=0$

4. Prove that div $\hat{e}_r=\frac{2}{r}$

Solution.

$\nabla\cdot\hat{e}_r=\nabla\cdot\left(\frac{\vec{r}}{r}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r}\right)=\sum\hat{i}\cdot\left(\frac{1}{r}\frac{\partial\vec{r}}{\partial x}+\vec{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right)=\sum\hat{i}\cdot\left(\frac{1}{r}\hat{i}+\vec{r}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right)$

$=\sum\left(\frac{1}{r}\hat{i}\cdot\hat{i}-\frac{1}{r^2}\cdot\frac{x}{r}\left(\hat{i}\cdot\hat{r}\right)\right)=\frac{1}{r}\cdot\Sigma1-\frac{1}{r^3}\Sigma x^2=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}$

5. Prove that vector $f(r)\vec{r}$ **is irrorational**

Solution.

A vector function is said to be irrotational if its curl is zero

$$\nabla\times\left(f(r)\vec{r}\right)=\sum\hat{i}\times\frac{\partial}{\partial x}\left(f(r)\vec{r}\right)=\sum\hat{i}\times\left(f'(r)\frac{\partial r}{\partial x}\vec{r}+f(r)\frac{\partial\vec{r}}{\partial x}\right)=\sum\hat{i}\times\left(f'(r)\frac{x}{r}\vec{r}+f(r)\hat{i}\right)$$

$$=\frac{f'(r)}{r}\Sigma x\hat{i}\times\hat{r}+\Sigma f(r)\Sigma\hat{i}\times\hat{i}=\frac{f'(r)}{r}\vec{r}\times\vec{r}+f(r)\Sigma\hat{i}\times\hat{i}=0$$

Since, curl of $f(r)\vec{r}$ is zero, hence $f(r)\vec{r}$ is irrotational.

6. Prove that $\nabla^2\left(\frac{1}{r}\right)=0$

Solution.

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right),\ \nabla\left(\frac{1}{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\sum\hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)=\sum\hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right)=-\frac{1}{r^3}\Sigma x\hat{i}=-\frac{\vec{r}}{r^3}$$

$$\nabla\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right)=\nabla\cdot\left(-\frac{\vec{r}}{r^3}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(-\frac{\vec{r}}{r^3}\right)=-\sum\hat{i}\cdot\left(\frac{1}{r^3}\frac{\partial\vec{r}}{\partial x}+\vec{r}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right)$$

$$=-\sum\hat{i}\cdot\left(\frac{1}{r^3}\hat{i}+\vec{r}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right)=-\sum\left(\frac{1}{r^3}\left(\hat{i}\cdot\hat{i}\right)-\frac{3x}{r^5}\cdot\left(\hat{i}\cdot\vec{r}\right)\right)=-\frac{1}{r^3}\Sigma 1+\frac{3}{r^5}\Sigma x^2=-\frac{3}{r^3}+\frac{3}{r^5}\cdot r^2=0$$

7. Prove that div grad $r^n=n(n+1)r^{n-2}$

Solution.

grad $r^n=\sum\hat{i}\frac{\partial}{\partial x}r^n=\sum\hat{i}nr^{n-1}\frac{\partial r}{\partial x}=\sum\hat{i}nr^{n-1}\frac{x}{r}=nr^{n-2}\Sigma x\hat{i}=nr^{n-2}\vec{r}$

div grad $r^n=\sum div\left(nr^{n-2}\vec{r}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(nr^{n-2}\vec{r}\right)=n\sum\hat{i}\cdot\left(r^{n-2}\frac{\partial\vec{r}}{\partial x}+\vec{r}\frac{\partial}{\partial x}\left(r^{n-2}\right)\right)$

$$=n\sum\hat{i}\cdot\left(r^{n-2}\hat{i}+\vec{r}(n-2)r^{n-3}\frac{\partial r}{\partial x}\right)=nr^{n-2}\Sigma\hat{i}\cdot\hat{i}+n\Sigma\hat{i}\cdot\left((n-2)r^{n-3}\frac{x}{r}\vec{r}\right)$$

$$=3nr^{n-2}+n(n-2)r^{n-4}\Sigma x\left(\hat{i}\cdot\vec{r}\right)=3nr^{n-2}+n(n-2)r^{n-1}\Sigma x^2=3nr^{n-2}+n(n-2)r^{n-2}$$

$$=\left(n^2+n\right)r^{n-2}=n(n+1)r^{n-2}$$

8. Prove that $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi$

Solution.

$$\nabla^2(\phi\psi)=\nabla\cdot\left(\nabla(\phi\psi)\right)=\nabla\cdot(\psi\nabla\phi+\phi\nabla\psi)=\nabla\cdot(\psi\nabla\phi)+\nabla\cdot(\phi\nabla\psi)=\psi\nabla^2\phi+2\nabla\phi\cdot\nabla\psi+\phi\nabla^2\psi$$

9. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenaidal

Solution.

$\vec{A}\times\vec{B}$ are irrotational So, $\nabla\times\vec{A}=0$ & $\nabla\times\vec{B}=0$

Now, $\nabla\left(\vec{A}\cdot\vec{B}\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)=\sum\left[\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]$

$$=\sum\left[\vec{B}\cdot\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)-\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)\right]=\vec{B}\cdot\sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}-\sum\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)=\vec{B}\cdot curl\,\vec{A}-\vec{A}\cdot\sum\hat{i}\times\frac{\partial\vec{B}}{\partial x}$$

$=\vec{B}\cdot curl\,\vec{A}-\vec{A}curl\,\vec{B}=0$ Since $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=0$

Hence, $\vec{A}\times\vec{B}$ is solenoidal.

10. If *f* and *g* are two scalar point function prove that div $\left(f\nabla g\right)=f\nabla^2 g+\nabla f\cdot\nabla g$ **.**

Solution.

We can use a vector identity $\nabla\cdot\left(\phi\vec{f}\right)=\nabla\phi\cdot\vec{f}+\phi\nabla\cdot\vec{f}$, Where ϕ is a scalar function & $\vec{f}$ is a vector function. So, $\nabla\cdot\left(f\nabla g\right)=\nabla f\cdot\nabla g+f\nabla\cdot\left(\nabla g\right)=\nabla f\cdot\nabla g+f\nabla^2 g$

Aliter. $f\nabla g=f\left(\sum\hat{i}\frac{\partial}{\partial x}g\right)=f\frac{\partial g}{\partial x}\hat{i}+f\frac{\partial g}{\partial y}\hat{j}+f\frac{\partial g}{\partial z}\hat{k}$

$$\nabla\cdot\left(f\nabla g\right)=\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)\hat{i}+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$=\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}$$

$$=f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial x}\right)$$

$$=f\nabla^2 g+\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}\hat{j}+\frac{\partial g}{\partial z}\hat{k}\right)=f\nabla^2 g+\nabla f\cdot\nabla g$$

11. Prove that div $\left(\vec{A}\times\vec{r}\right)=\vec{r}\cdot curl\,\vec{A}$ **when** $\vec{A}$ **is a constant vector**

Solution.

Using identity $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot curl\,\vec{A}-\vec{A}\cdot curl\,\vec{B}$, $\nabla\cdot\left(\vec{A}\times\vec{r}\right)=\vec{r}\cdot curl\,\vec{A}-\vec{A}\cdot curl\,\vec{r}$

$=\vec{r}\cdot curl\,\vec{A}$ (as curl $\vec{r}=0$)

12. If $\vec{a}$ is a constant vector, prove that

div $\left\{r^n\left(\vec{a}\times\vec{r}\right)\right\}=0$

Solution.

$$\nabla\cdot\left(r^n\left(\vec{a}\times\vec{r}\right)\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(r^n\left(\vec{a}\times\vec{r}\right)\right)=\sum\hat{i}\cdot\left(nr^{n-1}\frac{\partial r}{\partial x}\left(\vec{a}\times\vec{r}\right)+r^n\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)\right)$$

$$=nr^{n-1}\left(\Sigma xi\right)\cdot\left(\vec{a}\times\vec{r}\right)+r^n\Sigma\hat{i}\cdot\left(\vec{a}\times\hat{i}\right)=nr^{n-2}\vec{r}\cdot\left(\vec{a}\times\vec{r}\right)+r^n\Sigma\hat{i}\cdot\left(\vec{a}\times\hat{i}\right)=0$$

13. Prove that

Solution.

$$\nabla\cdot\left(\phi\nabla\psi-\psi\nabla\phi\right)=\phi\nabla^2\psi-\psi\nabla^2\phi$$

$$\nabla\cdot\left(\phi\nabla\psi\right)=\phi\nabla\cdot\left(\nabla\psi\right)+\nabla\phi\cdot\nabla\psi=\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot\left(\psi\nabla\phi\right)=\psi\nabla\cdot\left(\nabla\phi\right)+\nabla\psi\cdot\nabla\phi=\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi$$

$$\nabla\cdot\left(\phi\nabla\psi-\psi\nabla\phi\right)=\left(\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi\right)-\left(\psi\nabla^2\phi+\nabla\psi\cdot\nabla\phi\right)=\phi\nabla^2\psi-\psi\nabla^2\phi$$

14. If $\vec{a}$ and $\vec{b}$ are constant vectors, prove that

(i) div $\left[\left(\vec{r}\times\vec{a}\right)\times\vec{b}\right]=-2\vec{b}\cdot\vec{a}$ **(ii) curl** $\left[\left(\vec{r}\times\vec{a}\right)\times\vec{b}\right]=\vec{b}\times\vec{a}$

Solution.

(i) div $\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\nabla\cdot\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\cdot\left[\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)\times\vec{b}\right]$

$=\sum\hat{i}\cdot\left[(\hat{i}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\cdot\left[(\hat{i}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\hat{i}\right]=\sum\left[(\hat{i}\cdot\vec{b})(\hat{i}\cdot\vec{a})-(\vec{a}\cdot\vec{b})(\hat{i}\cdot\hat{i})\right]$

$=\sum a_x b_x-(\vec{a}\cdot\vec{b})\Sigma 1=\vec{a}\cdot\vec{b}-3(\vec{a}\cdot\vec{b})=-2\vec{a}\cdot\vec{b}=-2\vec{b}\cdot\vec{a}$

(ii) curl $\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\times\frac{\partial}{\partial x}\left[(\vec{r}\times\vec{a})\times\vec{b}\right]=\sum\hat{i}\times\left[\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)\times\vec{b}\right]=\sum\hat{i}\times\left[(\hat{i}\times\vec{a})\times\vec{b}\right]$

$=\sum\hat{i}\times\left[(\hat{i}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\hat{i}\right]=\sum(\hat{i}\cdot\vec{b})(i\times\vec{a})-(\vec{a}\cdot\vec{b})(\hat{i}\times\hat{i})=\sum b_x\left(a_y\hat{k}-a_z\hat{j}\right)$

$=\left(b_xa_y\hat{k}-b_xa_z\hat{j}\right)+\left(b_ya_z\hat{i}-b_ya_x\hat{k}\right)+\left(b_za_x\hat{j}-b_za_y\hat{i}\right)$

$=\left(b_ya_z-b_za_y\right)\hat{i}+\left(b_za_x-b_xa_z\right)\hat{j}+\left(b_xa_y-b_ya_x\right)\hat{k}=\vec{b}\times\vec{a}$

15. If $\vec{r}$ denotes the position vector of a point and if $\hat{r}$ be the unit vector in direction of $\vec{r}$, $r=|\vec{r}|$, determine of grad $\left(r^{-1}\right)$ in terms of $\hat{r}$ and *r*.

Solution.

$\nabla\left(\frac{1}{r}\right)=\Sigma\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\Sigma\hat{i}\left(-\frac{1}{r^2}\right)\cdot\frac{\partial r}{\partial x}=-\frac{1}{r^3}\Sigma x\hat{i}=-\frac{\vec{r}}{r^3}$, $\nabla\left(\frac{1}{r}\right)=\frac{\hat{r}}{r^2}$, Where, $\hat{r}$ is a unit vector in direction of $\vec{r}$.

16. For any constant vector $\vec{a}$, show that the vector represented by curl $(\vec{a}\times\vec{r})$ is always parallel to the vector $\vec{a}\cdot\vec{r}$ being the position vector of a point (x,y,z), measured from the origin.

Solution.

Curl $(\vec{a}\times\vec{r})=\Sigma\hat{i}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)=\Sigma\hat{i}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)=\Sigma\hat{i}\times(\vec{a}\times\hat{i})=\Sigma\hat{i}\times(\vec{a}\times\hat{i})=\Sigma(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}$

$=\vec{a}\Sigma 1-\Sigma(\hat{i}\cdot\vec{a})\hat{i}=3\vec{a}-\vec{a}=2\vec{a}$. So, Curl $(\vec{a}\times\vec{r})$ is always parallel to vector $\vec{a}$.

17. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ find the value(s) of *n* in order that $r^n\vec{r}$ may be (i) solenoidal, (ii) irroataional.

Solution.

(i) For vector to be solenoidal, its divergence should be zero.

$\nabla\cdot\left(r^n\vec{r}\right)=\Sigma\hat{i}\cdot\frac{\partial}{\partial x}\left(r^n\vec{r}\right)=\Sigma\left[(\hat{i}\cdot\vec{r})nr^{n-1}\frac{\partial r}{\partial x}+r^n\hat{i}\cdot\frac{\partial\vec{r}}{\partial x}\right]=\Sigma\left[nr^{n-1}\frac{x^2}{r}+r^n(\hat{i}\cdot\hat{i})\right]$

$=nr^{n-2}\Sigma x^2+r^n\Sigma 1=nr^n+3r^n=(n+3)r^n$. So, $r^n\vec{r}$ will solenoidal if $n=-3$.

(ii) For vector to be irrotational, its curl must be zero.

Curl $r^n\vec{r}=\Sigma\hat{i}\times\frac{\partial}{\partial x}\left(r^n\vec{r}\right)=\Sigma\left[(\hat{i}\times\vec{r})nr^{n-1}\frac{\partial r}{\partial x}+r^n\left(\hat{i}\times\frac{\partial\vec{r}}{\partial x}\right)\right]=\Sigma\left[(\hat{i}\times\vec{r})nr^{n-2}x+r^n(\hat{i}\times\hat{i})\right]$

$=\Sigma\left[nr^{n-2}(x\hat{i}\times\vec{r})+0\right]=nr^{n-2}\left[(\Sigma xi)\times\vec{r}\right]=nr^{n-2}\left[\vec{r}\times\vec{r}\right]=0$

So, $r^n\vec{r}$ will be irrotational for any value of *n*.

18. If $u\vec{f}=\nabla v$, when *u, v* are scalar fields and $\vec{f}$ is a vector, find the value of $\vec{f}$ .curl $\vec{f}$.

Solution.

Given $u\vec{f} = \nabla V$, Taking curl on both sides, $\nabla\times\left(u\vec{f}\right) = \nabla\times\left(\nabla V\right) \Rightarrow \nabla u\times\vec{f} + u\nabla\times\vec{f} = 0$ (as curl of gradient = 0). Taking dot product with $\vec{f}$

$\vec{f}\cdot\left(\nabla u\times\vec{f}\right) + u\vec{f}\cdot\nabla\times\vec{f} = 0$. Since $\vec{f}\cdot\left(\nabla u\times\vec{f}\right) = 0$. So, $\vec{f}\cdot\nabla\times\vec{f} = 0$

19. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$. If a scalar field φ and a vector field $\vec{u}$ satisfy $\nabla\varphi = \nabla\times\vec{u} + f(r)\vec{r}$ where f is an arbitrary differentiable function, then show that $\nabla^2\varphi = rf'(r) + 3f(r)$.

Solution.

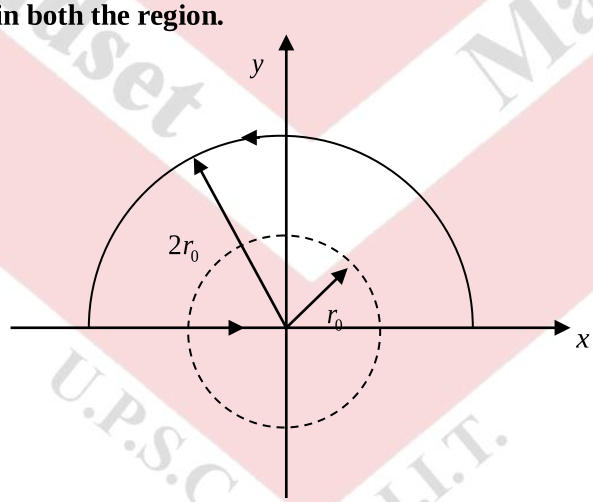
$$\nabla\phi = \nabla\times\vec{u} + f(r)\vec{r} \quad \nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\cdot\left(\nabla\times\vec{u}\right) + \nabla\cdot\left(f(r)\vec{r}\right) = 0 + \sum\hat{i}\cdot\frac{\partial}{\partial x}\left(f(r)\vec{r}\right)$$

$$= \sum\left(\hat{i}\cdot\vec{r}\right)f'(r)\frac{x}{r} + \sum\hat{i}f(r)\cdot\frac{\partial\vec{r}}{\partial x} = \frac{f'(r)}{r}\sum x^2 + f(r)\sum\hat{i}\cdot\hat{i} = rf'(r) + 3f(r)$$

20. A vector field is given by

$$\vec{F}(r) = \begin{cases} a\left(x\hat{j} - y\hat{i}\right) & \text{for}\left(x^2+y^2\right) \le r_0^2 \text{ (region-I)} \\ ar_0^2\dfrac{\left(x\hat{j} - y\hat{i}\right)}{\left(x^2+y^2\right)} & \text{for}\left(x^2+y^2\right) > r_0^2 \text{ (region-II)} \end{cases}$$

Here *a* and r_0 are two constants.
Find the curl of this field in both the region.



Solution.

$\vec{F}(r) = \alpha\left(-y\hat{i} + x\hat{j}\right)$ for $r \le r_0$ (Region I), $\alpha r_0^2\dfrac{\left(-y\hat{i} + x\hat{j}\right)}{x^2+y^2}$ for $r > r_0$ (Region II)

In regions I, $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\alpha y & \alpha x & 0 \end{vmatrix} = 2\alpha\hat{k}$

In region II $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\alpha r_0^2 y}{x^2+y^2} & \frac{\alpha r_0^2 x}{x^2+y^2} & 0 \end{vmatrix} = 0$

21. If $\vec{r}$ is the position vector of the point (x, y, z) w.r.t. origin.

Prove that

$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Find $f(r)$ such that $\nabla^2 f(r) = 0$.

Solution.

$\nabla^2 f(r) = \nabla \cdot \nabla f(r)$, $\nabla f(r) = \sum \hat{i} \frac{\partial}{\partial x} f(r) = \sum \hat{i} f'(r) \frac{\partial r}{\partial x} = \sum \hat{i} f'(r) \frac{x}{r} = \frac{f'(r)}{r} \sum \hat{i} x$

$= \frac{f'(r)}{r} \vec{r}$

$\nabla^2 f(r) = \nabla \cdot \left(\frac{f'(r)}{r} \vec{r} \right) = \sum \hat{i} \cdot \frac{\partial}{\partial x} \left(\frac{1}{r} f'(r) \vec{r} \right)$

$= \sum \hat{i} \cdot \left[\frac{\partial}{\partial x} \left(\frac{1}{r} \right) f'(r) \vec{r} + \frac{1}{r} \frac{\partial}{\partial x} \left(f'(r) \right) \vec{r} + \frac{1}{r} f'(r) \frac{\partial \vec{r}}{\partial x} \right]$

$= \sum \hat{i} \cdot \left[-\frac{1}{r^2} \cdot \frac{x}{r} f'(r) \vec{r} + \frac{1}{r} f''(r) \frac{\partial r}{\partial x} \vec{r} + \frac{1}{r} f'(r) \hat{i} \right]$

$= \sum \left[-\frac{f'(r)}{r^3} \cdot x (\hat{i} \cdot \vec{r}) + \frac{1}{r} f''(r) \frac{x}{r} \cdot (\hat{i} \cdot \vec{r}) + \frac{1}{r} f'(r) \hat{i} \cdot \hat{i} \right]$

$= -\frac{f'(r)}{r^3} \Sigma x^2 + \frac{f''(r)}{r^2} \Sigma x^2 + \frac{1}{r} f'(r) \Sigma 1 = -\frac{f'(r)}{r} + f''(r) + \frac{3}{r} f'(r) = f''(r) + \frac{2}{r} f'(r)$

Now, let us find $f(r)$ such that $\nabla^2 f(r) = 0$,Let $g(r) = f'(r)$

Now, $\nabla^2 f(r) = 0 \Rightarrow f''(r) + \frac{2}{r} f'(r) = 0 \Rightarrow g'(r) + \frac{2}{r} g(r) = 0 \Rightarrow \frac{dg}{dr} + \frac{2}{r} g = 0 \Rightarrow \frac{dg}{g} + 2\frac{dr}{r} = 0$

Integrating

$\int \frac{dg}{g} + 2\int \frac{dr}{r} = \text{constant} \Rightarrow gr^2 = C_1 \because g(r) = \frac{C_1}{r^2}\ \frac{df}{dr} = \frac{C_1}{r^2}, f = \int \frac{C_1}{r^2} dr + C_2$

$f(r) = -\frac{C_1}{r} + C_2$

22. Evaluate $div\{\vec{a} \times (\vec{r} \times \vec{a})\}$ where $\vec{a}$ is a constant vector.

Solution.

$$\nabla\cdot\left\{\vec{a}\times(\vec{r}\times\vec{a})\right\}=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left\{\vec{a}\times(\vec{r}\times\vec{a})\right\}=\sum\hat{i}\cdot\left\{\vec{a}\times\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)\right\}$$

$$=\sum\hat{i}\cdot\left\{\vec{a}\times(\hat{i}\times\vec{a})\right\}=\sum\hat{i}\cdot\left\{(\vec{a}\cdot\vec{a})\hat{i}-(\vec{a}\cdot\hat{i})\vec{a}\right\}=\sum\left[(\vec{a}\cdot\vec{a})(\hat{i}\cdot\hat{i})-(\vec{a}\cdot\hat{i})(\vec{a}\cdot\hat{i})\right]=a^2\Sigma1-\Sigma a_x^2$$

$$=3a^2-a^2=2a^2$$

23. If $\vec{f}=\frac{1}{r}\vec{r}$; find grad$\left(div\,\vec{f}\right)$.

Solution.

$$div\,\vec{f}=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\vec{r}\right)=\sum\hat{i}\cdot\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\vec{r}+\frac{1}{r}\frac{\partial\vec{r}}{\partial x}\right)=\sum\hat{i}\cdot\left(-\frac{1}{r^2}\cdot\frac{x}{r}\vec{r}+\frac{1}{r}\hat{i}\right)$$

$$=-\frac{1}{r^3}\sum x(\hat{i}\cdot\vec{r})+\sum\frac{1}{r}(\hat{i}\cdot\hat{i})=-\frac{1}{r^3}\sum x^2+\frac{1}{r}\sum1=-\frac{1}{r}+\frac{3}{r}=\frac{2}{r}$$

grad (div $\vec{f}$) = $\text{grad}\left(\frac{2}{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{2}{r}\right)=\sum\hat{i}\left(-\frac{2}{r^2}\right)\frac{\partial r}{\partial x}=-\frac{2}{r^3}\cdot\sum x\hat{i}=-\frac{2}{r^3}\cdot\vec{r}$

24. Prove that curl $(\vec{a}\times\vec{r})r^n=(n+2)r^n\vec{a}-nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$.

Solution.

$$\text{curl}(\vec{a}\times\vec{r})r^n=\sum\hat{i}\times\frac{\partial}{\partial x}\left[(\vec{a}\times\vec{r})r^n\right]=\sum\hat{i}\times\left[\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)r^n+(\vec{a}\times\vec{r})\frac{\partial}{\partial x}(r^n)\right]$$

$$=\sum\hat{i}\left[(\vec{a}\times\hat{i})r^n+(\vec{a}\times\vec{r})nr^{n-1}\frac{\partial r}{\partial x}\right]=\sum\hat{i}\times(\vec{a}\times\hat{i})r^n+\sum\hat{i}\times(\vec{a}\times\vec{r})nr^{n-1}\frac{x}{r}$$

$$=r^n\Sigma\hat{i}\times(\vec{a}\times\hat{i})+r^{n-2}\Sigma x\hat{i}\times(\vec{a}\times\vec{r})=r^n\Sigma\left[(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}\right]+nr^{n-2}\left[\vec{r}\times(\vec{a}\times\vec{r})\right]$$

$$=r^n\left[\vec{a}\Sigma1-\Sigma(\hat{i}\cdot\vec{a})\hat{i}\right]+nr^{n-2}\left[(\vec{r}\cdot\vec{r})\vec{a}-(\vec{r}\cdot\vec{a})\vec{r}\right]=3r^n\vec{a}-r^n\vec{a}+nr^n\vec{a}-nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$$

$$=(n+2)r^n\vec{a}-nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$$

25. Prove that div $\left\{\frac{f(r)}{r}\vec{r}\right\}=\frac{1}{r^2}\frac{d}{dr}(r^2f)$

Solution.

$$div\left\{\frac{f(r)}{r}\vec{r}\right\}=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left\{\frac{1}{r}f(r)\vec{r}\right\}=\sum\hat{i}\cdot\left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)f(r)\vec{r}+\frac{1}{r}\frac{\partial}{\partial x}(f(r))\vec{r}+\frac{1}{r}f(r)\frac{\partial\vec{r}}{\partial x}\right\}$$

$$=\sum\hat{i}\cdot\left\{-\frac{1}{r^2}\cdot\frac{x}{r}f(r)\vec{r}+\frac{1}{r}f'(r)\frac{x}{r}\cdot\vec{r}+\frac{1}{r}f(r)\hat{i}\right\}$$

$$=\sum\left\{-\frac{f(r)}{r^3}x(\hat{i}\cdot\vec{r})+\frac{1}{r^2}f'(r)x(\hat{i}\cdot\vec{r})+\frac{1}{r}f(r)(\hat{i}\cdot\hat{i})\right\}=-\frac{f(r)}{r^3}\Sigma x^2+\frac{1}{r^2}f'(r)\Sigma x^2+\frac{1}{r}f(r)\Sigma1$$

$$=-\frac{f(r)}{r}+f'(r)+3\frac{f(r)}{r}=f'(r)+2\frac{f(r)}{r}=\frac{1}{r^2}\left[r^2f'(r)+2r\,f(r)\right]=\frac{1}{r^2}\frac{d}{dr}(r^2f)$$

26. Prove that $\nabla\cdot\left\{r\nabla\left(\frac{1}{r^3}\right)\right\}=\frac{3}{r^4}$

Solution.

$$\nabla\left(\frac{1}{r^3}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)=\sum\hat{i}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)=\sum\hat{i}\left(-\frac{3x}{r^5}\right)=-\frac{3}{r^5}\Sigma\hat{i}x=-\frac{3\vec{r}}{r^5}.r\nabla\left(\frac{1}{r^3}\right)=-\frac{3\vec{r}}{r^4}$$

$$\nabla\cdot\left(r\nabla\left(\frac{1}{r^3}\right)\right)=\sum\hat{i}\cdot\frac{\partial}{\partial x}\left(-3\frac{\vec{r}}{r^4}\right)$$

$$=-3\sum\hat{i}\cdot\left\{\frac{\partial}{\partial x}\left(\frac{1}{r^4}\right)\vec{r}+\frac{1}{r^4}\cdot\frac{\partial\vec{r}}{\partial x}\right\}=-3\sum\left\{-\frac{4}{r^5}\cdot\frac{\partial r}{\partial x}\left(\hat{i}\cdot\vec{r}\right)+\frac{1}{r^4}\left(\hat{i}\cdot\hat{i}\right)\right\}=-3\sum\left\{-\frac{4}{r^6}\cdot x^2+\frac{1}{r^4}\right\}$$

$$=-3\left\{-\frac{4}{r^6}\Sigma x^2+\frac{1}{r^4}\Sigma 1\right\}=-3\left\{-\frac{4}{r^4}+\frac{3}{r^4}\right\}=\frac{3}{r^4}$$

27. Prove that

$$\vec{b}\cdot\nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{r}\right)\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

Solution.

$$\nabla\left(\frac{1}{r}\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\sum\hat{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right)=\sum\hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right)=-\frac{1}{r^2}\Sigma\hat{i}x=-\frac{\vec{r}}{r^3}$$

$$\vec{a}\cdot\nabla\left(\frac{1}{r}\right)=-\frac{(\vec{a}\cdot\vec{r})}{r^3},\ \nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{r}\right)\right)=\sum\hat{i}\frac{\partial}{\partial x}\left(-\frac{(\vec{a}\cdot\vec{r})}{r^3}\right)$$

$$=-\sum\hat{i}\left(\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)(\vec{a}\cdot\vec{r})+\frac{1}{r^3}\left(\vec{a}\cdot\frac{\partial\vec{r}}{\partial x}\right)\right)=-\sum\hat{i}\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}(\vec{a}\cdot\vec{r})+\frac{1}{r^3}\left(\vec{a}\cdot\hat{i}\right)\right)$$

$$=\frac{3}{r^5}(\vec{a}\cdot\vec{r})\left(\Sigma\hat{i}x\right)-\frac{1}{r^3}\Sigma\left(\vec{a}\cdot\hat{i}\right)\hat{i}=\frac{3}{r^5}(\vec{a}\cdot\vec{r})\vec{r}-\frac{1}{r^3}\vec{a}$$

$$\vec{b}\cdot\nabla\left(\vec{a}\cdot\nabla\left(\frac{1}{r}\right)\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{1}{r^3}\left(\vec{a}\cdot\vec{b}\right)$$

28. If $\vec{a}$ is a constant vector, prove that

$$\textbf{curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=-\frac{\vec{a}}{r^3}+\frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$$

Solution.

$$\text{curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=\sum\hat{i}\times\frac{\partial}{\partial x}\left[(\vec{a}\times\vec{r})\frac{1}{r^3}\right]=\sum\hat{i}\times\left[\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)\frac{1}{r^3}+(\vec{a}\times\vec{r})\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right]$$

$$=\sum\hat{i}\times\left[\left(\vec{a}\times\hat{i}\right)\frac{1}{r^3}+(\vec{a}\times\vec{r})\left(-\frac{3}{r^4}\cdot\frac{\partial r}{\partial x}\right)\right]=\frac{1}{r^3}\Sigma\hat{i}\times\left(\vec{a}\times\hat{i}\right)-\frac{3}{r^5}\Sigma\hat{i}x\times(\vec{a}\times\vec{r})$$

$$=\frac{1}{r^3}\left(\Sigma\left(\hat{i}\cdot\hat{i}\right)\vec{a}-\Sigma\left(\hat{i}\cdot\vec{a}\right)\hat{i}\right)-\frac{3}{r^5}\cdot\vec{r}\times(\vec{a}\times\vec{r})=\frac{1}{r^3}(3\vec{a}-\vec{a})-\frac{3}{r^5}\left[(\vec{r}\cdot\vec{r})\vec{a}-(\vec{a}\cdot\vec{r})\vec{r}\right]$$

$$=\frac{2\vec{a}}{r^3}-\frac{3}{r^5}\left[r^2\vec{a}-(\vec{a}\cdot\vec{r})\vec{r}\right]=-\frac{\vec{a}}{r^3}+\frac{3}{r^5}(\vec{a}\cdot\vec{r})\vec{r}$$

29. Evaluate $\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r^2}\right)\right]$.

Solution.

$$\nabla\cdot\left(\frac{\vec r}{r^2}\right)=\sum\hat i\cdot\frac{\partial}{\partial x}\left(\frac{\vec r}{r^2}\right)=\sum\hat i\cdot\left[\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)\vec r+\frac{1}{r^2}\frac{\partial\vec r}{\partial x}\right]=\sum\hat i\cdot\left[-\frac{2}{r^3}\cdot\frac{x\vec r}{r}+\frac{1}{r^2}\hat i\right]$$

$$=\sum\left[-\frac{2x}{r^4}\cdot\left(\hat i\cdot\vec r\right)+\frac{1}{r^2}\hat i\cdot\hat i\right]=\sum\left[-\frac{2x^2}{r^4}+\frac{1}{r^2}\right]$$

$$=-\frac{2}{r^4}\Sigma x^2+\frac{1}{r^2}\Sigma 1=-\frac{2}{r^2}+\frac{3}{r^2}=\frac{1}{r^2}$$

Now, $\nabla^2\left[\nabla\cdot\left(\frac{\vec r}{r^2}\right)\right]=\nabla^2\left(\frac{1}{r^2}\right)=\nabla\cdot\nabla\left(\frac{1}{r^2}\right)$, $\nabla\left(\frac{1}{r^2}\right)=\sum\hat i\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)=\sum\hat i\left(-\frac{2}{r^3}\cdot\frac{\partial r}{\partial x}\right)$

$$=-\frac{2}{r^4}\sum x\hat i=-\frac{2\vec r}{r^4}$$

$$\nabla\cdot\nabla\left(\frac{1}{r^2}\right)=\nabla\cdot\left(-\frac{2\vec r}{r^4}\right)=\sum\hat i\cdot\frac{\partial}{\partial x}\left(-\frac{2\vec r}{r^4}\right)=-2\sum\hat i\cdot\left[\frac{\partial}{\partial x}\left(\frac{1}{r^4}\right)\vec r+\frac{1}{r^4}\frac{\partial\vec r}{\partial x}\right]$$

$$=-2\sum\hat i\cdot\left[-\frac{4}{r^5}\frac{\partial r}{\partial x}\vec r+\frac{1}{r^4}\hat i\right]=-2\sum\left[-\frac{4x}{r^6}\cdot\left(\hat i\cdot\vec r\right)+\frac{1}{r^4}\hat i\cdot\hat i\right]=-2\sum\left[-\frac{4x^2}{r^6}+\frac{1}{r^4}\right]$$

$$=-2\left[-\frac{4}{r^6}\Sigma x^2+\frac{1}{r^4}\Sigma 1\right]=-2\left[-\frac{4}{r^4}+\frac{3}{r^4}\right]=\frac{2}{r^4}$$

30. Expand

(i) $\nabla\left(\vec a\cdot\frac{\vec r}{r^n}\right)$ **(ii)** $\nabla\cdot\left(\vec a\times\frac{\vec r}{r^n}\right)$ **(iii)** $\nabla\times\left(\vec a\times\frac{\vec r}{r^n}\right)$

where $\vec a$ is a constant vector.

Solution.

(i) $$\nabla\left(\vec a\cdot\frac{\vec r}{r^n}\right)=\sum\hat i\frac{\partial}{\partial x}\left(\vec a\cdot\frac{\vec r}{r^n}\right)=\sum\hat i\left[\vec a\cdot\frac{\partial}{\partial x}\left(\frac{\vec r}{r^n}\right)\right]=\sum\hat i\left[\vec a\cdot\left(\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)\vec r+\frac{1}{r^n}\frac{\partial\vec r}{\partial x}\right)\right]$$

$$=\sum\hat i\left[\vec a\cdot\left(-\frac{n}{r^{n+1}}\cdot\frac{\partial r}{\partial x}\vec r+\frac{1}{r^n}\hat i\right)\right]=\sum\hat i\left[-\frac{n}{r^{n+2}}x\left(\vec a\cdot\vec r\right)+\frac{1}{r^n}\left(\vec a\cdot\hat i\right)\right]$$

$$=-\frac{n}{r^{n+2}}\left(\vec a\cdot\vec r\right)\left(\Sigma\hat i x\right)+\frac{1}{r^n}\Sigma\left(\vec a\cdot\hat i\right)\hat i=-\frac{n\left(\vec a\cdot\vec r\right)}{r^{n+2}}\vec r+\frac{1}{r^n}\vec a$$

(ii) $$\nabla\cdot\left(\vec a\times\frac{\vec r}{r^n}\right)=\sum\hat i\cdot\frac{\partial}{\partial x}\left(\vec a\times\frac{\vec r}{r^n}\right)=\sum\hat i\cdot\left[\vec a\times\frac{\partial}{\partial x}\left(\frac{\vec r}{r^n}\right)\right]=\sum\hat i\cdot\left[\vec a\times\left(\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right)\cdot\vec r+\frac{1}{r^n}\frac{\partial\vec r}{\partial x}\right)\right]$$

$$=\sum\hat i\cdot\left[\vec a\times\left(-\frac{n}{r^{n+1}}\cdot\frac{\partial r}{\partial x}\vec r+\frac{1}{r^n}\hat i\right)\right]=\sum\hat i\cdot\left[\vec a\times\left(-\frac{nx}{r^{n+2}}\vec r+\frac{1}{r^n}\hat i\right)\right]$$

$$=\sum\hat i\cdot\left[\vec a\times\left(-\frac{nx}{r^{n+2}}\right)\vec r\right]+\sum\hat i\cdot\left(\vec a\times\frac{1}{r^n}\hat i\right)=-\frac{n}{r^{n+2}}\Sigma\hat i x\cdot\left(\vec a\times\vec r\right)+0=-\frac{n}{r^{n+2}}\vec r\cdot\left(\vec a\times\vec r\right)$$

(iii) $$\nabla\times\left(\vec a\times\frac{\vec r}{r^n}\right)=\sum\hat i\times\frac{\partial}{\partial x}\left(\vec a\times\frac{\vec r}{r^n}\right)=\sum\hat i\times\left[\vec a\times\frac{\partial}{\partial x}\left(\frac{\vec r}{r^n}\right)\right]$$

$$= \sum \hat{i} \times \left[\vec{a} \times \left(\frac{\partial}{\partial x} \left(\frac{1}{r^n} \right) \vec{r} + \frac{1}{r^n} \frac{\partial \vec{r}}{\partial x} \right) \right] = \sum \hat{i} \times \left[\vec{a} \times \left(-\frac{n}{r^{n+1}} \cdot \frac{\partial r}{\partial x} \vec{r} + \frac{1}{r^n} \hat{i} \right) \right]$$

$$= \sum \hat{i} \times \left[\vec{a} \times \left(-\frac{n}{r^{n+2}} x\vec{r} + \frac{1}{r^n} \hat{i} \right) \right] = \sum \hat{i} \times \left[\vec{a} \times \left(-\frac{n}{r^{n+2}} x \right) \vec{r} \right] + \sum \hat{i} \times \left(\vec{a} \times \frac{1}{r^n} \hat{i} \right)$$

$$= -\frac{n}{r^{n+2}} \sum \hat{i} x \times \left(\vec{a} \times \vec{r} \right) + \frac{1}{r^n} \sum \hat{i} \times \left(\vec{a} \times \hat{i} \right) = -\frac{n}{r^{n+2}} \vec{r} \times \left(\vec{a} \times \vec{r} \right) + \frac{1}{r^n} \Sigma \left[\left(\hat{i} \cdot \hat{i} \right) \vec{a} - \left(\hat{i} \cdot \vec{a} \right) \hat{i} \right]$$

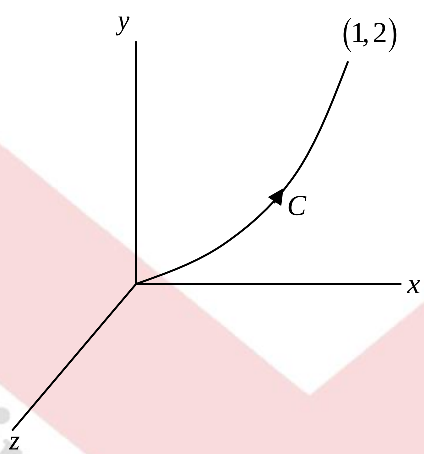
$$= -\frac{n}{r^{n+2}} \left[\left(\vec{r} \cdot \vec{r} \right) \vec{a} - \left(\vec{r} \cdot \vec{a} \right) \vec{r} \right] + \frac{1}{r^n} \left[\vec{a} \Sigma 1 - \Sigma \left(\hat{i} \cdot \vec{a} \right) \hat{i} \right] = -\frac{n}{r^{n+2}} \left[\left(r^2 \vec{a} - \left(\vec{r} \cdot \vec{a} \right) \vec{r} \right) \right] + \frac{1}{r^n} \left[3\vec{a} - \vec{a} \right]$$

$$= -\frac{n}{r^n} \vec{a} + n \frac{\left(\vec{r} \cdot \vec{a} \right)}{r^{n+2}} \vec{r} + \frac{2\vec{a}}{r^n} = \frac{\left(2 - n \right) \vec{a}}{r^n} + n \frac{\left(\vec{r} \cdot \vec{a} \right)}{r^{n+2}} \vec{r}$$

CHAPTER 4
LINE INTEGRAL

SOLVED EXAMPLES

1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ determine the value of $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0,0)$ to $(1,2)$.



Solution.

The curve lies in xy plane, so $z = 0$. z can never be taken as independent variable z is a dependent variable. Now, out of x and y, and one variable can be taken as independent.

Suppose x is taken as independent variable $y = 2x^2, dy = 4xdx$

$\vec{F}\cdot d\vec{r} = 3xydx - y^2dy = 6x^3dx - 4x^4\cdot 4xdx = \left(6x^3 - 16x^5\right)dx$

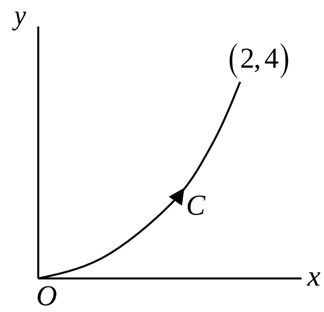
So, the line integral $\int_C \vec{f}\cdot d\vec{r}$ reduces to a definite integral. $\int_0^1 \left(6x^3 - 16x^5\right)dx = 6\left.\frac{x^4}{4}\right|_0^1 - 16\left.\frac{x^6}{6}\right|_0^1 = -\frac{7}{6}$

If y is taken as independent variable then x can be expressed in terms of y as $x = \sqrt{\frac{y}{2}}$

$dx = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y}}dy$ So, $\vec{f}\cdot d\vec{r} = 3xydx - y^2dy = 3y\sqrt{\frac{y}{2}}\cdot\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y}}dy - y^2dy = \left(\frac{3}{4}y - y^2\right)dy$

So, the line integral $\int \vec{f}\cdot d\vec{r}$ reduces to a definite integral $\int_0^2\left(\frac{3}{4}y - y^2\right)dy = \left.\frac{3}{8}y^2 - \frac{y^3}{3}\right|_0^2 = -\frac{7}{6}$

2. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(x^2 - y^2\right)\hat{i} + xy\hat{j}$ and curve C is arc of the curve $y = x^2$ from $(0,0)$ to $(2,4)$.



Solution.

$\vec{F} \cdot d\vec{r} = \left(x^2 - y^2\right)dx + xydy$ Taking x as independent variable, $y = x^2, dy = 2xdx$

$\vec{F} \cdot d\vec{r} = \left(x^2 - y^2\right)dx + xydy = \left(x^2 - x^4\right)dx + x^3 \cdot 2xdx = \left(x^2 + x^4\right)dx$

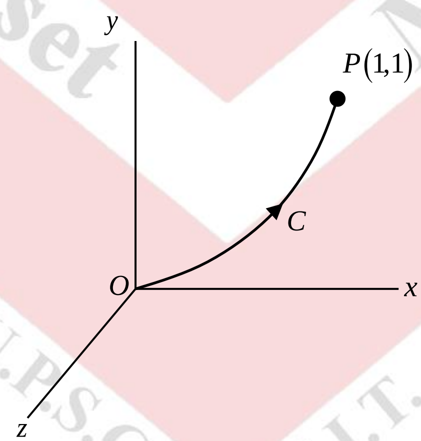
So, the line integral $\int \vec{F} \cdot d\vec{r}$ reduces to a definite integral $\int_0^2 \left(x^2 + x^4\right)dx = \left.\frac{x^3}{3} + \frac{x^5}{5}\right|_0^2 = \frac{136}{15}$

3. Find the work done when a force

$\vec{F} = \left(x^2 - y^2 + x\right)\hat{i} - \left(2xy + y\right)\hat{j}$

moves a particle in xy plane from $(0,0)$ to $(1,1)$ along the parabola $y^2 = x$.

Solution.



Here, on the curve C, y can be taken as independent variable and $x = y^2, dx = 2ydy$

work done in moving a particle by displacement $d\vec{r}$ $dW = \vec{F} \cdot d\vec{r} = \left(x^2 - y^2 + x\right)dx - \left(2xy + y\right)dy$

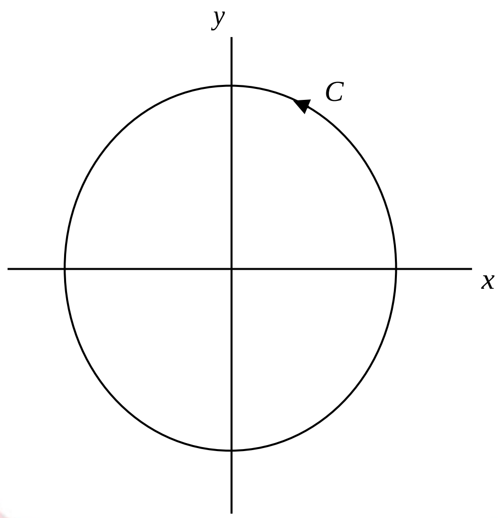
$= \left(y^4 - y^2 + y^2\right) \cdot 2ydy - \left(2y^2 \cdot y + y\right)dy = \left(2y^5 - 2y^3 - y\right)dy$

Hence, work done is moving a particle from O to P is given by

$W = \int_0^1 \left(2y^5 - 2y^3 - y\right)dy = \left. 2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right|_0^1 = -\frac{2}{3}$

4. Evaluate $\oint xdy - ydx$ around a circle $x^2 + y^2 = r^2$

Solution.



Let C denotes the circle. The parametric equations of circle is $x = r\cos\theta \; y = r\sin\theta$
Here, x and y have been expressed in terms of parameter which varies from 0 to 2π as one traverses the circle.

$$x = r\cos\theta \Rightarrow dx = -r\sin\theta d\theta,\; y = r\sin\theta \Rightarrow dy = r\cos\theta d\theta$$

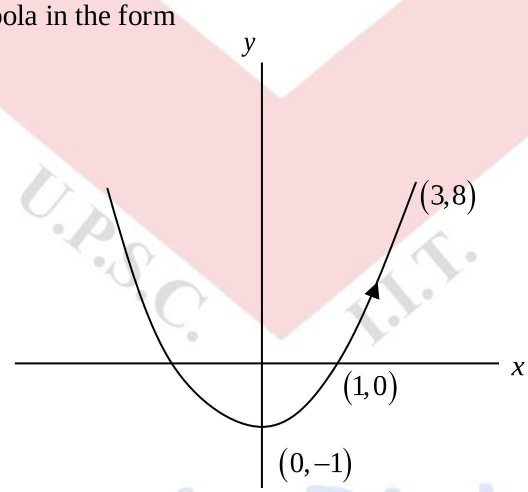
$$xdy - ydx = r\cos\theta r\cos\theta d\theta - r\sin\theta(-r\sin\theta)d\theta = r^2 d\theta$$

So, $\oint_C xdy - ydx = r^2 \oint d\theta = 2\pi r^2$ .Here, r is a constant, because integral is carried over a circle.

5. Calculate the work done when a force $\vec{F} = xy\hat{i} + \left(x^2 + y^2\right)\hat{j}$ moves a particle in xy plane from $(1,0)$ to $(3,8)$ along the curve C, $y = x^2 - 1$.

Solution.

The curve C is $y = x^2 - 1$. Since, this is quadratic in x and linear in y with no xy terms. This is a parabola. Let us put this parabola in the form



$$(x-\alpha)^2 = 4a(y-\beta),\; C:(x-0)^2 = (y+1)$$

This is a parabola with vertex at $(0,-1)$ and axis parallel to y axis.

On curve C, let us take x as independent variable. The dependent variable y can be written in terms of x as $y = x^2 - 1$, $dy = 2xdx$

work done is moving a particle by displacement $d\vec{r}$, $dW = \vec{F}\cdot d\vec{r}$

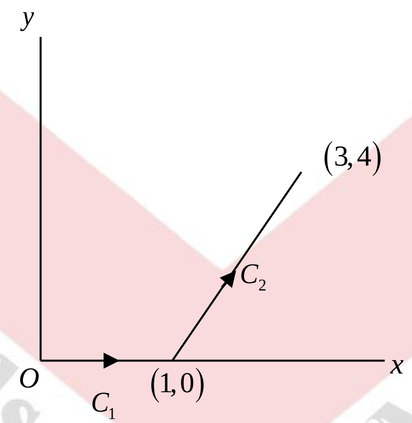
$= xydx + \left(x^2+y^2\right)dy = x\left(x^2-1\right)dx + \left(x^2+\left(x^2-1\right)^2\right)2xdx = \left(2x^5 - x^3 + x\right)dx$

So, work done is moving a particle from $(1,0)$ to $(3,8)$ along a curve C.

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_1^3 \left(2x^5 - x^3 + x\right)dx = \left(2\cdot\frac{x^6}{6} - \frac{x^4}{4} + \frac{x^2}{2}\right)\Bigg|_1^3 = 227$$

6. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x+2y)\hat{i} + (2y-x)\hat{j}$ and C is curve in xy plane consisting of the straight lines from $(0,0)$ to $(1,0)$ and then to $(3,4)$.

Solution.



The curve C consists of two pieces of smooth curves C_1 and C_2.

C_1 is straight line from $(0,0)$ to $(1,0)$ i.e. $y=0$

C_2 is straight line from $(1,0)$ to $(3,4)$

i.e. $y - 0 = \left(\frac{4-0}{3-1}\right)\cdot(x-1)$ or, $y = 2x-2$

So, along $C_1, y=0, dy=0$ (x is an independent variable), $\vec{F}\cdot d\vec{r} = xdx$

Along $C_2; y = 2x-2, dy = 2dx$ (let us take x as independent variables).

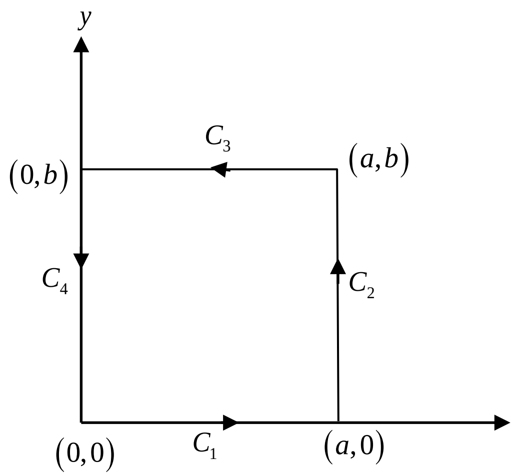
$\vec{F}\cdot d\vec{r} = (x+2y)dx + (2y-x)dy$

on C_2, $\vec{F}\cdot d\vec{r} = \left(x+2(2x-2)\right)dx + \left(2(2x-2)-x\right)\cdot 2dx = (11x-12)dx$

So, $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} = \int_0^1 xdx + \int_1^3 (11x-12)dx = \frac{x^2}{2}\Big|_0^1 + \left(\frac{11}{2}x^2 - 12x\right)\Big|_1^3 = 20.5$

7. Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(x^2+y^2\right)\hat{i} - 2xy\hat{j}$, where curve C is a rectangle in the xy plane bounded by $y=0, x=a, y=b, x=0$.

Solution.



The curve C as shown in figure 4.11 consists of four pieces of smooth curves C_1, C_2, C_3 & C_4.

$\vec{F}\cdot d\vec{r} = \left(x^2 + y^2\right)dx - 2xydy$

On $C_1, y=0, dy=0, \vec{F}\cdot d\vec{r} = x^2dx$, On $C_2, x=a, dx=0, \vec{F}\cdot d\vec{r} = -2aydy$

On $C_3, y=b, dy=0, \vec{F}\cdot d\vec{r} = \left(x^2+b^2\right)dx$, On $C_4, x=0, dx=0, \vec{F}\cdot d\vec{r} = 0$

$$\oint \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r} = \int_0^a x^2dx + \int_0^b -2aydy + \int_a^0 \left(x^2+b^2\right)dx + \int_b^0 0\cdot dy$$

$$= \left.\frac{x^3}{3}\right|_0^a + \left[-ay^2\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0 = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2$$

8. Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ **along the curve** $x = t^2+1, y = 2t^2, z = t^3$ **from** $t=1$ **to** $t=2$**.**

Solution.

On curve C, the coordinates x, y, z are expressed in terms of parameter t.

$x = t+1, dx = 2tdt$, $y = 2t^2, dy = 4tdt$, $z = t, dz = 3t^2dt$

t varies from $t=1$ to $t=2$.

$\vec{F}\cdot d\vec{r} = 3xydx - 5zdy + 10xdz = 3\left(t^2+1\right)\cdot 2t^2\cdot 2tdt - 5t^3 4tdt + 10\left(t^2+1\right)\cdot 3t^2dt$

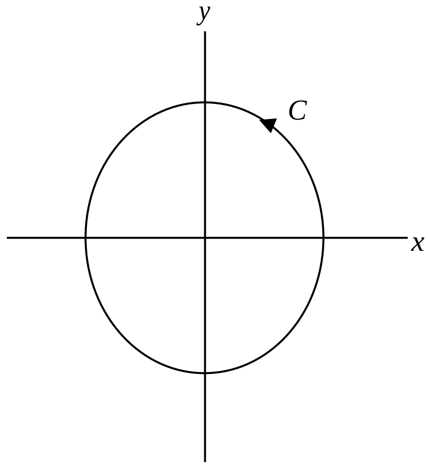
$= \left(12t^5 + 10t^4 + 12t^3 + 30t^2\right)dt$

So, total work done, $W = \int_C \vec{F}\cdot d\vec{r} = \int_1^2 \left(12t^5 + 10t^4 + 12t^3 + 30t^2\right)dt$

$$= \left.\left(12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right)\right|_1^2 = 303$$

9. Find the work done in moving a particle once around a circle C in the xy **plane if the circle has a centre at the origin and radius 2 and if the force field** $\vec{F}$ **is given by**

Solution.



Equation of circle as shown in figure 4.12 is written in parametric form as

$x = 2\cos\theta \Rightarrow dx = -2\sin\theta d\theta$, $y = 2\sin\theta \Rightarrow dy = 2\cos\theta d\theta$, $z = 0 \Rightarrow dz = 0$

x, y, z are expressed in terms of parameter θ.

$$\vec{F}\cdot d\vec{r} = (2x - y + 2z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$$

$$= (4\cos\theta - 2\sin\theta)(-2\sin\theta)d\theta + (2\cos\theta + 2\sin\theta)(2\cos\theta)d\theta + (6\cos\theta - 4\sin\theta)\cdot 0$$

$$= (4 - 4\sin\theta\cos\theta)d\theta$$

θ varies from 0 to 2π .So, total work done $W = \int_0^{2\pi}(4 - 4\sin\theta\cos\theta)d\theta = 4\theta - 2\sin^2\theta\Big|_0^{2\pi} = 8\pi$

10. If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int \vec{F}\cdot d\vec{r}$ where C is a straight line joining $(0,0,0)$ to $(1,1,1)$.

Solution.

Equation of straight line joining $(0,0,0)$ to $(1,1,1)$ is given by $\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$, where t is parameter.

In parametric form equation of curve is given by

$x = t \Rightarrow dx = dt$, $y = t \Rightarrow dy = dt$ $z = t \Rightarrow dz = dt$

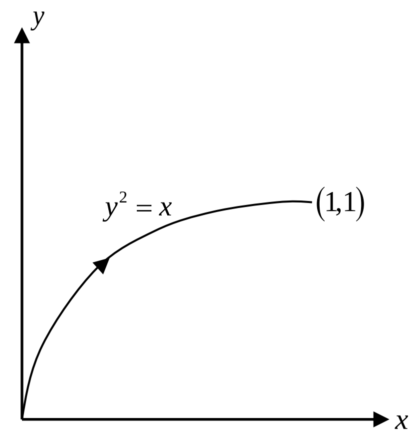
t varies from 0 to 1.

$$\vec{F}\cdot d\vec{r} = (3x^2 + 6y)dx - 14yzdy + 20xz^2dz = (20t^3 - 11t^2 + 6t)dt \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (20t^3 - 11t^2 + 6t)dt$$

$$= \left(5t^4 - \frac{11}{3}t^3 + 3t^2\right)\Bigg|_0^1 = \frac{13}{3}$$

11. Integrate the function $\vec{F} = x^2\hat{i} - xy\hat{j}$ from the point $(0,0)$ to $(1,1)$ along the parabola $y^2 = x$.

Solution.



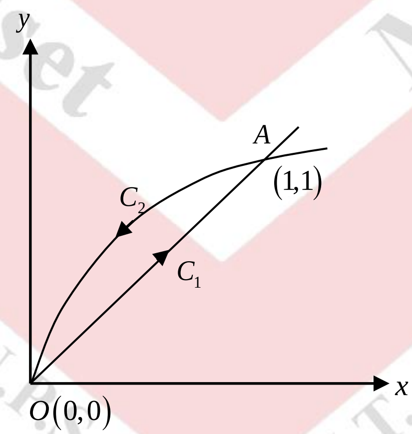
Here the curve C is parabola $y^2 = x$ as shown in figure 4.13. On C, y can be taken as independent variable. The dependent variable x can be written in terms of y as

$x = y^2$, $dx = 2ydy$, $\vec{F}\cdot d\vec{r} = x^2dx - xydy = y^4 \cdot 2ydy - y^2 \cdot ydy = \left(2y^5 - y^3\right)dy$

So, the line integral $\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(2y^5 - y^3\right)dy = \frac{1}{3}y^6 - \frac{1}{4}y^4\Big|_0^1 = \frac{1}{12}$

12. Find the value of $\int_C \left[\left(x + y^2\right)dx + \left(x^2 - y\right)dy\right]$ taken in the counter-clockwise sense along the closed curve C formed by straight line $y = x$ and curve $y^3 = x^2$.

Solution.



The curve C consists of chord OA and curved part AO as shown in figure 4.14.

Equation of OA is $y = x$ and curved part is $y^3 = x^2$.

Along chord OA, x can be taken as independent variable and $y = x$.

$\vec{F}\cdot d\vec{r} = \left(x + y^2\right)dx + \left(x^2 - y\right)dy = \left(x + x^2\right)dx + \left(x^2 - x\right)dx = 2x^2dx$

Along OA, x varies from 0 to 1. On curved part AO, let y be taken as independent variable & dependent variable x can be put as $x = y^{3/2}, dx = \frac{3}{2}y^{1/2}dy$.

$d\vec{r} = \left(x + y^2\right)dx + \left(x^2 - y\right)dy = \left(y^{3/2} + y^2\right)\frac{3}{2}y^{1/2}dy + \left(y^3 - y\right)dy = \left(y^3 + \frac{3}{2}y^{5/2} + \frac{3}{2}y^2 - y\right)dy$

y varies from 1 to 0.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 2x^2dx + \int_1^0 \left(y^3 + \frac{3}{2}y^{5/2} + \frac{3}{2}y^2 - y \right) dy$$

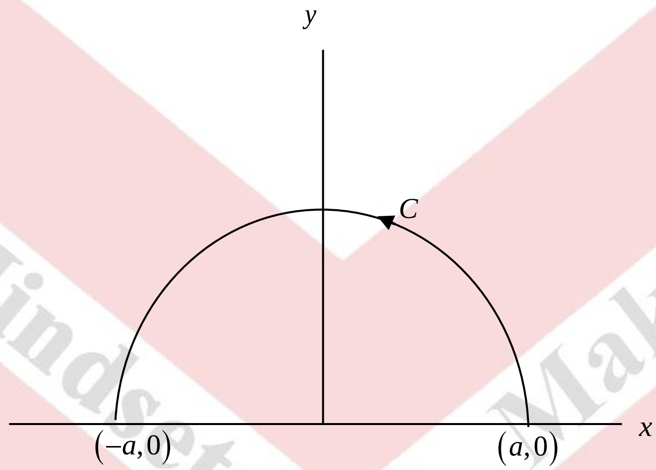
$$= \frac{2}{3}x^3\Big|_0^1 + \frac{1}{4}y^4 + \frac{3}{7}y^{7/2} + \frac{1}{2}y^3 - \frac{1}{2}y^2\Big|_1^0 = -\frac{1}{84}$$

Note: If the integral is carried out in clockwise direction. The answer will differ only in sign.

$\oint_C \vec{F}\cdot d\vec{r}$ in clockwise direction $= \dfrac{1}{84}$.

13. Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \dfrac{y^2}{x^2+y^2}\hat{i} - \dfrac{x^2}{x^2+y^2}\hat{j}$, where C is the semi-circle $r = \sqrt{a^2 - x^2}$.

Solution.



The curve C shown in Figure is the semi-circle $y = \sqrt{a^2 - x^2}$

The equation can be written in parametric form as $x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta$ $y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$. θ varies from 0 to π

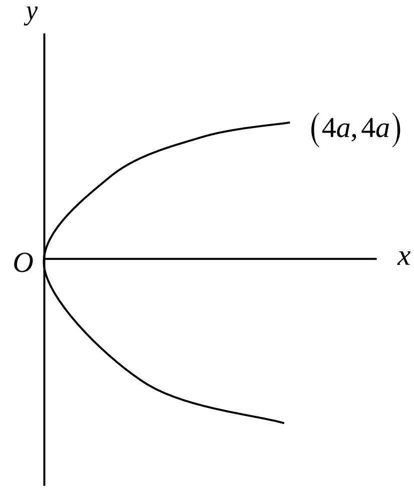
$$\vec{F}\cdot d\vec{r} = \frac{y^2dx - x^2dy}{x^2+y^2} = \frac{a^2\sin^2\theta(-a\sin\theta)d\theta - (a^2\cos^2\theta)\cdot a\cos\theta d\theta}{a^2} = -a\left(\sin^3\theta + \cos^3\theta\right)d\theta$$

$$\int_C \vec{F}\cdot d\vec{r} = -a\int_0^\pi \left(\sin^3\theta + \cos^3\theta\right)d\theta = -a\int_0^\pi \sin^3\theta d\theta - a\int_0^\pi \cos^3\theta d\theta$$

$$= -2a\int_0^{\pi/2} \sin^3\theta d\theta - 0\left(\text{Since, } \int_0^\pi \cos^3\theta d\theta = 0\right) = -2a\frac{\sqrt{2}\sqrt{1/2}}{2\sqrt{5/2}} = -\frac{4a}{3}$$

14. Evaluate $\int_C \dfrac{dx}{x+y}$ where C is the curve $y^2 = 4ax$ from $(0,0)$ to $(4a,4a)$.

Solution.



The equation of curve (as shown in the figure 4.16) can be written in parametric form as
$x = at^2 \Rightarrow dx = 2atdt$, $y = 2at \Rightarrow dy = 2adt$. Parameter, t varies from 0 to 2.

The integral $\int_0 \frac{dx}{x+y} = \int_0^2 \frac{2atdt}{at^2+2at} = 2\int_0^2 \frac{dt}{t+2} = 2\log(t+2)\Big|_0^2 = 2\log 2$

15. Evaluate $\int_C (ydx - xdy)$, where C is arc of cycloid $x = 2(\theta - \sin\theta), y = 2(1-\cos\theta)$ joining the points $(0,0)$ and $(4\pi, 0)$.

Solution.
The parametric equation of curve C as shown in figure 4.17 is given as
$x = 2(\theta - \sin\theta) \Rightarrow dx = 2(1-\cos\theta)d\theta$, $y = 2(1-\cos\theta) \Rightarrow dy = 2\sin\theta d\theta$

θ is the parameter since (x, y) varies from $(0,0)$ to $(4\pi, 0)$. So, θ will vary from 0 to 2π.

The integrand $ydx - xdy$

$= 2(1-\cos\theta)\cdot 2(1-\cos\theta)d\theta - 2(\theta - \sin\theta)\cdot 2\sin\theta d\theta$

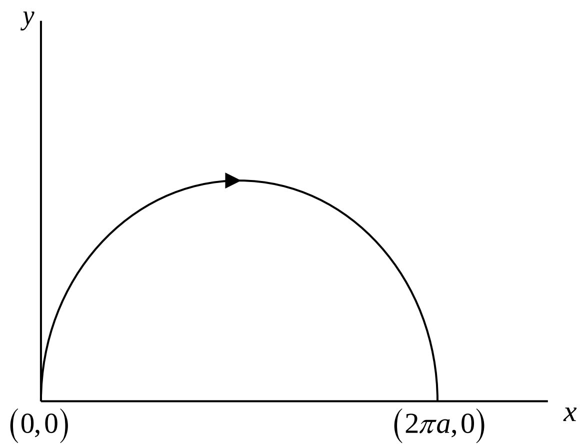
$= 4(2 - 2\cos\theta - \theta\sin\theta)d\theta$

So, $\int_C ydx - xdy = 4\int_0^{2\pi}(2 - 2\cos\theta - \theta\sin\theta)d\theta$

$= 8\int_0^{2\pi} d\theta - 8\int_0^{2\pi}\cos\theta d\theta - 4\int_0^{2\pi}\theta\sin\theta d\theta = 16\pi - 8[\sin\theta]_0^{2\pi} - 4[-\theta\cos\theta + \sin\theta]_0^{2\pi} = 24\pi$

16. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (2a - y)\hat{i} - (a - y)\hat{j}$ where C is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ from $(0,0)$ to $(2\pi a, 0)$.

Solution.



The equation of cycloid is written in parametric form as

$x = a(\theta - \sin\theta) \Rightarrow dx = a(1-\cos\theta)d\theta$, $y = a(1-\cos\theta) \Rightarrow dy = a\sin\theta d\theta$

where θ is the parameter varying from 0 to 2π.

On C: $\vec{F}\cdot d\vec{r} = (2a-y)dx - (a-y)dy = a(1+\cos\theta)\cdot a(1-\cos\theta)d\theta - a\cos\theta\cdot a\sin\theta d\theta$

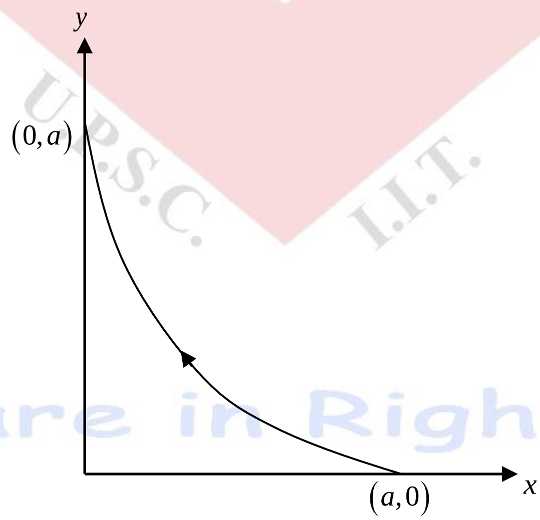
$= a^2\left(1-\cos^2\theta - \sin\theta\cos\theta\right)d\theta$

So, the line integral $\int_C \vec{F}\cdot d\vec{r} = a^2\int_0^{2\pi}\left(1-\cos^2\theta-\sin\theta\cos\theta\right)d\theta$

$$= a^2\int_0^{2\pi} d\theta - a^2\int_0^{2\pi}\cos^2\theta d\theta - a^2\int_0^{2\pi}\sin\theta\cos\theta d\theta = \pi a^2$$

17. Evaluate $\int_C \frac{x^2dy - y^2dx}{x^{5/3}+y^{5/3}}$ where C is the quarter of the astroid $x = a\cos^3 t, y = a\sin^3 t$ from the point $(a,0)$ to the point $(0,a)$.

Solution.



The parametric equation of the astroid as shown in figure 4.18 is given as

$x = a\cos^3 t \Rightarrow dx = -3a\cos^2 t\sin t dt$, $y = a\sin^3 t \Rightarrow dy = 3a\sin^2 t\cos t dt$

(x, y) varies from $(a, 0)$ to $(0, a)$. So, t varies from 0 to $\pi/2$.

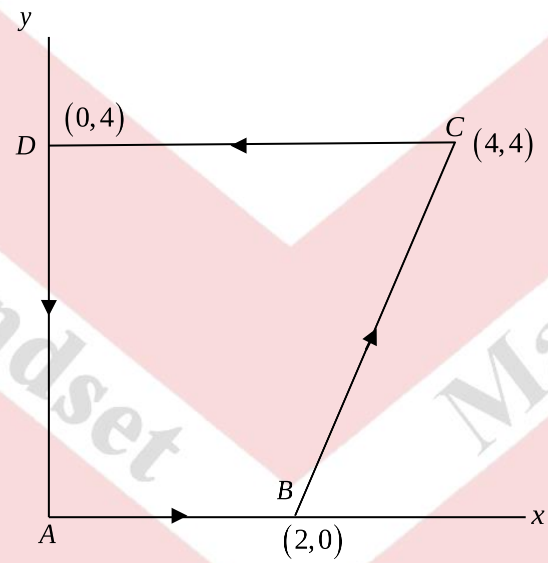
The integrand $\dfrac{x^2dy - y^2dx}{x^{5/3} + y^{5/3}} = \dfrac{a^2\cos^6 t\left(3a\sin^2 t\cos t\right)dt - \left(a^2\sin^6 t\right)\cdot\left(-3a\cos^3 t\sin t\right)dt}{a^{5/3}\left(\cos^5 t + \sin^5 t\right)}$

$= 3a^{4/3}\sin^3 t\cos^2 tdt$

The line integral reduces to $3a^{4/3}\int_0^{\pi/2}\sin^2 t\cos^2 tdt = 3a^{4/3}\dfrac{\boxed{3/2}\boxed{3/2}}{2\boxed{3}} = \dfrac{3\pi a^{4/3}}{16}$

18. Find the value of $\int_C\left(x^2 + y^2\right)dy$ taken in the counter clockwise sense along the quadrilateral with vertices $(0,0),(2,0),(4,4),(0,4)$.

Solution.



The quadrilateral as shown in Figure 4.19 consists of four pieces of smooth curves AB. BC, CD & DA.

On $AB, y = 0, dy = 0, \left(x^2 + y^2\right)dy = 0$. On $BC, y - 0 = \dfrac{4-0}{4-2}(x-2)$, $y = 2x - 4, dy = 2dx$

$\left(x^2 + y^2\right)dy = \left(x^2 + (2x-4)^2\right)2dx = \left(10x^2 - 32x + 32\right)dx$

x varies from 2 to 4. On $CD, y = 4, dy = 0$, $\left(x^2 + y^2\right)dy = 0$

On $DA, x = 0, dx = 0$, $\left(x^2 + y^2\right)dy = y^2dy$, y varies from 4 to 0

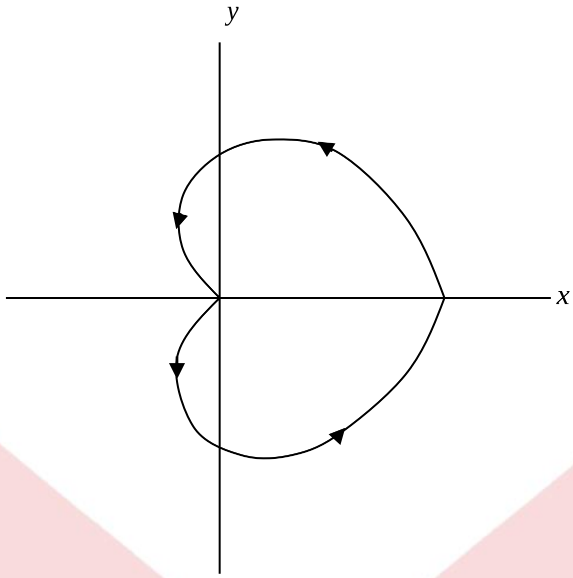
So, the line integral

$\int_C\left(x^2 + y^2\right)dy = \int_{AB}\left(x^2 + y^2\right)dy + \int_{BC}\left(x^2 + y^2\right)dy + \int_{CD}\left(x^2 + y^2\right)dy + \int_{DA}\left(x^2 + y^2\right)dy$

$= 0 + \int_2^4\left(10x^2 - 32x + 32\right)dx + 0 + \int_4^0 y^2dy = \left(\dfrac{10}{3}x^3 - 16x^2 + 32x\right)\Bigg|_2^4 + \dfrac{y^3}{3}\Bigg|_4^0 = \dfrac{112}{3}$

19. Evaluate $\int_C xy^2dy - x^2ydx$ taken in the counter clockwise sense along the cardioid $r = a(1 + \cos\theta)$.

Solution.



The curve C as shown in figure 4.20 cardiod whose equation is

$$r = a(1+\cos\theta),\ x = r\cos\theta = a(1+\cos\theta)\cos\theta = a\left(\cos\theta+\cos^2\theta\right)$$

$$dx = a(-\sin\theta - 2\cos\theta\sin\theta)d\theta$$

$$y = r\sin\theta = a(1+\cos\theta)\sin\theta = a(\sin\theta+\sin\theta\cos\theta)$$

$$dy = a\left(\cos\theta+\cos^2\theta-\sin^2\theta\right)d\theta$$

The integrand $\left(xy^2dy - x^2ydx\right)$

$$= r^3\cos\theta\sin^2\theta a\left(\cos\theta+\cos^2\theta-\sin^2\theta\right)d\theta - r^3\cos^2\theta\sin\theta a(\sin\theta-2\cos\theta\sin\theta)d\theta$$

$$= ar^2\cos\theta\sin^2\theta\left(\cos\theta+\cos^2\theta-\sin^2\theta+\cos\theta+2\cos^2\theta\right)d\theta$$

$$= a^1(1+\cos\theta)^3\cos\theta\sin^2\theta\left(4\cos^2\theta+2\cos\theta-1\right)d\theta$$

$$= a^4\left[\cos^6\theta\sin^2\theta+14\cos^5\theta\sin^2\theta+17\cos^4\theta\sin^2\theta+7\cos^3\theta\sin^2\theta-\cos^2\theta\sin^2\theta-\cos\theta\sin^2\theta\right]$$

The line integral

$$\oint\left(xy^2dy - x^2ydx\right)$$

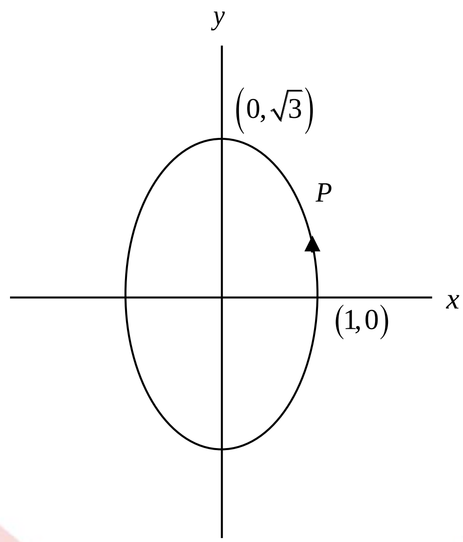
$$= a^4\int_0^{2\pi}\left(\cos^6\theta\sin^2\theta+14\cos^5\theta\sin^2\theta+17\cos^4\theta\sin^2\theta+7\cos^3\theta\sin^2\theta-\cos^2\theta\sin^2-\cos\theta\sin^2\theta\right)d\theta$$

$$= \frac{35}{16}a^4\pi$$

20. A particle moves counterclockwise along the curve $3x^2+y^2=3$ from $(1,0)$ to a point P, under the action of the force

$$\vec{F}(x,y) = \frac{x}{y}\hat{i}+\frac{y}{x}\hat{j}.$$

Prove that there are two possible locations of P such that work done by $\vec{F}$ is 1.
Solution.



$\frac{x^2}{1}+\frac{y^2}{3}=1$, Point on ellipse is represented as $\left(\cos\theta,\sqrt{3}\sin\theta\right)$

$\int \vec{F}\cdot d\vec{r}=\int\left(\frac{x}{y}\hat{i}+\frac{y}{x}\hat{j}\right)\cdot\left(dx\hat{i}+dy\hat{j}\right)=\int\frac{x}{y}dx+\frac{y}{x}dy$

$=\int_0^{\theta}\frac{\cos\theta}{\sqrt{3}\sin\theta}(-\sin\theta)d\theta+\frac{\sqrt{3}\sin\theta}{\cos\theta}\cdot\sqrt{3}\cos\theta d\theta=\int_0^{\theta}\left(-\frac{1}{\sqrt{3}}\cos\theta+3\sin\theta\right)d\theta$

$=-\frac{1}{\sqrt{3}}\sin\theta-3\cos\theta\Big|_0^{\theta}=-\frac{1}{\sqrt{3}}\sin\theta-3\cos\theta+3$

Work done in equal to 1. So, $-\frac{1}{\sqrt{3}}\sin\theta-3\cos\theta+3=1\Rightarrow\frac{1}{\sqrt{3}}\sin\theta+3\cos\theta=2$

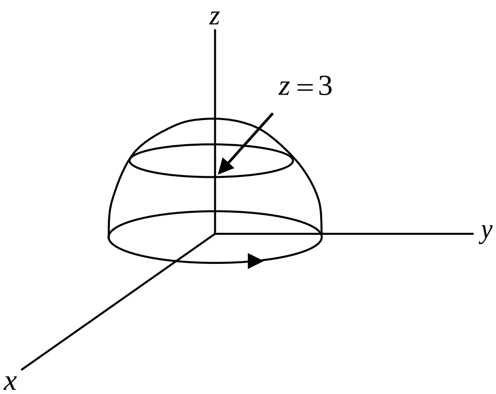
$\Rightarrow\left(\frac{1}{\sqrt{3}}\sin\theta\right)^2=\left(2-3\cos\theta\right)^2\Rightarrow\frac{1}{3}\sin^2\theta=4+9\cos^2\theta-12\cos\theta$

$\Rightarrow 28\cos^2\theta-36\cos\theta+11=0\Rightarrow\left(2\cos\theta-1\right)\left(14\cos\theta-11\right)=0$. $\cos\theta=\frac{1}{2},\frac{11}{14}$

So, there are two value of θ i.e., two possible location of P such that the work done by $\vec{F}$ is 1.

21. Find the circulation of the field $\vec{F}=-x^2y\hat{i}+xy^2\hat{j}+\left(y^3-x^3\right)\hat{k}$ around the curve C, where C is the intersection of the sphere $x^2+y^2+z^2=25$ and the plane $z=3$. The orientation of the curve C is counterclockwise when viewed from above.

Solution.



$\vec{F} = -x^2 y\hat{i} + xy^2\hat{j} + \left(y^3 - x^3\right)\hat{k}$ .C is the curve of intersection of surfaces

$x^2 + y^2 + z^2 = 25, z = 3$ .So, $x^2 + y^2 = 16$

$\vec{F}\cdot d\vec{r} = x^2 y dx + xy^2 dy + \left(y^3 - x^3\right)dz$ .For curve C, $z = 3, dz = 0$, $\oint \vec{F}\cdot d\vec{r} = \int -x^2 y dx + xy^2 dy$

Let $x = 4\cos\theta, y = 4\sin\theta$

$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(256\cos^2\theta\sin^2\theta d\theta + 256\cos^2\theta\sin^2\theta\right)d\theta = 512\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta$$

$$= 512\times 4\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = 2048\frac{\Gamma\frac{3}{2}\Gamma\frac{3}{2}}{2\Gamma 3} = 128\pi$$

22. If $\phi = 2x^2yz, \vec{F} = xy\hat{i} - z^2y\hat{j} + x^2\hat{k}$ and C is the curve $x = 2t, y = t^2, z = t^3$ from $t = 0$ and $t = 1$. Evaluate the line integrals (a) $\int_C \phi d\vec{r}$ (b) $\int_C \vec{F}\times d\vec{r}$.

Solution.

(a) Along $C, \phi = 2x^2yz = 2\left(2t\right)^2\cdot t^2\cdot t^3 = 8t^7$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = 2t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $d\vec{r} = \left(2\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right)dt$

$$\int_C \phi d\vec{r} = \int_0^1 8t^7\left(2\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right)dt = \hat{i}\int_0^1 16t^7 dt + \hat{j}\int_0^1 16t^8 dt + \hat{k}\int_0^1 24t^9 dt = 2\hat{i} + \frac{16}{9}\hat{j} + \frac{12}{5}\hat{k}$$

(b) Along C, $\vec{F} = xy\hat{i} - z^2y\hat{j} + x^2\hat{k} = 2t^3\hat{i} - t^8\hat{j} + 4t^2\hat{k}$

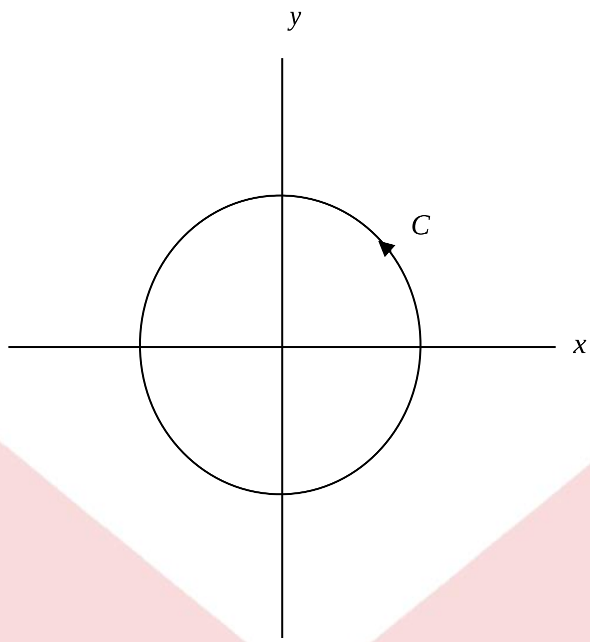
$$\vec{F}\times d\vec{r} = \left(2t^3\hat{i} - t^8\hat{j} + 4t^2\hat{k}\right)\times\left(2\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2t^3 & -t^8 & 4t^2\\ 2 & 2t & 3t^2\end{vmatrix}$$

$$= \left(-3t^{10} - 8t^3\right)\hat{i} + \left(8t^2 - 6t^3\right)\hat{j} + \left(4t^4 + 2t^8\right)\hat{k}$$

$$\int_C \vec{F}\times d\vec{r} = \hat{i}\int_0^1\left(-3t^{10} - 8t^3\right)dt + \hat{j}\int_0^1\left(8t^2 - 6t^3\right)dt + \hat{k}\int_0^1\left(4t^4 + 2t^8\right)dt = -\frac{47}{11}\hat{i} + \frac{5}{3}\hat{j} + \frac{46}{45}\hat{k}$$

23. Find the work done in moving the particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1, z = 0$ under the field of force given by $\vec{F} = \left(2x + y + z\right)\hat{i} + \left(x + y - z^2\right)\hat{j} + \left(3x - 2y + 4z\right)\hat{k}$.

Solution.



Work done moving the particle by distance dr

$\vec{F}\cdot d\vec{r}=\left(2x+y+z\right)dx+\left(x+y-z^2\right)dy+\left(3x-2y+4z\right)dz$

The curve C is ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.The equation of ellipse is given by $x=5\cos\theta, y=4\sin\theta, z=0$

$dx=-5\sin\theta d\theta, dy=4\cos\theta d\theta, dz=0$

$\vec{F}\cdot d\vec{r}=\left(10\cos\theta+4\sin\theta\right)\left(-5\sin\theta\right)d\theta+\left(5\cos\theta+4\sin\theta\right)4\cos\theta d\theta$

$=\left(-34\sin\theta\cos\theta+20\cos^2\theta-20\sin^2\theta\right)d\theta$

On C,θ varies from 0 to 2π

So, work done in moving a particle around the ellipse.So, $W=\oint\vec{F}\cdot d\vec{r}$

$=\int_0^{2\pi}\left(-34\sin\theta\cos\theta+20\cos^2\theta-20\sin^2\theta\right)d\theta$

$=-34\int_0^{2\pi}\sin\theta\cos\theta d\theta+20\int_0^{2\pi}\cos^2\theta d\theta-20\int_0^{2\pi}\sin^2\theta d\theta=0$

24. Evaluate $\int_C\vec{F}\cdot d\vec{r}$ **where** $\vec{F}=c\left[-3a\sin^2\theta\cos\theta\hat{i}+a\left(2\sin\theta-3\sin^2\theta\right)\hat{j}+b\sin2\theta\hat{k}\right]$ **and the curve C is given by** $\vec{r}=a\cos\theta\hat{i}+a\sin\theta\hat{j}+b\theta\hat{k},\ \theta$ **varying from** $\frac{\pi}{4}$ **to** $\frac{\pi}{2}$**.**

Solution.

$\vec{r}=a\cos\theta\hat{i}+a\sin\theta\hat{j}+b\theta\hat{k}$, $d\vec{r}=\left(-a\sin\theta\hat{i}+a\cos\theta\hat{j}+b\hat{k}\right)d\theta$

$\vec{F}\cdot d\vec{r}=c\left[3a^2\sin^3\theta\cos\theta+a^2\left(2\sin\theta-3\sin^2\theta\right)\cos\theta+b^2\sin2\theta\right]d\theta$

The line integral

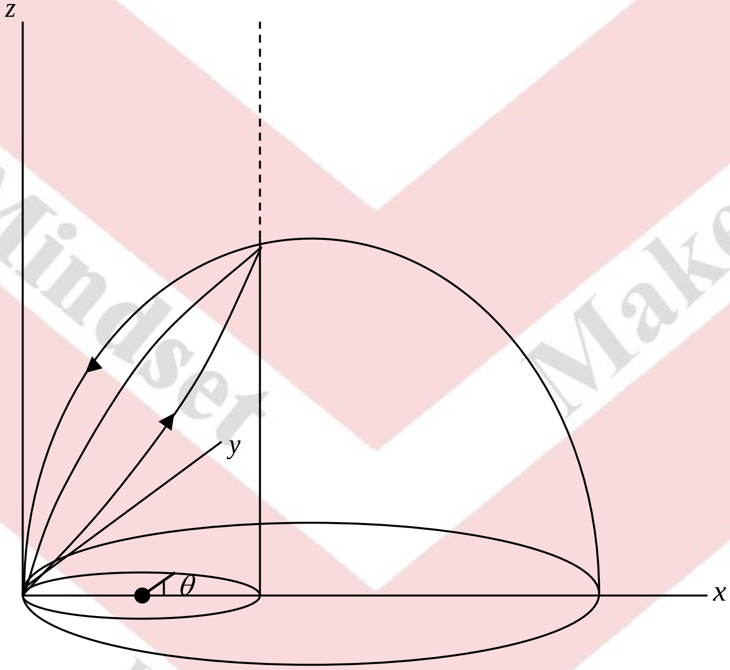
$$\int_C \vec{F}\cdot d\vec{r} = 3a^2c\int_{\pi/4}^{\pi/2}\sin^3\theta\cos\theta d\theta + a^2c\int_{\pi/4}^{\pi/2}\left(2\sin\theta - 3\sin^2\theta\right)\cos\theta d\theta + b^2c\int_{\pi/4}^{\pi/2}\sin 2\theta d\theta$$

$$= 3a^2c\left[\frac{\sin^4\theta}{4}\right]_{\pi/4}^{\pi/2} + a^2c\left[\sin^2\theta - \sin^3\theta\right]_{\pi/4}^{\pi/2} - \frac{b^2c}{2}\left[\cos 2\theta\right]_{\pi/4}^{\pi/2} = \frac{9}{16}a^2c + a^2c\left[\frac{1}{2} - \frac{1}{2\sqrt{2}}\right] + \frac{b^2c}{2}$$

$$= \left(\frac{17}{16} - \frac{1}{2\sqrt{2}}\right)a^2c + \frac{b^2c}{2}$$

25. Evaluate $\int_C \left(y^2+z^2\right)dx + \left(z^2+x^2\right)dy + \left(x^2+y^2\right)dz$ where C is the part for which $z \geq 0$ of the intersection of the surfaces $x^2+y^2+z^2 = 4x, x^2+y^2 = 2x$ and curve beings at the origin and runs at first in the positive octant.

Solution.



The C is the intersection of the two surfaces

$\left(x-1\right)^2 + y^2 = 1$ (Cylinder), $z^2 = 2x$ (Parabolic cylinder)

The parametric equation of C is given as

$x = 1+\cos\theta = 2\cos^2\theta/2$, $dx = -2\sin{\theta/2}\cos{\theta/2}\,d\theta$, $y = \sin\theta = 2\sin{\theta/2}\cos{\theta/2}$

$dy = \cos\theta d\theta$, $z = \sqrt{2\left(1+\cos\theta\right)} = 2\cos{\theta/2}$, $dz = -\sin{\theta/2}\,d\theta$

$$\left(y^2+z^2\right)dx + \left(z^2+x^2\right)dy + \left(x^2+y^2\right)dz$$

$$= \left(4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)\left(-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta\right) + \left(4\cos^2\frac{\theta}{2} + 4\cos^4\frac{\theta}{2}\right)\cos\theta d\theta$$

$$+\left(4\cos^4\frac{\theta}{2} + 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)d\theta$$

So, the line integral becomes

$$\int_{-\pi}^{\pi}\left(y^2+z^2\right)dx+\left(z^2+x^2\right)dy+\left(x^2+y^2\right)dz$$

$$=-\int_{-\pi}^{\pi}4\left(\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)\sin\theta d\theta+4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\cos\theta d\theta-4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

The first and third integral vanishes since, the integrand is an odd function
So, integral reduces to

$$I=4\int_{-\pi}^{\pi}\cos^2\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\cos\theta d\theta=\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}\right)\left(2+2\cos^2\frac{\theta}{2}\right)\cos\theta d\theta$$

$$=\int_{-\pi}^{\pi}(1+\cos\theta)\cdot(3+\cos\theta)\cos\theta d\theta=\int_{-\pi}^{\pi}\cos^3\theta+4\cos^2\theta+3\cos\theta d\theta$$

$$=\int_{-\pi}^{\pi}\cos^3\theta d\theta+4\int_{-\pi}^{\pi}\cos^2\theta d\theta+3\int_{-\pi}^{\pi}\cos\theta d\theta=2\int_{0}^{}\cos^3\theta d\theta+16\int_{0}^{\pi/2}\cos^2\theta d\theta+6\int_{0}^{\pi/2}\cos\theta d\theta$$

$$=0+16\cdot\frac{\pi}{4}+0=4\pi$$

26. Evaluate the following integrals along segment of straight line joining the given points

(i) $\int xdx+ydy+(x+y-1)dz$ **from** $(1,1,1)$ **to** $(2,3,4)$

(ii) $\int\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2-x-y+2z}}$ **from** $(1,1,1)$ **to** $(4,4,4)$

Solution.

(i) The curve C is a line joining $(1,1,1)$ to $(2,3,4)$

$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}=t$ (parameter)

The parametric form of line is given as

$x=1+t, y=2t+1, z=3t+1, dx=dt, dy=2dt, dz=3dt$

t varies from 0 to 1]

The line integral

$$I=\int xdx+ydy+(x+y-1)dz=\int d\left(\frac{x^2+y^2}{2}\right)+\int(x+y-1)dz=\left.\frac{x^2+y^2}{2}\right|_{(1,1,1)}^{(2,3,4)}+\int_0^1(3t+1)3dt$$

$$=\frac{1}{2}+3\left[\frac{3}{2}t^2+t\right]_0^1=\frac{1}{2}+\frac{15}{2}=8$$

(ii) The curve is straight line from $(1,1,1)$ to $(4,4,4)$ given by

$x=t+1, dx=dt$, $y=t+1, dy=dt$, $z=t+1, dz=dt$

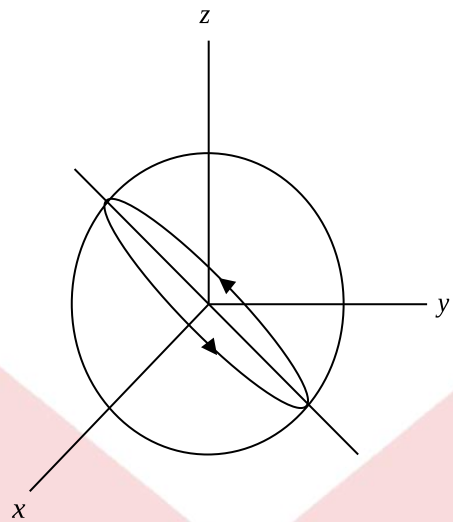
t varies from 0 to 3

The integral reduces to

$$\int_C\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2-x-y+2z}}=\int_0^3\frac{(t+1)dt+(t+1)dt+(t+1)dt}{\sqrt{3(t+1)^2}}=\sqrt{3}\int_0^3 dt=3\sqrt{3}$$

27. Find the integral $\int_C (y+z)\,dx+(z+x)\,dy+(x+y)\,dz$ where C is the circle $x^2+y^2+z^2=a^2, x+y+z=0$.

Solution.



$$\oint (y+z)\,dx+(z+x)\,dy+(x+y)\,dz$$
$$=\oint ydx+zdx+zdy+xdy+xdz+ydz$$
$$=\oint d\left(xy+yz+zx\right)=0$$

The integral is an exact differential

So, $\oint \vec{F}\cdot d\vec{r}=0$.

28. Evaluate $\int_C x^2y^3dx+dy+zdz$ where C is the circle $x^2+y^2=R^2, z=0$.

Solution.

The curve C is the circle $x^2+y^2=R^2, z=0$

$x=R\cos\theta, dx=R\sin\theta d\theta$

$y=R\sin\theta, dy=R\cos\theta d\theta$

$$I=\oint\left(x^2y^3dx+dy+zdz\right)$$
$$=\int_0^{2\pi} R^2\cos^2\theta\cdot R^3\sin^3\theta\left(-R\sin\theta\right)d\theta+\oint d\left(y+\frac{z^2}{2}\right)$$
$$=-R^6\int_0^{2\pi}\cos^2\theta\sin^4\theta d\theta+0$$
$$=-4R^6\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta$$
$$=-4R^6\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}=-\frac{\pi R^6}{8}$$

29. Evaluate $\int_C \vec{A}\cdot d\vec{r}$ along the curve $x^2+y^2=1, z=1$ from $(0,1,1)$ to $(1,0,1)$ if $\vec{A}=(yz+2x)\hat{i}+xz\hat{j}+(xy+2z)\hat{k}$.

Solution.

The curve C is the circle of radius 1 with the centre at $(0,0,1)$ lying in a plane parallel to xy plane.

$\vec{F}\cdot d\vec{r}=(yz+2x)dx+xzdy+(xy+2z)dz$

$=d\left(xyz+x^2+z^2\right)$

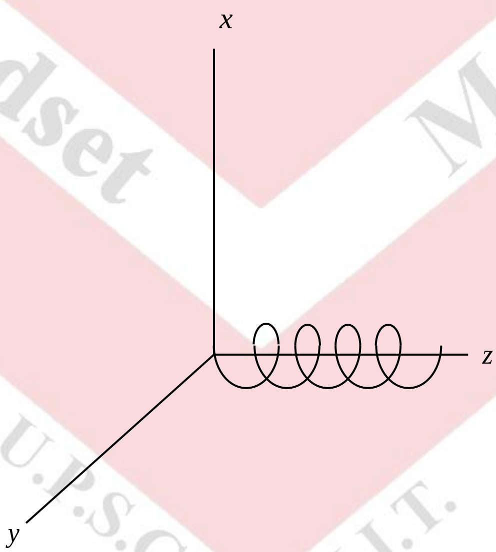
$\vec{F}\cdot d\vec{r}$ is an exact differential. So, line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of curve joining initial and final points

$\int_C \vec{F}\cdot d\vec{r}=\int d\left(xyz+x^2+z^2\right)$

$=\left[xyz+x^2+z^2\right]_{(0,1,1)}^{(1,0,1)}=1$

30. Evaluate $\int_C yzdx+zxdy+xydz$ where C is the arc of curve $x=b\cos t, y=b\sin t, z=\frac{at}{2\pi}$ from the point it intersects $z=0$ to the point it intersects $z=a$.

Solution.



The curve C is a spiral given by

$x=b\cos t, y=b\sin t, z=\frac{at}{2\pi}$

Since, z varies from $z=0$ to $z=a$, hence, t varies from 0 to 2π

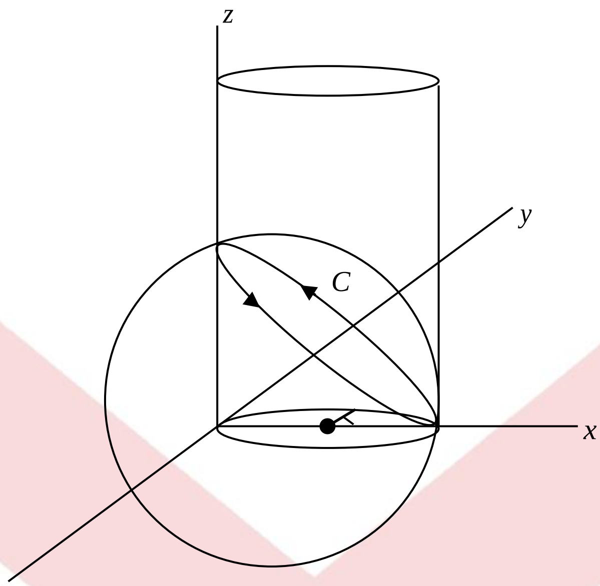
The line integral

$\int_C (yzdx+zxdy+xydz)$

$=\int_C d(xyz)=[xyz]$

$=\left[\frac{ab^2}{2\pi}t\sin t\cos t\right]_0^{2\pi}=0$

31. Evaluate $\int_C y^2dx + z^2dy + x^2dz$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax\left(a > 0, z \geq 0\right)$ integrated anticlockwise when viewed from the origin.
Solution.



The curve C is the curve of intersection of

$$x^2 + y^2 = ax \Rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}, x^2 + y^2 + z^2 = a^2$$

$$x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow z^2 + ax = a^2$$

$$\Rightarrow z^2 = -a\left(x - a\right)$$

Let $x = \frac{a}{2} + \frac{a}{2}\cos\theta \quad \Rightarrow dx = -\frac{a}{2}\sin\theta d\theta$

$$y = \frac{a}{2}\sin\theta \quad \Rightarrow dy = \frac{a}{2}\cos\theta d\theta$$

$$z^2 = a\left(a - x\right)$$

$$= a\left(\frac{a}{2} - \frac{a}{2}\cos\theta\right)$$

$$= a^2 \sin^2\frac{\theta}{2}$$

So, $z = a\sin\frac{\theta}{2} \Rightarrow dz = \frac{a}{2}\cos\frac{\theta}{2}d\theta$

32. θ varies from 0 to 2π.
The line integral

$$I = \int y^2dx + z^2dy + x^2dz$$

$$= \int_0^{2\pi} \frac{a^2}{4}\sin^2\theta\left(-\frac{a}{2}\sin\theta d\theta\right)d\theta + \int_0^{2\pi} \frac{a^2}{2}\left(1 - \cos\theta\right)\frac{a}{2}\cos\theta d\theta + \int_0^{2\pi} \frac{a^2}{4}\left(1 + \cos\theta\right)^2 \frac{a}{2}\cos\theta/_2\, d\theta$$

$$= 0 + \frac{a^3}{4}\int_0^{2\pi}\left(\cos\theta - \cos^2\theta\right)d\theta + \frac{a^3}{2}\int_0^{2\pi}\cos^5 \theta/2 \, d\theta$$

$$= \frac{a^3}{2}\int_0^{\pi}\cos\theta d\theta - \frac{a^3}{2}\int_0^{\pi}\cos^2\theta d\theta + \frac{a^3}{2}\int_0^{2\pi}\cos^5\frac{\theta}{2}d\theta$$

$$= -a^3\int_0^{\pi/2}\cos^2\theta d\theta + a^3\int_0^{\pi}\cos^5\phi d\phi \quad \left(\phi = 0/2\right)$$

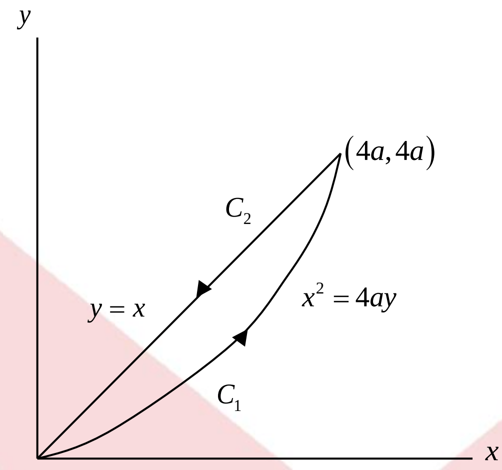
$$= -\frac{a^3\pi}{4} + 0$$

$$= -\frac{a^3\pi}{4}$$

5. Green's Theorem

1. Verify Green's theorem in the plane for $\oint_C \left(xy+x^2\right)dx+x^2dy$ where C is the closed curve of the region bounded by $y=x$ and $x^2=4ay$.

Solution.



Here $Mdx+Ndy$

$=\left(xy+x^2\right)dx+x^2dy$

$M=xy+x^2 \Rightarrow \frac{\partial M}{\partial y}=x$, $N=x^2 \Rightarrow \frac{\partial N}{\partial x}=2x$

Let us first evaluate the double integral over Region R bounded by $x^2=4ay$ (curve C_1) & $y=x$ (curve C_2) as

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\int_0^{4a}\int_{x^2/4a}^{x} xdydx=\int_0^{4a} x\left(x-\frac{x^2}{4a}\right)dx=\frac{x^3}{3}-\frac{x^4}{16a}\Big|_0^{4a}=\frac{16a^3}{3}$$

Now let us evaluate the line integral $\oint Mdx+Ndy$ on closed curve C. The curve C is a piecewise smooth curve consisting of C_1 and C_2. On $C_1, y=\frac{x^2}{4a}$, $dy=\frac{x}{2a}dx$

$$Mdx+Ndy=\left(xy+x^2\right)dx+x^2dy=\left(\frac{x^3}{4a}+x^2\right)dx+x^2\frac{x}{2a}dx=\left(\frac{3}{4}\cdot\frac{x^3}{a}+x^2\right)dx$$

x varies from 0 to $4a$ on C_1. So, $\int_{C_1} Mdx+Ndy=\int_0^{4a}\left(\frac{3x^3}{4a}+x^2\right)dx=\frac{3}{16a}x^4+\frac{x^3}{3}\Big|_0^{4a}$

$$=8a^3+\frac{64a^3}{3}=\frac{208a^3}{3}$$

On C_2, $y = x, dy = dx$

$Mdx + Ndy = \left(xy + x^2\right)dx + x^2 dy\ = 3x^2 dx$

x varies from $4a$ to 0.

So, $\int\limits_{C_2} Mdx + Ndy = \int\limits_{4a}^{0} 3x^2 dx = x^3\Big|_{4a}^{0} = -64a^3$. So, $\int\limits_{C} Mdx + Ndy = \int\limits_{C_1} Mdx + Ndy + \int\limits_{C_2} Mdx + Ndy$

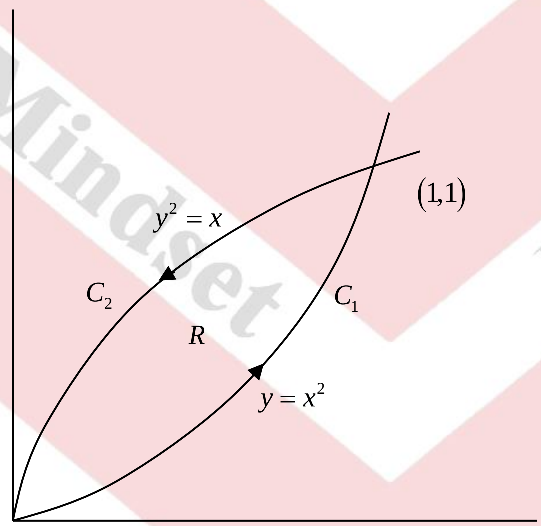
$= \frac{208}{3}a^3 - 64a^3 = \frac{16}{3}a^3$

Since, $\iint\limits_{R}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \oint\limits_{C} Mdx + Ndy$

So, Green's theorem is verified.

2. Verify Green's theorem in the plane for $\oint\limits_{C}\left(2xy - x^2\right)dx + \left(x^2 + y^2\right)dy$ where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in positive sense.

Solution.



Here, $Mdx + Ndy = \left(2xy - x^2\right)dx + \left(x^2 + y^2\right)dy$

$M = 2xy - x^2 \Rightarrow \frac{\partial M}{\partial y} = 2x$, $N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

So, the double integral $\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$ over region R bounded by $y = x^2$ (curve C_1) and $y^2 = x$ (curve C_2) is zero as integer and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$.

Now, let us evaluate the line integral over a closed curve C. The curve C is a piece wise smooth curve consisting of C_1 & C_2.

On C_1, $y = x^2, dy = 2xdx$ (Taking x as independent variable).

$$Mdx + Ndy = \left(2xy - x^2\right)dx + \left(x^2 + y^2\right)dy = \left(2x^3 - x^2\right)dx + \left(x^2 + x^4\right)\cdot 2xdx = \left(2x^5 + 4x^3 - x^2\right)dx$$

x varies from 0 to 1 on C_1

So, $\int_{C_1} Mdx + Ndy = \int_0^1 \left(2x^5 + 4x^3 - x^2\right)dx = \left.\frac{x^6}{3} + x^4 - \frac{x^3}{3}\right|_0^1 = 1$

On C, $x = y, dx = 2ydy$

Taking y as independent variable

$$Mdx + Ndy = \left(2xy - x^2\right)dx + \left(x^2 + y^2\right)dy = \left(2y^2y - y^4\right)\cdot 2ydy + \left(y^4 + y^2\right)dy$$

$$= \left(-2y^5 + 5y^4 + y^2\right)dy$$

y varies from 1 to 0 on C_2

$$\int_{C_2} Mdx + Ndy = \int_1^0 \left(-2y^5 + 5y^4 + y^2\right)dy = \left.-\frac{y^6}{3} + y^5 - \frac{y^3}{3}\right|_1^0 = -1$$

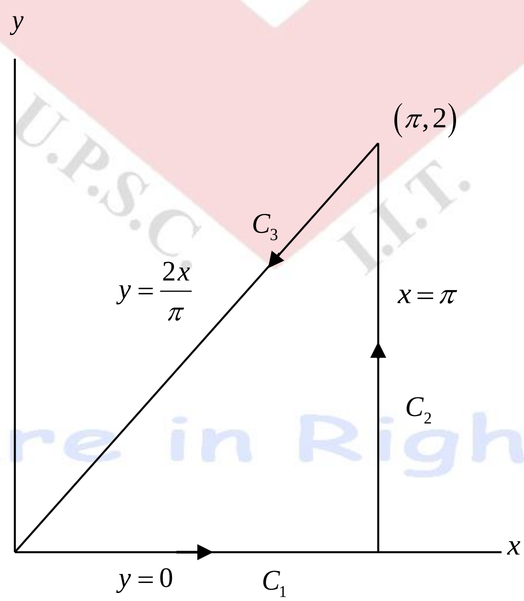
So, $\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy = 0$

Since, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \oint_C Mdx + Ndy$

So, Green's Theorem is verified.

3. Apply Green's theorem in the plane to evaluate $\oint_C \left\{\left(y - \sin x\right)dx + \cos x dy\right\}$ **where C is the triangle enclosed by the lines** $y = 0, x = \pi, \pi y = 2x$.

Solution.



Here, $Mdx + Ndy = (y - \sin x)dx + \cos x dy$

So, $M = y - \sin x,\ \dfrac{\partial M}{\partial y} = 1,\ N = \cos x,\ \dfrac{\partial N}{\partial x} = -\sin x$

According to Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

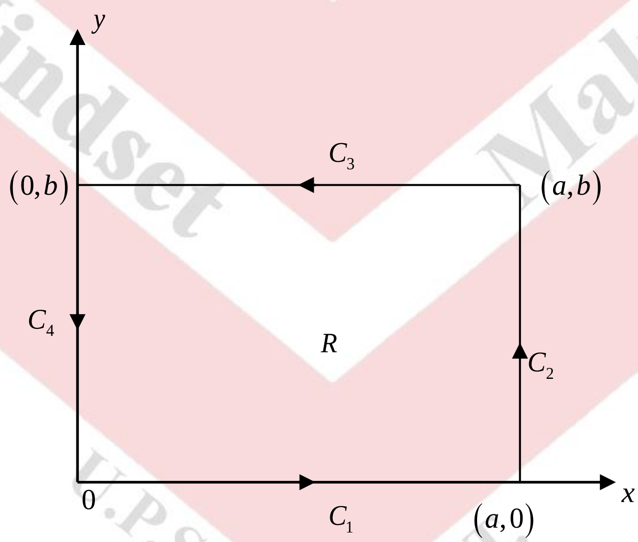
where R is the region enclosed by the piece wise smooth curve C consisting of curve $C_1(y=0)$, curve $C_2(x=\pi)$ curve $C_3(\pi y = 2x)$ as shown in Figure 5.4.

So, $\iint \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right) dxdy = \int_0^2 \int_{\pi y/2}^{\pi} (-\sin x - 1) dxdy = \int_0^2 [\cos x - x]_{\pi y/2}^{\pi} dy$

$$= \int_0^2 \left(-1 - \pi - \cos\frac{\pi y}{2} + \frac{\pi y}{2} \right) dy = -(1+\pi)y - \frac{2}{\pi}\sin\frac{\pi y}{2} + \frac{\pi y^2}{4} \Bigg|_0^4 = -2 - \pi$$

4. If $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j}$, find the value of $\oint (x^2 - y^2)dx + 2xydy$ around the rectangular boundary $x = 0, x = a, y = 0$ and $y = b$.

Solution.



Here the curve C is a piecewise smooth curve consisting of $C_1(y=0), C_2(x=a), C_3(y=b)$ & $C_4(x=0)$.

The region bounded by C is shown in figure 5.5.

$$\oint \vec{F} \cdot d\vec{r} = \oint (x^2 - y^2)dx + 2xydy = \oint Mdx + Ndy$$

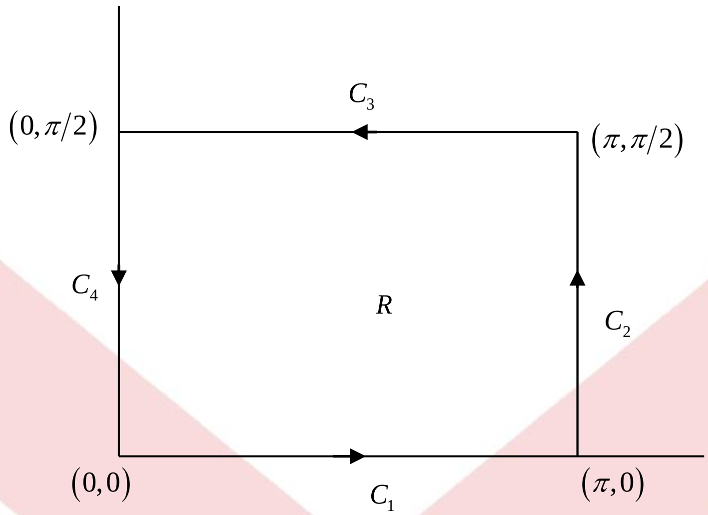
Here, $M = x^2 - y^2,\ \dfrac{\partial M}{\partial y} = -2y,\ N = 2xy,\ \dfrac{\partial N}{\partial x} = 2y$

Applying Green's theorem

$$\oint Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = 4\int_0^b \int_0^a ydxdy = 4a\int_0^b ydy = 2ab^2$$

5. Evaluate $\oint_C e^{-x}\sin y dx + e^{-x}\cos y dy$ by Green's theorem in plane where C is the rectangle with vertices $(0,0),(\pi,0),(\pi,\pi/2),(0,\pi/2)$.

Solution.



The curve C is a piecewise smooth curve consisting of $C_1(y=0), C_2(x=\pi), C_3(y=\pi/2)$ & $C_4(x=0)$.

The region R bounded by C is as shown in figure.

$$\oint_C Mdx + Ndy = \oint_C e^{-x}\sin y dx + e^{-x}\cos y dy$$

Here $M = e^{-x}\sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

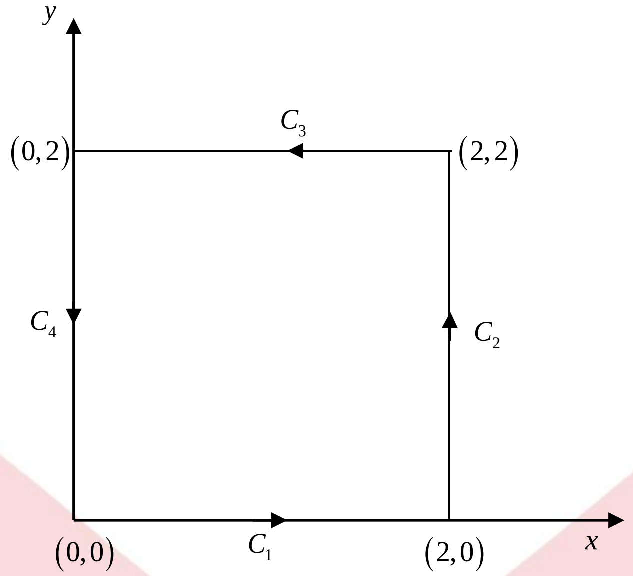
Applying Green's theorem

$$\oint Mdx + Ndy = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dydx = -2\int_0^{\pi}\int_0^{\pi/2} e^{-x}\cos y dydx = -2\int_0^{\pi} e^{-x}\left[\sin y\right]_0^{\pi/2} dx$$

$$= -2\int_0^{\pi} e^{-x}dx = 2e^{-x}\Big|_0^{\pi} = 2\left(e^{-\pi}-1\right)$$

6. Verify Green's theorem in the plane for $\oint\left(x^2 - x^3\right)dx + \left(y^2 - 2xy\right)dy$ where C is the square with vertices $(0,0),(2,0),(2,2),(0,2)$.

Solution.



$M = x^2 - xy^3 \Rightarrow \dfrac{\partial M}{\partial y} = -3xy^2$, $N = y^2 - 2xy \Rightarrow \dfrac{\partial N}{\partial x} = -2y$

Let us first evaluate the double integral over region R bounded by curves $C_1(y=0)$, $C_2(x=2)$, $C_3(y=2)$, $C_4(x=0)$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dydx = \int_0^2\int_0^2 \left(3xy^2 - 2y\right) dydx = \int_0^2 \left[xy^3 - y^2\right]_0^2 dx = 4\int_0^2 (2x-1)\,dx = 4\left[x^2 - x\right]_0^2 = 8$$

Now, let us evaluate the line integral as closed curve C. The curve C is a piecewise smooth curve consisting of C_1, C_2, C_3 & C_4.On $C_1, y=0, dy=0$

$Mdx + Ndy = x^2dx$

x varies from 0 to 2 on C_1, $\displaystyle\int_{C_1} Mdx + Ndy = \int_0^2 x^2dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$

On $C_2, x=2, dx=0$, $Mdx + Ndy = \left(y^2 - 4y\right)dy$

y varies from 0 to 2 on C_2, $\displaystyle\int_{C_2} Mdx + Ndy = \int_0^2 \left(y^2 - 4y\right)dy = \left.\frac{y^3}{3} - 2y^2\right|_0^2 = -\frac{16}{3}$

On C_3, $y=2, dy=0$, $Mdx + Ndy = \left(x^2 - 8x\right)dx$

x varies from 2 to 0 on C_3, $\displaystyle\int_{C_3} Mdx + Ndy = \int_2^0 \left(x^2 - 8x\right)dx = \left.\frac{x^3}{3} - 4x^2\right|_2^0 = \frac{40}{3}$

On $C_4, x=0, dx=0$, $Mdx + Ndy = y^2dy$

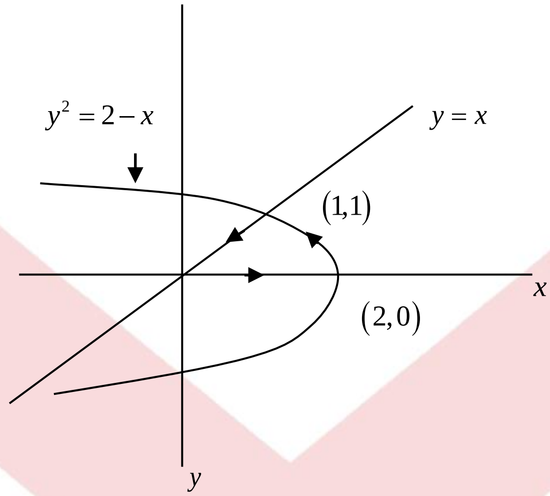
y varies from 2 to 0 on C_4, $\displaystyle\int_{C_4} Mdx + Ndy = \int_2^0 y^2dy = \left.\frac{y^3}{3}\right|_2^0 = -\frac{8}{3}$

$$\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy + \int_{C_3} Mdx + Ndy + \int_{C_4} Mdx + Ndy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = 8$$

Since, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy$. So, Green's theorem is verified.

7. Use Green's theorem to evaluate the integral $\oint_C x^2 dx + \left(x + y^2\right) dy$, where C is the closed curve given by $y = 0, y = x$ and $y^2 = 2 - x$ in the first quadrant, oriented counter clockwise.

Solution.



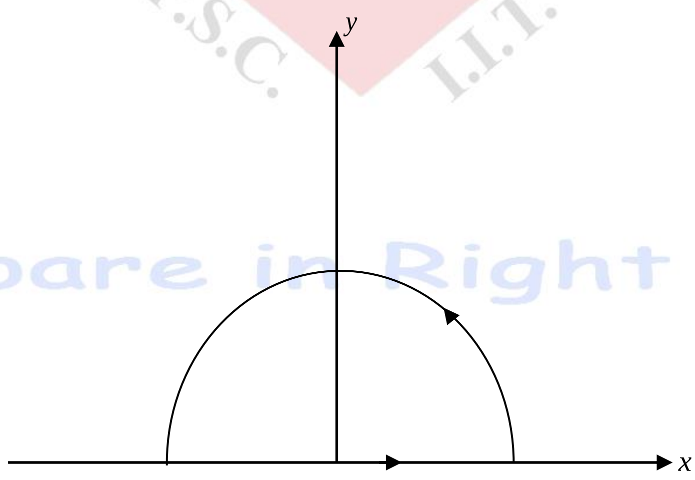
The given integral is
$\oint x^2 dx + \left(x + y^2\right) dy = \oint Mdx + Ndy$. So, $M = x^2;\ N = x + y^2$

According to Green's theorem $\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

So, $\oint x^2 dx + \left(x + y^2\right) dy = \iint dxdy = \int_0^1 \int_y^{2-y^2} dxdy = \int_0^1 \left(2 - y^2 - y\right) dy = \left[2y - \frac{y^3}{3} - \frac{y^2}{2}\right]_0^1 = \frac{7}{6}$

8. Let $\vec{F} = \left(x^2 - xy^2\right)\hat{i} + y^2 \hat{j}$. Using Green's theorem, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the positively oriented closed curve which is the boundary of the region enclosed by the x-axis and the semi-circle $y = \sqrt{1 - x^2}$ in the upper half plane.

Solution.



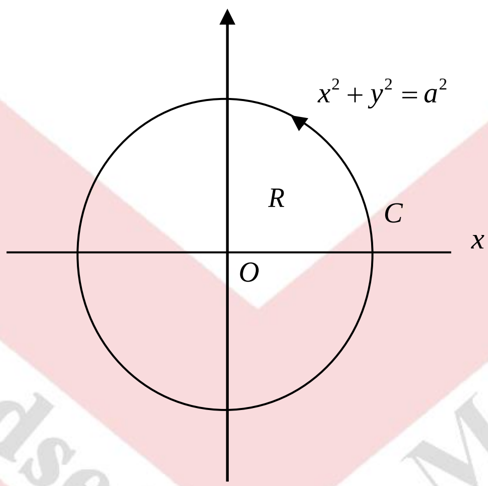
$\vec{F} = \left(x^2 - xy^2\right)\hat{i} + y^2\hat{j}$. So, $\vec{F}\cdot d\vec{r} = \left(x^2 - xy^2\right)\hat{i} + y^2\hat{j}$

According to Green's theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 2xydydx = \int_{-1}^{1} x\left[y^2\right]_0^{\sqrt{1-x^2}} dx$

$= \int_{-1}^{1} x\left(1-x^2\right)dx = 0\left(\int_{-a}^{a} f(x) = 0 \text{ if } f(x) \text{ is odd function}\right)$

9. Evaluate by Green's theorem $\oint (\cos x \sin y - xy)dx + \sin x \cos y dy$ where C is the circle $x^2 + y^2 = a^2$.

Solution.



The given integral is
$\oint_C (\cos x \sin y - xy)dx + \sin x \cos y dy$

Where curve C is a circle of radius a and centered at origin enclosing region R as shown in Figure 5.10.

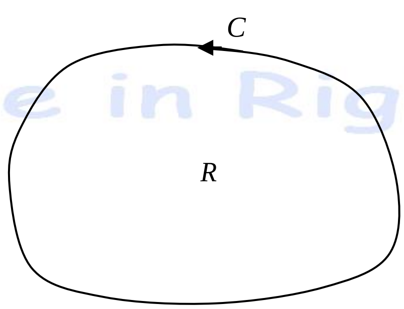
Here $M = \cos x \sin y - xy \Rightarrow \frac{\partial M}{\partial y} = \cos x \cos y - x$, $N = \sin x \cos y \Rightarrow \frac{\partial N}{\partial x} = \cos x \cos y$

Using Green's theorem

$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint xdxdy = \int_0^a\int_0^{2\pi} r\cos\theta rd\theta dr = \int_0^a r^2\left[\sin\theta\right]_0^{2\pi} dr = 0$

10. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$. Hence find the area of the ellipse $x = a\cos\theta, y = b\sin\theta$.

Solution.



According to Green's theorem, if R is a plane region bounded by a simple closed curve C according to the Green's Theorem

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint Mdx + Ndy$$

Let us put $M = -y/2, N = x/2$

So, $\frac{1}{2}\oint xdy - ydx = \iint \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{2}\right) dxdy = \iint dxdy$ = Area of region R bounded by C.

So, area of region bounded by simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$

For as ellipse, $x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta$

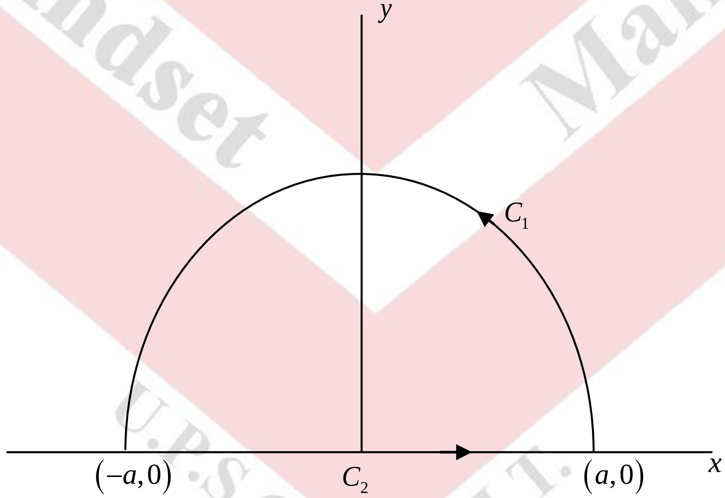
$y = a\sin\theta \Rightarrow dy = a\cos\theta d\theta$

$xdy - ydx = a\cos\theta b\cos\theta - b\sin\theta(-a\sin\theta)d\theta = abd\theta$

So, area of region bounded by ellipse $= \frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi} abd\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab$

11. Apply Green's theorem in the plane to evaluate $\oint_C \left(2x^2 - y^2\right)dx + \left(x^2 + y^2\right)dy$ where C is the boundary of the surface enclosed by the x-axis and the semi circle $y = \sqrt{a^2 - x^2}$.

Solution.



Curve C is a piecewise smooth curve consisting of semi circle $C_1\left(y = \sqrt{a^2 - x^2}\right)$ & part of x axis C_2 $(y = 0)$.

Region R is bounded by curve C as shown in Figure 5.12.

$$\oint \left(2x^2 - y^2\right)dx + \left(x^2 + y^2\right)dy = \oint Mdx + Ndy$$

Here $M = 2x^2 - y^2, \frac{\partial M}{\partial y} = -2y$, $N = x^2 + y^2$, $\frac{\partial N}{\partial x} = 2x$

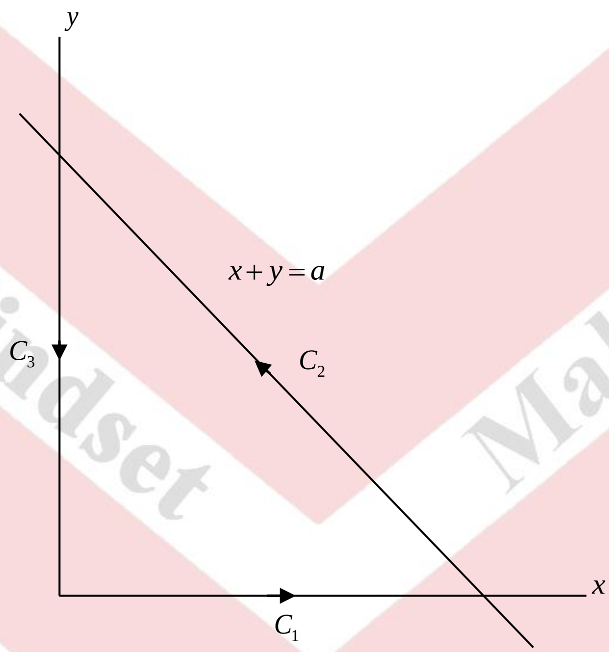
According to Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = 2\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (x+y)\,dxdy$

$= 2\iint ydxdy$

(Since, $\int_{-a}^{a} f(x)dx = 0$ if $f(-x) = f(2y)$ so, $\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdx = 0$) $= 2\int_0^a \int_0^\pi r\sin\theta \cdot rd\theta dr = 4\int_0^a r^2 dr = \frac{4a^3}{3}$

12. Verify Green's theorem in the plane for $\oint (3x^2 - 8y^2)dx + (2y - 3xy)dy$ **where C is the boundary of region bounded by** $x = 0, y = 0, x + y = a$**.**

Solution.



The given integral is

$\oint (3x^2 - 8y^2)dx + (2y - 3xy)dy = \oint Mdx + Ndy$

Here, $M = 3x^2 - 8y^2$, $\frac{\partial M}{\partial y} = -16y$, $N = 2y - 3xy$, $\frac{\partial N}{\partial x} = -3y$

C is a piecewise smooth curve which consists of $C_1(y=0), C_2(x+y=a)$ & $C_3(x=0)$ bounding region R

Let us first evaluate the double integral

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = 13\int_0^a \int_0^{a-x} ydydx = \frac{13}{2}\int_0^a \left[y^2 \right]_0^{a-x} dx = \frac{13}{2}\int_0^a (a-x)^2 dx = -\frac{13}{6}(a-x)^3 \Big|_0^a = \frac{13}{6}a^3$

Now, let us evaluate the line integral

On $C_1, y = 0, dy = 0, Mdx + Ndy = 3x^2 dx$

x varies from 0 to a. $\int_{C_1} Mdx + Ndy = 3\int_0^a x^2 dx = a^3$

On $C_2, y = a - x,\ dy = -dx$

$$Mdx+Ndy=\left(3x^2-8y^2\right)dx+\left(2y-3xy\right)dy$$

$$=\left(3x^2-8\left(a-x\right)^2\right)dx+\left(2\left(a-x\right)-3x\left(a-x\right)\right)dy=\left(-8x^2+19ax+2x-8a^2-2a\right)dx$$

x varies from a to 0. $\int_{C_2} Mdx+Ndy=\int_a^0\left(-8x^2+19ax+2x-8a^2-2a\right)dx$

$$=-\frac{8x^3}{3}+\frac{19a}{2}x^2+x^2-\left(8a^2+2a\right)x\Big|_a^0=\frac{7}{6}a^3+a^2$$

On $C_3, x=0, dx=0, Mdx+Ndy=2ydy$

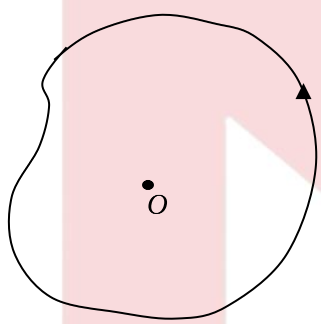
y varies from a to 0. $\int_{C_3} Mdx+Ndy=2\int_a^0 ydy=-a^2$

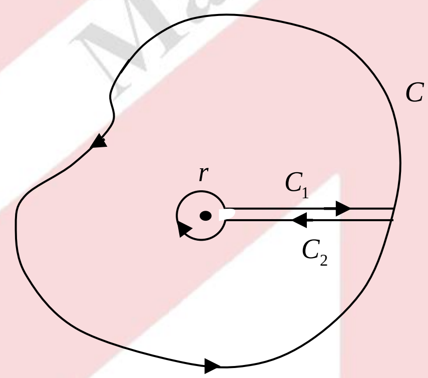
So, $\oint Mdx+Ndy=\int_{C_1} Mdx+Ndy+\int_{C_2} Mdx+Ndy+\int_{C_3} Mdx+Ndy=a^3+\left(\frac{7}{6}a^3+a^2\right)-a^2=\frac{13}{6}a^3$

Hence, $\oint Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$. So, Green' theorem is verified.

13. Evaluate the line integral $\oint_C \frac{xdy-ydx}{x^2+y^2}$ taken in the positive direction over any closed continuous curve C with the origin inside it.

Solution.





The given integral is

$\oint \frac{xdy-ydx}{x^2+y^2}=\oint Mdx+Ndy$. Here, $M=\frac{-y}{x^2+y^2}, N=\frac{x}{x^2+y^2}$

Since, M & N are not continuous at origin O. Hence, Green's theorem will not hold good for the given curve C.

Let us enclose the origin by a circle Γ of radius $\in$

Consider the region R enclosed by curve C' made of C, C_2, Γ, C_1.

M and N are continuous function of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R.

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)-\frac{\left(x^2+y^2\right)-y\cdot 2y}{\left(x^2+y^2\right)^2}=-\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{\left(x^2+y^2\right)-x\cdot 2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

So, line integral $\oint Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

$\Rightarrow \oint_C Mdx+Ndy=\int_C Mdx+Ndy+\int_{C_2} Mdx+Ndy+\int_\Gamma Mdx+Ndy+\int_{C_1} Mdx+Ndy=0$

But $\oint_{C_1} Mdx+Ndy=-\int_{C_2} Mdx+Ndy \Rightarrow \oint_C Mdx+Ndy=\int_C Mdx+Ndy+\int_\Gamma Mdx+Ndy=0$

So, $\int_C Mdx+Ndy=-\int_\Gamma Mdx+Ndy$ ....(1)

In the figure curve Γ is oriented in negative direction.

On the curve Γ, $x=\in\cos\theta\Rightarrow dx=-\in\sin\theta d\theta, y=\in\sin\theta\Rightarrow dy=\in\cos\theta d\theta$

θ varies from 2π to 0.

$$\int_\Gamma\frac{xdy-ydx}{x^2+y^2}=\int_{2\pi}^{0}\frac{\in\cos\theta\cdot\in\cos\theta d\theta-\in\sin\theta\left(-\in\sin\theta\right)d\theta}{\in^2}=\int_{2\pi}^{0}d\theta=-2\pi$$

So, from (1) $\int_C Mdx+Ndy=-\int_\Gamma Mdx+Ndy=2\pi$

14. Using the line integral, compute the area of the loop of Descarte's folium $x^3+y^3=3xy$.

Solution.

Putting $y=tx$ in the equation of folium $x^3+y^3=3xy$

$$x=\frac{3t}{1+t^3};y=\frac{3t^2}{1+t^3}$$

Let $t=\frac{y}{x}=\tan\theta$ where θ varies from 0 to $\pi/2$.

So, t varies from 0 to ∞.

$$dx=\frac{3\left(1-2t^3\right)}{\left(1+t^3\right)^2}dt$$

$$dy=\frac{3\left(2t-t^4\right)}{\left(1+t^3\right)^2}dt$$

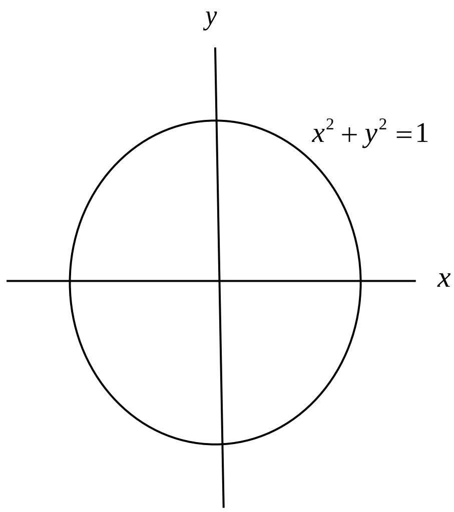
Area of loop $A=\frac{1}{2}\int_C xdy-ydx$

$$=\frac{9}{2}\int_0^\infty\frac{t^2dt}{\left(1+t^3\right)^2}=\frac{3}{2}$$

15. Verify the Green's theorem

$\int_C\left(1-x^2\right)ydx+\left(1+y^2\right)xdy$ **where C is** $x^2+y^2=1$.

Solution.



Here $\oint_C Mdx + Ndy = \oint_C (1-x^2)\,ydx + (1+y^2)\,xdy$

So, $M = (1-x^2)\,y \Rightarrow \dfrac{\partial M}{\partial y} = 1 - x^2$. $N = (1+y^2)\,x \Rightarrow \dfrac{\partial N}{\partial x} = 1 + y^2$

Let us first evaluate the double integral over region R bounded by the curve $C\left(x^2 + y^2 = 1\right)$

$$\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint\left(y^2 + x^2\right)dxdy = \int_0^1\int_0^{2\pi} r^3 dr d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{\pi}{2}$$

Now let us evaluate the line integral $\oint Mdx + Ndy$ as the closed curve $C\left(x^2 + y^2 = 1\right)$.

On C, $x = \cos\theta \Rightarrow dx = -\sin\theta d\theta$, $y = \sin\theta \Rightarrow dy = \cos\theta d\theta$

θ varies from 0 to 2π.

$$\oint Mdx + Ndy = \int_0^{2\pi}\left(1-\cos^2\theta\right)\sin\theta\left(-\sin\theta\right)d\theta + \left(1+\sin\theta\right)\cos\theta\cos\theta d\theta$$

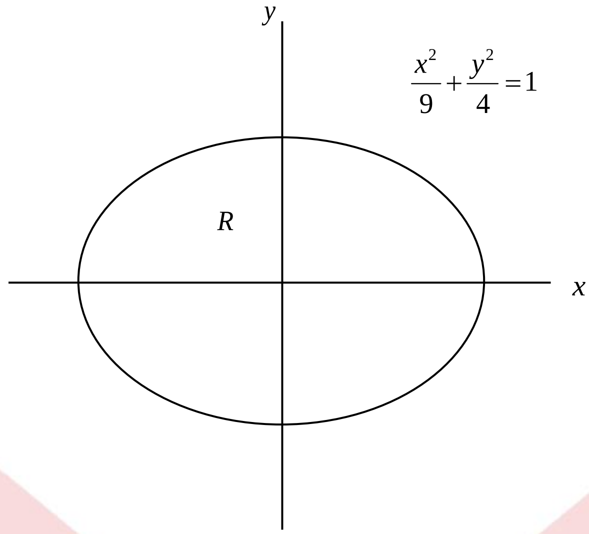
$$= \int_0^{2\pi}\left(-\sin^2\theta + \cos^2\theta + 2\sin^2\theta\cos^2\theta\right)d\theta = -\int_0^{2\pi}\left(-\sin^2\theta + \cos^2\theta + 2\sin^2\theta\cos^2\theta\right)d\theta$$

$$= -\int_0^{2\pi}\sin\theta d\theta + \int_0^{2\pi}\cos\theta d\theta + 2\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta = -\pi + \pi + 8\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = 8\frac{\Gamma(3/2)\Gamma(3/2)}{2\Gamma(3)} = \frac{\pi}{2}$$

Since, $\iint_R\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)dxdy = \oint Mdx + Ndy$. Hence, Green's theorem is verified.

16. Verify Green's theorem in the plane for $\oint_C (xy + x + y)\,dx + (xy + x - y)\,dy$ where C is the closed curve $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

Solution.



Given line integral is

$$\oint_C Mdx + Ndy = \oint (xy + x + y)\,dx + (xy + x - y)\,dy$$

$$M = xy + x + y \quad \frac{\partial M}{\partial y} = x + 1,\ N = xy + x - y \quad \frac{\partial N}{\partial x} = y + 1$$

Let us first evaluate the double integral over region R bounded by $C = \frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint (y - x)\, dxdy = \int_{-3}^{3} \int_{-\frac{2}{3}\sqrt{9-x^2}}^{\frac{2}{3}\sqrt{9-x^2}} ydydx - \int_{-2}^{2} \int_{-\frac{3}{2}\sqrt{4-y^2}}^{\frac{3}{2}\sqrt{4-y^2}} xdxdy$$

$$= 0 - 0 = 0 \left(\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f(x) \text{ is odd} \right)$$

Now, let us evaluate the line integral $\oint Mdx + Ndy$ on the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

On C, $x = 3\cos\theta \Rightarrow dx = -3\sin\theta d\theta$, $y = 2\sin\theta \Rightarrow dy = 2\cos\theta d\theta$

θ varies from 0 to 2π.

$$Mdx + Ndy = (xy + x + y)\,dx + (xy + x - y)\,dy$$

$$= (6\cos\theta\sin\theta + 3\cos\theta + 2\sin\theta)(-3\sin\theta)\,d\theta + (6\cos\theta\sin\theta + 3\cos\theta - 2\sin\theta)(2\cos\theta d\theta)$$

$$= \left(12\cos^2\theta\sin\theta - 18\cos\theta\sin^2\theta - 5\cos\theta\sin\theta - 6\sin^2\theta + 6\cos^2\theta\right) d\theta$$

So, the line integral

$$\oint Mdx + Ndy = 12\int_0^{2\pi} \cos^2\theta \sin\theta d\theta + 18\int_0^{2\pi} \cos\theta \sin^2\theta d\theta - 5\int_0^{2\pi} \cos\theta \sin\theta d\theta$$

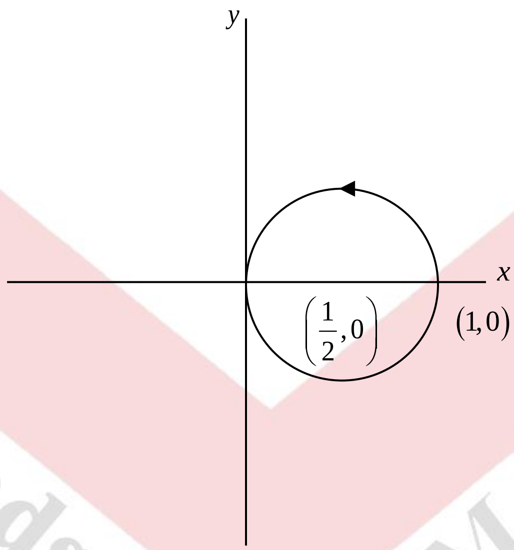
$$-6\int_0^{2\pi} \sin^2\theta d\theta + 6\int_0^{2\pi} \cos^2\theta d\theta = 0$$

Here, we used $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x) = 0$ if $f(2a-x) = -f(x)$

$$\int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} \sin^2\theta d\theta = \frac{\pi}{4}$$

17. Verify the Green's theorem $\oint_C (xy+x+y)dx + (xy+x-y)dy$ where C is the circle $x^2+y^2=x$.

Solution.



The given line integral

$$\oint Mdx + Ndy = \int (xy+x+y)dx + (xy+x-y)dy$$

$$M = (xy+x+y), \frac{\partial M}{\partial y} = x+1, N = (xy+x-y), \frac{\partial N}{\partial x} = y+1$$

Let us first evaluate the line integral over region R bounded by $C: x^2+y^2=x$ as shown in Figure 5.18.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint (y-x)dxdy = \int_0^{1}\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} ydydx - \iint xdxdy = 0 - \iint r\cos\theta rdrd\theta$$

$= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r^2\cos\theta drd\theta$ [Equation of C in polar coordinate is $r = \cos\theta$] $= -\int_{-\pi/2}^{\pi/2} \left.\frac{r^3}{3}\right|_0^{\cos\theta} \cos\theta d\theta$

$$= -\frac{1}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = -\frac{2}{3}\int_0^{\pi/2} \cos^4\theta d\theta = -\frac{2}{3}\cdot\frac{\lceil 5/2 \lceil 1/2}{2\lceil 3} = -\frac{\pi}{8}$$

Now, let us evaluate the line integral $\oint Mdx + Ndy$ over the curve

$$C: x^2+y^2 = x \Rightarrow (x-1/2)^2 + y^2 = 1/4$$

On C, $x = \frac{1}{2} + \frac{1}{2}\cos\theta \Rightarrow dx = -\frac{1}{2}\sin\theta d\theta$, $y = \frac{1}{2}\sin\theta \Rightarrow dy = \frac{1}{2}\cos\theta d\theta$

$$Mdx + Ndy = (xy+x+y)dx + (xy+x-y)dy$$

$$=\left(\left(\frac{1}{2}+\frac{1}{2}\cos\theta\right)\frac{1}{2}\sin\theta+\frac{1}{2}+\frac{1}{2}\cos\theta+\frac{1}{2}\sin\theta\right)\left(-\frac{1}{2}\sin\theta d\theta\right)$$

$$+\left(\left(\frac{1}{2}+\frac{1}{2}\cos\theta\right)\frac{1}{2}\sin\theta+\frac{1}{2}+\frac{1}{2}\cos\theta-\frac{1}{2}\sin\theta\right)\frac{1}{2}\cos\theta d\theta$$

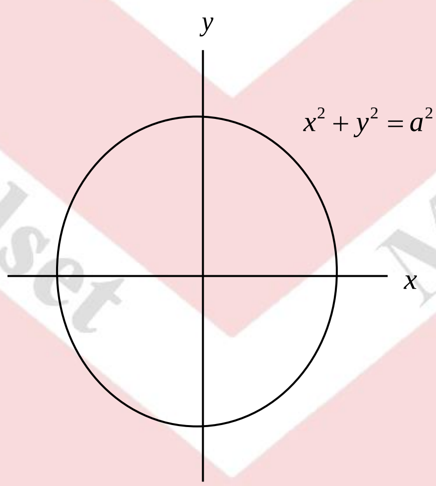
$$=\left(-\frac{3}{8}\sin^2\theta+\frac{1}{4}\cos^2\theta-\frac{1}{8}\cos\theta\sin^2\theta+\frac{1}{8}\cos^2\theta\sin\theta-\frac{1}{8}\cos\theta\sin\theta+\frac{1}{4}\cos\theta-\frac{1}{4}\sin\theta\right)d\theta$$

So, $$\oint_C Mdx+Ndy=-\frac{3}{8}\int_0^{2\pi}\sin^2\theta d\theta+\frac{1}{4}\int_0^{2\pi}\cos^2\theta d\theta-\frac{1}{8}\int_0^{2\pi}\cos\theta\sin^2\theta d\theta$$

$$+\frac{1}{8}\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta-\frac{1}{8}\int_0^{2\pi}\cos\theta\sin\theta d\theta+\frac{1}{4}\int_0^{2\pi}\cos\theta d\theta-\frac{1}{4}\int_0^{2\pi}\sin\theta d\theta=-\frac{\pi}{8}$$

18. Evaluate the line integral $\int_C\left(yx^3+xe^y\right)dx+\left(xy^3+ye^y-2y\right)dy$ using Green's theorem where C is a circle of radius a.

Solution.



The region enclosed by a circle of radius a as shown in figure.

$$\int_C\left(yx^3+xe^y\right)dx+\left(xy^3+ye^y-2y\right)dy=\int_C Mdx+Ndy$$

Here, $M=yx^3-xe^x\Rightarrow\frac{\partial M}{\partial y}=x^3+xe^x, N=xy^3+ye^x-2y\Rightarrow\frac{\partial N}{\partial x}=y^3$

Applying Green's theorem

$$\oint Mdx+Ndy=\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\iint\left(y^3-x^3-xe^y\right)dxdy$$

Changing to polar coordinates

$$=\int_0^a\int_0^{2\pi}\left(r^3\sin^3\theta-r^3\cos^3\theta-r\cos\theta e^{r\sin\theta}\right)rd\theta dr$$

$$= \int_0^a r^4 \left(\int_0^{2\pi} \sin^3 \theta d\theta \right) dr - \int_0^a r^4 \left(\int_0^{2\pi} \cos^3 \theta d\theta \right) dr - \int_0^a r \left(\int_0^{2\pi} e^{r\sin\theta} r \cos\theta d\theta \right) dr$$

$$= 0 - 0 - \int_0^a r \left[e^{r\sin\theta} \right]_0^{2\pi} dr = 0$$

19. Evaluate $\oint_C \frac{xdy - ydx}{x^2 + 4y^2}$ **round the circle** $x^2 + y^2 = a^2$ **in the positive direction using Green's theorem.**

Solution.

The given line integral is

$$\oint_C \frac{xdy - ydx}{x^2 + 4y^2} = \oint Mdx + Ndy$$

Comparing the two integrals,

$$M = -\frac{y}{x^2 + 4y^2}, \frac{\partial M}{\partial y} = -\frac{x^2 - 4y^2}{\left(x^2 + 4y^2\right)^2} = \frac{-x^2 + 4y^2}{\left(x^2 + 4y^2\right)^2}, \ N = \frac{x}{x^2 + 4y^2}, \frac{\partial N}{\partial x} = \frac{-x^2 + 4y^2}{\left(x^2 + 4y^2\right)^2}$$

The curve C is the circle of radius a. R is the region enclosed by the circle $x^2 + y^2 = a^2$. M and N are not continuous at origin. So, the Green's theorem will not hold good for the given line integral. Proceeds similarly as done in question (13).

$$\int_C Mdx + Ndy = -\int_\Gamma Mdx + Ndy$$

$$= -\int_\Gamma \frac{xdy - ydx}{x^2 + 4y^2} = -\int_{2\pi}^{0} \frac{\in \cos\theta \in \cos\theta - \in \sin\theta\left(- \in \sin\theta\right) d\theta}{\in^2 \cos^2\theta + 4 \in^2 \sin^2\theta} \quad \text{(put } x = \in \cos\theta, y = \in \sin\theta\text{)}$$

$$= \int_0^{2\pi} \frac{1}{\cos^2\theta + 4\sin^2\theta} d\theta = \int_0^{2\pi} \frac{\sec^2\theta}{1 + 4\tan^2\theta} d\theta$$

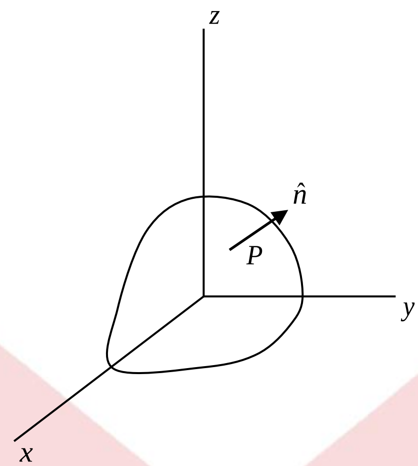
$$= 2\int_0^{\pi} \frac{\sec^2\theta}{1 + 4\tan^2\theta} d\theta = 4\int_0^{\pi/2} \frac{\sec^2\theta}{1 + 4\tan^2\theta} d\theta$$

$$= 4\int_0^{\infty} \frac{dt}{1 + 4t^2} = 4\frac{1}{2} \cdot \tan^{-1} 2t \Big|_0^{\infty} = \pi$$

6. Surface Integral

1. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ **where** $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ **and S is that part of the surface of the sphere** $x^2 + y^2 + z^2 = a^2$ **which lies in the first octant.**

Solution.

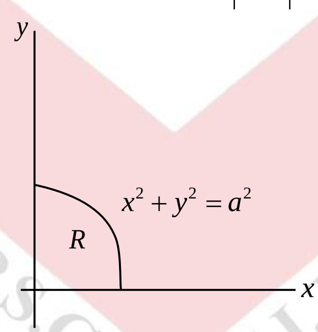


The surface of sphere $x^2 + y^2 + z^2 = a^2$ is shown in figure

The sphere belongs to a family of level surface given by $S = x^2 + y^2 + z^2 = c$

So, the unit vector $\hat{n}$ at any point P is given by, $\hat{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$, $\hat{n}\cdot\hat{k} = \dfrac{z}{a}$

$$\vec{F}\cdot\hat{n} = \left(yz\hat{i} + zx\hat{j} + xy\hat{k}\right)\cdot\frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{a} = \frac{3xyz}{a},\quad dS = \frac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|} = \frac{a}{z}dxdy$$



$$\vec{F}\cdot\hat{n}dS = \frac{3xyz}{a}\cdot\frac{a}{z}\cdot dxdy = 3xydxdy$$

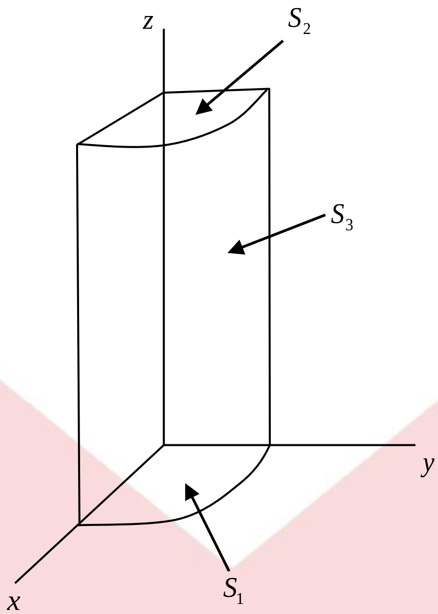
$\int_S \vec{F}\cdot\hat{n}dS = 3\iint_R xy\,dx\,dy$ (The region of integration of double integration given by R)

$$= 3\int_0^{\pi/2}\int_0^a r^3\cos\theta\sin\theta dr\,d\theta = 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \cos\theta\sin\theta d\theta = \frac{3a^4}{4}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

$$= \frac{3a^4}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3}{8}a^4$$

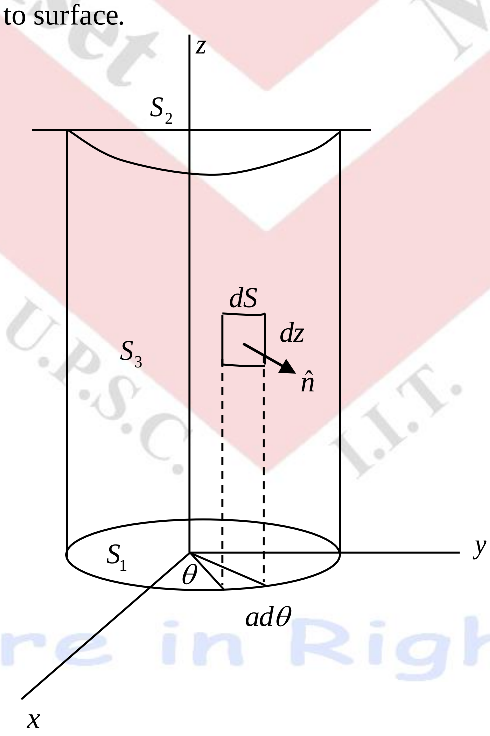
2. Evaluate $\int_S \vec{F}\cdot\hat{n}dS$ **where** $\vec{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ **and S is the surface of the cylinder** $x^2 + y^2 = a^2$ **along with the bases included in the first octant between** $z = 0$ **&** $z = b$**.**

Solution.



The cylinder is a piecewise smooth surface consisting of S_1, S_2 and S_3 where S is lower base $z = 0$, S_2 is upper base $z = b, S_3$ is the curved surface of cylinder, as shown in figure 6.4 & figure 6.5.

$\hat{n}$ is an outward drwan normal to surface.



$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1} \vec{F}\cdot\hat{n}\,dS + \int_{S_2} \vec{F}\cdot\hat{n}\,dS + \int_{S_4} \vec{F}\cdot\hat{n}\,dS$$

On $S_1, \hat{n} = -\hat{k}, z = 0, dS = dxdy$

$\vec{F}\cdot\hat{n} = \vec{F}\cdot(-k) = 3y^2 z = 0$ (as $z = 0$ on S_1). So, $\int\limits_{S_4} \vec{F}\cdot\hat{n}\, dS = 0$

On $S_2, \hat{n} = \hat{k}, z = b, dS = dxdy$, $\vec{F}\cdot\hat{n} = 3y^2 z = -3by^2$

So, $\int\limits_{S_4} \vec{F}\cdot\hat{n}\, dS = -3b\iint y^2 dxdy = -3b\int\limits_0^{\pi/2}\int\limits_0^{a} r^3 \sin^2\theta dr\, d\theta = -3b\int\limits_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin^2\theta\, d\theta$

$= -\frac{3}{4}ba^4\int\limits_0^{\pi/2}\sin^2\theta\, d\theta = -\frac{3}{16}\pi a^4 b$

The curved surface S_3 belongs to family of level surface $S \equiv x^2 + y^2 =$ constant

The unit normal vector to the surface S_3 is given by $= \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j}}{a}$

For S_3, $\vec{F}\cdot\hat{n} = \left(z\hat{i} + x\hat{j} - 3y^2 z\hat{k}\right)\cdot\frac{\left(x\hat{i} + y\hat{j}\right)}{a} = \frac{1}{a}(zx + xy)$

$dS = ad\theta\, dz$. On $S_3, x = a\cos\theta, y = a\sin\theta$

So, $\vec{F}\cdot\hat{n} = \frac{1}{a}\left[az\cos\theta + a^2\sin\theta\cos\theta\right] = z\cos\theta + a\sin\theta\cos\theta$

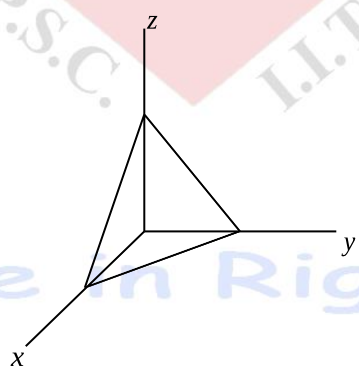
The surface integral becomes $\int\limits_{S_3}\vec{F}\cdot\hat{n}\, dS = \int\limits_0^b\int\limits_0^{\pi/2}\left(z\cos\theta + a\sin\theta\cos\theta\right)ad\theta dz$

$= a\int\limits_0^b\left[z\sin\theta + \frac{a}{2}\sin^2\theta\right]_0^{\pi/2} dz = a\int\limits_0^b\left(z + \frac{a}{2}\right)dz = a\left[\frac{z^2}{2} + \frac{a}{2}z\right]_0^b = \frac{ab}{2}(a + b)$

$\int\limits_S\vec{F}\cdot\hat{n}\, dS = \int\limits_{S_1}\vec{F}\cdot\hat{n}\, dS + \int\limits_{S_2}\vec{F}\cdot\hat{n}\, dS + \int\limits_{S_3}\vec{F}\cdot\hat{n}\, dS = -\frac{3}{18}\pi a^4 b + \frac{ab}{2}(a + b)$

3. Evaluate $\int\limits_S\vec{F}\cdot\hat{n}\, dS$, where $\vec{F} = \left(x + y^2\right)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $x + 2y + 3z = 6$ in the first octant.

Solution.



The plane belongs to the family of level surface given by $S = x + 2y + 3z$ = constant

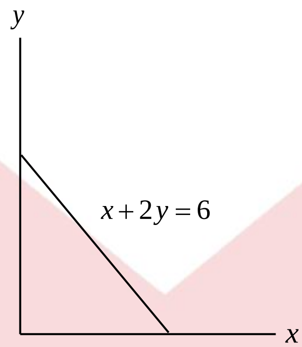
A unit vector normal to the surface is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}, \ \vec{F} \cdot \hat{n} = \frac{1}{\sqrt{14}}\left[\left(x + y^2\right) - 4x + 6yz\right]$$

$$= \frac{1}{\sqrt{14}}\left[x + y^2 - 4x + 2y\left(6 - x - 2y\right)\right]\left(z = \frac{1}{3}\left(6 - x - 2y\right)\right) = \frac{1}{\sqrt{14}}\left(12y - 3x - 3y^2 - 2xy\right)$$

$$dS = \frac{dxdy}{\left|\hat{n} \cdot \hat{k}\right|} = \frac{\sqrt{14}}{3} \cdot dxdy, \vec{F} \cdot \hat{n} dS = \frac{1}{3}\left(12y - 3x - 3y^2 - 2xy\right) dxdy$$

So, $\int\limits_S \vec{F} \cdot \hat{n}\, dS = \frac{1}{3}\int\limits_0^6 \int\limits_0^{\frac{6-x}{2}} \left(2y - 3x - 3y^2 - 2xy\right) dydx$ (The region of double integration is shown in figure

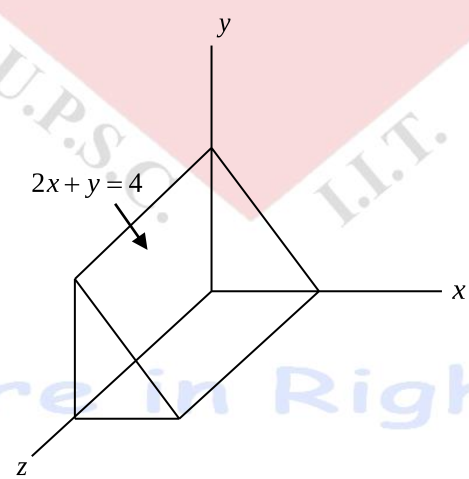


$$= \frac{1}{3}\int\limits_0^6 \left[6y^2 - 3xy - y^3 - xy^2\right]_0^{\frac{6-x}{2}} \cdot dx = \frac{1}{3}\int\limits_0^6 \left(-\frac{x^3}{8} + \frac{15}{4}x^2 - \frac{45}{2}x + 27\right) dx$$

$$= \frac{1}{3}\left[-\frac{x^4}{32} + \frac{5x^3}{4} - \frac{45}{4}x^2 + 27x\right]_0^6 = 4.5$$

4. Evaluate $\int\limits_S \vec{F} \cdot \hat{n}\, dS$ where $\vec{F} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x + y = 4$ in the first octant cut off by the plane $z = 4$.

Solution.



The surface of the plane $2x + y = 4$ belongs to family of level surface $S = 2x + y =$ octant.

A unit vector normal to the surface, $\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i}+\hat{j}}{\sqrt{5}}$

The integral $\vec{F}\cdot\hat{n} = \left(y\hat{i}+2x\hat{j}-z\hat{k}\right)\cdot\left(\frac{2\hat{i}+\hat{j}}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(2y+2x\right) = \frac{2}{\sqrt{5}}\left(x+y\right)$

$\hat{n}\cdot\hat{j} = \frac{1}{\sqrt{5}}\left(2i+\hat{j}\right)\cdot\hat{j} = \frac{1}{\sqrt{5}}$

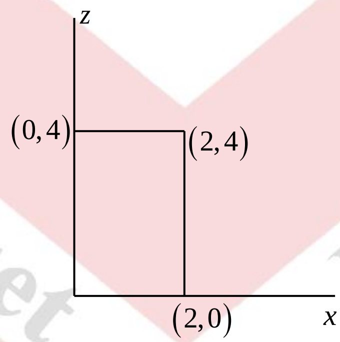
Now, taking projection of the surface on xz plane as shown in figure 6.9.

$$dS = \frac{dxdz}{\left|\hat{n}\cdot\hat{j}\right|} = \sqrt{5}dxdz$$

$$\vec{F}\cdot\hat{n}dS = \frac{2}{\sqrt{5}}\left(x+y\right)\sqrt{5}dxdz = 2\left(x+y\right)dxdz$$

$= 2\left(x+4-2x\right)dxdz$ ($y = 4-2x$ from the equation of surface)

$= 2\left(4-x\right)dxdz$

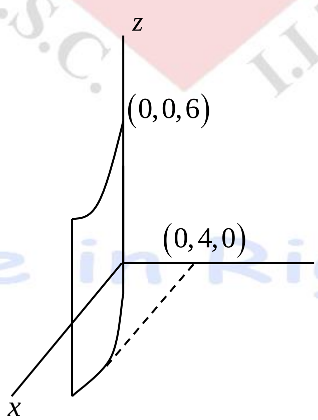


So, Surface integral becomes

$$\int_S \vec{F}\cdot\hat{n}\,dS = 2\int_0^4\int_0^2\left(4-x\right)dxdz = 2\int_0^4 4x-\frac{x^2}{2}\Big|_0^2\,dz = 12\int_0^4 dz = 48$$

5. If $\vec{F} = 2y\hat{i}-z\hat{j}+x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 4x$ in the first octant bounded by the planes $y = 4$ and $z = 6$ then evaluate $\int_S \vec{F}\cdot\hat{n}dS$.

Solution.



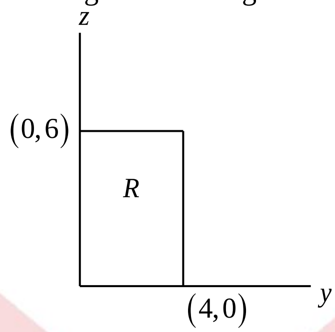
The parabolic surface as shown in fig. 6.10 belong to family of level surface $S = 4x-y^2$ = constant.

The unit normal vector to the parabolic cylinder is given by

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i} - y\hat{j}}{\sqrt{y^2+4}}, \quad \vec{F}\cdot\hat{n} = \left(2y\hat{i} - z\hat{j} + x^2\hat{k}\right)\cdot\frac{\left(2\hat{i} - y\hat{j}\right)}{\sqrt{y^2+4}} = \frac{4y+4z}{\sqrt{y^2+4}}$$

$$\hat{n}\cdot\hat{i} = \frac{2}{\sqrt{y^2+4}}, \quad dS = \frac{dydz}{|\hat{n}\cdot\hat{i}|} = \frac{1}{2}\sqrt{y^2+4}\,dydz, \quad \vec{F}\cdot\hat{n}dS = \frac{1}{2}\left(4y+yz\right)dydz$$

So, the surface integral reduces to double integral whose region of integration R is given in fig. 6.11



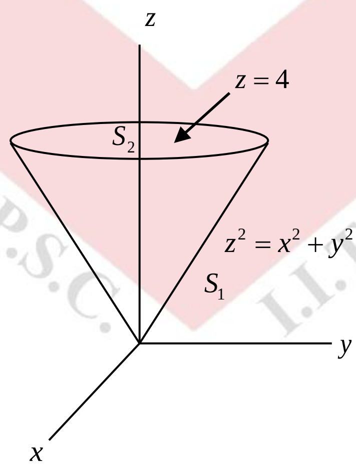
$$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{2}\iint_R \left(4y+yz\right)dydz$$

Region R is the projection of parabolic cylinder on yz plane

$$\int_S \vec{F}\cdot\hat{n}\,dS = \frac{1}{2}\int_0^6\int_0^4 \left(4y+yz\right)dydz = \frac{1}{2}\int_0^6 \left.2y^2 + \frac{y^2z}{2}\right|_0^4 dz = \int_0^6 \left(16+4z\right)dz = 16z + 2z^2\Big|_0^6 = 168$$

6. Evaluate $\int_S \vec{F}\cdot\hat{n}\,dS$ over the entire surface of the region above xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$ if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Solution.



The conical surface S, as shown in the fig. 6.12 belongs to a family of level surface given by $S = x^2 + y^2 - z^2$ = constant.

The unit normal vector to cone is given by $\hat{n} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$$\vec{F}\cdot\hat{n}=\left(4xz\hat{i}+xyz^2\hat{j}+3zk\hat{k}\right)\cdot\frac{\left(x\hat{i}+y\hat{j}-z\hat{k}\right)}{\sqrt{x^2+y^2+z^2}}=\frac{4x^2z+xy^2z^2-3z^2}{\sqrt{x^2+y^2+z^2}}$$

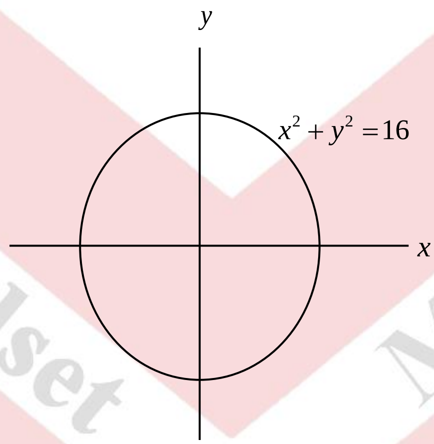
$$\hat{n}\cdot\hat{k}=\frac{-z}{\sqrt{x^2+y^2+z^2}}\ ,\ dS=\frac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|}=\frac{\sqrt{x^2+y^2+z^2}}{z}dxdy$$

$$\vec{F}\cdot\hat{n}dS=\frac{1}{z}\left(4x^2z+xy^2z^2-3z^2\right)dxdy=\left(4x^2+xy^2z-3z\right)dxdy=\left(4x^2+xy^2z-3z\right)dxdy$$

$$=\left(4x^2+xy^2\sqrt{x^2+y^2}-3\sqrt{x^2+y^2}\right)dxdy$$

So, $\int_S \vec{F}\cdot\hat{n}\,dS=\iint_R\left(4x^2+xy^2\sqrt{x^2+y^2}-3\sqrt{x^2+y^2}\right)dxdy$

(R is the region of integration given by projection of cone on xy plane as shown in fig)



$$=4\iint x^2dxdy-3\iint\sqrt{x^2+y^2}\,dxdy\left(\text{as}\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}xy^2\sqrt{x^2-y^2}\,dxdy=0\right)$$

$$=4\int_0^{2\pi}\int_0^4 r^3\cos^2\theta drd\theta-3\int_0^{2\pi}\int_0^4 r^2drd\theta=4\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^4\cos^2\theta d\theta-3\int_0^{2\pi}\left.\frac{r^3}{3}\right|_0^4 d\theta=256\int_0^{2\pi}\cos^2\theta d\theta-64\int_0^{2\pi}d\theta$$

$$=256\pi-128\pi=128\pi$$

On S_2, $\hat{n}=\hat{k},\ dS=dxdy$, $\vec{F}\cdot\hat{n}=3z=12$

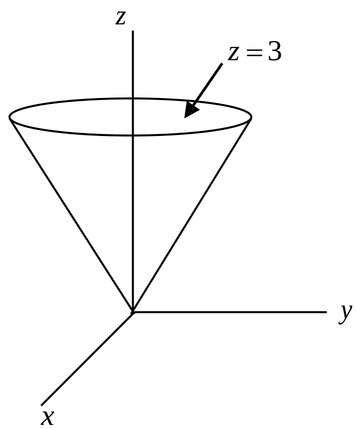
$$\int_{S_2}\vec{F}\cdot\hat{n}\,dS=12\iint dxdy=192\pi$$

So, $\int_S\vec{F}\cdot\hat{n}\,dS=\int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS=128\pi+192\pi=320\pi$

7. Evaluate $\iint\left(x^2+y^2\right)dS$ where S is the surface of the cone $z^2=x^2+y^2$ bounded by $z=0$ & $z=3$

.

Solution.



Upper part of a cone is given by $z=\sqrt{x^2+y^2}$ as shown in fig. 6.14.

It belongs to family of level surface given by $S:\sqrt{x^2+y^2}-z$ = constant.

Outward drawn unit normal vector is given by

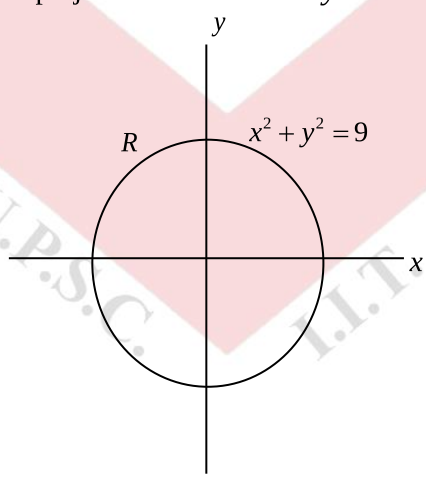
$$\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{\frac{x}{\sqrt{x^2+y^2}}\hat{i}+\frac{y}{\sqrt{x^2+y^2}}\hat{j}-\hat{k}}{\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}}, \quad |\hat{n}\cdot\hat{k}|=\frac{1}{\sqrt{2}}, \quad dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\sqrt{2}dxdy$$

S is a piecewise smooth surface consisting of conical part

$S_1:\sqrt{x^2+y^2}-z=0$ and $S_2:z=3$ as shown in fig. On S_1, $dS=\sqrt{2}dxdy$

So, $\int_{S_1}\left(x^2+y^2\right)dS=\iint_R\left(x^2+y^2\right)\sqrt{2}dxdy$

The region of double integration R is projection of cone $x^2+y^2=z^2$ on the xy plane as shown in fig



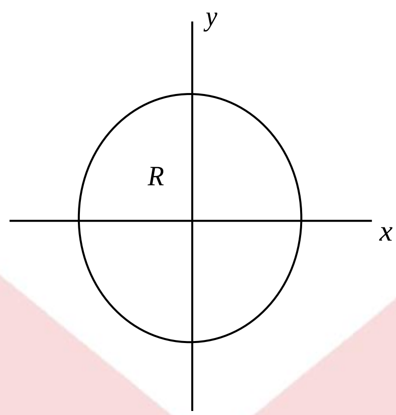
$$=\sqrt{2}\int_0^{2\pi}\int_0^3 r^2\cdot r\,dr\,d\theta=\sqrt{2}\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^3 d\theta=\frac{81\sqrt{2}}{4}\int_0^{2\pi}d\theta=\frac{81\sqrt{2}}{2}\pi$$

On $S_2, z=3$, $dS=dxdy=rdrd\theta$, $\int_{S_2}\left(x^2+y^2\right)dS=\int_0^{2\pi}\int_0^3 r^2 rd\theta dr=\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^3 d\theta=\frac{81}{4}\int_0^{2\pi}d\theta=\frac{81}{2}\pi$

So, $\int_S \left(x^2+y^2\right)dS = \int_{S_1}\left(x^2+y^2\right)dS + \int_{S_2}\left(x^2+y^2\right)dS = \frac{81\sqrt{2}}{2}\pi + \frac{81}{2}\pi = \frac{81}{2}\pi\left(\sqrt{2}+1\right).$

8. Evaluate the surface integral $\int_S \frac{dS}{r}$ where S is the portion of the surface of hyperbolic paraboloid $z = xy$ cut by the cylinder $x^2+y^2=1$ and r is the distance from a point on the surface to z axis.

Solution.



Surface of hyperbolic paraboloid belongs to the family of level surface $S: xy - z =$ constant.

The unit normal vector to surface is given by $\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{y\hat{i} + x\hat{j} - \hat{k}}{\sqrt{x^2+y^2+1}}$

$\left|\hat{n}\cdot\hat{k}\right| = \frac{1}{\sqrt{x^2+y^2+1}}$, $dS = \frac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|} = \sqrt{x^2+y^2+1}\,dxdy$

So, the surface integral reduces to a double integral

$$I = \int_S \frac{dS}{r} = \iint_R \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}dxdy$$

where R is the region of the integration of double integral as shown in fig. 6.16 which is projection of surface on xy plane.

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rd\,\theta\,dr$

$$I = \iint_R \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}dxdy = \int_0^1\int_0^{2\pi}\sqrt{1+r^2}\,d\theta dr = 2\pi\left[\frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\log\left(r+\sqrt{1+r^2}\right)\right]_0^1$$

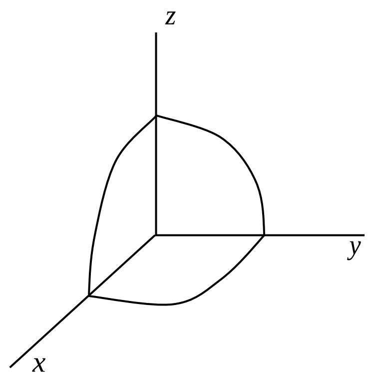
$$= \pi\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$$

9. Evaluate

$$I = \iint xdydz + dzdx + xz^2dxdy$$

where S is the part of sphere $x^2+y^2+z^2=a^2$ in the first octant.

Solution.



S is the part of sphere $x^2+y^2+z^2=a^2$ lying the first octant as shown in fig.

S belongs to family of level surface given by $S: x^2+y^2+z^2$ = constant

Outward drawn unit normal vector to S,, $\hat{n}=\dfrac{\nabla S}{|\nabla S|}=\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$, $\left|\hat{n}\cdot\hat{k}\right|=\dfrac{z}{a}$. $dS=\dfrac{dxdy}{\left|\hat{n}\cdot\hat{k}\right|}=\dfrac{a}{z}\cdot dxdy$

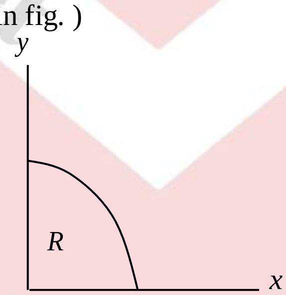
The given integral can be written as

$$\iint xdydz+dzdx+xz^2dxdy=\int\left(x\hat{i}+\hat{j}+xz^2\hat{k}\right)\cdot\hat{n}dS=\int\vec{F}\cdot\hat{n}\,dS$$

Where $\vec{F}=x\hat{i}+\hat{j}+xz^2\hat{k}$, $\vec{F}\cdot\hat{n}=\left(x\hat{i}+\hat{j}+xz^2\hat{k}\right)\cdot\left(\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)=\dfrac{1}{a}\left(x^2+y+xz^3\right)$

$$\int_S\vec{F}\cdot\hat{n}\,dS=\iint\frac{\left(x^2+y+xz^3\right)}{z}dxdy=\iint_R\left[\frac{x^2+y}{\sqrt{a^2-x^2-y^2}}+x\left(a^2-x^2-y^2\right)\right]dxdy$$

(R is the region of integration as shown in fig.)



$$=\iint_R\frac{x^2dxdy}{\sqrt{a^2-x^2-y^2}}+\iint_R\frac{y}{\sqrt{a^2-x^2-y^2}}dydx+\iint_R x\left(a^2-x^2-y^2\right)dxdy$$

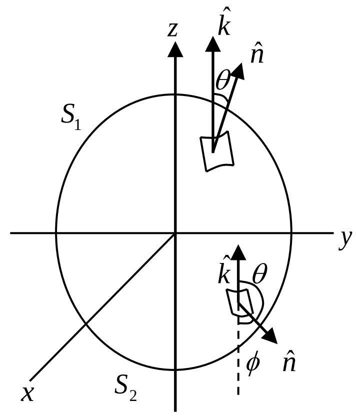
$$=\int_0^a\int_0^{\pi/2}\frac{r^3\cos^2\theta}{\sqrt{a^2-r^2}}d\theta dr+\int_0^a\int_0^{\pi/2}\frac{r^2\sin\theta}{\sqrt{a^2-r^2}}d\theta dr+\int_0^a\int_0^{\pi/2}r^2\left(a^2-r^2\right)\cos\theta d\theta dr$$

$$=\frac{\pi}{4}\int_0^a\frac{r^3}{\sqrt{a^2-r^2}}dr+\int_0^a\frac{r^2}{\sqrt{a^2-r^2}}dr+\int_0^a\left(a^2r^2-r^4\right)dr=\frac{\pi a^3}{6}+\frac{a^2\pi}{4}+\left(a^2\frac{r^3}{3}-\frac{r^5}{5}\right)_0^a$$

$$=\frac{\pi a^3}{6}+\frac{\pi a^2}{4}+\frac{2a^5}{15}$$

10. Evaluate the surface integral $\iint z\cos\theta dS$ over the surface or sphere $x^2+y^2+z^2=a^2$ where θ is the inclination of normal at any point of the sphere with the z axis.

Solution.



S is the surface of sphere consisting of

upper hemisphere $S_1 : z = \sqrt{a^2 - x^2 - y^2}$ and

lower hemisphere $S_2 : z = \sqrt{a^2 - x^2 - y^2}$ as shown in fig. 6.19.

Over $S_1, dS\cos\theta = dxdy, z = \sqrt{a^2 - x^2 - y^2}$

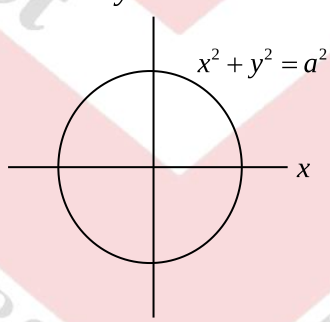
$z\cos\theta\, dS = \sqrt{a^2 - x^2 - y^2}\,dxdy$

Over $S_2, dS\cos\theta = dS\cos(\pi - \phi) = -dS\cos\phi = -dxdy$

$z = -\sqrt{a^2 - x^2 - y^2}$

$z\cos\theta dS = \sqrt{a^2 - x^2 - y^2}\,dxdy$

Since projection of S_1 and S_2 is same i.e. $x^2 + y^2 = a^2$



$\int\limits_{S_1} z\cos\theta dS = \int\limits_{S_2} z\cos\theta dS$ So, $\int\limits_{S} z\cos\theta dS = \int\limits_{S_1} z\cos\theta\, dS + \int\limits_{S_2} z\cos\theta\, dS$

$= 2\iint\limits_{R} \sqrt{a^2 - x^2 - y^2}\,dxdy$ (R is the region of integration as shown in fig. $= 2\int\limits_0^a \int\limits_0^{2\pi} \sqrt{a^2 - r^2}\,rd\theta\, dr$

$= 4\pi \int\limits_0^a \sqrt{a^2 - r^2}\,r\, dr = \dfrac{4\pi}{3}a^3$

11. Evaluate $\int\limits_S x\, dS$ where S is the entire surface of solid bounded by the cylinder $x^2 + y^2 = a^2$ and $z = 0,\ z = x + 2.$

Solution.

S is piece wise smooth surface consisting of

S_1 : Base of cylinder, $z = 0$, S_2 : roof of cylinder, $z = x + 2$, S_3 : curved surface of cylinder $x^2 + y^2 = a^2$

On S_1, $dS = dxdy$, $\int_{S_1} x\,dS = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0$

S_2 belongs to family of level surface given by $S_2 : z - x$ = constant.

So, outwards drawn unit normal to S_2, $\hat{n} = \dfrac{-\hat{i} + \hat{k}}{\sqrt{2}}$

On S_2, $dS = \dfrac{dxdy}{|\hat{n} \cdot \hat{k}|} = \sqrt{2}\,dxdy$, So, $\int_{S_2} x\,dS = \sqrt{2} \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} xdxdy = 0$

On S_3, $dS = ad\theta dz$, $x = a\cos\theta$, $y = a\sin\theta$

z varies from 0 to $x + 2$ i.e. 0 to $2 + a\cos\theta$

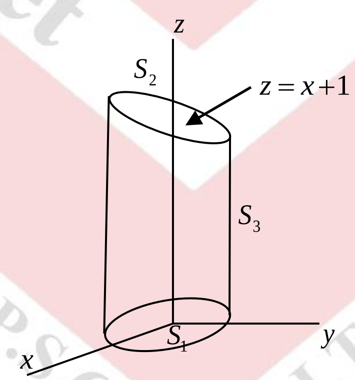
$$\int_{S_3} x\,dS = \int_0^{2\pi} \int_0^{2+a\cos\theta} a\cos\theta a\,dzd\theta = a^2 \int_0^{2\pi} \cos\theta \cdot (2 + a\cos\theta)\,d\theta = 2a^2 \int_0^{2\pi} \cos\theta d\theta + a^3 \int_0^{2\pi} \cos^2\theta d\theta$$

$$= \pi a^3 \quad \left(\int_0^{2\pi} \cos\theta d\theta = 0 \right)$$

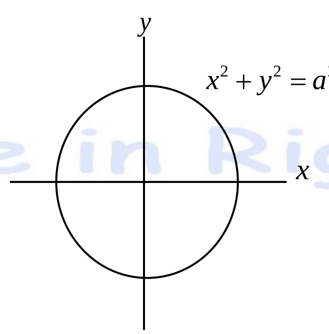
So, $\iint_S \vec{F} \cdot \hat{n}\,dS = \int_{S_1} \vec{F} \cdot \hat{n}\,dS + \int_{S_2} \vec{F} \cdot \hat{n}\,dS + \int_{S_3} \vec{F} \cdot \hat{n}\,dS = \pi a^3$

12. Evaluate $\iint \vec{F} \cdot \hat{n}\,dS$ where S is the entire surface of the solid formed by $x^2 + y^2 = a^2, z = x + 1$ and $\hat{n}$ is the outward drawn unit normal and the vector function $\vec{F} = 2x\hat{i} - 3y\hat{j} + z\hat{k}$.

Solution.



S is the pricewise smooth surface consisting of $S_1 : z = 0$, $S_2 : z = x + 1$ and $S_3 : x^2 + y^2 = a^2$ (curved surface) as shown in fig. 6.21.



On $S_1, z=0,\ \hat{n}=-\hat{k},\ \vec{F}\cdot\hat{n}=-z=0$ So, $\int_{S_1}\vec{F}\cdot\hat{n}dS=0$

On $S_2, z=x+2$, $\hat{n}=\dfrac{-\hat{i}+\hat{k}}{\sqrt{2}}$ (as done in previous question) , $\vec{F}\cdot\hat{n}=\dfrac{1}{\sqrt{2}}(-2x+z)=\dfrac{1}{\sqrt{2}}(-x-1)$

$$dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\sqrt{2}dxdy,\ \vec{F}\cdot\hat{n}dS=(1-x)dxdy,\ \int_{S_2}\vec{F}\cdot\hat{n}dS=\iint_R(1-x)dxdy$$

$$=\iint dxdy-\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}xdxdy$$

(R is the region of double integration as shown in fig)

$$=\iint dxdy\ \left(\text{as}\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}xdxdy=0\right)=\pi a^2$$

S_3 belong to family of level surface $S_3: x^2+y^2$ = constant.

Outward drawn unit normal vector., $\hat{n}=\dfrac{\nabla S_3}{|\nabla S_3|}=\dfrac{x\hat{i}+y\hat{j}}{a}$

On S_3 $\vec{F}\cdot\hat{n}=\dfrac{1}{a}\left(2x^2-3y^2\right),\ x=a\cos\theta, y=a\sin\theta$

$$dS=ad\theta\,dz,\ \vec{F}\cdot\hat{n}dS=\left(2x^2-3y^2\right)ad\theta\,dz=a^3\left(2\cos^2\theta-3\sin^2\theta\right)dzd\theta$$

z varies from 0 to $x+1$, i.e. 0 to $1+a\cos\theta$

$$\int_{S_3}\vec{F}\cdot\hat{n}\,dS=\int_0^{2\pi}\int_0^{(1+a\cos\theta)}a^3\left(2\cos^2\theta-3\sin^2\theta\right)dzd\theta$$

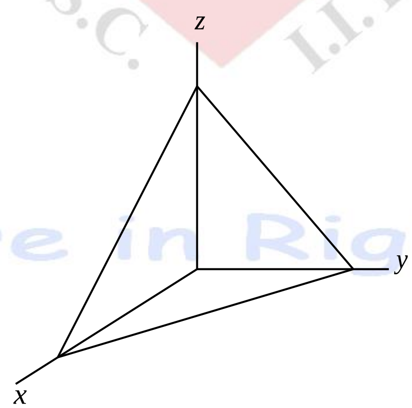
$$=a^3\int_0^{2\pi}\left(2\cos^2\theta-3\sin^2\theta\right)d\theta-a^4\int_0^{2\pi}\left(2\cos^2\theta-3\sin^2\theta\right)\cos\theta d\theta$$

$$=-\pi a^3$$

So, $\iint_S\vec{F}\cdot\hat{n}\,dS=\int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS+\int_{S_3}\vec{F}\cdot\hat{n}\,dS=0+\pi a^2-\pi a^3=\pi a^2(1-a)$

13. Evaluate $\int_S xyz\,dS$ over the portion of $x+y+z=a,\ a>0$, lying in the first octant.

Solution.

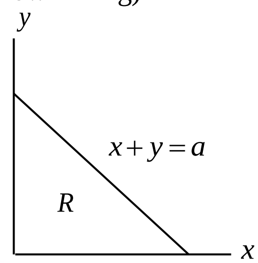


S is the surface given by $x+y+z=a$ in the first octant. It belong to family of level surface given by $S: x+y+z$ = constant as shown in fig.

Unit normal vector to the surface S, $\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$, $\hat{n}\cdot\hat{k}=\frac{1}{\sqrt{3}}$

$dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\sqrt{3}dydx$. So, $\int_S xyz\,dS=\iint_R xyz\sqrt{3}dydx=\sqrt{3}\int_0^a\int_0^{a-x} xy(a-x-y)\,dydx$

(R is the region of double integration as shown in fig)

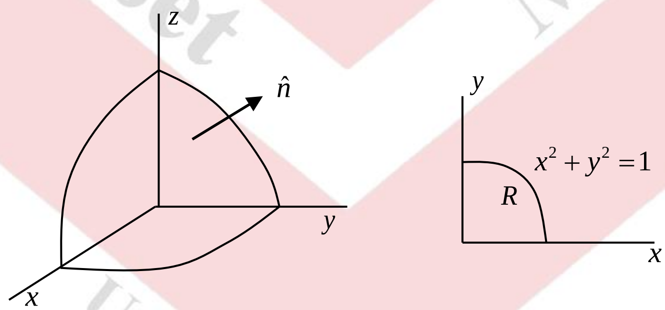


$$=\sqrt{3}\int_0^a x(a-x)\frac{y^2}{2}-\frac{xy^3}{3}\Big|_0^{a-x} dx \;:\; (z=a-x-y)$$

$$=\sqrt{3}\int_0^a \frac{x}{2}(a-x)^3-\frac{x}{3}(a-x)^3\,dx=\frac{\sqrt{3}}{6}\int_0^a x(a-x)^3\,dx=\frac{1}{40\sqrt{3}}a^5$$

14. Evaluate $\int x\,dS$ where S is the portion of the sphere $x^2+y^2+z^2=1$ lying in the first octant.

Solution.

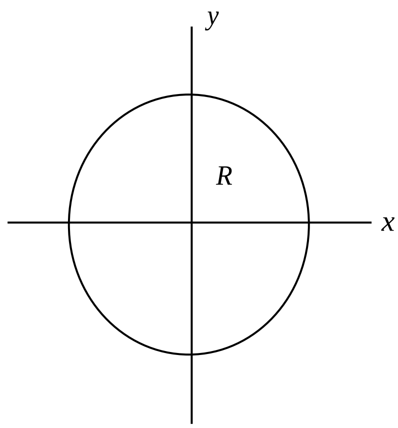


S is the surface of sphere lying in the first octant as shown in fig. 6.25 and belongs to family of level surface $S: x^2+y^2+z^2$ = constant.

An outward drawn unit normal vector to S. $\hat{n}=\frac{\nabla S}{|\nabla S|}=x\hat{i}+y\hat{j}+z\hat{k}$, $\hat{n}\cdot\hat{k}=z$

$dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\frac{1}{z}dxdy=\frac{1}{\sqrt{1-x^2-y^2}}dxdy$, $\int_S x\,dS=\iint_R \frac{x}{\sqrt{1-x^2-y^2}}$

(R is the region of double integration as shown in fig.)



$$= \int_0^1 \int_0^{\pi/2} \frac{r^2 \cos\theta}{\sqrt{1-r^2}} d\theta dr = \int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr = \frac{\pi}{4}$$

15. Evaluate the integral $\int_S \sqrt{1-x^2-y^2}\,dS$ where S is the hemisphere $z=\sqrt{1-x^2-y^2}$.

Solution.

S is the surface of hemisphere $z=\sqrt{1-x^2-y^2}$

An outward drawn unit normal vector to S, $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$, $dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{dxdy}{z}$

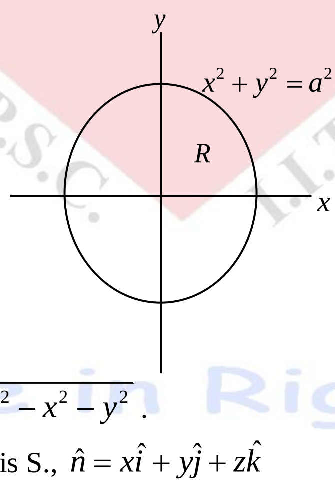
$$\int_S \sqrt{1-x^2-y^2}\,dS = \iint_R \sqrt{1-x^2-y^2}\,\frac{dxdy}{z} \quad \left(z=\sqrt{1-x^2-y^2}\right)$$

$= \iint_R dxdy$ = Area of region R

(R is the region of double integration as shown in fig.)

16. Evaluate the integral $\int_S x^2 y^2 dS$ where S is the hemisphere $z=\sqrt{a^2-x^2-y^2}$.

Solution.



S is the surface of hemisphere $z=\sqrt{a^2-x^2-y^2}$.

An outward drawn unit normal vector is S., $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$

$$dS = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \frac{dxdy}{z}$$

$$\int_S x^2y^2 dS = \iint_R \frac{x^2y^2}{z} dxdy$$

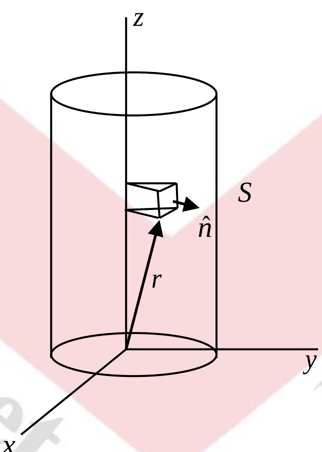
(R is the region of double integration as shown in fig.)

$$= \iint_R \frac{x^2y^2}{\sqrt{a^2-x^2-y^2}} \cdot dxdy = \int_0^a \int_0^{2\pi} \frac{r^5 \sin^2\theta \cos^2\theta}{\sqrt{a^2-r^2}} d\theta dr = 4\int_0^a \int_0^{\pi/2} \frac{r^5}{\sqrt{a^2-r^2}} \sin^2\theta \cos^2\theta d\theta dr$$

$$= 4\int_0^a \frac{r^5}{\sqrt{a^2-r^2}} \frac{\Gamma(3/2)\Gamma(3/2)}{2\Gamma(3)} dr = \frac{\pi}{4}\int_0^1 \frac{r^5}{\sqrt{a^2-r^2}} dr = \frac{2}{15}\pi a^6$$

17. Evaluate $\int \frac{dS}{r^2}$ where S is the cylinder $x^2+y^2=a^2$ bounded by the plane $z=0$ and $z=b$ and r is the distance between a point on the surface and the origin.

Solution.

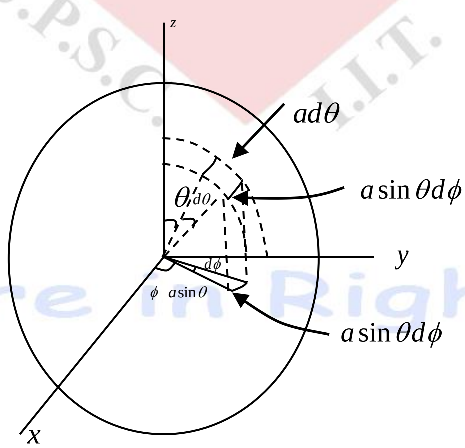


S is the surface of cylinder lying between $z=0$ and $z=b$ on S as shown in fig. 6.29,

$$dS = ad\theta\, dz,\ r=\sqrt{a^2+z^2},\ I=\int \frac{dS}{r^2} = \int_0^b \int_0^{2\pi} \frac{ad\theta dz}{\left(a^2+z^2\right)} = 2\pi a\int_0^b \frac{dz}{z^2+a^2} = 2\pi \tan^{-1}\frac{z}{a}\Big|_0^b = 2\pi \tan^{-1}\frac{b}{a}$$

18. Evaluate $\iint x^3dydz+y^3dzdx+z^3dxdy$ where S is the outer surface of the sphere $x^2+y^2+z^2=a^2$.

Solution.



S is the outer surface of sphere $x^2+y^2+z^2=a^2$ as shown in fig. Normal to the outer surface

$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$. On a surface of sphere $dS = a d\theta \cdot a\sin\theta d\phi = a^2 \sin\theta d\theta d\phi$

$$\iint \left(x^3 dydz + y^3 dzdx + z^3 dxdy \right) = \iint \left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k} \right) \cdot \left(dy\,dz\hat{i} + dz\,dx\hat{j} + dx\,dy\hat{k} \right)$$

$$= \int \left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k} \right) \cdot \hat{n}\,dS = \iint \frac{\left(x^4 + y^4 + z^4\right)}{a} a^2 \sin\theta d\theta d\phi = a\iint \left(x^4 + y^4 + z^4\right) \sin\theta d\theta d\phi$$

$$= a\int_0^{2\pi}\int_0^{\pi} \left(a^4 \sin^5\theta\cos^4\phi + a^4\sin^5\theta\sin^4\phi + a^4\cos^4\theta\sin\theta\, d\phi \right)$$

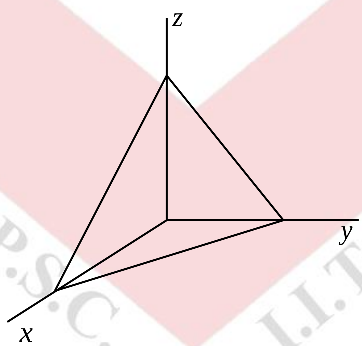
$\left(z = a\cos\theta, x = a\sin\theta\cos\phi, y = a\sin\theta\sin\phi \text{ on S} \right)$

$$\left[\begin{array}{l} \int_0^{\pi} \sin^5\theta d\theta = 2\int_0^{\pi/2} \sin^5\theta d\theta = 2\cdot\frac{\Gamma(3)\Gamma(1/2)}{2\Gamma(7/2)} = \frac{16}{15} \\ \int_0^{\pi} \cos^4\theta\sin\theta d\theta = 2\int_0^{\pi/2} \cos^4\theta\sin\theta d\theta = \frac{2\Gamma(5/2)\Gamma(1)}{2\Gamma(7/2)} = \frac{2}{5} \end{array} \right] = a^5\cdot\frac{16}{15}\cdot\left[\int_0^{2\pi} \left(\sin^4\phi + \cos^4\phi \right) d\phi \right] + a^5\cdot\frac{2}{5}\int_0^{2\pi} d\phi$$

$$\left[\begin{array}{l} \int_0^{2\pi} \sin^4\phi d\phi = 4\int_0^{\pi/2} \sin^4\phi d\phi = 4\cdot\frac{\Gamma(5/2)\Gamma(1/2)}{2\cdot\Gamma(3)} = \frac{3}{4}\pi \\ \int_0^{2\pi} \cos^4\phi d\phi = 4\int_0^{\pi/2} \cos^4\phi d\phi = 4\cdot\frac{\Gamma(5/2)\Gamma(1/2)}{2\cdot\Gamma(3)} = \frac{3}{4}\pi \end{array} \right] = a^5\cdot\frac{16}{15}\left[\frac{3}{4}\pi + \frac{3}{4}\pi \right] + \frac{2}{5}a^5\cdot 2\pi = \frac{12}{5}\pi a^5$$

19. Evaluate $\iint_y \left(xz\,dx\,dy + xy\,dy\,dz + yz\,dz\,dx \right)$ where S is the outer side of the pyramid formed by the planer $x = 0, y = 0, z = 0$ and $x + y + z = a$.

Solution.



S is the piece wise smooth surface formed by

$S_1 : x = 0.\ S_2 : y = 0,\ S_3 : z = 0, S_4 = x + y + z = a$ as shown in fig. 6.31.

$$\iint_S xzdxdy + xydydz + yzdzdx = \iint \left(xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k} \right) \cdot \left(dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k} \right)$$

$$= \int_S \left(xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k} \right) \cdot \hat{n}\,dS$$

$$\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$$

Here, $\iint_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1} \vec{F}\cdot\hat{n}\,dS + \int_{S_2} \vec{F}\cdot\hat{n}\,dS + \int_{S_3} \vec{F}\cdot\hat{n}\,dS + \int_{S_4} \vec{F}\cdot\hat{n}\,dS$

On $S_1 : x = 0, \hat{n} = -\hat{i}, \vec{F} \cdot \hat{n} = -xy = 0$, $\int_{S_1} \vec{F} \cdot \hat{n}\, dS = 0$

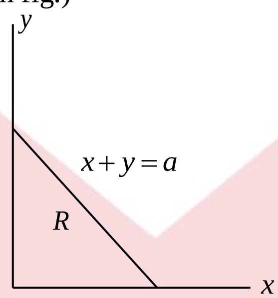
On $S_2 : y = 0, \hat{n} = -\hat{j}, \vec{F} \cdot \hat{n} = -yz = 0$, $\int_{S_2} \vec{F} \cdot \hat{n}\, dS = 0$

On $S_3 : z = 0, \hat{n} = -\hat{k}, \vec{F} \cdot \hat{n} = -xz = 0$, $\int_{S_3} \vec{F} \cdot \hat{n} = 0$

S_4 belongs to family of level surface , $S_4 : x + y + z$ = constant, $\hat{n} = \dfrac{\nabla S_4}{|\nabla S_4|} = \dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

$\vec{F} \cdot \hat{n} = \dfrac{1}{\sqrt{3}}(xy + yz + zx)$, $dS = \dfrac{dxdy}{|\hat{n} \cdot \hat{k}|} = \sqrt{3}\, dx\, dy$, $\int_{S_4} \vec{F} \cdot \hat{n}\, dS = \iint_R (xy + yz + zx) dydx$

(R is the region of integration as shown in fig.)

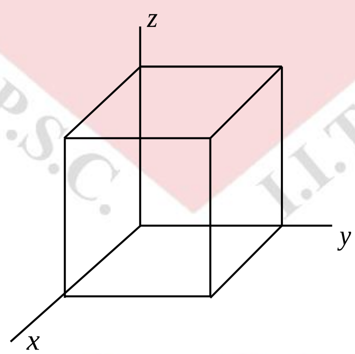


$$= \iint \left[xy + (x+y)(a - x - y) \right] dy\, dx = \int_0^a \int_0^{a-x} \left(ax + ay - x^2 - y^2 - xy \right) dydx$$

$$= \int_0^a axy + \frac{ay^2}{2} - x^2y^2 - \frac{y^3}{3} - \frac{xy^3}{2} \Bigg|_0^{a-x} dx = \int_0^a a^2x - 2ax^2 + x^3 + \frac{1}{6}(a-x)^3\, dx = \frac{1}{8}a^4$$

20. Evaluate the surface integral $\iint \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \cdot \hat{n}\, dS$ where S is the positive side of the cube formed by the plane $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$.

Solution.



S is piece wise smooth surface consisting of
$S_1 : x = 0, S_2 : y = 0, S_3 : z = 0; S_4 : x = 1, S_5 : y = 1, S_6 : z = 1$ as shown in fig.

On $S_1 : x = 0, dS = dydz, \hat{n} = -\hat{i}, \vec{F} \cdot \hat{n} = -x = 0$, $\int_{S_1} \vec{F} \cdot \hat{n}\, dS = 0$

On $S_2 : y=0, dS = dxdz, \hat{n} = -\hat{j}, \vec{F}\cdot\hat{n} = -y = 0$, $\int_{S_2} \vec{F}\cdot\hat{n}\,dS = 0$

On $S_3 : z=0, dS = dxdy, \hat{n} = -\hat{k}, \vec{F}\cdot\hat{n} = -z = 0$, $\int_{S_3} \vec{F}\cdot\hat{n}\,dS = 0$

On $S_4 : x=1, dS = dydz, \hat{n} = \hat{i}, \vec{F}\cdot\hat{n} = x = 1$, $\int_{S_4} \vec{F}\cdot\hat{n}\,dS = \iint dydz = 1$

On $S_5 : y=1, dS = dxdz, \hat{n} = \hat{j}, \vec{F}\cdot\hat{n} = y = 1$, $\int_{S_5} \vec{F}\cdot\hat{n}\,dS = \iint dxdz = 1$

On $S_6 : z=1, dS = dxdy, \hat{n} = \hat{k}, \vec{F}\cdot\hat{n} = z = 1$, $\int_{S_6} \vec{F}\cdot\hat{n}\,dS = \iint dxdy = 1$

So, $\oiint \vec{F}\cdot\hat{n}dS = \int_{S_1} \vec{F}\cdot\hat{n}dS + \int_{S_2} \vec{F}\cdot\hat{n}dS + \int_{S_3} \vec{F}\cdot\hat{n}dS + \int_{S_4} \vec{F}\cdot\hat{n}dS + \int_{S_5} \vec{F}\cdot\hat{n}dS + \int_{S_6} \vec{F}\cdot\hat{n}dS$

21. Evaluate $\int_S (x\cos\alpha + y\cos\beta + z\cos\gamma)dS$ where $\cos\alpha, \cos\beta, \cos\gamma$ are directional cosines of the outward drawn normal to the surfaces where S is the outer surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lying above the xy plane.

Solution.

S is the outer surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lying above the xy plane.

An outward drawn unit normal vector to S is given as

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

$$dS = \frac{dydz}{|\hat{n}\cdot\hat{i}|} = \frac{dydz}{\cos\alpha} \Rightarrow dydz = dS\cos\alpha$$

Similarly, $dxdy = dS\cos\gamma$

$$dxdz = dS\cos\beta$$

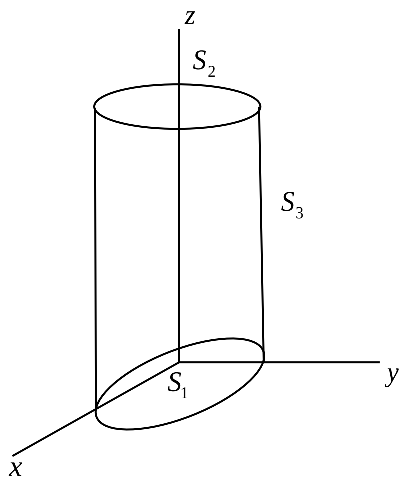
$$I = \int_S (x\cos\alpha + y\cos\beta + z\cos\gamma)dS = \iint xdydz + ydxdz + zdxdy$$

$$\left(\iint xdydz = \iint ydxdz = \iint zdxdy = \text{volume of ellipsoid in the above } xy \text{ plane} = \frac{2\pi}{3}abc\right)$$

So, $\int_S (x\cos\beta + y\cos\beta + z\cos\gamma)dS = 3\times\frac{2\pi}{3}abc = 2\pi\,abc$

22. Evaluate $\int_S (x+y+z)(ax+by+cz)dS$ where S is the surface of region $x^2+y^2\le 1, 0\le z\le 1$.

Solution.

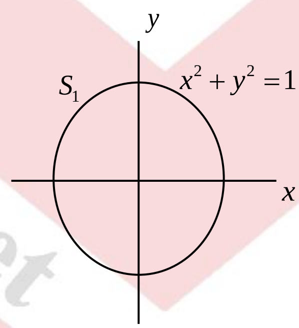


S is the surface bounding the region $x^2+y^2\le 1$ & $0\le z\le 1$

S is a piece wise smooth surface consisting of

S_1 : lower base $z=0$

S_2 : upper base $z=1, S_3$: curved surface of cylinder, $x^2+y^2=1$ as shown in fig.



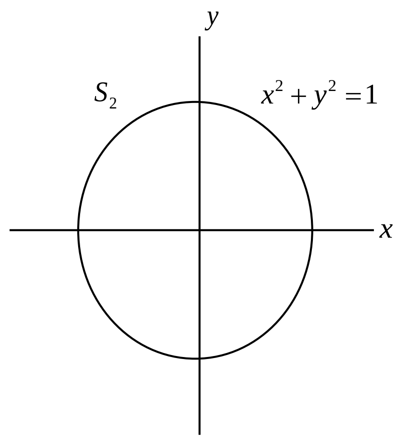
On $S_1: z=0, dS=dxdy$ $\int\limits_{S_1}(x+y+z)(ax+by+cz)\,dS=\iint(x+y)(ax+by)\,dxdy$

$$=\iint\left(ax^2+(a+b)xy+by^2\right)dxdy=\int_0^{2\pi}\int_0^1\left(ar^2\cos^2\theta+(a+b)r^2\sin\theta\cos\theta+br^2\sin^2\theta\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(a\cos^2\theta+(a+b)\sin\theta\cos\theta+b\sin^2\theta\right)\cdot\left.\frac{r^4}{4}\right|_0^1 d\theta$$

$$=\frac{1}{4}\cdot\int_0^{2\pi}\left(a\cos^2\theta+b\sin^2\theta+(a+b)\sin\theta\cos\theta\,d\theta\right)$$

$$=\frac{1}{4}a\int_0^{2\pi}\cos^2\theta d\theta+\frac{b}{4}\cdot\int_0^{2\pi}\cos^2\theta d\theta+\frac{(a+b)}{4}\cdot\int_0^{2\pi}\sin\theta\cos\theta d\theta=(a+b)\frac{\pi}{4}$$



On $S_2 : z=1, dS = dxdy$

$$=\iint (x+y+z)(ax+by+c)\,dxdy$$

$$=\iint (x+y)(ax+by)\,dxdy+\int_{-1}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}}(a+c)x\,dxdy+\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}}(b+c)y\,dydx+c\iint dxdy$$

$$=(a+b)\frac{\pi}{4}+c\pi$$

On $S_3 : x=\cos\theta, y=\sin\theta, dS = d\theta\, dz$,

$$\int_{S_3}(x+y+z)(ax+by+cz)\,dS$$

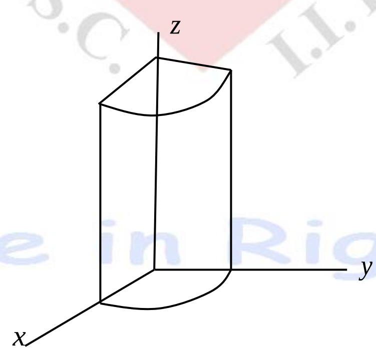
$$=\int_0^1\int_0^{2\pi}(\cos\theta+\sin\theta+z)(a\cos\theta+b\sin\theta+cz)\,d\theta dz=\int_0^1\left((a+b)\pi+2\pi cz^2\right)dz=(a+b)\pi+\frac{2c\pi}{3}$$

$$\int_S (x+y+z)(ax+by+cz)\,dS=\int_{S_1}+\int_{S_2}+\int_{S_3}$$

$$=(a+b)\frac{\pi}{4}+(a+b)\frac{\pi}{4}+c\pi+(a+b)\pi+\frac{2\pi c}{3}=\frac{3}{2}(a+b)\pi+\frac{5c\pi}{3}$$

23. Find the value of surface integral $\iint yzdxdy+xzdydz+xydxdz$ where S is the outer side of the surface formed by the cylinder $x^2+y^2=4$ and the planes $x=0, y=0, z=0$ & $z=2$.

Solution.



S is a piece wise smooth surface bounded by $S_1 : x=0, S_2 : y=0, S_3 : z=0$ & $S_4 : x^2+y^2=4$.

$$\iint yzdxdy + xzdydz + xydxdz = \oiint_S \left(xz\hat{i} + xy\hat{j} + yz\hat{k}\right)\cdot \hat{n}dS = \oiint \vec{F}\cdot\hat{n}dS$$

On $S_1, \hat{n} = -\hat{i}, dS = dydz, x = 0, \vec{F}\cdot\hat{n} = xz = 0$. So, $\int_{S_1} \vec{F}\cdot\hat{n}\,dS = 0$

On $S_2, y = 0, \hat{n} = -\hat{j}, dS = dxdz, \vec{F}\cdot\hat{n} = xy = 0$. So, $\int_{S_2} \vec{F}\cdot\hat{n}\,dS = 0$

On $S_3, z = 0, \hat{n} = -\hat{k}, dS = dxdy, \vec{F}\cdot\hat{n} = yz = 0$. So, $\int_{S_3} \vec{F}\cdot\hat{n}\,dS = 0$

On $S_4, x^2 + y^2 = 4, \hat{n} = \dfrac{x\hat{i} + y\hat{j}}{2}, x = 2\cos\theta, y = 2\sin\theta$

So, $\vec{F}\cdot\hat{n} = \dfrac{x^2z + xy^2}{2} = \dfrac{4z\cos^2\theta + 8\cos\theta\sin^2\theta}{2} = 2z\cos^2\theta + 4\cos\theta\sin^2\theta$

$dS = 2d\theta dz$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,dS = \iint\left(2z\cos^2\theta + 4\cos\theta\sin^2\theta\right)2d\theta dz = 4\int_0^2\int_0^{\pi/2}\left(z\cos^2\theta + 2\cos\theta\sin^2\theta\right)d\theta dz$$

$$= 4\int_0^2\left(2\cdot\frac{\pi}{4} + \frac{2}{3}\right)dz = 4\left[\frac{\pi}{8}z^2 + \frac{2}{3}z\right]_0^2 = 4\left(\frac{\pi}{2} + \frac{4}{3}\right)$$

So, $\oiint_S \vec{F}\cdot\hat{n}\,dS = 4\left(\dfrac{\pi}{2} + \dfrac{4}{3}\right)$

7. Gauss Divergence Theorem

1. Green's Theorem. Let ϕ and ψ are scalar point function which together with their derivatives in any direction are uniform and continuous within the region V bounded by closed surface S then

$$\int_C \left(\phi\nabla^2\phi - \psi\nabla^2\phi\right) d\tau = \oiint_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\hat{n}\, dS$$

Proof: By Gauss Divergence theorem

$$\oiint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau$$

Let $\vec{F} = \phi\nabla\psi - \psi\nabla\phi$

$$\nabla\cdot\vec{F} = \nabla\cdot\left(\phi\nabla\psi\right) - \nabla\cdot\left(\psi\nabla\phi\right) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi^2\psi\nabla^2\phi - \nabla\psi\cdot\nabla\phi = \phi\nabla^2\psi - \psi\nabla^2\phi$$

So, $\oiint_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\hat{n}\, dS = \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\tau$ ....(1)

Since, $\nabla\psi = \frac{\partial\psi}{\partial n}\hat{n}$, $\nabla\phi = \frac{\partial\phi}{\partial n}\hat{n}$

So, (1) can be written as

$$\oiint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS = \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\tau \qquad ....(2)$$

Note: Harmonic function: A scalar function ϕ is said to be harmonic function if it satisfies Laplace's equation $\nabla^2\phi = 0$

If ϕ and ψ both are harmonic, i.e. $\nabla^2\phi = \nabla^2\psi = 0$ equation (2) reduces to

$$\oiint \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS = 0.$$

2. Prove that

$$\int_V \nabla\phi d\tau = \int_S \phi\hat{n}\, dS$$

Proof: Let $\vec{F} = \phi\vec{C}$ where $\vec{C}$ is any arbitrary constant non zero vector

$$\nabla\cdot\vec{F} = \nabla\phi\cdot\vec{C} + \phi\nabla\cdot\vec{C} = \nabla\phi\cdot\vec{C} \quad \left(\text{as} \nabla\cdot\vec{C} = 0\right)$$

Applying Divergence theorem

$$\oiint_S \vec{F}\cdot\hat{n}\, dS = \int_V \nabla\cdot\vec{F}d\tau$$

Where S is bounding surface of V.

$$\oiint \phi\vec{C}\cdot\hat{n}dS = \int \nabla\cdot\left(\phi\vec{C}\right)d\tau \Rightarrow \vec{C}\cdot\oiint \phi\hat{n}dS = \vec{C}\cdot\int\nabla\phi d\tau \Rightarrow \vec{C}\cdot\left[\int\nabla\phi d\tau - \oiint\phi\hat{n}\, dS\right] = 0$$

Since, $\vec{C}\cdot\left[\int\nabla\phi d\tau - \oiint\phi\hat{n}\, dS\right]$ is zero for any arbitrary non-zero vector $\vec{C}$.

So, $\int\nabla\phi d\tau - \oiint\phi\hat{n}\, dS = 0$. Hence, $\int\nabla\phi d\tau = \oiint\phi\hat{n}\, dS$

3. Prove that $\int\nabla\times\vec{g}\, d\tau = \oiint\hat{n}\times\vec{g}\, dS$.

Proof: Let $\vec{F} = \vec{g}\times\vec{C}$ where C is any arbitrary non-zero vector.

$$\nabla\cdot\vec{F} = \nabla\cdot\left(\vec{g}\times\vec{C}\right) = \vec{C}\cdot\text{curl}\,\vec{g} - \vec{g}\cdot\text{curl}\,\vec{C}$$

$= \vec{C} \cdot \text{curl}\, \vec{g} \quad (\because \ \ \text{curl}\, \vec{C} = 0)$

Applying Divergence theorem

$\oiint \vec{F} \cdot \hat{n}\, dS = \int \nabla \cdot \vec{F} d\tau \Rightarrow \oiint \vec{g} \times \vec{C} \cdot \hat{n}\, dS = \int \vec{C} \cdot \text{curl}\, \vec{g}\, d\tau \Rightarrow \oiint (\hat{n} \times \vec{g}) \cdot \vec{C}\, dS = \int \vec{C} \cdot \text{curl}\, \vec{g}\, d\tau$

$\left(\therefore (\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}\right)$

$\Rightarrow \ \vec{C} \cdot \left[\int \text{curl}\, \vec{g}\, d\tau - \oiint \hat{n} \times \vec{g}\, dS \right] = 0$

Since, $\vec{C} \cdot \left[\int \text{curl}\, \vec{g}\, d\tau - \oiint \hat{n} \times \vec{g}\, dS \right]$ is zero for any arbitrary non-zero vector $\vec{C}$,

So, $\int \text{curl}\, \vec{g}\, d\tau - \oiint \hat{n} \times \vec{g}\, dS = 0$. So, $\int_V \nabla \times \vec{g} d\tau = \oiint_S \hat{n} \times \vec{g}\, dS$

Solved Examples

1. Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the inward unit normal vector to S. Then $\oiint_S \vec{F} \cdot \hat{n} dS$ is equal to?

$\oiint \vec{F} \cdot \hat{n} dS = -\oiint \vec{F} \cdot n' dS$

Where $\hat{n}'$ is outward drawn unit normal vector to S i.e. $\hat{n} = -\hat{n}'$

$= -\int \nabla \cdot \vec{F} d\tau$ (Gauss Divergence theorem) $= -6 \times$ volume of sphere (Since, $\nabla \cdot \vec{F} = 6$) $= -8\pi$

2. Let S be a closed surface for which $\iint_S \vec{r} \cdot \hat{n}\, d\sigma = 1$. Then the volume enclosed by the surface is ?

$\oiint \vec{r} \cdot \hat{n} dS = 1 \Rightarrow \ \int \nabla \cdot \vec{r} d\tau = 1$ (Using Gauss Divergence theorem)

$\Rightarrow \ 3\int d\tau = 1$ (Since, $\nabla \cdot \vec{r} = 3$). Volume $V = \int d\tau = \frac{1}{3}$

3. Let $V = \left\{ (x, y, z) \in \mathbf{R}^2, \frac{1}{4} \le x^2 + y^2 + z^2 \le 1 \right\}$ and $\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^2}$ for $(x, y, z) \in V$. Let $\hat{n}$ denote the outward unit normal vector to the boundary of V and S denotes the part $\left\{ (x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = \frac{1}{4} \right\}$ of the boundary of V. Then $\int_S \vec{F} \cdot \hat{n} dS$ is equal to?

Outward unit normal to boundary of V.

$\hat{n} = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{1/2} = -2(x\hat{i} + y\hat{j} + z\hat{k})$

$\int \vec{F} \cdot \hat{n} dS = -2\int \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^2} \cdot (x\hat{i} + y\hat{j} + z\hat{k}) dS = -2\int \frac{1}{x^2 + y^2 + z^2} dS$

$= -8\int dS \quad \left(\text{Since}, x^2 + y^2 + z^2 = \frac{1}{4} \text{ on S} \right) = -8 \times 4\pi \cdot \frac{1}{4} = -8\pi$

4. The value of the integral $\oiint_S \vec{F}\cdot\hat{n}dS$, where $\vec{F}=3x\hat{i}+2y\hat{j}+z\hat{k}$ and S is the closed surface given by the planes $x=0, x=1, y=0, y=2, z=0$ and $z=3$ is ?

By divergence theorem

$$\oiint \vec{F}\cdot\hat{n}dS=\int\nabla\cdot\vec{F}d\tau=6\int_0^1\int_0^2\int_0^3 dx\,dy\,dz=36$$

5. For any closed surface S, the surface integral $\oiint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$ is equal to?

By divergence theorem

$$\oiint \text{curl}\,\vec{F}\cdot\hat{n}dS=\int\text{div}\left(\text{curl}\,\vec{F}\right)d\tau=0 \text{ since, } \nabla\cdot\left(\nabla\times\vec{F}\right)=0$$

6. For any closed surface S, the integral $\oiint_S \vec{r}\cdot\hat{n}\,dS$ is equal to ?

By Divergence theorem

$$\oiint \vec{r}\cdot\hat{n}\,dS=\int\nabla\cdot\vec{r}d\tau=\int 3d\tau \quad \left(\nabla\cdot\vec{r}=3\right)=3\times\text{volume enclosed by surface S}=3V$$

7. If $\vec{F}=ax\hat{i}+by\hat{j}+cz\hat{k}, a,b,c$ are constants, then the integral $\oiint_S \vec{F}\cdot\hat{n}\,dS$, S as a sphere of radius r is equal to?

By Divergence theorem

$$\oiint_S \vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\vec{F}d\tau \text{ Where V is bounding surface of volume S.}$$

Let $\vec{F}=ax\hat{i}+by\hat{j}+cz\hat{k}$

$$\oiint \vec{F}\cdot\hat{n}\,dS=\oiint\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)\cdot\hat{n}dS=\int\nabla\cdot\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)d\tau \text{ (By Gauss divergence theorem)}$$

$$=\left(a+b+c\right)\int d\tau=\left(a+b+c\right)\times\text{volume of sphere of radius } r=\left(a+b+c\right)\frac{4}{3}\pi r^3$$

8. If $\hat{n}$ is the outward drawn unit normal vector to S then the integral $\int_V div\,\hat{n}\,d\tau$ is equal to ?

By Divergence theorem

$$\int\nabla\cdot\vec{F}d\tau=\oiint\vec{F}\cdot\hat{n}dS$$

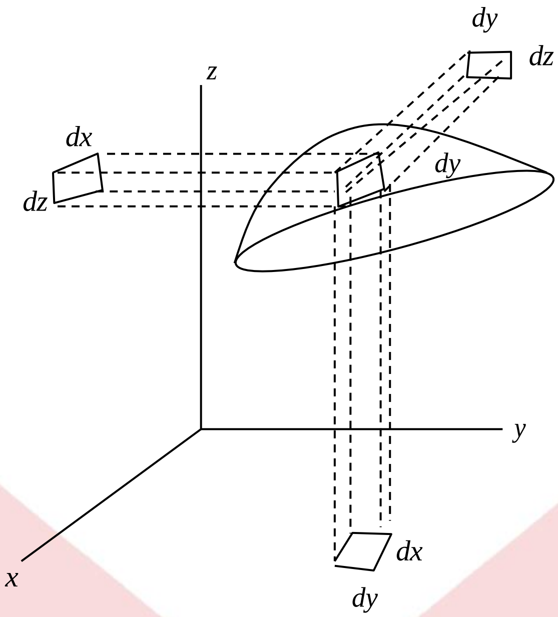
So, $\int_V\nabla\cdot\hat{n}d\tau=\oiint_S\hat{n}\cdot\hat{n}dS=\oiint_S dS=S$.

9. Let S be the surface of the cube bounded by $x=-1, y=-1, z=-1, x=1, y=1, z=1$. The integral $\oiint \vec{r}\cdot\hat{n}\,dS$ is equal to ?

Using Divergence theorem

$$\oiint\vec{r}\cdot\hat{n}\,dS=\int\nabla\cdot\vec{r}d\tau=3\int d\tau=3\int_{-1}^1\int_{-1}^1\int_{-1}^1 dxdydz=3\times 8\int_0^1\int_0^1\int_0^1 dxdydz=24$$

10. S be the surface of sphere $x^2+y^2+z^2=9$. The integral $\iint_S\left[\left(x+z\right)dydz+\left(y+z\right)dzdx+\left(x+y\right)dxdy\right]$ is equal to?



The surface element

$\hat{n}dS = dydz\,\hat{i} + dxdz\,\hat{j} + dxdy\,\hat{k}$

So, $\iint\left[(x+z)\,dydz + (y+z)\,dzdx + (x+y)\,dxdy\right]$

$= \iint\left[(x+z)\hat{i} + (y+z)\,\hat{j} + (x+y)\,\hat{k}\right]\cdot\left[dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}\right]$

$= \oint\!\!\oint (x+z)\hat{i} + (y+z)\,\hat{j} + (x+y)\hat{k}\cdot\hat{n}\,dS$

$= \int\nabla\cdot\left((x+z)\hat{i} + (y+z)\,\hat{j} + (x+y)\hat{k}\right)d\tau = \int 2d\tau = 2\times\frac{4}{3}\cdot\pi(3)^3 = 72\pi$

11. For any closed surface S, the integral $\oint\!\!\oint \hat{n}\,dS$ is equal to

Let $\vec{C}$ be any arbitrary constant non zero vector

By using divergence theorem $\oint\!\!\oint_S \vec{C}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{C}d\tau = 0 \Rightarrow \ \vec{C}\cdot\oint\!\!\oint \hat{n}dS = 0$

Since, $\vec{C}\cdot\oint\!\!\oint \hat{n}dS$ is zero for any arbitrary vector $\vec{C}$ Hence, $\oint\!\!\oint \hat{n}dS = 0$

12. For any closed surface S, the integral $\oint\!\!\oint \vec{r}\times\hat{n}\,dS$ is equal to ?

Let $\vec{C}$ be any arbitrary constant vector

The integral $\vec{C}\cdot\oint\!\!\oint \vec{r}\times\hat{n}\,dS = \oint\!\!\oint \vec{C}\cdot(\vec{r}\times\hat{n})\,dS = \oint\!\!\oint (\vec{C}\times\vec{r})\cdot\hat{n}\,dS = \int\nabla\cdot(\vec{C}\times\vec{r})\,d\tau$ (using divergence theorem)

$= \int\left(\vec{r}\cdot\nabla\times\vec{C} - \vec{C}\cdot\nabla\times\vec{r}\right)d\tau = 0$ (Since, $\nabla\times\vec{C} = 0$ & $\nabla\times\vec{r} = 0$)

Since, $\vec{C}\cdot\oint\!\!\oint \vec{r}\times\hat{n}\,dS$ is zero for any arbitrary vector $\vec{C}$. So, $\oint\!\!\oint \vec{r}\times\hat{n}dS = 0$

13. For any closed surface S, the integral $\oint\!\!\oint \nabla\phi\times\hat{n}\,dS$ is equal to ?

Let $\vec{C}$ be any arbitrary constant vector

The integral $\vec{C}\cdot\oint\!\!\oint \nabla\phi\times\hat{n}\,dS = \oint\!\!\oint \vec{C}\cdot\nabla\phi\times\hat{n}\,dS = \oint\!\!\oint_S (\vec{C}\times\nabla\phi)\cdot\hat{n}dS$

$= \int_V \nabla \cdot \left(\vec{C} \times \nabla \phi \right) d\tau$ (Using divergence theorem) $= \int_V \left(\nabla \phi \cdot \text{curl}\, \vec{C} - \vec{C} \cdot \text{curl}\, \nabla \phi \right) d\tau = 0$ (as Curl of gradient = 0)

Since, $\vec{C} \cdot \oint\!\!\oint \nabla \phi \times \hat{n} dS$ is zero for any arbitrary vector $\vec{C}$ ..Hence, $\oint\!\!\oint \nabla \phi \times \hat{n} dS = 0$

14. Let $\vec{a}$ be a constant vector and V is the volume enclosed by the closed surface S. The integral $\oint\!\!\oint \hat{n} \times \left(\vec{a} \times \vec{r} \right) dS$ is equal to ?

Using $\oint\!\!\oint \hat{n} \times \vec{F} dS = \int \nabla \times \vec{F} d\tau$. Putting $\vec{F} = \vec{a} \times \vec{r}$

$$\oint\!\!\oint_S \hat{n} \times \left(\vec{a} \times \vec{r} \right) dS = \int_V \nabla \times \left(\vec{a} \times \vec{r} \right) d\tau \qquad \text{....(1)}$$

$$\nabla \times \left(\vec{a} \times \vec{r} \right) = \sum \hat{i} \times \frac{\partial}{\partial x} \left(\vec{a} \times \vec{r} \right) = \sum \hat{i} \times \left(\vec{a} \times \frac{\partial \vec{r}}{\partial x} \right) = \sum \hat{i} \times \left(\vec{a} \times \hat{i} \right) = \sum \left(\hat{i} \cdot \hat{i} \right) \vec{a} - \left(\hat{i} \cdot \vec{a} \right) \hat{i}$$

$= 3\vec{a} - \vec{a} = 2\vec{a}$

Equation (1) reduces to

$$\oint\!\!\oint_S \hat{n} \times \left(\vec{a} \times \vec{r} \right) dS = \int_V 2\vec{a} d\tau = 2\vec{a} \int_V d\tau = 2V\vec{a}$$

15. If ϕ is harmonic in V then for any closed surface bonding V, the initial $\oint\!\!\oint \frac{\partial \phi}{\partial n} dS$ is equal to ?

$$\oint\!\!\oint_S \frac{\partial \phi}{\partial n} dS = \oint\!\!\oint_S \frac{\partial \phi}{\partial n} \hat{n} \cdot \hat{n} dS = \oint\!\!\oint_S \nabla \phi \cdot \hat{n} dS = \int_V \nabla \cdot \left(\nabla \phi \right) d\tau$$ (Using Gauss divergence theorem)

$= \int \nabla^2 \phi d\tau = 0$ (as ϕ is harmonic i.e. $\nabla^2 \phi = 0$)

16. Let vector $\vec{B}$ is always normal to a given closed surface S. For a region V bounded by S, the integral $\int_V \nabla \times \vec{B} d\tau$ is equal to?

We have $\int \nabla \times \vec{B} d\tau = \oint\!\!\oint \hat{n} \times \vec{B} dS$

Since, $\vec{B}$ is parallel to normal $\hat{n}$

So, $\hat{n} \times \vec{B} = 0$. So, $\int \nabla \times \vec{B} d\tau = \oint\!\!\oint \hat{n} \times \vec{B} dS = 0$

17. $\vec{F} = \left(2x + 5z \right) \hat{i} - \left(x^2 z + y \right) \hat{j} + \left(y^2 + 2z \right) \hat{k}$ the value of integral $\oint\!\!\oint_S \vec{F} \cdot \hat{n} dS$ where S the surface of sphere having centre at $(2,3,1)$ and radius a is equal to ?

By Gauss Divergence theorem

$$\oint\!\!\oint \vec{F} \cdot \hat{n}\, dS = \int \nabla \cdot \vec{F} d\tau = \iiint \nabla \cdot \left(\left(2x + 5z \right) \hat{i} - \left(x^2 z + y \right) \hat{j} + \left(y^2 + 2z \right) \hat{k} \right) dxdydz$$

$= 3 \iiint dxdydz = 3 \times$ volume of sphere of radius a $= 3 \times \frac{4}{3} \pi a^3 = 4\pi a^3$

18. If S be any closed surface enclosing a volume V and $\vec{F} = 2x\hat{i} + 3y\hat{j} + 7z\hat{k}$. Then, the value of surface integral $\oint\!\!\oint \vec{F} \cdot \hat{n}\, dS$ is equal to ?

Using Gauss Divergence theorem

$$\oint\!\!\oint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} d\tau = 12 \int_V d\tau = 12V$$

19. If S is the surface of sphere $(x-1)^2+(y-2)^2+(z-3)^2=1$ endosing volume V, and $\vec{F}=x\hat{i}+y\hat{j}+2z\hat{k}$ then the value of integral $\oiint \vec{F}\cdot\hat{n}\,dS$ is equal to ?

V is volume enclosed by sphere S, given by

$(x-1)^2+(y-2)^2+(z-3)^2=1$

$$\oiint_S \vec{F}\cdot\hat{n}dS=\int_V \nabla\cdot\vec{F}d\tau=4\int d\tau=4\times \text{ volume of sphere of radius } 1=4\times\frac{4}{3}\pi=\frac{16}{3}\pi$$

<u>Assignment</u>

1. Prove that

$\oiint_S \phi\vec{A}\cdot\hat{n}\,dS=\int_V \nabla\phi\cdot\vec{A}d\tau+\int_V \phi\nabla\cdot\vec{A}d\tau$**, where V is volume of region enclosed by closed surface S.**

Solution.

By Gauss Divergence theorem $\oiint_S \vec{F}\cdot\hat{n}dS=\int_V \nabla\cdot\vec{F}d\tau$ Let $\vec{F}=\phi\vec{A}$

$\nabla\cdot\vec{F}=\nabla\cdot\left(\phi\vec{A}\right)=\nabla\phi\cdot\vec{A}+\phi\nabla\cdot\vec{A}$

So, $\oiint_S \phi\vec{A}\cdot\hat{n}dS=\int_V \nabla\cdot\left(\phi\vec{A}\right)d\tau=\int_V\left(\nabla\phi\cdot\vec{A}+\phi\nabla\cdot\vec{A}\right)=\int_V \nabla\phi\cdot\vec{A}d\tau+\int_V \phi\nabla\cdot\vec{A}d\tau$

2. If $\vec{F}=\nabla\phi$ and $\nabla^2\phi=0$, show that for a closed surface $\oiint \phi\vec{F}\cdot\hat{n}dS=\int F^2 d\tau$.

Solution.

Using Gauss divergence theorem

$\oiint_S\left(\phi\vec{F}\cdot\hat{n}\,dS\right)=\int_V \nabla\cdot\left(\phi\vec{F}\right)d\tau=\int_V\left(\nabla\phi\cdot\vec{F}+\phi\nabla\cdot\vec{F}\right)d\tau=\int_V\left(\nabla\phi\cdot\nabla\phi+\phi\nabla\cdot\nabla\phi\right)d\tau$

$=\int_V\left(\nabla\phi\right)^2 d\tau+\int_V \phi\nabla^2\phi d\tau=\int_V F^2 d\tau+0=\int_V F^2 d\tau$

3. If ϕ is harmonic in V, then $\oiint_S \phi\frac{\partial\phi}{\partial n}dS=\int_V\left(\nabla\phi\right)^2 d\tau$.

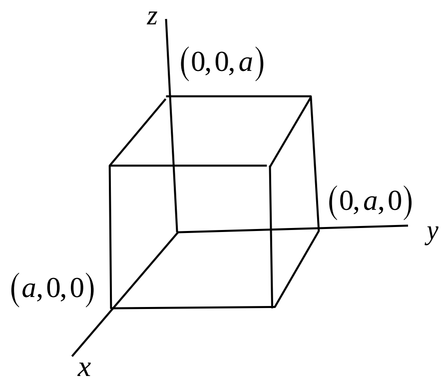
Solution.

$\nabla\phi=\frac{\partial\phi}{\partial n}\hat{n}$, $\oiint \phi\frac{\partial\phi}{\partial n}dS=\oiint \phi\frac{\partial\phi}{\partial n}\hat{n}\cdot\hat{n}dS=\oiint \phi\nabla\cdot\hat{n}\,dS=\int\nabla\cdot\left(\phi\nabla\phi\right)d\tau=\int\left(\nabla\phi\cdot\Delta\phi+\phi\nabla^2\phi\right)d\tau$

$=\int\left(\nabla\phi\right)^2 d\tau \quad \left(\because \nabla^2\phi=0\right)$

4. Verify divergence theorem for $\vec{F}=4xz\hat{i}-y^2\hat{j}+yz\hat{k}$ taken over the cube bounded by $x=0, y=0$, $z=0, x=a, y=a, z=a$.

Solution.



Let us first find the volume integral

$$\int_V \nabla\cdot\vec{F}d\tau=\int_0^a\int_0^a\int_0^a(4z-y)\,dx\,dy\,dz=\int_0^a\int_0^a(4z-y)[x]_0^a\,dy\,dz=a\int_0^a 4yz-\frac{y^2}{2}\bigg|_0^a dz$$

$$=a^2\int_0^a\left(4z-\frac{a}{2}\right)dz=a^2\left[2z^2-\frac{az}{2}\right]_0^a=\frac{3}{2}a^4$$

The region V is bounded by S. S is a piecewise smooth surface consisting of $S_1(x=0),S_2(x=a)$, $S_3(y=0),S_4(y=a),S_5(z=0),S_6(z=a)$

$$\iint\vec{F}\cdot\hat{n}dS=\int_{S_1}\vec{F}\cdot\hat{n}\,dS+\int_{S_2}\vec{F}\cdot\hat{n}\,dS+\int_{S_3}\vec{F}\cdot\hat{n}\,dS+\int_{S_4}\vec{F}\cdot\hat{n}dS+\int_{S_5}\vec{F}\cdot\hat{n}dS+\int_{S_6}\vec{F}\cdot\hat{n}dS \quad ....(1)$$

On $S_1, x=0,\hat{n}=-\hat{i},\vec{F}\cdot\hat{n}=0,dS=dydz$. So, $\int_{S_1}\vec{F}\cdot\hat{n}dS=0$

On $S_2, x=a,\hat{n}=\hat{i},\vec{F}\cdot\hat{n}=4az,dS=dydz$

So,

$$=\int_0^a 4az[y]_0^a\,dz=4a^2\int_0^a zdz=4a^4$$

On $S_3, y=0,\hat{n}=-\hat{j},dS=dxdz,\vec{F}\cdot\hat{n}=0$. So, $\int_{S_3}\vec{F}\cdot\hat{n}dS=0$

On $S_4, y=a,\hat{n}=\hat{j},\vec{F}\cdot\hat{n}=-a^2,dS=dxdz$. So, $\int_{S_4}\vec{F}\cdot\hat{n}dS=-\int_0^a\int_0^a a^2dxdz=-a^4$

On $S_5, z=0,\hat{n}=-\hat{k},\vec{F}\cdot\hat{n}=0,dS=dxdy$..So, $\int_{S_5}\vec{F}\cdot\hat{n}dS=0$

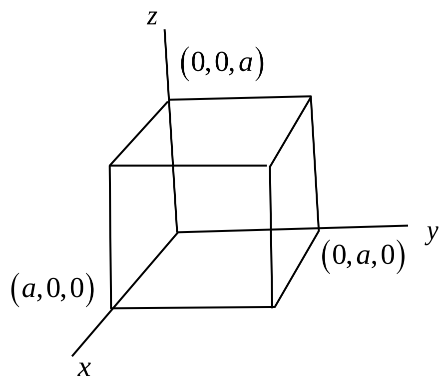
On $S_6, z=a,\hat{n}=\hat{k},\vec{F}\cdot\hat{n}=ay,dS=dxdy$. $\int_{S_6}\vec{F}\cdot\hat{n}dS=\int_0^a\int_0^a ay\,dxdy=\frac{a^4}{2}$

From (1)

$\iint\vec{F}\cdot\hat{n}dS=0+2a^4+0-a^4+0+\frac{a^4}{2}=\frac{3a^4}{2}$, Hence, $\iint_S\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\vec{F}d\tau$

5. Verify divergence theorem for $\vec{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ **taken over the rectangular parallelepiped** $0\le x\le a,0\le y\le a,0\le z\le a$ **.**

Solution.



Let us first calculate the volume integral

$\vec{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}, \nabla\cdot\vec{F} = 2\left(x+y+z\right)$

The volume integral $\int_V \nabla\cdot\vec{F}d\tau = 2\int_0^a\int_0^a\int_0^a \left(x+y+z\right)dxdydz = 2\int_0^a\int_0^a \left[\frac{x^2}{2} + x\left(y+z\right)\right]_0^a dydz$

$= 2a\int_0^a\int_0^a \left(\frac{a}{2} + \left(y+z\right)\right)dydz = 2a\int_0^a \frac{ay}{2} + \frac{y^2}{2} + zy\Big|_0^a dz$

$= 2a^2\int_0^a \left(\frac{a}{2} + \frac{a}{2} + z\right)dz = 2a^2\left[az + \frac{z^2}{2}\right]_0^a = 3a^4$

The surface S enclosing volume V consists of six pieces of smooth surfaces, $S_1\left(x=0\right), S_2\left(x=a\right), S_3$ $\left(y=0\right), S_4\left(y=a\right), S_5\left(z=0\right), S_6\left(z=a\right)$.

$\oint_S \vec{F}\cdot\hat{n}\,dS = \int_{S_1}\vec{F}\cdot\hat{n}\,dS + \int_{S_2}\vec{F}\cdot\hat{n}\,dS + \int_{S_3}\vec{F}\cdot\hat{n}\,dS + \int_{S_4}\vec{F}\cdot\hat{n}\,dS + \int_{S_5}\vec{F}\cdot\hat{n}\,dS + \int_{S_6}\vec{F}\cdot\hat{n}\,dS$

On $S_1, x=0, \hat{n} = -\hat{i}, dS = dydz, \vec{F}\cdot\hat{n} = yz$,, $\int_{S_1}\vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a yzdydz = \frac{a^4}{4}$

On $S_2, x=a, \hat{n} = \hat{i}, dS = dydz, \vec{F}\cdot\hat{n} = \left(a^2 - yz\right)$, $\int_{S_2}\vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a \left(a^2 - yz\right)dydz$

$= \int_0^a\int_0^a a^2dydz - \int_0^a\int_0^a yzdydz = a^4 - \frac{a^4}{4} = \frac{3}{4}a^4$

On $S_3, y=0, \hat{n} = -\hat{j}, dS = dxdz, \vec{F}\cdot\hat{n} = zx$, $\int_{S_3}\vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a zxdxdz = \frac{a^4}{4}$

On $S_4, y=a, \hat{n} = \hat{j}, dS = dxdz, \vec{F}\cdot\hat{n} = \left(a^2 - zx\right)$, $\int_{S_4}\vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a \left(a^2 - zx\right) = \frac{a^4}{4} = \frac{3}{4}a^4$

On $S_5, z=0, \hat{n} = -\hat{k}, dS = dxdy, \vec{F}\cdot\hat{n} = xy$, $\int_{S_5}\vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a xy\,dxdy = \frac{a^4}{4}$

On $S_6, z=a, \hat{n} = \hat{k}, dS = dxdy, \vec{F}\cdot\hat{n} = a^2 - xy$, $\int_{S_6}\vec{F}\cdot\hat{n}\,dS = \int_0^a\int_0^a \left(a^2 - xy\right)dxdy = \frac{3}{4}a^4$

So, $\oiint \vec{F}\cdot\hat{n}\,dS = \frac{a^4}{4}+\frac{3a^4}{4}+\frac{a^4}{4}+\frac{3a^4}{4}+\frac{a^4}{4}+\frac{3a^4}{4} = 3a^{-1}$. Hence, $\oiint \vec{F}\cdot\hat{n}\,dS = \int \nabla\cdot F\cdot d\tau$

6. Evaluate

$\iint x^2dydz + y^2dzdx + 2z(xy-x-y)dxdy$

where S is the surface of the cube

$0\le x\le a, 0\le y\le a, 0\le z\le a$

Solution.

$\hat{n}dS = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$x^2dydz + y^2dzdx + 2z(xy-x-y)dxdy = \left(x^2\hat{i}+y^2\hat{j}+2z(xy-x-y)\hat{k}\right)\cdot\hat{n}dS$

So, $\iint x^2dydz + y^2dzdx + 2z(xy-x-y)dxdy = \int_S \left(x^2\hat{i}+y^2\hat{j}+2z(xy-x-y)\hat{k}\right)\cdot\hat{n}\,dS$

$= \int \nabla\cdot\left(x^2\hat{i}+y^2\hat{j}+2z(xy-x-y)\hat{k}\right)d\tau$ (By Gauss Divergence theorem)

$= 2\int_0^a\int_0^a xydxdy = \frac{a^2}{2}$

7. Use divergence theorem to evaluate $\iint_S x^3dydz + x^2ydzdx + x^2zdxdy$ **where S is the sphere** $x^2+y^2+z^2=1$.

Solution.

$I\iint x^3dydz + x^2ydzdx + x^2zdxdy = \oiint\left(x^3\hat{i}+x^2y\hat{j}+x^2z\hat{k}\right)\cdot\hat{n}dS$

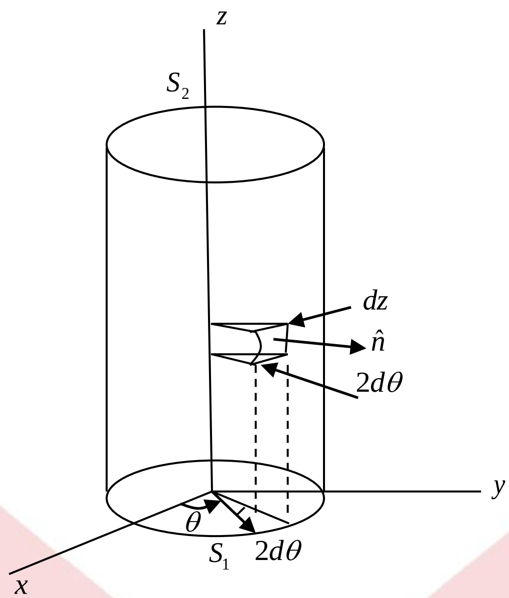
$= \int\nabla\cdot\left(x^3\hat{i}+x^2y\hat{j}+x^2z\hat{k}\right)d\tau$ (By Gauss Divergence theorem) $= 5\iiint x^2dxdydz$

$= 5\iiint r^2\sin^2\theta\cos^2\phi\cdot r^2\sin\theta drd\theta d\phi = 5\int_0^1\int_0^\pi\int_0^{2\pi} r^4\sin^3\theta\cos^2\phi\cdot d\phi d\theta dr = 5\pi\int_0^1\int_0^\pi r^4\sin^3\theta d\theta dr$

$= 10\pi\int_0^1\int_0^{\pi/2} r^4\sin^3\theta d\theta dr = 10\pi\cdot\frac{2}{3}\int_0^1 r^4dr\left[\int_0^{\pi/2}\sin^3\theta d\theta = \frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\Gamma\frac{5}{2}} = \frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} = \frac{2}{3}\right] = \frac{4}{3}\pi$

8. Verify the divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ **taken over the region bounded by** $x^2+y^2=4, z=0$ **and** $z=3$.

Solution.



Let us first calculate the volume integral

$\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, $\nabla\cdot\vec{F} = (4-4y+2z)$

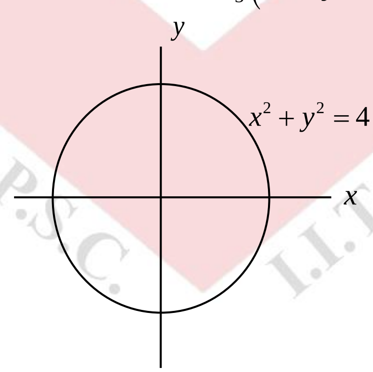
$$\int_0 \nabla\cdot\vec{F}d\tau = \iint\int_0^3 (4-4y+2z)dzdydx = \iint\left[(4-4y)z + z^2\right]_0^3 dydx = \iint(21-12y)dydx$$

The region of double integral is shown in Figure 7.5

$$\iint(21-12y)dydx = 21\iint dydx - 12\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} ydydx = 21\iint dydx - 0 \qquad \left(\because \int_{-a}^{a} f(x)dx = 0 \text{ if } f \text{ in odd}\right)$$

$= 84\pi$

This volume V is bounded by the surface S which is a piecewise smooth surface consisting of lower base $S_1(z=0)$, upper base $S_2(z=3)$ and curved surface $S_3\left(x^2+y^2=4\right)$.



On $S_1, z=0, dS = dxdy, \hat{n} = -\hat{k}, \vec{F}\cdot\hat{n} = 0$, $\int_{S_1}\vec{F}\cdot\hat{n}dS = 0$

On $S_2, z=3, dS = dxdy, \hat{n} = \hat{k}, \vec{F}\cdot\hat{n} = z^2 = 9$, $\int_{S_2}\vec{F}\cdot\hat{n}dS = 9\int_{S_2} dS = 9\times$ Area of circle of radius 2

$= 36\pi$

On $S_3 x = 2\cos\theta, y = 2\sin\theta$

Equation of S_3 belongs to family of level surface $S: x^2+y^2$ =constant

An outward drawn unit normal vector $\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j}}{2}$, $\vec{F} \cdot \hat{n} = \left(2x^2 - y^3\right)$

$$= 8\cos^2\theta - \sin^3\theta$$
$$dS = 2d\theta dz$$

$$\int_{\pi_3} \vec{F} \cdot \hat{n} dS = 16\int_0^{2\pi}\int_0^3 \left(\cos^2\theta - \sin^3\theta\right) dz d\theta = 48\int_0^{2\pi}\left(\cos^2\theta - \sin^3\theta\right) d\theta$$

$$= 48\int_0^{2\pi}\cos^2\theta d\theta - 48\int_0^{2\pi}\sin^3\theta d\theta \quad \left(\int_0^{2\pi}\sin^3\theta d\theta = 0\right) = 48\pi$$

The surface integral over S

$$\iint_S \vec{F} \cdot \hat{n} dS = \int_{S_1} \vec{F} \cdot \hat{n} dS + \int_{S_2} \vec{F} \cdot \hat{n} dS + \int_{S_3} \vec{F} \cdot \hat{n} dS = 0 + 36\pi + 48\pi = 84\pi$$

Hence, $\iint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} d\tau$

9. Using divergence theorem, evaluate $\iint_S \vec{A} \cdot \hat{n} dS$ where $A = x^3\hat{i} + y^3\hat{j} + z^3 k$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution.

Using divergence theorem

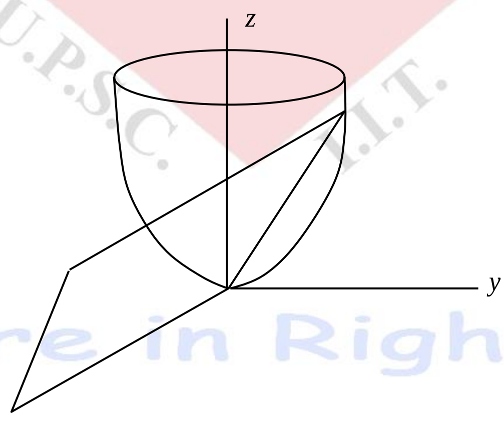
$$\iint_S \vec{A} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{A} d\tau = 3\int\left(x^2 + y^2 + z^2\right) d\tau = 3\int_0^{2\pi}\int_0^{\pi}\int_0^a r^2 r^2 \sin\theta dr d\theta d\phi$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\sin\theta\left[\frac{r^5}{5}\right]_0^a d\theta d\phi = \frac{3}{5}a^5\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = \frac{3}{5}a^5\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= \frac{6}{5}a^5\int_0^{2\pi} d\phi = \frac{12}{5}\pi a^5$$

10. Use divergence theorem to evaluate $\iint \vec{V} \cdot \hat{n} dS$ where $\vec{V} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$ and is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

Solution.

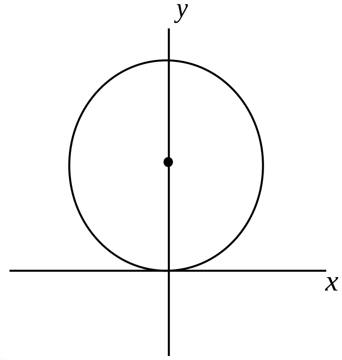


Applying Gauss divergence theorem

$\oiint \vec{V}\cdot\hat{n}dS=\int\nabla\cdot\vec{V}d\tau=\int d\tau$ (This region of volume integration is as shown in Figure) $=\iint\int_{x^2+y^2}^{4y} dzdxdy$

$=\iint\left(4y-x^2-y^2\right)dxdy$

The region of integration of double integration in the projection of region V on xy plane as shown in Figure



$x^2+y^2=4y\Rightarrow x^2+\left(y-2\right)^2=4$

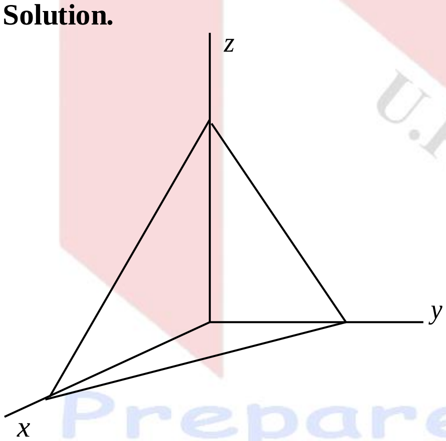
In polar form, $r=4\sin\theta$, $I=\iint\left(4y-x^2-y^2\right)dxdy=\int_0^{\pi}\int_0^{4\sin\theta}\left(4r\sin\theta-r^2\right)rdrd\theta$

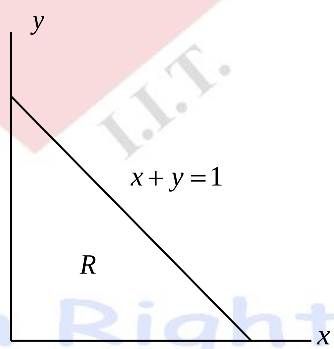
$=\int_0^{\pi}\left[\frac{4}{3}r^3\sin\theta-\frac{r^4}{4}\right]_0^{4\sin\theta}d\theta=\frac{64}{3}\int_0^{\pi}\sin^4\theta d\theta=\frac{128}{3}\int_0^{\pi/2}\sin^4\theta d\theta$

$=\frac{128}{3}\cdot\frac{3\pi}{16}=8\pi$ $\left[\int_0^{\pi/2}\sin^4\theta d\theta=\frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{2\Gamma 3}=\frac{\frac{3}{2}\cdot\frac{1}{2}\pi}{2\times2\times1}\right]$

11. Verify Gauss divergence theorem for $\vec{F}=xy\hat{i}+z^2\hat{j}+2yz\hat{k}$ **on the tetrahedron** $x=y=z=0, x+y+z=1$

Solution.





Let us find volume integral $\int_V\nabla\cdot\vec{F}d\tau$

V is the region bounded by $x=0, y=0, z=0$ and $x+y+z=1$ as shown in Figure

$\vec{F}=xy\hat{i}+z^2\hat{j}+2yz\hat{k}$, $\nabla\cdot\vec{F}=3y$

$$\int \nabla\cdot\vec{F}d\tau = 3\iint\int_0^{1-x-y} ydzdxdy = 3\iint_R y(1-x-y)\,dxdy$$

Where R is the region of double integral obtained by taking projection of V on the xy plane as shown in Figure

$$=3\int_0^1\int_0^{1-x}\left(y(1-x)-y^2\right)dydx = 3\int_0^1\left[(1-x)\frac{y^2}{2}-\frac{y^3}{3}\right]_0^{1-x}dx = \frac{1}{2}\int_0^1(1-x)^3\,dx = -\frac{1}{2}\frac{(1-x)^4}{4}\Bigg|_0^1 = \frac{1}{8}$$

The volume V is bounded by surface S. S is a piecewise smooth surface consisting of $S_1(x=0)$, $S_2(y=0), S_3(z=0), S_4(x+y+z=1)$, On $S_1, x=0, \hat{n}=-\hat{i}, dS=dydz, \vec{F}\cdot\hat{n}=0$

$\int_{S_1}\vec{F}\cdot\hat{n}dS=0$. On $S_2, y=0, dS=dxdz, \hat{n}=-\hat{j}, \vec{F}\cdot\hat{n}=-z^2$, $\int_{S_2}\vec{F}\cdot\hat{n}dS=\int_0^1\int_0^{1-x}z^2dzdx=-\int_0^1\frac{z^3}{3}\Bigg|_0^{(1-x)}dx$

$$=-\frac{1}{3}\int_0^1(1-x)^3\,dx=\frac{1}{12}(1-x)^4\Bigg|_0^1=-\frac{1}{12}$$

On $S_3, z=0, dS=dxdy, \hat{n}=-\hat{k}, \vec{F}\cdot\hat{n}=0$, $\int_{S_3}\vec{F}\cdot\hat{n}dS=0$

On S_4, equation of S_4 belongs to family of level surface given by , $S: x+y+z=$ constant

Outward drawn unit normal to S_4, $\hat{n}=\frac{\nabla S}{|\nabla S|}=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

$$\vec{F}\cdot\hat{n}=\frac{1}{\sqrt{3}}\left(xy+z^2+2yz\right)=\frac{1}{\sqrt{3}}\left(xy+(1-x-y)^2+2y(1-x-y)\right)=\frac{1}{\sqrt{3}}\left(x^2-y^2+xy-2x+1\right)$$

$dS=\frac{dxdy}{|\hat{n}\cdot\hat{k}|}=\sqrt{3}dxdy$. So, $\int_{S_4}\vec{F}\cdot\hat{n}dS=\int_0^1\int_0^{1-x}\left(x^2-y^2+xy-2x+1\right)dydx$

(The region of double integration is given by projection of V on xy plane as shown in Figure)

$$=\int_0^1\left(x^2-2x+1\right)y-\frac{y^3}{3}+\frac{xy^2}{2}\Bigg|_0^{1-x}dx=\int_0^1\left(\frac{2}{3}(1-x)^3+\frac{x(1-x)^2}{2}\right)dx$$

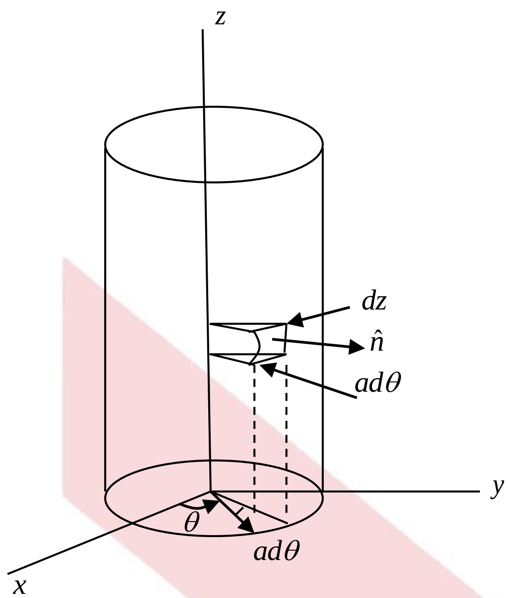
$$=-\frac{1}{6}(1-x)^4\Bigg|_0^1+\frac{1}{2}\left(\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}\right)\Bigg|_0^1=\frac{1}{6}+\frac{1}{2}\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right)=\frac{5}{24}$$

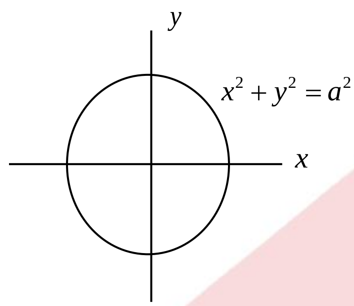
So, $\oiint_S\vec{F}\cdot\hat{n}dS=\int_{S_1}\vec{F}\cdot\hat{n}dS+\int_{S_2}\vec{F}\cdot\hat{n}dS+\int_{S_3}\vec{F}\cdot\hat{n}dS+\int_{S_4}\vec{F}\cdot\hat{n}dS=0+\left(-\frac{1}{12}\right)+0+\frac{5}{24}=\frac{1}{8}$

Hence, $\oiint_S\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\vec{F}d\tau$

12. Using Divergence theorem evaluate $I=\iint x^3dydz+x^2ydzdx+x^2zdxdy$ where S is the closed surface bounded by the planes $z=0, z=b$ and the cylinder $x^2+y^2=a^2$.

Solution.





$$I=\iint x^3dydz+x^2ydzdx+x^2zdxdy=\oiint\left(x^3\hat{i}+x^2y\hat{j}+x^2z\hat{k}\right)\cdot\hat{n}dS$$

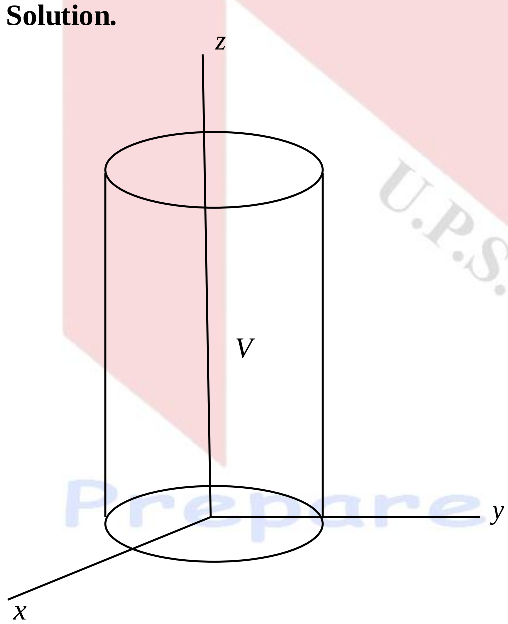
$$=\int\nabla\cdot\left(x^3\hat{i}+x^2y\hat{j}+x^2z\hat{k}\right)d\tau=5\iint\int_0^b x^2dzdxdy=5b\iint_R x^2dxdy$$

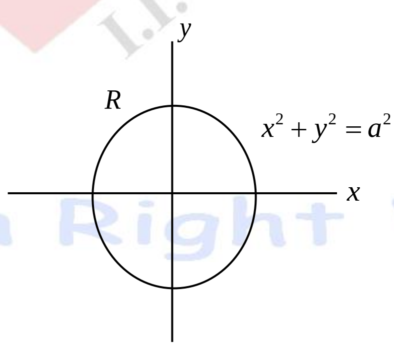
(This region of double integral R is given by projection cylinder on xy plane as shown in Figure)

$$=5b\int_0^{2\pi}\int_0^a r^2\cos^2\theta rdrd\theta=5b\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^a\cos^2\theta d\theta=\frac{5}{4}a^4b\int_0^{2\pi}\cos^2\theta d\theta=\frac{5}{4}\pi a^4b$$

13. If $\vec{F}=x\hat{i}-y\hat{j}+\left(z^2-1\right)\hat{k}$ find the value of $\oiint\vec{F}\cdot\hat{n}dS$ where S in the closed surface bounded by the planes $z=0,z=b$ and the cylinder $x^2+y^2=a^2$.

Solution.





By Gauss Divergence theorem

$\oiint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau$, $\vec{F} = x\hat{i} - y\hat{j} + \left(z^2 - 1\right)\hat{k}$, $\nabla\cdot\vec{F} = 2z$

$\int \nabla\cdot F d\tau = \iint\int_0^b 2zdzdxdy = \iint z^2\Big|_0^b dxdy = b^2\iint_R dxdy$

(The region of integration R is projection of volume region V on xy plane as shown in Figure)

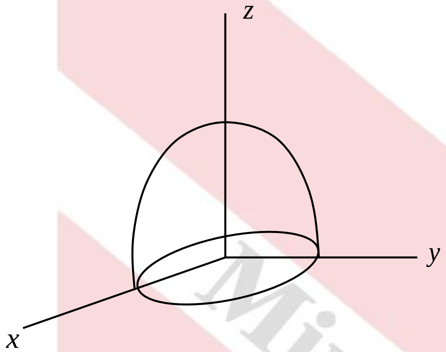
$= b^2 \times$ area of circle of radius $a = \pi a^2 b^2$

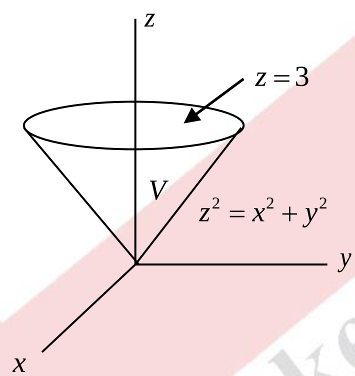
14. Evaluate

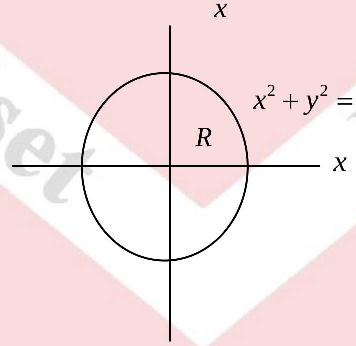
$\oiint\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane bounded by this plane.

Solution.







By divergence theorem

$\oiint\left(y^2z\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot\hat{n}dS = \int\nabla\cdot\left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)\cdot d\tau = \iiint 2zy^2dxdydz$

$= \iiint 2r\cos\theta\cdot r^2\sin^2\theta\sin^2\phi\cdot r^2\sin\theta drd\theta d\phi = 2\int_0^a\int_0^{\pi/2}\int_0^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi d\phi d\theta dr$

$= 2\pi\int_0^a\int_0^{\pi/2} r^5\sin^3\theta\cos\theta dr = 2\pi\int_0^a r^5\frac{\sin^4\theta}{4}\Big|_0^{\pi/2} dr = \frac{\pi}{2}\int_0^a r^5dr = \frac{1}{12}\pi a^6$

15. Evaluate $\oiint\vec{F}\cdot\hat{n}dS$ over the entire surface of the region above the xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 3$ if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$.

Solution.

By Gauss Divergence

$\oiint_S \vec{F}\cdot\hat{n}dS = \int_V \nabla\cdot\vec{F}d\tau = \int_V \nabla\cdot\left(4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}\right)d\tau$

(V is volume enclosed by cone $z^2 = x^2 + y^2$ and the plane $z = 3$ as shown in Figure)

$$= \iint \int_{\sqrt{x^2+y^2}}^{3} \left(4z + xz^2 + 3\right) dzdxdy = \iint_R 2z^2 + x\frac{z^3}{3} + 3z \Big|_{\sqrt{x^2+y^2}}^{3} dxdy$$

(The region of double integration R is projection of volume V on xy plane as shown Figure)

$$\iint \left[2\left(9 - x^2 - y^2\right) + \frac{x}{3}\left(27 - \left(x^2 + y^2\right)^{3/2}\right) + 3\left(3 - \sqrt{x^2 + y^2}\right) \right] dx\, dy$$

$$= \int_0^3 \int_0^{2\pi} \left[\left(27 - 2r^2 - 3r\right) + \frac{1}{3} r \cos\theta \left(27 - r^3\right) \right] r d\theta\, dr = 2\pi \int_0^3 \left(27r - 2r^3 - 3r^2\right) dr$$

$$= 2\pi \left[\frac{27}{2} r^2 - \frac{r^4}{2} - r^3 \right]_0^3 = 108\pi$$

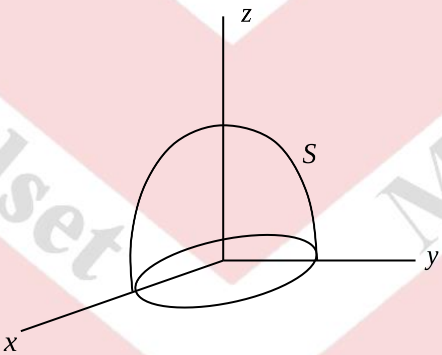
16. Evaluate by divergence theorem the integral

$$\iint_3 xz^2 dydz + \left(x^2 y - z^3\right) dzdx + \left(2xy + y^2 z\right) dxdy$$

Where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$.

Solution.

The surface is shown in Figure



$$\hat{n}\, dS = dydz\hat{i} + dxdz\hat{j} + dxdy\hat{k}$$

$$= \iint_S xz^2 dydz + \left(x^2 y - z^3\right) dzdx + \left(2xy + y^2 z\right) dxdy = \oiint_S \left(xz^2 \hat{i} + \left(x^2 y - z^3\right) \hat{j} + \left(2xy + y^2 z\right) \hat{k} \right) \cdot \hat{n} dS$$

S is the surface of hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ as shown in Figure .

$$\int_V \nabla \cdot \left(xz^2 \hat{i} + \left(x^2 y - z^3\right) \hat{j} + \left(2xy + y^2 z\right) \hat{k} \right) d\tau$$

(By Gauss Divergence theorem $\oiint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot F d\tau$)

$$= \iiint \left(z^2 + x^2 + y^2\right) dxdydz = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^2 \cdot r^2 \sin\theta drd\theta d\phi = \int_0^{2\pi} \int_0^{\pi/2} \frac{r^5}{5} \Big|_0^a \sin\theta d\theta d\phi$$

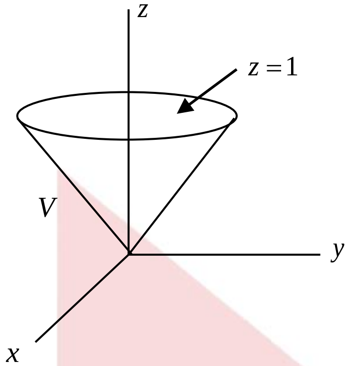
$$= \frac{a^5}{5} \int_0^{2\pi} \int_0^{\pi/2} \sin\theta d\theta d\phi = \frac{a^5}{5} \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi/2} d\phi = \frac{a^5}{5} \int_0^{2\pi} d\phi = \frac{2\pi a^5}{5}$$

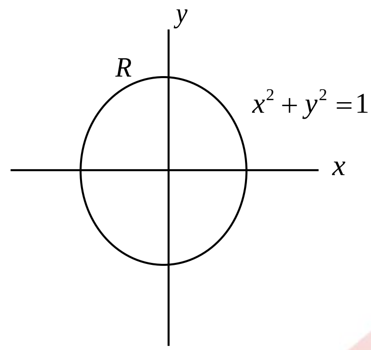
17. By using Gauss Divergence theorem,

Evaluate $\oiint \left(x\hat{i} + y\hat{j} + z^2\hat{k} \right) \cdot \hat{n}\, dS$

where S is the closed surface bounded by cone $x^2+y^2=z^2$ and the plane $z=1$.

Solution.





Using Gauss Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,dS=\int_V \nabla\cdot\vec{F}\,d\tau$$

$$\int_S\left(x\hat{i}+y\hat{j}+z^2\hat{k}\right)\cdot\hat{n}dS=\int_V\nabla\cdot\left(x\hat{i}+y\hat{j}+z^2k\right)d\tau=2\iint\int_{\sqrt{x^2+y^2}}^{1}(z+1)\,dzdxdy$$

(V is volume enclosed by cone $x^2+y^2=z^2$ & the plane $z=1$ as shown in Figure)

$$=2\iint_R \left.\frac{z^2}{2}+z\right|_{\sqrt{x^2+y^2}}^{1} dxdy$$

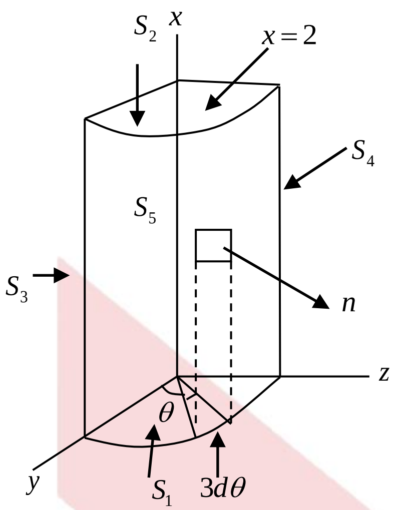
(The region of integration of double integral R is the projection of volume V on xy plane as shown in Figure)

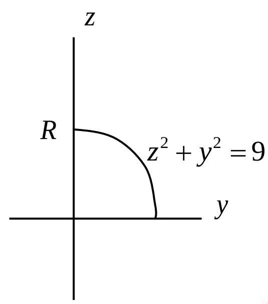
$$=\iint\left(1-x^2-y^2\right)+2\left(1-\sqrt{x^2+y^2}\right)dxdy=\int_0^{2\pi}\int_0^{1}\left(3-2r-r^2\right)rdrd\theta$$

$$=\int_0^{2\pi}\left.\frac{3}{2}r^2-\frac{2r^3}{3}-\frac{r^4}{4}\right|_0^1 d\theta=\frac{7}{12}\int_0^{2\pi}d\theta=\frac{7\pi}{6}$$

18. Verify divergence theorem for $\vec{F}=2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ & $x=2$.

Solution.





Let us first find the volume integral $\int_V \nabla\cdot F d\tau$, V is the volume enclosed by surface $y^2+z^2=9$ & $x=2$ in first octant as shown in Figure .

$\vec{F}=2x^2y\hat{i}-y^2\hat{j}+4xz^2\hat{k}$, $\nabla\cdot\vec{F}=4xy-2y+8z$, $\int_V \nabla\cdot F d\tau=\iint\int_0^2(4xy-2y+8xz)dxdydz$

$$=\iint_R 2x^2y-2xy+4x^2z\Big|_0^2 dydz$$

(R is the projection of V is xy plane as shown in Figure).

$$=4\int_0^3\int_0^{\pi/2}(r\cos\theta+4r\sin\theta)rd\theta dr=4\int_0^3 r^2\left[\sin\theta-4\cos\theta\right]_0^{\pi/2}dr=20\int_0^3 r^2dr=180$$

Now, let us calculate the surface integral over S. S is a piecewise smooth surface consisting of $S_1(x=0), S_2(x=2), S_3(z=0), S_4(y=0), S_5(y^2+z^2=9)$

On $S_1, x=0, dS=dydz, \hat{n}=-\hat{i}, \vec{F}\cdot\hat{n}=0$, So, $\int_{S_1}\vec{F}\cdot\hat{n}dS=0$

On $S_2, x=2, dS=dydz, \hat{n}=\hat{i}, \vec{F}\cdot\hat{n}=8y$, So, $\int_{S_2}\vec{F}\cdot\hat{n}dS=8\iint ydydz=8\int_0^3\int_0^{\pi/2}r\cos\theta rd\theta dr$

$$=8\int_0^3 r^2dr=72$$

On $S_3, z=0, dS=dxdy, \hat{n}=-\hat{k}, \vec{F}\cdot\hat{n}=0$, So, $\int_{S_3}\vec{F}\cdot\hat{n}dS=0$

On $S_4, y^2+z^2=9, dS=3d\theta dx$, $\hat{n}=\dfrac{y\hat{j}+z\hat{k}}{3}, \vec{F}\cdot\hat{n}=\dfrac{1}{3}\left(4xz^3-y^3\right)$

Let $y=3\cos\theta, z=3\sin\theta$, $\vec{F}\cdot\hat{n}=9\left(4x\sin^3\theta-\cos^3\theta\right)$, $\vec{F}\cdot\hat{n}dS=27\left(4x\sin^3\theta-\cos^3\theta\right)d\theta dx$

$$\int_{S_5}\vec{F}\cdot\hat{n}dS=27\int_0^2\int_0^{\pi/2}\left(4x\sin^3\theta-\cos^3\theta\right)d\theta dx$$

$$\left[\int_0^{\pi/2} \sin^3\theta d\theta = \int_0^{\pi/2} \cos^3\theta d\theta = \frac{\Gamma 2\,\Gamma\tfrac{1}{2}}{2\Gamma\tfrac{5}{2}} = \frac{\Gamma 2\,\Gamma\tfrac{1}{2}}{2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\tfrac{1}{2}} = \frac{2}{3}\right] = 18\int_0^2 (4x-1)\,dx = 18\left[2x^2 - x\right]_0^2 = 108$$

So, surface integral $\iint_S \vec{F}\cdot\hat{n}dS$ is give as

$$\iint_S \vec{F}\cdot\hat{n}dS = \int_{S_1}\vec{F}\cdot\hat{n}dS + \int_{S_2}\vec{F}\cdot\hat{n}dS + \int_{S_3}\vec{F}\cdot\hat{n}dS + \int_{S_4}\vec{F}\cdot\hat{n}dS + \int_{S_5}\vec{F}\cdot\hat{n}dS = 0+72+0+0+108$$

$$=180$$

So, $\iint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau$. Hence, Gauss divergence theorem is verified.

19. Evaluate by using Gauss divergence theorem

(i) $\iint_S \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{1/2} dS$

(ii) $\iint_S \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{-1/2} dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$**.**

Solution.

S is the ellipsoid belonging to family to level surface as shown in Figure 7.22.

$S : ax^2 + by^2 + cz^2$ = constant

The outward drawn unit normal vector $\hat{n}$ to S is given by $\hat{n} = \dfrac{ax\hat{i} + by\hat{j} + cz\hat{k}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$

(i) $\iint \left(a^2x^2 + b^2y^2 + c^2z\right)^{1/2} dS = \iint \vec{F}\cdot\hat{n}dS$

Comparing the integrals $\vec{F}\cdot\hat{n} = \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{1/2}$

$$\vec{F}\cdot\frac{\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)}{a^2x^2 + b^2y^2 + c^2z^2} = \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{1/2}$$

$$\vec{F}\cdot\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right) = a^2x^2 + b^2y^2 + c^2z^2$$

For using Gauss Divergence theorem, $\vec{F}$ should continuous and should have continuous partial derivatives in region V enclosed by ellipsoid S. The surface $\vec{F}$ can be taken as

$\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

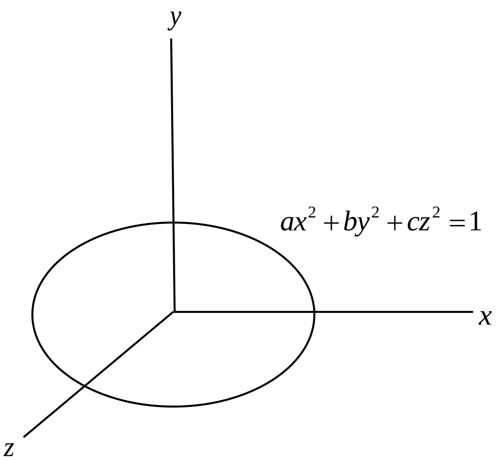
So, $\iint_S \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{1/2} dS = \iint_S \left(ax\hat{i} + by\hat{j} + c^2\hat{k}\right)\cdot\hat{n}dS$

$$= \int_V \nabla\cdot\left(ax\hat{i} + by\hat{j} + cz\hat{k}\right)d\tau$$

According to Gauss Divergence theorem

$$\iint \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau = (a+b+c)\int_V d\tau = (a+b+c)\times \text{volume of ellipsoid} = \frac{4\pi(a+b+c)}{3\sqrt{abc}}$$

(ii) $\iint \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{-1/2} dS = \iint \vec{F}\cdot\hat{n}dS$



Comparing the integral

$$\vec{F}\cdot\hat{n}=\left(a^2x^2+b^2y^2+c^2z^2\right)^{-1/2}$$

$$\Rightarrow \vec{F}\cdot\frac{\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}=\frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \Rightarrow \vec{F}\cdot\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)=1$$

The function $\vec{F}$ can be taken as

$\vec{F}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{F}\cdot\hat{n}=\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)=ax^2+by^2+cz=1$ (on $S, ax^2+by^2+cz^2=1$), $\iint_S\left(ax^2+by^2+cz\right)dS=\iint\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{n}dS$

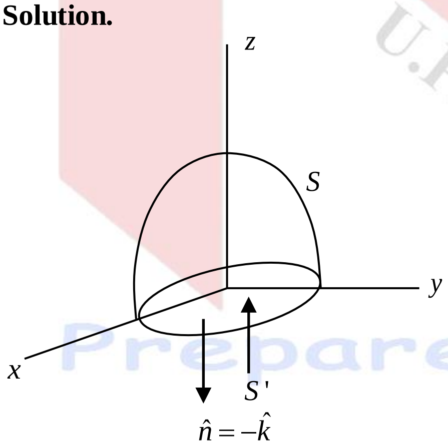
$$=\int\nabla\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)d\tau=3\int d\tau$$

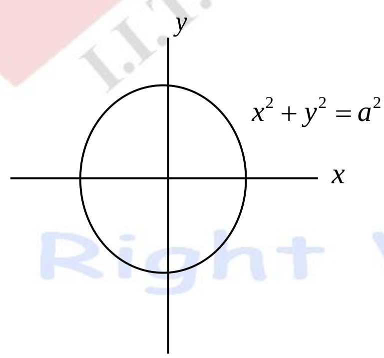
$$=3\times\text{volume of ellipsoid}=\frac{4\pi}{\sqrt{abc}}$$

Note. While evaluating surface integration, we can incorporate the equation of surface.

20. If $\vec{F}=\left(x^2+y-4\right)\hat{i}+3xy\,\hat{j}+\left(2xz+z^2\right)\hat{k}$**. Evaluate** $\int_S\left(\nabla\times\vec{F}\right)\cdot\hat{n}dS$ **where S is the surface of the sphere** $x^2+y^2+z^2=a^2$ **above** xy **plane.**

Solution.





The surface S is sphere $x^2+y^2+z^2=a^2$ above xy plane as shown in Figure 7.23.

So, S is a open surface. But, Gauss theorem applies only to surface integral on closed surface. Had the surface S been closed, the integral $\int_S \nabla\times\vec{F}\cdot\hat{n}dS$ would have been zero because

$$\int_S \nabla\times\vec{F}\cdot\hat{n}dS=\int\nabla\cdot\left(\nabla\times\vec{F}\right)d\tau=0$$

Since, divergence of curl $\vec{F}$ will be zero.

S is open surface. Here we will make use of the fact that $\int\nabla\times\vec{F}\cdot\hat{n}dS$ over the closed surface will be zero.

Now, let us consider a closed surface Σ consisting of hemispherical part $S: x^2+y^2+z^2=a^2$ above xy plane and base of hemisphere $S': z=0$.

We have to find $\int_S \nabla\times\vec{F}\cdot\hat{n}dS$

Now, $\oiint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS=\int\nabla\cdot\left(\nabla\times\vec{F}\right)d\tau=0$

$$\Rightarrow \oiint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS=\int_S \nabla\times\vec{F}\cdot\hat{n}dS+\int_S \nabla\times\vec{F}\cdot\hat{n}dS=0$$

$$\int_S \nabla\times\vec{F}\cdot\hat{n}dS-\int_S \nabla\times\vec{F}\cdot\hat{n}dS$$

Now, it is easier to evaluate the surface integral over the plane surface i.e. over the base of hemisphere $S': z=0$.

On $S', z=0, dS=dxdy, \hat{n}=-\hat{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}$$

$$=-2z\hat{j}+\left(3y-1\right)\hat{k}$$

So, $\nabla\times\vec{F}\cdot\hat{n}=\left(-2z\hat{j}+\left(3y-1\right)\hat{k}\right)\cdot\left(-\hat{k}\right)=-\left(3y-1\right)$

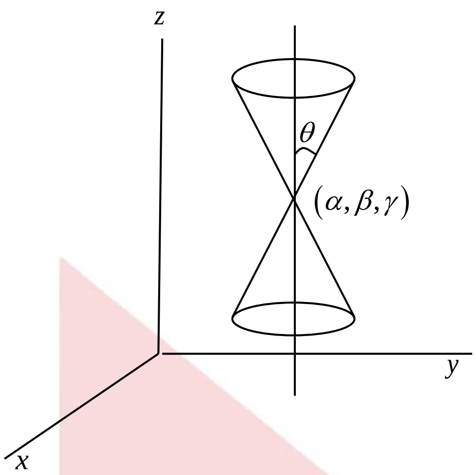
So, $\int_S \nabla\times\vec{F}\cdot\hat{n}dS=-\int_S \nabla\times\vec{F}\cdot\hat{n}dS=\iint\left(3y-1\right)dydx=3\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y\,dydx-\iint dydx$

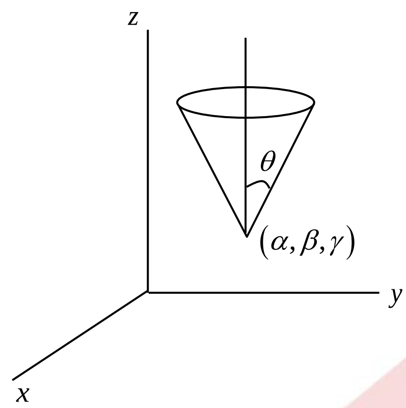
$=0$ - Area of base $=-\pi a^2$

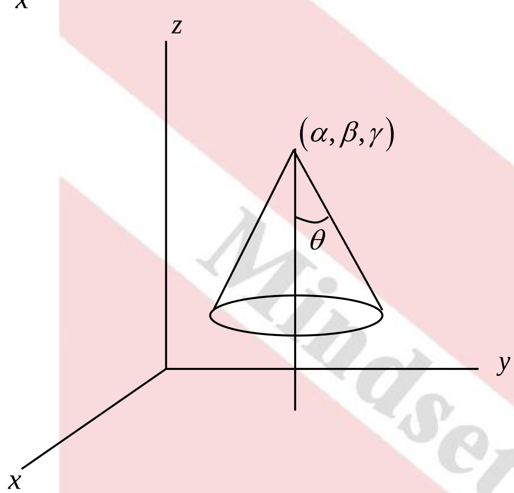
Note: In this problem, we have converted as integral over a curved surface to integral over a plane surface.

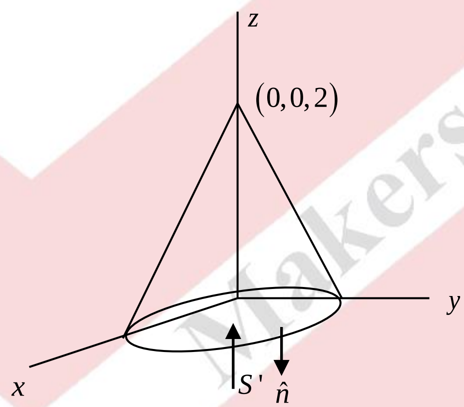
21. Evaluate $\int_S\left(\nabla\times\vec{F}\right)\cdot\hat{n}dS$ where $\vec{F}=\left(x-z\right)\hat{i}+\left(x^3+yz\right)\hat{j}-3xy^2\hat{k}$ and S is the surface of the cone $z=2-\sqrt{x^2+y^2}$ above the xy plane.

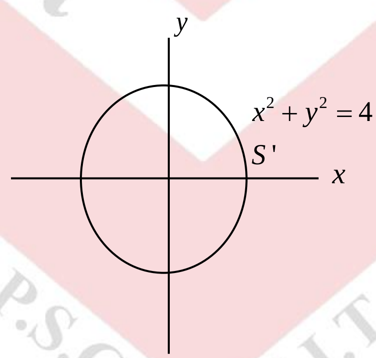
Solution.











The general equation of cone with axis parallel to z axis and vertex at (α, β, γ) with semi vertical angle θ is given by.

$(z-\gamma)\tan^2\theta = (x-\alpha)^2 + (y-\beta)^2$

The cone given by above equation is shown in Figure 7.25.

$(z-\gamma)\tan\theta = +\sqrt{(x-\alpha)^2 + (y-\beta)^2}$ denotes part of cone above the vertex (α, β, γ) as shown in Figure 7.26.

$(z-\gamma)\tan\theta = -\sqrt{(x-\alpha)^2 + (y-\beta)^2}$ denotes the part of cone below the vertex (α, β, γ) as shown in Figure 7.27.

Equation of cone given here is

$z = 2 - \sqrt{x^2 + y^2}$

$(z-2) = -\sqrt{x^2 + y^2}$

Comparing this equation with standard equation of cone

$(z-\gamma)\tan\theta = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$

The vertex is $(0,0,2)$ and semi vertical angle is θ. It represents part of cone below the vertex. Here also, we will make use of the fact that the surface integral $\int \nabla \times \vec{F} \cdot \hat{n} dS$ over the closed surface will be zero as $\oiint \nabla \times \vec{F} \cdot \hat{n} dS = \int \nabla \cdot (\nabla \times \vec{F}) d\tau = 0$

Since $\nabla \cdot (\nabla \times \vec{F}) = 0$

Let us consider a closed piecewise smooth surface Σ consisting of two surface.

S :Part of cone $z = 2 - \sqrt{x^2 + y^2}$ lying above xy plane, S': base of cone, bounded by $x^2 + y^2 = 4, z = 0$
.

The surface integral

$\oiint_\Sigma \nabla \times \vec{F} \cdot \hat{n} dS = \int \nabla \cdot (\nabla \times \vec{F}) d\tau = 0$

$\Rightarrow \oiint_\Sigma \nabla \times \vec{F} \cdot \hat{n} dS = \int_S \nabla \times \vec{F} \cdot \hat{n} dS + \int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS = 0$

$\Rightarrow \int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - z & x^3 + yz & -3xy^2 \end{vmatrix}$$

$= (-6xy - y)\hat{i} + (-1 + 3y^2)\hat{j} + (3x^2)\hat{k}$

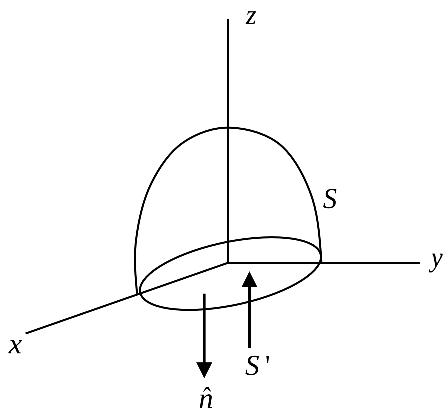
$(\nabla \times \vec{F}) \cdot \hat{n} = -3x^2 \qquad (\hat{n} = -\hat{k})$

So, $\int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS$

$$= 3\iint x^2 dxdy = 3\int_0^{2\pi}\int_0^2 r^2 \cos^2\theta r\, dr\, d\theta = 3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 \cos^2\theta d\theta = 12\int_0^{2\pi} \cos^2\theta d\theta = 12\pi$$

22. If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\int_S \nabla \times \vec{F} \cdot \hat{n} dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

Solution.



Similar to previous problem, here also use will make use of the fact that the surface integral $\int \nabla \times \vec{F} \cdot \hat{n}\, dS$ evaluated over a closed surface is equal to zero because the divergence of function $\nabla \times \vec{F}$ is always zero. So, applying Gauss divergence theorem

$\int \nabla \times \vec{F} \cdot \hat{n} dS$ evaluated over a closed surface is zero. Consider a closed piecewise smooth surface Σ consisting of two surfaces.

(i) S :spherical part $x^2 + y^2 + z^2 = a^2$ above xy plane

(ii) S': base of sphere $x^2 + y^2 + z^2 = a^2$, bounded by circle $x^2 + y^2 = a^2$ in xy plane

Applying Gauss Divergence theorem

$$\oint_{\Sigma} \nabla \times \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \left(\nabla \times \vec{F}\right) d\tau = 0$$

$$\Rightarrow \oint_{\Sigma} \nabla \times \vec{F} \cdot \hat{n} dS = \int_S \nabla \times \vec{F} \cdot \hat{n} dS + \int_S \nabla \times \vec{F} \cdot \hat{n} dS = 0$$

$$\Rightarrow \int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS$$

On $S', dS = dxdy, z = 0, \hat{n} = -\hat{k}$

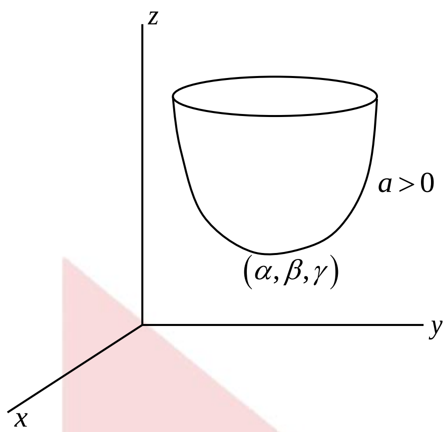
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix} = x\hat{i} + y\hat{j} - 2z\hat{k}$$

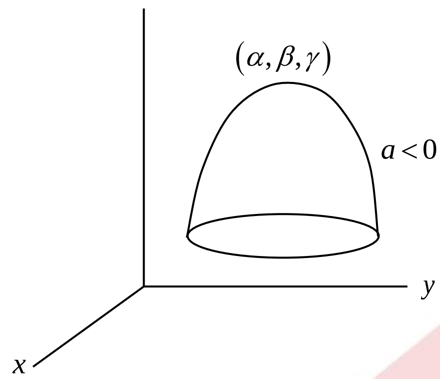
$$\nabla \times \vec{F} \cdot \hat{n} = 2z = 0$$

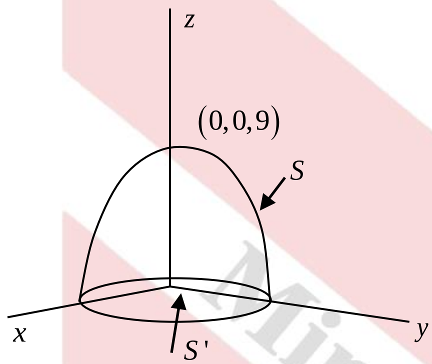
$\int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS = 0$, So, $\int_S \nabla \times \vec{F} \cdot \hat{n} dS = -\int_{S'} \nabla \times \vec{F} \cdot \hat{n} dS = 0$. So, $\int_S \nabla \times \vec{F} \cdot \hat{n} dS = 0$

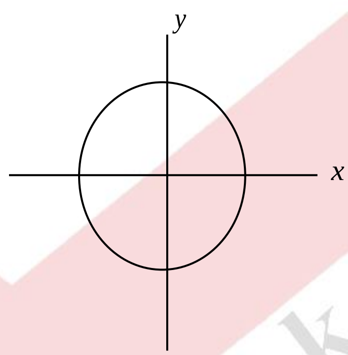
23. Evaluate $\int_S \left(\nabla \times \vec{F}\right) \cdot \hat{n} dS$ where $\vec{F} = \left(x^2 + y - 4\right)\hat{i} + 3xy\hat{j} + \left(2xz + z^2\right)\hat{k}$ and S is the surface of paraboloid with axis parallel to z axis $z = 9 - \left(x^2 + y^2\right)$.

Solution.









The standard equation of parabola is given by

$$(z-\gamma)=a\left[(x-\alpha)^2+(y-\beta)^2\right]$$

where (α,β,γ) is the vertex of paraboloid

Comparing given equation of paraboloid $z=9-\left(x^2+y^2\right)$ (Figure 7.31) with standard equation. The vertex is $(0,0,9)$. Here also, we will make use of fact that the integral $\int \nabla\times\vec{F}\cdot\hat{n}dS$ is equal to zero for a closed surface as shown in Figure.

Consider a closed piecewise smooth surface Σ consisting of paraboloid S and base of paraboloid $x^2+y^2=9$ as shown in Figure 7.34.

Using Gauss Divergence theorem,

$$\oiint_{\Sigma}\nabla\times\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\left(\nabla\times\vec{F}\right)d\tau=0 \Rightarrow \oiint_{\Sigma}\nabla\times\vec{F}\cdot\hat{n}dS=\int_S\nabla\times\vec{F}\cdot\hat{n}dS+\int_S\nabla\times\vec{F}\cdot\hat{n}dS=0$$

So, $\int_S\nabla\times\vec{F}\cdot\hat{n}dS=-\int_{S'}\nabla\times\vec{F}\cdot\hat{n}dS$, On $S',\hat{n}=-\hat{k},dS=dxdy$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}=-2z\hat{j}+(3y-1)\hat{k}$$

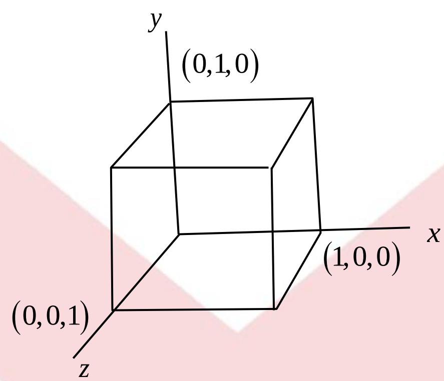
$$\nabla\times\vec{F}\cdot\hat{n}=-(3y-1),\ \int_S\nabla\times\vec{F}\cdot\hat{n}dS=-\int_{S'}\nabla\times\vec{F}\cdot\hat{n}dS=\iint(3y-1)\,dydx$$

$$= 3\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} ydydx - \iint dydx$$

= - Area of base of paraboloid $= -9\pi$

24. Let $\phi(x,y,z) = e^x \sin y$. Evaluate the surface integral $\iint_S \frac{\partial \phi}{\partial n} d\sigma$, where S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ and $\frac{\partial \phi}{\partial n}$ is the directional derivative of ϕ in the direction of the unit outward normal to S. Verify the divergence theorem.

Solution.



$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}$$

$$\nabla \phi = e^x \sin y \hat{i} + e^x \cos y \hat{j}$$

S is a piecewise smooth surface as shown in Figure 7.35 consisting of following surface

$x = 0, \quad x = 1; \; y = 0, \quad y = 1 \; ; z = 0, \quad z = 1$

On $x = 0, \hat{n} = -\hat{i}, dS = dydz$, $\iint \frac{\partial \phi}{\partial n} d\sigma = -\int_0^1\int_0^1 \sin y dydz = (\cos 1 - 1)$

On $x = 1; \hat{n} = \hat{i}; \; dS = dydz$, $\iint \frac{\partial \phi}{\partial n} \cdot d\sigma = \int_0^1\int_0^1 e \sin y \, dydz = -e(\cos 1 - 1)$

On $y = 0, \hat{n} = -\hat{j}, dS = dxdz$, $\iint \frac{\partial \phi}{\partial n} d\sigma = -\iint e^x dxdz = -\int_0^1\int_0^1 e^x dxdz = -(e-1)$

On $y = 1, \hat{n} = \hat{j}, dS = dxdz$, $\iint \frac{\partial \phi}{\partial n} d\sigma = \int_0^1\int_0^1 e^x \cos 1 dxdz = \cos 1 (e-1)$

On $z = 0, \hat{n} = -\hat{k}, dS = dxdy$, $\iint \frac{\partial \phi}{\partial n} d\sigma = 0$

On $z = 1, dS = dxdz \; z = 1, \hat{n} = \hat{k} = dS = dxdy$, $\iint \frac{\partial \phi}{\partial n} d\sigma = 0$

So, $\oiint \frac{\partial \phi}{\partial n} \partial\sigma = (\cos 1 - 1) - e(\cos 1 - 1) - (e-1) + \cos 1 (e-1) = 0$

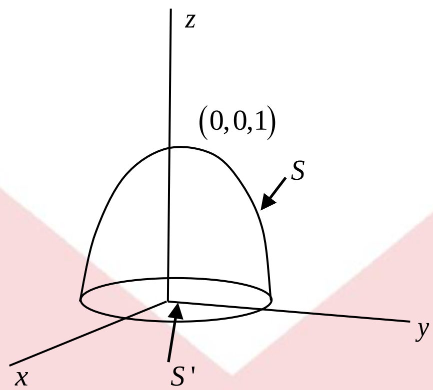
Using Divergence Theorem

$$\oiint_S \nabla\phi\cdot\hat{n}dS = \int_V \nabla^2\phi d\tau = 0$$

Since, $\nabla^2\phi = \nabla\cdot\left(e^x \sin y\hat{i} + e^x \cos y\hat{j}\right) = e^x \sin y - e^x \sin y = 0$.Hence, Gauss Divergence theorem is verified.

25. Let S be the surface $\left\{(x,y,z)\in\mathbf{R}^3 : x^2+y^2+2z=2, z\geq 0\right\}$, and let $\hat{n}$ be the outward unit normal to S. If $\vec{F} = y\hat{i} + xz\hat{j} + \left(x^2+y^2\right)\hat{k}$, then evaluate the integral $\int_S \vec{F}\cdot\hat{n}dS$.

Solution.



$S : x^2 + y^2 = -2(z-1)$

is a paraboloid with vertex at $(0,0,1)$ as shown is Figure 7.36

$\vec{F} = y\hat{i} + xz\hat{j} + \left(x^2+y^2\right)\hat{k}$

$\nabla\cdot\vec{F} = 0$

Consider a closed surface S which consists of two piecewise smooth surface S and S", where S' is base of Paraboloid and S is paraboloid

$$\oiint_\Sigma \vec{F}\cdot\hat{n}dS = \int \nabla\cdot\vec{F}d\tau = 0$$

$$\oiint_\Sigma \vec{F}\cdot\hat{n}dS = \int_S \vec{F}\cdot\hat{n}dS + \int_{S'} \vec{F}\cdot\hat{n}dS = 0$$

$$\int_S \vec{F}\cdot\hat{n}dS = -\int_{S''} \vec{F}\cdot\hat{n}dS$$

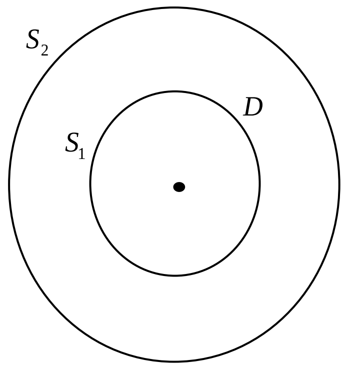
For $S', \hat{n} = -\hat{k}$ $\quad dS = dxdy$

So, $$\int_S \vec{F}\cdot\hat{n}dS = -\int_{S''} \vec{F}\cdot\hat{n}dS = -\iint\left(y\hat{i} + xz\hat{j} + \left(x^2+y^2\right)\hat{k}\right)\cdot\left(-\hat{k}\right)dxdy = \iint\left(x^2+y^2\right)dxdy$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} r^2 r d\theta\, dr = \int_0^{2\pi}\left|\frac{r^4}{4}\right|_0^{\sqrt{2}} d\theta = 2\pi$$

26. Let D be the region bounded by the concentric spheres $S_1 : x^2+y^2+z^2 = a^2$ and $S_2 : x^2+y^2+z^2 = b^2$ where $a<b$. Let $\hat{n}$ be the unit normal to S_1 directed away from the origin. If $\nabla^2\varphi = 0$ in D and $\varphi = 0$ on S_2, then show that $\int_D |\nabla\phi|^2 dV + \int_{S_1}\phi(\nabla\phi)\cdot\hat{n}dS = 0$.

Solution.



Let us consider a surface Σ consisting of S and S' enclosing a volume D as shown is Figure 7.37.
According to Gauss Divergence theorem

$$\iint_{\Sigma} \phi\nabla\phi\cdot\hat{n}dS = \int_D \nabla\cdot(\phi\nabla\phi)dV$$

$$= \int_D \nabla\phi\cdot\nabla\phi dV + \int_D \phi\nabla^2\phi dV$$

$$= \int_D |\nabla\phi|^2 dV \ \text{(as } \nabla^2\phi = 0 \text{ in D)}$$

Now, $\iint_{\Sigma} \phi\nabla\phi\cdot\hat{n}dS = \int_{S_1} \phi\nabla\phi\cdot\hat{n}dS + \int_{S_2} \phi\nabla\phi\cdot\hat{n}dS$

$$= \int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}dS + 0 \ \text{(as } \phi = 0 \text{ on } S_2\text{)}$$

Here $\hat{n}$ is outward drawn normal i.e. pointing towards origin

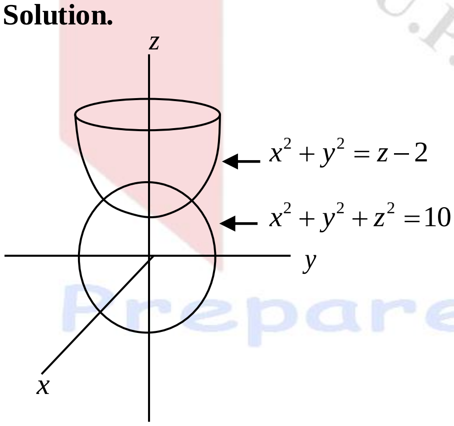
So, $\int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}dS = -\int_{S_1} \phi\cdot\nabla\phi\cdot\hat{n}'dS$

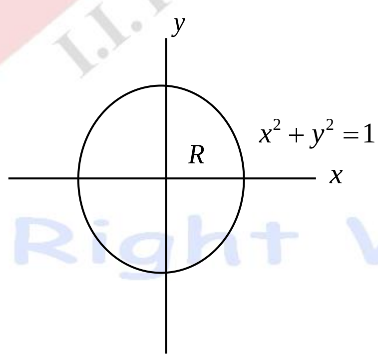
where $\hat{n}' = -\hat{n}$ is unit normal to S_1 directed away from the origin

So, $\int_D |\nabla\phi|^2 dV + \int_{S_1} \phi(\nabla\phi)\cdot\hat{n}'dS = 0$

27. Using Gauss's divergence theorem, evaluate the integral $\int_S \vec{F}\cdot\hat{n}dS$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + 4yz\hat{k}$, S is the surface of the solid bounded by the sphere $x^2+y^2+z^2=10$ and the paraboloid $x^2+y^2=z-2$, and $\hat{n}$ is the outward unit normal vector to S.

Solution.





$\vec{F} = 4xz\hat{i} - y^2\hat{j} + 4yz\hat{k}$

$\nabla \cdot \vec{F} = 4z + 2y$

Using Gauss divergence theorem

$$\iint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} d\tau = \iint \int_{x^2+y^2+2}^{\sqrt{10-x^2-y^2}} (4z+2) dzdydx \ (d\tau = dxdydz) = 2\iint \left[z^2 + yz \right]_{x^2+y^2+2}^{\sqrt{10-x^2-y^2}} dxdy$$

$$= 2\iint \left(6 - 5(x^2+y^2) - (x^2+y^2)^2 + y\left(\sqrt{10-x^2-y^2} - x^2 - y^2 - 2\right)\right) dxdy$$

Surfaces bounding the volume are $x^2+y^2+z^2 = 10$ & $x^2+y^2 = z-2$ as shown in Figure 7.38.

So, curve of intersection of surfaces is given as

$x^2 + z - 2 = 10 \Rightarrow z = 3$

$$\left.\begin{array}{l} x^2+y^2 = 1 \\ z = 3 \end{array}\right\} \text{Curve of intersection}$$

Putting $x = r\cos\theta, y = r\sin\theta, dxdy = rd\theta dr$

(r is the region of integration of double integration)

$$\iint_S \vec{F} \cdot \hat{n} dS = 2\int_0^1 \int_0^{2\pi} \left[\left(6 - 5r^2 - r^4\right) + r\sin\theta\left(\sqrt{10-r^2} - \left(r^2+2\right)\right) \right] rd\theta dr$$

$$= 2\int_0^1 \int_0^{2\pi} \left(6 - 5r^2 - r^4\right) rd\theta dr + 2\int_0^1 \int_0^{2\pi} r^2 \left(\sqrt{10-r^2} - r^2 - 2\right) \sin\theta d\theta dr$$

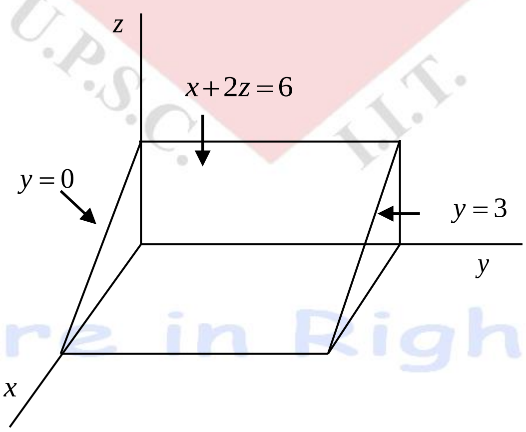
Now, $\int_0^{2\pi} \sin\theta d\theta = 0$ So, integral of second term

$$2\int_0^1 \int_0^{2\pi} r^2 \left(\sqrt{10-r^2} - r^2 - 2\right) \sin\theta d\theta dr = 0$$

$$\iint \vec{F} \cdot \hat{n} dS = 4\pi \int_0^1 \left[6r - 5r^3 - r^5 \right] dr = 4\pi \left[3r^2 - \frac{5r}{4} - \frac{r^6}{6} \right]_0^1 = \frac{19}{3}\pi$$

28. Let W be the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + 2z = 6$. Let S be the boundary of this region. Using Gauss divergence theorem, evaluate $\int_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and $\hat{n}$ is the outward unit normal vector to S.

Solution.



Using Gauss Divergence theorem

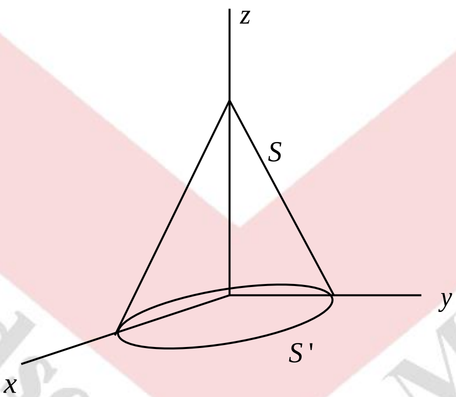
$$\oiint \vec{F}\cdot\hat{n}dS=\int\nabla\cdot Fd\tau=\iiint\left(2y+z^2+x\right)dxdydz=\iint\int_0^3\left(x+2y+z^2\right)dydxdz$$

$$=\iint xy+y^2+z^2y\Big|_0^3 dxdz=\int_0^6\int_0^{\frac{6-x}{2}}\left(3x+3z^2+9\right)dzdx=\int_0^6 3xz+9z+z^3\Big|_0^{\frac{6-x}{2}}dx$$

$$=\frac{1}{8}\int_0^6\left(-x^3+6x^2-72x+512\right)dx=\frac{1}{8}\left[-\frac{x^4}{4}+2x^3-36x^2+512x\right]_0^6=235.5$$

29. If $\vec{F}=\left(x^2+y-4\right)\hat{i}+3x\hat{j}+\left(2xz+z^2\right)\hat{k}$, then evaluate the surface integral $\int_S\left(\nabla\times\vec{F}\right)\cdot\hat{n}\,dS$, where S is the surface of the cone $z=1-\sqrt{x^2+y^2}$ lying above the xy-plane and $\hat{n}$ is the unit normal to S making an acute angle with $\hat{k}$.

Solution.



Consider a closed surface Σ consisting of S & S' as shown in Figure 7.41. Where S is conical surface & S' is its base

$$\oiint_\Sigma \nabla\times\vec{F}\cdot\hat{n}dS=\int_V\nabla\cdot\left(\nabla\times\vec{F}\right)d\tau=0$$

$$\Rightarrow\int_S\nabla\times\vec{F}\cdot\hat{n}dS+\int_{S'}\nabla\times\vec{F}\cdot\hat{n}dS=0$$

$$\int_S\nabla\times\vec{F}\cdot\hat{n}dS=-\int_S\nabla\times\vec{F}\cdot\hat{n}dS$$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\ x^2+y-4&3x&2xz+z^2\end{vmatrix}$$

$$=-2z\hat{j}+2\hat{k}$$

For $S'\cdot\hat{n}=-\hat{k},\ dS=dxdy$

So, $\int_S\nabla\times\vec{F}\cdot\hat{n}dS=-\int\nabla\times\vec{F}\cdot\hat{n}dS$

$$=\iint 2dxdy$$

$$=2\pi$$

30. For the vector field $\vec{V}=xz^2\hat{i}-yz^2\hat{j}+z\left(x^2-y^2\right)\hat{k}$

(a) Calculate the volume integral of the divergence of $\vec{V}$ over the region defined by $-a \le x \le a, -b \le y \le b$ and $0 \le z \le c$.

(b) Calculate the flux of $\vec{V}$ out of the region through the surface at $z = c$. Hence deduce the net flux through the rest of the boundary of the region.

Solution.

$\vec{V} = xz^2\hat{i} - yz^2\hat{j} + z\left(x^2 - y^2\right)\hat{k}$

div $\vec{V} = z^2 - z^2 + x^2 - y^2$

$= \left(x^2 - y^2\right)$

Volume integral of divergence of $\vec{V}$ over the region defined by $-a \le y \le a, -b \le y \le b$ and $0 \le z \le c$.

$$\iiint \nabla \cdot \vec{V}\, dx\, dy\, dz = \int_{-a}^{a}\int_{-b}^{b}\int_{0}^{c}\left(x^2 - y^2\right) dxdydz = c\int_{-a}^{a}\int_{-b}^{b}\left(x^2 - y^2\right) dxdy = c\int_{-a}^{a}\left|x^2 y - \frac{y^3}{3}\right|_{-b}^{b} dx$$

$$= 2c\int_{-a}^{a}\left(bx^2 - \frac{b^3}{3}\right)dx = 2c\left|\frac{bx^3}{3} - \frac{b^3 x}{3}\right|_{-a}^{a} = \frac{4abc}{3}\left(a^2 - b^2\right)$$

(b) For $z = c, \hat{n} = \hat{k}$

$\vec{V} \cdot \hat{n} = z\left(x^2 - y^2\right) = c\left(x^2 - y^2\right)$ for $z = c$

$dS = dxdy$, Flux across $z = c$,

$$\int \vec{V} \cdot \hat{n} dS = c\int_{-a}^{a}\int_{-b}^{b}\left(x^2 - y^2\right)dxdy = c\int_{-a}^{a}\left[x^2 y - \frac{y^3}{3}\right]_{-b}^{b} dx = 2c\int_{-a}^{a}\left(x^2 b - \frac{b^3}{3}\right)dx = 2c\left[\frac{x^3 b}{3} - \frac{b^3 x}{3}\right]_{-a}^{a}$$

$$= \frac{4abc}{3}\left(a^2 - b^2\right)$$

Flux across the closed surface

$$\oiint \vec{V} \cdot \hat{n}\, dS = \int_{r} \nabla \cdot V d\tau = \frac{4abc}{3}\left(a^2 - b^2\right)$$

This is equal to flux through the surface at $z = c$ .So, flux through rest of the boundary of the region = 0

31. Using Divergence theorem, evaluate $\int_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounded by the region $x^2 + y^2 = 4, z = 0, z = 3$.

Solution.

From Divergence Theorem

$$\int_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} d\tau,\ \nabla \cdot \vec{F} = \left(4 - 4y + 2z\right), \int_V \nabla \cdot \vec{F} d\tau = \iint \int_{z=0}^{3}\left(4 - 4y + 2z\right)dxdydz$$

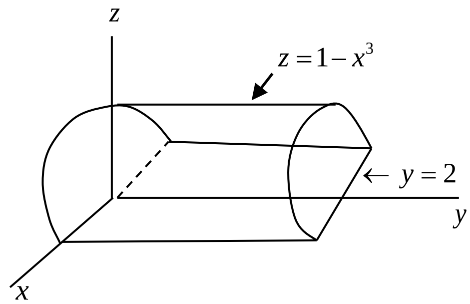
$$= \iint 4z - 4yz + z^2\Big|_0^3 dxdy = \iint\left(21 - 12y\right)dxdy = 21\iint dxdy - 12\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} ydxdy$$

$$= 21\iint dxdy - 0 \qquad \left[\because \int_{-a}^{a} f\left(x\right)dx = 0 \text{ if } f\left(x\right) \text{is odd function}\right]$$

$= 21 \times$ area of circle of radius $2 = 84\pi$

32. Let S be the boundary of the region consisting of the parabolic cylinder $z = 1 - x^2$ and the planes $y = 0, y = 2$ and $z = 0$. Evaluate the integral $\oiint_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = xy\hat{i} + \left(y^2 + e^{xz^2}\right)\hat{j} + \sin(xy)\hat{k}$ and $\hat{n}$ is the outward drawn unit normal to S.

Solution.



The surface S is shown Figure 7.42

$$\oiint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} d\tau = \iint \int_0^2 3y \, dx \, dz \, dy = 3 \iint \frac{y^2}{2}\Big|_0^2 dx \, dz$$

$$= 6 \int_{-1}^{1} \int_0^{1-x^2} dx \, dz = 6 \int_{-1}^{1} \left(1 - x^2\right) dx = 6\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 12 \times \frac{2}{3} = 8$$

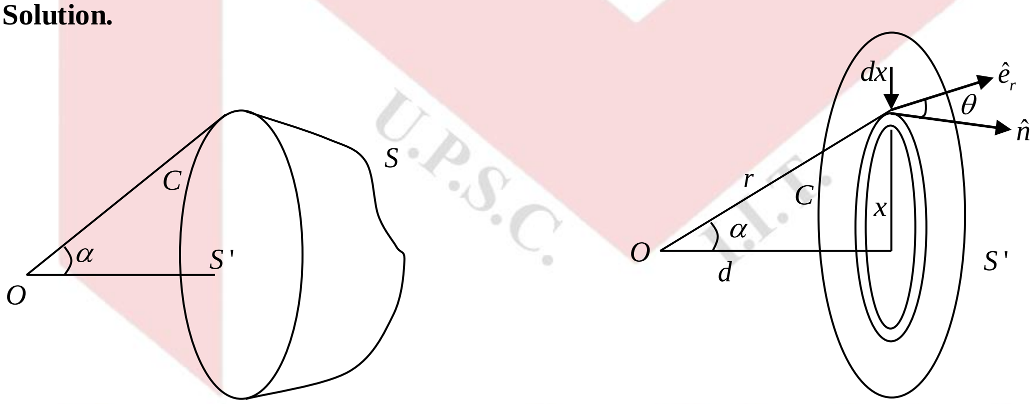
33. If $\vec{V} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$ and S is the surface of the closed cylinder $x^2 + y^2 = 16, z = 0$ and $z = 4$, evaluate the integral $\iint_S \vec{V} \cdot \hat{n} dS$.

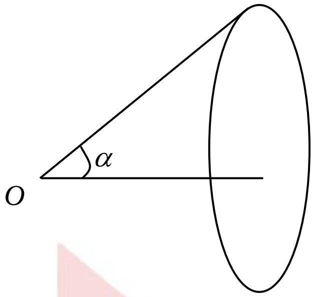
Solution.

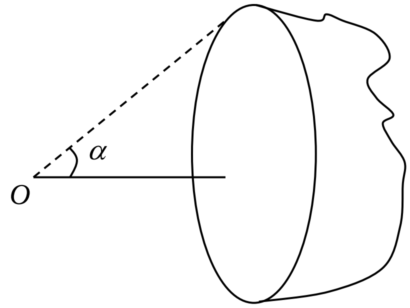
$\oiint \vec{V} \cdot \vec{n} dS = \int \nabla \cdot \vec{V} d\tau$ (Gauss Divergence Theorem) $= \int \left(2xz + 1 - 2xz\right) d\tau$ = volume of cylinder $= 64\pi$

34. Evaluate the solid angle subtended by any arbitrary open surface bounded by circle C at any point O.

Solution.







The solid angle subtended by S at O is same as solid angle subtended by a plane surface S'. Let us evaluate the solid angle subtended at point O by surface S'.
Consider a ring of radius x and thickness dx, the surface element

$$dS = 2\pi x dx$$

Solid angle subtended by surface elements dS at O.

$$d\Omega = \frac{\vec{r}\cdot\hat{n}dS}{r^3} = \frac{\hat{e}_r\cdot\hat{n}}{r^2}dS = \frac{2\pi x\cos\theta}{\left(x^2+d^2\right)}dx = \frac{2\pi dx}{\left(x^2+d^2\right)^{3/2}}dx$$

So, Solid angle subtended by S' at point O.

$$\Omega = \int d\Omega = 2\pi d\int_0^R \frac{x}{\left(x^2+d^2\right)^{3/2}}dx$$ (R is the radius of circle)

Let $x^2+d^2=t^2$

$\Rightarrow xdx = tdt$,

$$\Omega = 2\pi d\int\frac{tdt}{t^3} = 2\pi d\left[-\frac{1}{t}\right] = 2\pi d\left[-\frac{1}{\sqrt{x^2+d^2}}\right]_0^R = 2\pi d\left[\frac{1}{d}-\frac{1}{\sqrt{R^2+d^2}}\right]$$

$$= 2\pi\left[1-\frac{d}{\sqrt{R^2+d^2}}\right] = 2\pi\left(1-\cos\alpha\right)$$

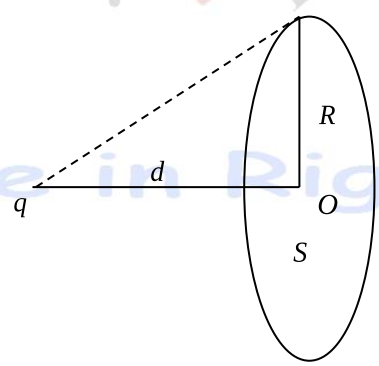
So, the solid angle subtended by a disc of radius R at any point O lying on a perpendicular axis passing through its centre is given as

$$\Omega = 2\pi\left(1-\cos\alpha\right)$$

Since, the solid angle is same for any surface bounded by C. So, the solid angle subtended by any arbitrary surface S bounded by a circle C is equal to $\Omega = 2\pi\left(1-\cos\alpha\right)$.

35. Using result obtained in previous problem, find the flux of electrostatics field across the disc of radius R due to point charge q placed at distance d from its centre.

Solution.



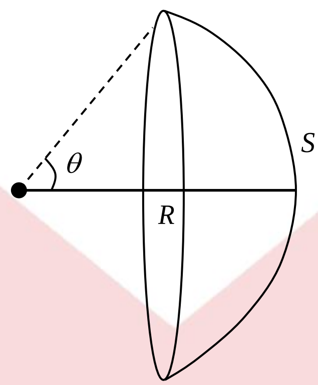
The flux of electrostatic field $\vec{E}$ across the disc S is given as

$$\phi = \int \vec{E}\cdot\hat{n}dS = \int \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}\cdot\hat{n}dS$$ where $\hat{n}$ is unit normal vector to S.

$$= \frac{q}{4\pi\epsilon_0}\int\frac{\vec{r}}{r^3}\cdot\hat{n}dS = \frac{q}{4\pi\epsilon_0}\times$$ Solid angle subtended by S at the position of q.

$$= \frac{q}{4\pi\epsilon_0}\cdot 2\pi\left(1-\cos\alpha\right) = \frac{q}{2\epsilon_0}\left(1-\frac{d}{\sqrt{R^2+d^2}}\right)$$

36. Evaluate the solid angle subtended by a part of sphere at centre O as shown in figure.
Solution.



The bounding curve of S is a circle.
So, solid angle subtended by S at centre O is same as solid angle subtended by a plane surface enclosed by circle C.

$$\Omega = 2\pi\left(1-\cos\alpha\right)$$

Note. If the surface of previous problem is hemisphere then solid angle subtended by hemisphere at its centre O.

$$\Omega = 2\pi\left(1-\cos\frac{\pi}{2}\right) \quad \left(\theta = \frac{\pi}{2}\ \text{for hemisphere}\right)$$

$$= 2\pi$$

For sphere, $\theta = \pi$

So, Solid angle subtended sphere at its centre

$$\Omega = 2\pi\left(1-\cos\pi\right)$$

$$= 4\pi$$

VECTOR CALCULUS: UPSC PREVIOUS YEARS QUESTION(CSE & IFoS)

VECTOR: BASICS & TRIPLE PRODUCT

1. VECTOR DIFFERENTIATION
   - GRADIENT, DIRECTIONAL DERIVATIVES
   - DIVERGENCE
   - CURL
2. VECTOR INTEGRATION-
   - LINE, SURFACE AND VOLUME INTEGRALS
3. THREE IMPORTANT THEOREMS
   - GREEN'S THEOREM
   - GAUSS' DIVERGENCE THEOREM
   - STOKE'S THEOREMS
4. SOME OTHER TOPICS
   - CURVATURE & TORSION
   - CURVILINEAR COORDINATES.

INTRODUCTION:VECTOR ANALYSIS

Q1. Prove that the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ can form the sides of a triangle. Find the lengths of the medians of the triangle.
[5b UPSC CSE 2016]

Q2. Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$. **[8c 2016 IFoS]**

Q3. For three vectors show that: $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = 0$.
[5e 2014 IFoS]

1. VECTOR DIFFERENTIAL CALCULUS

Q1. The position vector of a moving point at time t is $\bar{r} = \sin t\hat{i} + \cos 2t\hat{j} + \left(t^2 + 2t\right)\hat{k}$. Find the components of acceleration $\bar{a}$ in the directions parallel to the velocity vector $\bar{v}$ and perpendicular to the plane of $\bar{r}$ and $\bar{v}$ at time $t = 0$.
[5e UPSC CSE 2017]

Q2. If

$$\vec{A} = x^2yz\hat{i} - 2xz^3\vec{j} + xz^2\vec{k}$$
$$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$$

find the value of $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)$ at $(1,0,-2)$. **[5e UPSC CSE 2012]**

Q3. For two vectors $\bar{a}$ and $\bar{b}$ given respectively by
$\bar{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\bar{b} = \sin t\hat{i} - \cos t\hat{j}$ determine:
(i) $\frac{d}{dt}\left(\bar{a}\cdot\bar{b}\right)$ and (ii) $\frac{d}{dt}\left(\bar{a}\times\bar{b}\right)$ **[5e UPSC CSE 2011]**

Q4. The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}$$

At time $t = 1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. **[8d 2011 IFoS]**

GRADIENT, DIRECTIONAL DERIVATIVES

Q1. Prove that for a vector $\vec{a}$,
$\nabla\left(\vec{a}\cdot\vec{r}\right) = \vec{a}$; where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = |\vec{r}|$.
Is there any restriction on $\vec{a}$?
Further, show that

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = \frac{3\left(\vec{a}\cdot\vec{r}\right)\left(\vec{b}\cdot\vec{r}\right)}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3}$$

Give an example to verify the above. **[5e 2020 IFoS]**

Q2. Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1,1,1)$. **[5e UPSC CSE 2019]**

Q3. Find the angle between the tangent at a general point of the curve whose equations are $x = 3t, y = 3t^2, z = 3t^3$ and the line $y = z - x = 0$. **[5b UPSC CSE 2018]**

Q4. Find $f(r)$ such that $\nabla f = \frac{\vec{r}}{r^5}$ and $f(1) = 0$. **[8a UPSC CSE 2016]**

Q5. Find the angle between the surfaces $x^2 + y^2 + z^2 - 9 = 0$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.

[5e UPSC CSE 2015]

Q6. Find the value of λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2 y + z^3 = 4$ may intersect orthogonally at $(1, -1, 2)$. **[6c UPSC CSE 2015]**

Q7. A curve in space is defined by the vector equation $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$. Determine the angle between the tangents to this curve at the points $t = +1$ and $t = -1$. **[8b UPSC CSE 2013]**

Q8. If $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = yz + zx + xy$ prove that grad u, grad v and grad w are coplaner. **[5e 2012 IFoS]**

Q9. Examine whether the vectors ∇_u, ∇_v and ∇_w are coplaner, where u, v and w are the scalar functions defined by:

$u = x + y + z$,

$v = x^2 + y^2 + z^2$

and $w = yz + zx + xy$. **[8a UPSC CSE 2011]**

Q10. Find the directional derivative of $f(x, y) = x^2 y^3 + xy$ at the point $(2, 1)$ in the direction of a unit vector which makes an angle of $\pi/3$ with the x-axis.

[1e UPSC CSE 2010]

Q11. Find the directional derivation of $\vec{V}^2$, where, $\vec{V} = xy^2\vec{i} + zy^2\vec{j} + xz^2\vec{k}$ at the point $(2, 0, 3)$ in the direction of the outward normal to the surface $x^2 + y^2 + z^2 = 14$ at the point $(3, 2, 1)$.

[5f 2010 IFoS]

Q12. Find the directional derivative of -

(i) $4xz^3 - 3x^2y^2z^2$ at $(2, -1, 2)$ along z-axis;

(ii) $x^2yz + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. **[5f UPSC CSE 2009]**

DIVERGENCE

Q1. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $f(r)$ is differentiable, show that $\text{div}\left[f(r)\vec{r} \right] = rf'(r) + 3f(r)$.

Hence or otherwise show that $\text{div}\left(\frac{\vec{r}}{r^3} \right) = 0$. **[5e 2018 IFoS]**

Q2. Calculate $\nabla^2(r^n)$ and find its expression in terms of *r* and *n*, *r* being the distance of any point (x, y, z) from the origin, *n* being a constant and ∇^2 being the Laplace operator.

[8a UPSC CSE 2013]

Q3. Prove that $\text{div}\left(f\vec{V} \right) = f\left(\text{div}\vec{V} \right) + (\text{grad } f)\cdot\vec{V}$ where *f* is a scalar function.

[6c UPSC CSE 2010]

Q4. Show that, $\nabla^2 f(r) = \left(\frac{2}{r} \right) f'(r) + f''(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. **[4. 8a 2010 IFoS]**

Q5. Show that $\text{div}\left(\text{grad}\, r^n \right) = n(n+1)r^{n-2}$ where $r = \sqrt{x^2 + y^2 + z^2}$. **[5e UPSC CSE 2009]**

CURL

Q1. For what value of a,b,c is the vector field $\bar{V}=(4x-3y+az)\hat{i}+(bx+3y+5z)\hat{j}+(4x+cy+3z)\hat{k}$ irrotational? Hence, express $\bar{V}$ as the gradient of a scalar function ϕ. Determine ϕ. **[5c UPSC CSE 2020]**

Q2. Let $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$. Show that curl $(\text{curl}\,\vec{v})=\text{grad}(\text{div}\,\vec{v})-\nabla^2\vec{v}$. **[8a UPSC CSE 2018]**

Q3. Show that $\vec{F}=(2xy+z^3)\hat{i}+x^2\hat{j}+3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$. **[6c 2018 IFoS]**

Q4. For what values of the constants a,b and c the vector $\bar{V}=(x+y+az)\hat{i}+(bx+2y-z)\hat{j}+(-x+cy+2z)\hat{k}$ is irrotational. Find the divergence in cylindrical coordinates of this vector with these values. **[5d UPSC CSE 2017]**

Q5. A vector field is given by $\vec{F}=(x^2+xy^2)\hat{i}+(y^2+x^2y)\hat{j}$. Verify that the field $\vec{F}$ is irrorational or not. Find the scalar potential. **[7c UPSC CSE 2015]**

Q6. Examine if the vector field defined by $\vec{F}=2xyz^3\hat{i}+x^2z^3\hat{j}+3x^2yz^2\hat{k}$ is irrorational. If so, find the scalar potential ϕ such that $\vec{F}=\text{grad}\,\phi$. **[6d 2015 IFoS]**

Q7. For the vector $\bar{A}=\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\bar{A}=\nabla\phi$. **[6d 2014 IFoS]**

Q8. $\vec{F}$ being a vector, prove that $\text{curl}\,\text{curl}\,\vec{F}=\text{grad}\,\text{div}\,\vec{F}-\nabla^2\vec{F}$ where $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$.

[5c 2013 IFoS]

Q9. If u and v are two scalar fields and $\vec{f}$ is a vector field, such that $u\vec{f}=\text{grad}\,v$, find the value of $\vec{f}\cdot\text{curl}\,\vec{f}$. **[5f UPSC CSE 2011]**

Q10.If $\vec{r}$ be the position vector of a point, find the value(s) of n for which the vector $r^n\vec{r}$ is (i) irrotational, (ii) solenoidal. **[8c UPSC CSE 2011]**

Q11. Prove the vector identity:
$\text{curl}(\vec{f}\times\vec{g})=\vec{f}\,\text{div}\,\vec{g}-\vec{g}\,\text{div}\,\vec{f}+(\vec{g}\cdot\nabla)\vec{f}-(\vec{f}\cdot\nabla)\vec{g}$ and verify it for the vectors $\vec{f}=x\hat{i}+z\hat{j}+y\hat{k}$ and $\vec{g}=y\hat{i}+z\hat{k}$. **[8b 2011 IFoS]**

Q12. Show that the vector field defined by the vector function $\vec{V}=xyz(yz\vec{i}+xz\vec{j}+xy\vec{k})$ is conservative. **[1f UPSC CSE 2010]**

Q13. Show that $\vec{F} = \left(2xy + z^3\right)\vec{i} + x^2\vec{j} + 3z^2x\vec{k}$ is a conservative field. Find its scalar potential and also the work done in moving a particle from $(1,-2,1)$ to $(3,1,4)$. **[8a 2010 IFoS]**

2. VECTOR INTRGRAL CALCULUS

Q1. For the vector function $\bar{A}$, where $\bar{A}=\left(3x^2+6y\right)\hat{i}-14yz\hat{j}-14yz\hat{j}+20xz^2\hat{k}$, calculate $\int_C \bar{A}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths:

(i) $x=t, y=t^2, z=t^3$

(ii) Straight lines joining $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

(iii) Straight line joining $(0,0,0)$ to $(1,1,1)$

Is the result same in all the cases? Explain the reason. **[6b UPSC CSE 2020]**

Q2. Find the circulation of $\vec{F}$ round the curve C, where $\vec{F}=\left(2x+y^2\right)\hat{i}+\left(3y-4x\right)\hat{j}$ and C is the curve $y=x^2$ from $(0,0)$ to $(1,1)$ and the curve $y^2=x$ from $(1,1)$ to $(0,0)$.

[6b UPSC CSE 2019]

Q3. Evaluate $\int_{(0,0)}^{(2,1)}\left(10x^4-2xy^3\right)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$.

[5e 2019 IFoS]

Q4. Evaluate $\int_C e^{-x}\left(\sin y dx+\cos y dy\right)$, where C is the rectangle with vertices $(0,0),(\pi,0),\left(\pi,\frac{\pi}{2}\right),\left(0,\frac{\pi}{2}\right)$. **[8c UPSC CSE 2015]**

Q5. If $\vec{A}=2y\vec{i}-z\vec{j}-x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4, z=6$, evaluate the surface integral, $\iint_S \vec{A}\cdot\hat{n}\overrightarrow{dS}$. **[8c 2010 IFoS]**

Q6. Find the work done in moving the particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1, z=0$ under the field of force given by $\vec{F}=\left(2x-y+z\right)\hat{i}+\left(x+y-z^2\right)\hat{j}+\left(3x-2y+4z\right)\hat{k}$. **[8a UPSE CSE 2009]**

3.THREE IMPORTANT THEOREMS

GREEN'S THEOREM

Q1. Let $\vec{F} = xy^2\vec{i} + (y+x)\vec{j}$. Integrate $(\nabla \times \vec{F})\cdot\vec{k}$ over the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$ using Green's theorem.

[8c UPSE CSE 2018]

Q2. Using Green's theorem, evaluate the $\int_C F(\bar{r})\cdot d\bar{r}$ counterclockwise where $F(\bar{r}) = (x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}$ and $d\bar{r} = dx\hat{i} + dy\hat{j}$ and the curve C is the boundary of the region $R = \{(x,y) | 1 \le y \le 2 - x^2\}$. **[8c UPSE CSE 2017]**

Q3. Verify Green's theorem in the plane for $\oint_C \left[(xy+y^2)dx + x^2dy\right]$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. **[8b UPSE CSE 2013]**

Q4. Find the value of the line integral over a circular path given by $x^2+y^2=a^2, z=0$, where the vector field, $\vec{F} = (\sin y)\vec{i} + x(1+\cos y)\vec{j}$.

[8b 2012 IFoS]

Q5. Verify Green's theorem in the plane for $\oint_C \left[(3x^2-8y^2)dx + (4y-6xy)dy\right]$, where C is the boundary of the region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$.

[8c 2011 IFoS]

Q6. Verify Green's theorem for $e^{-x}\sin y dx + e^{-x}\cos y dy$ the path of integration being the boundary of the square whose vertices are $(0,0),(\pi/2,0),(\pi/2,\pi/2)$ and $(0,\pi/2)$. **[8c UPSE CSE 2010]**

Q7. Use Green's theorem in a plane to evaluate the integral, $\int_C \left[(2x^2-y^2)dx + (x^2+y^2)dy\right]$, where C is the boundary of the surface in the *xy*-plane enclosed by, $y = 0$ and the semi-circle $y = \sqrt{1-x^2}$.

[8b 2012 IFoS]

GAUSS' DIVERGENCE THEOREM

Q1. Given a portion of a circular disc of radius 7 units and of height 1.5 units such that $x, y, z \geq 0$. Verify Gauss Divergence Theorem for the vector field $\vec{f} = (z, x, 3y^2 z)$ over the surface of the above mentioned circular disc.
[7c 2020 IFoS]

Q2. State Gauss divergence theorem. Verify this theorem for $\vec{F} = 4x\hat{i} - 2y^2 j + z^2\hat{k}$, taken over the region bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$.
[8c UPSC CSE 2019]

Q3. If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, then evaluate $\iint_S \left[(x+z)dydz + (y+z)dzdx + (x+y)dxdy\right]$ using Gauss' divergence theorem.

[6d UPSC CSE 2018]

Q4. Evaluate the integral: $\iint_S \bar{F} \cdot \hat{n} ds$ where $\bar{F} = 3xy^2\hat{i} + (yx^2 - y^3)\hat{j} + 3zx^2\hat{k}$ and S is a surface of the cylinder $y^2 + z^2 \leq 4, -3 \leq x \leq 3$, using divergence theorem.
[8c UPSC CSE 2017]

Q5. If E be the solid bounded by the *xy* plane and the paraboloid $z = 4 - x^2 - y^2$, then $\iint_S \bar{F} \cdot dS$ where S is the surface bounding the volume E and $\bar{F} = (zx \sin yz + x^3)\hat{i} + \cos yz\hat{j} + (3zy^2 - e^{\lambda^2 + y^2})\hat{k}$. **[5e 2016 IFoS]**

Q6. Using divergence theorem, evaluate $\iint_S (x^3 dydz + x^2 ydzdx + x^2 zdydx)$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. **[7b 2015 IFoS]**

Q7. Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0, z = 3$. **[8c 2014 IFoS]**

Q8. By using Divergence Theorem of Gauss, evaluate the surface integral $\iint (a^2x^2 + b^2y^2 + c^2z^2)^{-\frac{1}{2}} dS$, where S is the surface of the elliposid $ax^2 + by^2 + cz^2 = 1, a, b$ and c being all positive constants. **[8c UPSC CSE 2013]**

Q9. Evaluate $\int_S \vec{F} \cdot d\vec{s}$, where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and s is the surface bounding the region $x^2 + y^2 = 4, z = 0$ and $z = 3$. **[6b 2013 IFoS]**

Q10. Verify the Divergence theorem for the vector function $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c$. **[8b 2013 IFoS]**

Q11. Verify Gauss' Divergence Theorem for the vector $\vec{v} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ taken over the cube $0 \leq x, y, z \leq 1$. **[8d UPSC CSE 2011]**

Q12. Use the divergence theorem to evaluate $\iint_S \vec{V}\cdot\vec{n}dA$ where $\vec{V}=x^2z\vec{i}+y\vec{j}-xz^2\vec{k}$ and S is the boundary of the region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4y$.

[7c UPSC CSE 2010]

Q13. Use divergence theorem to evaluate, $\iint_S \left(x^3dy\,dz+x^2ydz\,dx+x^2z\,dy\,dx\right)$ where S is the sphere, $x^2+y^2+z^2=1$. **[8b 2010 IFoS]**

Q14. Using divergence theorem, evaluate $\iint_S \vec{A}\cdot d\vec{S}$ where $\vec{A}=x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$. **[8b UPSC CSE 2009]**

STOKE'S THEOREM

Q1. Verify the Stokes' theorem for the vector field $\bar{F} = xy\hat{i} + yz\hat{j} + xz\hat{k}$ on the surface S which is the part of the cylinder $z = 1 - x^2$ for $0 \le x \le 1, -2 \le y \le 2$; S is oriented upwards.

[7a UPSC CSE 2020]

Q2. Evaluate the surface integral $\iint_S \nabla \times \bar{F} \cdot \hat{n} dS$ for $\bar{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the *xy*-plane. **[8b UPSC CSE 2020]**

Q3. Evaluate by Stokes' theorem $\oint_C e^x dx + 2y dy - dz$, where C is the curve $x^2 + y^2 = 4, z = 2$.

[8c UPSC CSE 2019]

Q4. Verify Stokes's theorem for $\bar{V} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **[6c 2019 IFoS]**

Q5. Evaluate the line integral $\int_C -y^3 dx + x^3 dy + z^3 dz$ using Stokes's theorem. Here C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. The orientation on C corresponds to counterclockwise motion in the *xy*-plane.
[8b UPSC CSE 2018]

Q6. Using Stoke's theorem evaluate
$\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz]$, where C is the boundary of the triangle with vertices at $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$. **[6c 2017 IFoS]**

Q7. Evaluate
$\iint_S (\nabla \times \vec{f}) \cdot \hat{n} dS$, where S is the surface of the cone, $z = 2 - \sqrt{x^2 + y^2}$ above xy-plane and $\vec{f} = (x - z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$. **[7d 2017 IFoS]**

Q8. Prove that $\oint_C f d\vec{r} = \iint_S d\vec{S} \times \nabla f$. **[8b UPSC CSE 2016]**

Q9. Evaluate $\iint_S (\nabla \times \bar{f}) \cdot \hat{n} dS$ for $\bar{f} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane. **[6d 2016 IFoS]**

Q10. State Stokes' theorem. Verify the Stokes' theorem for the function $\bar{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one. **[7a 2016 IFoS]**

Q11. If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. **[8b 2015 IFoS]**

Q12. Evaluate by Stokes' theorem $\int_{\Gamma}(y\,dx+z\,dy+x\,dz)$ where Γ is the curve given by $x^2+y^2+z^2-2ax-2ay=0, x+y=2a$ starting from $(2a,0,0)$ and then going below the z-plane.

[6c UPSC CSE 2014]

Q13. Evaluate $\iint_S \nabla\times\bar{A}\cdot\vec{n}dS$ for $\bar{A}=(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}$ and S is the surface of hemisphere $x^2+y^2+z^2=16$ above xy plane. **[7b 2014 IFoS]**

Q14. Use Stokes' theorem to evaluate the line integral $\int_C(-y^3dx+x^3dy-z^3dz)$, where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$. **[8d UPSC CSE 2013]**

Q15. If $\vec{F}=y\vec{i}+(x-2xz)\vec{j}-xy\vec{k}$, evaluate $\iint_S(\vec{\nabla}\times\vec{F})\cdot\vec{n}d\vec{s}$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **[8c UPSC CSE 2012]**

Q16. Find the value of $\iint_S(\vec{\nabla}\times\vec{F})\cdot d\vec{s}$ taken over the upper portion of the surface $x^2+y^2-2ax+az=0$ and the bounding curve lies in the plane $z=0$, when $\vec{F}=(y^2+z^2-x)\vec{i}+(z^2+x^2-y^2)\vec{j}+(x^2+y^2-z^2)\vec{k}$. **[6b 2012 IFoS]**

Q17. If $\bar{u}=4y\hat{i}+x\hat{j}+2z\hat{k}$, calculate the double integral $\iint(\nabla\times\bar{u})\cdot d\vec{s}$ over the hemisphere given by $x^2+y^2+z^2=a^2,\ z\geq 0$. **[8b UPSC CSE 2011]**

Q18. Evaluate the line integral $\oint_C(\sin x dx+y^2dy-dz)$, where C is the circle $x^2+y^2=16, z=3$, by using Stokes' theorem. **[5e 2011 IFoS]**

Q19. Find the value of $\iint_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ taken over the upper portion of the surface $x^2+y^2-2ax+az=0$ and the bounding curve lies in the plane $z=0$, when $\vec{F}=(y^2+z^2-x^2)\hat{i}+(z^2+x^2-y^2)\hat{j}+(x^2+y^2-z^2)\hat{k}$. **[8c UPSC CSE 2009]**

CURVATURE & TORSION

Q1. A tangent is drawn to a given curve at some point of constant. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{\left[25\kappa^2\tau^2\left(1+25\kappa^2\right)+\left\{\kappa+5\frac{d\kappa}{ds}+25\kappa^3\right\}\right]^{1/2}}{\left(1+25\kappa^2\right)^{3/2}}$$

Find the curvature and torsion of the curve $\vec{r}=t\hat{i}+t^2\hat{j}+t^3\hat{k}$. **[6c 2020 IFoS]**

Q2. Find the radius of curvature and radius of torsion of the helix $x=a\cos u$, $y=a\sin u, z=au\tan\alpha$. **[7b UPSC CSE 2019]**

Q3. Let $\bar{r}=\bar{r}(s)$ represent a space curve. Find $\frac{d^3\bar{r}}{ds^3}$ in terms of $\bar{T},\bar{N}$ and $\bar{B}$, where $\bar{T},\bar{N}$ and $\bar{B}$ represent tangent, principal normal and binormal respectively. Compute $\frac{d\bar{r}}{ds}\cdot\left(\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}\right)$ in terms of radius of curvature and the torsion. **[5d 2019 IFoS]**

Q4. Derive the Frenet-Serret formulae. Verify the same for the space curve $x=3\cos t, y=3\sin t, z=4t$. **[7c 2019 IFoS]**

Q5. Find the curvature and torsion of the curve $\vec{r}=a(u\sin u)\vec{i}+a(1-\cos u)\vec{j}+bu\vec{k}$.

[7b UPSC CSE 2018]

Q6. Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of) a circle. **[7d 2018 IFoS]**

Q7. For a curve lying on a sphere of radius a and such that the torsion is never 0, show that

$\left(\frac{1}{\kappa}\right)^2+\left(\frac{\kappa'}{\kappa^2\tau}\right)^2=a^2$ **[8c 2018 IFoS]**

Q8. Find the curvature vector and its magnitude at any point $\vec{r}=(\theta)$ of the curve $\bar{r}=(a\cos\theta, a\sin\theta, a\theta)$. Show that the locus of the feet of the perpendicular from the origin to the tangent is a curve that completely lies on the hyperboloid $x^2+y^2-z^2=a^2$.

[7a UPSC CSE 2017]

Q9. Find the curvature and torsion of the circular helix $\vec{r}=a(\cos\theta,\sin\theta,\theta\cot\beta)$, β is the constant angle at which it cuts its generators. **[8c 2017 IFoS]**

Q10. If the tangent to a curve makes a constant angle α, with a fixed lines, then prove that $\kappa\cos\alpha\pm\tau\sin\alpha=0$. Conversely, if $\frac{\kappa}{\tau}$ is constant, then show that the tangent makes a constant angle with a fixed direction. **[8d 2017 IFoS]**

Q11. For the cardioid $r = a(1+\cos\theta)$, show that the square of the radius of curvature at any point (r,θ) is a proportional to r. Also find the radius of curvature if $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$. **[8d UPSC CSE 2016]**

Q12. Find the curvature and torsion of the curve $x = a\cos t, y = a\sin t, z = bt$.
[5c 2015 IFoS]

Q13. Find the curvature vector at any point of the curve $\vec{r}(t) = t\cos t\hat{i} + t\sin t\hat{j},\ 0 \le t \le 2\pi$. Give its magnitude also. **[5e UPSC CSE 2014]**

Q14. Show that the curve $\vec{x}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} + \left(\frac{1-t^2}{t}\right)\hat{k}$ lies in a plane. **[5e UPSC CSE 2013]**

Q15. Derive the Frenet-Serret formulae. Define the curvature and torsion for a space curve. Compute them for the space curve $x = t, y = t^2, z = \frac{2}{3}t^3$. Show that the curvature and torsion are equal for this curve. **[8a UPSC CSE 2012]**

Q16. Find the curvature, torsion and the relation between the arc length S and parameter u for the curve: $\vec{r} = \vec{r}(u) = 2\log_e u\hat{i} + 4u\hat{j} + (2u^2+1)\hat{k}$. **[8a 2011 IFoS]**

Q17. Find κ/τ for the curve
$\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + bt\vec{k}$. **[1c UPSC CSE 2010]**

CURVILINEAR COORDINATES

Q1. Derive expression of ∇f in terms of spherical coordinates.
Prove that $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$ for any two vector point functions $f(r,\theta,\phi)$ and $g(r,\theta,\phi)$. Construct one example in three dimensions to verify this identity. **[8a 2020 IFoS]**

Q2. Derive $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)$ in spherical coordinates. **[8c 2019 IFoS]**

Q3. For what values of the constants a, b and c the vector $\bar{V} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$ is irrotational. Find the divergence in cylindrical coordinates of this vector with these values.
[5d UPSC CSE 2017]