LINEAR ALGEBRA PYQs Updated till Year 2022

1. SYSTEM OF LINEAR EQUATIONS
MATRICES – BASICS, RANK OF A MATRIX, NORMAL FORM, SPECIAL MATRICS

2. VECTOR SPACES & SUBSPACES
LINEAR DEPENDENCE AND INDEPENDENCE
BASIS & DIMENSIONS

3. LINEAR TRANSFORMATIONS, RANK AND NULLITY, ALGEBRA OFLINEAR TRANSFORMATIONS

4. MATRIX OF LT, LT WHEN MATRIX IS GIVEN

5. EIGEN VALLUES, EIGEN VECTORS
CAYLEY- HAMILTON THEOREM
MINIMAL POLYNOMIAL
DIAGONALIZATION OF MATRIX
ORTHOGONAL AND UNITARY MATRICES
CONGRUENCE AND SIMILARITY
BILINEAR AND QUADRATIC FORMS

CHAPTER 1. MATRICES- BASICS

Q1. Let

$$A=\begin{bmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ 4 & 1 & 8\end{bmatrix} \text{ and } B=\begin{bmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{bmatrix}$$

(i) Find AB

(ii) Find det (A) and det (B)

(iii) Solve the following system of linear equations:

$x+2z=3, 2x-y+3z=3, 4x+y+8z=14$ **[UPSC CSE 2020]**

Q2. Using elementary row operations, reduce the matrix

$$A=\begin{bmatrix}2 & 1 & 3 & 0\\ 3 & 0 & 2 & 5\\ 1 & 1 & 1 & 1\\ 2 & 1 & 1 & 3\end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX=b$, where $X=(x,y,z,u)^T$ and $b=(2,1,0,4)^T$. **[IFoS 2019]**

Q3. Given that Adj $A=\begin{bmatrix}2 & 2 & 0\\ 2 & 5 & 1\\ 0 & 1 & 1\end{bmatrix}$ and det A = 2. Find the matrix A. **[IFoS 2018]**

Q4. Using elementary row operations, find the inverse of $A=\begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}$.

[UPSC CSE 2016]

Q5. Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$ **[IFoS 2015]**

Q6. Let A be a non-singular, $n \times n$ square matrix. Show that A. $(\text{adj}\, A) = |A| \cdot I_n$.

Hence show that $|\text{adj}(\text{adj}\, A)| = |A|^{(n-1)^2}$. **[UPSC CSE 2011]**

Q7. Find a Hermitian and a skew-Hermitian matrix each whose sum is the matrix

$$\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$$ **[UPSC CSE 2009]**

RANK OF A MATRIX

Q1. By applying elementary row operations on the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

reduce it to a row-reduced echelon matrix. Hence find the rank of A.

[IFoS 2020]

Q2. Let

$$A = \begin{pmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{pmatrix}$$

(i) Find the rank of matrix A

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \;\middle|\; A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$$ **[UPSC CSE 2019]**

Q3. Let A be a 3×2 matrix and B a 2×3 matrix. Show that $C = A \cdot B$ is a singular matrix.

[UPSC CSE 2018]

Q4. Reduce the following matrix to a row-reduced echelon form and hence find its rank:

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$ **[IFoS 2017]**

Q5. Reduce the following matrix to row echelon form and hence find its rank:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$ **[UPSC CSE 2015]**

Q6. Using elementary row or column operations, find the rank of matrix.

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$ **[UPSC CSE 2014]**

Q7. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$$ **[UPSC CSE 2013]**

Q8. Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$ **[IFoS 2010]**

NORMAL FORM

Q1. For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that $PAQ = I$. Hence find A^{-1}. **[IFoS 2016]**

Q2. Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P^t H\bar{P}$ is diagonal and also find its signature. **[IFoS 2013]**

Q3. Let

$$H = \begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}$$

be a Hermitian matrix. Find a non-singular matrix P such that $D = P^T H\bar{P}$ is a diagonal.

[UPSC CSE 2012]

SYSTEM OF LINEAR EQUATIONS

Q1. Find all solutions to the following systems of equations by row-reduced method:

$x_1 + 2x_2 - x_3 = 2$

$2x_1 + 3x_2 + 5x_3 = 5$

$-x_1 - 3x_2 + 8x_3 = -1$. **[2a UPSC CSE 2022]**

Q2. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

(i) Find AB

(ii) Find det (A) and det (B)

(iii) Solve the following system of linear equations:

$x + 2z = 3, \quad 2x - y + 3z = 3, \quad 4x + y + 8z = 14$ **[UPSC CSE 2020]**

Q3. If

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

then show that $AB = 6I_3$. Use this result to solve the following system of equations:

$2x + y + z = 5$

$x - y = 0$

$2x + y - z = 1$ **[UPSC CSE 2019]**

Q4. Using elementary row operations, reduce the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equation $AX = b$, where $X = (x, y, z, u)^T$ and $b = (2, 1, 0, 4)^T$. **[IFoS 2019]**

Q5. For the system of linear equations

$x + 3y - 2z = -1$

$5y + 3z = -8$

$x - 2y - 5z = 7$

determine which of the following statements are true and which are false:

(i) The system has no solution

(ii) The system has a unique solution

(iii) The system has infinitely many solutions **[UPSC CSE 2018]**

Q6. Consider the following system of equations in x, y, z:

$x + 2y + 2z = 1$

$x+ay+3z=3$
$x+11y+az=b$

(i) For which values of a does the system have a unique solution?
(ii) For which pair of values (a,b) does the system have more than one solution?

[UPSC CSE 2017]

Q7. Using elementary row operations, find the condition that the linear equations

$x-2y+z=a$
$2x+7y-3z=b$
$3x+5y-2z=c$

have a solution. **[UPSC CSE 2016]**

Q8. Find the condition on a,b and c so that the following system in unknowns x,y and z has a solution

$x+2y-3z=a$
$2x+6y-11z=b$
$x-2y+7z=c$ **[IFoS 2015]**

Q9. Investigate the value of λ and μ so that the equations $x+y+z=6$, $x+2y+3z=10, x+2y+\lambda z=\mu$ have (1) no solution, (2) a unique solution, (3) an infinite number of solution. **[UPSC CSE 2014]**

Q10. Find the inverse of the matrix:

$$A=\begin{bmatrix}1 & 3 & 1\\ 2 & -1 & 7\\ 3 & 2 & -1\end{bmatrix}$$

by using elementary row operations. Hence solve the system of linear equations

$x+3y+z=10$
$2x-y+7z=21$
$3x+2y-z=4$ **[UPSC CSE 2013]**

Q11. Discuss the consistency and the solutions of the equations $x+ay+az=1, ax+y+2az=-4, ax-ay+4z=2$ for different values of a. **[IFoS 2013]**

Q12. Show that there are three real values of λ for which the equations: $(a-\lambda)x+by+cz=0, bx+(c-\lambda)y+az=0,\quad cx+ay+(b-\lambda)z=0$ are simultaneously true and that the product of these values of λ is

$$D=\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$$ **[IFoS 2012]**

Q13. Let

$$A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & 6 & 7\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\ B=\begin{bmatrix}2\\ 6\\ 5\end{bmatrix}$$

Solve the system of equations given by AX = B

Using the above, also solve the system of equations $A^T X = B$ where A^T denotes the transpose of matrix A. **[UPSC CSE 2011]**

CHAPTER 2. VECTOR SPACES & SUBSPACES

Q1. Let U and W be subspaces of a vector space V and $x, y \in V$. Then prove that $x+U \subseteq y+W$ iff $U \subseteq W$ and $x-y \in W$. [**1aIFoS 2022]**

Q2. Consider the set V of all $n \times n$ real magic squares. Show that V is a vector space over R. Give examples of two distinct 2×2 magic squares.

[UPSC CSE 2020]

Q4. Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where
(i) W contains all 2×2 matrices with zero determinant.
(ii) W consists of all 2×2 matrices A such that $A^2 = A$. **[UPSC CSE 2017]**

Q4. Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors $(1,2,3)$ and $(1,-1,1)$.

[UPSC CSE 2014]

Q5. Let be the vector space of all 2×2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

[UPSC CSE 2012]

Q6. Show that the set of all functions which satisfy the differential equation $\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$ is a vector space. **[IFoS 2012]**

LINEAR COMBINATION, LINEAR DEPENDENCE AND INDEPENDENCE

Q1. Express the vector $(1,2,5)$ as a linear combination of the vectors $(1,1,1),(2,1,2)$ and $(3,2,3)$, if possible justify your answer. **[UPSC CSE 2020]**

Q2. Express basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ as linear combinations of $\alpha_1 = (2,-1)$ and $\alpha_2 = (1,3)$. **[UPSC CSE 2018]**

Q3. If $(n+1)$ vectors $\alpha_1, \alpha_2, ..., \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, ..., \alpha_n$; proved $\alpha_1, \alpha_2, ..., \alpha_n$ form a linearly independent set.

[IFoS 2018]

Q4. Given that the set $\{u, v, w\}$ is linearly independent, examine the sets
(i) $\{u+v, v+w, w+u\}$
(ii) $\{u+v, u-v, u-2v+2w\}$
for linear independence. **[IFoS 2017]**

Q5. The vectors $V_1 = (1,1,2,4), V_2 = (2,-1,-5,2), V_3 = (1,-1,-4,0)$ and $V_4 = (2,1,1,6)$ are linearly independent. Is it true? Justify your answer. **[UPSC CSE 2015]**

Q6. Show that the vectors $X_1=(1,1+i,i), X_2=(i,-i,1-i)$ and $X_3=(0,1-2i,2-i)$ in C^3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers. **[UPSC CSE 2013]**

Q7. Show that the vectors $(1,1,1),(2,1,2)$ and $(1,2,3)$ are linearly independent in $R^{(3)}$. Let $T:\mathbf{R}^{(3)}\to\mathbf{R}^{(3)}$ be a linearly transformation defined by $T(x,y,z)=(x+2y+3z, x+2y+5z, 2x+4y+6z)$. Show that the images of above vectors under T are linearly dependent. Give the reason for the same. **[UPSC CSE 2011]**

Q8. In the *n*-space $\mathbf{R}^n$, determine whether or not the set $\{e_1-e_2, e_2-e_3, \dots, e_{n-1}-e_n, e_n-e_1\}$ is linearly independent. **[UPSC CSE 2010]**

BASIS & DIMENSIONS

Q1. Prove that any set of *n* linearly independent vectors in a vector space V of dimension *n* constitutes a basis for V. [**1aUPSC CSE2022]**.

Q2. Let the set $P=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\middle|\begin{matrix}x-y-z=0 \text{ and}\\ 2x-y+z=0\end{matrix}\right\}$

be the collection of vectors of a vector space $\mathbf{R}^3(\mathbf{R})$. Then

(i) prove that P is subspace of $\mathbf{R}^3$.

(ii) find a basis and dimension of P**. [Q3UPSC CSE 2022]**

Q3. Show that $S=\{(x,2y,3x): x, y \text{ are real numbers}\}$ is a subspace of $R^3(R)$. Find two bases of S. Also find the dimension of S. [**2cUPSC CSE2021]**

Q4.Express the polynomial $f(x)=x^2+4x-3$ over R as linear combination of polynomials $e_1=x^2-2x+5$, $e_2=2x^2-3x, e_3=x+3$. Also, show that the set $\{e_1,e_2,e_3\}$ forms a basis of all quadratic polynomials over R. **[2aIFoS 2021]**

Q5. Let

$$A=\begin{pmatrix}5 & 7 & 2 & 1\\ 1 & 1 & -8 & 1\\ 2 & 3 & 5 & 0\\ 3 & 4 & -3 & 1\end{pmatrix}$$

(i) Find the rank of matrix A.

(ii) Find the dimension of the subspace

$$V=\left\{(x_1,x_2,x_3,x_4)\in\mathbf{R}^4 \middle| A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=0\right\}$$ **[UPSC CSE 2019]**

Q6. Consider the vectors $x_1=(1,2,1,-1), x_2=(2,4,1,1), x_3=(-1,-2,0,-2)$ and $x_4=(3,6,2,0)$ in $\mathbf{R}^4$. Justify that the linear span of the set $\{x_1,x_2,x_3,x_4\}$ is a subspace of $\mathbf{R}^4$, defined as

$\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbf{R}^4 : 2\xi_1-\xi_2=0, 2\xi_1-3\xi_3-\xi_4=0\}$

Can this subspace be written as $\{(\alpha,2\alpha,\beta,2\alpha,-3\beta):\alpha,\beta\in\mathbf{R}\}$? What is the dimension of this subspace? **[IFoS 2019]**

Q7. Show that the vectors $\alpha_1=(1,0,-1),\alpha_2=(1,2,1),\alpha_3=(0,-3,2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1,\alpha_2,\alpha_3$. **[IFoS 2018]**

Q8. Suppose U and W are distinct four dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of subspace $U\cap W$.

[UPSC CSE 2017]

Q9. If

$W_1=\{(x,y,z)|x+y-z=0\}$

$W_2=\{(x,y,z)|3x+y-2z=0\}$

$W_3=\{(x,y,z)|x-7y+3z=0\}$

then find $\dim(W_1\cap W_2\cap W_3)$ and $\dim(W_1+W_2)$. **[UPSC CSE 2016]**

Q10. Find the dimension of the subspace of $\mathbf{R}^4$, spanned by the set

$\{(1,0,0,0),(0,1,0,0),(1,2,0,1),(0,0,0,1)\}$

Hence find its basis. **[UPSC CSE 2015]**

Q11. Suppose U and W are distinct four-dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimension of $U\cap W$. **[IFoS 2015]**

Q12. Let V and W be the following subspaces of R^4:

$V=\{(a,b,c,d):b-2c+d=0\}$ and

$W=\{(a,b,c,d):a=d,b=2c\}$.

Find a basis and the dimension of (i) V, (ii) W, (iii) $V\cap W$. **[UPSC CSE 2014]**

Q13. Show that $u_1=(1,-1,0),u_2=(1,1,0)$ and $u_3=(0,1,1)$ form a basis for $\mathbf{R}^3$. Express $(5,3,4)$ in terms of u_1,u_2 and u_3. **[IFoS 2014]**

Q14. Let V be an n-dimensional vector space and $T:V\to V$ be an invertible linear operator. If $\beta=\{X_1,X_2,....,X_n\}$ is a basis of V, show that $\beta'=\{TX_1,TX_2,....,TX_n\}$ is also a basis of V.

[UPSC CSE 2013]

Q15. Find the dimension and a basis of the solution space W of the system

$x+2y+2z-s+3t=0, x+2y+3z+s+t=0, 3x+6y+8z+s+5t=0$ **[UPSC CSE 2013]**

Q16. Prove or disprove the following statement:

If $B=\{b_1,b_2,b_3,b_4,b_5\}$ is a basis for $\mathbf{R}^5$ and V is a two-dimensional substance $\mathbf{R}^5$, then V has a basis made of just two members of B. **[UPSC CSE 2012]**

Q17. Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation:

$x_1+2x_2+3x_3-2x_4+4x_5=0$

$2x_1+4x_2+8x_3+x_4+9x_5=0$

$3x_1+6x_2+13x_3+4x_4+14x_5=0$ **[UPSC CSE 2012]**

Q18. Let $V = \mathbf{R}^3$ and $\alpha_1 = (1,1,2), \alpha_2 = (0,1,3)$, $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

[UPSC CSE 2012]

Q19. Show that the subspaces of $\mathbf{R}^3$ spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace. **[UPSC CSE 2011]**

Q20. Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbf{R}$. Let $W = \{A \in V \mid \text{Trace } A = 0\}$.Show that W is a subspace of V. Find a basis of W and dimension of W.

[IFoS 2011]

Q21. Let $V = \{(x, y, z, u) \in \mathbf{R}^4 : y + z + u = 0\}$, $W = \{(x, y, z, u) \in \mathbf{R}^4 : x + y = 0, z = 2u\}$ be two subspaces of $\mathbf{R}^4$. Find base for V, W, V+W and $V \cap W$. **[IFoS 2011]**

Q22. Show that the set

$P[t] = \{at^2 + bt + c / a, b, c \in \mathbf{R}\}$ forms a vector space over the field $\mathbf{R}$. Find a basis for this vector space. What is the dimension of this vector space? **[IFoS 2010]**

Q23. Show that the vectors

$\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$ form a basis for $\mathbf{R}^3$. Find the components of $(1,0,0)$ w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_2\}$. **[IFoS 2010]**

CHAPTER 3. LINEAR TRANSFORMATION

RANGE SPACE & NULL SPACE, RANK AND NULLITY

Q1. Let $v_1=(1,1,-1), v_2=(4,1,1), v_3=(1,-1,2)$ be a basis of $\mathbf{R}^3$ and let $T:\mathbf{R}^3\to\mathbf{R}^2$ be the linear transformation such that $Tv_1=(1,0)$, $Tv_2=(0,1), Tv_3=(1,1)$. Describe the linear transformation T. [**1bIFoS 2022]**

Q2. Let $M_2(R)$ be the vector space of all 2×2 real matrices. Let $B=\begin{bmatrix}1 & -1\\ -4 & 4\end{bmatrix}$. Suppose $T:M_2(R)\to M_2(R)$ is a linear transformation defined by $T(A)=BA$. Find the rank and nullity of T. Find a matrix A which maps to the null matrix.

[UPSC CSE 2020]

Q3. Let F be a subfield of complex numbers and T a function from $F^3\to F^3$ defined by $T(x_1,x_2,x_3)=(x_1+x_2+3x_3, 2x_1-x_2, -3x_1+x_2-x_3)$. What are the conditions on a,b,c such that (a,b,c) be in the null space of T? Find the nullity of T.

[UPSC CSE 2020]

Q4. Consider the matrix mapping $A:R^4\to R^3$, where $A=\begin{pmatrix}1 & 2 & 3 & 1\\ 1 & 3 & 5 & -2\\ 3 & 8 & 13 & -3\end{pmatrix}$. Find a basis and dimension of the image of A and those of the kernel A.

[UPSC CSE 2017]

Q5. Show that the mapping $T:V_2(\bar{R})\to V_3(\bar{R})$ defined as $T(a,b)=(a+b,a-b,b)$ is a linear transformation. Find the range, rank and nullity of T. **[IFoS 2014]**

Q6. Let V be the vector space of 2×2 matrices over $\mathbf{R}$ and let $M=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}$. Let $F:V\to V$ be the linear map defined by $F(A)=MA$. Find a basis and the dimension of

(i) the kernel of W of F

(ii) the image U of F **[IFoS 2013]**

Q7. Let F be a subfield of complex numbers and T a function from $F^3\to F^3$ defined by $T(x_1,x_2,x_3)=(x_1+x_2+3x_3, 2x_1-x_2, -3x_1+x_2-x_3)$. What are the conditions on (a,b,c) such that (a,b,c) be in the null space of T? Find the nullity of T.

[IFoS 2013]

Q8. Let $T:\mathbf{R}^3\to\mathbf{R}^3$ be the linear transformation defined by $T(\alpha,\beta,\gamma)=(\alpha+2\beta-3\gamma, 2\alpha+5\beta-4\gamma, \alpha+4\beta+\gamma)$. Find a basis and the dimension of the image of T and the kernel of T. **[UPSC CSE 2012]**

Q9. Find the nullity and a basis of the null space of the linear transformation $A:R^{(4)}\to R^{(4)}$ given by the matrix

$$A=\begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$$ **[UPSC CSE 2011]**

Q10. Find the linear transformation from $\mathbf{R}^3$ into $\mathbf{R}^3$ which has its range the subspace spanned by $(1,0,-1),(1,2,2)$. **[IFoS 2011]**

Q11. What is the null space of the differentiation transformation

$$\frac{d}{dx}:P_n \to P_n$$

where P_n is the space of all polynomials of degree $\leq n$ over the real numbers? What is the null space of the second derivative as a transformation of P_n? What is the null space of the kth derivative?

[UPSC CSE 2010]

***ALGEBRA OF LTs**

Q1. Let T be a linear transformation from a vector space V over reals into V such that $T-T^2=I$. Show that T is invertible. **[UPSC CSE 2010]**

CHAPTER 4. TO FIND MATRIX OF LT

Q1. Let $T:\mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-3\\2\\8\end{pmatrix}$. Find $T\begin{pmatrix}2\\4\end{pmatrix}$. **[1bUPSC CSE 2022]**

Q2. Find the matrix associated with the linear operator on $V_3(R)$ defined by $T(a,b,c)=(a+b,a-b,2c)$ with respect to the ordered basis $B=\{(0,1,1),(1,0,1),(1,1,0)\}$. **[1bUPSC CSE 2021]**

Q3. Let $T:\mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(x,y,z)=(2x,-3y,x+y)$ and $B_1=\{(-1,2,0),(0,1,-1),(3,1,2)\}$ be a basis of $\mathbf{R}^3$. Find the matrix representation of T relative to the basis B_1. **[UPSC CSE 2020]**

Q4. Let $T:\mathbf{R}^2 \to \mathbf{R}^2$ be a linear map such that $T(2,1)=(5,7)$ and $T(1,2)=(3,3)$. If A is the matrix corresponding to T with respect to the standard bases e_1,e_2, then find Rank (A).

[UPSC CSE 2019]

Q5. Let $T:R^3 \to R^3$ be a linear operator on R^3 defined by $T(x,y,z)=(2y+z,x-4y,3x)$. Find the matrix of T in the basis $\{(1,1,1),(1,1,0),(1,0,0)\}$. **[IFoS 2019]**

Q6. Let $T:V_2(R)\to V_2(R)$ be a linear transformation defined by $T(a,b)=(a,a+b)$. Find the matrix of T, taking (e_1,e_2) as a basis for the domain and $\{(1,1),(1,-1)\}$ as a basis for the range.

[IFoS 2018]

Q7. If $M_2(R)$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T:M_2(R)\to P_2(x)$, such that $T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)=a+c+(a-d)x+(b+c)x^2$, with respect to the standard bases of $M_2(R)$ and $P_2(x)$. Further find the null space of T.

[UPSC CSE 2016]

Q8. If $T:P_2(x)\to P_3(x)$ is such that $T(f(x))=f(x)+5\int_0^x f(t)\,dt$, then choosing $\{1,1+x,1-x^2\}$ and $\{1,x,x^2,x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T.

[UPSC CSE 2016]

Q9. Let $T:\mathbf{R}^3\to\mathbf{R}^4$ be given by $T(x,y,z)=(2x-y,2x+z,x+2z,x+y+z)$. Find the matrix of T with respect to standard basis of $\mathbf{R}^3$ and $\mathbf{R}^4$ (i.e., $(1,0,0),(0,1,0)$ etc.). Examine if T is a linear map. **[IFoS 2016]**

Q10. Let T be a linear map such that $T:V_3 \to V_2$ defined by $T(e_1)=2f_1-f_2, T(e_2)=f_1+2f_2, T(e_3)=0f_1+0f_2$, where e_1,e_2,e_3 and f_1,f_2 standard basis in V_3 and V_2. Find the matrix of T relative to these basis. Further take two other basis $B_1[(1,1,0)(1,0,1)(0,1,1)]$ and $B_2[(1,1)(1,-1)]$. Obtain the matrix T_1 relative B_1 and B_2.

[IFoS 2016]

Q11. Let $V=R^3$ and $T\in A(V)$, for all $a_i \in A(V)$, be defined by $T(a_1,a_2,a_3)=(2a_1+5a_2+a_3,-3a_1+a_2-a_3,-a_1+2a_2+3a_3)$. What is the matrix T relative to the basis $V_1=(1,0,1)$ $V_2=(-1,2,1)$ $V_3=(3,-1,1)$? **[UPSC CSE 2015]**

Q12. Let G be the linear operator on $\mathbf{R}^3$ defined by $G(x,y,z)=(2y+z,x-4y,3x)$. Find the matrix representation of G relative to the basis $S=\{(1,1,1),(1,1,0),(1,0,0)\}$. **[IFoS 2015]**

Q13. Consider the linear mapping $F:\mathbf{R}^2 \to \mathbf{R}^2$ given as $F(x,y)=(3x+4y,2x-5y)$ with usual basis. Find the matrix associated with the linear transformation relative to the basis $S=\{u_1,u_2\}$ where $u_1=(1,2), u_2=(2,3)$. **[IFoS 2014]**

Q14. Let P_n denote the vector space of all real polynomials of degree atmost n and $T:P_2 \to P_3$ be a linear transformation given by $T(p(x))=\int_0^x p(t)\,dt,\ \ p(x)\in p_2$. Find the matrix of T with respect to the bases $\{1,x,x^2\}$ and $\{1,x,1+x^2,1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T. **[UPSC CSE 2013]**

Q15. Consider the linear mapping $f:\mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y)=(3x+4y,2x-5y)$. Find the matrix A relative to the basis $\{(1,0),(0,1)\}$ and the matrix B relative to the basis $\{(1,2),(2,3)\}$.

[UPSC CSE 2012]

Q16. Let $f:\mathbf{R} \to \mathbf{R}^3$ be a linear transformation defined by $f(a,b,c)=(a,a+b,0)$. Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1,e_2,e_3) and the basis (e_1',e_2',e_3') where $e_1'=(1,1,0), e_2'=(0,1,1)$, $e_3'=(1,1,1)$. Also, show that there exists an invertible matrix P such that $B=P^{-1}AP$. **[IFoS 2012]**

Q16. Find the matrix representation of linear transformation T on $V_3(\mathbf{R})$ defined as $T(a,b,c)=(2b+c,a-4b,3a)$ corresponding to the basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$. **[IFoS 2012]**

TO FIND LT WHEN MATRIX IS GIVEN

Q1. If $A=\begin{bmatrix}1 & -1 & 2\\ -2 & 1 & -1\\ 1 & 2 & 3\end{bmatrix}$ is the matrix representation of a linear transformation $T:P_2(x)\to P_2(x)$ with respect to the bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1,1+x,1+x^2\}$, then find T. **[UPSC CSE 2016]**

Q2. Let $M=\begin{pmatrix}4 & 2 & 1\\ 0 & 1 & 3\end{pmatrix}$. Find the unique linear transformation $T:\mathbf{R}^3\to\mathbf{R}^3$ so that M is the matrix of T with respect to the basis $\beta=\{v_1=(1,0,0), v_2=(1,1,0), v_3=(1,1,1)\}$ of $\mathbf{R}^3$ and $\beta'=\{w_1=(1,0), w_2=(1,1)\}$ of $\mathbf{R}^2$. Also find $T(x,y,z)$. **[UPSC CSE 2010]**

CHAPTER 5. CAYLEY- HAMILTON THEOREM

Q1. If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, then show that $A^2=A^{-1}$ (without finding A^{-1}).

[1aUPSC CSE 2022]

Q2. Using the Cayley-Hamilton theorem, find the inverse of the matrix $A=\begin{bmatrix}2 & -1 & 3\\ 1 & 0 & -2\\ 4 & 2 & 1\end{bmatrix}$. **[4bIFoS 2021]**

Q3. Verify Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$, hence find its inverse. Also, express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.

[IFoS 2020]

Q4. State the Cayley-Hamilton theorem. Use this theorem to find A^{100}, where $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$ **[UPSC CSE 2019]**

Q5. State the Cayley-Hamilton theorem. Verify this theorem for the matrix $A=\begin{bmatrix}1 & 0 & 2\\ 0 & -1 & 1\\ 0 & 1 & 0\end{bmatrix}$. Hence find A^{-1}. **[IFoS 2017]**

Q6. If $A=\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$, the find $A^{14}+3A-2I$. **[UPSC CSE 2016]**

Q7. For the matrix $A=\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4+3A^3-9A^2$.

[IFoS 2016]

Q8. If matrix $A=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then find A^{30}. **[UPSC CSE 2015]**

Q9. Verify Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and hence find its inverse. Also, find the matrix represented by $A^5-4A^4-7A^3+11A^2-A-10I$.

[UPSC CSE 2014]

Q10. For the matrix $A=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that $A^n=A^{n-2}+A^2-I,\ n\geq 3$.

[IFoS 2014]

Q11. Find the characteristic equation of the matrix $A=\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$. **[IFoS 2013]**

Q12. Verify Cayley-Hamilton theorem for the matrix $A=\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.

[IFoS 2012]

Q13. Verify the Cayley-Hamilton theorem for the matrix

$$A=\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$$

Using this, show that A is non-singular and find A^{-1}. **[UPSC CSE 2011]**

Q14. Find the characteristic polynomial of the matrix

$$A=\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10}. **[IFoS 2011]**

Q15. Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley-Hamilton theorem for this matrix and hence find its inverse. **[IFoS 2010]**

MINIMAL POLYNOMIAL

Q16. Find the minimal polynomial of matrix $A=\begin{pmatrix}4 & -2 & 2\\ 6 & -3 & 4\\ 3 & -2 & 3\end{pmatrix}$. **[IFoS 2015]**

EIGEN VALLUES, EIGEN VECTORS

Q1. Find a linear map $T:\mathbf{R}^2\to\mathbf{R}^2$ which rotates each vector of $\mathbf{R}^2$ by an angle θ. Also, prove that for $\theta=\frac{\pi}{2}$, T has no eigenvalue in $\mathbf{R}$.

[4aUPSC CSE 2022]

Q2. The eigenvalues of a real symmetric matrix A are -1, 1 and -2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-110)^T,(001)^T$ and $\frac{1}{\sqrt{2}}(-1-10)^T$ respectively. Find the matrix A^4. **[IFoS 2019]**

Q3. Consider the singular matrix

$$A=\begin{bmatrix}-1 & 3 & -1 & 1\\ -3 & 5 & 1 & -1\\ 10 & -10 & -10 & 14\\ 4 & -4 & -4 & 8\end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1100)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers p,q,r that satisfy the matrix equation $A^4+pA^3+qA^2+rA=0$. **[IFoS 2019]**

Q4. Prove that the eigenvalues of a Hermitian matrix are all real. **[IFoS 2018]**

Q5. Prove that distinct non-zero eigenvectors of a matrix are linearly independent. **[UPSC CSE 2017]**

Q6. If $A=\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, then find the eigenvalues and eigenvectors of A.

[UPSC CSE 2016]

Q7. Prove that eigenvalues of a Hermitian matrix are all real.

[UPSC CSE 2016]

Q8. Find the eigen values and eigen vectors of the matrix:

$\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$. **[UPSC CSE 2015]**

Q9. Let $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$. Find the eigen values of A and the corresponding eigen vectors.

[UPSC CSE 2014]

Q10. Prove that the eigen values of a unitary matrix have absolute value 1.

[UPSC CSE 2014]

Q11. Let $B=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over:

(i) the real find R

(ii) the complex field C **[IFoS 2014]**

Q12. Let A be a square matrix and A* be its adjoint, show that the eigenvalues of matrices AA* and A*A are real. Further show that trace (AA*) = trace (A*A). **[UPSC CSE 2013]**

Q13. Let $A=\begin{bmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{bmatrix}$ where $\omega(\neq 1)$ is a cube root of unity. If $\lambda_1,\lambda_2,\lambda_3$ denote the eigenvalues of A^2, show that $|\lambda_1|+|\lambda_2|+|\lambda_3|\leq 9$. **[UPSC CSE 2013]**

Q14. Let A be a Hermetian matrix having all distinct eigenvalues $\lambda_1,\lambda_2,....,\lambda_n$. If $X_1,X_2,....,X_n$ are corresponding eigenvectors then show that the $n\times n$ matrix C whose kth column consists of the vector X_k is non singular. **[UPSC CSE 2013]**

Q15. If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A. **[UPSC CSE 2012]**

Q16. Let $A=\begin{bmatrix}2 & -2 & 2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of the matrix B^3 where $B=C^{-1}AC$. **[UPSC CSE 2011]**

Q17. If $\lambda_1,\lambda_2,\lambda_3$ are the eigenvalues of the matrix

$$A=\begin{pmatrix}26 & -2 & 2\\ 2 & 21 & 4\\ 4 & 2 & 28\end{pmatrix}$$

show that $\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}\leq\sqrt{1949}$.

[UPSC CSE 2010]

Q18. Let A and B be $n\times n$ matrices over reals. Show that I - BA is invertible if I - AB is invertible. Deduce that AB and BA have the same eigenvalues.

[UPSC CSE 2010]

DIAGONALIZATION OF MATRIX

Q1. Are the matrices $A=\begin{bmatrix}2 & 4\\ 0 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$ similar? Justify your answer.

[IFoS 2022]

(b) Given the matrix $A=\begin{bmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$, find a similarity transformation that diagonalises the matrix A.

[2aIFoS 2021]

Q2. Let $A=\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

[UPSC CSE 2017]

Q3. Find the eigenvalues and the corresponding eigenvectors for the matrix $A=\begin{bmatrix}0 & -2\\ 1 & 3\end{bmatrix}$. Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $D=P^{-1}AP$. **[IFoS 2017]**

Q4. Examine whether the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix P such that $p^{-1}AP$ is a diagonal matrix.

[IFoS 2014]

Q5. Let

$$A=\begin{pmatrix}1 & -3 & 3\\ 0 & -5 & 6\\ 0 & -3 & 4\end{pmatrix}$$

Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. **[IFoS 2011]**

Q6. Let $A=\begin{pmatrix}5 & -6 & -6\\ -1 & 4 & 2\\ 3 & -6 & -4\end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

[IFoS 2010]

CONGRUENCE AND SIMILARITY

Q1. When is a matrix A said to be similar to another matrix B?
Prove that
(i) if A is similar to B, then B is similar to A.
(ii) two similar matrices have the same eigenvalues.
Further, by choosing appropriately the matrices A and B, show that the converse of (ii) above may not be true. **[IFoS 2020]**

Q2. Show that if A and B are similar $n\times n$ matrices, then they have the same eigenvalues.

[UPSC CSE 2018]

Q3. Show that the matrices

$A=\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 3\\ 0 & 2 & 2\\ 3 & 2 & 0\end{bmatrix}$ are congruent. **[IFoS 2018]**

Q16. Show that similar matrices have the same characteristic polynomial.

[UPSC CSE 2017]

Q17. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigen values of a $n \times n$ square matrix. A with corresponding eigen vectors $X_1, X_2, ..., X_n$. If B is a matrix similar to A show that the eigen values of B are same as that of A. Also find the relation between the eigen vectors of B and eigen vectors of A. **[UPSC CSE 2011]**

BILINEAR AND QUADRATIC FORMS

Q1. Reduce the following quadratic form over the real field $\mathbf{R}$ to orthogonal form:

$q(x,y,z) = x^2 + 5y^2 - 4z^2 + 2xy - 4xz$. **[3aIFoS 2022]**

Q2. Consider the following quadratic form:

$q(x,y,z) = 2x^2 + 2y^2 + 6z^2 + 2xy - 6yz - 6zx$

where (x,y,z) are the coordinates of the vector X with respect to the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of R^3. Find the expression of $q(x,y,z)$ with respect to the basis

$B=\left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$.

Is q positive definite? Justify your answer. **[1aIFoS 2021]**

Q3. Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. **[IFoS 2016]**

Q4. Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form $q(x,y) = 2x^2 - 4xy + 5y^2$. **[IFoS 2013]**

Q5. Find an orthogonal transformation to reduce the quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical form. **[IFoS 2011]**

Q6. Determine whether the quadratic form $q = x^2 + y^2 + 2xz + 4yz + 3z^2$ is positive definite.

[IFoS 2010]

SPECIAL MATRICES,ORTHOGONAL AND UNITARY MATRICES

Q1. Prove that the product of two Hermitian matrices A, B is Hermitian if and only if A and B commute. Give an example of a pair of 3×3 symmetric matrices such that their product is again symmetric (do not consider only diagonal matrices) and also check whether they commute or not. **[1bIFoS 2021]**

Q2. Prove that the eigen vectors, corresponding to two distinct eigen values of a real symmetric matrix, are orthogonal. **[3cUPSC CSE 2021]**

(ii) For two square matrices A and B of order 2, show that trace (AB) = trace (BA). Hence show that $AB - BA \neq I_2$, where I_2 is an identity matrix of order 2.

Q3. Reduce the following matrix to a row-reduced echelon form and hence also, find its rank:

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

(ii) Find the eigen values and the corresponding eigen vectors of the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, over the complex-number field. **[4aUPSC CSE 2021]**

Q4. Define an $n \times n$ matrix as $A = I - 2u \cdot u^T$, where u is a unit column vector.
(i) Examine if A is a symmetric.
(ii) Examine if A is orthogonal.
(iii) Show that trace $(A) = n - 2$

(iv) Find $A_{3\times3}$, when $u = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$. **[UPSC CSE 2020]**

Q5. If A is a skew-symmetric matrix and I+A be a non-singular matrix, then show that $(I - A)(I + A)^{-1}$ is orthogonal. **[IFoS 2020]**

Q6. Let A and B be two orthogonal matrices of same order and det A + det B = 0. Show that A+B is a singular matrix. **[UPSC CSE 2019]**

Q7. Let A be a square matrix of order 3 such that each of its diagonal element is 'a' and each of its off-diagonal elements is 1. If B = bA is orthogonal, determine the values of a and b. **[IFoS 2017]**

Q8. Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

[IFoS 2015]

Q9. Prove that the eigen values of a unitary matrix have absolute value 1.
[UPSC CSE 2014]

DIFFERENTIAL CALCULUS

- FUNCTIONS OF ONE VARIABLES:
- LIMIT, CONTINUITY AND DIFFERENTIABILITY, INDETERMINATE FORMS
- MEAN VALUE THEOREMS - ROLL'S, LAGRANGE'S, CAUCHY'S, TAYLOR'S
- FUNCTIONS OF TWO VARIABLES:
- LIMIT CONTINUITY AND DIFFERENTIABILITY
- PARTIAL DERIVATIVES
- APPLICATIONS- MAXIMA-MINIMA, TOTAL DIFFERENTIATION
- LAGRANGE'S MULTIPLYER'S METHOD
- JACOBIAN

CHAPTER 1- FUNCTIONS OF ONE VARIABLES: LIMIT, CONTINUITY AND DIFFERENTIABILITY, INDETERMINATE FORMS

Updated

Q1 Evaluate $\lim_{x\to\infty}\left(e^x+x\right)^{\frac{1}{x}}$. **[UPSC CSE 2022Q1(c)]**

Q3(ii) If $\int_0^x f(t)\,dt = x+\int_x^1 tf(t)\,dt$, then find the value of $f(1)$. **[UPSC CSE 2021]**

Q1(c) Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$. **[IFoS 2022]**

Q1(c) $\Delta(x)=\begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

where f is a real valued differentiable function and α is a constant. Find $\lim_{x\to 0}\frac{\Delta(x)}{x}$. **[UPSC CSE 2021]**

Q1. Evaluate $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$. **[1c UPSC CSE 2020]**

Q2. Given that $f(x+y)=f(x)f(y)$, $f(0)\neq 0$, for all real *x*, *y* and $f'(0)=2$. Show that for all real *x*, $f'(x)=2f(x)$. Hence find $f(x)$. **[1c 2020 IFoS]**

Q3. Evaluate

$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$ **[4c(i) 2020 IFoS]**

Q4. Let $f:\left[0,\frac{\pi}{2}\right]\to \mathbf{R}$ be a continuous function such that

$f(x)=\frac{\cos^2 x}{4x^2-\pi^2}$, $0\le x<\frac{\pi}{2}$

Find the value of $f\left(\frac{\pi}{2}\right)$. **[1a UPSC CSE 2019]**

Q5. Is $f(x)=|\cos x|+|\sin x|$ differentiable at $x=\frac{\pi}{2}$? If yes, then find its derivative at $x=\frac{\pi}{2}$. If no, then give a proof of it. **[2a UPSC CSE 2019]**

Q6. Determine if $\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2}$ exists or not. If the limit exists, then find its value.

[1c UPSC CSE 2018]

Q7. Evaluate the following limit:

$\underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ **[1c UPSC CSE 2015]**

Q8. Let $f(x)$ be a real-valued function defined on the interval $(-5,5)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt$ for all $x\in(-5,5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $\left(f^{-1}\right)'(2)$. **[1c 2015 IFoS]**

Q9. If $\sqrt{x+y}+\sqrt{y-x}=c$, find $\frac{d^2y}{dx^2}$.**[3b 2015 IFoS]**

Q10. Evaluate $\lim_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x}-\frac{3}{x^4}\right)$. **[4d 2015 IFoS]**

Q11. Show that the function given by

$$f(x)=\begin{cases}\frac{x\left(e^{1/x}-1\right)}{\left(e^{1/x}+1\right)}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous but not differentiable at $x=0$. **[1c 2014 IFoS]**

Q12. Evaluate:

$\lim_{x\to 0}\left(\frac{e^{ax}-b^{bx}+\tan x}{x}\right)$. **[1a 2013 IFoS P-2]**

Q13. Show that the function defined as

$$f(x)=\begin{cases}\frac{\sin 2x}{x} & \text{when } x\neq 0\\ 1 & \text{when } x=0\end{cases}$$

has removable discontinuity at the origin. **[3d 2012 IFoS]**

Q14. Evaluate:

(i) $\lim_{x\to 2}f(x)$, where $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ \pi, & x=2\end{cases}$. **[3a(i) UPSC CSE 2011]**

Q15. Let the function *f* be defined by

$$f(t)=\begin{cases}0, \text{ for } t<0\\ t, \text{ for } 0\leq t\leq 1\\ 4, \text{ for } t>1\end{cases}$$

(i) Determine the function $F(x)=\int_0^x f(t)\,dt$

(ii) Where is F non-differentiable? Justify your answer. **[1d 2011 IFoS]**

Q16. Let *f* be a function defined on $\mathbf{R}$ such that

$f(x+y)=f(x)+f(y),\ x,y\in\mathbf{R}$. **[1d 2010 IFoS]**

If *f* is differentiable at one point of $\mathbf{R}$, then prove that *f* is differentiable on $\mathbf{R}$.

Q17. Let

$$f(x)=\begin{cases}\frac{|x|}{2}+1 \text{ if } x<1\\ \frac{x}{2}+1 \text{ if } 1\le x<2\\ -\frac{|x|}{2}+1 \text{ if } 2\le x\end{cases}$$

What are the points of discontinuity of *f*, if any? What are the points where *f* is not differentiable if any? Justify yours answers. **[1d UPSC CSE 2009 P-2]**

Q18. Obtain the values of the constants, *a*, *b* and *c* for which the function defined by

$$f(x)=\begin{cases}\frac{\sin(a+1)x+\sin x}{x}, & x<0\\ \frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}}, & x>0\end{cases}$$

is continuous at $x=0$. **[IFoS 2008 1(c)]**

Q19. Prove that function

$$f(x)=\begin{cases}\frac{x\left(e^{1/x}-1\right)}{e^{rx+1}}, & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous but not differentiable at $x=0$.

CHAPTER-2. MEAN VALUE THEOREMS - ROLL'S, LAGRANGE'S, CAUCHY'S AND TAYLOR'S MVT

Q1(d) Show that between any two roots of $e^x \cos x = 1$, there exists at least one root $e^x \sin x - 1 = 0$.

[UPSC CSE 2021]

Q2. (a) Does $f(x) = x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$ satisfy the conditions of the mean value theorem? If yes, then justify your answer and find $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}\left(a = \frac{1}{2}, b = 3\right)$. **[IFoS 2021]**

Q1. Find the Taylor's series expansion for the function $f(x) = \log(1+x), -1 < x < \infty$, about $x = 2$ with Lagrange's form of remainder after 3-terms.

[1d 2020 IFoS]

Q2. Justify by using Rolle's theorem or mean value theorem that there is no number k for which the equation $x^3 - 3x + k = 0$ has two distinct solutions in the interval $[-1, 1]$. **[1d 2019 IFoS]**

Q3. If $f : [a,b] \to R$ be continuous in $[a,b]$ and derivable in (a,b), where $0 < a < b$, show that for $c \in (a,b)$ $f(b) - f(a) = cf'(c)\log(b/a)$. **[1c 2018 IFoS]**

Q4. If ϕ and ψ be two functions derivable in $[a,b]$ and $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a,b]$, there lies exactly one root of $\psi(x) = 0$. **[2c 2018 IFoS]**

Q5. Using the Mean Value Theorem, show that

(i) $f(x)$ is constant in $[a,b]$, if $f'(x) = 0$ in $[a,b]$

(ii) $f(x)$ is a decreasing function in (a,b), if $f'(x)$ exists and is < 0 everywhere in (a,b).

[1c 2017 IFoS]

Q6. Show that $\frac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$. **[1b 2016 IFoS]**

Q7. Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on $[2,3]$, where the tangent is parallel to the chord joining the end points of the curve. **[2c 2016 IFoS]**

Q8. Prove that between two real roots of $e^x \cos x + 1 = 0$, a real root of $e^x \sin x + 1 = 0$ lies.

[1c UPSC CSE 2014]

Q9. Find C of the Mean value theorem, if $f(x) = x(x-1)(x-2), a = 0, b = \frac{1}{2}$ and C has usual meaning. **[1e 2013 IFoS]**

Q10. Prove that if $a_0, a_1, a_2, ..., a_a$ are the real numbers such that $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+......+\frac{a_{n-1}}{2}+a_n=0$ then there exists at least one real number x between 0 and 1 such that $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0$. **[3a 2013 IFoS]**

Q11. Let f be a function on $\mathbf{R}$ such that f$f(0)=-3$ and $f'(x)\le 5$ for all values of x in $\mathbf{R}$. How large can $f(2)$ possibly be? **[1d UPSC CSE 2011]**

Q12. A twice-differentiable function $f(x)$ is such that $f(a)=0=f(b)$ and $f(c)>0$ for $a<c<b$. Prove that there is at least one point $\xi, a<\xi<b$, for which $f''(\xi)<0$.

[1c UPSC CSE 2010]

Q13. Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$. **[1c 2010 IFoS]**

Q14. Suppose that f'' is continuous on $[1,2]$ and that f has three zeros in the interval $(1,2)$. Show that f'' has at least one zero in the interval $(1,2)$. **[1c UPSC CSE 2009]**

Q15. If $f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h)\ 0<\theta<1$

Find θ, when $h=1$ and $f(x)=(1-x)^{5/2}$. **[1c 2009 IFoS]**

MAXIMA- MINIMA

Q1. Consider the function $f(x)=\int_0^x (t^2-5t+4)(t^2-5t+6)\,dt$

(i) Find the critical points of the function $f(x)$

(ii) Find the points at which local minimum occurs

(iii) Find the points at which local maximum occurs

(iv) Find the number of zeros of the function $f(x)$ in $[0,5]$.

[3a UPSC CSE 2020]

Q2. Find the maximum and the minimum value of the function $f(x)=2x^3-9x^2+12x+6$ on the interval $[2,3]$. **[3a UPSC CSE 2019]**

Q3. Find the shortest distance from the point $(1,0)$ to the parabola $y^2=4x$.

[2b UPSC CSE 2018]

Q4. Show that the maximum rectangle inscribed in a circle is a square.

[1a 2018 IFoS]

Q5. A conical tent is of given capacity. For the least amount of Canvas required, for it, find the ratio of its height to the radius of its base. **[2b UPSC CSE 2015]**

Q6. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

[3a UPSC CSE 2014]

Q7. Find the difference between the maximum and the minimum of the function $\left(a-\frac{1}{a}-x\right)\left(4-3x^2\right)$ where a is a constant and greater than zero. **[1c 2009 IFoS]**

Q8. A wire of length b is cut into two parts which are bent in the form of a square and a circle respectively. Find the minimum value of the sum of the areas so formed. **[3c 2008 IFoS]**

Q9. Find the maximum and minimum values of x^3+y^3 where $ax^2+2hxy+by^2=1$.

Q10. A cone is circumscribed to a sphere of radius r; show that when the volume of the cone is a minimum, its altitude is $4r$ and its semi-vertical angle $\sin^{-1}\frac{1}{3}$. **[Madras 1953; P.U. 1930]**

Q11. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r/\sqrt{3}$. **[P.U. 1934]**

Q12. Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

Q13. Show that x^x is minimum for $x=e^{-1}$

Q14. Investigate for maximum and minimum values the function $\sin x+\cos 2x$.

CHAPTER 3. FUNCTIONS OF TWO VARIABLES

LIMIT, CONTINUITY AND DIFFERENTIABILITY

Q2(b) Given that $f(x,y)=|x^2-y^2|$. Find $f_{xy}(0,0)$ and $f_{yz}(0,0)$. Hence show that $f_{xy}(0,0)=f_{yz}(0,0)$. **[UPSC CSE 2021]**

Q1. Let $f:D(\subseteq \mathbf{R}^2)\to \mathbf{R}$ be a function and $(a,b)\in D$. If $f(x,y)$ is continuous at (a,b) then show that the functions $f(x,b)$ and $f(a,y)$ are continuous at $x=a$ and at $y=b$ respectively.

[1b UPSC CSE 2019]

Q2. Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x-y}, & (x,y)\neq(1,-1),(1,1)\\ 0, & (x,y)=(1,1),(1,-1)\end{cases}$$

is continuous and differentiable at $(1,-1)$. **[1b P-2 UPSC CSE 2019]**

Q3. Let

$f(x,y)=xy^2$, if $y>0$

$=-xy^2$, if $y\le 0$

Determine which of $\dfrac{\partial f}{\partial x}(0,1)$ and $\dfrac{\partial f}{\partial y}(0,1)$ exists and which does not exist.

[3b UPSC CSE 2018]

Q4. Consider the function *f* defined by

$$f(x,y)=\begin{cases}xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{where } x^2+y^2\neq 0\\ 0, & \text{where } x^2+y^2=0\end{cases}$$

Show that $f_{xy}\neq f_{yx}$ at $(0,0)$. **[2b 2018 P-2 IFoS]**

Q5. If

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases},$$

calculate $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at $(0,0)$. **[3c UPSC CSE 2017]**

Q6. Let

$$f(x,y)=\begin{cases}\dfrac{2x^4y-5x^2y^2+y^5}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Find a $\delta>0$ such that $|f(x,y)-f(0,0)|<.01$, whenever $\sqrt{x^2+y^2}<\delta$.

[3b UPSC CSE 2016]

Q7. For the function

$$f(x,y)=\begin{cases}\dfrac{x^2-x\sqrt{y}}{x^2+y}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Examine the continuity and differentiability. **[4d UPSC CSE 2015]**

Q8. Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Also, discuss the continuity of f_{xy} and f_{yx} at $(0,0)$. **[3b UPSC CSE 2013]**

Q9. Define a function *f* of two real variables in the *xy*-plane by

$$f(x,y)=\begin{cases}\dfrac{x^3\cos\frac{1}{y}+y^3\cos\frac{1}{x}}{x^2+y^2} & \text{for } x,y\neq 0\\ 0, & \text{otherwise}\end{cases}$$

Check the continuity and differentiability of *f* at $(0,0)$. **[1a UPSC CSE 2012]**

Q10. Find $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y}{x^3+y^3}$ if it exists. **[1c UPSC CSE 2011]**

Q11. Let $f:\mathbf{R}^2\to\mathbf{R}$ be defined as

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Is *f* continuous at $(0,0)$? Compute partial derivatives of *f* at any point (x,y) if exist.

[3b UPSC CSE 2009]

Q12. A function $f(x,y)$ is defined as follows:

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Show that $f_{xy}(0,0)=f_{yx}(0,0)$. **[2017 4b IFoS]**

Q13. Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$ given that

$$f(x,y)=\begin{cases}x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y} & \text{if } xy\neq 0\\ 0, & \text{otherwise}\end{cases}$$

. **[3a 2017 P-2 IFoS]**

Q14. Examine if the function $f(x,y)=\dfrac{xy}{x^2+y^2},(x,y)\neq(0,0)$ and $f(0,0)=0$ is continuous at $(0,0)$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points other than origin. **[2016 1c IFoS]**

Q15. Examine the continuity of $f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y)\neq(0,0)\\ \dfrac{1}{2}, & (x,y)=(0,0)\end{cases}$ at the point $(0,0)$.

[3b 2016 P-2 IFoS]

Q16. Obtain $\dfrac{\partial^2 f(0,0)}{\partial x\partial y}$ and $\dfrac{\partial^2 f(0,0)}{\partial y\,\partial x}$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy(3x^2-2y^2)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Also, discuss the continuity of $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at $(0,0)$.

[3b P-2 UPSC CSE 2014]

Q17. Let $f(x,y)=\begin{cases}\dfrac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 1, & \text{if } (x,y)=(0,0)\end{cases}$

Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ through $f(x,y)$ is not continuous at $(0,0)$.

[2b P-2 UPSC CSE 2012]

Q18. Show that the function defined by

$$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat.

[2011 1c IFoS]

Q19. Let

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Show that:

(i) $f_{xy}(0,0)\neq f_{yx}(0,0)$

(ii) f is differentiable at $(0,0)$ **[2010 3c IFoS]**

PARTIAL DERIVATIVES

Q1. If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y},\ x\neq y$ then show that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=(1-4\sin^2 u)\sin 2u$.

[3c P-2 UPSC CSE 2020]

Q2. If $u = u(y-z, z-x, x-y)$, then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

If $u(x,y,z) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

[1b 2020 P-2 IFoS]

Q3. If

$$u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$$

then show that $\sin^2 u$ is a homogeneous function of x and y of degree $-\frac{1}{6}$. Hence show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right).$$ **[4c(i) UPSC CSE 2019]**

Q4. If $f = f(u,v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$ **[2018 3b IFoS]**

Q5. If $u(x,y) = \cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}, 0 < x < 1,\ 0 < y < 1$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

[3c 2016 P-2 IFoS]

Q6. If the three thermodynamic variables P,V,T are connected by a relation, $f(P,V,T) = 0$. Show that, $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$. **[2012 1c IFoS]**

Q7. If $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\left(\frac{n}{2\mu}\right)}$. **[2012 1d IFoS]**

Q8. If

$u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$.

[2012 P-2 IFoS]

Q9. If $f(x,y)$ is a homogeneous function of degree n in x and y, and has continuous first and second-order partial derivatives, then show that

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

(ii) $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$ **[4b UPSC CSE 2010]**

CHAPTER-4. APPLICATIONS- MAXIMA-MINIMA, TOTAL DIFFERENTIATION

Q1. Using differentials, find an approximate value of $f(4\cdot1, 4\cdot9)$ where $f(x,y)=(x^3+x^2y)^{\frac{1}{2}}$.

[2c P-2 UPSC CSE 2019]

Q2. Find the maxima and minima for the function
$f(x,y)=x^3+y^3-3x-12y+20$
Also find the saddle points (if any) for the function. **[2017 2c IFoS]**

Q3. Find the relative maximum and minimum values of the function
$f(x,y)=x^4+y^4-2x^2+4xy-2y^2$. **[3b P-2 UPSC CSE 2016]**

Q4. Find the maxima and minima of the function $f(x,y)=x^3+y^3-3x-12y+20$.

[2016 1c P-2 IFoS]

Q5. Locate the stationary points of the function $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature. **[2013 2b IFoS]**

Q6. Find the points of local extrema and saddle points of the function f of two variables defined by
$f(x,y)=x^3+y^3-63(x+y)+12xy$. **[3a UPSC CSE 2012]**

Q7. Let $f(x,y)=y^2+4xy+3x^2+x^3+1$. At what points will $f(x,y)$ have a maximum or minimum? **[3c P-2 UPSC CSE 2013]**

Q8. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. **[2012 3b IFoS]**

Q9. Show that the function defined by
$f(x,y,z)=3\log(x^2+y^2+z^2)-2x^2-2y^3-2z^3$, $(x,y,z)\neq(0,0,0)$ has only one extreme value, $\log\left(\frac{3}{e^2}\right)$. **[2011 3d IFoS]**

Q10. Find the maxima, minima and saddle points of the surface $Z=(x^2-y^2)e^{(-x^2-y^2)/2}$.

[3d P-2 UPSC CSE 2010]

Q11. If $x=3\pm0.01$ and $y=4\pm0.01$, with approximately what accuracy can you calculate the polar coordinates r and θ of the point $P(x,y)$? Express your estimates as percentage changes of the values that r and θ have at the point $(3,4)$.

CHAPTER 5. ASYMPTOTES, CURVE TRACING

Q4(b) Trace the curve $y^2x^2 = x^2 - a^2$, where a is a real constant.

[UPSC CSE 2022]

Q1. Find all the asymptotes of the curve $(2x+3)y = (x-1)^2$.

[1d UPSC CSE 2020]

Q2. Find the asymptotes of the curve $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$. **[3a 2020 IFoS]**

Q3. Find all the asymptotes of the curve $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$.

[4d 2013 IFoS]

Q4. Determine the asymptotes of the curve

$x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$.

[3b 2009 IFoS]

CHAPTER 6. LAGRANGE'S MULTIPLYER'S METHOD

Q2(b) Using Lagrange's undetermined multipliers method, find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **[IFoS 2022]**

Q2(b) A wire length l is cut into two parts which are bent in the form of a square and a circle respectively. Using Lagrange's method of undetermined multipliers, find the least value of the sum of the areas so formed.

[UPSC CSE 2022]

Q2(b) Find the shortest distance between the line $y = 10 - 2x$ and the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

using Lagrange's method of multipliers. **[IFoS 2021]**

Q1. Find an extreme value of the function $u = x^2 + y^2 + z^2$, subject to the condition $2x + 3y + 5z = 30$, by using Lagrange's method of undetermined multiplier. **[4c UPSC CSE 2020]**

Q2. Using Lagrange's multiplier, show that the rectangular solid of maximum value which can be inscribed in a sphere is a cube. **[2b 2020 IFoS]**

Q3. Find the extreme values of $f(x,y,z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.

[3a 2020 P-2 IFoS]

Q4. Find the maximum value of $f(x,y,z) = x^2y^2z^2$ subject to the subsidiary condition $x^2 + y^2 + z^2 = c^2, (x,y,z > 0)$. **[4a P-2 UPSC CSE 2019]**

Q5. Determine the extreme values of the function $f(x,y) = 3x^2 - 6x + 2y^2 - 4y$ in the region $\{(x,y) \in \mathbf{R}^2 : 3x^2 + 2y^2 \le 20\}$. **[2019 2a IFoS]**

Q6. Show that the maximum rectangle inscribed in a circle is a square.

[2018 1a IFoS]

Q7. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$.

[2018 3a P-2 IFoS]

Q8. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the condition $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$. **[2017 3b P-2 IFoS]**

Q9. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $x+y-z=0$. **[3a UPSC CSE 2016]**

Q10. Using Lagrange's method of multipliers, find the point on the plane $2x+3y+4z=5$ which is closest to the point $(1,0,0)$. **[2016 3b IFoS]**

Q11. A conical tent is of given capacity. For the least amount of Canvas required, for it, find the ratio of its height to the radius of its base.

[2b UPSC CSE 2015]

Q12. Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point $(2,1,3)$?

[3b UPSC CSE 2015]

Q13. Find the absolute maximum and minimum values of the function $f(x,y)=x^2+3y^2-y$ over the region $x^2+2y^2\le 1$. **[4b P-2 UPSC CSE 2015]**

Q14. A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensions of the box so that the total surface is a minimum.

[2015 3c IFoS]

Q15. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

[3a UPSC CSE 2014]

Q16. Find the maximum or minimum values of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$. Interpret the result geometrically.

[3b UPSC CSE 2014]

Q17. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $xyz=a^3$ the method of Lagrange multipliers.

[4b P-2 UPSC CSE 2014]

Q18. If $xyz=a^3$ then show that the minimum value of $x^2+y^2+z^2$ is $3a^2$.

[2014 2b IFoS]

Q19. Using Lagrange's multiplier method find the shortest distance between the line $y=10-2x$ and the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. **[3a UPSC CSE 2013]**

Q20. Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ from the origin, by the method of Lagrange's multipliers.

[2d P-2 UPSC CSE 2012]

Q21. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$. **[3b UPSC CSE 2011]**

Q22. Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2+8xy+7y^2=225$.

[2d P-2 UPSC CSE 2011]

Q23. Show that a box (rectangular parallelepiped) surface area is a cube.

[2b UPSC CSE 2010]

Q24. A rectangular box open at the top is to have a surface area of 12 square units. Find the dimensions of the box so that the volume is maximum.

[2010 2a P-2 IFoS]

Q25. Find the extreme value of xyz if $x+y+z=a$. **[2010 3b IFoS]**

Q26. A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x,y,z) on the probe surface is given by $T(x,y,z)=8x^2+4yz-16z+600$. Find the hottest point on the probe surface.

[3c UPSC CSE 2009]

CHAPTER 7. JACOBIAN

Q1(d) If $x+y+z=u, y+z=uv, z=uvw$, then determine $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$. [**IFoS 2022**

3.(a) (i) If $u=x^2+y^2, v=x^2-y^2$, where $x=r\cos\theta, y=r\sin\theta$, then find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

[UPSC CSE 2021]

Q1. Using the Jacobian method, show that if $f'(x)=\dfrac{1}{1+x^2}$ and $f(0)=0$, then $f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right)$. **[4c (ii) UPSC CSE 2019]**

Q2. Show that the functions $u=x+y+z,\ v=xy+yz+zx$ and $w=x^3+y^3+z^3-3xyz$ are dependent and find the relation between them. **[2018 4b IFoS]**

Q3. Let $u(x,y)=ax^2+2hxy+by^2$ and $v(x,y)=Ax^2+2Hxy+By^2$. Find the Jacobian $J=\dfrac{\partial(u,v)}{\partial(x,y)}$, and hence show that u,v are independent unless $\dfrac{a}{A}=\dfrac{b}{B}=\dfrac{h}{H}$.

[2017 1d IFoS]

Q4. Show that the functions:

$u=x^2+y^2+z^2$

$v=x+y+z$

$w=yz+zx+xy$

are not independent of one another. **[2012 1d P-2 IFoS]**

Q5. The roots of the equation in λ $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ are u,v,w. Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=-2\dfrac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$.

Q6. Show that the function u,v,w given by $u=\dfrac{x}{y-z}, v=\dfrac{y}{z-x}$ and $\omega=\dfrac{z}{x-y}$ are not independent of one another. Also find the relation between them.

INTEGRAL CALCULUS

1. SUMMATION OF SERIES
2. DEFINITE INTEGRALS
3. BETA GAMMA FUNCTIONS
4. DOUBLE INTEGRALS, TRIPLE INTEGRALS
5. SURFACE AREAS AND VOLUMES
6. IMPROPER AND INFINITE INTEGRALS
   MISCELLANEOUS

CHAPTER 1. INTEGRAL AS A LIMIT OF SUM

Q1. Find the limit $\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=0}^{n-1}\sqrt{n^2-r^2}$. **[1d UPSC CSE 2018]**

Q2. Define a sequence S_n of real numbers by

$$S_n=\sum_{i=1}^{n}\frac{\left(\log(n+i)-\log n\right)^2}{n+i}$$

Does $\lim_{n\to\infty} S_n$ exist? If so, compute the value of this limit and justify your answer.

[3b UPSC CSE 2012]

CHAPTER 2. DEFINITE INTEGRALS

Q1. Show that $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log_e\left(1+\sqrt{2}\right)$. **[4b P-2 UPSC CSE 2020]**

Q2. Evaluate $\int_0^1\tan^{-1}\left(1-\frac{1}{x}\right)dx$. **[2a UPSC CSE 2020]**

Q3. Evaluate the following integral:

$$\int_{\pi/6}^{\pi/3}\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}dx$$. **[1d UPSC CSE 2015]**

Q4. Evaluate

$$\int_0^1\frac{\log_e(1+x)}{1+x^2}dx$$. **[1d UPSC CSE 2014]**

Q5. For $x>0$, let $f(x)=\int_1^x\frac{\ln t}{1+t}dt$. Evaluate $f(e)+f\left(\frac{1}{e}\right)$. **[1d 2015 IFoS]**

Q6. Evaluate $\int_0^1\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)dx$. **[1c UPSC CSE 2013]**

Q7. Evaluate

$$\int_0^{\pi/2}\frac{x\sin x\cos x\,dx}{\sin^4 x+\cos^4 x}$$. **[4a 2013 IFoS]**

Q8. $\int_0^1 \ell nx\,dx$. **[3a(ii) UPSC CSE 2011]**

CHAPTER 3. BETA GAMMA FUNCTIONS

Q3(iii) Express $\int_a^b (x-a)^m (b-x)^n \, dx$ in terms of Beta function. **[UPSC CSE 2021]**

(c) Using Beta and Gamma functions, evaluate the following integrals:

(i) $\int_0^2 x\left(8-x^3\right)^{1/3} dx$

(ii) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}}$. **[IFoS 2021]**

Q1. $\int_{-\infty}^{\infty} xe^{-x^2} dx$. **[4c(ii) 2020 IFoS]**

Q2. Show that

$$\int_0^{\pi/x} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, \quad p,q>-1$$

Hence evaluate the following integrals:

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x dx$

(ii) $\int_0^1 x^3 \left(1-x^2\right)^{5/2} dx$

(iii) $\int_0^1 x^4 (1-x)^3 dx$. **[2b 2017 IFoS]**

Q3. Evaluate:

$I = \int_0^1 \sqrt[3]{x \log\left(\frac{1}{x}\right)} dx$. **[1c UPSC CSE 2016]**

Q4. Show that the integral $\int_0^{\infty} e^{-x} x^{\alpha-1} dx, \ \alpha > 0$ exists, by separately taking the case for $\alpha \geq 1$ and $0 < \alpha < 1$. **[4b 2016 IFoS]**

Q5. Prove that

$\sqrt{2z} = \frac{2^{2z-1}}{\sqrt{\pi}} \sqrt{z} \sqrt{z+\frac{1}{2}}$. **[4c 2016 IFoS]**

Q6. Evaluate the integral

$I = \int_0^{\infty} z^{ax^2} dx$ using Gamma function. **[3c 2014 IFoS]**

Q7. Find all the real values of p and q so that the integral $\int_0^1 x^p \left(\log \frac{1}{x}\right)^q dx$ converges.

[3c P-1 UPSC CSE 2012]

Q8. Evaluate the following in terms of Gamma function:

$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}dx$. **[4d 2012 IFoS]**

CHAPTER 4. DOUBLE INTEGRALS, TRIPLE INTEGRALS

Q1(d) Evaluate $\iint_R x^2 dx\,dy$, where R is the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x, y=0$ and $x=8$. **[IFoS 2021]**

Q1. Evaluate the integral $\int_0^a \int_{x/a}^x \frac{xdy\,dx}{x^2+y^2}$. **[4b UPSC CSE 2018]**

Q2. Evaluate $\iint_R \left(x^2+xy\right)dx\,dy$ over the region R bounded by $xy=1, y=0$, $y=x$ and $x=2$.

[2018 3d IFoS]

Q3. Show that

$$\iint_R x^{m-1}y^{n-1}(1-x-y)^{t-1}\,dx\,dy=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{r(l+m+n)};\ l,m,n>0$$

taken over R: the triangle bounded by $x=0, y=0, x+y=1$. **[IFoS 2018 4a P-2]**

Q4. Prove that $\frac{\pi}{3} \le \iint_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \le \pi$ where D is the unit disc.

[4d UPSC CSE 2017]

Q5. Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy$, by changing to polar coordinates.

Hence show that $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$. **[2017 3c IFoS]**

Q6. Evaluate $\int_{x=0}^{\infty}\int_{y=0}^{x} xe^{-x^2/y}dy\,dx$. **[IFoS 2017 4c P-2]**

Q7. Evaluate $\iint_R f(x,y)\,dx\,dy$ over the rectangle $R=[0,1;0,1]$ where

$$f(x,y)=\begin{cases} x+y, & \text{if } x^2<y<2x^2 \\ 0, & \text{elsewhere}\end{cases}$$. **[4c UPSC CSE 2016]**

Q8. After changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy}\sin nx\,dx\,dy$, show that

$\int_0^\infty \frac{\sin nx}{x}dx=\frac{\pi}{2}$.

[2016 2a IFoS]

Q9. Evaluate the integral $\int_0^2 \int_0^{y^2/2} \frac{y}{\left(x^2+y^2+1\right)^{\frac{1}{2}}}dx\,dy$. **[IFoS 2016 3d P-2]**

Q10. Evaluate the integral
$$\iint_R (x-y)^2 \cos^2(x+y)\,dx\,dy$$
where R is the rhombus with successive vertices as $(\pi,0)(2\pi,\pi)(\pi,2\pi)(0,\pi)$.

[3d UPSC CSE 2015]

Q11. Evaluate $\iint_R \sqrt{|y-x^2|}\,dx\,dy$ where $R=[-1,1;0,2]$. **[4a UPSC CSE 2015]**

Q12. By using the transformation $x+y=u, y=uv$, evaluate the integral $\iint\{(xy(1-x-y))\}^{1/2}\,dx\,dy$ taken over the area enclosed by the straight lines $x=0, y=0$ and $x+y=1$. **[2c UPSC CSE 2014]**

Q13. Evaluate $\iint_R y\frac{\sin x}{x}\,dx\,dy$ over R where $R=\{(x,y): y\le x\le \pi/2, 0\le y\le \pi/2\}$.

[2014 1d IFoS]

Q14. Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$ over the region R bounded between $0\le x\le \frac{y^2}{2}$ and $0\le y\le 2$. **[2014 4c IFoS]**

Q15. Change the order of integration and evaluate $\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy$.

[IFoS 2014 3b P-2]

Q16. Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$. **[3c UPSC CSE 2013]**

Q17. Evaluate the integral $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dydx$ by changing the order of integration.

[1c UPSC CSE 2013]

Q18. Evaluate
$$\iint \sqrt{4x^2-y^2}\,dx\,dy$$
over the triangle formed by the straight lines $y=0, x=1, y=x$.

[4a P-2 UPSC CSE 2011]

Q19. Evaluate $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$. **[3d UPSC CSE 2010]**

Q20. Evaluate
$$\iint_D (x-y+1)\,dxdy$$
where R is the region inside the unit square in which $x+y\ge \frac{1}{2}$.

[3b P-2 UPSC CSE 2010]

Q21. Evaluate
$$I=\iint_S xdydz+dzdx+xz^2dxdy$$

where S is the outer side of the part of the sphere $x^2+y^2+z^2=1$ in the first octant.

[4b UPSC CSE 2009]

TRIPLE INTEGRALS

Q1. Consider the three-dimensional region R bounded by $x+y+z=1, y=0, z=0$. Evaluate $\iiint\limits_R \left(x^2+y^2+z^2\right)dx\,dy\,dz$. **[2c UPSC CSE 2015]**

Q2. Let D be the region determined by the inequalities $x>0, y>0, z<8$ and $z>x^2+y^2$. Compute
$\iiint\limits_D 2x\,dx\,dy\,dz$. **[3b UPSC CSE 2010]**

CHAPTER 5. SURFACE AREAS AND VOLUMES

Q3(b) Use double integration to calculate the area common to the circle x^2+y^2-4 and the parabola $y^2=3x$. **[UPSC CSE 2022]**

Q4(b) Show that the entire area of the Astroid: $x^{2/3}+y^{2/3}=a^{2/3}$ is $\frac{3}{8}\pi a^2$.

[UPSC CSE 2021]

Q4(c) Find the whole area included between the curve $x^2y^2=a^2\left(y^2-x^2\right)$ and its asymptotes. **[IFoS 2021]**

Q1. Find the volume lying inside the cylinder $x^2+y^2-2x=0$ and outside the paraboloid $x^2+y^2=2z$, while bounded by *xy*-plane. **[1c UPSC CSE 2019]**

Q2. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the *x*-axis. Find the volume of the solid of revolution.

[2c UPSC CSE 2018]

Q3. Find the volume of the solid above the *xy*-plane and directly below the portion of the elliptic paraboloid $x^2+\frac{y^2}{4}=z$ which is cut off by the plane $z=9$.

[2a UPSC CSE 2017]

Q4. Find the volume of the region common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

[IFoS 2017 3d P-2]

Q5. Find the area enclosed by the curve in which the plane $z=2$ cuts the ellipsoid
$\frac{x^2}{25}+y^2+\frac{z^2}{5}=1$. **[2015 2d IFoS]**

Q6. Compute the double integral which will give the area of the region between the *y*-axis, the circle $(x-2)^2+(y-4)^2=z^2$ and the parabola $2y=x^2$. Compute the integral and find the area.

[IFoS 2015 4a P-2]

Q7. Find the area of the region between the x-axis and $y=(x-1)^3$ from $x=0$ to $x=2$.

[2013 4a P-2 IFoS]

Q8. Compute the volume of the solid enclosed between the surfaces $x^2+y^2=9$ and $x^2+z^2=9$.

[4a UPSC CSE 2012]

Q9. Find by triple Integration the volume cut off from the cylinder $x^2+y^2=ax$ by the planes $z=mx$ and $z=nx$. **[2012 4a IFoS]**

Q10. Find the volume of the solid bounded above by the parabolic cylinder $z=4-y^2$ and bounded below by the elliptic paraboloid $z=x^2+3y^2$.

[3a P-2 UPSC CSE 2012]

Q11. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$ above the xy-plane and inside the cylinder $x^2+y^2=2x$. **[3c UPSC CSE 2011]**

Q12. Show that the area of the surface of the sphere $x^2+y^2+z^2=a^2$ cut off by $x^2+y^2=ax$ is $2(\pi-2)a^2$. **[3c UPSC CSE 2011]**

CENTROID

Q3(b) Find the centre of mass of a solid bounded below by $x^2+y^2\le 4, z=0$ above by the paraboloid $z=4-x^2-y^2$. Take the density of the solid as uniform.

[IFoS 2022]

Q1. Find the centroid of the solid generated by revolving the upper half of the cardioid $r=a(1+\cos\theta)$ bounded by the line $\theta=0$ about the initial line. Take the density of the solid as uniform. **[4b 2019 IFoS]**

RECTIFICATION, CURVATURE

Q2. Obtain the area between the curve $r=3(\sec\theta+\cos\theta)$ and its asymptote $x=3$.

[3c 2016 IFoS]

CONVERGENCE OF IMPROPER AND INFINITE INTEGRALS

Q4.(a) Show that the improper integral

$$\int_0^\infty x^{m-1}e^{-x}dx$$

is convergent if $m>0$. **[IFoS 2022]**

Q1(d) Examine the convergence of $\int_0^2 \frac{dx}{(2x-x^2)}$. **[UPSC CSE 2022]**

Q1. Discuss the convergence $\int_1^2 \frac{\sqrt{x}}{l_n x}dx$. **[4c P-2 UPSC CSE 2019]**

Q2. Show that the integral $\int_0^{\pi/2} \log\sin x\,dx$ is convergent and hence evaluate it.

[2b P-2 UPSC CSE 2019]

Q3. Show that the improper integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$ is convergent.

[3d P-2 UPSC CSE 2018]

Q4. Examine if the improper integral $\int_0^3 \frac{2x\,dx}{\left(1-x^2\right)^{2/3}}$ exists**. [4c UPSC CSE 2017]**

Q5. Prove that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent but not absolutely convergent.

[3c P-2 UPSC CSE 2017]

Q6. Evaluate the integral $\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$. **[4a P-2 UPSC CSE 2016]**

Q7. Test the convergence of the improper integral $\int_1^\infty \frac{dx}{x^2\left(1+e^{-x}\right)}$.

[1b P-2 UPSC CSE 2014]

Q8. Test for convergence the integral $\int_0^\infty \sqrt{xe^{-x}}\,dx$. **[3b 2011 IFoS]**

Q9. Examine the convergence of

$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ and evaluate, if possible. **[3a P-2 UPSC CSE 2011]**

Q10. Does the integral $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}dx$ exist? If so, find its value.

[1d UPSC CSE 2010]

Q11. Discuss the convergence of the integral

$\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$. **[3a UPSC CSE 2010]**

VECTOR CALCULUS
UPSC PREVIOUS YEARS QUESTION(CSE &IFoS)

VECTOR: BASICS & TRIPLE PRODUCT
1. VECTOR DIFFERENTIATION
GRADIENT, DIRECTIONAL DERIVATIVES
DIVERGENCE
CURL
2. VECTOR INTEGRATION-
LINE, SURFACE AND VOLUME INTEGRALS
3. THREE IMPORTANT THEOREMS
GREEN'S THEOREM
GAUSS' DIVERGENCE THEOREM
STOKE'S THEOREMS
4. SOME OTHER TOPICS
CURVATURE & TORSION
CURVILINEAR COORDINATES.

INTRODUCTION:VECTOR ANALYSIS

Q1. Prove that the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{c} = 4\hat{i} - 2\hat{j} - 6\hat{k}$ can form the sides of a triangle. Find the lengths of the medians of the triangle.
[5b UPSC CSE 2016]

Q2. Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$. **[8c 2016 IFoS]**

Q3. For three vectors show that: $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = 0$.
[5e 2014 IFoS]

1. VECTOR DIFFERENTIAL CALCULUS

Q1. The position vector of a moving point at time t is $\bar{r} = \sin t\hat{i} + \cos 2t\hat{j} + (t^2 + 2t)\hat{k}$. Find the components of acceleration $\bar{a}$ in the directions parallel to the velocity vector $\bar{v}$ and perpendicular to the plane of $\bar{r}$ and $\bar{v}$ at time $t = 0$.
[5e UPSC CSE 2017]

Q2. If
$\vec{A} = x^2yz\hat{i} - 2xz^3\vec{j} + xz^2\vec{k}$
$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$
find the value of $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B})$ at $(1, 0, -2)$. **[5e UPSC CSE 2012]**

Q3. For two vectors $\bar{a}$ and $\bar{b}$ given respectively by
$\bar{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\bar{b} = \sin t\hat{i} - \cos t\hat{j}$ determine:
(i) $\frac{d}{dt}(\bar{a} \cdot \bar{b})$ and (ii) $\frac{d}{dt}(\bar{a} \times \bar{b})$ **[5e UPSC CSE 2011]**

Q4. The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r}=\left(t^2-2t\right)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}$$

At time $t=1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. **[8d 2011 IFoS]**

GRADIENT, DIRECTIONAL DERIVATIVES

Q5(e) Show that $\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r}\right)\right]=\frac{2}{r^4}$, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. **[UPSC CSE 2021]**

Q5(e) Determine constants a,b,c so that the directional derivative of $\phi(x,y,z)=axy^2+byz+cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude 88 in a direction parallel to z-axis. **[IFoS 2022]**

Q1. Prove that for a vector $\vec{a}$,

$\nabla(\vec{a}\cdot\vec{r})=\vec{a}$; where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k},\ r=|\vec{r}|$.

Is there any restriction on $\vec{a}$?

Further, show that

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$$

Give an example to verify the above. **[5e 2020 IFoS]**

Q2. Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t, y=t^2, z=t^3$ at the point $(1,1,1)$. **[5e UPSC CSE 2019]**

Q3. Find the angle between the tangent at a general point of the curve whose equations are $x=3t, y=3t^2, z=3t^3$ and the line $y=z-x=0$. **[5b UPSC CSE 2018]**

Q4. Find $f(r)$ such that $\nabla f=\frac{\vec{r}}{r^5}$ and $f(1)=0$. **[8a UPSC CSE 2016]**

Q5. Find the angle between the surfaces $x^2+y^2+z^2-9=0$ and $z=x^2+y^2-3$ at $(2,-1,2)$. **[5e UPSC CSE 2015]**

Q6. Find the value of λ and μ so that the surfaces $\lambda x^2-\mu yz=(\lambda+2)x$ and $4x^2y+z^3=4$ may intersect orthogonally at $(1,-1,2)$. **[6c UPSC CSE 2015]**

Q7. A curve in space is defined by the vector equation $\vec{r}=t^2\hat{i}+2t\hat{j}-t^3\hat{k}$. Determine the angle between the tangents to this curve at the points $t=+1$ and $t=-1$. **[8b UPSC CSE 2013]**

Q8. If $u=x+y+z,\ v=x^2+y^2+z^2,\ w=yz+zx+xy$ prove that grad u, grad v and grad w are coplaner. **[5e 2012 IFoS]**

Q9. Examine whether the vectors ∇_u, ∇_v and ∇_w are coplaner, where u,v and w are the scalar functions defined by:

$u=x+y+z$,

$v=x^2+y^2+z^2$

and $w=yz+zx+xy$. **[8a UPSC CSE 2011]**

Q10. Find the directional derivative of $f(x,y)=x^2y^3+xy$ at the point $(2,1)$ in the direction of a unit vector which makes an angle of $\pi/3$ with the x-axis.
[1e UPSC CSE 2010]

Q11. Find the directional derivation of $\vec{V}^2$, where, $\vec{V}=xy^2\vec{i}+zy^2\vec{j}+xz^2\vec{k}$ at the point $(2,0,3)$ in the direction of the outward normal to the surface $x^2+y^2+z^2=14$ at the point $(3,2,1)$.

[5f 2010 IFoS]

Q12. Find the directional derivative of -
(i) $4xz^3-3x^2y^2z^2$ at $(2,-1,2)$ along z-axis;
(ii) $x^2yz+4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i}-\hat{j}-2\hat{k}$. **[5f UPSC CSE 2009]**

DIVERGENCE

Q1. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $f(r)$ is differentiable, show that $\text{div}\left[f(r)\vec{r}\right]=rf'(r)+3f(r)$.

Hence or otherwise show that $\text{div}\left(\frac{\vec{r}}{r^3}\right)=0$. **[5e 2018 IFoS]**

Q2. Calculate $\nabla^2(r^n)$ and find its expression in terms of r and n, r being the distance of any point (x,y,z) from the origin, n being a constant and ∇^2 being the Laplace operator.

[8a UPSC CSE 2013]

Q3. Prove that $\text{div}(f\vec{V})=f(\text{div}\vec{V})+(\text{grad } f)\cdot\vec{V}$ where f is a scalar function.

[6c UPSC CSE 2010]

Q4. Show that, $\nabla^2 f(r)=\left(\frac{2}{r}\right)f'(r)+f''(r)$, where $r=\sqrt{x^2+y^2+z^2}$.

[4. 8a 2010 IFoS]

Q5. Show that $\text{div}(\text{grad } r^n)=n(n+1)r^{n-2}$ where $r=\sqrt{x^2+y^2+z^2}$.

[5e UPSC CSE 2009]

CURL

Q5(e) Show that $\vec{A}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ is irrotational. Also find ϕ such that $\vec{A}=\nabla\phi$. **[UPSC CSE 2022]**

Q7.(a) Derive vector identity for divergence of cross product of two vector point functions. Given a relation between linear and angular velocity as $\vec{v}=\vec{\omega}\times\vec{r}$.
If $\vec{\omega}$ is constant, then show that
(i) curl $\vec{v}=2\vec{\omega}$
(ii) div $\vec{v}=0$.
(b) Given that $y_1=x^2$ is a solution of the differential equation.

[IFoS 2022]

Q5(e) If

$\vec{F} = \left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y} \right)\hat{i} + \left(z\frac{\partial \phi}{\partial x} - x\frac{\partial \phi}{\partial z} \right)\hat{j} + \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \right)\hat{k}$, then prove that $\vec{F} - (\vec{r} \times \nabla\phi) = \vec{F}\cdot\vec{r} = \vec{F}\cdot\nabla\phi = 0$. **[IFoS 2021]**

Q1. For what value of a,b,c is the vector field $\vec{V} = (4x-3y+az)\hat{i} + (bx+3y+5z)\hat{j} + (4x+cy+3z)\hat{k}$ irrotational? Hence, express $\vec{V}$ as the gradient of a scalar function ϕ. Determine ϕ. **[5c UPSC CSE 2020]**

Q2. Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Show that curl $(\text{curl}\,\vec{v}) = \text{grad}(\text{div}\,\vec{v}) - \nabla^2\vec{v}$.

[8a UPSC CSE 2018]

Q3. Show that $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$. **[6c 2018 IFoS]**

Q4. For what values of the constants a,b and c the vector $\vec{V} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$ is irrotational. Find the divergence in cylindrical coordinates of this vector with these values.

[5d UPSC CSE 2017]

Q5. A vector field is given by $\vec{F} = (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}$. Verify that the field $\vec{F}$ is irrorational or not. Find the scalar potential. **[7c UPSC CSE 2015]**

Q6. Examine if the vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrorational. If so, find the scalar potential ϕ such that $\vec{F} = \text{grad}\,\phi$.

[6d 2015 IFoS]

Q7. For the vector $\bar{A} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\bar{A} = \nabla\phi$. **[6d 2014 IFoS]**

Q8. $\vec{F}$ being a vector, prove that $\text{curl}\,\text{curl}\,\vec{F} = \text{grad}\,\text{div}\,\vec{F} - \nabla^2\vec{F}$ where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

[5c 2013 IFoS]

Q9. If u and v are two scalar fields and $\vec{f}$ is a vector field, such that $u\vec{f} = \text{grad}\,v$, find the value of $\vec{f}\cdot\text{curl}\,\vec{f}$. **[5f UPSC CSE 2011]**

Q10.If $\vec{r}$ be the position vector of a point, find the value(s) of n for which the vector $r^n\vec{r}$ is (i) irrotational, (ii) solenoidal. **[8c UPSC CSE 2011]**

Q11. Prove the vector identity:
$\text{curl}\left(\vec{f}\times\vec{g}\right) = \vec{f}\,\text{div}\,\vec{g} - \vec{g}\,\text{div}\,\vec{f} + (\vec{g}\cdot\nabla)\vec{f} - \left(\vec{f}\cdot\nabla\right)\vec{g}$ and verify it for the vectors $\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$ and $\vec{g} = y\hat{i} + z\hat{k}$. **[8b 2011 IFoS]**

Q12. Show that the vector field defined by the vector function $\vec{V} = xyz\left(yz\vec{i} + xz\vec{j} + xy\vec{k}\right)$ is conservative. **[1f UPSC CSE 2010]**

Q13. Show that $\vec{F} = (2xy+z^3)\vec{i} + x^2\vec{j} + 3z^2x\vec{k}$ is a conservative field. Find its scalar potential and also the work done in moving a particle from $(1,-2,1)$ to $(3,1,4)$.

[8a 2010 IFoS]

2. VECTOR INTRGRAL CALCULUS

Q6(c) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is an arbitrary closed curve in the *xy*-plane and $\vec{F}=\frac{y\hat{i}+x\hat{j}}{x^2+y^2}$. **[UPSC CSE 2021]**

Q1. For the vector function $\bar{A}$, where $\bar{A}=(3x^2+6y)\hat{i}-14yz\hat{j}-14yz\hat{j}+20xz^2\hat{k}$, calculate $\int_C \bar{A}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths:

(i) $x=t, y=t^2, z=t^3$

(ii) Straight lines joining $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

(iii) Straight line joining $(0,0,0)$ to $(1,1,1)$

Is the result same in all the cases? Explain the reason. **[6b UPSC CSE 2020]**

Q2. Find the circulation of $\vec{F}$ round the curve C, where $\vec{F}=(2x+y^2)\hat{i}+(3y-4x)\hat{j}$ and C is the curve $y=x^2$ from $(0,0)$ to $(1,1)$ and the curve $y^2=x$ from $(1,1)$ to $(0,0)$.

[6b UPSC CSE 2019]

Q3. Evaluate $\int_{(0,0)}^{(2,1)}(10x^4-2xy^3)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$.

[5e 2019 IFoS]

Q4. Evaluate $\int_C e^{-x}(\sin y dx+\cos y dy)$, where C is the rectangle with vertices $(0,0),(\pi,0),\left(\pi,\frac{\pi}{2}\right),\left(0,\frac{\pi}{2}\right)$. **[8c UPSC CSE 2015]**

Q5. If $\vec{A}=2y\vec{i}-z\vec{j}-x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4, z=6$, evaluate the surface integral, $\iint_S \vec{A}\cdot\hat{n}\overrightarrow{dS}$. **[8c 2010 IFoS]**

Q6. Find the work done in moving the particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1, z=0$ under the field of force given by $\vec{F}=(2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$. **[8a UPSE CSE 2009]**

3.THREE IMPORTANT THEOREMS

GREEN'S THEOREM

Q6(c) Verify Green's theorem in the plane for $\oint_C(3x^2-8y^2)dx+(4y-6xy)dy$, where C is the boundary curve of the region defined by $x=0, y=0$, $x+y=1$.

[UPSC CSE 2022]

Q1. Let $\vec{F}=xy^2\vec{i}+(y+x)\vec{j}$. Integrate $(\nabla\times\vec{F})\cdot\vec{k}$ over the region in the first quadrant bounded by the curves $y=x^2$ and $y=x$ using Green's theorem.

[8c UPSE CSE 2018]

Q2. Using Green's theorem, evaluate the $\int_C F(\bar{r})\cdot d\bar{r}$ counterclockwise where $F(\bar{r}) = (x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}$ and $d\bar{r} = dx\hat{i} + dy\hat{j}$ and the curve C is the boundary of the region $R = \{(x,y)|1 \le y \le 2-x^2\}$. **[8c UPSE CSE 2017]**

Q3. Verify Green's theorem in the plane for $\oint_C \left[(xy+y^2)dx + x^2dy\right]$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. **[8b UPSE CSE 2013]**

Q4. Find the value of the line integral over a circular path given by $x^2+y^2=a^2, z=0$, where the vector field, $\vec{F} = (\sin y)\vec{i} + x(1+\cos y)\vec{j}$.

[8b 2012 IFoS]

Q5. Verify Green's theorem in the plane for $\oint_C \left[(3x^2-8y^2)dx + (4y-6xy)dy\right]$, where C is the boundary of the region enclosed by the curves $y=\sqrt{x}$ and $y=x^2$.

[8c 2011 IFoS]

Q6. Verify Green's theorem for $e^{-x}\sin y dx + e^{-x}\cos y dy$ the path of integration being the boundary of the square whose vertices are $(0,0),(\pi/2,0),(\pi/2,\pi/2)$ and $(0,\pi/2)$. **[8c UPSE CSE 2010]**

Q7. Use Green's theorem in a plane to evaluate the integral, $\int_C \left[(2x^2-y^2)dx + (x^2+y^2)dy\right]$, where C is the boundary of the surface in the *xy*-plane enclosed by, $y=0$ and the semi-circle $y=\sqrt{1-x^2}$.

[8b 2012 IFoS]

GAUSS' DIVERGENCE THEOREM

Q8(c) Using Gauss' divergence theorem, evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ and S is the cylinder formed by the surfaces $z=0, z=1, x^2+y^2=4$. **[UPSC CSE 2022]**

7.(a) Verify Gauss divergence theorem for $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$. **[UPSC CSE 2021]**

Q1. Given a portion of a circular disc of radius 7 units and of height 1.5 units such that $x,y,z \ge 0$. Verify Gauss Divergence Theorem for the vector field $\vec{f} = (z, x, 3y^2z)$ over the surface of the above mentioned circular disc.

[7c 2020 IFoS]

Q2. State Gauss divergence theorem. Verify this theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, taken over the region bounded by $x^2+y^2=4, z=0$ and $z=3$.

[8c UPSC CSE 2019]

Q3. If S is the surface of the sphere $x^2+y^2+z^2=a^2$, then evaluate $\iint_S \left[(x+z)dydz + (y+z)dzdx + (x+y)dxdy\right]$ using Gauss' divergence theorem.

[6d UPSC CSE 2018]

Q4. Evaluate the integral: $\iint_S \bar{F}\cdot\hat{n}ds$ where $\bar{F}=3xy^2\hat{i}+\left(yx^2-y^3\right)\hat{j}+3zx^2\hat{k}$ and S is a surface of the cylinder $y^2+z^2\le 4, -3\le x\le 3$, using divergence theorem.

[8c UPSC CSE 2017]

Q5. If E be the solid bounded by the *xy* plane and the paraboloid $z=4-x^2-y^2$, then $\iint_S \bar{F}\cdot dS$ where S is the surface bounding the volume E and $\bar{F}=\left(zx\sin yz+x^3\right)\hat{i}+\cos yz\hat{j}+\left(3zy^2-e^{\lambda^2+y^2}\right)\hat{k}$. **[5e 2016 IFoS]**

Q6. Using divergence theorem, evaluate $\iint_S\left(x^3dydz+x^2ydzdx+x^2zdydx\right)$ where S is the surface of the sphere $x^2+y^2+z^2=1$. **[7b 2015 IFoS]**

Q7. Verify the divergence theorem for $\bar{A}=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ over the region $x^2+y^2=4$, $z=0, z=3$. **[8c 2014 IFoS]**

Q8. By using Divergence Theorem of Gauss, evaluate the surface integral $\iint\left(a^2x^2+b^2y^2+c^2z^2\right)^{-\frac{1}{2}}dS$, where S is the surface of the elliposid $ax^2+by^2+cz^2=1, a, b$ and c being all positive constants. **[8c UPSC CSE 2013]**

Q9. Evaluate $\int_S \vec{F}\cdot d\vec{s}$, where $\vec{F}=4x\vec{i}-2y^2\vec{j}+z^2\vec{k}$ and s is the surface bounding the region $x^2+y^2=4, z=0$ and $z=3$. **[6b 2013 IFoS]**

Q10. Verify the Divergence theorem for the vector function $\vec{F}=\left(x^2-yz\right)\vec{i}+\left(y^2-xz\right)\vec{j}+\left(z^2-xy\right)\vec{k}$ taken over the rectangular parallelopiped $0\le x\le a,\ 0\le y\le b,\ 0\le z\le c$. **[8b 2013 IFoS]**

Q11. Verify Gauss' Divergence Theorem for the vector $\vec{v}=x^2\hat{i}+y^2\hat{j}+z^2\hat{k}$ taken over the cube $0\le x, y, z\le 1$. **[8d UPSC CSE 2011]**

Q12. Use the divergence theorem to evaluate $\iint_S \vec{V}\cdot\vec{n}dA$ where $\vec{V}=x^2z\vec{i}+y\vec{j}-xz^2\vec{k}$ and S is the boundary of the region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4y$.

[7c UPSC CSE 2010]

Q13. Use divergence theorem to evaluate, $\iint_S\left(x^3dy\,dz+x^2ydz\,dx+x^2z\,dy\,dx\right)$ where S is the sphere, $x^2+y^2+z^2=1$. **[8b 2010 IFoS]**

Q14. Using divergence theorem, evaluate $\iint_S \vec{A}\cdot d\vec{S}$ where $\vec{A}=x^3\hat{i}+y^3\hat{j}+z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$. **[8b UPSC CSE 2009]**

STOKE'S THEOREM

Q6(c) Given that C is a curve of the intersection of the cylinder $x^2+y^2=4$ and the plane $x+y+z=2$ and C is described counterclockwise. Verify Stokes' theorem for the line integral $\int_C -y^3dx+x^3dy-z^3dz$. **[IFoS 2022]**

Q8(c) Using Stokes' theorem, evaluate $\iint_S \left(\nabla\times\vec{F}\right)\cdot\hat{n}dS$, where $\vec{F}=\left(x^2+y-4\right)\hat{i}+3xy\hat{j}+\left(2xy+z^2\right)\hat{k}$ and S is the surface of the paraboloid $z=4-\left(x^2+y^2\right)$ above the *xy*-plane. Here, $\hat{n}$ is the unit outwar d normal vector on S. [**UPSC CSE 2021]**

Q7. (a) Verify Stores' theorem for $\vec{F}=x\hat{i}+z^2\hat{j}+y^2\hat{k}$ over the plane surface: $x+y+z=1$ lying in the first octant. **[UPSC CSE 2022]**

Q1. Verify the Stokes' theorem for the vector field $\bar{F}=xy\hat{i}+yz\hat{j}+xz\hat{k}$ on the surface S which is the part of the cylinder $z=1-x^2$ for $0\le x\le 1,-2\le y\le 2$; S is oriented upwards.

[7a UPSC CSE 2020]

Q2. Evaluate the surface integral $\iint_S \nabla\times\bar{F}\cdot\hat{n}dS$ for $\bar{F}=y\hat{i}+\left(x-2xz\right)\hat{j}-xy\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the *xy*-plane.

[8b UPSC CSE 2020]

Q3. Evaluate by Stokes' theorem $\oint_C e^x dx+2ydy-dz$, where C is the curve $x^2+y^2=4, z=2$. **[8c UPSC CSE 2019]**

Q4. Verify Stokes's theorem for $\bar{V}=\left(2x-y\right)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary. **[6c 2019 IFoS]**

Q5. Evaluate the line integral $\int_C -y^3dx+x^3dy+z^3dz$ using Stokes's theorem. Here C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$. The orientation on C corresponds to counterclockwise motion in the *xy*-plane.

[8b UPSC CSE 2018]

Q6. Using Stoke's theorem evaluate

$\oint_C \left[\left(x+y\right)dx+\left(2x-z\right)dy+\left(y+z\right)dz\right]$, where C is the boundary of the triangle with vertices at $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$. **[6c 2017 IFoS]**

Q7. Evaluate

$\iint_S \left(\nabla\times\vec{f}\right)\cdot\hat{n}dS$, where S is the surface of the cone, $z=2-\sqrt{x^2+y^2}$ above xy-plane and $\vec{f}=\left(x-z\right)\hat{i}+\left(x^3+yz\right)\hat{j}-3xy^2\hat{k}$. **[7d 2017 IFoS]**

Q8. Prove that $\oint_C fd\vec{r}=\iint_S d\vec{S}\times\nabla f$. **[8b UPSC CSE 2016]**

Q9. Evaluate $\iint_S \left(\nabla\times\bar{f}\right)\cdot\hat{n}dS$ for $\bar{f}=\left(2x-y\right)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by its projection on the xy plane. **[6d 2016 IFoS]**

Q10. State Stokes' theorem. Verify the Stokes' theorem for the function $\bar{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one.

[7a 2016 IFoS]

Q11. If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. **[8b 2015 IFoS]**

Q12. Evaluate by Stokes' theorem $\int_\Gamma (y\,dx + z\,dy + x\,dz)$ where Γ is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$ starting from $(2a, 0, 0)$ and then going below the z-plane.

[6c UPSC CSE 2014]

Q13. Evaluate $\iint_S \nabla \times \bar{A} \cdot \bar{n}\, dS$ for $\bar{A} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xy plane. **[7b 2014 IFoS]**

Q14. Use Stokes' theorem to evaluate the line integral $\int_C (-y^3 dx + x^3 dy - z^3 dz)$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. **[8d UPSC CSE 2013]**

Q15. If $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$, evaluate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\vec{s}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. **[8c UPSC CSE 2012]**

Q16. Find the value of $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s}$ taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when $\vec{F} = (y^2 + z^2 - x)\vec{i} + (z^2 + x^2 - y^2)\vec{j} + (x^2 + y^2 - z^2)\vec{k}$. **[6b 2012 IFoS]**

Q17. If $\bar{u} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$, calculate the double integral $\iint (\nabla \times \bar{u}) \cdot d\vec{s}$ over the hemisphere given by $x^2 + y^2 + z^2 = a^2,\ z \geq 0$. **[8b UPSC CSE 2011]**

Q18. Evaluate the line integral $\oint_C (\sin x dx + y^2 dy - dz)$, where C is the circle $x^2 + y^2 = 16, z = 3$, by using Stokes' theorem. **[5e 2011 IFoS]**

Q19. Find the value of $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when $\vec{F} = (y^2 + z^2 - x^2)\hat{i} + (z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - z^2)\hat{k}$. **[8c UPSC CSE 2009]**

CURVATURE & TORSION

Q8(c) If a curve in a space is represented by $\vec{r}=\vec{r}(t)$, then derive expressions of its torsion and curvature in terms of $\dot{\vec{r}}$, $\ddot{\vec{r}}$ and $\dddot{\vec{r}}$. Find the curvature and torsion of the curve given by $\vec{r}=(at-a\sin t, a-a\cos t, bt)$. IFoS 2022

Q1. A tangent is drawn to a given curve at some point of constant. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{\left[25\kappa^2\tau^2\left(1+25\kappa^2\right)+\left\{\kappa+5\frac{d\kappa}{ds}+25\kappa^3\right\}\right]^{1/2}}{\left(1+25\kappa^2\right)^{3/2}}$$

Find the curvature and torsion of the curve $\vec{r}=t\hat{i}+t^2\hat{j}+t^3\hat{k}$. **[6c 2020 IFoS]**

Q2. Find the radius of curvature and radius of torsion of the helix $x=a\cos u$, $y=a\sin u, z=au\tan\alpha$. **[7b UPSC CSE 2019]**

Q3. Let $\bar{r}=\bar{r}(s)$ represent a space curve. Find $\frac{d^3\bar{r}}{ds^3}$ in terms of $\bar{T},\bar{N}$ and $\bar{B}$, where $\bar{T},\bar{N}$ and $\bar{B}$ represent tangent, principal normal and binormal respectively. Compute $\frac{d\bar{r}}{ds}\cdot\left(\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}\right)$ in terms of radius of curvature and the torsion. **[5d 2019 IFoS]**

Q4. Derive the Frenet-Serret formulae. Verify the same for the space curve $x=3\cos t, y=3\sin t, z=4t$. **[7c 2019 IFoS]**

Q5. Find the curvature and torsion of the curve $\vec{r}=a(u\sin u)\vec{i}+a(1-\cos u)\vec{j}+bu\vec{k}$.

[7b UPSC CSE 2018]

Q6. Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of) a circle. **[7d 2018 IFoS]**

Q7. For a curve lying on a sphere of radius a and such that the torsion is never 0, show that

$\left(\frac{1}{\kappa}\right)^2+\left(\frac{\kappa'}{\kappa^2\tau}\right)^2=a^2$ **[8c 2018 IFoS]**

Q8. Find the curvature vector and its magnitude at any point $\vec{r}=(\theta)$ of the curve $\bar{r}=(a\cos\theta, a\sin\theta, a\theta)$. Show that the locus of the feet of the perpendicular from the origin to the tangent is a curve that completely lies on the hyperboloid $x^2+y^2-z^2=a^2$.

[7a UPSC CSE 2017]

Q9. Find the curvature and torsion of the circular helix $\vec{r}=a(\cos\theta,\sin\theta,\theta\cot\beta)$, β is the constant angle at which it cuts its generators. **[8c 2017 IFoS]**

Q10. If the tangent to a curve makes a constant angle α, with a fixed lines, then prove that $\kappa\cos\alpha \pm \tau\sin\alpha = 0$. Conversely, if $\frac{\kappa}{\tau}$ is constant, then show that the tangent makes a constant angle with a fixed direction. **[8d 2017 IFoS]**

Q11. For the cardioid $r = a(1+\cos\theta)$, show that the square of the radius of curvature at any point (r,θ) is a proportional to *r*. Also find the radius of curvature if $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$. **[8d UPSC CSE 2016]**

Q12. Find the curvature and torsion of the curve $x = a\cos t, y = a\sin t, z = bt$.
[5c 2015 IFoS]

Q13. Find the curvature vector at any point of the curve $\vec{r}(t) = t\cos t\hat{i} + t\sin t\hat{j},\ 0 \le t \le 2\pi$. Give its magnitude also. **[5e UPSC CSE 2014]**

Q14. Show that the curve $\vec{x}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} + \left(\frac{1-t^2}{t}\right)\hat{k}$ lies in a plane.

[5e UPSC CSE 2013]

Q15. Derive the Frenet-Serret formulae. Define the curvature and torsion for a space curve. Compute them for the space curve $x = t, y = t^2, z = \frac{2}{3}t^3$. Show that the curvature and torsion are equal for this curve. **[8a UPSC CSE 2012]**

Q16. Find the curvature, torsion and the relation between the arc length S and parameter u for the curve: $\vec{r} = \vec{r}(u) = 2\log_e u\hat{i} + 4u\hat{j} + (2u^2+1)\hat{k}$. **[8a 2011 IFoS]**

Q17. Find κ/τ for the curve
$\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + bt\vec{k}$. **[1c UPSC CSE 2010]**

CURVILINEAR COORDINATES

Q1. Derive expression of ∇f in terms of spherical coordinates.
Prove that $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$ for any two vector point functions $f(r,\theta,\phi)$ and $g(r,\theta,\phi)$. Construct one example in three dimensions to verify this identity. **[8a 2020 IFoS]**

Q2. Derive $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$ in spherical coordinates. **[8c 2019 IFoS]**

Q3. For what values of the constants a, b and c the vector $\bar{V} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$ is irrotational. Find the divergence in cylindrical coordinates of this vector with these values.

[5d UPSC CSE 2017]

ORDINARY DIFFERENTIAL EQUATIONS (ODE)

1. FORMATION OF DE
2. HOMOGENEOUS DE, LINEAR DE
3. EXACT DIFF EQUATIONS, INTEGRATING FACTORS
4. EXACT DIFF EQUATIONS, INTEGRATING FACTOR
5. ORTHOGONAL TRAJECTORIES
6. LINEAR DE WITH CONSTANT COEFFICIENTS
7. HOMOGENEOUS LINEAR DE
8. LINEAR DE OF SECOND ORDER- CHANGING DEPENDENT VARIABLE, CHANGING INDEPENDENT VARIABLE
9. LINEAR DE OF SECOND ORDER - VARIATION OF PARAMETER METHOD
*UNDETERMINED COEFFICIENT METHOD
10. SIMULTANEOUS LINEAR DE WITH CONSTANT COEFFICIENTS
11. LAPLACE AND INVERSE TRANSFORMS
12. SOLUTION OF SECOND ORDER LINEAR DE WITH CONSTANT COEFF BY LAPLACE TRANSFORMS

FORMATION OF DE

Q1. A snowball of radius $r(t)$ melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process. **[8b 2018 IFoS]**

Q2. Find the differential equation representing all the circles in the $x-y$ plane.

[5a UPSC CSE 2017]

Q3. If the growth rate of the population of bacteria at any time t is propositional to the amount present at that time and population doubles in one week, then how much bacterias can be expected after 4 weeks?

[6a(ii) UPSC CSE 2017]

4. Obtain the curve which passes through $(1,2)$ and has a slope $=\frac{-2xy}{x^2+1}$.

Obtain one asymptote to the curve.**[5a 2016 IFoS]**

5. Find the curve for which the part of the tangent cut-off by the axes is bisected at the point of tangency.

[5b UPSC CSE 2014]

6. Find the family of curves whose tangents form an angle $\pi/4$ with hyperbolas $xy=0$.

[5a 2011 IFoS]

HOMOGENEOUS DE, LINEAR DE

Q1. Solve:

$\frac{dy}{dx}=\frac{4x+6y+5}{3y+2x+4}$ **[7b 2018 IFoS]**

Q2. Solve:
$\frac{dy}{dx}=\frac{1}{1+x^2}\left(e^{\tan^{-1}x}-y\right)$ **[5c UPSC CSE 2016]**

3. Solve the differential equation
$\frac{dy}{dx}-y=y^2(\sin x+\cos x)$. **[8b 2016 IFoS]**

4. Solve the differential equation:
$x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$. **[5a UPSC CSE 2015]**

5. Solve the following differential equation:
$\frac{dy}{dx}=\frac{2y}{x}+\frac{x^3}{y}+x\tan\frac{y}{x^2}$. **[6a 2014 IFoS]**

6. Solve:
$\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$. **[5a 2013 IFoS]**

7. Solve
$\frac{dy}{dx}=\frac{2xye^{(x/y)^2}}{y^2\left(1+e^{(x/y)^2}\right)+2x^2e^{(x/y)^2}}$. **[5a UPSC CSE 2012]**

8. Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$. **[5a 2012 IFoS]**

9. Obtain the solution of the ordinary differential equation $\frac{dy}{dx}=(4x+y+1)^2$, if $y(0)=1$

[5a UPSC CSE 2011]

10. Solve the following differential equation
$\frac{dy}{dx}=\sin^2(x-y+6)$. **[6a 2010 IFoS]**

EXACT DIFF EQUATIONS, INTEGRATING FACTORS

Q5.(a) Show that the general solution of the differential equation $\frac{dy}{dx}+Py=Q$ can be written in the form $y=\frac{Q}{P}-e^{-\int P\,dx}\left\{C+\int e^{\int P\,dx}d\left(\frac{Q}{P}\right)\right\}$, where P, Q are non-zero functions of x and C, an arbitrary constant. **[UPSC CSE 2022]**

Q5(b) Solve the differential equation
$y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)$. **[IFoS 2021]**

1. Solve the following differential equation:
$x\cos\left(\frac{y}{x}\right)(ydx+xdy)=y\sin\left(\frac{y}{x}\right)(xdy-ydx)$. **[5a UPSC CSE 2020]**

2. Solve the initial value problem:
$\left(2x^2+y\right)dx+\left(x^2y-x\right)dy=0,\ y(1)=2$. **[5a 2020 IFoS]**

3. Solve the differential equation
$\left(2y\sin x+3y^4\sin x\cos x\right)dx-\left(4y^3\cos^2 x+\cos x\right)dy=0$. **[5a UPSC CSE 2019]**

4. Find α and β such $x^{\alpha}y^{\beta}$ is an integrating factor of $\left(4y^2+3xy\right)dx-\left(3xy+2x^2\right)dy=0$ and solve the equation.**[7d UPSC CSE 2018]**

5. Find $f(y)$ such that $\left(2xe^y+3y^2\right)dy+\left(3x^2+f(y)\right)dx=0$ is exact and hence solve. **[8d UPSC CSE 2018]**

6. Solve the differential equation $\left(y^2+2x^2y\right)dx+\left(2x^3-xy\right)dy=0$. **[6a 2018 IFoS]**

7. Solve $\left\{y(1-x\tan x)+x^2\cos x\right\}dx-xdy=0$. **[6a UPSC CSE 2016]**

8. $\left(2xy^4e^y+2xy^3+y\right)dx+\left(x^2y^4e^y-x^2y^2-3x\right)dy=0$. **[5b UPSC CSE 2015]**
Solve the differential equation

9. Find the constant a so that $(x+y)^a$ is the Integrating factor of $\left(4x^2+2xy+6y\right)dx+\left(2x^2+9y+3x\right)dy=0$ and hence solve the differential equation.
[6a UPSC CSE 2015]

10. Justify that a differential equation of the form:
$\left[y+xf\left(x^2+y^2\right)\right]dx+\left[yf\left(x^2+y^2\right)-x\right]dy=0$
where $f\left(x^2+y^2\right)$ is an arbitrary function of $\left(x^2+y^2\right)$, is not an exact differential equation and $\dfrac{1}{x^2+y^2}$ is an integrating factor for it. Hence solve this differential equation for $f\left(x^2+y^2\right)=\left(x^2+y^2\right)^2$.**[5a UPSC CSE 2014]**

11. Find the sufficient condition for the differential equation $M(x,y)dx+N(x,y)dy=0$ to have an integrating factor as a function of $(x+y)$. What will be the integrating factor in that case? Hence find the integrating factor for the differential equation
$\left(x^2+xy\right)dx+\left(y^2+xy\right)dy=0$,
and solve it.**[8a UPSC CSE 2014]**

12. y is a function of x, such that the differential coefficient $\dfrac{dy}{dx}$ is equal to $\cos(x+y)+\sin(x+y)$. Find out a relation between x and y, which is free from any derivative/differential.
[5a UPSC CSE 2013]

13. Solve the differential equation
$\left(5x^3+12x^2+6y^2\right)dx+6xydy=0$. **[6a UPSC CSE 2013]**

14. Show that the differential equation $(2xy\log y)dx+\left(x^2+y^2\sqrt{y^2+1}\right)dy=0$ is not exact. Find an integrating factor and hence, the solution of the equation.
[6a UPSC CSE 2012]

15. Show that the differential equation $(3y^2-x)+2y(y^2-3x)y'=0$ admits an integrating factor which is a function of $(x+y^2)$. Hence solve the equation.

[5b UPSC CSE 2010]

16. (a) Verify that

$$\frac{1}{2}(Mx+Ny)d(\log_e(xy))+\frac{1}{2}(Mx-Ny)d\left(\log_e\left(\frac{x}{y}\right)\right)=M\,dx+N\,dy$$

Hence show that –

(i) if the differential equation $M\,dx+N\,dy=0$ is homogeneous, then $(Mx+Ny)$ is an integrating factor unless $Mx+Ny\equiv 0$;

(ii) if the differential equation $M\,dx+N\,dy=0$ is not exact but is of the form $f_1(xy)\,y\,dx+f_2(xy)\,x\,dy=0$ then $(Mx-Ny)^{-1}$ is an integrating factor unless $Mx-Ny\equiv 0$.

[6a UPSC CSE 2010]

17. Show that $\cos(x+y)$ is an integrating factor of

$y\,dx+\left[y+\tan(x+y)\right]dy=0$

Hence solve it.**[5a 2010 IFoS]**

18. Consider the differential equation

$y'=\alpha x,\ x>0$

where α is a constant. Show that –

(i) if $\phi(x)$ is any solution and $\psi(x)=\phi(x)e^{-\alpha x}$, then $\psi(x)$ is a constant;

(ii) if $\alpha<0$, then every solution tends to zero as $x\to\infty$.**[5a UPSC CSE 2010]**

ODE OF FIRST ORDER BUT HIGHER DEGREE, CLAIRAUT'S FORM, SINGULAR SOLUTIONS

UPDATED

Q5. (a) Find the general solution of the differential equation:

$p^2\cos^2 y+p\sin x\cos x\cos y-\sin y\cos^2 x=0$, where $p\equiv\frac{dy}{dx}$. **[IFoS2022]**

Q6.(a) Solve the differential equation:

$4(xp^2+yp)=y^4$, where $p\equiv\frac{dy}{dx}$. **[IFoS 2022]**

Q8.(a)(i). Find the general and singular solutions of the differential equation: $(x^2-a^2)p^2-2xyp+y^2+a^2=0$, where $p=\frac{dy}{dx}$. Also give the geometric relation between the general and singular solutions. **[UPSC CSE 2022]**

Q8.(a)(ii) Solve the following differential equation:

$(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$. **[UPSC CSE 2022]**

Q7(b) Find all possible solutions of the differential equation:

$y^2 \log y = xy\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$.**[UPSC CSE 2021]**

Q7(b) (i) Reduce the differential equation $axyp^2 + \left(x^2 - ay^2 - b\right)p - xy = 0$, $\left(p = \frac{dy}{dx}\right)$ to Clairaut's form and find the general solution.

(ii) Find the singular solution of the differential equation

$9p^2(2-y)^2 = 4(3-y), \left(p = \frac{dy}{dx}\right)$. **[IFoS 2021]**

1. Find the general and singular solutions of the differential equation $9p^2(2-y)^2 = 4(3-y)$, where $p = \frac{dy}{dx}$.**[8a(ii) UPSC CSE 2020]**

2. Reduce the differential equation

$xp^2 - 2yp + x + 2y = 0, \left(p = \frac{dy}{dx}\right),$

To Clairaut's form and obtain its complete primitive. Also, determine a singular solution of the given differential equation.**[8b 2020 IFoS]**

3. Obtain the singular solution of the differential equation

$\left(\frac{dy}{dx}\right)^2 \left(\frac{y}{x}\right)^2 \cot^2\alpha - 2\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \operatorname{cosec}^2\alpha = 1$

Also find the complete primitive of the given differential equation. Give the geometrical interpretations of the complete primitive and singular solution.

[8a UPSC CSE 2019]

4. Solve the differential equation $(px - y)(py + x) = h^2 p$, where $p = y'$.

[5b 2019 IFoS]

5. Solve:

$\left(\frac{dy}{dx}\right)^2 y + 2\frac{dy}{dx}x - y = 0$. **[6a UPSC CSE 2018]**

6. (i) Consider the differential equation $xy\,p^2 - \left(x^2 + y^2 - 1\right)p + xy = 0$ where $p = \frac{dy}{dx}$. Substituting $u = x^2$ and $v = y^2$ reduce the equation to Clairaut's form in terms of u, v and $p' = \frac{dv}{du}$. Hence, or otherwise solve the equation.**[6b(i) UPSC CSE 2017]**

7. Solve the differential equation

$\left(\frac{dy}{dx}\right)^2 + 2\cdot\frac{dy}{dx}\cdot y\cot x = y^2$. **[6a 2017 IFoS]**

8. Solve the differential equation

$e^{3x}\left(\frac{dy}{dx} - 1\right) + \left(\frac{dy}{dx}\right)^3 e^{2y} = 0$. **[7a 2017 IFoS]**

9. Obtain the singular solution of the differential equation

$y^2 - 2pxy + p^2\left(x^2 - 1\right) = m^2, p = \frac{dy}{dx}$.**[7c 2016 IFoS]**

10. Solve the differential equation

$x = py - p^2$ where $p = \frac{dy}{dx}$.**[7d UPSC CSE 2015]**

11. Reduce the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0, p = \frac{dy}{dx}$ to Clairaut's form. Hence, find the singular solution of the equation.**[5a 2015 IFoS]**

12. Solve the differential equation:

$y = 2px + p^2y,\ p = \frac{dy}{dx}$

and obtain the non-singular solution.**[5a 2014 IFoS]**

13. Solve and find the singular solution of $x^3p^2 + x^2py + a^3 = 0$.**[5b 2012 IFoS]**

14. Solve $x = y\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$. **[8a 2012 IFoS]**

15.(a) Obtain Clairaut's form of the differential equation

$\left(x\frac{dy}{dx} - y\right)\left(y\frac{dy}{dx} + x\right) = \mathrm{a}^2\frac{dy}{dx}$

Also find its general solution.**[6a UPSC CSE 2011]**

16. Solve:

$p^2 + 2py\cot x = y^2$, where $p = \frac{dy}{dx}$.**[6a 2011 IFoS]**

ORTHOGONAL TRAJECTORIES

Q5(b) Show that the orthogonal trajectories of the system of parabolas : $x^2 = 4a(y+a)$ belong to the same system. **[UPSC CSE 2022]**

8.(a)(i) Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1;\ a > b > 0$ are constants and λ is a parameter. Show that the given family of curves is self orthogonal. **[UPSC CSE 2021]**

1. Find the orthogonal trajectories of the family of circles passing through the points $(0,2)$ and $(0,-2)$.

[5b UPSC CSE 2020]

2. Suppose that the streamlines of the fluid flow are given by a family of curves $xy = c$. Find the equipotential lines that is, orthogonal trajectories of the family of curves representing the streamlines.

[5b UPSC CSE 2017]

3. Show that the family of parabolas $y^2 = 4cx + 4c^2$ is self-orthogonal.

[5d UPSC CSE 2016]

4. Obtain the equation of the orthogonal trajectory of the family of curves represented by $r^n = a\sin n\theta, (r,\theta)$ being plane polar coordinates.

[5b UPSC CSE 2013]

5. Find the orthogonal trajectories of the family of curves $x^2 + y^2 = ax$.

[5b UPSC CSE 2012]

6. Determine the orthogonal trajectory of a family of curves represented by the polar equation
$r = a(1-\cos\theta), (r,\theta)$ being the plane polar coordinates of any point.

[5b UPSC CSE 2011]

LINEAR DE WITH CONSTANT COEFFICIENTS

UPDATED

Q5(b) Solve the differential equation:
$(D^2-1)y = e^x(1+x^2)$, where $D \equiv \frac{d}{dx}$. **[IFoS 2022]**

5.(a) Solve the differential equation:
$\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$. **[UPSC CSE 2021]**

Q5.(a) Solve the differential equation
$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$. **[IFoS 2021]**

1. Solve the differential equation
$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 16x - 12e^{2x}$. **[5b 2020 IFoS]**

2. Determine the complete solution of the differential equation
$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x$. **[5b UPSC CSE 2019]**

3. Solve the differential equation $(D^2+1)y = x^2\sin 2x;\ D \equiv \frac{d}{dx}$. **[5a 2019 IFoS]**

4. Find the general solution of the differential equation
$\ddot{x} + 4x = \sin^2 2t$
Hence find the particular solution satisfying the conditions
$x\left(\frac{\pi}{8}\right) = 0$ and $\dot{x}\left(\frac{\pi}{8}\right) = 0$. **[7a 2019 IFoS]**

5. Solve: $y'' - y = x^2e^{2x}$. **[5a UPSC CSE 2018]**

6. Solve: $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$. **[5c UPSC CSE 2018]**

7. Solve: $y'' + 16y = 32\sec 2x$. **[6c UPSC CSE 2018]**

8. Solve the initial value problem
$y'' - 5y' + 4y = e^{2t}$
$y(0) = \frac{19}{12}, y'(0) = \frac{8}{3}$. **[7c UPSC CSE 2018]**

9. Find the complementary function and particular integral for the equation
$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$
and hence the general solution of the equation. **[5a 2018 IFoS]**

10. (ii) Solve the following initial value differential equations:
$20y'' + 4y' + y = 0,\ y(0) = 3\cdot 2$ and $y'(0) = 0$. **[6b(ii) UPSC CSE 2017]**

11. Solve $(2D^3-7D^2+7D-2)y=e^{-8x}$ where $D=\frac{d}{dx}$.**[5a 2017 IFoS]**

12. Find a particular integral of $\frac{d^2y}{dx^2}+y=e^{x/2}\sin\frac{x\sqrt{3}}{2}$.**[5a UPSC CSE 2016]**

13. Solve the dE to get the particular integral of $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=x^2\cos x$.

[5b 2016 IFoS]

14. Solve: $(D^4+D^2+1)y=e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right)$, where $D\equiv\frac{d}{dx}$.**[6c 2015 IFoS]**

15. Solve: $\frac{d^4y}{dx^4}-16y=x^4+\sin x$. **[5b 2014 IFoS]**

16. Solve the D.E.

$\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x$. **[7c 2014 IFoS]**

17. Solve

$(D^3+1)y=e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$ where $D=\frac{d}{dx}$.**[7a 2013 IFoS]**

18. Find the general solution of the equation $y'''-y''=12x^2+6x$.

[6b UPSC CSE 2012]

19. Solve $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=x^2\cos x$.**[6d 2012 IFoS]**

20.Obtain the general solution of the second order ordinary differential equation

$y''-2y'+2y=x+e^x\cos x$,

where dashes denote derivatives w.r. to x.**[6b UPSC CSE 2011]**

21. Solve:

$(D^4+D^2+1)y=ax^2+be^{-x}\sin 2x$, where $D\equiv\frac{d}{dx}$.**[6c 2011 IFoS]**

22. Solve:

$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$. **[5b 2010 IFoS]**

23. Solve:

$\left(\frac{d}{dx}-1\right)^2\left(\frac{d^2}{dx^2}+1\right)^2y=x+e^x$. **[6c 2010 IFoS]**

HOMOGENEOUS LINEAR DE

Q8(b) Solve the differential equation:

$x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=15x^4+8x^3$. **[IFoS 2022]**

Q7(b) Find the general solution of the differential equation

$(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$. **[IFoS 2021]**

1. (i) Solve the following differential equation:
$(x+1)^2 y''-4(x+1)y'+6y=6(x+1)^2+\sin\log(x+1)$. **[8a(i) UPSC CSE 2020]**

2. Solve the differential equation
$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$. **[7b 2020 IFoS]**

3. Solve: $(1+x)^2 y''+(1+x)y'+y=4\cos(\log(1+x))$. **[7a UPSC CSE 2018]**

4. Solve the differential equation
$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$. **[5b 2017 IFoS]**

5. Find the general solution of the equation
$x^2\frac{d^3y}{dx^3}-4x\frac{d^2y}{dx^2}+6\frac{dy}{dx}=4$.**[6c UPSC CSE 2016]**

6. Solve:
$x^4\frac{d^6y}{dx^4}+6x^3\frac{d^3y}{dx^3}+4x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\cos(\log_e x)$. **[8d UPSC CSE 2015]**

7. Solve the differential equation $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}$.**[5e 2015 IFoS]**

8. Solve the differential equation:
$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8y=65\cos(\log_e x)$. **[6b UPSC CSE 2014]**

9. Find the general solution of the equation
$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\ln x\sin(\ln x)$. **[6c UPSC CSE 2013]**

10. Solve:
$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=(1-x)^{-2}$. **[8d 2012 IFoS]**

11. Solve:
$\left[x^4D^4+6x^3D^3+9x^2D^2+3xD+1y\right]=(1+\log x)^2$, where $D\equiv\frac{d}{dx}$.**[6b 2011 IFoS]**

LINEAR DE OF SECOND ORDER- CHANGING DEPENDENT VARIABLE, CHANGING INDEPENDENT VARIABLE

1. Find one solution of the differential equation
$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$.
by inspection and using that solution determine the other linearly independent solution of the given equation. Obtain the general solution of the given differential equation.**[6a 2020 IFoS]**

2. Solve the differential equation
$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\sin^2 x$. **[6c(i) UPSC CSE 2019]**

3. Find the general solution of the differential equation
$(x-2)y''-(4x-y)y'+(4x-6)y=0$. **[8a 2019 IFoS]**
4. (i) Solve the differential equation:
$x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^3\sin(x^2)$. **[7b(i) UPSC CSE 2017]**
5. Solve $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^3\sin x$ by changing the independent variable.
[6a 2015 IFoS]
6. Solve the following differential equation:
$x\frac{d^2y}{dx^2}-2(x+1)\frac{dy}{dx}+(x+2)y=(x-2)e^{2x}$, when e^x is a solution to its corresponding homogeneous differential equation.**[7a UPSC CSE 2014]**
7. Solve the differential equation
$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\sin 2x$.
by changing the dependent variable.**[6a 2013 IFoS]**
8. Solve the ordinary differential equation
$x(x-1)y''-(2x-1)y'+2y=x^2(2x-3)$. **[6c UPSC CSE 2012]**
9. Solve:
$x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0$. **[6a 2012 IFoS]**
10. Solve:
$\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+5y=\sec x\cdot e^x$. **[5b 2011 IFoS]**
11. Find the general solution of
$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2+1)y=0$. **[6b 2010 IFoS]**

LINEAR DE OF SECOND ORDER - VARIATION OF PARAMETER METHOD

Q7(b) Given that $y_1=x^2$ is a solution of the differential equation,
$x^3\frac{d^2y}{dx^2}-(x^2+3x)\frac{dy}{dx}+6y=0$,
find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. **[IFoS 2022]**
Q8(a)(ii) Find the general solution of the differential equation:
$x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=0$.

Hence, solve the differential equation: $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$ by the method of variation of parameters. **[UPSC CSE 2021]**

1. Using the method of variation of parameters, solve the differential equation $y''+(1-\cot x)y'-y\cot x=\sin^2 x$, if $y=e^{-x}$ is one solution of CF.

[6a UPSC CSE 2020]

2. Find the linearly independent solutions of the corresponding homogeneous differential equation of the equation $x^2y''-2xy'+2y=x^3\sin x$ and then find the general solution of the given equation by the method of variation of parameters.**[7a UPSC CSE 2019]**

3. Solve by the method of variation of parameters the differential equation

$x''(t)-\frac{2x(t)}{t^2}=t$, where $0<t<\infty$. **[6a 2019 IFoS]**

4. Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\log x\,(x>0)$ by the method of variation of parameters.

[5b 2018 IFoS]

5. Solve the following differential equation using method of variation of parameters:

$\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=44-76x-48x^2$. **[7b(ii) UPSC CSE 2017]**

6. Solve $\frac{d^2y}{dx^2}+4y=\tan 2x$ by using the method of variation of parameter.

[8a 2017 IFoS]

7. Using the method of variation of parameters, solve the differential equation

$(D^2+2D+1)y=e^{-x}\log(x),\ \left[D\equiv\frac{d}{dx}\right]$. **[6b UPSC CSE 2016]**

8. Using the method of variation of parameters, solve

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$. **[6b 2016 IFoS]**

9. Solve by the method of variation of parameters:

$\frac{dy}{dx}-5y=\sin x$. **[6a UPSC CSE 2014]**

10. Solve by the method of variation of parameters:

$y''+3y'+2y=x+\cos x$. **[6c 2014 IFoS]**

11. Using the method of variation of parameters, solve the differential equation

$\frac{d^2y}{dx^2}+a^2y=\sec ax$.**[6b UPSC CSE 2013]**

12. Apply the method of variation of parameters to solve

$\frac{d^2y}{dx^2}-y=2(1+e^x)^{-1}$. **[8a 2013 IFoS]**

13. (c) Using the method of variation of parameters, solve the second order differential equation

$\frac{d^2y}{dx^2}+4y=\tan 2x$. **[6c UPSC CSE 2011]**

14. Solve by the method of variation of parameters the following equation

$\left(x^2-1\right)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=\left(x^2-1\right)^2$. **[6d 2010 IFoS]**

UNDETERMINED COEFFICIENT METHOD

Q8(b) Solve the following differential equation by using the method of variation of parameters:
$\left(x^2-1\right)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=\left(x^2-1\right)^2$, given that $y=x$ is one solution of the reduced equation. **[UPSC CSE 2022]**

8.(a) Use the method of undetermined coefficients to find the particular solution of $y''+y=\sin x+\left(1+x^2\right)e^x$ and hence find its general solution.

[8a UPSC CSE 2010]

SIMULTANEOUS LINEAR DE WITH CONSTANT COEFFICIENTS

1. Solve the following simultaneous linear differential equations:
$(D+1)y=z+e^x$ and $(D+1)z=y+e^x$ where y and z are functions of independent variable x and $D\equiv\dfrac{d}{dx}$. **[6a(i) UPSC CSE 2017]**

LAPLACE AND INVERSE TRANSFORMS

1. Find the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$. Prove that the Laplace transform of $t^{n+\frac{1}{2}}$, where $n\in N$, is
$\dfrac{\Gamma\left(n+1+\frac{1}{2}\right)}{s^{n+1+\frac{1}{2}}}$. **[6c(ii) UPSC CSE 2019]**

2. Find the Laplace transform of $f(t)=\dfrac{1}{\sqrt{t}}$. **[5d(i) UPSC CSE 2018]**

3. Find the inverse Laplace transform of $\dfrac{5s^2+3x-16}{(s-1)(s-2)(s+3)}$. **[5d(ii) UPSC CSE 2018]**

4. Obtain Laplace Inverse transform of
$\left\{\ell n\left(1+\dfrac{1}{s^2}\right)+\dfrac{s}{s^2+25}e^{-\pi x}\right\}$. **[7a(i) UPSC CSE 2015]**

SOLUTION OF SECOND ORDER LINEAR DE WITH CONSTANT COEFF BY LAPLACE TRANSFORMS

Q7(b) Solve the following initial value problem by using Laplace's transformation
$\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+2y=h(t)$, where

$h(t)=\begin{cases}2, & 0<t<4,\\ 0, & t>4,\end{cases} \quad y(0)=0, y'(0)=0$.

[UPSC CSE 2022]

Q5(b) Solve the initial value problem:

$\frac{d^2y}{dx^2}+4y=e^{-2x}\sin 2x;\ y(0)=y'(0)=0$

using Laplace transform method. **[UPSCCSE 2021]**

1. Using Laplace transform, solve the initial value problem $ty''+2ty'+2y=2$; $y(0)=1$ and $y'(0)$ is arbitrary. Does this problem have a unique solution?

[7b UPSC CSE 2020]

2. Solve the following initial value problem using Laplace transform:

$\frac{d^2y}{dx^2}+9y=r(x),\ \ y(0)=0,\ y'(0)=4$

where $r(x)=\begin{cases}8\sin x \text{ if } & 0<x<\pi\\ 0 \quad \text{if} & x\geq\pi\end{cases}$. **[8b UPSC CSE 2017]**

3. Using Laplace transformation, solve the following:

$y''-2y'-8y=0,\ \ y(0)=3,\ \ y'(0)=6$. **[6d UPSC CSE 2016]**

4. Using Laplace transform, solve

$y''+y=t,\ y(0)=1,\ y'(0)=-2$. **[7a(ii) UPSC CSE 2015]**

5. Solve the initial value problem

$\frac{d^2y}{dt^2}+y=8e^{-2t}\sin t,\ \ \ y(0)=0,\ y'(0)=0$

by using Laplace-transform.**[8c UPSC CSE 2014]**

6. By using Laplace transform method, solve the differential equation $(D^2+n^2)x=a\sin(nt+\alpha),\ D^2\equiv\frac{d^2}{dt^2}$ subject to the initial conditions $x=0$ and $\frac{dx}{dt}=0$, at $t=0$, in which a, n and α are constants.**[6d UPSC CSE 2013]**

7. Using Laplace transforms, solve the initial value problem

$y''+2y'+y=e^{-t},\ y(0)=-1,\ y'(0)=1$. **[5c UPSC CSE 2012]**

8. Use Laplace transform method to solve the following initial value problem:

$\frac{d^2x}{dt^2}-2\frac{dx}{dt}+x=e^t,\ x(0)=2$ and $\left.\frac{dx}{dt}\right|_{t=0}=-1$.**[6d UPSC CSE 2011]**

Linear Algebra + Differential Equations

8. Show that the set of solutions of the homogeneous linear differential equation $y'+p(x)y=0$ on an interval $I=[a,b]$ forms a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

[7a UPSC CSE 2010]

3D ANALYTICAL GEOMETRY

1. 2D GEOMETRY
2. D.R.'s, D.C.'s
3. PLANE
4. STRAIGHT LINE
   SHORTEST DISTANCE
   SKEW LINES
5. SPHERE
6. CONE& CYLINDER
7. CONICOID- ELLIPSOID, HYPERBOLOID, PARABOLOID
8. GENERATING LINES
9. REDUCTION OF GENERAL EQUATION OF SECOND DEGREE

CHAPTER 1. 2D GEOMETRY

Q1. If the straight lines, joining the origin to the points of intersection of the curve $3x^2-xy+3y^2+2x-3y+4=0$ and the straight line $2x+3y+k=0$, are at right angles, then show that $6k^2+5k+52=0$. **[1e 2020 IFoS]**

Q2. If the coordinates of the points A and B are respectively $(b\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ and if the line joining A and B is produced to the point $M(x,y)$ so that $AM:MB=b:a$, then show that $x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=0$. **[1e 2019 IFoS]**

Q3. If the point $(2,3)$ is the mid-point of a chord of the parabola $y^2=4x$, then obtain the equation of the chord. **[1d 2016 IFoS]**

Q4. A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$.

[(2b) 2016 IFoS]

Q5. Find the locus of the poles of chords which are normal to the parabola $y^2=4ax$.

[(4c) 2015 IFoS]

Q6. The tangent at $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.

[(1e) 2015 IFoS]

Q7. Prove that two of the straight lines represented by the equation $x^3+bx^2y+cxy^2+y^3=0$ will be at right angles, if $b+c=-2$.

[(1e) UPSC CSE 2012]

Q8. Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord. **[(4c) 2011 IFoS]**

CHAPTER 2. DIRECTION COSINES, DIRECTION RATIOS

Q3(c) Show that the straight lines whose direction cosines are given by the equations $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ (where a,b,c,u,v,w are constants) are parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$ and perpendicular if $a^2(v+w)+b^2(w+u)+c^2(u+v)=0$. **[IFoS 2021]**

Q1. Prove that the angle between two straight lines whose direction cosines are given by $l+m+n=0$ and $fmn+gnl+hlm=0$ is $\frac{\pi}{3}$, if $\frac{1}{f}+\frac{1}{g}+\frac{1}{h}=0$.

[2c 2020 IFoS]

Q2. A line makes a angle $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube. Show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$. **[(2c) 2019 IFoS]**

Q3. Find the angle between the lines whose direction cosines are given by the relations $l+m+n=0$ and $2lm+2\ln-mn=0$. **[(3d) 2017 IFoS]**

CHAPTER 3. PLANE

Q1(e) A variable plane is at a constant distance of 6 units from the origin and meets the axes in $A:(a,0,0), B:(0,b,0)$ and $C:(0,0,c)$. Find the locus of the centroid of the triangle ABC. **[IFoS 2022]**

Q1(e) Find the equation of the plane passing through the points $(1,-1,1)$ and $(-2,1,-1)$ and perpendicular to the plane $2x+y+z+5=0$. **[IFoS 2021]**

Q1. A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, which is fixed. The plane through P and perpendicular to OP meets the axes in A, B, C respectively. The planes through A, B, C parallel yz, zx and xy planes respectively intersect at Q. Prove that the locus of Q is

$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$. **[3c 2020 IFoS]**

Q2. Find the equation of the plane parallel to $3x-y+3z=8$ and passing through the point $(1,1,1)$.

[(4d) UPSC CSE 2018]

Q3. Show that the angles between the planes given by the equation $2x^2-y^2+3z^2-xy+7zx+2yz=0$ is $\tan^{-1}\frac{\sqrt{50}}{4}$. **[(2d) 2017 IFoS]**

Q4. Find the equations of the planes parallel to the plane $3x-2y+6z+8=0$ and at a distance 2 from it. **[(1e) 2017 IFoS]**

Q5. Obtain the equation of the plane passing through the points $(2,3,1)$ and $(4,-5,3)$ parallel to x-axis. **[(3c(i) UPSC CSE 2015]**

Q6. Find the equation of the plane containing the straight line $y+z=1, x=0$ and parallel to the straight line $x-z=1, y=0$. **[(3d) 2015 IFoS]**

Q7. Find the equation of the plane which passes through the points $(0,1,1)$ and $(2,0,-1)$, and is parallel to the line joining the points $(-1,1,-2),(3,-2,4)$. Find also the distance between the line and the plane. **[(1d) UPSC CSE 2013]**

Q8. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Prove that the locus of the centroid of the tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$. **[(1e) 2011 IFoS]**

Q9. If a plane cuts the axes in A, B, C and (a,b,c) are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$. **[(1e) 2010 IFoS]**

CHAPTER 4. STRAIGHT LINE

Q4(c) Find equation of the plane containing the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7},$$

$$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}.$$

Also find the point of intersection of the given lines. **[UPSC CSE 2021]**

Q1. Show that the lines

$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$

intersect. Find the coordinates of the point of intersection and the equation of the plane containing them. **[(1e) UPSC CSE 2019]**

Q2. Find the projection of the straight line $\frac{x-1}{2}=\frac{y-1}{3}=\frac{z+1}{-1}$ on the plane $x+y+2z=6$.

[(1e) UPSC CSE 2018]

Q3. Verify if the lines:

$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$

are coplanar. If yes, then find the equation of the plane in which they lie.

[(3c(ii) UPSC CSE 2015]

Q4. Find the equations of the straight line through the point $(3,1,2)$ to intersect the straight line $x+4=y+1=2(z-2)$ and parallel to the plane $4x+y+5z=0$.

[(1e) UPSC CSE 2011]

SHORTEST DISTANCE

Q2(c) Obtain the coordinates of the points where the shortest distance line between the straight lines $\frac{x-3}{-1}=\frac{y-2}{2}=\frac{z-2}{-1}$; $\frac{x-2}{2}=\frac{y+3}{3}=\frac{z+2}{2}$ meets them. Also find the magnitude of the shortest distance and the equation of the shortest distance line between the straight lines mentioned above.

[IFoS 2022]

Q1. Show that the shortest distance between the straight lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ is $3\sqrt{30}$. Find also the equation of the line of shortest distance. **[(3c) 2019 IFoS]**

Q2. Find the shortest distance between the lines

$a_1x+b_1y+c_1z+d_1=0$

$a_2x+b_2y+c_2z+d_2=0$

and the *z*-axis. **[(2d) UPSC CSE 2018]**

Q3. Find the shortest distance between the skew lines:

$\frac{x-3}{3}=\frac{8-y}{1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. **[(1e) UPSC CSE 2017]**

Q4. Find the shortest distance and the equation of the line of the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$.

[(4d) 2017 IFoS]

Q5. Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{4}=z-3$ and $y-mx=z=0$. For what value of *m* will the two lines intersect?

[(1e) UPSC CSE 2016]

Q6. Find the magnitude and the equations of the line of shortest distance between the lines

$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$

$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **[(4c) 2013 IFoS]**

Q7. If 2C is the shortest distance between the lines

$\frac{x}{l}-\frac{z}{n}=1,\ y=0$ and $\frac{y}{m}+\frac{z}{n}=1,\ x=0$ then show that $\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}=\frac{1}{c^2}$.

[(3c) 2012 IFoS]

SKEW LINES

Q1. Find the surface generated by a line which intersects the lines $y=a=z$, $x+3z=a=y+z$ and parallel to the plane $x+y=0$. **[(4a) UPSC CSE 2016]**

Q2. Find the locus of the variable straight line that always intersects $x=1, y=0;\ y=1, z=0;\ z=1, x=0$. **[(4b) 2015 IFoS]**

Q3. Prove that the locus of a variable line which intersects the three lines: $y=mx,\ z=c;\ y=-mx, z=-c;\ y=z, mx=-c$ is the surface $y^2-m^2x^2=z^2-c^2$.

[(1e) 2014 IFoS]

Q4. Find the surface generated by the straight line which intersects the lines $y=z=a$ and $x+3z=a=y+z$ and is parallel to the plane $x+y=0$.

[(1d) 2013 IFoS]

CHAPTER 5. SPHERE

Q(c) Find the equation of the sphere of smallest possible radius which touches the straight lines: $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$. **[UPSC CSE 2022]**

Q3.(b) A sphere of constant radius *r* passes through the origin O and cuts the axes at the points A, B and C. Find the locus of the foot of the perpendicular drawn from O to the plane ABC. **[UPSC CSE 2021]**

Q4.(a) Find the equation of the sphere passing through the points $(1,1,2)$, $(1,-1,2)$ and having centre on the line $x+y-z-1=0=2x+y-z-2$. **[IFoS 2021]**

Q1(e) A variable plane passes through a fixed point (a,b,c) and meets the axes at points A, B and C respectively. Find the locus of the centre of the sphere passing through the points O, A, B and C, O being the origin.

[UPSC CSE 2022]

Q1. The plane $x+2y+3z=12$ cuts the axes of coordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC.

[(2c)(i) UPSC CSE 2019]

Q2. Find the equation of the sphere in *xyz*-plane passing through the points $(0,0,0),(0,1,-1),(-1,2,0)$ and $(1,2,3)$. **[(3d) UPSC CSE 2018]**

Q3. Find the equation of the tangent plane that can be drawn to the sphere $x^2+y^2+z^2-2x+6y+2z+8=0$, through the straight line $3x-4y-8=0=y-3z+2$.

[(3a) 2018 IFoS]

Q4. A plane passes through a fixed point (a,b,c) and cuts the axes at the points A, B, C respectively. Find the locus of the centre of the sphere which passes through the origin O and A, B, C.

[(2b) UPSC CSE 2017]

Q5. Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$. Find the point of contact. **[(2c) UPSC CSE 2017]**

Q6. Find the equation of the sphere which passes through the circle $x^2+y^2=4$; $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

[(1d) UPSC CSE 2016]

Q7. Obtain the equation of the sphere on which the intersection of the plane $x+2y+2z=0$ with the sphere which has $(0,1,0)$ and $(3,-5,2)$ as the end points of its diameter is a great circle.

[(3d) 2016 IFoS]

Q8. For what positive value of a, the plane $ax-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and hence find the point of contact.

[(1e) UPSE CSE 2015]

Q9. Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point $(2,1,3)$?

[(3b) UPSE CSE 2015]

Q10. Find the co-ordinates of the points on the sphere $x^2+y^2+z^2-4x+2y=4$, the tangent planes at which are parallel to the plane $2x-y+2z=1$.

[(4a(i) UPSE CSE 2014]

Q11. Prove that every sphere passing through the circle $x^2+y^2-2ax+r^2=0, z=0$ cut orthogonally every sphere through the circle $x^2+z^2=r^2, y=0$.

[(2c) 2014 IFoS]

Q12. A moving plane passes through a fixed point $(2,2,2)$ and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C.

[(3b) 2014 IFoS]

Q13. A sphere S has points $(0,1,0),(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle. **[(1e) UPSC CSE 2013]**

Q14. Show that the three mutually perpendicular tangent lines can be drawn to the sphere $x^2+y^2+z^2=r^2$ from any point on the sphere $2\left(x^2+y^2+z^2\right)=3r^2$.

[(4a) UPSC CSE 2013]

Q15. Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere $x^2+y^2+2fx+2gy+2hz=0$ if $af+bg+ch=0$.

[(4b) 2012 IFoS]

Q16. Show that the equation of the sphere which touches the sphere $4\left(x^2+y^2+z^2\right)+10x-25y-2z=0$ at the point $(1,2,-2)$ and passes through the point $(-1,0,0)$ is $x^2+y^2+z^2+2x-6y+1=0$. **[(1f) UPSC CSE 2011]**

Q17. Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,1,-1)$. **[(3b) UPSC CSE 2011]**

Q18. Show that the plane $x+y-2z=3$ cuts the sphere $x^2+y^2+z^2-x+y=2$ in a circle of radius 1 and find the equation of the sphere which has this circle as a great circle. **[(1e) UPSC CSE 2010]**

Q19. Show that every sphere through the circle $x^2+y^2-2ax+r^2=0,\ z=0$ cuts orthogonally every sphere through the circle $x^2+z^2=r^2, y=0$.

[(3c) UPSC CSE 2010]

Q20. Find the equations of the spheres passing through the circle $x^2+y^2+z^2-6x-2z+5=0, y=0$ and touching the plane $3y+4z+5=0$.

[(1f) 2010 IFoS]

CHAPTER 6. CONE& CYLINDER

Q4(c) If the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2=0$ in perpendicular generators, then prove that $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$. **[UPSC CSE 2022]**

2.(a) Show that the planes, which cut the cone $ax^2+by^2+cz^2=0$ in perpendicular generators, touch the cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$. **[UPSC CSE 2021]**

(c) Find the equation of the cone whose vertex is $(1,2,1)$ and which passes through the circle $x^2+y^2+z^2=5, x+y-z=1$. **[IFoS 2021]**

Q1. A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes at the points A, B and C. Prove that the circle ABC lies on the cone

$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$. **[(4c) 2019 IFoS]**

Q2. Find the equation of the cone with $(0,0,1)$ as the vertex and $2x^2-y^2=4,\ z=0$ as the guiding curve. **[(4c) UPSC CSE 2018]**

Q3. Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios $(1,-2,2)$. **[(4c) 2017 IFoS]**

Q4. A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C. **[(4d) 2016 IFoS]**

Q5. Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$, represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$. **[(4a(ii) UPSC CSE 2014]**

Q6. Prove that the equation $4x^2-y^2+z^2-3yz+2xy+12x-11y+6z+4=0$ represents a cone with vertex at $(-1,-2,-3)$. **[(3d) 2014 IFoS]**

Q7. A cone has for its guiding curve the circle $x^2+y^2+2ax+2by=0, z=0$ and passes through a fixed point $(0,0,c)$. If the section of the cone by the plane $y=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle $x^2+y^2+z^2+2ax+2by=0$, $2ax+2by+cz=0$. **[(4b) UPSC CSE 2013]**

Q8. A variable plane is parallel to the plane

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$. **[(4b) UPSC CSE 2012]**

Q9. Prove that the second degree equation

$x^2-2y^2+3z^2+5yz-6zx-4xy+8x-19y-2z-20=0$ represents a cone whose vertex is $(1,-2,3)$. **[(4a) 2010 IFoS]**

<u>Intersection of a plane and cone</u>

Q1.Find the equations of the straight lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. Find the angle between the two straight lines.

[(4a) 2018 IFoS]

Q2. Examine whether the plane $x+y+z=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines. **[(1e) UPSC CSE 2014]**

Q3. Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. **[(4b) 2014 IFoS]**

Q4. Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. **[(1e) 2012 IFoS]**

Q5. Show that the cone $yz+zx+xy=0$ cuts the sphere $x^2+y^2+z^2=a^2$ in two equal circles, and find their area. **[(4b) UPSC CSE 2011]**

<u>Three mutually perpendicular generators</u>

Q1. If the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, then find the equations of the other two generators. **[(3c) UPSC CSE 2020]**

Q2. Show that the cone $3yz-2zx-2xy=0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$ is a generator belonging to one such set, find the other two. **[(4b) UPSC CSE 2016]**

Q3. If $6x=3y=2z$ represents one of the three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$ then obtain the equations of the other two generators. **[(2d) UPSC CSE 2015]**

Q4. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two.

[(4c) 2010 IFoS]

<u>Enveloping cone</u>

Q1. Prove that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has the vertex at $(2,4,1)$ in a rectangular hyperbola.

[(2c) (ii) UPSC CSE 2019]

<u>CYLINDER</u>

Q3(c) Find the equation of the cylinder whose generators intersect the curve, $2x^2+3y^2=4z, x-y+2z=3$ and are parallel to the line $3x=-2y=4z$. **[IFoS 2022]**

Q1(e) Find the equation of the cylinder whose generators are parallel to the line $x=-\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is $x^2+2y^2=1, z=0$. **[UPSC CSE 2021]**

Q1. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is $x^2+y^2=4, z=2$. **[(2c) UPSC CSE 2020]**

Q2. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is $x^2+y^2=4, z=2$. **[(2a) 2018 IFoS]**

Q3. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$. **[(4a) 2011 IFoS]**

CHAPTER 7. CONICOID- ELLIPSOID, HYPERBOLOID

Q2(c) If P, Q, R; P', Q', R' are feet of the six normals drawn from a point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the plane PQR is represented by $lx+my+nz=p$, show that the plane P'Q'R' is given by $\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0$. **[UPSC CSE 2022]**

Q1. Find the equations of the tangent plane to the ellipsoid $2x^2+6y^2+3z^2=27$ which passes through the line $x-y-z=0=x-y+2z-9$. **[(1e) UPSC CSE 2020]**

Q2. Let P be the vertex of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. If the section of this cone made by the plane $z=0$ is a rectangular hyperbola, then find the locus of P. **[4a 2020 IFoS]**

Q3. Find the length of the normal chord through a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the *xy*-plane, then P lies on the cone $\frac{x^2}{a^6}\left(2c^2-a^2\right)+\frac{y^2}{b^6}\left(2c^2-b^2\right)+\frac{z^2}{c^4}=0$. **[(4b) UPSC CSE 2019]**

Q4. Find the equations of the tangent planes to the ellipsoid $2x^2+6y^2+3z^2=27$ which pass through the line $x-y-z=0=x-y+2z-9$. **[(1d) 2018 IFoS]**

Q5. Find the locus of the point of intersection of three mutually perpendicular tangent planes to $ax^2+by^2+cz^2=1$. **[(3d) UPSC CSE 2017]**

Q6. Find the locus of the point of intersection of three mutually perpendicular tangent planes to the coincoid $ax^2+by^2+cz^2=1$. **[(4d) UPSC CSE 2016]**

Q7. Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2+by^2+cz^2=1$ at, its points of intersection with the plane $lx+my+nz=p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$ **[(4b) UPSC CSE 2014]**

Q8. Find the equations to the tangent planes to the surface $7x^2-3y^2-z^2+21=0$, which pass through the line $7x-6y+9=0, z=3$. **[(3d) 2013 IFoS]**

Q9. A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2+4y^2+9z^2=36$. Find its equation. **[(4c) 2012 IFoS]**

Q10. Find the equations of the tangent plane to the ellipsoid $2x^2+6y^2+3z^2=27$ which passes through the line $x-y-z=0=x-y+2z-9$. **[(2c) 2012 IFoS]**

Q11. Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere. **[(4a) UPSC CSE 2011]**

Q12. Find the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$. **[(4b) 2011 IFoS]**

Q13. Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$. **[(4d) 2010 IFoS]**

Q14. If the feet of three normals drawn from a point P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in the lane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, prove that the feet of the other three normals lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1=0$. **[(4b) 2010 IFoS]**

PARABOLOID

Q1, Prove that, in general, three normals can be drawn from a given point to the parabholoid $x^2+y^2=2az$, but if the point lies on the surface $27a\left(x^2+y^2\right)+8\left(a-z\right)^3=0$ then two of the three normals coincide.

[(3b) UPSC CSE 2019]

Q2. Find the equation of the tangent plane at point $(1,1,1)$ to the conicoid $3x^2-y^2=2z$. **[(1d) UPSC CSE 2017]**

Q3. Two perpendicular tangent planes to the paraboloid $x^2+y^2=2z$ intersect in a straight line in the plane $x=0$. Obtain the curve to which this straight line touches. **[(4c) UPSC CSE 2015]**

Q4. Show that the locus of a point from which the three mutually perpendicular tangent lines can be drawn to the paraboloid $x^2+y^2+2z=0$ is $x^2+y^2+4z=1$.

[(4c) UPSC CSE 2012]

Q5. Tangent planes to two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid. **[(4d) 2011 IFoS]**

Q6. Show that the plane $3x+4y+7z+\frac{5}{2}=0$ touches the paraboloid $3x^2+4y^2=10z$ and find the point of contact. **[(2c) UPSC CSE 2010]**

CHAPTER 8. GENERATING LINES

Q1. Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$. **[(4b) UPSC CSE 2020]**

Q2. Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point $(1,1,1)$.

[(3c) UPSC CSE 2018]

Q3. Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$. **[(4c) 2018 IFoS]**

Q4. Find the equations of the two generating lines through any point $(a\cos\theta, b\sin\theta, 0)$, of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$, of the hyperboloid by the plane $z=0$.

[(4c) UPSC CSE 2014]

Q5. A variable generator meets two generators of the system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2c^2=1$ in P and P'. Prove that $BP\cdot B'P'=a^2+c^2$.

[(4c) UPSC CSE 2013]

Q6. Show that the generators through any one of the ends of the an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2+b^2=6c^2$. Find also the condition for the generators to be perpendicular to each other.

[(4c) UPSC CSE 2011]

Q7. Find the vertices of the skew quadrilateral formed by the four generators of the hyperboloid $\frac{x^2}{4}+y^2-z^2=49$ passing through $(10,5,1)$ and $(14,2,-2)$.]

[(4c) UPSC CSE 2010]

CHAPTER 9. REDUCTION OF GENERAL EQUATION OF SECOND DEGREE

Q4(c) Reduce the equation, $3x^2+6yz-y^2-z^2-6x+6y+2z+2=0$ to a canonical form and mention the name of the surface it represents. **[IFoS 2022]**

Q1. Reduce the following equation to the standard form and hence determine the nature of the conicoid: $x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$.

[(4a) UPSC CSE 2017]

Q2. Reduce the following equation to its canonical form and determine the nature of the conic $4x^2+4xy+y^2-12x-6y+5=0$. **[(3b) 2013 IFoS]**

STATICS
Previous Years Questions Chapterwise(UPSC CSE/IFoS)

Chapter-1	**Catenary**
Chapter-2	**Forces in Three Dimensions**
Chapter-3	**Stable and Unstable Equilibrium**
Chapte-4	**Moments, Equilibrium**
Chapter-5	**Friction**

1. COMMON CATANORY

6.(a) A cable of weight *w* per unit length and length $2l$ hands from two points P and Q in the same horizontal line. Show that the span of the cable is $2l\left(1-\frac{2h^2}{3l^2}\right)$, where h is the sag in the middle of the tightly stretched position.

[UPSC CSE 2022]

Q1. Derive intrinsic equation
$x = c\log(\sec\psi + \tan\psi)$
of the common category, where symbols have usual meanings.
Prove that the length of an endless chain, which will hang over a circular pulley of radius '*a*' so as to be in contact with $\frac{2}{3}$ of the circumference of the pulley, is

$a\left\{\frac{4\pi}{3} + \frac{3}{\log\left(2+\sqrt{3}\right)}\right\}$. **[7a 2020 IFoS]**

Q2. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu\log\frac{1+\left(1+\mu^2\right)^{\frac{1}{2}}}{\mu}$$

where μ is the coefficient of friction. **[7a 2018 IFoS]**

Q3. Find the length of an endless chain which will hang over a circular pulley of radius '*a*' so as to be in contact with the two-thirds of the circumference of the pulley. **[8a UPSC CSE 2015]**

Q4. Determine the length of an endless chain which will hang over a circular pulley of radius *a* so as to be in contact with two-thirds of the circumference of the pulley. **[7a 2015 IFoS]**

Q5. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the radio of the maximum span to the length of the chain is $\mu\log\left[\frac{1+\sqrt{1+\mu^2}}{\mu}\right]$ where μ is the coefficient of friction. **[7c UPSC CSE 2012]**

Q6. A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120\cosh^{-1}\left(\frac{5}{3}\right)$ and also find the sag. **[5d 2011 IFoS]**

Q7. A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is

$\frac{1}{2}\left[P\cdot\frac{1}{h}+W\cdot\frac{h^2+l^2}{2hl}\right]$. **[7a 2010 IFoS]**

2. FORCES IN THREE DIMENSIONS

Q1. The forces P, Q and R act along three straight lines $y=b, z=-c, z=c, x=-a$ and $x=a, y=-b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action.

[6b 2015 IFoS]

3. STABLE, UNSTABLE & NEUTRAL EQUILIBRIUM

Q7(c) Suppose a cylinder of any cross-section is balanced on another fixed cylinder, the contact of curved surfaces being rough and the common tangent line horizontal. Let ρ and ρ' be the radii of curvature of the two cylinders at the point of contact and h be the height of centre of gravity of the upper cylinder above the point of contact. Show that the upper cylinder is balanced in stable equilibrium if $h<\frac{\rho\rho'}{\rho+\rho'}$. **[UPSC CSE 2022]**

Q8.(a) A bucket is in the form of a frustum of a cone and is filled with water of density ρ. If the bottom and top ends of the bucket have radii a and $b$respectively and h is the height of the bucket, then find the resultant vertical thrust on the curved surface of the bucket. Is that thrust equal to $\frac{1}{3}\pi\rho gh(b-a)(b+2a)$? **[IFoS 2022]**

Q1. A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius a. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$.

[6a UPSC CSE 2019]

Q2. A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ for equilibrium. If ϕ be less than this value, is the equilibrium stable? **[6c UPSC CSE 2017]**

Q3. A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is 90°. **[5d 2017 IFoS]**

Q4. A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}\times$radius of hemisphere. **[7a 2014 IFoS]**

Q5. A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium.

[5e 2013 IFoS]

Q6. A heavy hemispherical shell of radius a has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius b at the highest point. Prove that if $\frac{b}{a} > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

[7b UPSC CSE 2012]

Q7. A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability. **[5c 2010 IFoS]**

4. VIRTUAL WORK

Q8(b) A chain of n equal uniform rods is smoothly joined together and suspended from its one end A_1. A horizontal force $\vec{P}$ is applied to the other end A_{n+1} of the chain. Find the inclinations of the rods to the downward vertical line in the equilibrium configuration. **[UPSC CSE 2022]**

Q5(c) Two rods LM and MN are joined rigidly at the point M such that $(LM)^2+(MN)^2=(LN)^2$ and they are hanged freely in equilibrium from a fixed point L. Let ω be the weight per unit length of both the rods which are uniform. Determine the angle, which the rod LM makes with the vertical direction, in terms of lengths of the rods. **[UPSC CSE 2021]**

Q5(d) Four light rods are joined smoothly to form a quadrilateral ABCD. Let P and Q be the mid-points of an opposite pair of rods and these points are connected by a string in a state of tension T. Let R and S be the mid-points of the other opposite pair of rods and these points are connected by a light rod in a state of thrust X. Show that $T\cdot(RS)=X\cdot(PQ)$. **[IFoS 2021]**

Q1. A square framework formed of uniform heavy rods of equal weight W joined together, is hung up by one corner. A weight W is suspended from each of the three lower corners, and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.

[7c UPSC CSE 2020]

Q2. A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely joined together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance 2b apart. A weight W is suspended from A. Find the trust in the rod BC. **[7c 2018 IFoS]**

Q3. A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\frac{2W\left(2b^2-a^2\right)}{b\sqrt{4b^2-a^2}}$. **[6b 2017 IFoS]**

Q4. Two equal uniform rods AB and AC, each of length *l*, are freely joined at A and rest on a smooth fixed vertical circle of radius *r*. If 2θ is the angle between the rods, then find the relation between *l*, *r* and θ, by using the principle of virtual work. **[5d UPSC CSE 2014]**

Q5. A regular pentagon ABCDE, formed of equal heavy uniform bars joined together, is suspended from the joint A, and is maintained in form by a light rod joining the middle points of BC and DE. Find the stress in this rod.

[7c UPSC CSE 2014]

Q6. Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely joined at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Find the tension in the string. **[7c UPSC CSE 2013]**

Q7. A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$a\left(1+\frac{W}{2\pi\lambda}\cos\alpha\right)$. **[8c 2012 IFoS]**

Q8. One end of a uniform rod AB, of length 2*a* and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2\sin^{-1}\left(\frac{3}{4}\right)$. **[7a 2011 IFoS]**

Q9. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by using the principle of virtual work that

$\tan\phi=\frac{3}{8}+\tan\theta$. **[8b UPSC CSE 2010]**

5. MOMENTS, EQUILIBRIUM OF CO-PLANAR FORCES

6.(a) A heavy string, which is not of uniform density, is hung up from two points. Let T_1, T_2, T_3 be the tensions at the intermediate points A, B, C of the catenary respectively where its inclinations to the horizontal are in arithmetic progression with common difference β. Let ω_1 and ω_2 be the weights of the parts AB and BC of the string respectively. Prove that

(i) Harmonic mean of T_1, T_2 and $T_3 = \dfrac{3T_2}{1+2\cos\beta}$

(ii) $\dfrac{T_1}{T_3} = \dfrac{\omega_1}{\omega_2}$

[UPSC CSE 2021]

Q5(c) Three forces P, Q and R act along the sides BC, CA and AB of ΔABC in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$\dfrac{1+\cos\alpha}{P} + \dfrac{1+\cos\beta}{Q} + \dfrac{1+\cos\gamma}{R} = 0$. **[IFoS 2022]**

Q7(c) PR and QR are two equal heavy strings tied together at R and carrying a weight W at R. P and Q are two points in the same horizontal line and 2a is the distance between them. l is the length of each string and $h$is the depth of R below PQ. Prove that

(i) $l^2 - h^2 = 2c^2\left(\cosh\dfrac{a}{c} - 1\right)$,

(ii) Tension at P or $Q = \dfrac{1}{2h}\left\{lW + \left(l^2 + h^2\right)w\right\}$,

where α, β, γ are the interior angles subtended at A, B, C respectively.

[IFoS 2022]

Q1. A uniform rod, in vertical position, can turn freely about one of its ends and is pulled aside from the vertical by a horizontal force acting at the other end of the rod and equal to half its weight. At what inclination to the vertical will the rod rest? **[5d UPSC CSE 2020]**

Q2. A beam AD rests on two supports B and C, where AB = BC = CD. It is found that the beam will tilt when a weight of p kg is hung from A or when a weight of q kg is hung from D. Find the weight of the beam.

[6c UPSC CSE 2020]

Q3. A cylinder of radius 'r', whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length l and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. Prove that the reaction of the cylinder is $\dfrac{W\sqrt{5}}{2}$ and the pressure on the wall is $\dfrac{W}{2}$. Also, prove that the ratio of radius of the cylinder to the length of the beam is $5+\sqrt{5}:4\sqrt{2}$. **[5d 2020 IFoS]**

Q4. A 2 meters rod has a weight of 2N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hung loads of 3N, 7N and 10N respectively. Find the point at which the rod must be supported if it is to remain horizontal. **[5c 2019 IFoS]**

Q5. A uniform rod AB of length 2*a* movable about a hinge at A rests with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, prove that the magnitude of reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$ where W is the weight of the rod. **[7a UPSC CSE 2016]**

Q6. Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a light rod AB. If the strings OA and OB make angles α and β with the rod AB, show that the angle θ which the rod makes with the vertical is given by

$\tan\theta = \dfrac{P+Q}{P\cos\alpha - Q\cot\beta}$. **[7b UPSC CSE 2016]**

Q7. A square ABCD, the length of whose sides is *a*, is fixed in a vertical plane with two of its sides horizontal. An endless string of length $l(>4a)$ passes over four pegs at the angles of the board and through a ring of weight W which is hanging vertically. Show that the tension of the string is $\dfrac{W(l-3a)}{2\sqrt{l^2-6la+8a^2}}$.

[7c UPSC CSE 2016]

Q8. A weight W is hanging with the help of two strings of length *l* and 2*l* in such a way that the other ends A and B of those strings lie on a horizontal line at a distance 2*l*. Obtain the tension in the two strings. **[5c 2016 IFoS]**

Q9. A rod of 8 kg is movable in a vertical plane about a hinge at one end, another end is fastened a weight equal to half of the rod, this end is fastened by a string of length *l* to a point at a height *b* above the hinge vertically. Obtain the tension in the string. **[5d UPSC CSE 2015]**

Q10. A ladder of weight W rests with one end against a smooth vertical wall and the other end rests on a smooth floor. If the inclination of the ladder to the horizon is 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from slipping down.

[7b UPSC CSE 2011]

Q11. AB is a uniform rod, of length 8*a*, which can turn freely about the end A, which is fixed C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium.

Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$W, where W is the weight of the end. **[7b 2011 IFoS]**

Q12. A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass *m* slides down its slant face which is inclined at an angle α to the horizontal plane. Prove that the acceleration of the wedge is, $\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$. **[7c 2010 IFoS]**

6. FRICTION

Q5(c) A body of weight w rests on a rough inclined plane of inclination θ, the coefficient of friction, μ, being greater than $\tan\theta$. Find the work done in slowly dragging the body a distance 'b' up the plane and then dragging it back to the starting point, the applied force being in each case parallel to the plane.

[UPSC CSE 2022]

Q1. One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. When the rod is on the point of sliding, then $AC^2 - AB^2 = BC^2$. If θ is the angle between AB and the horizontal line, then prove that the coefficient of friction is $\frac{\cot\theta}{2+\cot^2\theta}$.

[5c UPSC CSE 2019]

Q2. A uniform rod of weight W is resting against an equally rough horizon and a wall, at and angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W\tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α? **[7b 2016 IFoS]**

Q3. Two equal ladders of weight 4 kg each are placed so as to lean at A against each other with their ends resting on a rough floor, given the coefficient of friction is μ. The ladders at A make an angle 60° with each other. Find what weight on the top would cause them to slip. **[6b UPSC CSE 2015]**

Q4. A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coefficient of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is: $\sin^{-1}\left(\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2}\right)$. **[8a 2014 IFoS]**

Q5. The base of an inclined plane is 4 metres in length and the height is 3 metres. A force of 8 kg acting parallel to the plane will just prevent a weight of 20 kg from sliding down. Find the coefficient of friction between the plane and the weight. **[5d UPSC CSE 2013]**

Q6. A uniform ladder rests at an angle of 45° with the horizontal with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ and μ' are the coefficients of limiting friction between the ladder and the ground and wall respectively, then find the minimum horizontal force required to move the lower end of the ladder towards the wall. **[7b UPSC CSE 2013]**

Q7. Two bodies of weight w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other. **[6c 2013 IFoS]**

Q8. A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by $\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$,

μ, being the coefficient of friction. **[7b 2012 IFoS]**

3. VIRTUAL WORK

Q1. A square framework formed of uniform heavy rods of equal weight W joined together, is hung up by one corner. A weight W is suspended from each of the three lower corners, and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.

[7c UPSC CSE 2020]

Q2. A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely joined together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance 2b apart. A weight W is suspended from A. Find the trust in the rod BC. **[7c 2018 IFoS]**

Q3. A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\frac{2W\left(2b^2-a^2\right)}{b\sqrt{4b^2-a^2}}$. **[6b 2017 IFoS]**

Q4. Two equal uniform rods AB and AC, each of length *l*, are freely joined at A and rest on a smooth fixed vertical circle of radius *r*. If 2θ is the angle between the rods, then find the relation between *l*, *r* and θ, by using the principle of virtual work. **[5d UPSC CSE 2014]**

Q5. A regular pentagon ABCDE, formed of equal heavy uniform bars joined together, is suspended from the joint A, and is maintained in form by a light rod joining the middle points of BC and DE. Find the stress in this rod.

[7c UPSC CSE 2014]

Q6. Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely joined at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Find the tension in the string. **[7c UPSC CSE 2013]**

Q7. A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$a\left(1+\frac{W}{2\pi\lambda}\cos\alpha\right)$. **[8c 2012 IFoS]**

Q8. One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2\sin^{-1}\left(\frac{3}{4}\right)$. **[7a 2011 IFoS]**

Q9. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by using the principle of virtual work that $\tan\phi=\frac{3}{8}+\tan\theta$. **[8b UPSC CSE 2010]**

DYNAMICS

UPSC PREVIOUS YEARS QUESTIONS CHAPTERWISE(UPSC CSE/IFoS)

1. KINEMATICSIN TWO DIMENSIONS
2. RECTILINEAR MOTION
   SIMPLE HARMONIC MOTION
   PLANATORY MOTION
3. CONSTRAINED MOTION
4. MOTION IN A RESISTING MEDIUM
5. CENTRAL ORBITS
6. PROJECTILES
7. WORK, ENERGY AND IMPULSE

1. KINEMATICS IN TWO DIMENSIONS

(LINEAR MOTION, ANGULAR VELOCITY, RADIAL & TRANSVERSE VELOCITIES & ACC (POLAR COORDINATES), TANGENT AND NORMAL VELOCITIES (INTRINSIC COORDINATES))

Q1. If the radial and transverse velocities of a particle are proportional to each other, then prove that the path is an equiangular spiral. Further, if radial acceleration is proportional to transverse acceleration, then show that the velocity of the particle varies as some power of the radius vector.

[5c 2020 IFoS]

Q2. A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet?

[6a 2016 IFoS]

Q3. A particle is acted on a force parallel to the axis of *y* whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to *x*-axis at the point where $y = a$. Prove that it will describe a catenary. **[8d 2016 IFoS]**

Q4. A particle is acted on by a force parallel to the axis of *y* whose acceleration (always towards the axis of *y*) is μy^{-2} and when $y = a$, it is projected parallel to the axis of *x* with velocity $\sqrt{\frac{2\mu}{a}}$. Find the parametric equation of the path of the particle. Here μ is a constant. **[8b UPSC CSE 2014]**

Q5. The velocity of a train increases from 0 to *v* at a constant acceleration f_1, then remains constant for an interval and again decreases to 0 at a constant retardation f_2. If the total distance described is *x*, find the total time taken.

[5c UPSC CSE 2011]

CHAPTER-2. RECTILINEAR MOTION

SHM, ELASTIC STRINGS

Q1. A particle moving along the y-axis has an acceleration Fy towards the origin, where F is a positive and even function of y. The periodic time, when the particle vibrates between $y=-a$ and $y=a$, is T. Show that

$$\frac{2\pi}{\sqrt{F_1}} < T < \frac{2\pi}{\sqrt{F_2}}$$

where F_1 and F_2 are the greatest and the least values of F within the range $[-a, a]$. Further, show that when a simple pendulum of length l oscillates through 30° on either side of the vertical line, T lines between $2\pi\sqrt{l/g}$ and $2\pi\sqrt{l/g}\sqrt{\pi/3}$.

[7c UPSC CSE 2019]

Q2. A particle moving with simple harmonic motion in a straight line has velocities v_1 and v_2 at distance x_1 and x_2 respectively from the centre of its path. Find the period of its motion. **[6b UPSC CSE 2018]**

Q3. If the velocities in a simple harmonic motion at distances a, b and c a fixed point on the straight line which is not the centre of force, are u, v and w respectively, show that the periodic time T is given by

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$. **[5c 2018 IFoS]**

Q4. Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and C_2 and C_2 are the statistical extension due to these weights and g is the acceleration due to gravity. Show that $g = \frac{4\pi^2(C_1 - C_2)}{T_1^2 - T_2^2}$. **[6b 2018 IFoS]**

Q5. A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position $P(OP=b)$ with velocity v in the direction OP. Prove that the time that elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$.

[5c 2017 IFoS]

Q6. A body moving under SHM has an amplitude 'a' and time period 'T'. If the velocity is trebled, when the distance from mean position $\frac{2}{3}a$, the period being unaltered, find the new amplitude. **[5c UPSC CSE 2015]**

Q7. A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point. **[5b 2015 IFoS]**

Q8. A particle is performing a simple harmonic motion (S.H.M.) of period T about a centre O with amplitude a and it passes through a point P, where $OP = b$

in the direction OP. Prove that the time which elapses before it returns to P to $\frac{T}{\pi}\cos^{-1}\left(\frac{b}{a}\right)$. **[5c UPSC CSE 2014]**

Q9. A body is performing S.H.M. in a straight line OPQ. Its velocity is zero at points P and Q whose distances from O are x and y respectively and its velocity is v at the mid-point between P and Q. Find the time of one complete oscillation. **[5c UPSC CSE 2013]**

Q10. A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P, where $OP = b$ with velocity v in the direction of OP. Find the time which elapses before it returns to P.

[5b 2013 IFoS]

Q11. A particle is thrown over a triangle from one end of horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that, $\tan\theta = \tan\theta_1 + \tan\theta_2$.

[5d 2010 IFoS]

KEPLER'S LAWS OF PLANETARY MOTION

Q1. A particle starts at a great distance with velocity V. Let p be the length of the perpendicular from the centre of a star on the tangent to the initial path of the particle. Show that the least distance of particle from the centre of the star is λ, where $V^2\lambda = \sqrt{\mu^2 + p^2V^4} - \mu$. Here μ is a constant.

[7c UPSC CSE 2020]

Q2. Prove that the path of a planet, which is moving so that its acceleration is always directed to a fixed point (star) and is equal to $\frac{\mu}{(\text{distance})^2}$, is a conic section. Find the conditions under which the path becomes (i) ellipse, (ii) parabola and (iii) hyperbola. **[8b UPSC CSE 2019]**

Q3. A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that the velocity then is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis, e is the eccentricity and T is the periodic time.

[7b 2017 IFoS]

Q4. The apses of a satellite of the Earth at distance r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2. **[5c 2011 IFoS]**

CHAPTER 3. CONSTRAINED MOTION

Q6(b) A particle of mass m, which is attached to one end of a light string whose other end is fixed at a point O, describes a circular motion in a horizontal plane about the vertical axis through O. Prove that the particle moves in a conical pendulum only if $g < l\omega^2$, where l is the length of the string and ω being angular velocity. Further, a particle of mass m is attached to the middle of a light string of length $2l$, one end of which is fastened to a fixed point and the other end to a smooth ring of mass M which slides on a smooth vertical rod. If the particle describes a horizontal circle with uniform angular velocity ω about the rod, then prove that the inclination of both portions of the string to the vertical is

$\cos^{-1}\left\{\frac{(m+2M)g}{ml\omega^2}\right\}$. **[IFoS 2022]**

Q7(c) A heavy particle hangs by an inextensible string of length *a* from a fixed point and is then projected horizontally with a velocity $\sqrt{2gh}$. If $\frac{5a}{2}>h>a$, then prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$ from the point of projection. Also, prove that the greatest height ever reached by the particle above the point of projection is $\frac{(4a-h)(a+2h)^2}{27a^2}$.

[UPSC CSE 2021]

Q1. A fixed wire is in the shape of the cardiod $r=a(1+\cos\theta)$, the initial line being the downward vertical. A small ring of mass *m* can slide on the wire and is attached to the point $r=0$ of the cardiod by an elastic string of natural length *a* and modulus of elasticity 4 mg. The string is released from rest when the string is horizontal. Show by using the laws of conservation of energy that $a\theta^2(1+\cos\theta)-g\cos\theta(1-\cos\theta)=0$, *g* being the acceleration due to gravity.

[5c UPSC CSE 2017]

Q2. A particle is free to move on a smooth vertical circular wire of radius *a*. At time $t=0$ it is projected along the circle from its lowest point A with velocity just sufficient to carry it to the highest point B. Find the time T at which the reaction between the particle and the wire is zero.

[7c UPSC CSE 2017]

Q3. A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

[6b UPSC CSE 2010]

CHAPTER 4. MOTION THROUGH RESISTING MEDIUM

Q1. A particle of mass *m* is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time *t*.

[7c 2015 IFoS]

Q2. A particle is projected vertically upwards with a velocity *u*, in a resisting medium which produces a retardation kv^2 when the velocity is *v*. Find the height when the particle comes to rest above the point of projection.

[7c 2013 IFoS]

Q3. A particle is projected with a velocity *v* along a smooth horizontal plane in a medium whose resistance per unit mass is double the cube of the velocity. Find the distance it will describe in time *t*. **[8c 2013 IFoS]**

CHAPTER 5. CENTRAL ORBITS

Q5(d) If a planet, which revolves around the Sun in a circular orbit, is suddenly stopped in its orbit, then find the time in which it would fall into the Sun. Also, find the ratio of its falling time to the period of revolution of the planet.

[UPSC CSE 2021]

Q1. A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force O, it reaches a point B distant 'b' from O. Find the time in reaching from A to B and the velocity at B. When will the particle reach at the centre? **[6b 2020 IFoS]**

Q2. Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force). **[6b 2019 IFoS]**

Q3. A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$. **[8b 2017 IFoS]**

Q4. A particle moves with a central acceleration which varies inversely as the cube of the distance. If it is projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, then find the equation to the path. **[5e UPSC CSE 2016]**

Q5. A particle moves in a straight line. Its acceleration is directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then find time, the particle will arrive at O. **[8c UPSC CSE 2016]**

Q6. A mass starts from rest at a distance 'a' from the centre of force which attracts inversely as the distance. Find the time of arriving at the centre.

[6d UPSC CSE 2015]

Q7. A particle moves in a plane under a force, towards a fixed centre, proportional to the distance. If the path of the particle has two apsidal distance $a, b(a > b)$, then find the equation of the path. **[8b UPSC CSE 2015]**

Q8. A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path. **[8c 2015 IFoS]**

Q9. A particle whose mass is m, is acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$. **[5c 2014 IFoS]**

Q10. A particle moves with an acceleration $\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a from the origin, find its velocity when its distance from the origin is $\frac{a}{2}$. **[5d UPSC CSE 2012]**

Q11. A particle is moving with central acceleration $\mu\left[r^5-c^4r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)}c^3$, show that its path is a curve, $x^4+y^4=c^4$. **[7a 2012 IFoS]**

Q12. A particle of mass m moves on straight line under an attractive force mn^2x towards a point O on the line, where x is the distance from O. If $x=a$ and $\frac{dx}{dt}=u$ when $t=0$, find $x(t)$ for any time $t>0$. **[7c(ii) UPSC CSE 2011]**

Q13. A particle moves with a central acceleration $\mu\left(r^5-9r\right)$, being projected from an apse at a distance $\sqrt{3}$ with velocity $3\sqrt{(2\mu)}$. Show that its path is the curve $x^4+y^4=9$. **[7b UPSC CSE 2010]**

Q14. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is, $\sin^{-1}\left[\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}\right]$. **[7b 2010 IFoS]**

CHAPTER 6. PROJECTILES

Q5(d) A projectile is fired from a point O with velocity $\sqrt{2gh}$ and hits a tangent at the point $P(x,y)$ in the plane, the axes OX and OY being horizontal and vertically downward lines through the point O, respectively. Show that if the two possible directions of projection be at right angles, then $x^2=2hy$ and then one of the possible directions of projection bisects the angle POX.

[UPSC CSE 2022]

Q8(b) Describe the motion and path of a particle of mass m which is projected in a vertical plane through a point of projection with velocity u in a direction making an angle θ with the horizontal direction. Further, if particles are projected from that point in the same vertical plane with velocity $4\sqrt{g}$, then determine the locus of vertices of their paths. **[UPSC CSE 2021]**

Q5(c) A particle is projected in a direction making an angle α with the horizon. It passes through the two points (x_1, y_1) and (x_2, y_2). Prove that

$\tan\alpha = \dfrac{y_1 R}{Rx_1 - x_1^2} = \dfrac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)}$, where R denotes the horizontal range. **[IFoS 2021]**

Q1. A short projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark $h$metres above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection. **[8b 2019 IFoS]**

Q2. A particle projected from a given point on the ground just clears a wall of height h at a distance d from the point projection. If the particle moves in a vertical plane and if the horizontal range is R, find the elevation of the projection. **[1e UPSC CSE 2018]**

Q3. From a point in a smooth horizontal plane, a particle is projected velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cos\beta = 2\tan(\alpha - \beta)$. **[5d 2016 IFoS]**

Q4. A particle is projected from the base of a hill whose slope is that of right circular cone, whose axis is vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If the semivertical angle of the cone is 30°, h is height, determine the initial velocity u of the projection and its angle of projection. **[7b UPSC CSE 2015]**

Q5. A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is

$\dfrac{2u}{g\sqrt{(1+3\sin^2\beta)}}$, range on the plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$ and the vertical height of the point struck is $\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$ above the point of projection. **[6c 2012 IFoS]**

Q6. A projectile aimed at a mark which is in the horizontal plane through the point of projection, falls x meter short of it when the angle of projection is α and goes y meter beyond when the angle of projection is β. If the velocity of projection is assumed same in all cases, find the correct angle of projection.

[5d UPSC CSE 2011]

Q7. If v_1, v_2, v_3 are the velocities at three points A, B, C of the path of a projectile, where the inclinations to the horizon are $\alpha, a-\beta, a-2\beta$ and if t_1, t_2 are the times of describing the arcs AB, BC respectively, prove that

$v_3 t_1 = v_1 t_2$ and $\dfrac{1}{v_1} + \dfrac{1}{v_3} = \dfrac{2\cos\beta}{v_2}$. **[5d UPSC CSE 2010]**

CHAPTER 7. WORK, POWER & ENERGY

Q5(d) A person is drawing water from a well with a light bucket which leaks uniformly. The bucket weighs 50 kg when it is full. When it arrives at the top, half of the water remains inside. If the depth of the water level in the well from the top is 30 m, then find the work done in raising the bucket to the top from the water level. **[IFoS 2022]**

Q1. A light rigid rod ABC has three particles each of mass *m* attached to it at A, B and C. The rod is struck by a blow P at right angles to it at a point distant from A equal to BC. Prove that the kinetic energy set up is $\frac{1}{2}\frac{P^2}{m}\frac{a^2-ab+b^2}{a^2+ab+b^2}$, where $AB=a$b and $BC=b$. **[5e UPSC CSE 2020]**

Q2. A four-wheeled railway truck has a total mass M, the mass and radius of gyration of each pair of wheels and axle are *m* and *k* respectively, and the radius of each wheel is *r*. Prove that if the truck is propelled along a level tack by a force P, the acceleration is $\frac{P}{M+\frac{2mk^2}{r^2}}$, and find the horizontal force exerted on each axle by the truck. The axle friction and wind resistance are to be neglected. **[8c UPSC CSE 2020]**

Q3. The force of attraction of a particle by the earth is inversely proportional to the square of its distance from the earth's centre. A particle, whose weight on the surface of the earth is W, falls to the surface of the earth from a height $3h$ above it. Show that the magnitude of work done by the earth's attraction force is $\frac{3}{4}hW$, where *h* is the radius of the earth. **[5d UPSC CSE 2019]**

Q4. A spherical shot of W gm weight and radius *r* cm, lies at the bottom of cylindrical bucket of radius R cm. The bucket is filled with water up to a depth of *h* cm $(h \geq 2r)$. Show that the minimum amount of work done in lifting the shot just clear of the water must be $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ cm gm. W' gm is the weight of water displaced by the shot. **[8a UPSC CSE 2017]**

Q5. An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2-\frac{1}{2}\right)$. **[6b 2014 IFoS]**

Q6. A particle of mass 2.5 kg hangs at the end of a string, 0.9 m long, the other end of which is attached to a fixed point. The particle is projected horizontally with a velocity 8 m/sec. Find the velocity of the particle and tension in the string when the string is (i) horizontal (ii) vertically upward. **[7a UPSC CSE 2013]**

Q7. A heavy ring of mass *m*, slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant *a* from the rod and has a mass $M(>m)$ fastened to its other end. Show that if the ring be dropped from a point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2Mma}{M^2-m^2}$ before coming to rest. **[7a UPSC CSE 2012]**

Q8. A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height *h* is

$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$. **[5c 2012 IFoS]**

Q9. A mass of 560 kg. moving with a velocity of 240 m/sec strikes a fixed target and is brought to rest in $\frac{1}{100}$ sec. Find the impulse of the blow on the target and assuming the resistance to be uniform throughout the time taken by the body in coming to rest, find the distance through which it penetrates.

[7a UPSC CSE 2011]

Q10. After a ball has been falling under gravity for 5 seconds it passes through a pane of glass and loses half its velocity. If it now reaches the ground in 1 second, find the height of glass above the ground. **[7c(i) UPSC CSE 2011]**

Q11. The position vector $\vec{r}$ of a particle of mass 2 units at any time *t*, referred to fixed origin and axes, is

$$\vec{r}=\left(t^2-2t\right)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}$$

At time $t=1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. **[8d 2011 IFoS]**

Mentor: Upendra Singh
Designation
Guide at Insights IAS and Nirman IAS (Prev.)
Sr. Faculty at Physics By Fiziks (Higher Studies(Mathematics) since year 2013)
Chairman: Patiyayat Farmer Producer Company Limited.
Associated in Policy making for Rural upliftment/education with Govt. Uttar Pradesh.
Educational Background
Alumnus, IIT Delhi (2011 Batch, AIR49)
UPSC CSE Journey- On the basis of one of top scores multiple times given by UPSC, I'm confident about each stage of CSE. Preliminary (Developed special strategies KNOWLEDGE+STRATEGY), Mains_ Good command over Essay and GS4+GS2, optional mathematics), Personality(Interview) - Developed it through experiences and learnings through reading books from multiple dimensions during UPSC journey, Part of Interview board for Higher studies.

Message from Upendra Sir

HOW TO CHOOSE OPTIONAL FOR UPSC CSE (Main Exam)

(I had mathematics optional and one of my close friend took history optional) .Since years, I've been teaching higher mathematics, guiding UPSC aspirants too. So I guess, can answer you for the asked question. What research needed to be done before choosing optional for UPSC(CSE)? There are few parameters on which we need to have idea

1- My own interest/ my core subject in UG or PG.

2- Resources/guidance/coaching availability for that subject.

3- Pattern of marks in previous years- For this, we should read some blogs written by selected persons with that optional. Advice- don't rely on many youTube videos, those may be overhyped.

4- Duration to cover the syllabus- There are some optionals which take less time to cover. example- Hindi literature takes less time to finish as compared to history . But here we must aware about scores low and high frequently.

5- Management- Let's say if you choose mathematics, obviously it'll have no link with general studies. You need a different mindset to prepare Mathematics Optional. You need lot of practice. It may exhaust you and you can compromise with your other subjects. The most important is - in main exam hectic schedule. Mathematics may be a reason of anxiety and restlessness. That 5 days gap seems long but during mains preparation we're exhausted and literally tired to grasp mathematics revision. So untill unless, you don't have good background and interest in mathematics, you should avoid it. Solution- during preparation , you have to develop traits and qualities to handle/manage it. You can talk to me at the mentoring number.

How to Prepare for UPSC (logically with Mathematics Optional)

Mindset Makers for UPSC (Prepare in Right Way)

As an UPSC aspirant with **Science optional (For example Mathematics)**, We need completely a different approach. The interesting part is that if you have clear vision/mindset with such optional then you have high probability to secure the top rank. So is there any sigma rule for such strategy! The answer is – Yes in some sense. You must keep yourself isolated from the thought process where people have negative views about such optionals.

Yes it happens in this UPSC Preparation zone because there is a definite gap of mentors with equal interest in General Studies papers, Essay paper and such optionals. Mentors who are very good at GS and essay , they may not be good science optionals and who are teaching science optional they are experts of their domains mainly. Keeping all such obseravtions in mind , Mindset Makers tries to make easy this preparation with such optional combinations.

For this to happen, we are working in completely offline coaching module at- **YouTube (Upendra Singh : Mindset Makers for UPSC)**. A lecture Series with well planned course structure is going on . Here we teach Mathematics Optional and also do the mindset making for GS and Essay simultaneously with a scientific approach required for this optional combination. You must attend at least one lecture on youtube platform to feel the difference.

Be Part of Disciplined Learning.