STRENGTHENING BRAINS

IAS/IFoS

STATICS

Systematically designed
mathematics optional book
Concepts, Examples & PYQs Analysis

Written by: Upendra Singh
Alumnus: IIT Delhi, Sr. Faculty in Higher Mathematics (2013 onwards) , Asso. Policy Making (UP Govt.), Chairman: Patiyayat FPC Ltd.

STATICS for UPSC CSE/IFoS Mathematics Optional

Common Catenary

Stable & Unstable Equilibrium

Principle of Virtual Work

Exam point:

1. **For CSE:** Make sure you have all formulae at one place. You have solved all examples.

→ **Mindset Making:**

- Read the question
- Get keywords
- Use formula based on that keyword.

→ Required chapters are: Only Three.

Namely: 1. Common Catenary

2. Stable and Unstable Equilibrium

3. Principle of principle (Virtual) Work.

- Bit understanding from centre of Gravity.

2. **For IFoS:** Above three chapters + centre of Gravity and Forces in 3D also required.

Year	Common Catenary	Stable/unstable Equilibrium	Principle of Virtual work
2010	0	0	1
2011	0	0	1
2012	1	1	
2013	1	1	1
2014	0	0	2
2015	1	0	2
2016	1	1	1
2017	0	1	0
2018	0	0	0
2019	0	1	1

COMMON CATENARY

To understand common catenary let's have some other terms-

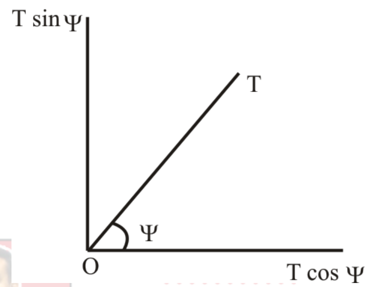
Flexible String: A string which offers no resistance to bending at any point. e.g. A chain whose links are quite small and perfectly smooth can be regarded as a flexible string.

Note: In case of flexible string, the resultant action along (across) any section of the string, consists of a single force action along the tangent to the curve formed by the string.

Explanation: Because any normal section is small and so the string may be considered as a curved line.

→ Let's use this **uniform flexible** string of chain:-

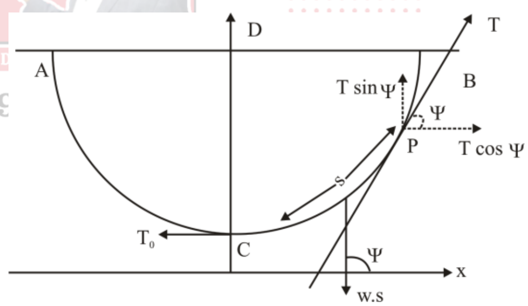
When it hangs freely between two points (these points; need not be in the same vertical line) under the action of gravity, Then this is called a '**common catenary**'.



(Provided that the weight per unit length of the string or chain is constant)

Note that if the weight per unit length of the string is not constant; then the above system will be called Catenary but not the common catenary.

➢ Points need not be in the same horizontal line.



Mathematics behind Common Catenary

1. Intrinsic Equation of Common Catenary

Base Understanding:

❖ An equation involving the arc length(s) and angle of tangent Ψ is called the intrinsic equation. (Differential calculus, Topic: Tangents and normal)

Vertical & Horizontal component of Tension T.

Now, let's try to find intrinsic equation for a common catenary.
Step (i) Practice for drawing this system (multiple times)

DC = Sag. Of catenary

P: Some arbitrary point (x,y) of Catenary.

We have considered the cartesian coordinate system as base for common catenary.

Y-axis: Called **the axis of common catenary.**

X-axis: **Called the directrix.**

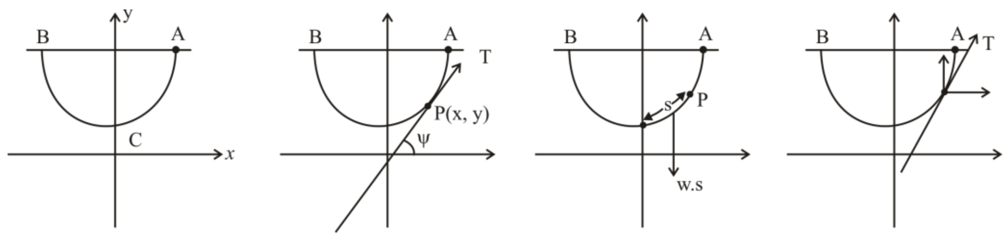
w: **weight per** unit length of string.

'Vertex' of catenary : C: The **lowest** point of catenary

T : Tangential (Tension) at point P

T_o : Horizontal tension at C.

'Span' = AB ; A & B : Support point for string



Figures: Just for Practice

Step (ii) : Consider the **equilibrium** of the **portion of CP** :

It is due to the horizontal tension To at the point C, the tension T at the point P and the weight w.s acting vertically downwards through the centre of gravity of arc CP.

$\therefore$ String CP is in equilibrium under the action of there forces (**all lying in one plane),** the line of action of the weight w.s must pass through the intersection of the line of action To and T.

Mathematics : $T\cos\Psi = To$ ...(1)

$T\sin\Psi = w.s$ ...(2)

$$\therefore \frac{To}{w.s} = \frac{\cos\Psi}{sn\Psi} \Rightarrow s = \frac{To}{w}\tan\Psi$$

$\therefore$ Intrinsic Equation of Common Catenary $\& = c\tan\Psi$

Hence $c = \dfrac{T_o}{w}$

c ; called the parameter of common catenary

Extra: Notice from (1) & (2)

$$\tan\Psi = \frac{w.s}{T_o} = \frac{Vertical\ component}{Horizontal\ component}$$

We'll use this in some questions.

2. Cartesian Equation of Common Catenary

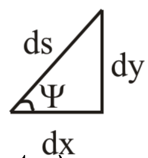
$$y = c \cosh \frac{x}{c}$$

Base understanding

for Arc length; $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(calculus : Rectification Chapter)

$\frac{dx}{ds} = \cos\Psi$, $\frac{dy}{dx} = \tan\Psi$



[small arc ds]

(Differential Calculus: Tangents & Normal Chapter)

Solving differential equation (basic)

Step (i) We know that $s = c \tan \Psi$

$\therefore \; s = c.\frac{dy}{dx}$

(To use $\frac{ds}{dx}$) differentiating w.r.t. x

$$\frac{ds}{dx} = c.\frac{d^2y}{dx^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = c.\frac{d^2y}{dx^2}$$

This is just a differential equation

let's solve it Let $\frac{dy}{dx}$ = p

$$\therefore \sqrt{1+p^2} = c\frac{dp}{dx}$$

On integrating $\sin h^{-1}\; p = \frac{x}{c} + k$

k: integration constant

$\therefore$ At the lowest point C of catenary;

$$x = 0 \qquad \therefore\; dx = 0$$

and $p = \frac{dy}{dx} = 0$

Therefore $k_1 = 0$,

$\therefore$ $\frac{x}{c} = \sin h^{-1} p$ $\Rightarrow p = \sin h \frac{x}{c}$

$$\Rightarrow \frac{dy}{dx} = \sin h \frac{x}{c}$$

On integrating

$$y = c \cos h \frac{x}{c} + k_1$$

at C; $x = 0$, $y = c$ $\therefore k_1 = 0$

$$y = c \cos h \frac{x}{c}$$

Relation between x & s.

$$s = c \sin h \frac{x}{c}$$

3. **Span of Common Catenary:**

Half span $x = c \log\left(sec\,\Psi + \tan\Psi\right)$

Full (Total) span $= 2x = 2c \log\left(\sec\Psi + \tan\Psi\right)$

How? $\because s = c \tan \Psi$

$$\therefore \frac{ds}{dx} = c \sec^2\Psi \frac{d\Psi}{dx}$$

Using $\frac{ds}{dx} = \sec\Psi$, we get ; $\sec\Psi = c\sec^2\Psi \frac{d\Psi}{dx}$

On integrating, we get

$$x = c\log\left(\sec\Psi + \tan\Psi\right) + k$$

$\therefore$ at C; $x = 0$ and $\Psi = 0$ $\therefore k = 0$

Therefore $x = c\log\left(\sec\Psi + \tan\Psi\right)$

Exam point: Whenever in the question, the keyword is **span**.

Just Target to use above formula. (Means try to find Ψ and c from the given information).

Sometimes the relation between x and s is also used in this form.

$$\therefore\ x = c\log(\sec\Psi + \tan\Psi)$$

$$x = c\log\left(\sqrt{1+\tan^2\Psi} + \tan\Psi\right) \quad \text{and } s = c\tan\Psi$$

$$\therefore x = c\log\left(\sqrt{1+\frac{s^2}{c^2}} + \frac{s}{c}\right)$$

$$x = c\log\left\{\frac{s+\sqrt{(s^2+c^2)}}{c}\right\}$$

(4) Relation between y and Ψ.

Relation between s and y.

Relation between x, y and s.

→ $\because\ s = c\tan\Psi$

$$\therefore \frac{ds}{dy} = c\sec^2\Psi\frac{d\Psi}{dy}$$

Using $\dfrac{dy}{ds} = \sin\Psi$; (Differential Calculus), we get $\quad \operatorname{cosec}\Psi = c\sec^2\Psi\dfrac{d\Psi}{dy}$

$$\therefore y = c\sec\Psi$$

→ $\because$ For common catenary;

$$\therefore T\cos\Psi = T_o = w.c$$

$$\therefore T = w.\, c\sec\Psi$$

$$\therefore\ T = w.y \qquad\qquad \therefore\ y = c\sec\Psi$$

→ For a common catenary

$$\cosh\frac{x}{c}.\cos\Psi = 1;\ \text{using } y = c\cosh\frac{x}{c}, \qquad y = c\sec\Psi$$

→ For a common catenary ; $\qquad x = c\log\left(\dfrac{y+s}{c}\right)$

$$y + s = c.e^{x/c}$$

For proof.

Using $y = c \sec \Psi$, $s = c \tan \Psi$

$$x = c\log(\sec\Psi + \tan\Psi)$$

$$\rightarrow y^2 = c^2 + s^2$$

Just use : $y = c.\sec\Psi$, $s = c\tan\Psi$

For IF$_o$S exam: (Add these two)

1. Approximation of common catenary

Here we consider approximation of the common catenary to the **parabola** and **exponential** curve, depending upon certain conditions.

Case (i) When $\frac{x}{c}$ is small.

$$\because y = c\cos h\frac{x}{c} = c\frac{1}{2}\left(e^{x/c} + e^{-x/c}\right)$$

$$\because \cos hQ = \frac{e^Q + e^{-Q}}{2}$$

On using expansion for $e^{x/c}$, $e^{-x/c}$

We get

$$y = c\left\{1 + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \ldots\right\}$$

$\because$ x/c is small $\therefore$ Higher powers of $\frac{x}{c}$ can be neglected.

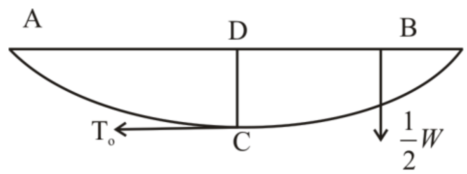
$$y = c + \frac{x^2}{2c} \text{ Parabola.}$$

Observation. The above expression shows as long as x is small and c large the catenary coincides very nearly with a parabola having its vertex at the point (o,c) and latus rectum equal to 2c or $2T_o/w$.

Case (ii) When x is large in this case $e^{-x/c}$ becomes very small.

$$\therefore y = \frac{1}{2}c.e^{x/c}$$

2. Sag of a tightly stretches wire



let A and B be two points in a same horizontal line between which a wire is tightly stretched.

l; length of the wire

w ; weight of the wire

To; Horizontal tension

k = sag DC

h = span AB

Whatever rules we have discussed earlier; we can think those here too.

$T_o k = \frac{1}{2}W.\frac{1}{4}l$ approximately

[for portion CB (taking moments about B)]

$$T_o = \frac{l}{8k}W$$

We now proceed to calculate the increase in the length of the wire on account of the sag in middle.

for this let's take $s = c\sin h\frac{x}{c}\ldots(1)$

$\because$ radius of curvature of the catenary is $\rho\sec^2\Psi = c$, c will be large if ρ is large.

$\therefore$ If the catenary is flat near the vertex. It follows that $\frac{x}{c}$ will be small for tightly stretched wire.

$$\therefore s = c\left\{\frac{x}{c} + \frac{1}{3!}\left(\frac{x}{c}\right)^3\right\}$$

$$s = x + \frac{x^3}{6c^2}$$

Hence $s - x$ = CB – DB

$\frac{x^3}{6c^2}$ approximately

$$\frac{w^2x^2}{6T_o^{\,2}} \quad , \qquad \because c=\frac{T_o}{w}$$

Now putting $x=\frac{1}{2}h$

The total increase, due to sagging in a span of length h is

$$2s-h = 2.\frac{w^2.\frac{1}{6}h^3}{6T_o^{\,2}}$$

$$2x-h = \frac{w^2h^3}{24T_o^{\,2}}$$

Examples (arranged in the same order as of required for concept building)

Example.1. A uniform string of weight W is suspended from two points at the same level and a weight P is attached to its lowest point. If α and β are now the inclinations at the highest and the lowest points, prove that $\frac{\tan\alpha}{\tan\beta}=1+\frac{W}{P}$

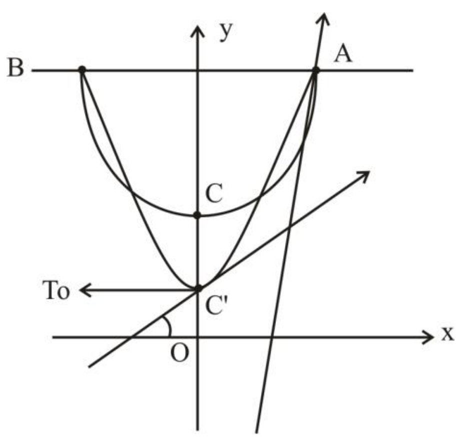
Solutions: Let's break the question:

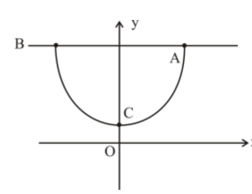
Keywords: Uniform String ; suspended ; A & B

Step (i) Common Catenary

A & B at same level

W : weight of string ACB





- ❖ tangent at A makes angle α
- ❖ tangent at C’ makes angle β

AC’B is the string’s position on attaching the weight P. to Point C.

Step (iii) Mathematics

We know that

$$\tan\psi = \frac{\text{vertical componet of tension}}{\text{Horizontal component of tension}}$$

at a point at which tension (tangent) makes an angle ψ with the x – axis

$$\therefore \qquad \tan\alpha = \frac{\frac{1}{2}(P+W)}{T_o}; \qquad \text{At the point A}$$

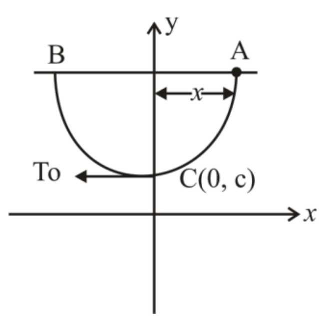
(Weight of strings half portion CA and half of attached weight)

$$\tan\beta = \frac{\frac{1}{2}P}{T_o}; \qquad \text{At the point C'}$$

Therefore $\dfrac{\tan\alpha}{\tan\beta} = \dfrac{1+W}{P}$

Ex. 2 A uniform chain of length ℓ is suspended from two points A, B in the same horizontal line. If the tension at A is n times that at the lowest point, show that the span AB is $\dfrac{\ell}{\sqrt{n^2-1}} \log\left\{n+\sqrt{(n^2-1)}\right\}$.

Sol. Common catenary, Length of chain = ℓ □ A & B are in same horizontal line



Given, $T_A = nT_o$, ...(1) Where T is tension at A, T_o is at the point C.

Keyword: Span

$\therefore$ Click $x = c\log(\sec\psi + \tan\psi)$

$\downarrow$

We need c, ψ

Target (i): Finding ψ

$\because$ On resolving tensions horizontal and vertical components; then for equilibrium

$T\cos\psi = T_o$

Using (1); $nT_o \cos\psi = T_o$

$\therefore$ $\sec\psi = n$ ...(2)

$\tan\psi = \sqrt{n^2 - 1}$ ...(3)

Target (ii) Findig c;

We know that for a common catenary

$s = c\tan\psi$

$\therefore$ $\dfrac{\ell}{2} = c\tan\psi$ $\because$ length of arc $CA = \dfrac{\ell}{2}$

$\Rightarrow$ $c = \dfrac{\ell}{2\sqrt{n^2 - 1}}$ ...(4)

$\therefore$ Using (2), (3), (4), we have

$$x = \frac{\ell}{2\sqrt{n^2 - 1}}\left\{n + \sqrt{n^2 + 1}\right\}$$

$\therefore$ Required Span = 2x $= \dfrac{\ell}{\sqrt{n^2 - 1}}\left\{n + \sqrt{n^2 + 1}\right\}$

Ex. 3 The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the maximum span to the length of the chain is $\mu\log\left\{\dfrac{1 + (1 + \mu^2)^{3/2}}{\mu}\right\}$.

Sol. Keywords: Common Catenary

<u>Slide</u>, <u>rough</u> $\therefore$ friction

Span (maximum)

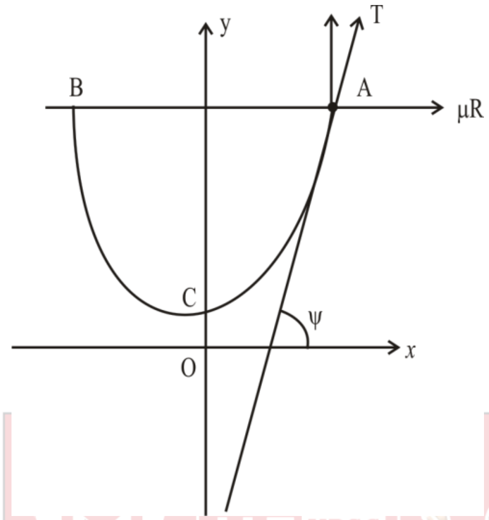
↓

For the maximum span, the points A and B of the chain must be in equilibrium.

↓

Under three forces:

Normal Reaction R, force of friction μR, Tension T



Target (i) : finding ψ:

On resolving forces vertically and horizontally,

$T\cos\psi = \mu R$ …(1)

$T\sin\psi = R$ …(2)

$\therefore$ $\tan\psi = \dfrac{1}{\mu}$ $\therefore$ $\sec\psi = \sqrt{1+\dfrac{1}{\mu^2}}$

Target (ii) : Finding c (Somewhere talked about length)

$\because$ $s = c\tan\psi$

$\therefore$ at A $\dfrac{\ell}{2} = c\tan\psi$ $\therefore$ $c = \dfrac{1}{2}\mu\ell$

$\therefore$ Maximum span AB $= 2 \times \dfrac{1}{2}\mu\ell\log\left\{\sqrt{1+\dfrac{1}{\mu^2}}+\dfrac{1}{\mu}\right\}$

$\because x = c\log(\sec\psi + \tan\psi)$

$\therefore$ Required ratio $= \dfrac{Maximum\,span}{length\,of\ chain}$

$$= \mu \log \left\{ \frac{1+\left(1+\mu^2\right)^{1/2}}{\mu} \right\}$$

Example4:- A heavy chain of length $2l$ has one end tied at A and other is attached to a small heavy ring which can slide on a rough horizontal rod, which passes through A. If the weight of ring is n times the weight of the chain, show that its greatest possible distance from A is $\dfrac{2l}{\lambda}\log\left\{\sqrt{\left(1+\lambda^2\right)}+\lambda\right\}$, where $\dfrac{1}{\lambda}=\mu\left(2n+1\right)$

Solution:- Let W be the weight of the chain go that the weight of the ring is nW .

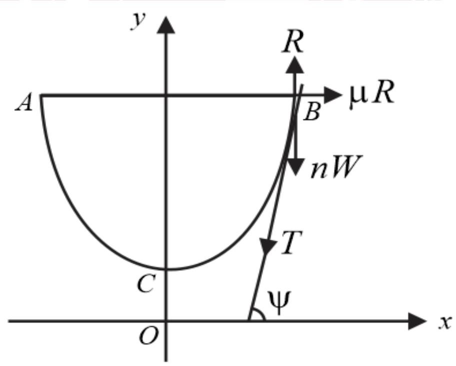
For the greatest possible distance of B from the fixed point A, the point B must be in equilibrium now the point B is in equilibrium under the action of the following forces:

(i) Normal reaction R

(ii) Force of friction μR

(iii) Weight of the ring nW

(iv) Tension T



Resolving the forces horizontal and vertically, we have $T\cos\psi = \mu R$ (1)

And $nW + T\sin\psi = R$ (2)

Where ψ is the angle made by the tangent at B with the x-axis.

Using the value of R from (2) in (1), we have

$T\cos\psi = \mu\left(nW + T\sin\psi\right)$ (3)

$= \mu\left(n.2lw + wl\right)$, where $w\left(=W/2l\right)$ is the weight per unit length of the chain and $T\sin\psi = wl$

Since $T\cos\psi = T_0 = wc,$ (3) reduces to $wc = \mu\left(2n+1\right)lw$, which gives $c = \mu\left(2n+1\right)l$

But we are given that $\frac{1}{\lambda}=\mu(2n+1)$. Therefore, $c=l/\lambda$ (4)

Now using the relation $s=c\tan\psi$ at B, we have

$l=c\tan\psi$ i.e. $l=\frac{l}{\lambda}\tan\psi$ using (4)

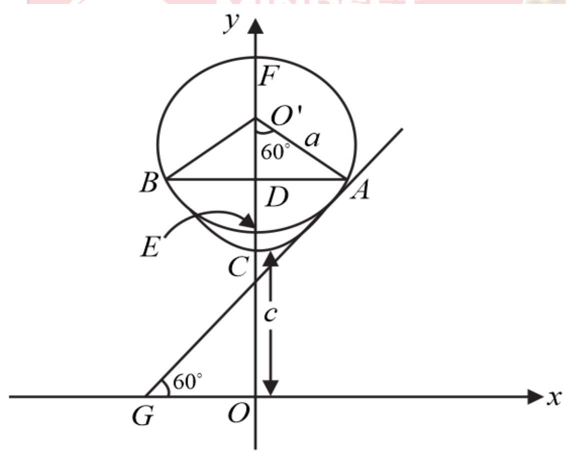
i.e. $\tan\psi=\lambda$ so that $\sec\psi=\sqrt{(1+\lambda^2)}$

Hence the greatest possible distance of B from A

$$=2c\log(\sec\psi+\tan\psi)=\frac{2l}{\lambda}\log\left\{\sqrt{(1+\lambda^2)}+\lambda\right\}$$

Example5:- Show that the length of an endless chain which will hang over a circular pulley of radius 'a'so as to be in contact with two-third of the circumference of the pulley is $a\left\{\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right\}$

Solution:- Let $AFBEA$ be a circular pulley of radius a. An endless chain $AFBCA$ is hanging on this pulley so as to be in contact with two-third of the circumference of the pulley.



Therefore, chain $AFB=\frac{2}{3}(2\pi)=\frac{4}{3}\pi a$ (1)

Since AEB forms one-third circumference of the pulley, the angle $AO'B$ subtended by this part at the centre O' of the pulley is equal to $\frac{1}{3}\times 2\pi$, i.e. 120°. Therefore,

$$\angle AO'D=\frac{1}{2}(\angle AO'B)=60^\circ$$

And hence $\angle O'AD=90^\circ-60^\circ=30^\circ$

Now $\angle AGO=\angle BAG=90^\circ-\angle O'AD=90^\circ-30^\circ=60^\circ$

In the right-angled $\Delta O'AD$, we have

$$AD = AO'\sin 60^\circ = \frac{1}{2}\sqrt{3}\,a \qquad (2)$$

But $AD = c\log\left(\sec\psi + \tan\psi\right)$

$$= c\log\left(\sec 60^\circ + \tan 60^\circ\right) = c\log\left(2+\sqrt{3}\right) \qquad (3)$$

Equating the two values of AD obtained in (2) and (3), we find that

$\frac{1}{2}\sqrt{3}a = c\log\left(2+\sqrt{3}\right)$, which gives $c = \frac{a\sqrt{3}}{2\log\left(2+\sqrt{3}\right)}$

Now using the formula $s = c\tan\psi$, we find that the length of chain $ACB = 2c\tan 60^\circ$

$$= 2\times\frac{a\sqrt{3}}{2\log\left(2+\sqrt{3}\right)}\times\sqrt{3} = \frac{3a}{\log\left(2+\sqrt{3}\right)}$$

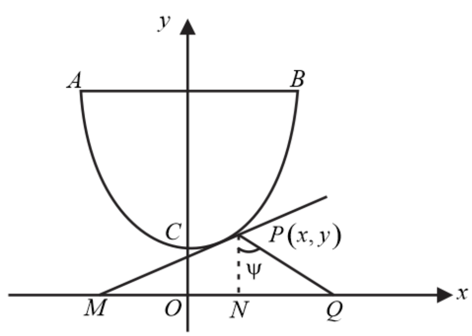
$=$ chain $ACB +$ chain AFB

$$= \frac{3a}{\log\left(2+\sqrt{3}\right)} + \frac{4}{3}\pi a$$

$$= a\left\{\frac{3}{\log\left(2+\sqrt{3}\right)} + \frac{4\pi}{3}\right\}$$

Example6:- If the normal at any point P of a common catenary meets the directrix at Q, then prove that $PQ = \rho$ (radius of curvature).

Solution:- Let PM be the tangent at any point $P(x, y)$ of a common catenary and let the normal at P meet the directrix at the point Q. Also let ψ be the angle made by PM with the directrix and PN be the perpendicular from P on the directrix.



We see that $\angle QPN = 90^\circ - \angle NPM$

$= 90^\circ - \left(90^\circ - \psi\right)$

$= \psi$

Now in the right-angled ΔPNQ, we find that $\frac{PN}{PQ} = \cos\psi$

i.e. $\dfrac{y}{PQ} = \cos\psi$ i.e. $PQ = y\sec\psi$

Since $y = c\sec\psi$, it follows that $PQ = c\sec^2\psi$ (1)

But we know that $s = c\tan\psi$. Differentiating with respect to ψ, this gives $\dfrac{ds}{d\psi} = c\sec^2\psi$

, i.e. $\rho = c\sec^2\psi$

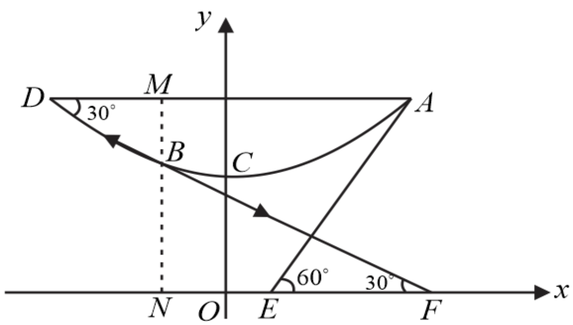
(2)

Since $\rho = ds/d\psi$

Comparing (1) and (2), we conclude that $PQ = \rho$.

Example7:- A heavy uniform chain AB hangs freely under gravity with A fixed and B attracted by a string BD to a fixed point D at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angle of 60° and 30° respectively with the horizontal. Prove that the ratio of these lengths is $\left(\sqrt{3}-1\right):1$.

Solution:- Clearly, heavy chain AB will form a part of the catenary as shown in the figure below.



Let C be the lowest point of this catenary and c be its parameter.

The point B is in equilibrium due to the tension in the string BD in the direction of BD and the tension in the chain in the direction of BF.

It is given that $\angle AEF = 60^\circ$ and $\angle BFO = 30^\circ$. Since DA is parallel to the x-axis, we find that $\angle MDB = 30^\circ$. Now chain $AB =$ chain $AC +$ chain CB

$= c\tan 60^\circ + c\tan 30^\circ$ using $y = c\sec\psi$

$$= c\left(2 - \frac{2}{\sqrt{3}}\right) = \frac{2\left(\sqrt{3}-1\right)c}{\sqrt{3}}$$

Now in the right-angled ΔBMD ,we have

$$BD = BM\,\mathrm{cosec}\,30^\circ = \frac{2\left(\sqrt{3}-1\right)c}{\sqrt{3}} \times 2 = \frac{4\left(\sqrt{3}-1\right)c}{\sqrt{3}}$$

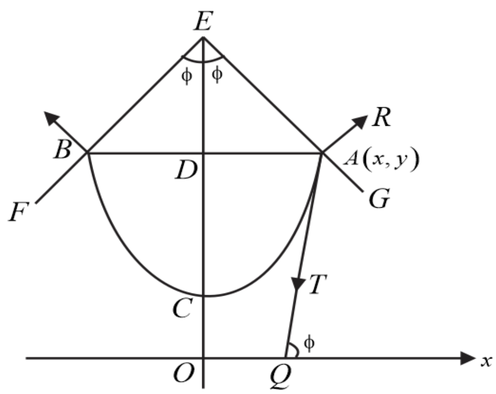
Hence $\dfrac{\text{length of the string } BD}{\text{length of the chain } AB} = \dfrac{4\left(\sqrt{3}-1\right)c}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{4c} = \sqrt{3}-1$

Thus the ratio of string BD and chain AB is $\left(\sqrt{3}-1\right):1$

Example8:- The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles ϕ to the vertical. Prove that the sag in the middle is $\frac{l}{2}\tan\frac{\phi}{2}$.

Solution:- Let EBF and EAG be two smooth rods, which are inclined in opposite directions at an angle ϕ to the vertical EO.

ACB is a uniform chain of length l, whose end links slide on these rods. The ends of the chain are in equilibrium due to the tension T and normal reaction R.



Now, $\angle AQx = \angle BAQ = 90^\circ - \angle DAE = 90^\circ - (90^\circ - \phi) = \phi$

Therefore, $\psi = \phi$

Now using the relation $s = c\tan\psi$, we have $\frac{1}{2}l = \tan\phi$, which gives $c = \frac{1}{2}l\cot\phi$ (1)

Hence sag $CD = DO - CO = y - c$

$$= c\sec\phi - c = c(\sec\phi - 1)$$

$= \frac{1}{2}l\cot\phi(\sec\phi - 1)$, using (1)

$$\frac{l}{2}\left(\frac{1-\cos\phi}{\sin\phi}\right) = \frac{l}{2}.\frac{2\sin^2\frac{1}{2}\phi}{2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}$$

$$\frac{l}{2}\tan\frac{\phi}{2}.$$

Example9:- if α and β be the inclinations to the horizon of the tangent of the extremities of a portion of common catenary and l is the length of the portion, show that the height of one extremity above the other is $l\sin\left(\frac{\alpha+\beta}{2}\right) / \cos\left(\frac{\alpha-\beta}{2}\right)$, the two extremities being on one side of the vertex of the catenary.

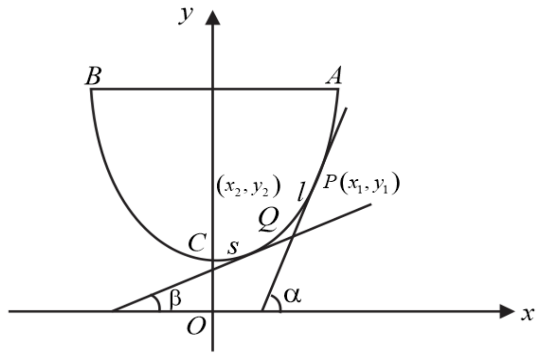
Solution:- Let PQ be a portion of a catenary such that arc length $PQ = l$. Also, let α and β be the inclinations to the horizon of the tangents at the extremities P and Q of this portion.

With C as the lowest point of the catenary, let s be arc length CQ.

Let (x_1, y_1) and (x_2, y_2) be the Cartesian co-ordinates of the points P and Q, respectively. Then using the formula $y = c\sec\psi$ we have

$$y_1 = c\sec\alpha \quad \text{(at P)} \qquad (1)$$

And $$y_2 = c\sec\beta \quad \text{(at Q)} \qquad (2)$$



Subtracting (2) from (1), we obtain

$$y_1 - y_2 = c(\sec\alpha - \sec\beta) \qquad (3)$$

Which represents the height of one extremity (P) above the other (Q).

We need to eliminate c from (3). For this using the formula $s = c\tan\psi$, we get

$$s + l = c\tan\alpha \quad \text{(at P)} \qquad (4)$$

And $$s = c\tan\beta \quad \text{(at Q)} \qquad (5)$$

Subtracting (5) from (4), we obtain

$$l = c(\tan\alpha - \tan\beta) \qquad (6)$$

Now dividing (3) by (6), we finally have

$$\frac{y_1 - y_2}{l} = \frac{\sec\alpha - \sec\beta}{\tan\alpha - \tan\beta}$$

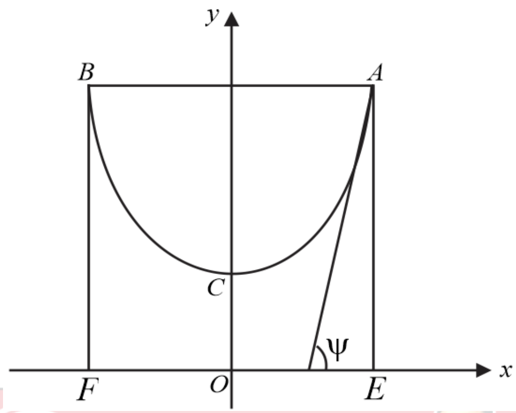
$$= \frac{\dfrac{1}{\cos\alpha} - \dfrac{1}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}} = \frac{\cos\beta - \cos\alpha}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$= \frac{\cos\beta - \cos\alpha}{\sin(\alpha - \beta)} = \frac{2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)}{2\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)}$$

This gives $y_1 - y_2 = l\sin\left(\dfrac{\alpha+\beta}{2}\right) / \cos\left(\dfrac{\alpha-\beta}{2}\right)$, the desired result.

Example10:- A heavy uniform string hangs over two smooth pegs in the same horizontal line. If the length of each portion which hangs freely is n times the length between the pegs, probe that the ratio of the whole length of string is to the distance between the pegs as $k:\log k$ where $k=\left(\dfrac{2n+1}{2n-1}\right)^{1/2}$

Solution:- Let $2l$ be the length of the portion of the string ACB which forms a catenary between the smooth pegs A and B. it is given that the length of each of the portions BF and AE which hang freely is n times the length between the pegs. So $BF=2nl$ and $AE=2nl$ (1)



For equilibrium, we see that the tension at $A=w.\,AE$, where w is the weight per unit length of the string.

But we know that $T=wy$

Therefore, $w.AE=wy$ i.e. $AE=y$ (2)

This shows that the end directrix of the catenary as shown in the adjacent figure.

Now equating the two values of AE from (1) and (2), we get $y=2nl$, i.e. $c\sec\psi=2n.c\tan\psi$, since at $A, y=c\sec\psi$ and $l=c\tan\psi$ (using $s=c\tan\psi$)

i.e. $\dfrac{2n}{1}=\dfrac{\sec\psi}{\tan\psi}$ **(note)**

Applying the componendo and dividend, it gives $\dfrac{2n+1}{2n-1}=\dfrac{\sec\psi+\tan\psi}{\sec\psi-\tan\psi}$,

i.e. $k^2=\dfrac{\sec\psi+\tan\psi}{\sec\psi-\tan\psi}\times\dfrac{\sec\psi+\tan\psi}{\sec\psi+\tan\psi}$, since $k=\left(\dfrac{2n+1}{2n-1}\right)^{1/2}$

$$=\frac{(\sec\psi+\tan\psi)^2}{\sec^2\psi-\tan^2\psi}=(\sec\psi+\tan\psi)^2$$

Therefore, $k=\sec\psi+\tan\psi$ (3)

Now $\dfrac{\textit{whole length of the string}}{\textit{distance between the pegs}}=\dfrac{2y+2l}{AB}$

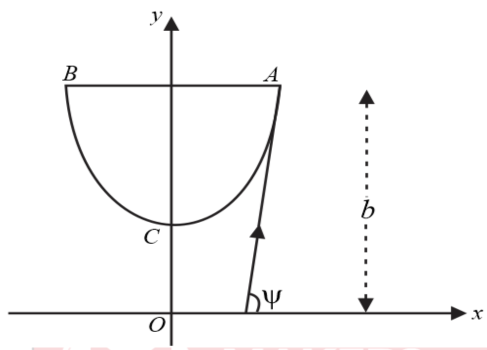
$=\dfrac{2.c\sec\psi+2.c\tan\psi}{2c\log(\sec\psi+\tan\psi)}$, since span $AB=2c\log(\sec\psi+\tan\psi)$

$$= \frac{\sec\psi + \tan\psi}{\log(\sec\psi + \tan\psi)} = \frac{k}{\log k}, \text{ using (3)}$$

Hence the ratio of the whole length of string is to the distance between the pegs as $k : \log k$.

Example11:- A given length $2s$ of a uniform chain has to be hung between two points in the same horizontal level and the tension has not to exceed the weight of the length b of the chain. Show that the greatest span is $\left(b^2 - s^2\right)^{1/2} \log\left(\frac{b+s}{b-s}\right)$

Solution:- Let w be the weight per unit length of the string. If T denote the tension at the point A, we are given that $T_{\max} = wb$ (1)



But we know that $T = wy$. (2)

From (1) and (2), we get that $y = b$ when the tension at A is maximum so that the span AB is the greatest.

Now putting $y = b$ in the relation $y^2 = c^2 + s^2$, we obtain $b^2 = c^2 + s^2$, which gives $c = \left(b^2 - s^2\right)^{1/2}$.

Since are $CA = s$ (given), from the relation $s = c\tan\psi$, we have

$$\tan\psi \frac{s}{c} = \frac{s}{\left(b^2 - s^2\right)^{1/2}}, \text{ using (1)} \qquad (2)$$

Also, using the relation $y = c\sec\psi$ at the point A, we have $b = c\sec\psi$ i.e.

$$\sec\psi = \frac{b}{c} = \frac{b}{\left(b^2 - s^2\right)^{1/2}} \qquad (3)$$

Using (1), (2) and (3), we finally see that:

The greatest span $AB = 2c\log(\sec\psi + \tan\psi)$

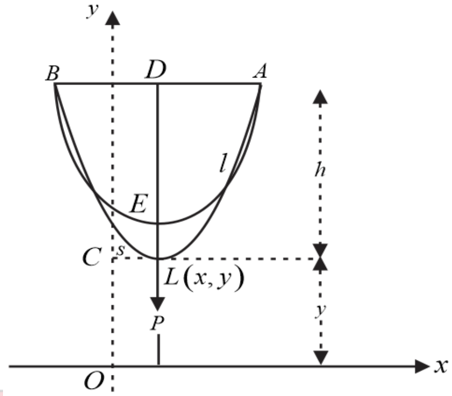
$$= 2\left(b^2 - s^2\right)^{1/2} \log\left\{\frac{b}{\left(b^2 - s^2\right)^{1/2}} + \frac{s}{\left(b^2 - s^2\right)^{1/2}}\right\}$$

$$= 2\left(b^2 - s^2\right)^{1/2} \log\left\{\frac{b+s}{\left(b^2 - s^2\right)^{1/2}}\right\}$$

$$=2\left(b^2-s^2\right)^{1/2}\log\left\{\frac{(b+s)^{1/2}}{(b-s)^{1/2}}\right\}=\left(b^2-s^2\right)^{1/2}\log\left(\frac{b+s}{b-s}\right)$$

Example12:- A uniform chain of length $2l$ and weight $2W$ is suspended from two points in the same horizontal line. A load W is now suspended from the middle point of the chain and the depth of the point below the horizontal line is h. Show that the terminal tension is: $\frac{W}{2}\left(\frac{h^2+2l^2}{hl}\right)$

Solution:- Let AEB be the catenary formed by the chain of length $2l$ and $LD=h$



Let the co-ordinates of the point L be (x,y). This is the new position of the point E.

When a load W is suspended from E, two catenaries are formed. Let one of them be ALC, where $CL=s$ and $AL=l$

At the point L, we have $y^2=c^2+s^2$ (1)

Whereas at the point A, we have $(y+h)^2=c^2+(s+l)^2$ (2)

Subtracting (1) from (2), we have $h^2+2hy=l^2+2sl$

This gives $y=\dfrac{l^2+h^2+2sl}{2h}$

If w be the weight per unit length, then the tension T at the point A is given by $T=w(y+h)$

$$=w\left(\frac{l^2-h^2+2sl}{2h}+h\right)=\frac{w}{2}\left(\frac{2sl}{h}+\frac{h^2+l^2}{h}\right)$$

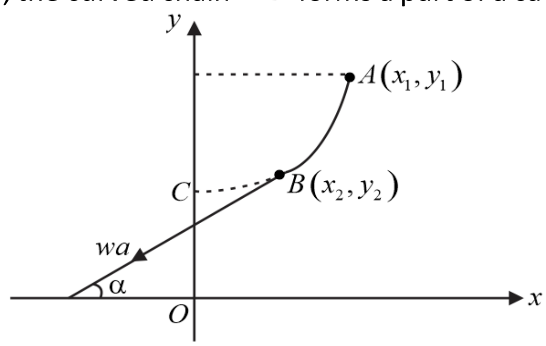
$$=\frac{1}{2}\left(\frac{2wsl}{h}+\frac{W}{l}.\frac{h^2+l^2}{h}\right)\text{, since } w=\frac{W}{l}$$

But $W=2T\sin\psi=2ws$, at the point L. Therefore $T=\dfrac{1}{2}\left(\dfrac{Wl}{h}+\dfrac{W}{l}+\dfrac{h^2+l^2}{h}\right)$

$$=\frac{W}{2}\left(\frac{l}{h}+\frac{h^2+l^2}{hl}\right)=\frac{W}{2}\left(\frac{h^2+2l^2}{hl}\right)$$

Example13:- A string of length l is attached to a fixed point A and other end B is pulled with a force 'wa' inclined at an angle α to the horizon, w being the weight peer unit length of the string. Show that the vertical distance of A, above B is $\sqrt{\left(l^2+a^2+2la\sin\alpha\right)}-a$

Solution:- Let AB be a string of length l and w be its weight per unit length. The end A of the string is fixed whereas the end B is pulled with a force wa inclined at an angle α to the horizon (i.e. the x-axis). Clearly, the curved chain AB forms a part of a catenary with C as its lowest point.



Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the points A and B, respectively.

If T denote the tension at B, we have

$$T = wa \qquad (1)$$

Also, using the relation $T = wy$ at B, we have

$$T = wy_2 \qquad (2)$$

Comparing (1) and (2), we find that

$$y_2 = a \qquad (3)$$

Now using the formula $y = c\sec\psi$ at B, we obtain $y_2 = c\sec\alpha$ i.e. $a = c\sec\alpha$ which gives $c = a\sec\alpha$ (4)

Further, using the for the formula $s = c\tan\psi$ at B, we have $s = a\cos\alpha \,.\, \tan\alpha$ i.e. $s = a\sin\alpha$

Thus arc $CB = a\sin\alpha$ so that arc $CA = a\sin\alpha + l$.

We now use the formula $y^2 = c^2 + s^2$ at A to get $y_1^2 = (a\cos\alpha)^2 + (a\sin\alpha + l)^2$ using (4)

$$= a^2\cos^2\alpha + \left(a^2\sin^2\alpha + l^2 + 2al\sin\alpha\right)$$

$$= a^2\left(\cos^2\alpha + \sin\alpha\right) + l^2 + 2al\sin\alpha$$

$$= a^2 + l^2 + 2al\sin\alpha$$

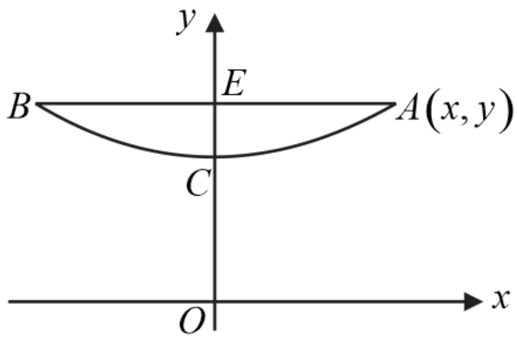
Whence $y_1 = \sqrt{\left(a^2 + l^2 + 2al\sin\alpha\right)}$. (5)

Finally, subtracting (3) from (5), the required vertical distance of A above B.

$$y_1 - y_2 = \sqrt{\left(a^2 + l^2 + 2al\sin\alpha\right)} - a.$$

Example14:- A uniform measuring chain of length l is tightly stretched over a river, the middle point just touching the surface of water, while each of the extremities has an elevation k above of surface. Show that the breadth of the river is nearly $\left(l-\frac{8k^2}{3l}\right)$

Solution:- Chain is tightly stretched over the river of breadth AB. Also ACB is the measuring chain of length l The sag $CE = k$.



Let the co-ordinates of the point A be (x, y).

Therefore, $\frac{l}{2} = c\sin h\frac{x}{c}$ at the point A, (since $s = c\sin h\frac{x}{c}$)

Expanding $\sin h(x/c)$ in the ascending powers of x/c, we get $\frac{l}{2} = c\left[\frac{x}{c}+\frac{1}{3!}\left(\frac{x}{c}\right)^3+...\right]$

Neglecting higher powers of x/c, we have $\frac{l}{2} = x+\frac{x^3}{6c^2}$, or $\frac{l-2x}{2} = \frac{x^3}{6c^2}$, or $l-2x = \frac{x^3}{3c^2}$

Sag $EC = k = y-c = c\cosh\frac{x}{c}-c = c\left(\cosh\frac{x}{c}-1\right)$

On expanding $\cosh\frac{x}{c}$ in ascending powers of $\frac{x}{c}$, we have

$$k = c\left[\left\{1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+....\right\}-1\right]$$

Neglecting higher powers of x/c other than $(x/c)^2$, we get $k = \frac{x^2}{2c}$ so that $c = \frac{x^2}{2k}$.

On putting this value of c in (1), we have $l - AB = \frac{4k^2}{3x} = \frac{4k^2}{3.\frac{l}{2}} = \frac{8k^2}{3l}$.

Since chain is tightly stretched, we have taken $x = l/2$.

Therefore, $AB = l-\frac{8k^2}{3l}$, i.e. width of the river $= l-\frac{8k^2}{3l}$.

Example15:- A telegraph wire stretched between two poles at distance a ft apart sags n ft in the middle. Prove that the tension at the end is approximately $w\left(\frac{a^2}{8n}+\frac{7n}{6}\right)$, where w is the weight per unit length of wire.

Solution:- We have $n = sag = y - c = c\left(\cosh\frac{x}{c}-1\right)$

Expanding $\cosh\frac{x}{c}$ in ascending, powers of $\frac{x}{c}$, we have

$$n = c\left[\left(1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\right)-1\right]$$

$=\frac{x^2}{2c}+\frac{x^4}{24c^3}$, neglecting higher powers of $\frac{x}{c}$

We have $n=\frac{a^2}{8c}+\frac{a^4}{21.16c^3}$, since $x=\frac{a}{2}$ (1)

Taking first approximation $n=\frac{a^2}{8c}$ or $c=\frac{a^2}{8n}$

Putting this value of c in the second term of R.H.S of equation (1) we get

$$n=\frac{a^2}{8c}+\frac{4n^3}{3a^2}+\ldots$$

Therefore, $n=\frac{a^2}{8c}+\frac{4n^3}{3a^2}$ nearly, which gives

$$\frac{a^2}{8c}=n-\frac{4n^3}{3a^2}=n\left(1-\frac{4n^2}{3a^2}\right)$$

So, $\frac{8c}{a^2}=\frac{1}{n}\left(1-\frac{4n^2}{3a^2}\right)^{-1}=\frac{1}{r}\left(1+\frac{4n^2}{3a^2}\right)$ nearly

Hence $c=\frac{a^2}{8n}\left(1+\frac{4n^2}{3a^2}\right)=\frac{a^2}{8n}+\frac{n}{6}$

Thus tension at the point of support

$=w(n+c)=w\left(n+\frac{a^2}{8n}+\frac{n}{6}\right)=w\left(\frac{a^2}{8n}+\frac{7n}{6}\right)$, on putting the value of c.

Therefore, tension at the pole is $w\left(\frac{a^2}{8n}+\frac{7n}{6}\right)$.

Example16:- A telegraph wire is made of given material and such a length l is stretched between two posts, distance d apart and of the same height, as will produce the least possible tension at the posts. Show that $l = \frac{d}{\lambda}\sinh\lambda$, where λ is given by the equation $\lambda \tan\mathrm{h}\,\lambda = 1$

Solution:- We know that $T = wy$

$$= wc\cosh\frac{d}{2c}\text{, since } x = d/2 \text{ at A}$$

Therefore, $T = wc\cosh\frac{d}{2c}$.

Differentiating it with respect to c, we have

$$\frac{dT}{dc} = w\left(\cosh\frac{d}{2c} - \frac{d}{2c}\sinh\frac{d}{2c}\right) \qquad (1)$$

Differentiating again with respect to c, we have

$$\frac{d^2T}{dc^2} = w\left(-\frac{d}{2c^2}\sinh\frac{d}{2c} + \frac{d}{2c^2}\sinh\frac{d}{2c} + \frac{d^2}{4c^3}\cosh\frac{d}{2c}\right)$$

Therefore, $\frac{d^2T}{dc^2} = w\frac{d^2}{4c^3}\cosh\frac{d}{2c} > 0$

Hence the tension at the point of support is minimum. For minimum tension $dT/dc = 0$, which gives $w\left(\cosh\frac{d}{2c} - \frac{d}{2c}\sinh\frac{d}{2c}\right) = 0$, from (1)

Therefore, $\frac{d}{2c}\tanh\frac{d}{2c} = 1$

On putting $d/2c = \lambda$, we have $\lambda = \tanh\lambda = 1$. Here $c = d/2\lambda$ (2)

We have $\frac{l}{2} = c\sinh\frac{d}{2c}$, since $s = c\sinh\frac{x}{c}$

On putting the value of c from (2) we have $l = \frac{d}{\lambda}\sinh\lambda$.

PREVIOUS YEARS QUESTIONS: IAS/IFoS (2008-2023)

SOLUTIONS HINT: Beauty of learning systematically this topic statics- No matter what book you follow, UPSC PYQs are always directly examples from book itself. As to avoid the documents to be lengthy and unnecessary repetition we have just put hints and mentioned the references in front of PYQs.

1.COMMON CATANORY

Q6(a) A cable of weight *w* per unit length and length $2l$ hands from two points P and Q in the same horizontal line. Show that the span of the cable is $2l\left(1-\frac{2h^2}{3l^2}\right)$, where h is the sag in the middle of the tightly stretched position.

UPSC CSE 2022

Q1. Derive intrinsic equation

$x = c\log(\sec\psi + \tan\psi)$

of the common category, where symbols have usual meanings.

Prove that the length of an endless chain, which will hang over a circular pulley of radius '*a*' so as to be in contact with $\frac{2}{3}$ of the circumference of the pulley, is

$a\left\{\frac{4\pi}{3}+\frac{3}{\log\left(2+\sqrt{3}\right)}\right\}$. **[7a 2020 IFoS]**

Solution Reference: part-1 Article 3 span of common catenary in theory part of this chapter. Part-2 Example 5

Q2. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu\log\frac{1+\left(1+\mu^2\right)^{\frac{1}{2}}}{\mu}$$

where μ is the coefficient of friction. **[7a 2018 IFoS]. Solution Ref. Example 3**

Q3. Find the length of an endless chain which will hang over a circular pulley of radius '*a*' so as to be in contact with the two-thirds of the circumference of the pulley. **[8a UPSC CSE 2015]. Solution Reference: Example 5**

Q4. Determine the length of an endless chain which will hang over a circular pulley of radius *a* so as to be in contact with two-thirds of the circumference of the pulley. **[7a 2015 IFoS]. Solution Reference: Example 5**

Q5. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the radio of the maximum span to the length of the chain is $\mu \log\left[\dfrac{1+\sqrt{1+\mu^2}}{\mu}\right]$ where μ is the coefficient of friction. **[7c UPSC CSE 2012]. Solution Reference: Example 3**

Q6. A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120\cosh^{-1}\left(\dfrac{5}{3}\right)$ and also find the sag. **[5d 2011 IFoS].**

Q7. A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be *h*. Show that each terminal tension is

$\dfrac{1}{2}\left[P\cdot\dfrac{1}{h}+W\cdot\dfrac{h^2+l^2}{2hl}\right]$. **[7a 2010 IFoS]. Solution Reference: Example 12**

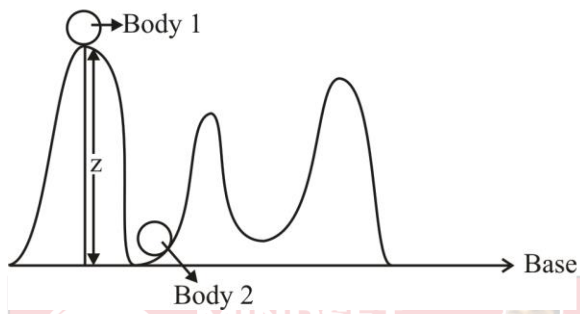
Step (i): Let's try to understand basic terms first and then the mathematics behind it.

Equilibrium :

Stable: After slight displacement, it comes into it's original position.

Unstable: After slight displacement, it does not return to it's original position.

Keywords: slight displacement from it's original position of equilibrium.



It can be seen that if body 1 is slightly displaced from it's position of equilibrium, the body may come to it's actual position but it is also possible that a further displacement it does not come back to it's position of equilibrium. The same we can think of body 2 too.

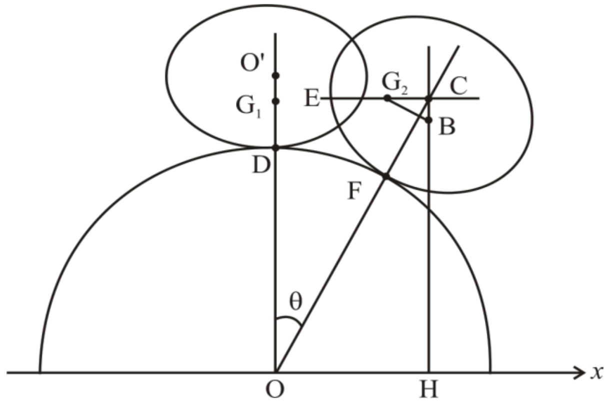
In case: body 1; body 1 is in unstable equilibrium.

Body 2 is in stable equilibrium.

The above story indicates that.

If height of centre of gravity of the body from the fixed (base); then it's position of equilibrium is unstable and if this height is minimum, then position of equilibrium is stable; Let's try to extend this understanding through mathematical ways.

Case (1): A body rests in equilibrium upon another body (which is fixed), the portions of two bodies in contact being spheres of radii r and R respectively, and the straight line joining the centres of the spheres being vertical; if the first body be slightly displaced; finding whether the equilibrium is stable or unstable, the bodies being rough enough to present sliding.



O; centre of spherical surface of lower body.

O'; centre of spherical surface of upper body.

G_1; centre of gravity of upper body (actual position).

G_2; centre of gravity of upper body after displacing.

Let's have the displacement ast DF(Arc length) = EF (Arc lenght)

OD = R, O'D = r

$EG_2 = h$, $\quad G_2B \perp CH$

Let $\angle DOF = \theta = \angle OCH$, $\quad \angle ECO = \phi$

$\therefore \quad \angle ECH = \theta + \phi$

$\because \quad$ Arc DF = Arc EF $\therefore \quad R \cdot \theta = r \cdot \phi \qquad \ldots(1)$

Let z be the height of G_2 above ox line.

$\therefore \quad z = CH - CB$ as $G_2B \parallel ox = OC\, cos\theta - CG_2\, cos(\theta + \phi)$

$= (R + r)cos\theta + (r - h)cos(\theta + \phi) \quad \ldots(2)$

$\because G_2C = r - h = (R + r)cos\theta + (r - h)cos\left(\frac{R+r}{r}\right)\theta \qquad \ldots(3)$

Using (1)

Differentiating (3) w.r.t θ, we get

$$\frac{dz}{d\theta} = -(R+r)sin\theta + (r-h)\left(\frac{R+r}{r}\right)sin\left(\frac{R+r}{r}\right)\theta$$

For equilibrium i.e. for maxima or minima of z, Putting $\frac{dz}{d\theta}=0$

$$\Rightarrow -(R+r)\sin\theta+(r-h)\left(\frac{R+r}{r}\right)\sin\left(\frac{R+r}{r}\right)\theta=0$$

We can observe that θ = 0° satisfies above equation.

$$\text{Now } \frac{d_2z}{d\theta^2}=-(R+r)\cos\theta+(r-h)\left(\frac{R+r}{r}\right)^2\cos\left(\frac{R+r}{r}\right)\theta \qquad \ldots(4)$$

$$\because \qquad \frac{d_2z}{d\theta^2}\Big|_{\theta=0}=-(R+r)+(r-h)\left(\frac{R+r}{r}\right)^2$$

$$=\left(\frac{R+r}{r}\right)^2\left(\frac{-r^2}{(R+r)}+r-h\right)=\frac{(R+r)^2}{r^2}\left(\frac{rR}{R+r}-h\right)$$

Clearly: $\frac{d_2z}{d\theta^2}>0 \text{ if } \frac{r\cdot R}{R+r}>h \Rightarrow \frac{1}{h}>\frac{1}{R}+\frac{1}{r}$ $(Minima)$

And $\frac{d_2z}{d\theta^2}<0 \text{ if } \frac{r\cdot R}{R+r}<h \Rightarrow \frac{1}{h}<\frac{1}{R}+\frac{1}{r}$ $(Maxima)$

Thus the equilibrium is

Stable when $\frac{1}{h}>\frac{1}{R}+\frac{1}{r}$

Unstable when $\frac{1}{h}<\frac{1}{R}+\frac{1}{r}$

Now for $\because$ $\frac{d_2z}{d\theta^2}\Big|_{\theta=0}=0$ *i.e.* $h=\frac{rR}{R+r}$

So Let's check the sign of $\frac{d_3z}{d\theta^3}$ at θ = 0

$$\frac{d_3z}{d\theta^3}=(R+r)\sin\theta-(r-h)\left(\frac{R+r}{r}\right)^3\sin\left(\frac{R+r}{r}\right)\theta$$

And $\frac{d_3z}{d\theta^3}\Big|_{\theta=0}=0 \therefore$ $\frac{d_4z}{d\theta^4}=(R+r)\cos\theta-(r-h)\left(\frac{R+r}{r}\right)^4\cos\left(\frac{R+r}{r}\right)\theta$

$$\frac{d_4 z}{d\theta^4}\Big|_{\theta=0} = (R+r)\left\{1-(r-h)\left(\frac{R+r}{r^2}\right)\left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1-\left(\frac{R+r}{r^2}\right)^2\right\} \qquad \because\ h=\frac{rR}{r+R}$$

$$= (R+r)\left\{1-\left(1+\frac{R}{r}\right)^2\right\}$$

= a negative quantity

$\therefore$ z is maximum so equilibrium is unstable.

Therefore, the equilibrium is –

Stanble when	$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$
Unstable when	$\frac{1}{h} \leq \frac{1}{r} + \frac{1}{R}$

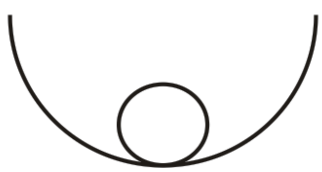
❖ **Deductions:** If the upper body has a plane surface in contact with the lower body i.e. r $\to\infty$ the equilibrium is stable or unstable according as h < or > R.

($\because$ $\frac{1}{h} > \frac{1}{\infty} + \frac{1}{R}$, $\frac{1}{h} \leq \frac{1}{\infty} + \frac{1}{R}$)

❖ Similarly we can think of if lower body has plane surface in contact the R $\to \infty$

Exam Suggestion: In the exam, Proofs are not being asked. Only; we need to remember above formula. For three cases. (See examples, how)

Case (ii):



A body rests in equilibrium inside another concave fixed body, the portions of two bodies in contact being spheres of radii r and R respectively, and the straight line joining the centres of the spheres being vertical. If the first body be slightly displaced. Discuss the stability of equilibrium the bodies being rough enough to prevent sliding.

Proof: Similar way as for case (ii);

The equilibrium is

Stabel when $\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$,

Unstable when $\frac{1}{h} < \frac{1}{r} - \frac{1}{R}$,

If $\frac{1}{h} = \frac{1}{r} - \frac{1}{R}$ then the equilibrium is stable when R > 2r and unstable when R < 2r

Case (iii): A body rests in equilibrium upon another body (which is fixed) and the portions of two bodies in contact have radii of curvature ρ_1 and ρ_2 respectively. The C.G. of the first body is at height h above the point of contact and the common normal makes an angle $\propto$ with the vertical, then equilibrium is-

Stabel if $h < \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \cos\alpha$

Unstable if $h > \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \cos\alpha$

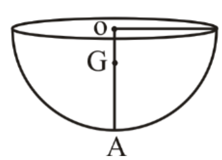
In case $\frac{1}{h} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$, the equilibrium is neutral.

Note: Now; In questions, we need to observe.

- ❖ Contact surfaces
- ❖ **Height of C.G. (For this we need to remember C.G. for some bodies in particular.)**

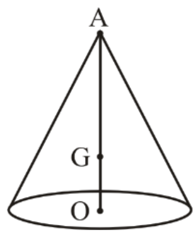
Example:

1. Hemisphere OG = $\frac{3a}{8}$



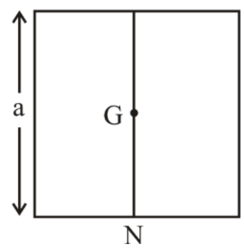
Where ‘a’ is radius:OA

2. Cone; OG = $\frac{\ell}{4}$

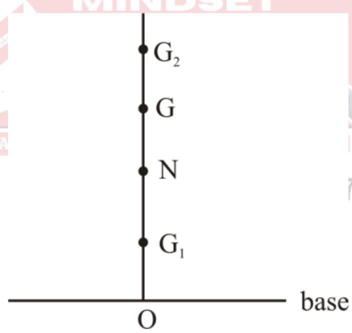


Where OA = ℓ

3. **Square;** NG = $\frac{a}{2}$



4. Calculating C.G. of a system of two bodies.



Let y_2 is height of C.G. of upper body with C.G.G_2

y_1 is height of C.G. of lower body with C.G.G_2

Then C.G. of the system of two bodies

$$z = h = \bar{y} = \frac{w_1 y_1 + w_2 y_2}{w_1 + w_2}$$

Where w_1: weight of lower body.

w_2: weight of upper body.

Use:

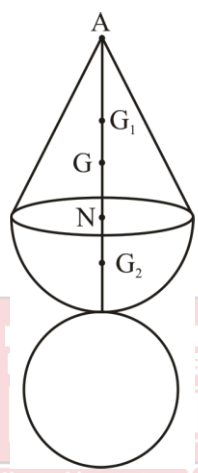
$$w_1 = mg = = \left(\frac{4}{3}\pi r^3 \times \delta\right) g \text{ for sphere}$$

Homogeneous body/bodies of same substance means the density ρ is same.

For cone and hemisphere

$$\therefore \qquad \bar{y} = \frac{\frac{1}{2}\pi r^2 \ell\left(r + \frac{\ell}{4}\right) + \frac{2\pi}{3} r^3\left(r - \frac{3r}{8}\right)}{\frac{\pi}{3}\pi r^2 \ell + \frac{2}{3}\pi r^3}$$

For



Here $NG_1 = \frac{\ell}{4}$

$NG_2 = \frac{3r}{8}$

Radius of hemisphere : r

Note: We have to deal different types of problems:

Type(i): Curved surfaces related problems Hemisphere, sphere, cone, elliptical (The lower body is a curved surface or the upper)

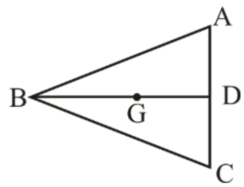
Type (ii): Problems related to rods

Type (iii): Problems related to rectangular laminas

Type (iv): Miscellaneous problems.

Revision for exam hall (Recalling : geometry applied in questions)

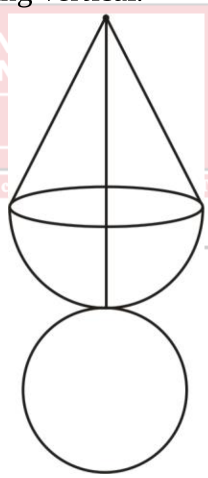
- A hemisphere rests on sphere : When curved surface on the sphere. When the flat surface on the sphere.
- A lamina in the form of an isosceles triangle whose vertical angle is α, placed on sphere. Is flat surface on sphere.



$$BG = \frac{2}{3} BD$$

Square on sphere; maximum till : $\frac{\pi}{4}$

- Hemisphere has a solid right cone on its base and hemisphere rests on the convex side of a fixed sphere; the axis of cone being vertical.

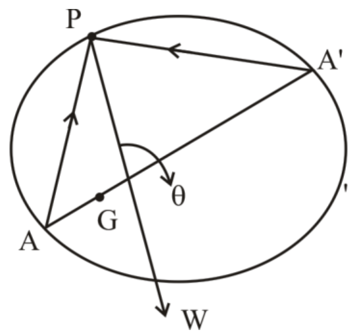


$$\bar{y} = \frac{w_1 y_1 + w_2 y_2}{w_1 + w_2}$$

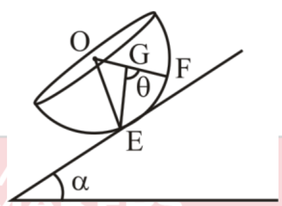
quadratic equation in ℓ; ℓ cannot be negative.

- Hemisphere lying in a fixed spherical shell. Particle is attached to the upper end.

- A thin hemispherical bowl; on the highest point of fixed sphere, Inside the bowl, a small sphere. Keyword. Using ‘moment’
- A rod; string: Slung over a peg considering ellipse.
  - The focal distance of any point P on the ellipse is constant and is equal to the length of its major axis.

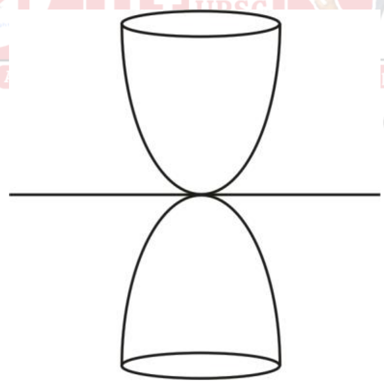


  - Using $\frac{\ell}{r} = 1 + cos\theta$; equation of ellipse. Where $b^2 = a^2(1 - e^2)$

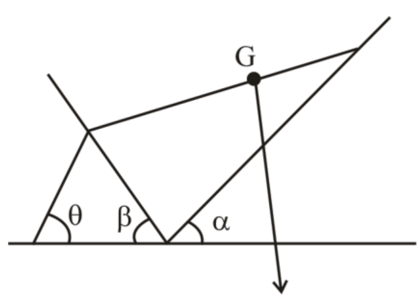


  -

- Paraboloid frustum



- Rods



Sine rule

$sin(\theta + \beta)$

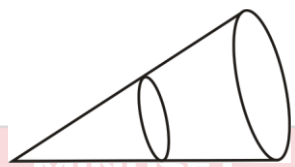
$sin(\alpha - \beta)$

$sin\{\pi - (\alpha + \beta)\}$

- Depth of C.G. ; max depth stable,
  Min. depth unstable.
- String and rod : $r^2 = 4a^2 + i^2 - 4a\cos\theta$

- Square lamina ; Properties of squares (diagonals) rectangles, rhombus.
- Miscellaneous;
  - Isosceles triangular lamina in contact with two smooth pegs.

    Sine rule $\dfrac{PG}{\sin A} = \dfrac{AQ}{\sin APQ}$
  - Solid circular cone



- Four uniform rods
- Three equal spheres on a smooth table, elastic band.
- String, pulley, weight.

Example1:- A hemisphere rests in equilibrium on a sphere of equal radius: show that the equilibrium is unstable when the curved surface rest on the sphere and stable when the flat surface of hemisphere rests on the sphere.

Solution:- Case : 1:- When the curved surface rests on sphere. Suppose that radius of the sphere is a, C.G. is at the point G; N, the point of contact. We know that $OG = 3a/8$

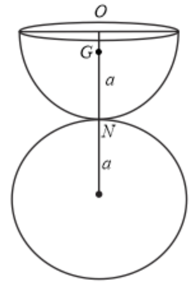


Fig. 1

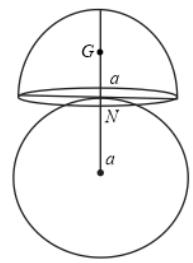


Fig. 2

$h =$ height of C.G above N

$= ON - OG$

$$= a - \frac{3a}{8} = \frac{5a}{8}$$

Also radius of lower body $= R = a$ radius of upper body $= r = a$.

Applying Art. 39 the equilibrium is unstable if

$$\frac{1}{h} \le \frac{1}{r} + \frac{1}{R}$$

i.e. $\frac{8}{5}a < \frac{1}{a} + \frac{1}{a} = \frac{2}{a}$

i.e. $\frac{8}{5} < 2$, which is true.

Hence the equilibrium is unstable.

Case: 2:- When the flat surface rests on the sphere.

In this case of plane face of the upper body is in contact with the lower sphere, so $r = \infty$,

$R = a \quad h = NG = 3a/8$

The equilibrium is stable if

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

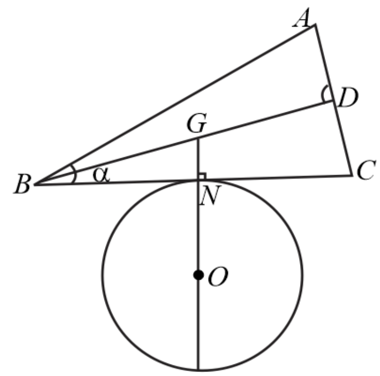
i.e. $\frac{8}{3a} > \frac{1}{\infty} + \frac{1}{a} = \frac{1}{a}$

i.e. $\frac{8}{3} > 1$ which is true.

So, the equilibrium is stable.

Example2:- A lamina in the form of an isosceles triangle whose vertical angle is α, is placed on a sphere of radius r so what its plane is vertical and one of the equal sides is in contact with the sphere. Show that if triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < 3r/a$, where a is one of the equal sides.

Solution:- ABC is the triangular lamina with equal sides BA and BC such that



$BA = BC = a,\ \angle ABC = \alpha$.

BD is the bisector of $\angle ABC$, $\angle ADB = 90^\circ$

So, $\angle ABD = \frac{1}{2}\alpha,\ \angle BNG = 90^\circ$

G being the C.G. of the lamina, then $BG = \frac{2}{3}DB = \frac{2}{3}a\cos(\alpha/2)$ as $AB = a$ and

$h = GN = BG = \sin\alpha/2$

$= \frac{2}{333}a\cos\alpha/2\sin\alpha/2$

$= \frac{a}{3}\sin\alpha$

$R =$ radius of the lower body $= r$ (given)

$r =$ radius of the upper body (Flat surface) $= \infty$

The equilibrium is stable if

$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

If $\frac{3}{a\sin\alpha} > \frac{1}{\infty} + \frac{1}{r}$

Example3:- A heavy cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to presents sliding and if the side of the cube is $\pi r/2$, show that the cube can rock through a right angle without falling.

Solution:- As shown in the fig. 1 ABCD is a uniform cube, O is the centre of the given sphere $AB = r\pi/2$, G is the C.G. of the cube, so $h = GN = r\pi/4$, $R =$ radius of the lower surface $= r$.

The surface of cube in contact with the sphere is plane AB.

So, $r = \infty$, the radius of the lower body (sphere) $= r$.

Applying Art. 39, the equilibrium is stable if

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}, \text{ i.e. } \frac{4}{r\pi} > \frac{1}{\infty} + \frac{1}{r}$$

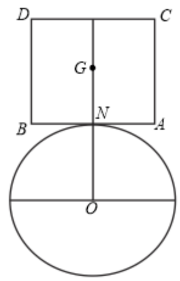


Fig. 1

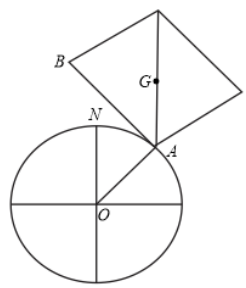


Fig. 2

$4 > \pi$, which is true since $\pi = 22/7$ i.e.

Hence, the equilibrium is stable.

As the cube rocks clockwise, the C.G. of the cube will move towards right hand side. When the point A comes in contact with the surface of the sphere, in this position as shown in fig.2, the line GA becomes vertical. If the cube tilts further slightly the cube will fall down. Hence the cube will not fall down till the point A comes in contact with the surface of the sphere.

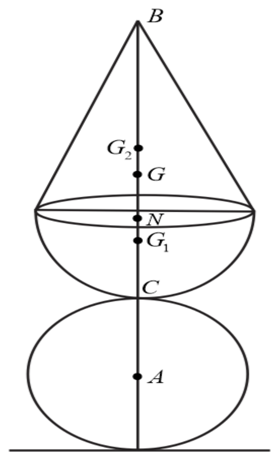
The arc $NA = r\theta = \frac{\pi r}{4}$ as $NA = \frac{1}{2}\frac{\pi r}{2} = \frac{\pi r}{4}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

It follow that the angle through which the cube can turn on one side is $\pi/4$. Similarly on the other side it can also turn through $\pi/4$. Therefore, the total angle through which the cube can rock (turn) without sliding is $\pi/4 + \pi/4 = \pi/2$.

Example4:- A solid homogenous hemisphere of radius r has a solid right cone of the same substance constricted on its base, the hemisphere rest on the convex side of a fixed sphere of radius R, the axis of the cone being vertical. Show that the greatest height of the cone consistent with stability for a small rolling displacement, is $\frac{r}{r+R}\left[\sqrt{(3R+r)(R-r)} - 2r\right]$

Solution:- As shown in the figure, suppose that G_1 and G_2 are the C.G. of the hemisphere and the cone respectively and that G, the C.G. of these combined bodies. Suppose that $NB = l$, $h = CG$. Given that $CN = r$, $AC = R$



Regarding C as origin and CG as y-axis and applying the formula $\bar{y}=\frac{w_1y_1+w_2y_2}{w_1+w_2}$ we get

$$h=\frac{\frac{\pi}{3}r^2l\left(r+\frac{l}{4}\right)+\frac{2\pi r}{3}\left(r-\frac{3r}{8}\right)}{\frac{\pi}{3}r^2l+\frac{2}{3}\pi r^3}$$

$NG_2 = l/4$, $NG_1 = 3r/8$

$$=\frac{l\left(r+\frac{1}{4}\right)+\frac{5}{4}r^2}{l+2r}$$

The equilibrium is stable if

$$\frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

Or if $\frac{l+2r}{l\left(r+\frac{l}{4}\right)+\frac{5}{4}r^2}>\frac{R+r}{Rr}$

Or if $l^2(r+R)+4r^2l+5r^3-3r^2R<0$ (1)

If l_1 and l_2 are the roots of the equation

$l^2(r+R)+4r^2l+5r^3-3r^2R=0$, then

$$l_1=\frac{-2r^2-r\{(r+3R)(R-r)\}^{1/2}}{r+R}$$

$$l_2=\frac{-2r^2+r\{(r+3r)(R-r)\}^{1/2}}{r+R}$$

In order to satisfy the inequality (1), l should be such that $l_1<l<l_2$.

But l_1 is a negative value and l cannot be negative, so $0<l<l_2$.

i.e. $$l<\frac{-2r^2+r\{(r+3R)(R-r)\}^{1/2}}{r+R}$$

$$= \frac{r}{r+R}\left[\left\{(r+3R)(R-r)\right\}^{1/2} - 2r\right]$$

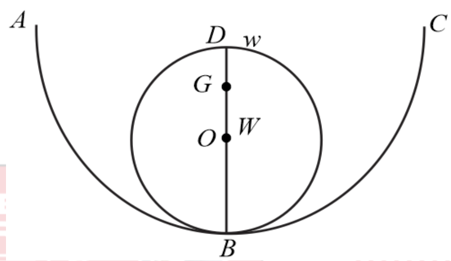
Hence for stability, $l < \frac{r}{r+R}\left[\left\{(r+3R)(R-r)\right\}^{1/2} - 2r\right]$

So, the greatest value of l consistent with stability of the equilibrium is

$\frac{r}{r+R}\left[\sqrt{\left\{(3R+r)(R-r)\right\}} - 2r\right]$.

Example5:- A sphere of weight W and radius a lies within a fixed spherical shell of radius b and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$ and that if $\frac{W}{w} = \frac{b-2a}{a}$, then the equilibrium is essentially stable.

Solution:- As shown in the fig. there is a spherical shell ABC within which there is a sphere with vertical diameter BD. A weight w is put at D, W is the weight of the sphere.



Radius of the shell $= b$, radius of the sphere $= OB = a$.

Suppose that G is the C.G. of the system containing weight w at D and the sphere.

$$h = BG = \frac{wBD + WBO}{w+W} = \frac{2aw + aW}{w+W} = \left(\frac{2w+W}{w+W}\right)a$$

Applying Art. 40, the equilibrium is stable

If $\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$

Or if $\frac{w+W}{(2w+W)a} > \frac{1}{a} - \frac{1}{b}$

Or if $(w+W)b > (b-a)(2w+W)$

Or if $\frac{W}{w} > \frac{b-2a}{a}$

Thus the equilibrium is stable if

$$\frac{W}{w} > \frac{b-2a}{a}$$

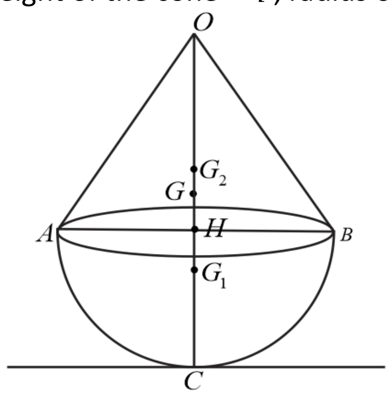
If $\frac{W}{w} = \frac{b-2a}{a}$ the equilibrium is stable if (Art. 40)

$$b > 2a \qquad \Rightarrow \quad b - 2a > 0$$

Concluding that $\frac{W}{w}$ is positive, which is true, therefore, if $\frac{W}{w} = \frac{b-2a}{a}$ then the equilibrium is essentially stable.

Example6:- A body consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table; show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.

Solution:- Suppose that the height of the cone $= l$; radius of the hemisphere $= r$.



The C.G.s of the hemisphere and cone are G_1 and G_2 respectively, and G, the C.G. of the combined system.

$HG_1 = \frac{3r}{8}$, $HG_2 = \frac{l}{4}$ If $h = HG$ then using the formula

$$\bar{y} = \frac{w_1 y_1 + w_2 y_2}{w_1 + w_2}$$

$$h = \frac{\frac{1}{3}\pi r^2 l\left(r + \frac{1}{4}\right) + \frac{2}{3}\pi r^3\left(r - \frac{3r}{8}\right)}{\frac{1}{3}\pi r^2 l + \frac{2}{3}\pi r^3}$$

$$= \frac{l\left(r + \frac{1}{4}\right) + \frac{5}{4}r^2}{1 + 2r}$$

Here $r = r,\ R = \infty$ the equilibrium is stable if $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

$$\Rightarrow \quad \frac{l + 2r}{l\left(r + \frac{1}{4}\right) + \frac{5}{4}r^2} > \frac{1}{r}$$

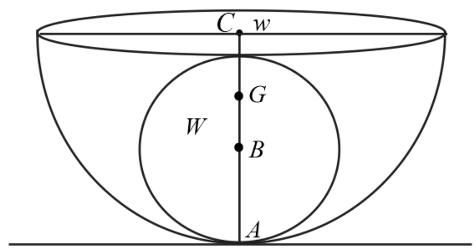
$$\Rightarrow \quad r(l + 2r) > l\left(r + \frac{1}{4}\right) + \frac{5}{4}r^2$$

$$\Rightarrow \quad l < r\sqrt{3}$$

Hence the greatest height of the cone. For stable equilibrium is $\sqrt{3}$ times the radius of the hemisphere.

Example7:- A solid sphere rests inside a fixed rough hemisphere bowl of twice its radius. Show that however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

Solution:- Suppose that B and C are the centre of the sphere and the hemispherical bowl respectively. $W=$ weight of the sphere; $w=$ weight attached to C



$AC=2AB=2r$ (say), so $R=2r$

If h is the C.G. of the system above the point of contact A,

$$h=\frac{Wr+w2r}{W+w}=\left(\frac{W+2w}{W+w}\right)r\ .$$

The equilibrium will be stable (using Theorem 3)

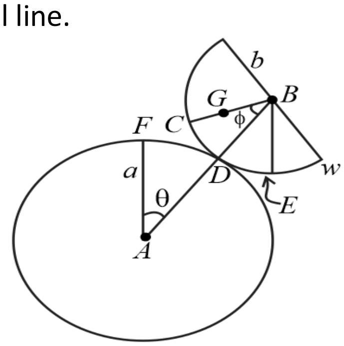
If $\frac{1}{h}>\frac{1}{r}-\frac{1}{R}=\frac{1}{r}-\frac{1}{2r}=\frac{1}{2r}$ (In usual notation)

$\Rightarrow \quad \frac{W+w}{(W+2w)r}>\frac{1}{2r} \quad \Rightarrow W>0$, which is true

Hence the equilibrium is stable.

Example8:- A thin hemispherical bowl of radius b and weight W rests in equilibrium on the highest point of a fixed sphere of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight w, show that the equilibrium is not unstable unless $w<W\left(\frac{a-b}{2b}\right)$

Solution:- The equilibrium position of the system is shown in the fig. A and B are the centres of the lower sphere and bowl respectively. Here bowl is slightly displaced. Initially, points C and F were coinciding. In the tilted position the weight w moves (slides) from C to its lowest position in which BE must be a vertical line.



If $\angle FAD = \theta$, $\angle CBD = \phi$, $BG = b/2$.

Since are $FD = arc\ CD$

$$\Rightarrow \quad a\theta = b\phi$$

The equilibrium will be stable if the moment of W acting at G about D the moment of w about D.

i.e. $\quad W\left\{\frac{b}{2}\sin(\theta+\phi) - b\sin\theta\right\} > wb\sin\theta$

$$\Rightarrow \quad W\left\{\frac{1}{2}\sin(1+a/b)\theta - \sin\theta\right\} > w\sin\theta$$

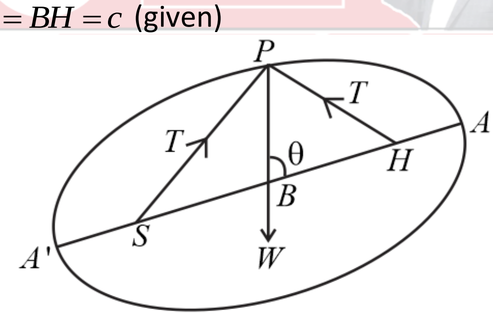
Since θ is very small, so using the property $\sin\theta = \theta$.

$$W\left\{\frac{1}{2}(a+b)\theta - b\theta\right\} > wb\theta$$

$$\Rightarrow \quad w < W\left(\frac{a-b}{2b}\right)$$

Example9:- A rod SH, of length $2c$ and whose centre of gravity G is at a distance d from its centre, has a string, of length $2c\sec\alpha$, tied to its two ends and the string is then slung over a small smooth peg. P; find the position of equilibrium and show that the position which is not vertical is unstable.

Solution:- Given that $PS + PH = 2c\sec\alpha$. Here B is the middle point of the rod SH and G, it $C.G.$ such that $BG = d$, $BS = BH = c$ (given)



We know that the sum of the focal distances of any point P on the ellipse is constant and is equal to the lengths of its major axis. So the peg P will be on the ellipse whose foci are S and H. Regarding AA' as major axis with centre (origin) B, if the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

Then $2a = PS + PH = 2c\sec\alpha$

$\Rightarrow \quad a = c\sec\alpha$ and $ac = BH = c \quad \Rightarrow ae = c$

But $b^2 = a^2(1-e^2) = c^2\sec^2\alpha - c^2 = c^2\tan^2\alpha$

$\Rightarrow \quad b = c\tan\alpha$

Using these values of a and b, the equation (1) becomes $\dfrac{x^2}{c^2\sec^2\alpha} + \dfrac{y^2}{c^2\tan^2\alpha} = 1$

$\Rightarrow \qquad x^2 \sin^2\alpha + y^2 = c^2 \tan^2\alpha$ referred to B as origin and $A'A$ as x-axis

Shifting the origin to the point $G(d,0)$ we get

$$(x+d)^2 \sin^2\alpha + y^2 = c^2\tan^2\alpha$$

Changing to polar coordinates, $(r\cos\theta + d)^2 \sin^2\alpha + r^2\sin^2\theta = c^2\tan^2\alpha$, where G is the pole and GH is initial line and $GP = r$, $\angle PGH = \theta$

$$\Rightarrow \quad r^2\cos^2\theta\cos^2\alpha - 2rd\sin^2\alpha\cos\theta + \left(c^2\tan^2\alpha - r^2 - d^2\sin^2\alpha\right) = 0 \qquad (2)$$

If we find the value of θ for which r is a maximum or minimum, and take the corresponding point P of the ellipse for the position of the peg. And set the rod to make PG vertical, we shall have the slant position of equilibrium.

The equation (2) is quadratic in $\cos\theta$.

$$\cos\theta = \frac{2rd\sin^2\alpha \pm \left[4r^2d^2\sin^4\alpha - 4r^2\cos^2\alpha\left(c^2\tan^2\alpha - r^2 - d^2\sin^2\alpha\right)\right]^{1/2}}{2r^2\cos^2\alpha}$$

$$= \frac{d\sin^2\alpha \pm \left[d^2\sin^4\alpha + \cos^2\alpha\left(d^2\sin^2\alpha + r^2 - c^2\tan^2\alpha\right)\right]^{1/2}}{r\cos^2\alpha}$$

But $\quad d^2\sin^2\alpha + \cos^2\alpha\left(d^2\sin\alpha + r^2 - c^2\tan^2\alpha\right)$

$$= d^2\sin\alpha\left(\sin^2\alpha + \cos^2\alpha\right) + r^2\cos^2\alpha - c^2\sin\alpha$$

$$= d^2\sin^2\alpha + r^2\cos^2\alpha - c^2\sin^2\alpha$$

$$r^2 = \cos^2\alpha\left(d^2 - c^2\right)\sin^2\alpha$$

So, $\quad \cos\theta = \dfrac{d\sin^2\alpha \pm \sqrt{r^2\cos^2\alpha + \left(d^2 - c^2\right)\sin^2\alpha}}{r\cos^2\alpha}$

The value of θ is real if $r^2\cos^2\alpha + \left(d^2 - c^2\right)\sin^2\alpha > 0$

$$\Rightarrow \quad r^2 > \left(c^2 - d^2\right)\tan^2\alpha$$

Since r cannot be negative, so

$$r^2 > \sqrt{\left(c^2 - d^2\right)}\tan\alpha$$

Therefore, the least value of r is $\sqrt{c^2 - d^2}\,\tan\alpha$ and when $r = \sqrt{c^2 - d^2}\,\tan\alpha$, then

$$\cos\theta = \frac{d\tan\alpha}{\sqrt{c^2 - d^2}}$$

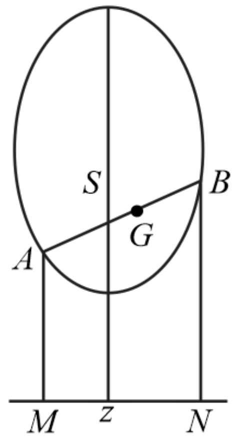
Since in this case r is minimum the C.G. of the rod is at its minimum depth below the peg. (Vertically) and therefore, the C.G. is at the maximum height above the horizontal, and so the equilibrium is unstable.

The order two positions of equilibrium are when P is at A or A' and the rod SH is then clearly adjusted to vertical.

Example10:- A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse, if the length of the beam be not less than the lotus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.

Solution:- Suppose that S is the focus and MN, the directrix of the ellipse, AB is the beam. Referring to S as pole, the equation to the ellipse is $\frac{l}{r}=1+e\cos\theta$ where, sz is the initial line.

By the definition of the ellipse.



$$As = eAM, BS = eBN$$

Hence,

$z =$ height of C.G. of rod AB above MN

$$=\frac{1}{2}(AM+BN)$$

$$=\frac{1}{2e}(AS+BS)$$

The equilibrium is stable if z is minimum

$\Rightarrow$ $AS+BS$ is minimum

$\Rightarrow$ Point A, B, S all lie on the same straight line

$\Rightarrow$ Beam AB must pass through the focus.

Thus when beam AB passes through the focus, the equilibrium is stable.

If $AB = AS+BS$

$$=\frac{l}{1+e\cos\theta}+\frac{l}{1+e\cos(\pi-\theta)}$$

$$=\frac{l}{1+e\cos\theta}+\frac{l}{1-e\cos\theta}$$

$$=\frac{2l}{1-e^2\cos^2\theta}$$

AB is minimum when $\cos\theta=0$, i.e. $\theta=\pi/2$ so when $\theta=\pi/2$, $AB=$ length of the latus rectum $=2l$

Hence, the minimum length of the rode $=$ length of latus rectum of the ellipse.

Example11:- A lamina in the form of a cycloid whose generating circle is of radius a, rests on the top of another cycloid whose generating circle is of radius b, their vertices being in contact and their axes vertical. If h be the height of C.G. of upper cycloid above its vertex, show that the equilibrium is stable only if $h < \frac{4ab}{a+b}$, and is unstable if $h \geq \frac{4ab}{a+b}$

Solution:- Cycloid $S = 4a\sin\psi$ (upper)

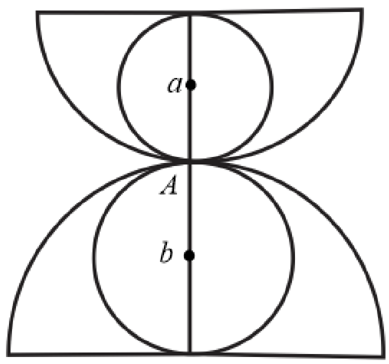
$$\rho = \frac{ds}{d\psi} = 4a\cos\psi$$

At vertex A (point of common contact)

$\psi = 0,\ \rho_1 = 4a$ at A.

Similarly, for lower cycloid ρ_2 at A is $4b$

Using Art. 41 deductions, the equilibrium is stable if



$\frac{1}{h} > \frac{1}{4a} + \frac{1}{4b}$ or $h < \frac{4ab}{a+b}$,and unstable if $h > \frac{4ab}{a+b}$.

Now, $$\frac{d}{ds}\left(\frac{1}{\rho_1}\right) + \frac{d}{ds}\left(\frac{1}{\rho_2}\right) = ..\left(\frac{1}{\rho_1^2}\frac{d\rho_1}{ds} + \frac{1}{p_2^2}\frac{d\rho_2}{ds}\right)$$

$$= \frac{1}{\rho_1^2}.4a\sin\psi + \frac{1}{\rho_2^2}.4b\sin\psi$$

$$= 4\left(\frac{a}{\rho_1^2} + \frac{b}{\rho_2^2}\right)\sin\psi$$

$= 0$, where $\psi = 0$, which gives no information.

Further, $$\frac{d^2}{ds^2}\left(\frac{1}{\rho_1}\right) + \frac{d^2}{ds^2}\left(\frac{1}{\rho_2^2}\right) = \frac{d}{ds}\left(-\frac{1}{\rho_1^2}\frac{d\rho_1}{ds}\right) + \frac{d}{ds}\left(-\frac{1}{\rho_2^2}\frac{d\rho_2}{ds}\right)$$

$$= \frac{d}{ds}\left(\frac{4a}{\rho_1^2}\sin\psi\right) + \frac{d}{ds}\left(\frac{4b}{\rho_2^2}\sin\psi\right)$$

$$= -2\times\frac{4a}{\rho_1^3}\frac{d\rho_1}{ds}\sin\psi + \frac{4a}{\rho_1^2}\cos\psi\frac{d\psi}{ds}$$

$$+4a\left(-\frac{2}{\rho_2^3}\frac{d\rho_2}{ds}\sin\psi + \frac{1}{\rho_2^2}\cos\psi\frac{d\psi}{ds}\right)$$

$$= \frac{8a \times 4a}{\rho_1^3} \sin^2 \psi + \frac{4a}{\rho_1^2} \cos\psi \frac{\sec\psi}{4a} + 4b \left(\frac{8b}{\rho_2^3} \sin^2 \psi + \frac{\cos\psi}{\rho_2^2} . \frac{1}{4b\cos\psi} \right)$$

$$= \frac{32a^2}{\rho_1^3} \sin^2 \psi + \frac{1}{\rho_1^2} + \frac{32b^2}{\rho_2^3} - \sin^2 \psi + \frac{1}{\rho_2^2}$$

So, $$\frac{d^2}{ds^2}\left(\frac{1}{\rho_1}\right) + \frac{d^2}{ds^2}\left(\frac{1}{\rho_2}\right) + \frac{(\rho_1+\rho_2)(\rho_2+2\rho_1)}{\rho_1^2 \rho_2^3}$$

$$= \frac{32a^2}{\rho_1^3} \sin^2 \psi + \frac{1}{\rho_1^2} + \frac{32a^2}{\rho_2^3} \sin^2 \psi + \frac{1}{\rho_2^2} + \frac{(\rho_1+\rho_2)(r_2+2\rho_1)}{\rho_1^2 \rho_2^3}$$

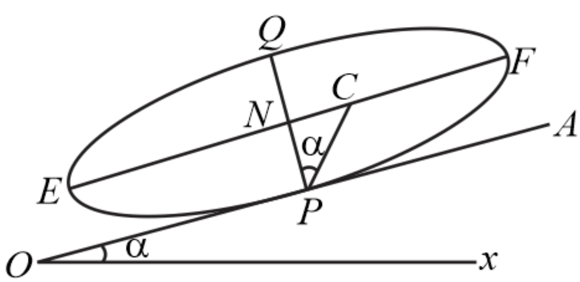
$$= \frac{1}{(4a)^2} + \frac{1}{(ab)^2} + \frac{4(a+b)(b+2a)4}{(4a)^2(ab)^2} \text{ when } \psi = 0 > 0.$$

Showing that the equilibrium is unstable.

Therefore, the equilibrium is unstable when $h \geq \frac{4ab}{a+b}$, stable when $h < \frac{4ab}{a+b}$

Example12:- An elliptic cylinder is placed with its axis horizontal on a rough plane inclined to the horizontal at an angle less than the angle of friction. Prove that the cylinder can not rest if the inclination of the plane exceeds $\sin^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$ and if the inclination is equal to $\sin^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$ the equilibrium is natural to first approximation.

Solution:- In the fig. the vertical cross-section of the inclined plane and the elliptical cylinder have been shown. OA is the inclined plane, ox is horizontal. $\angle Aox = \alpha$, the axis of cylinder is perpendicular to the plane of the paper. CP is vertical and NP is normal P is the point of contact. Regarding EF as the major axis and C as centre, the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Let the coordinate of P be $(a\cos\theta, b\sin\theta)$. Equation of the normal NP is

$ax = \sec\theta - by\cos ec\theta = a^2 - b^2$

Slope of $NP = m_1 = \frac{a}{b}\tan\theta$

Slope of $CP = m_2 = \frac{b}{a}\tan\theta$, α is the angle between PC and PN, so

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{a}{b}\tan\theta - \frac{b}{c}\tan\theta}{1 + \frac{a}{b}\tan\theta \frac{b}{a}\tan\theta}$$

$$= \frac{a^2 - b^2}{ab}\frac{\tan\theta}{1 + \tan^2\theta} = \frac{a^2 - b^2}{2ab}\sin 2\theta$$

$\Rightarrow$ $$\sin 2\theta = \frac{2ab}{a^2 - b^2}\tan\alpha$$

Since, the value of θ is real so, $|\sin 2\theta| \le 1$

$\Rightarrow$ $\frac{2ab\ \tan\alpha}{a^2 - b^2} \le 1$ as $\frac{ab\tan\alpha}{a^2 - b^2}$ is a $+ve$ quantity

$\Rightarrow$ $$\tan\alpha \le \frac{a^2 - b^2}{2ab} \Rightarrow \sin\alpha \le \frac{a^2 - b^2}{a^2 + b^2}$$

$\Rightarrow$ $\alpha \le \sin^{-1}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)$, which is the condition under which is the cylinder rests, Or. In other words, the cylinder cannot rest if $\alpha > \sin^{-1}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)$

Now consider the case when $\sin\alpha = \frac{a^2 - b^2}{a^2 + b^2}$

So that $\sin 2\theta = 1 \Rightarrow \theta = \pi/4$

$$h = CP = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{1/2}$$

$$= \left(\frac{a^2 + b^2}{2}\right)^{1/2} \text{ as } \theta = 45^\circ \qquad (1)$$

The parameter equation of the ellipse is

$x = a\cos\theta,\ \ y = b\sin\theta$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2\theta\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\operatorname{cosec}^2\theta\frac{1}{a\sin\theta} = -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

At $\theta = 45^\circ$

$$\frac{dy}{dx}=-\frac{b}{a},\frac{d^2ydx^2}{}=-2\sqrt{2}\,\frac{b}{a^2}\,.$$ None

$$\rho=\frac{\left\{1+\left(dy/dx\right)^2\right\}^{3/2}}{d^2y/dx^2}=-\frac{\left(1+b^2/a^2\right)^{3/2}}{2\sqrt{2}\left(b/a^2\right)}=-\frac{\left(a^2+b^2\right)^{3/2}}{2\sqrt{2}\,ab}$$

The equilibrium is natural if

$$\frac{1}{h}=\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sec\alpha$$

Here $\rho_1=$ radius of curvature of the inclined plane $=\infty$ and $\rho_2=\rho$. Hence

$$\frac{1}{h}=\frac{1}{\rho}\sec\alpha \;\Rightarrow h=\rho\cos\alpha$$

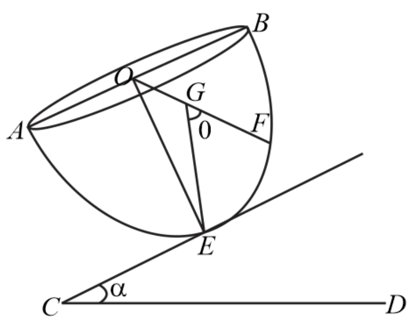
$$\Rightarrow \qquad h=\frac{\left(a^2+b^2\right)^{3/2}}{2\sqrt{2}\,ab}\,\frac{2ab}{a^2+b^2}=\left(\frac{a^2+b^2}{2}\right)^{1/2}$$

Which is true by the virtue of (1)

Hence, the equilibrium is natural.

Example13:- A solid hemisphere rests on a plane inclined to the horizon at an angle, $\alpha<\sin^{-1}(3/8)$ and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.

Solution:- CD is horizon CE is the inclined plane, $\angle ECD=\alpha=\angle OEG$. As shown in the figure. O I s the centre of the hemisphere, E is the point of contact, OE is normal to the inclined plane, G is the C.G. of the solid hemisphere such that $OG=3r/8$, where r = radius $=OE, GE$ is vertical.



Let $GE=h$. In ΔOEG, $\angle EGF=\theta$

$$\frac{EG}{\sin EOG}=\frac{OG}{\sin OEG}=\frac{OE}{\sin OGE}$$

$$\Rightarrow \qquad \frac{h}{\sin(\theta-\alpha)}=\frac{3r/8}{\sin\alpha}=\frac{r}{\sin\theta}$$

$$\Rightarrow \qquad h=\frac{3r\sin(\theta-\alpha)}{8\sin\alpha} \qquad (1)$$

For stable equilibrium, $h < \dfrac{\rho_1 \rho_2 \cos\alpha}{\rho_1 + \rho_2}$ or $\dfrac{1}{h} > \left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}\right)\sec\alpha$

Here $\rho_1 = r,\ \rho_2 = \infty$, so $h < r\cos\alpha$

Putting the value of h we have $\dfrac{3r\sin(\theta-\alpha)}{8\sin\alpha}, r\cos\alpha$

$$\Rightarrow \quad 3\sin(\theta-\alpha) < 8\sin\alpha\cos\alpha$$

$$\Rightarrow \quad 3(\sin\theta\cos\alpha - \cos\theta\sin\alpha) < 8\sin\alpha\cos\alpha \qquad (2)$$

From (1)

$$\sin\theta = \frac{8}{3}\sin\alpha, \cos\theta = \left(1 - \frac{64}{9}\sin^2\alpha\right)^{1/2} \qquad (3)$$

Putting these value of $\sin\theta$ and $\cos\theta$ in inequality (2), we have

$$3\left\{\frac{8}{3}\sin\alpha\cos\alpha - \left(1 - \frac{64}{9}\sin\alpha\right)^{1/2}\sin\alpha\right\} < 8\sin\alpha\cos\alpha$$

$$\Rightarrow \quad \left(1 - \frac{64}{9}\sin^2\alpha\right)^{1/2}\sin\alpha > 0$$

Since $\sin\alpha \neq 0$, so $1 - \dfrac{64}{9}\sin^2\alpha > 0 \ \Rightarrow \sin\alpha < \dfrac{3}{8}$

$$\Rightarrow \quad \alpha < \sin^{1}(3/8) \qquad (4)$$

For its truth, we see in equation (3)

$$\sin\theta = \frac{8}{3}\sin\alpha$$

For real value of θ, $\sin\theta < 1$

$$\Rightarrow \quad \frac{8}{3}\sin\alpha < 1 \Rightarrow \sin\alpha < \frac{8}{3}$$

So, result in inequality (4) is true. Therefore, the equilibrium is stable.

Example14:- A solid frustum of paraboloid of revolution, of height and latus rectum $4a$ rests with its vertex on the vertex of paraboloid of revolution whose latus rectum is $4b$. Show that the equilibrium is stable if $h < \dfrac{3ab}{(a+b)}$

Solution:- Regarding A as the origin the equation of the generating parabola of the paraboloid ACB is

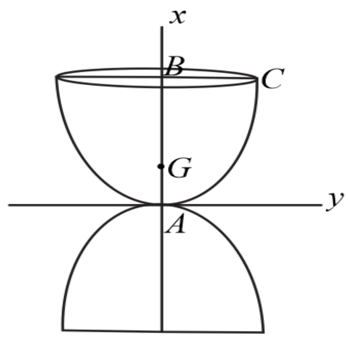
$$y^2 = 4ax$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2}.\frac{2a}{y} = -\frac{4a^2}{y^3}$$

$$\rho = \frac{\left\{1+\left(dy/dx\right)^2\right\}^{3/2}}{d^2y/dx^2} = \frac{\left(1+4a^2/y^2\right)^{3/2}}{-4a^2/y^3}$$

$$= -\frac{\left(y^2+4a^2\right)^{3/2}}{4a^2} = -\frac{\left(4ax+4a^2\right)^{1/3}}{4a^2}$$



The value of ρ at $A(0,0)$,

$$= -\frac{\left(0+4a^2\right)^{3/2}}{4a^2} = -2a$$

So, $\rho_1 = 2a$, since ρ (= radius of curvature) remains $+ve$

Similarly, $\rho_2 = 2b$,

Suppose that G is the C.G. of upper body, then

$$AG = \bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^h x\left(\pi y^2 dx\right)\rho}{\int_0^b \pi y^2 \rho dx}$$

$$= \frac{\int_0^b x^2 dx}{\int x\,dx} = \frac{2h}{3} = h_1 \text{ (say)}$$

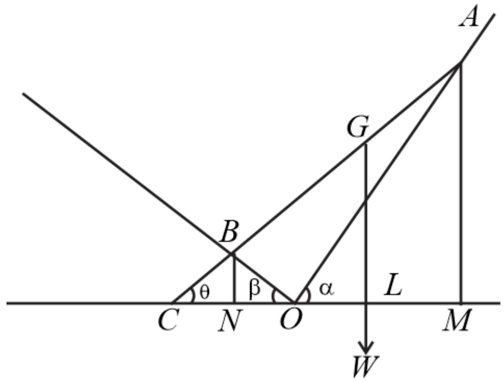
The equilibrium is stable if $h_1 < \dfrac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha$

$\Rightarrow \quad \dfrac{2h}{3} < \dfrac{2a\,2b\cos 0}{2a+2b}$, here $\alpha = 0$

$\Rightarrow \quad h < \dfrac{3ab}{a+b}$

Example15:- A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line, if the inclinations of the planes to the horizontal are α and $\beta(\alpha > \beta)$, show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by $\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$ and show that the beam is unstable in this position.

Solution:- As shown in the figure. Suppose that OA and OB are two inclined planes intersecting in a horizontal line through O and perpendicular to the plane of the paper. Let AB be the uniform rod resting on the planes and making $\angle\theta$ with the horizontal line MC so that $\angle AOM = \alpha$ m $\angle BOC = \beta$ $\angle BCM = \theta$. G is C.G. of the rod $AB = 2a$.



Applying sine formula in ΔOAB,

$$\frac{OA}{\sin OBA} = \frac{OB}{\sin OAB} = \frac{AB}{\sin AOB}$$

$$\Rightarrow \quad \frac{OA}{\sin(\theta+\beta)} = \frac{OV}{\sin(\alpha-\theta)} = \frac{2a}{\sin\{\pi-(\alpha+\beta)\}} = \frac{2a}{\sin(\alpha+\beta)}$$

$$\Rightarrow \quad OA = 2a\frac{\sin(\theta+\beta)}{\sin(\alpha+\beta)},\ OB = 2a\frac{\sin(\alpha-\theta)}{\sin(\alpha+\beta)}$$

$$z = GL = \frac{1}{2}(AM + Bn) = \frac{1}{2}\{OA\sin\alpha + OB\sin\beta\}$$

$$= \frac{2a}{2}\left\{\frac{\sin(\theta+\beta)}{\sin(\alpha+\beta)}\sin\alpha + \frac{\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta\right\}$$

$$= \frac{a}{\sin(\alpha+\beta)}\{\cos(\theta+\beta)\sin\alpha + \sin(\alpha-\theta)\sin\beta\}$$

$$\frac{dz}{d\theta} = \frac{a}{\sin(\alpha+\beta)}\{\cos(\theta+\beta)\sin\alpha - \cos(\alpha-\theta)\sin\beta\} \qquad (1)$$

$\frac{dz}{d\theta} = 0$ gives the position of equilibrium.

So, $\cos(\theta+\beta)\sin\alpha - \cos(\alpha-\theta)\sin\beta = 0$

$$\Rightarrow \quad (\cos\theta\cos\beta - \sin\theta\sin\beta)\sin\alpha - (\cos\alpha\cos\theta + \sin\alpha\sin\theta)\sin\beta = 0$$

$$\Rightarrow \quad 2\sin\theta\sin\alpha\sin\beta = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{1}{2}(\cot\beta = \cot\alpha) \qquad (2)$$

Which gives the position of equilibrium.

Differentiable (1) w.r.t θ

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha+\beta)}\left[-\sin(\theta+\beta)\sin\alpha - \sin(\alpha-\theta)\sin\beta\right]$$

$$= -\frac{a}{\sin(\alpha+\beta)}\left[\sin(\theta+\beta)\sin\alpha + \sin(\alpha-\theta)\sin\beta\right]$$

$$= -\frac{a}{\sin(\alpha+\beta)}[\sin\theta\cos\beta\sin\alpha + \cos\theta\sin\beta\sin\alpha +$$

$$\sin\alpha\cos\theta\sin\beta - \cos\alpha\sin\theta\sin\beta]$$

$$= -\frac{a}{\sin(\alpha+\beta)}\left[2\sin\alpha\sin\beta\cos\theta + (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\cos\theta\right]$$

$$= -\frac{2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[1 + \frac{1}{2}(\cot\beta - \cot\alpha)\tan\theta\right]$$

$$= -\frac{2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left(1+\tan^2\theta\right) \qquad \text{using (2)}$$

$$= -\frac{2a\sin\alpha\sin\beta\sec^2\theta}{\sin(\alpha+\beta)}$$

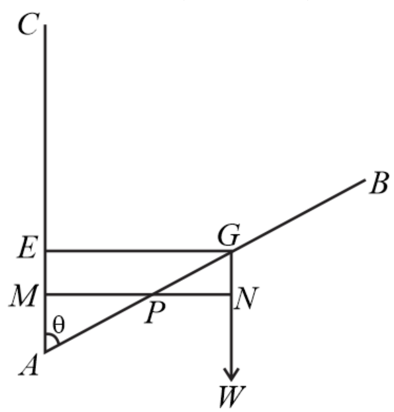
$= a$ negative quantity.

Since α, β, θ all are the acute angle and $\alpha + \beta < \pi$.

So, z is maximum, therefore the equilibrium is unstable.

Example16:- A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable.

Solution:- Suppose that AC is the wall and AB, the smooth rod with G as its C.G. and P s a peg whose distance from the wall is b. Let the rod resting on P make an angle θ with the vertical wall. Here the peg P is fixed and MPN is a horizontal fixed line. $AB = 2a$, $AG = a$. Let z be the height of C.G. of the rod above line MPN. So $(GN \,||\, CA)$



$z = GN = AE - AM$, as $EG \,||\, MN$

$= a\cos\theta - b\cot\theta$

$= -a\sin\theta + b\cos ec^2\theta$

$$\frac{d^2z}{dx^2} = -a\cos\theta - 2b\cos ec^2\theta\cot\theta$$
$$= -\left(a\cos\theta + 2b\cos es^2\theta\cot\theta\right)$$
$\frac{dz}{d\theta} = 0$ will give the position of equilibrium.

So, $\quad -a\sin\theta + b\cos ec^2\theta = 0$

$\Rightarrow \quad \sin\theta = (b/a)^{1/3}$

Which gives the position of equilibrium.

Now $\frac{d^2z}{dx^2} = -\left\{\frac{\left(a^{2/3}-b^{2/3}\right)^{1/2}}{a^{1/3}}a + 2b\left(\frac{a}{b}\right)^{2/3}\frac{\left(a^{2/3}-b^{2/3}\right)^{1/2}}{b^{1/3}}\right\}$

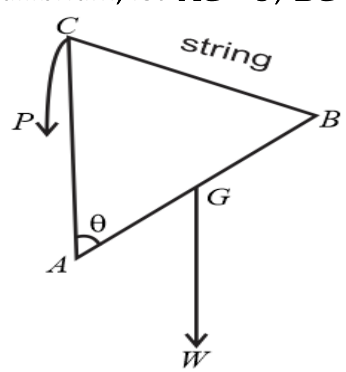
$$= -3a^{2/3}\left(a^{2/3}-b^{2/3}\right)^{1/2}$$

Since $a > b$, so $d^2z/d\theta^2$ is a negative quantity.

Hence z is maxim $\Rightarrow$ the equilibrium is unstable for $\sin\theta = (b/a)^{1/3}$

Example17:- A uniform heavy bar AB can move freely in a vertical plane about a hinges at A, and has a string attached to its end B which after passing over a small pulley at a point C vertically above A is attached to a weight. Show that the position of equilibrium in which AB is inclined to the vertical is an unstable one.

Solution:- Here AB is the uniform rod of weight W with G as its C.G. such that $AG = GB = a$, BCP is a string of length l such that $BC = r$ and $CP = l - r$, Also suppose that a weight P is suspended from the string at P in equilibrium, let $AC = s$, $BC = r$, $\angle BAC = \theta$. In ΔABC.



$$r^2 - 4a^2 + c^2 - 4ac\cos\theta \qquad (1)$$

$z =$ height of C.G. of weights P and W above A which is fixed.

$$= \frac{Wa\cos\theta + P(s-l+r)}{W+P}$$

Here r and θ are variables, $\frac{dz}{d\theta} = \frac{1}{W+P}\left[-Wa\sin\theta + \frac{dr}{d\theta}\right] \qquad (2)$

Differentiating I w.r.t $\theta, r\dfrac{dr}{d\theta} = 2ac\sin\theta$

So, $\dfrac{dz}{d\theta} = \dfrac{1}{W+P}\left[-Wa\sin\theta + P\dfrac{2ac}{r}\sin\theta\right]$ (3)

$\dfrac{dz}{d\theta} = 0$ gives the position of equilibrium,

$\Rightarrow \quad \left(-W + P\dfrac{2c}{r}\right)\sin\theta = 0$

$\Rightarrow \quad \sin\theta = 0$ or $r = \dfrac{2Pc}{W}$

If $\sin\theta = 0,\ \theta = 0 \Rightarrow$ the rod is in vertical position

Or $r = \dfrac{2Pc}{W}$, in this position of equilibrium, the rod is inclined.

Differentiating (3) w.r.t. θ, we have

$$\frac{d^2z}{d\theta^2} = \frac{1}{W+P}\left[-Wa\cos\theta + 2acP\left(-\frac{1}{r^2}\frac{dr}{d\theta}\sin\theta + \frac{1}{r}\cos\theta\right)\right]$$

$$= \frac{1}{W+P}\left[-Wa\cos\theta + 2acP\left(-\frac{1}{r^2}\frac{2ac}{r}\sin^2\theta + \frac{1}{r}\cos\theta\right)\right]$$

$$= \frac{1}{W+P}\left[-Wa\cos\theta - 4a^2c^2\frac{P}{r^3}\sin^2\theta + 2acP\frac{W}{2Pc}\cos\theta\right]$$

Putting the value of r

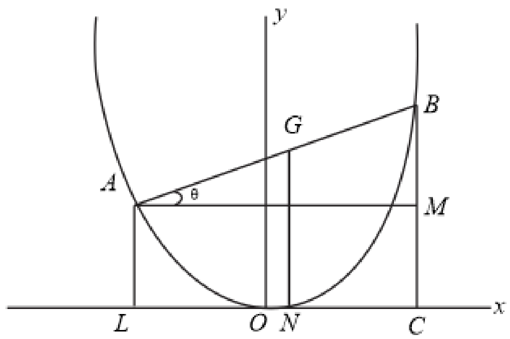
$$= -\frac{4a^2c^2P}{W+P}\frac{\sin^2\theta}{r^3}$$

$= a$ negative quantity as $\sin\theta > 0$ for $\theta < \pi$

So z is maximum when $r = 2Pc/W$, indicating that the equilibrium is unstable.

Example18:- A uniform rod of length $2l$, is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex down words, and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of equilibrium $bv\cos^2\theta = 2a/l$ and that, if these positions exist they are also stable. Show also that the positions in which the rod is horizontal are stable or rod is below or above the focus:

Solution:- Let AOB be a parabola whose equation is $x^2 = 4ay$, AB is the rod of tength $2l$ with its C.G. at G so that $AG = l$, AL, GN, BC are parallel to y-axis and AM is parallel to x-axis $\angle BAM = \theta$. Suppose that the coordinates of A and B are $2at_1, at_1^2$, and $\left(2at_1, at_2^2\right)$ respectively.



Here, $z =$ height of G above ox

$$= \frac{1}{2}(AL + BC)$$

$$= \frac{a}{2}\left(t_1^2 + t_2^2\right) \qquad (1)$$

In ΔABM,

$$\tan\theta = \frac{BM}{AM} = \frac{a\left(t_2^2 - t_1^2\right)}{2a\left(t_2 + t_1\right)} = \frac{1}{2}\left(t_2 + t_1\right)$$

$$\Rightarrow \quad t_2 - t_1 = 2\tan\theta \qquad (2)$$

And $\cos\theta = \dfrac{AM}{AB} = 2a\dfrac{\left(t_2 + t_1\right)}{2l} \Rightarrow t_2 + t_1 = \dfrac{l}{a}\cos\theta$ $\qquad (3)$

Squaring and adding (2) and (3)

$$2\left(t_1^2 + t_2^2\right) = 4\tan^2\theta + \frac{l^2}{a^2}\cos^2\theta \qquad (4)$$

So, $\quad z = \dfrac{a}{2\times 2}\left[4\tan^2\theta + \dfrac{l^2}{a^2}\cos^2\theta\right]$

$$= \frac{1}{4a}\left[4a^2 + \tan^2\theta + l^2\cos^2\theta\right]$$

$$\frac{dz}{d\theta} = \frac{1}{4a}\left[4a^2.2\tan\theta\sec^2\theta + l^2\left(-2\cos\theta\sin\theta\right)\right]$$

$$= \frac{1}{2a}\left[4a^2\tan\theta\sec^2\theta - l^2\sin\theta\cos\theta\right]$$

For the equilibrium, $dz/d\theta = 0$

$$\Rightarrow \quad 4a^2\tan\theta\sec^2\theta - l^2\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad \sin\theta\left(4a^2 - \sec^3\theta - l^2\cos\theta\right) = 0$$

Either $\sin\theta = 0$ or $\cos^2\theta = 2a/l$

$\sin\theta = 0 \Rightarrow \theta = 0$ gives the horizontal position.

But $\cos^2\theta = 2a/l$ gives the inclined position of equilibrium

$$\frac{d^2z}{d\theta^2} = \frac{1}{2a}\left[4a^2\left\{\sec^4\theta + 2\tan\theta\sec\theta\sec\theta\tan\theta\right\} - l^2\left(\cos^2\theta - \sin^2\theta\right)\right]$$

$$= \frac{1}{2a}\left[4a^2 - \sec^2\theta\left(\sec^2\theta + 2\tan^2\theta\right) + l^2 - 2l^2\cos^2\theta\right] \qquad (5)$$

$$= \frac{1}{2a}\left[4a^2 \frac{l}{2a}\left\{\frac{1}{2a} + \frac{2(l-2a)}{2a}\right\} + l^2 - 2l^2\frac{2a}{l}\right]$$

$$= \frac{2l^2}{a}\left(1 - \frac{2a}{l}\right) = \frac{2l^2}{a}\sin^2\theta$$

$=a$ Positive quantity

So, z is minimum when $\cos^2\theta = 2a/l$

Therefore, the equilibrium is stable

Now consider the case when $\theta = 0$, so putting $\theta = 0$ in (5).

$$\left.\frac{d^2z}{d\theta^2}\right|_{\theta=0} = \frac{1}{2a}\left[4a^2 + l^2 - 2l^2\right]$$

$$= \frac{1}{2a}\left(4a^2 - l^2\right)$$

$$= \frac{1}{2a}(2a+l)(2a-l)$$

$\left.\frac{d^2z}{d\theta^2}\right|_{\theta=0} > 0$ or < 0

According as, $2a - l > 0$ or $2a - l < 0$

According as semi-latus rectum $> l$ or $< l$

According as, the rod is below or above the focus in horizontal position.

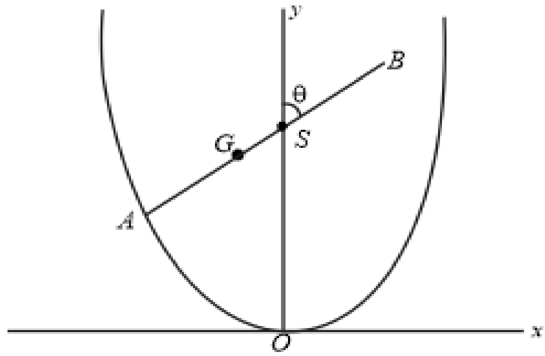
Therefore, the equilibrium is stable or unstable according as the rod is below or above the focus when it is in horizontal position.

Example19:- A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation. $\cos^4\left(\frac{\theta}{2}\right) = \frac{a}{2c}$ where $4a$ is the latus-rectum and $2c$, the length of the rod. Investigate also the stability of the equilibrium in this position.

Solution:- As shown in the fig. referred to the focus s as the pole and line SO as the initial line. The equation of the parabola is $\frac{2a}{r} = 1 + \cos\theta$

$$\Rightarrow \qquad r = a\cos^2(\theta/2) \qquad (1)$$

Where latus-rectum $= 4a$ rod $AB = 2c,\ G$ is its C.G. so $AG = c$



Let the polar coordinates of A be (r,θ) where $SA = r,\ \angle ASO = \theta,\ SG = r - c$

$z =$ the depth of G below S

(since S is fixed)

$$= GS\cos\theta = (r-c)\cos\theta$$

$$= \left[a\sec^2\left(\frac{\theta}{2}\right) - c\right]\left[2\cos^2\left(\frac{\theta}{2}\right) - 1\right]$$

$$= 2a + c - 2a\cos^2\left(\frac{\theta}{2}\right) - a\sec^2\left(\frac{\theta}{2}\right)$$

$$\frac{dz}{d\theta} = -2c\ 2\cos\left(\frac{\theta}{2}\right)\left[-\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\right] - a\,2\sec\left(\frac{\theta}{2}\right)\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right).\frac{1}{2}$$

$$= c\sin\theta - a\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)$$

If the equilibrium exists, then

$$\frac{dz}{d\theta} = 0$$

$$\Rightarrow \quad c\sin\theta - a\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{\theta}{2}\right)\left[2c\cos^4\left(\frac{\theta}{2}\right) - a\right] = 0$$

Either $\sin\theta/2 = 0 \ \Rightarrow\ \theta = 0$, the rod is vertical

Or $\quad \cos^4\left(\frac{\theta}{2}\right) = \frac{a}{2c}$

To test the nature of the equilibrium when $\cos^4\left(\frac{\theta}{2}\right) = \frac{a}{2c}$

$$\frac{d^2z}{d\theta^2} = c\cos\theta - \left[2\sec\left(\frac{\theta}{2}\right)\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right).\frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \sec^2\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\theta}{2}\right).\frac{1}{2}\right]$$

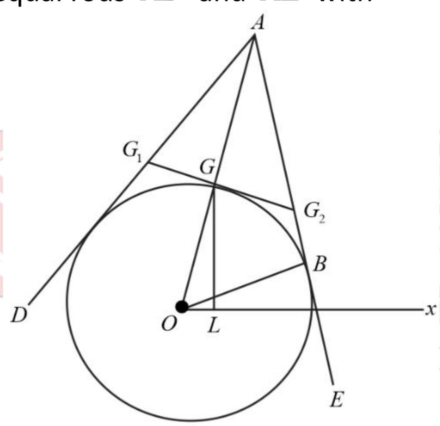
$$= c\cos\theta - \left(\sec^2\frac{\theta}{2}\tan^2\frac{\theta}{2} + \frac{1}{2}\sec^4\frac{\theta}{2}\right)a$$

$$= c\cos\theta - a\sec^4\frac{\theta}{2}\left(\sin^2\frac{\theta}{2}+\frac{1}{2}\right)$$
$$= c\cos\theta - a\frac{2c}{a}\left(\sin^2\frac{\theta}{2}+\frac{1}{2}\right)$$
$$= -\left\{c\left(1-\cos\theta\right)+2c\sin^2\frac{\theta}{2}\right\}$$
$= a$ negative quantity.

So, z is maximum i.e., the equilibrium is stable when $\cos^2\left(\theta/2\right)=a/2c$.

Example20:- Two equal uniform rods are firmly, joined at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that the are in a stable or unstable equilibrium according as the length of the rod is greater or less than $4r\cos ec\ \alpha$.

Solution:- Suppose that two equal rods AD and AE with



C.G.'s at G_1 and G_2 their lengths being $2b$, are resting in a vertical plane on a smooth sphere of radius r and of centre O. $\angle DAE=\alpha, AG_1=AG_2=b$. G is the C.G. of both rods. Line AO is the perpendicular bisector of the line G_1G_2 and also bisects the $\angle DAE$, ox is a horizontal line through O and let $\angle Aox=\theta$ in equilibrium.

$z =$ the height of C.G. G of both the rods above ox
$$= GL = OG\sin\theta$$
$$=\left(OA-AG\right)\sin\theta$$
$$=\left(OB\cos ec\frac{\alpha}{2}-AG_2\cos\frac{\alpha}{2}\right)\sin\theta,\ \angle ABO=90^\circ$$
$$=\left(r\cos ec\frac{\alpha}{2}-b\cos\frac{\alpha}{2}\right)\sin\theta$$
$$\frac{dz}{d\theta}=\left(r\cos es\frac{\alpha}{2}-b\cos\frac{\alpha}{2}\right)\cos\theta$$

In case of equilibrium $\frac{dz}{d\theta}=0$

$\Rightarrow \quad \cos\theta=0 \quad \Rightarrow \theta=\pi/2$

Which gives the position of equilibrium

To test the nature of equilibrium, $\frac{d^2z}{d\theta^2}=\left(r\cos ec\frac{\alpha}{2}-b\cos\frac{\alpha}{2}\right)(-\sin\theta)$

$$\frac{d^2z}{d\theta^2} \text{ at } \theta=\frac{\pi}{2}=-\left(r\cos ec\frac{\alpha}{2}-b\cos\frac{\alpha}{2}\right)$$

$$=\frac{1}{2}\cos\frac{\alpha}{2}\left(2b-4r\ \cos ec\,\alpha\right)$$

The equilibrium is stable or unstable if

$\frac{d^2z}{d\theta^2}$ at $\theta=\frac{\pi}{2}$ is positive or negative.

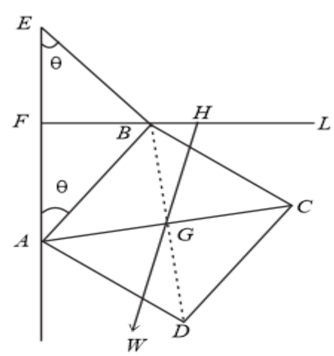
i.e. $\quad 2b-4r\ \cos es\ \alpha >$ or <0

i.e. $\quad 2b>$ or $<4r\ \cos ec\,\alpha$

i.e. $\quad$ length of the rod $>$ or $<4r\cos ec\,\alpha$.

Example21:- A square lamina rests with its plane perpendicular to smooth, one corner being attached to ta point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.

Solution:- Suppose that AFE is a wall and $ABCD$ is the square lamina inclined at an angle θ with the vertical such that $\angle BAF=\theta$. Let each side of the square be equal to $2b$. BE is the string of length $2b$. G is the C.G. of the lamina $ABCD$. FBH is a horizontal line $\angle ABG=45^\circ$ $\angle CBL=\theta$, so $\angle GBL=45^\circ+\theta$



Here $BG=2b\cos 45^\circ=\sqrt{2}\,b$

$z=$ the depth of G below E

$$=EF+HG$$

$$=2b\cos\theta+BG\sin\theta\left(\theta+45^\circ\right)$$

$$=b\left(2\cos\theta+\sqrt{2}\sin\left(\theta+45^\circ\right)\right)$$

$$=b\left(2\cos\theta+\sin\theta+\cos\theta\right)$$

$$= b(3\cos\theta + \sin\theta)$$

$$\frac{dz}{d\theta} = b(-3\sin\theta + \cos\theta)$$

For equilibrium, $\frac{dz}{d\theta} = 0$

$\Rightarrow \quad -3\sin\theta + \cos\theta = 0$

$\Rightarrow \quad \tan\theta = 1/3$

Now $\frac{d^2z}{d\theta} = b(-3\cos\theta - \sin\theta)$

$$= -b(-3\cos\theta - \sin\theta)$$

$$\frac{d^2z}{d\theta}\left(at\,\theta = \tan^{-1}1/3\right)$$

$$= -b\left(3\times\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{10}}\right) = -\sqrt{10}\,b$$

= negative quantity.

Which implies that z is maximum.

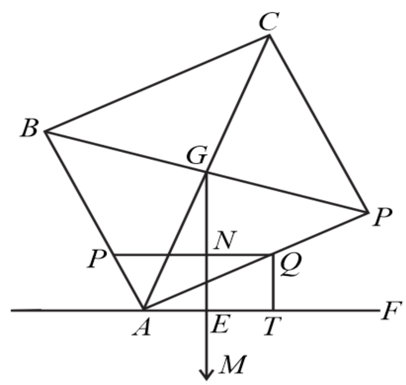
i.e. the depth of C.G. is maximum

i.e. the equilibrium is stable

Note:- The system is in equilibrium when $\theta = \tan^{-1}\left(\frac{1}{3}\right)$ but the above figure depicts the system tilted slightly from its equilibrium.

Example22:- A uniform square board of mass M is supported in a vertical plane on two smooth page at the same horizontal level. The distance between the page is a and the diagonal of the square is d where $d > 4a$. If one diagonal $i\varepsilon$ vertical and a mass m is attracted to its lower end, prove that the equilibrium is stable if $4am > M(d-4a)$

Solution:- Suppose that the uniform square board $ABCD$ is resting on the page P and Q distance a apart in vertical plane. Mass M at G (C.G. of the board) and a mass m at A are placed. $AC = d$ The system as shown in the fig. are is slightly displaced from its equilibrium. Let CA make angle θ with the horizontal in this position i.e. $\angle CAF = \theta$, AF and PQ are horizontal.



Now $GN = GE - NE = GA\sin\theta - AQ\sin\left(\theta - 45^\circ\right)$

$= \frac{d}{2}\sin\theta - PQ\cos\left(\theta-45^\circ\right)\sin\left(\theta-45^\circ\right)$ as $AQ = PQ\cos\left(\theta-45^\circ\right)$

$= \frac{d}{2}\sin\theta + \frac{a}{2}\cos 2\theta$

Again, $NE = QT = AQ\sin\left(\theta-45^\circ\right) = PQ = \cos\left(\theta-45^\circ\right)\sin\left(\theta-45^\circ\right)$

$= \frac{1}{2}a\sin\left(2\theta - \pi/2\right) = -\frac{a}{2}\cos 2\theta$

Let z be the height of the combined C.G. of M and m above PQ, then

$$z = \frac{MGN + m(-NE)}{M+m} = \frac{\frac{1}{2}M\left(d\sin\theta + a\cos 2\theta\right) + m\frac{a}{2}\cos 2\theta}{m+M}$$

$$= \frac{1}{2(m+M)}\left[M\left(d\sin\theta + \cos 2\theta\right) + ma\cos 2\theta\right]$$

$$= \frac{1}{2(m+N)}\left[Md\sin\theta + (m+M)a\cos 2\theta\right]$$

$$\frac{dz}{d\theta} = \frac{1}{2(m+M)}\left[Md\cos\theta - (m+M)a2\sin 2\theta\right]$$

For equilibrium, $\frac{dz}{d\theta} = 0$,

$\Rightarrow \quad Md\cos\theta - 2a(m+M)\sin 2\theta = 0$

$\Rightarrow \quad \cos\theta\left[Md - 4a(m+M)\sin\theta\right] = 0$

$\Rightarrow \quad \cos\theta = 0$ or $\sin\theta = \frac{Md}{4a(m+M)}$

$\Rightarrow \quad \theta = \frac{\pi}{2}$ or $\sin\theta = \frac{Md}{4a(m+M)}$

$\frac{d^2z}{d\theta^2}$ at $\theta = \frac{1}{2(m+M)}\left[-Md\sin - 4a(m+M)\right]$

$$\frac{d^2z}{d\theta^2} \text{ at } \theta=\frac{\pi}{2}=\frac{1}{2(m+M)}\left[-Md+4a(m+M)\right]$$

At $\theta=\pi/2$. The equilibrium is stable if

$$-Md+4a(m+M)>0$$

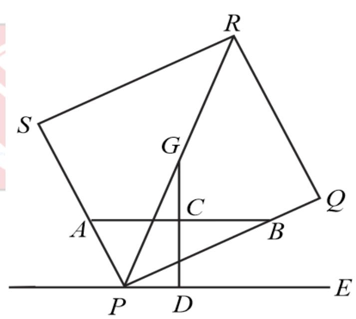
$\Rightarrow \quad 4am > Md-4aM = M(d-4a)$

$\Rightarrow \quad 4am > M(d-4a)$

In this case the diagonal AC is vertical.

Example23:- A square lamina rests in the vertical plane on two smooth page which are in the same horizontal line. Show that there is only one position of equilibrium unless the distance between the page is grater than one-quarter of the diagonal of the square, but that if this condition is satisfied, there may be there positions of equilibrium and that the symmetrical position will be stable, but the other two position of equilibrium will be unstable.

Solution:- Let the diagonal PR of the square lamina $PQRS$ resting on the page A and B distance a part inclined at angle θ to the horizontal. G is the C.G. of the lamina, PE is horizontal and $GD \perp PE$. $\angle RPF=\theta$.



Here AB is fixed.

$z=$ the height of the C.G. above AB line, $GC=GD-CFD$

$=PG\sin\theta - PB\sin(\theta-45^\circ)$

$$PR=d$$

$=\frac{d}{2}\sin\theta - AB\sin(\theta-45^\circ)\cos(\theta-45^\circ)$

$=\frac{d}{2}\sin\theta+\frac{a}{2}\cos 2\theta$

For equilibrium $\frac{dz}{d\theta}=0$,

$\Rightarrow \quad \frac{d}{2}\cos\theta - a\sin 2\theta=0$

$\Rightarrow \quad \cos\theta\left(\frac{d}{2}-2a\sin\theta\right)=0$

$\Rightarrow \quad \cos\theta=0$ or $\sin\theta=d/4a$

When $\cos\theta = 0$ i.e. $\theta = \pi/2$, diagonal RP is vertical.

$$\frac{d^2 z}{d\theta^2} = -\frac{d}{2}\sin\theta - 2a\cos 2\theta \qquad (1)$$

$$\frac{d^2 z}{d\theta^2}\left(at\ \theta = \pi/2\right) = -\frac{d}{2} + 2a = 2\left(a - d/4\right)$$

The equilibrium is

Stable when $a > \frac{d}{4} =$ one quarter of diagonal

Unstable when $a < \frac{d}{4}$,

Inclined position of the equilibrium when $\sin\theta = d/4a$ (from 1) gives us

$$\frac{d^2 z}{d\theta^2} = -\frac{d}{2}.\frac{d}{4a} - 2a\left(1 - 2\left(\frac{d}{4a}\right)^2\right)$$

$$= \frac{2}{a}\left(\left(\frac{d}{4}\right)^2 - a^2\right)$$

The equilibrium is unstable when $a > d/4$. Since for real values of θ, $|\sin\theta| < 1 \Rightarrow |d/4a| < 1 \Rightarrow d/4 < a$ so $(d/4) \not> a$ so, $(d/4) \not> a$.

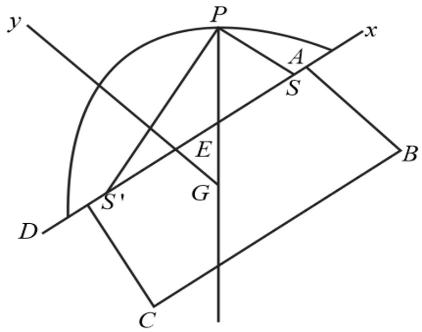
But, here s $\sin\theta = \sin(\pi - \theta)$ so there may be two positions when $\theta = \sin^{-1}(d/4a)$ or $\pi - \sin^{-1}(d/4a)$.

Let us summarize now that the equilibrium is unstable when $a < d/4$. One position only (diagonal is vertical) and when $a > d/4$, three position may arise;

1. Stable when $\theta = \pi/2$ i.e. diagonal is vertical
2. Unstable when $\theta = \sin^{-1}(d/4a)$, $\pi - \sin^{-1}(d/4a)$

Example24:- A rectangular picture hangs in a vertical position by means of a string. Of length l, which after passing over a smooth nail has its ends attached to two points symmetrically situated in the upper edge of the picture at a distance c apart. If the height of the picture be a, show that there is no position of equilibrium in which a side of the picture is inclined to the horizon if $la > c\sqrt{c^2 + a^2}$, whilst if $la < c\sqrt{c^2 + a^2}$ there are two such positions which are both stable. Show also that in the latter case the position in which the side is vertical is stable for some displacement and unstable for other displacements.

Solution:- Suppose that P is a fixed mail and $ABCD$ is



Rectangular picture hanging in a vertical plane by means of a string
$S'P + PS = l = 2a_1$ (say) (1)

E is the mid-point of the upper edge AD and $ES' = ES$ G is the C.G. of the picture $EG \perp AD$, PG is vertical. $SS' = c$, $AB = DC = a$. The equation (1) suggests that P lies on ellipse whose foci are S and S' and the length of whose semi-major axis is $a_1 = l/2$

Regarding AED as the x-axis and EY as the y-axis $(Ex \perp Ey)$, the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where b is the length of semi-major axis. So the coordinates of G are $(0, -a/2)$, the coordinates of P are $(a_1 \cos\theta, b\sin\theta)$. Using ellipse properties,

$$b^2 = a_1^2\left(1 - e^2\right) \quad (2)$$

Given $SS' = c \Rightarrow ES = c/2 = a_1 e$ making use of equation (2), we have

$$b^2 = \left(\frac{l}{2}\right)^2 \left\{1 - \left(\frac{c}{l}\right)^2\right\} = \frac{l^2 - c^2}{4} \Rightarrow \beta = \frac{\sqrt{t^2 - c^2}}{2}$$

Suppose that z is the depth of G below P, then

$$z = PG = \left\{\left(a_1 \cos\theta - 0\right)^2 + \left(b\sin\theta + a/2\right)^2\right\}^{1/2}$$

$$= \left\{a_1^2 \cos^2\theta + b^2 \sin^2\theta + ab\sin\theta + a^2/4\right\}^{1/2}$$

Suppose that

$$f(\theta) = z^2 = a_1^2 \cos^2\theta + b^2 \sin^2\theta + ab\sin\theta + \frac{a^2}{4}$$

Since z and z^2 are of the same nature, so we test $f(\theta)$

$$\frac{df(\theta)}{d\theta} = 2a_1^2 \cos\theta\sin\theta + 2b^2 \sin\theta\cos\theta + ab\cos\theta$$

Putting $\frac{df(\theta)}{d\theta} = 0$

$$\cos\theta\left\{2\left(b^2 - a_1^2\right)\sin\theta + ab\right\} = 0$$

Either $\cos\theta = 0 \Rightarrow \theta = \pi/2$,

or $\quad 2\left(b^2 - a_1^2\right)\sin\theta + ab = 0 \Rightarrow \sin\theta = \frac{ab}{2\left(a_1^2 - b^2\right)}$

$$\Rightarrow \quad \sin\theta = \frac{\frac{a}{2}\sqrt{l^2-c^2}}{2\left\{\left(\frac{l}{2}\right)^2-\left(\frac{l^2-c^2}{4}\right)\right\}} = \frac{a\sqrt{l^2-c^2}}{c^2} \qquad (3)$$

Since for real values of θ, $|\sin\theta|<1$

$$\Rightarrow \quad \frac{a\sqrt{l^2-c^2}}{c^2}<1 \;\Rightarrow a^2l^2<a^2c^2+c^4$$

$$\Rightarrow \quad al<c\sqrt{a^2+c^2}$$

Hence, if $al<c\sqrt{a^2+c^2}$, there may be three positions of equilibrium, namely, when

$$\theta=\sin^{-1}\left(\frac{a\sqrt{l^2-c^2}}{c^2}\right),\ \pi-\sin^{-1}\left(\frac{a\sqrt{l^2-c^2}}{c^2}\right),\ \frac{\pi}{2}$$

If $al>c\sqrt{a^2+c^2}$, there is only one position of equilibrium when $\theta=\pi/2$.

To test the nature of equilibrium

$$\frac{d^2f(\theta)}{d\theta^2}=2\left(b^2-a_1^2\right)\cos 2\theta-ab\sin\theta \qquad (4)$$

When $\theta=\pi/2$

$$\frac{d^2f(\theta)}{d\theta^2}=2\left(a_1^2-b^2\right)-ab$$

$$=2\left\{\frac{l^2}{4}-\left(\frac{l^2-c^2}{4}\right)\right\}-\frac{a}{2}\sqrt{l^2-c^2}$$

$$=\frac{1}{2}\left(c^2-a\sqrt{l^2-c^2}\right)$$

Hence, according to the theory, the equilibrium is at $\theta=\pi/2$

Stable when $c^2<a\sqrt{l^2-c^2}$ i.e. $al>c\sqrt{a^2+c^2}$

Unstable when $al<c\sqrt{a^2+c^2}$

Now consider the case when $\sin\theta=\dfrac{a\sqrt{l^2-c^2}}{c^2}$. The equation (4) can be written as

$$\frac{d^2f(\theta)}{d\theta^2}=2\left(b^2-a_1^2\right)\left(1-2\sin^2\theta\right)-ab\sin\theta$$

$$=\frac{1}{2c^2}\left[a^2\left(l^2-c^2\right)-c^4\right]$$

Since for real values of θ, $|\sin\theta|<1$

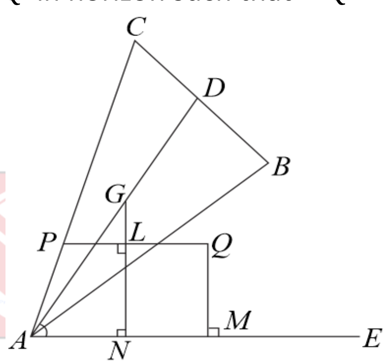
i.e. $\quad a\sqrt{l^2-c^2}<c^2$

So, when $a\sqrt{l^2-c^2}<c^2$ i.e. $al<c\sqrt{c^2+a^2}$, $\dfrac{d^2 f(\theta)}{d\theta^2}$ is negative, so the equilibrium is stable.

Significantly, the figure depicts the position of the system when displaced slightly from its equilibrium.

Example25:- A uniform isosceles triangular lamina ABC rests in equilibrium with its equal sides AB and AC in contact with two smooth page in the same horizontal line at a distance c apart. If the perpendicular AD upon BC is h show that there are three position of equilibrium, of which the one with AD vertical is stable and the other two are unstable if
$h<3c\cos ec\ A$; whilst if $h\geq 3c\cos ec\ A$ there is only one position of equilibrium which is unstable.

Solution:- Suppose that the uniform isosceles triangular lamina ABC rests in the vertical plane on two smooth pegs. P and Q in horizon such that $PQ=c$,



$AB=AC.\ AD\perp BC$. G is C.G. of the lamina, GN is vertical. As depicted in the figure, the situation is slightly displaced from its equilibrium and $\angle DAN=\theta$. The line AE is a horizontal line, $AG=2h/3$.

In ΔPAQ

$$\frac{PQ}{\sin A}=\frac{AQ}{\sin APQ}$$

$$\Rightarrow \quad \frac{C}{\sin A}=\frac{AQ}{\sin\left\{\pi-\left(\theta+\frac{A}{2}\right)\right\}}$$

$$\Rightarrow \quad AQ=\frac{\sin(\theta+A/2)}{\sin A}c.$$

Let z be the height of G above PQ then

$$z=GL=GN-QM$$

$$=AG\sin\theta-AQ\sin(\theta-A/2)$$

$$=\frac{2h}{3}\sin\theta-c\frac{\sin(\theta+A/2)}{\sin A}\sin(\theta-A/2)$$

$$= \frac{2h}{3}\sin\theta - \frac{c}{\sin A}\left(\sin^2\theta - \sin^2 A/2\right)$$

For equilibrium $dz/d\theta = 0$

So, $\frac{dz}{d\theta} = \frac{2h}{3}\cos\theta - \frac{2c}{\sin A}\sin\theta\cos\theta = 0$

Which gives $\cos\theta\left(\frac{2h}{3} - \frac{2c}{\sin A}\sin\theta\right) = 0$

Either $\cos\theta = 0 \Rightarrow \theta = \pi/2$ i.e. AD is vertical

Or $\sin\theta = \frac{h}{3c}\sin A$ (1)

To test the nature of equilibrium $\frac{d^2z}{d\theta^2} = -\frac{2h}{3}\sin\theta - \frac{2c}{\sin A}\left[\cos^2\theta - \sin^2\theta\right]$ (2)

When $\theta = \pi/2$

$$\frac{d^2z}{d\theta^2} = -\frac{2h}{3} - \frac{2c}{\sin A}(-1) = -\frac{2}{3}(h - 3c\cos ecA)$$

Hence, according to theory, the equilibrium is
Stable if $h < 3c \cos es A$ and unstable if $h > 3c \sec A$

Again, consider when $\sin\theta = \frac{h}{3c}\sin A$

Form the equation (2)

$$\frac{d^2z}{d\theta^2} = -\frac{2h}{3}\sin\theta + \frac{2c}{\sin A}\left(2\sin^2\theta - 1\right)$$

$$= -\frac{2h}{3}\left(\frac{h}{3c}\sin A\right) - \frac{2c}{\sin A} + \frac{4c}{\sin A}\left(\frac{h}{3c}\sin^2\right)^2$$

$$= \frac{2}{9c\sin A}\left(h^2\sin^2 A - 9c^2\right)$$

But for real values of θ, $|\sin\theta| < 1 \Rightarrow \frac{h}{3c}\sin A < 1$

$\Rightarrow \quad h < 3c \cos ec A$ (3)

Under the condition (3), $\frac{d^2z}{d\theta^2}$ is negative

But $\sin\theta = \sin(\pi - \theta)$

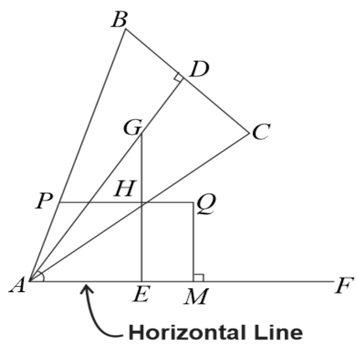
Hence the equilibrium is unstable in inclined position when

$\theta = \sin^{-1}\left(\frac{h}{3c}\sin A\right)$ or $\pi - \sin^{-1}\left(\frac{h}{3c}\sin A\right)$

Let us summarize that when $h < 3c \cos ecA$, the equilibrium is unstable, when $\theta = \sin^{-1}\left(\frac{h}{3c}\sin A\right)$ or $\pi - \sin^{-1}\left(\frac{h}{3c}\sin A\right)$ and stable when $\theta = \pi/2$; and when $h > 3c \cos ec A$, the equilibrium is unstable at $\theta = \pi/2$

Example26:- An isosceles triangle of angle 2α rests between two smooth pegs of the same level, distance $2c$ apart, if h be then distance of the C.G. from the vertex, and if $2c\sec\alpha < h < \dfrac{2c}{\sin\alpha\cos\alpha}$ then oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position in case when $h = \dfrac{2c}{\sin\alpha\cos\alpha}$

Solution:- Suppose that ABC is an isosceles triangle resting in vertical plane on two smooth pegs P and Q (in horizon) with its C.G. G at line AD bisector of the $\angle A$.



Here $PQ \,||\, AF$, $GE \perp AF$, $\angle BAC = 2\alpha$, $PQ = 2c$, $AG = h$, $AB = AC$. The figure shows the position of the system slightly displaced from its position of equilibrium. Let the line AD be inclined to AF at an angle θ. Suppose that z be the height of G above PQ, then

$z = GH = GE - QM$, as $HE = QM$

$$= h\sin\theta - AQ\sin(\theta-\alpha) \qquad (1)$$

For AQ, consider ΔAPQ

$$\frac{PQ}{\sin BAC} = \frac{AQ}{\sin APQ}$$

$$\Rightarrow \quad \frac{2c}{\sin 2\alpha} = \frac{AQ}{\sin(\pi-(\alpha+\theta))} = \frac{AQ}{\sin(\alpha+\theta)}$$

$$\Rightarrow \quad AQ = \frac{2c\sin(\alpha+\theta)}{\sin 2\alpha}$$

Making use of this value of AQ, we have from (1)

$$z = h\sin\theta - 2c\frac{\sin(\alpha+\theta)}{\sin 2\alpha}\sin(\theta-\alpha)$$

$$h = \sin\theta - \frac{2c}{\sin 2\alpha}\left(\sin^2\theta - \sin^2\alpha\right)$$

For equilibrium, $\dfrac{dz}{d\theta} = 0$

$$\Rightarrow \quad \frac{dz}{d\theta} = h\cos\theta - \frac{2c}{\sin 2\alpha} 2\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad \cos\theta\left(h - \frac{4c}{\sin 2\alpha}\sin\theta\right) = 0$$

$$\Rightarrow \quad \cos\theta = 0 \text{ or } \sin\theta = \frac{h\sin 2\alpha}{4c}$$

$$\Rightarrow \quad \theta = \frac{\pi}{2} \text{ or } \sin\theta = \frac{h\sin\alpha\cos\alpha}{2c}$$

But for real values of θ, $|\sin\theta| < 1$

i.e. $\left|\frac{h\sin\alpha\cos\alpha}{2c}\right| < 1 \Rightarrow h < \frac{2c}{\sin\alpha\cos\alpha}$

So, if $h > 2c\sec\alpha\operatorname{cosec}\alpha$, there is only one position of equilibrium with $\theta = \frac{\pi}{2}$, if $h < 2c\sec\alpha\operatorname{cosec}\alpha$, there are three positions of equilibrium obtained by $\theta = \frac{\pi}{2}, \sin^{-1}\left(\frac{h\sin\alpha\cos\alpha}{2c}\right), \ \pi - \sin^{-1}\left\{\frac{h\sin\alpha\cos\alpha}{2c}\right\}$

Now, $\frac{d^2z}{d\theta^2} = -h\sin\theta - \frac{2\times 2c}{\sin 2\alpha}\left(\cos^2\theta - \sin^2\theta\right)$

$$= -h\sin\theta - \frac{4c\left(1 - 2\sin^2\theta\right)}{\sin 2\alpha}$$

Case 1. When $\theta = \pi/2$, then

$\frac{d^2z}{d\theta^2} = -h + \frac{4c}{\sin 2\alpha} = -h + \frac{2c}{\sin\alpha\cos\alpha}$ then equilibrium is stable or unstable as $h <$ or $> \frac{2c}{\sin\alpha\cos\alpha}$ respectively.

If $h = \frac{2c}{(\sin\alpha\cos\alpha)}, \frac{d^2z}{d\theta^2}$ at $\theta = \pi/2 = 0$

$$\frac{d^3z}{d\theta^2} = -h\cos\theta + \frac{8c\times 2\sin\theta\cos\theta}{\sin 2\alpha}$$

At $\theta = \pi/2$, under $h = 2c\operatorname{cosec}\alpha\sec\alpha, d^3z/d\theta^3 = 0$.

Again $\frac{d^4z}{d\theta^4} = h\sin\theta + \frac{16c}{\sin 2\alpha}\left(\cos^2\theta - \sin^2\theta\right)$

Putting $\theta = \pi/2$ and $h = 2c\operatorname{cosec}\alpha\sec\alpha$

$$\frac{d^4z}{d\theta^{44}} = h + \frac{8c}{\sin\alpha\cos\alpha}(0-1) = \frac{2c}{\sin\alpha\cos\alpha} - \frac{8c}{\sin\alpha\cos\alpha}$$

$$= -\frac{6c}{\sin\alpha\cos\alpha}$$

$= -$ negative quantity as $\angle A$ is acute.
So, the equilibrium is unstable
Consider when $\sin\theta = h/2c\sin\alpha\cos\alpha$

$$\frac{d^2z}{d\theta^2} = -\frac{h^2}{2c}\sin\alpha\cos\alpha - \frac{4c}{\sin 2\alpha}\left(1 - 2\frac{h^2}{4c^2}\sin^2\alpha\cos^2\alpha\right)$$

$$= \frac{\sin\alpha\cos\alpha}{2c}\left(h^2 - \frac{4c^2}{\sin^2\alpha\cos^2\alpha}\right)$$

$=$ a negative quantity when $h < \dfrac{2c}{\sin\alpha\cos\alpha}$, since $\angle\alpha$ is acute.

Hence, the equilibrium in the inclined positions are unstable.

Now $\alpha < \theta \;\Rightarrow \sin\alpha < \sin\theta = \dfrac{h}{2c}\sin\alpha\cos\alpha$

$\Rightarrow$ $2c\sec\alpha < h$ since $\sin\alpha \neq 0$

Here $2c\sec\alpha < h,\; h < \dfrac{2c}{\sin\alpha\cos\alpha}$,

i.e. $2c\ \sec\alpha < h < \dfrac{2c}{\sin\alpha\cos\alpha}$

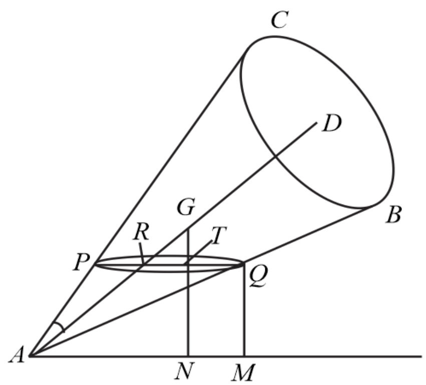
Thus the equilibrium in the inclined positions are unstable under the condition.
$2c\ \sec\alpha < h < 2c/(\sin\alpha\cos\alpha)$

Example27:- An isosceles triangular lamina of an angle 2α and height h rests between two smooth pegs at the same level, distance $2c$, apart prove that if $3c\ \sec\alpha < h < \dfrac{3c}{\sin\alpha\cos\alpha}$, then oblique positions of equilibrium exist, which are unstable. Discuss stability of the vertical positions.

Solution:- The question is same as question 26.

Example:- A smooth solid circular cone, of height h and vertical angle 2α is at test with its axis vertical in a horizontal circular hole of radius a. Show that if $16a > 3h\sin 2\alpha$, the equilibrium is stable and there are two other positions of unstable equilibrium and that if $16a < 3h\sin 2\alpha$, the equilibrium is unstable and the position in which the axis is vertical is the only position of equilibrium.

Solution:- Suppose that ABC is a solid circular cone with height $AD(=h)$ and G as C.G. is resting in a horizontal circular hole PQ of radius a. As shown in the figure. AD is perpendicular to $BC, AM \,||\, PQ, GN$ and QM are vertical, $AG = \dfrac{3}{4}AD$.



The figure shows the position of the system slightly displaced from its equilibrium. Let AD be inclined at angle θ with AM . Here PQ is fixed. Suppose that z is the height of G above PQ, so

$z = GT = GN - QM$

$$= AG\sin\theta - AQ\sin(\theta-\alpha) \qquad \angle BCA = 2\alpha$$

Now, in ΔPAQ

$$\frac{PQ}{\sin PAQ} = \frac{AQ}{\sin APQ}, \qquad \angle DAM = \theta$$

$$\Rightarrow \quad \frac{2a}{\sin 2\alpha} = \frac{AQ}{\sin(\pi-(\alpha+\theta))} = \frac{AQ}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AQ = 2a\frac{\sin(\theta+\alpha)}{\sin 2\alpha}$$

Making use of this result, we have

$$z = \frac{3}{4}h\sin\theta - 2a\frac{\sin(\theta+\alpha)}{\sin 2\alpha}\sin(\theta-\alpha)$$

$$= \frac{3}{4}h\sin\theta - \frac{2a}{\sin 2\alpha}\left(\sin^2\theta - \sin^2\alpha\right)$$

$$\frac{dz}{d\theta} = \frac{3}{4}h\cos\theta - \frac{2a}{\sin 2\alpha}.2\sin\theta\cos\theta$$

For equilibrium, putting $dz/d\theta = 0$,we have

$$\cos\theta\left(\frac{3h}{4} - \frac{4a}{\sin 2\alpha}\sin\theta\right) = 0$$

$$\Rightarrow \quad \cos\theta = 0 \text{ or } \sin\theta = \frac{3h\sin 2\alpha}{16a}$$

i.e. $\theta = \dfrac{\pi}{2}$ or $\sin^{-1}\left(\dfrac{3h\sin 2\alpha}{16a}\right)$ or $\pi - \sin^{-1}\left(\dfrac{3h\sin 2\alpha}{16a}\right)$

$$\frac{d^2z}{d\theta^2} = -\frac{3h}{4}\sin\theta - \frac{4a}{\sin 2\alpha}\left(\cos^2\theta - \sin^2\theta\right)$$

$$= -\frac{3h}{4}\sin\theta - \frac{4a}{\sin 2\alpha}\left(1 - 2\sin^2\theta\right)$$

To test the nature of equilibrium

Case: 1. $\theta = \pi/2$

$$\frac{d^2z}{d\theta^2} = -\frac{3h}{2} - \frac{4a}{\sin 2\alpha}(1-2) = \frac{1}{4\sin 2\alpha}(16a - 3h\sin 2\alpha)$$

Hence, the equilibrium is

Stable when $16a > 3h\sin 2\alpha$ and unstable when $16a < 3h\sin 2\alpha$

Case: 2. $\sin\theta = \dfrac{3h\sin 2\alpha}{16a}$

We know that for real values of θ, $|\sin\theta| < 1$.

$\Rightarrow$ Now $\dfrac{d^2z}{d\theta^2} = -\dfrac{3h}{4}\left(\dfrac{3h - \sin 2\alpha}{16a}\right) - \dfrac{4a}{\sin 2\alpha}\left\{1 - 2\left(\dfrac{3h - \sin 2\alpha}{16a}\right)^2\right\}$

$= \dfrac{1}{64a\sin 2\alpha}\left(9h^2\sin^2 2\alpha - 256a^2\right)$

= a negative quantity under the condition $3h\sin 2\alpha < 16a$.

Hence, the equilibrium is unstable.

Finally, let us summarize that

1. Under the condition $3h\sin 2\alpha < 16a$, the equilibrium is

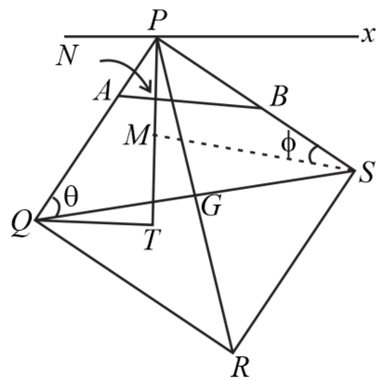
   Unstable when $\theta = \sin^{-1}\left(\dfrac{3h\sin 2\alpha}{16a}\right), \pi - \sin^{-1}\left(\dfrac{3h\sin 2\alpha}{16a}\right)$ in inclined position.

   Stable when $\theta = \pi/2$, in vertical position.

   Under this conditions, 3 positions of rest.
2. Under the condition $3h\sin 2\alpha > 16a$, the equilibrium is unstable at $\theta = \pi/2$, i.e., the only one vertical position of equilibrium.

Example28:- Four uniform rods, each of length $2a$, are hinged at their ends so as to form a rhombus and the system is hung over two smooth pegs in the same horizontal line at a distance $a\sqrt{2}$, the pegs being in contact with different rods. Show that the system is in equilibrium when the rhombus is a square, but that the equilibrium is not stable for all displacements.

Solution:- Suppose that A and B are two smooth pegs in a horizontal line such $AB = a\sqrt{2}$. Four rods PQ, QR, RS and SP in the form of a rhombus in the vertical plane is hanging over the pegs A and B. Length of each rod $.= 2a$. G is the C.G. of the system.



If the system being tilted slightly from it's equilibrium the rods PQ and PS are inclined at an angle θ and ϕ to the horizontal respectively.

i.e. $\angle PQT = \theta,\ \angle PSM = \phi\ \angle PBA$

$z =$ depth of G below AB

$=$ depth of G below $P -$ depth of AB below P

$= \frac{1}{2}$ (depth of $Q +$ depth of S) $-$ depth of AB below P

$$= \frac{1}{2}(PT + PM) - PN$$

$$= \frac{1}{2}(PQ\sin\theta + PS\sin\phi) - PB\sin\phi$$

Now for PB, consider the ΔPAB,

$$\frac{PB}{\sin\theta} = \frac{AB}{\sin APB} \Rightarrow \frac{PB}{\sin\theta} = \frac{a\sqrt{2}}{\sin\left[\pi - (\theta+\phi)\right]}$$

$$PB = \sqrt{2}\,a\frac{\sin\theta}{\sin(\theta+\phi)}$$

Making use of the result, $z = \frac{1}{2}\times 2a(\sin\theta + \sin\phi) - \sqrt{2}a\frac{\sin\theta\sin\phi}{\sin(\theta+\phi)}$

$$= a\left\{\sin\theta + \sin\phi - \sqrt{2}\frac{\sin\theta\sin\phi}{\sin(\theta+\phi)}\right\}$$

It is noteworthy that z is a function of two variables θ and ϕ. So we will apply the maxi and minima theory of two variables.

$$\frac{dz}{d\theta} = a\left\{\cos\theta - \sqrt{2}\sin\phi\frac{\cos\theta\sin(\theta+\phi) - \sin\theta\cos(\theta+\phi)}{\left\{\sin(\theta+\phi)^2\right\}}\right\}$$

$$= a\left[\cos\theta - \sqrt{2}\frac{\sin^2\phi}{\left\{\sin(\theta+\phi)^2\right\}}\right] \qquad (1)$$

And $\dfrac{dz}{d\phi} = a\left[\cos\phi - \dfrac{\sqrt{2}\sin^2\theta}{\{\sin(\theta+\phi)\}^2}\right]$ (2)

For equilibrium, $\dfrac{dz}{d\theta} = 0 = \dfrac{dz}{d\phi}$

$$\cos\theta - \frac{\sqrt{2}\sin^2\phi}{\{\sin(\theta+\phi)\}^2} = 0, \quad \cos\phi - \frac{\sqrt{2}\sin^2\theta}{\{\sin(\theta+\phi)\}^2} = 0 \qquad (3)$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin^2\phi} = \frac{\sqrt{2}}{\{\sin(\theta+\phi)\}^2} = \frac{\cos\phi}{\sin^2\theta}$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin^2\phi} = \frac{\cos\phi}{\sin^2\theta}$$

$$\Rightarrow \quad \theta = \phi$$

Putting $\theta = \phi$ in any of the equation (3) we have

$$\cos\theta - \sqrt{2}\,\frac{\sin^2\theta}{\sin^2 2\theta} = 0$$

$$\Rightarrow \quad \cos\theta = \sqrt{2}\,\frac{\sin^2\theta}{4\sin^2\theta\cos^2\theta} = \frac{1}{2\sqrt{2}\cos^2\theta} \text{ as } \theta \neq 0$$

$$\Rightarrow \quad \cos^3\theta = \frac{1}{2\sqrt{2}} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4} = \phi$$

Now $\angle QPS = \angle QPT + \angle TPS$

$$= \frac{\pi}{2} - \theta + \frac{\pi}{2} - \phi = \pi - (\theta+\phi)$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Thus the rhombus is a square.

To test the nature of equilibrium.

$$\frac{d^2z}{d\theta^2} = a\left[-\sin\theta + \sqrt{2}\,\frac{2\cos(\theta+\phi)}{\{\sin(\theta+\phi)\}^3}\sin^2\phi\right]$$

$$\frac{d^2z}{d\phi\, d\theta} = a\left[0 - \frac{\sqrt{2}\{2\sin\phi\cos\phi\sin^2(\theta+\phi) - 2\sin^2\phi\sin(\theta+\phi)\cos(\theta+\phi)\}}{\{\sin(\theta+\phi)^4\}}\right]$$

$$= -2a\sin\phi\left[\frac{\sqrt{2}\cos\phi\sin(\theta+\phi) - \sin\phi\cos(\theta+\phi)}{\{\sin(\theta+\phi)\}^3}\right]$$

$$\frac{d^2z}{d\phi^2} = a\left[-\sin\phi + 2\sqrt{2}\frac{\sin^2\theta\cos(\theta+\phi)}{\{\sin(\theta+\phi)\}^3}\right]$$

$$r = \left(\frac{d^2z}{d\theta^2}\right)_{\substack{\theta=\pi/4\\ \phi=\pi/4}} = a\left[-\frac{1}{\sqrt{2}} + 2\sqrt{2}\frac{(+1/2)(0)}{1}\right] = -\frac{a}{\sqrt{2}}$$

$$s = \left(\frac{d^2z}{d\theta\, d\phi}\right)_{\substack{\theta=\pi/4\\ \phi=\pi/4}} = -2a \times \frac{1}{\sqrt{2}}\left[\sqrt{2} \times \frac{1}{\sqrt{2}}\right] = -\sqrt{2a}$$

$$t = \left(\frac{d^2z}{d\phi^2}\right)_{\substack{\theta=\pi/4\\ \phi=\pi/4}} = a\left[-\frac{1}{\sqrt{2}} + 0\right] = -\frac{a}{\sqrt{2}}$$

So, $rt - s^2 = \left(-\frac{a}{\sqrt{2}}\right)\left(-\frac{a}{\sqrt{2}}\right) - \left(\sqrt{2}\,a\right)^2$

$= \frac{a^2}{2} - 2a^2 = -\frac{3}{2}a^2$

= a negative quaintly.

Hence z is neither a maximum nor a minimum when $\theta = \phi = \pi/4$. Here $z = z(\theta, \phi)$, $z + \delta z = z(\theta + \delta\theta, \phi + \delta\phi)$. Applying Taylor's theorem for function of the variables we have

$$z(\theta+\delta\theta, \phi+\delta\phi) - z(\theta,\phi)$$

$$= \delta\theta\left(\frac{\partial z}{\partial\theta}\right)_{\substack{\theta=\pi/4\\ \phi=\pi/4}} + \delta\phi\left(\frac{\partial z}{\partial\phi}\right)_{\substack{\theta=\pi/4\\ \phi=\pi/4}} + \frac{1}{2!}\left\{r(\delta\theta)^2 + 2\delta\theta\,\delta\phi s + (\delta\phi)^2 t\right\} + R_a + \ldots.$$

$$= \frac{1}{2!}\left(-\frac{a}{\sqrt{2}}(\delta\theta)^2 - \sqrt{2}a\,2\,\delta\theta\,\delta\phi - \frac{a}{\sqrt{2}}(\delta\phi)^2\right) + R_a \ldots..$$

$$\Rightarrow \quad \delta z = -\frac{a}{2}\left\{(\delta\theta)^2 + 4\,\delta\theta\,\delta\phi + (\delta\phi)^2\right\} + R_3$$

Now although z is neither a maximum nor a minimum where $\theta = \phi = \pi/4$, yet there is equilibrium because δz is then zero so far as terms of the first order in $\delta\theta$ and $\delta\phi$ (are zero). But as z is neither maximum nor minimum the equilibrium cannot be stated to be either stable or unstable universally. It is in fact stable with respect to some displacement and unstable with respect to other displacement. If for example we consider only such displacement as make $\delta\theta = \delta\phi$, then δz is certainty negative when $\delta\theta$, $\delta\phi$ are taken small enough. Thus C.G. is increased by the displacements and so then equilibrium is stable. If again we consider only such displacements as make $\delta\theta = -\delta\phi$ they make δz certainly positive then $\delta\theta$ and $\delta\phi$ are small enough. The C.G. is depressed by the displacement and so the equilibrium is unstable.

Example29:- Three equal spheres rest on a smooth table and are kept in position by a smooth elastic band in the plane of the centre, the band being unstretched when the spheres are in contact. A fourth equal sphere is placed above them. Prove that, if in a position of equilibrium the line joining the centre of the upper sphere to the centre of either of the lower spheres is inclined at an angle θ to the vertical, the equilibrium is stable for symmetrical displacements if $\sin^3\theta < 1/\sqrt{3}$.

Solution:- Let the three equal spheres of centres A, B and C be on the smooth table and a fourth sphere be placed on them. O is the foot of the normal from the centre D of the fourth sphere to the plane through A, B and C.

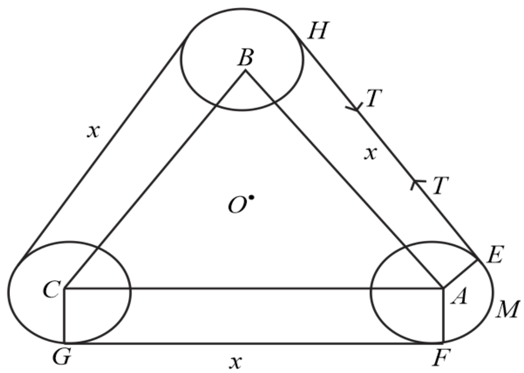


Fig. 1

Let θ be the inclination to the vertical of the line joining the centre of the upper sphere to what of one of the lower sphere, when then centre of the latter are at a distance x apart. Since ΔABC is a equilateral triangle, so in fig. 1

$$AO = \frac{2}{3} \text{ of the median}$$

$$= \frac{2}{3} AB \sin 60^\circ = \frac{\sqrt{3}}{3} x$$

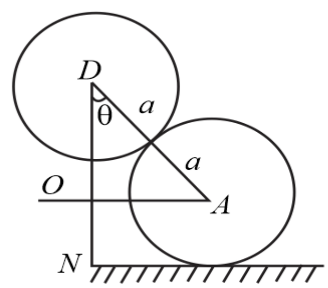


Fig. 2

In fig. 2, D is the centre of the fourth sphere in the equilibrium position O is foot of the perpendicular from D to the plane through A, B and C, OD is vertical and OA is horizontal so that ΔAOD is a right angled triangle at O. Let a be the radius of each sphere, we have

$$\sin\theta = \frac{OA}{AB} = \frac{\sqrt{3}\, x}{3.\ 2a}$$

$$\Rightarrow \qquad x = 2\sqrt{3}\,a\sin\theta \qquad (1)$$

In fig. 1, EH is tangent to the circles with centres A and B.

So, $\angle EAF = 120^\circ$, Arc $EMF = \dfrac{2\pi}{3}a$

The natural length of band (unstretched) $= 3\left(2a + \dfrac{2\pi}{3}a\right)$

$$= 2a(3+\pi)$$

The extended length of the band (as shown in fig. 1)

$$= 3x + 3\times\frac{2\pi}{3}a = 3x + 2\pi a$$

If λ be the coefficient of elasticity, the tension T of the band by Hook's Law is given by

$$T = \lambda\frac{extended\ length - natrual\ length}{natural\ length}$$

$$= \lambda\frac{3x + 2\pi a - 2a(3+\pi)}{2a(3+\pi)}$$

$$\Rightarrow \qquad T = \frac{3\lambda}{2(\pi+3)a}(x-2a)$$

Let W_1 be the weight of each sphere and δW be the element of work function; then we have for small displacements work done by the upper sphere $= -W_1\delta(a + 2a\cos\theta)$, (since $DN = DO + ON = a + 2a\cos\theta$), negative sign indicates that the distance ND is measured from N to D and force w acting from D towards N.

The work done by the tension $= -3T\delta x$

So, $\delta W = -W_1\delta(a + 2a\cos\theta) - 3T\delta x$

$$\Rightarrow \qquad \frac{dW}{d\theta} = -W_1\frac{d}{d\theta}(a + 2a\cos\theta) - 3T\frac{d}{d\theta}\left(2\sqrt{3}a\sin\theta\right)$$

$$= 2W_1 a\sin\theta - \frac{9\lambda(x-2a)}{2(\pi+3)a}2\sqrt{3}a\cos\theta$$

$$= 2W_1 a\sin\theta - \frac{9\sqrt{3}\lambda}{\pi+3}\left(2\sqrt{3}\,a\sin\theta - 2a\right)\cos\theta$$

$$= 2W_1 a\sin\theta - \frac{18\sqrt{3}\lambda}{\pi+3}\left(\sqrt{3}\sin\theta - 1\right)a\cos\theta$$

$$\frac{d^2W}{d\theta^2} = 2W_1 a\cos\theta - \frac{18\sqrt{3}\lambda a}{\pi+3}\left\{\sqrt{3}\left(\cos^2\theta - \sin^2\theta\right) + \sin\theta\right\} \qquad (2)$$

The position of equilibrium is given by $\dfrac{dW}{d\theta} = 0$

i.e. $\qquad W_1\sin\theta - \dfrac{9\sqrt{3}\lambda\left(\sqrt{3}\sin\theta - 1\right)\cos\theta}{\pi+3} = 0$

i.e. $$W_1 \sin\theta - \frac{9\sqrt{3}\,\lambda}{\pi+3}\left(\sqrt{3}\sin\theta\cos\theta - \cos\theta\right) \qquad (3)$$

For this value of θ, putting the value of W_1 from (3) in (2)

$$\frac{d^2W}{d\theta^2} = 2a\left[\frac{9\sqrt{3}\,\lambda}{\pi+3}\left(\sqrt{3}\sin\theta\cos\theta - \cos\theta\right)\right]\cot\theta$$

$$-\frac{18\sqrt{3}\,\lambda a}{\pi+3}\left\{\sqrt{3}\left(\cos^2\theta - \sin^2\theta\right) + \sin\theta\right\}$$

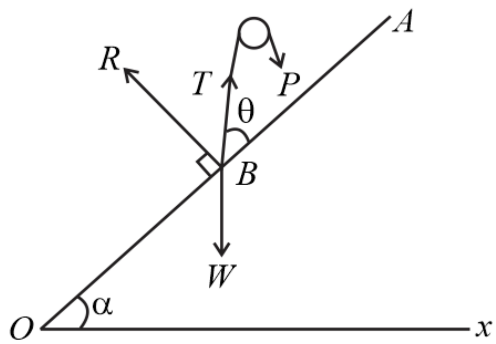
$$= \frac{18\sqrt{3}\,\lambda a}{\pi+3}\left[\sqrt{3}\cos^2\theta - \frac{\cos^2\theta}{\sin\theta} - \sqrt{3}\cos^2\theta + \sqrt{3}\sin^2\theta - \sin\theta\right]$$

$$= \frac{18\sqrt{3}\,\lambda a}{\pi+3}\,\frac{\left(\sqrt{3}\sin^3\theta - 1\right)}{\sin\theta}$$

If $\sin^3\theta < \frac{1}{\sqrt{3}}$ then $\frac{d^2W}{d\theta^2}$ is negative, the corresponding value of W is a maximum and the equilibrium is stable.

Example30:- A weight W is supported on a smooth inclined plane by a given weight P, connected with W by means of a string passing round a fixed pulley whose position is given. Find the position of equilibrium of W on the plane and show that it is stable.

Solution:- As shown in the figure, a weight W is placed at B on the plane OA inclined at angle α to the horizon ox. T is the tension in the string so $T = P$, since pulley is smooth.



Resolving the forces along the plane

$$P\cos\theta = W\sin\alpha$$

$$\cos\theta = \frac{W\sin\alpha}{P} = const.$$

This gives the position of equilibrium of the weight. If the body s slightly displaced in downward direction, θ decrease and hence $\cos\theta$ increases. Therefore the body tends to go up to resume its position of equilibrium. Furthermore if the body is displaced in the upward direction, θ increases which implies $\cos\theta$ decreases. Hence the body tends to get down the plane to resume its position of equilibrium.

Therefore the equilibrium is stable.

Example31:- Using the principle of conservation of energy, establish that the positions of maximum potential energy, are positions of unstable equilibrium and position of minimum potential energy are positions of stable equilibrium.

Proof:- The principle of conservation of energy states, "Potential energy + Kinetic energy = Constant, in case of a dynamical system". So whenever a body starts moving, it acquires kinetic energy and therefore loses potential energy. We will now use the principle to prove the result.

At first, if the potential energy of the system remains constant for small displacement, no work is done during this small displacement and the body is in equilibrium.

Now if the system be in such a position that its potential energy is maximum and if the system be slightly displaced from this position and then we make it free to move. During the move me the potential energy of the system decreases and kinetic energy increases (i.e. kinetic energy is positive). The kinetic energy, compels the system to move further away from the position maximum potential energy. Thus it shows that the equilibrium in the position of maximum potential energy, is an unstable one.

Conversely, if the system is in equilibrium in the position of minimum potential energy and if it is slightly displaced and then set free, the potential energy decreases. Since in this case the potential energy of the system cannot be decreased below minimum, so it will regain its original position. The position of minimum potential energy is therefore that of stable equilibrium.

PREVIOUS YEARS QUESTIONS IAS/IFoS (2008-2023)

STABLE, UNSTABLE & NEUTRAL EQUILIBRIUM

UPDATED Q7(c) Suppose a cylinder of any cross-section is balanced on another fixed cylinder, the contact of curved surfaces being rough and the common tangent line horizontal. Let ρ and ρ' be the radii of curvature of the two cylinders at the point of contact and *h* be the height of centre of gravity of the upper cylinder above the point of contact. Show that the upper cylinder is balanced in stable equilibrium if $h < \frac{\rho\rho'}{\rho+\rho'}$. **UPSC CSE 2022**

Q8.(a) A bucket is in the form of a frustum of a cone and is filled with water of density ρ. If the bottom and top ends of the bucket have radii *a* and *b* respectively and *h* is the height of the bucket, then find the resultant vertical thrust on the curved surface of the bucket. Is that thrust equal to $\frac{1}{3}\pi\rho gh(b-a)(b+2a)$? **IFoS 2022**

Q1. A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius *a*. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$. **[6a UPSC CSE 2019]**

Q2. A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ for equilibrium. If ϕ be less than this value, is the equilibrium stable? **[6c UPSC CSE 2017]**

Q3. A heavy uniform cube balances on the highest point of a sphere whose radius is *r*. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is 90°. **[5d 2017 IFoS]**

Q4. A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}\times$ radius of hemisphere. **[7a 2014 IFoS]**

Q5. A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium.

[5e 2013 IFoS]

Q6. A heavy hemispherical shell of radius a has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius b at the highest point. Prove that if $\frac{b}{a} > \sqrt{5} - 1$, the equilibrium is stable, whatever be the weight of the particle.

[7b UPSC CSE 2012]

Q7. A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability. **[5c 2010 IFoS]**

VIRTUAL WORK

Definition

Work. A force is said to do work when its point of application displaces from one position to another position.

Consider a force F acting on a particle at O in the direction OA and the particle is displaced from one position O to another position B. Let OB make an angle θ with OA, the direction of the force F.

Work done by the force $F = F \times OA = F \times OB \cos\theta$

$$= F \times \text{projection of } OB \text{ on } OA$$

Again work done by the force F

$= F \times OA = F \times OB \cos\theta = (F\cos\theta) \times OB$

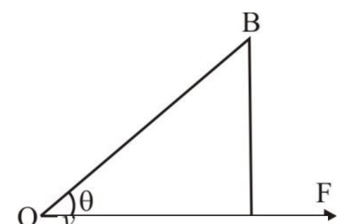
= Resolved part of the force in the direction of actual displacement × actual displacement.

So, the product of the force and the orthogonal projection of the displacement on the line of action of the force is said to be work done by the force.

or

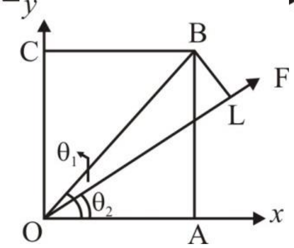
Product of resolved part of the force in the direction of actual displacement and the actual displacement is said to be work done by the force.

Work done is positive if it is in the direction of force. It is negative if it is in the direction opposite to the direction of the force. If the displacement is zero or it is in the direction perpendicular to the direction of the force, then the work done is zero.



Theorem 1. *The work done by a force in displacing a particle from one position to another position is equal to the algebraic sum of works done by the resolved parts of the forces.*

Proof. Let OX and OY be two mutually perpendicular axes. A force Facts at a particle placed at O. This force displaces the point



of application O to a point B. Let B be in- the plane of XOY. OB makes an angle θ_1, from the axis of X. Force F makes an angle θ_2, from this axis. Let OA and OC be the components of the displacement OB in the directions OX and OY respectively. F_1 and F_2, are the components of the forces along OX and OY respectively.

Now the work done by the force F

= Force F × displacement in the direction of the force F

$= F \times OL = F \times OB \cos(\theta_1 - \theta_2)$

$= F \times OB (\cos \theta_1 , \cos \theta_2 + \sin \theta_1, \sin \theta_2)$

$= (OB \cos \theta_1) (F \cos \theta_2) + (OB \sin \theta_1,) (F \sin \theta_2,)$

$= OA \times F_1 + OC \times F_2$

$= F_1 \times$ displacement in the direction of F_1

$+F_2, \times$ displacement in the direction of F_2

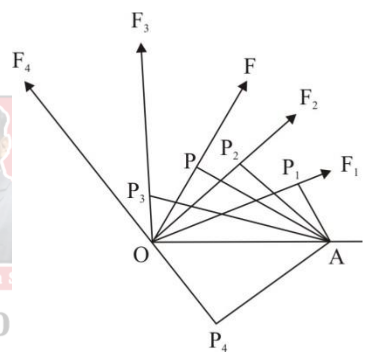
= Work done by the component F_1

+Work done by the component F_2

= Algebraic sum of work done by the components F, and F

= Algebraic sum of the work done by the resolved parts of the force F.

Theorem 2. *The algebraic sum of the works done by a number of coplanar forces acting on a particle, for any displacement of the particle, is equal to the work done by their resultant*



Proof .Let the forces F_1, F_2, F_3, F_4, ... act on particle at O. These forces displace the point of application from O to A. Forces F_1, F_2, F_3, F_4, ... make angles θ_1, θ_2, θ_3, θ_4, with OA respectively. Let F be the resultant of these forces, which makes an angle θ with OA.

The algebraic sum of the work done by the forces F_1, F_2, F_3. F_4, ...

= work done by Force F_1 + work done by the force F_2 + work done by the force $F3$ + work done by the force F_4 + ...

$= \quad = F_1 \times OP_1 + F_2 \times OP_2 + F_4 \times OP_4 + ...$

$= F_1 \times OA\cos\theta_1 + F_2 \times OA\cos\theta_2 + F_3 \times OA\cos\theta_3 \qquad + F_4 \times OA\cos\theta_4 + ...$

$= OA \times (F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3 + F_4 \cos\theta_4 + ...) = OA \times$ resolved part of the resultant along OA

$= F \times OP =$ work done by the resultant.

Virtual work and virtual Displacement.

Let a number of coplanar forces act on a particle. If the particle is an equilibrium under the action of the forces, then is no motion of the particle. So there is not actual displacement This type of displacement is called virtual displacement and the work done during his displacement is called virtual work.

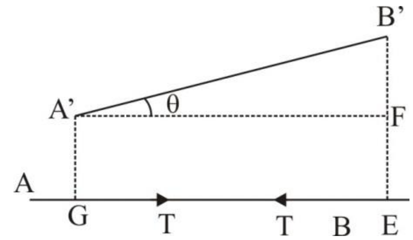
Principle of Virtual work for a system of Coplanar Forces Acting on a Particle.

Statement: *The necessary and sufficient condition that particle acted upon by a number of coplanar forces be in equilibrium is that sum of the virtual work done by the force in any small virtual displacement consistent with geometrical conditions of the system is zero.*

The tension of an inextensible string (non-extensible)

Let T be the tension in string AB.

This tension is replaced in two equal forces T, T acting inward in opposite direction. String AB is displaced to new position $A'B'$. Which makes an small angel θ with the direction of AB. Draw perpendicular $A'G$ from the point A' on AB and draw a perpendicular $B'E$ from B' on AB after producing it to point E.



Sum of the virtual work done by the tension T

$= T.\ AG - T.\ BE$

$= T.\ (AG + GB) - T.\ (GB + BE) = T.\ AB - T.\ GE$

$= T\ .\ AB - T\ .\ A'F = T\,(AB - A'B'\cos\theta)$

$= T.ab\{1-(1-\frac{\theta^2}{2!}+....)\}$

$(\because AB = A'B')$

$= 0$. since θ is very small.

Therefore work done = 0.

Forces which can be omitted in writing the equation of virtual work for a body in equilibrium.

(i) Tension of inextensible string or thrust in a light rod.

(ii) Reaction of any smooth surface with which the body is in contact.

iii) Internal action and reaction between parts of a same body.

(iv) Reaction at a fixed point or a fixed axis about which the body rotates.

Procedure of Solving the problems:

First of all draw the figure.

(i) If it is a string, replace the tension T by two equal forces T and T acting inward in opposite direction.

If l is the length of string in equilibrium. Then the virtual work done by the tension T is $-T\delta l$.

(ii) If it is a rod then tension T of the rod is replaced by two equal forces T and T acting outwards in opposite directions. If l is the length of the rod. Then the virtual work done by the thrust is $T\delta l$.

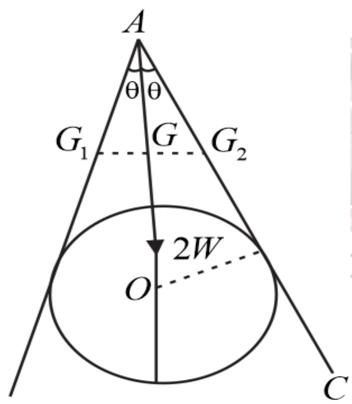
(iii) Distances of the action of forces are measured from a fixed line or a fixed point. If distance measured is in the direction of the force, then the virtual work done by the force is taken to be positive. If it is in opposite direction, then it is taken to be negative.

(iv) We equate the sum of the virtual work to zero.

(v) In this way the problem is solved.

Example1:- Two equal uniform rods AB and AC each of length $2b$ are freely joined at A and rest on a smooth vertical circle of radius a. Show that 2θ be the inclination between them, then $b\sin^3\theta = a\cos\theta$

Solution:- Let AB and AC be two rods resting on vertical circle of centre O. Since vertical circle is fixed. We will measure the distance from centre of the circle.



Let G_1 and G_2 be the centre of gravity of rod AB and AC respectively.

Let W is the weight of each rod.

Therefore, the weight $2W$ will act vertically downward from the point G. G is the middle point G_1, G_2.

A small displacement is given to the system; so that θ becomes $\theta + \delta\theta$.

$$OG = OA - GA = a\cos es\,\theta - b\cos\theta.$$

By the principle of virtual work $2W.\delta(OG) = 0$

Or $\delta(OG) = 0$. Since $W \neq 0$

Therefore, $\delta(a\ \cos ec\,\theta - b\cos\theta) = 0$ on putting the value of OG

Or $-a\ \cos ec\ \theta\ \cot\theta\ \delta\theta + b\sin\theta\,\delta\theta = 0$

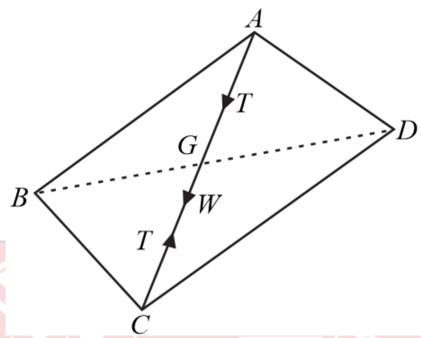
Or $\quad (-a\ \cos es\,\theta\cot\theta + b\sin\theta)\delta\theta = 0$.

But $\delta\theta \neq 0$ therefore, $\left(-a\ \text{cosec}\,\theta\cot\theta + b\sin\theta\right) = 0$

Therefore, $b\sin^3 = \theta = a\ \cos\theta$.

Example2:- Four uniform rods are freely. Joined at their extremities and form a parallelogram $ABCD$, which is suspended by the point A and is kept in shape of by a string AC. Prove that the tension of the string is equal to half of the whole weight.

Solution:- Let $ABCD$ is a parallelogram which is suspended from a point A. Point A and C are jointed by a string AC. Let G be the middle point of AC. Therefore, total weight W of these four rods will act. Vertically downwards from the point G. Replacing tension of the string AC by two forces T, T acting inward in opposite directions, distances are measured from a fixed point A. Let $AG = x$.



Therefore, $AC = 2x$; A virtual displacement is given to the system, so that x becomes $x + \delta x$.

Principle of virtual work $W.\ \delta(AG) - T.\ \delta(AC) = 0$

Or $\quad W.\ \delta(x) - T.\ \delta(2x) = 0$. On putting the value of AG and AC

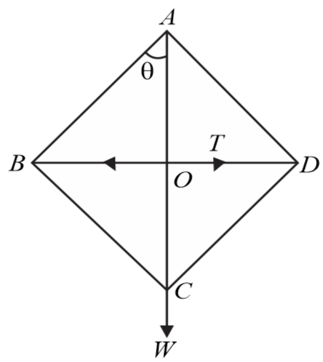
Or $\quad W\delta x - 2T\delta x = 0$

Or $\quad (W - 2T)\delta x = 0$; $\delta x \neq 0 \qquad \therefore W - 2T = 0$

Or $\quad T = \dfrac{W}{2}$ = half of the weight of the roads

Example3:- Five weightless rods of equal length are joined together so as to form a rhombus $ABCD$ with one diagonal BD. If a weight W be attached to C and the system be suspended from A show that there is a thrust in BD equal to $W/\sqrt{3}$.

Solution:- Let AB, BC, CD, DA and BD are five equal weightless rods. These rods are jointed and suspended from A. weights W is attached at C. Tension T in the rod BD is replaced by two forces T, T acting outward in opposite directions.



Let rod AB makes an angle θ with the vertical AC.

$$AC = 2a\ \cos\theta$$

$$BD = 2a\sin\theta$$

Where a is the length of each rod.

Y principle of virtual work [by small virtual displacement θ becomes $\theta + \delta\theta$]

$$W\delta(AC) + T\delta(BD) = 0$$

or $$W\delta(2a\cos\theta) + T\delta(2a\sin\theta) = 0$$

or $$(-2aW\sin\theta + 2aT\cos\theta)\delta\theta = 0.$$ But $\delta\theta \neq 0$

Therefore, $-2aW\sin\theta + 2aT\cos\theta = 0$

Which gives $T = W\tan\theta$

In equilibrium rod $AB = rod$, $AD = rod\ BD$, therefore, ΔABD is equilibrium triangle.

Therefore, $\angle BAD = 60^\circ$ or $\theta = 30^\circ$

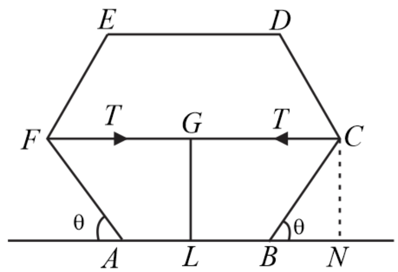
$\therefore\ T = W\ \tan 30^\circ = \dfrac{W}{\sqrt{3}}$

Thrust in $BD = \dfrac{W}{\sqrt{3}}$

Example4:- A regular hexagon $ABCDEF$ consists of six equal rods which are each of weight W and are freely joined together. The hexagon rests in a vertical plane and AB in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is $W\sqrt{3}$.

Solution:- Let each rod be of length $2a$. Replace tension of the string FC in two equal forces T, T acting inwards in opposite directions. Let the rod BC makes in an angle θ with the horizontal.

Therefore, $FC = 2a + 4a\cos\theta,\ GL = 2a\sin\theta$.



A small virtual displacement is given to the system so that θ becomes $\theta+\delta\theta$ and length l of the string becomes $l+\delta l$.

Therefore, equation of virtual work is $-T.\delta l-6W.\ \delta(GL)=0$

Or $\quad T\delta(2a+4a\cos\theta)+6W\delta(2a\sin\theta)=0$

Or $\quad 12aW\cos\theta\,\delta\theta-4aT\sin\theta\,\delta\theta=0$

Or $\quad (3W\cot\theta-T)\delta\theta=0$, since $\delta\theta\neq 0$

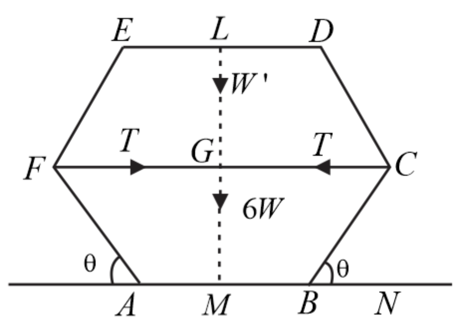
Therefore, $T=3W\cot\theta$ in equilibrium, $\theta=60^\circ$

$$\therefore T=3W\cot 60^\circ=\frac{3W}{\sqrt{3}} \qquad\qquad \therefore T=W\sqrt{3}$$

Example5:- A regular hexagon $ABCDEF$ is composed of six equal heavy rods jointed together and two opposite angle C and F are connected by a string, which is horizontal. AB being in contact with a horizontal plane. A weight W' is placed at the middle point of DE. If W be the weight of each rod, show that the tension in the string is $(3W+W')/\sqrt{3}$.

Solution:- Weight W' is placed at L, the middle point of the rod ED. The weight $6W$ will act at G, centre of gravity of hexagon. Let the rod BC makes an angle θ with the horizontal.

Length of each rod $=2a$.



A small displacement is given to the system, so that θ becomes $\theta+\delta\theta$

Then the equation of virtual work is $-6W\,\delta(GM)-W'\delta(LM)-T\,\delta(FC)=0$

$GM=2a\sin\theta$, $LM=4a\sin\theta$, $FC=2a+4a\cos\theta$.

Therefore, virtual work done by the forces

$$6W\,\delta(2a\sin\theta)+W'\delta(4a\sin\theta)+T\delta(2a+4\cos\theta)=0$$

Or $\quad 12aW\cos\theta\,\delta\theta+4aW'\cos\theta\,\delta\theta-4aT\sin\theta\,\delta\theta=0$

Or $\quad T\sin\theta - 3W\cos\theta - W'\cos\theta = 0$

Since $\delta\theta \neq 0$.

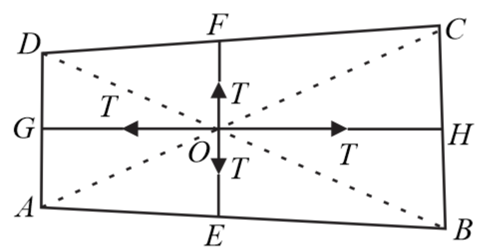
Therefore, $T = (3W + W')\cot\theta$ in equilibrium $\theta = 60^{\circ}$

$$\therefore T = (3W + W')\cot 60^{\circ}, \frac{(3W + W')}{\sqrt{3}}$$

Example6:- The middle points of opposite sides of a jointed quadrilateral are connected by light rods of lengths l and l'. If T and T' be the tensions in there rods, prove that $\frac{T}{l} + \frac{T'}{l'} = 0$.

Solution:- Let E, F, G, H be the middle points. Of the rods AB, CD, DA, and BC respectively.

Let T and T'be the tension in the rod EF and GH respectively. Replacing the tension by two forces acting outwards in opposite directions



A small virtual displacement is given to the system, which changes angles but not the lengths of sides. Therefore, the equation of the virtual work is

$$T\delta(EF) + T'\delta(GH) = 0 \qquad (1)$$

In the ΔAOB.

$$OA^2 + OB^2 - 2(OE^2 + AE^2)$$

$$2(OA^2 + OB^2) = EF^2 + AB^2 \qquad (2)$$

Or similarly,

$$2(OB^2 + OC^2) = GH^2 + BC^2 \qquad (3)$$

$$2(OC^2 + OD^2) = EF^2 + CD^2 \qquad (4)$$

$$2(OD^2 + OA^2) = GH^2 + DA^2 \qquad (5)$$

Subtracting (3) from (2), we have

$$2(OA^2 - OC^2) = EF^2 + AB^2 = GH^2 - BC^2 \qquad (6)$$

Subtracting (5) from (4), we have

$$2(OC^2 + OA^2) = EF^2 + CD^2 - GH^2 - DA^2 \qquad (7)$$

Adding (6) and (7)

$$0 = 2(EF^2 - GH^2) + AB^2 + CD^2 - BC^2 - DA^2$$

Taking differentials

$$2\left[2EF\,\delta(EF)-2GH\,\delta(GH)\right]=0$$

Since AB, BC, CA, DA are constant.

Therefore, $\delta(EF)=\dfrac{GH}{EF}\delta(GH)$ (8)

On putting the value of $\delta(EF)$ from (8) in (1), we have

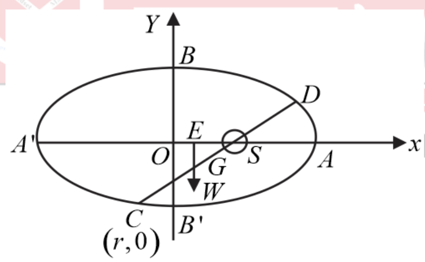
$$T\frac{GH}{EF}\delta(GH)+T'\delta(GH)=0 \text{ or } \left(\frac{T}{EF}+\frac{T'}{GH}\right).\,\delta(GH)=0$$

But $\delta(GH)\neq 0$. Therefore, $\dfrac{T}{EF}+\dfrac{T'}{GH}=0$ or $\dfrac{T}{l}+\dfrac{T'}{l'}=0$.

Example7:- A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal and rests with its lower end on the quadrant of the curve which is further removed from the focus .

Find its position of equilibrium and show that its length must at least be $\dfrac{a}{4}\left\{3+\sqrt{\left(1+8e^2\right)}\right\}$, where $2a$ in the length of major axis and e is the eccentricity.

Solution:- Let S be the pole. Equation of the ellipse in polar co-ordinates is $\dfrac{l}{r}=1-e\cos\theta$.



Let the co-ordinates of the point C be $(r,0)$, where angle $ESC=\theta$

Weight of the rod CD will act vertically downward from the point G.

Taking major axis AA' as a fixed line giving a small virtual displacement to the system so that θ becomes $\theta+\delta\theta$

Equation of the virtual work $W\,\delta(GE)=\theta$

$$\delta(GE)=0.$$

But $GE=GS\sin\theta=(CS-CG)\sin\theta(r-c)\sin\theta$ (1)

Where $2c$ is the length of the rod CD.

$r=\dfrac{l}{1-e\cos\theta}$ from the equation of ellipse, using this value of r.

$$GE=\left[\frac{l}{1-e\cos\theta}-c\right]\sin\theta, \text{ therefore } \delta\left[\frac{l}{1-e\cos\theta}-c\right]\sin\theta=0.$$

On putting the value of GE, we get the above result.

$$\left[\frac{l\cos\theta(1-e\cos\theta)-le\sin^2\theta}{(1-e\cos\theta)^2}-c\cos\theta\right]\delta\theta=0$$

But $\delta\theta\neq 0$

Therefore, $l\cos\theta-le-c\cos\theta(1-e\cos\theta)^2=0$ (2)

Length of the rod will be least if D coincides with S.

Therefore, $r=2c$. But $r=\dfrac{l}{1-e\cos\theta}$

Therefore, $r=2c=\dfrac{l}{1-e\cos\theta}$ (3)

Now putting the value of c from equation (3) in equation (2) we have

$$e\cos^2\theta+\cos\theta-2e=0$$

Which gives $\cos\theta=\dfrac{-1\pm\sqrt{(1+8e^2)}}{2e}$

Negative value of $\cos\theta$ is not admissible.

Therefore, $\cos\theta=\dfrac{-1+\sqrt{(1+8\theta^2)}}{23}$

Substituting this value of $\cos\theta$ in equation (3),

$$2c=\frac{l}{\left[1-\left\{\dfrac{-1+\sqrt{(1+8c^2)}}{2}\right\}\right]}$$

$$=\frac{2l}{\left\{3-\sqrt{(1+8e^2)}\right\}}\,\frac{\left\{3+\sqrt{(1+8e^2)}\right\}}{\left\{3+\sqrt{(1+8e^2)}\right\}}$$

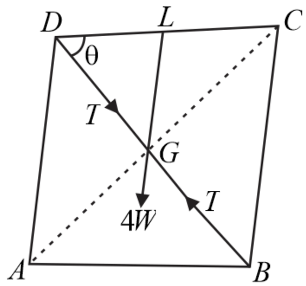
$$=\frac{2l\left\{3+\sqrt{(1+8e^2)}\right\}}{8-8e^2}=\frac{l\left\{3+\sqrt{(1+8e^2)}\right\}}{4(1-e^2)}$$

But $l=(1-e^2)a\ \therefore\ 2c=\dfrac{a}{4}\left\{3+\sqrt{(1+8e^2)}\right\}$

Hence required length of the rod $=\dfrac{a}{4}\left\{3+\sqrt{(1+8e^2)}\right\}$

Example8:- A string of length a, forms the shorter diagonal of a rhombus formed by four uniform rods, each of length of b and weight W. Which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is $\dfrac{2W\left(2b^2-a^2\right)}{b\left(4b^2-a^2\right)^{1/2}}$.

Solution:- Let the side CD of the rhombus be fixed in the horizontal position. BD is a string whose tension is T. Replacing the tension in two forces T, T inward in opposite directions.



Let the $\angle LDG = \theta$.

A small virtual displacement is given to system so that θ becomes $\theta + \delta\theta$.

Now the equation of virtual work is $4W\,\delta(LG) - T\delta(BD) = 0$

In the $\Delta DGC, 2\,DG = 2CD\cos\theta$ (1)

Therefore, $BD = 2b\cos\theta\,(CD = b)$

$$LG = GD\,\sin\theta = CD\,\cos\theta\sin\theta = b\cos\theta\sin\theta = \frac{b}{2}\sin 2\theta$$

Putting the value of LG and BD in equation (1), we have

$$4W\,\delta\left(\frac{b}{2}\sin 2\theta\right) - T\delta\left(2b\cos\theta\right) = 0$$

Or $\quad 4W\,b\cos 2\theta\,\delta\theta + 2bT\sin\theta.\delta\theta = 0$

Or $\quad \left(2W\cos 2\theta + T\sin\theta\right)\,\delta\theta = 0$ But $\delta\theta \neq 0$

$\therefore \quad 2W\cos 2\theta + T\sin\theta = 0$ or $T = -2W\dfrac{\cos 2\theta}{\sin\theta}$

Or $\quad T = 2W\dfrac{\left(\sin^2\theta - \cos^2\theta\right)}{\sin\theta}$ (2)

In the position of the equilibrium, from the triangle DGC, we have

$$\cos\theta = \frac{DG}{DC} = \frac{BD/2}{DC} = \frac{a/2}{b}$$

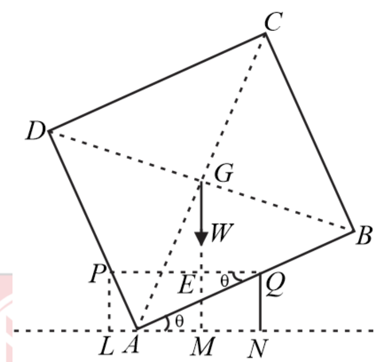
Or $\quad \cos\theta = \dfrac{a}{2b} \qquad \therefore \quad \sin\theta = \dfrac{\sqrt{\left(4b^2-a^2\right)}}{2b}$

On putting the value of $\sin\theta$ and $\cos\theta$, we get

$$T = 2W\left[\frac{2b^2 - a^2}{b\left(4b^2 - a^2\right)^{1/2}}\right]$$

Example9:- A square of side, $2a$ is placed with its plane vertical between two smooth pegs, which are in the same horizontal line at a distance c apart. Show that it will be in equilibrium when the inclination of one of its edges to the horizon is either $\frac{\pi}{4}$ or $\frac{1}{2}\sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right)$

Solution:- Let $ABCD$ be a square of weight W. The weight acts vertically downwards at the point C. G is the point inter-section of AC and BD. P and Q are two pegs. Let the side AB makes an angle θ with the horizontal.



In the $\Delta ANQ, \frac{QN}{AQ} = \sin\theta$, (1)

$\therefore\ QN = AQ\sin\theta$.

In the $\Delta PAQ, \frac{AQ}{PQ} = \cos\theta$

Or $\quad AQ = c\ \cos\theta$, since $PQ = c$

Putting this value of AQ in (1), $QN = c\cos\theta\sin\theta = \frac{1}{2}c\sin 2\theta$,

But $QN = EM$ $\qquad \therefore\ EM = \frac{1}{2}c\sin 2\theta$

In the $\Delta AMG, \frac{GM}{AG} = \sin\left(45^\circ + \theta\right)$

Or $\quad \frac{GM}{\sqrt{2a}} = \sin\left(45^\circ + \theta\right) \qquad \therefore\ GM = \sqrt{2}\,a\sin\left(45^\circ + \theta\right)$

Now $GE = GM - EM = \sqrt{2}\,a\sin\left(45^\circ + \theta\right) - \frac{1}{2}c\sin 2\theta$

Since pegs P, Q are fixed. Therefore distance of the force is measured in upward direction from PQ.

A small virtual displacement is given to the system, so that θ becomes $\theta + \delta\theta$.

Equation of virtual work $-W\delta(GE)=0$

$\therefore \quad \delta(GE)=0$

On putting the value of GE. We have

$$\delta\left[\sqrt{2}\,a\sin\left(45^\circ+\theta\right)-\frac{c}{2}\sin 2\theta\right]=0$$

Or $\quad \left[\sqrt{2}a\sin\left(45^\circ+\theta\right)-c\cos 2\theta\right]\delta\theta=0\;;\;\delta\theta\neq 0$

Therefore, $\sqrt{2}\,a\cos\left(45^\circ+\theta\right)-c\cos 2\theta=0$

Or $\quad \sqrt{2}\,a\left[\cos 45^\circ+\cos\theta-\sin 45^\circ\sin\theta\right]$

$$-c\left(\cos^2\theta-\sin^2\theta\right)=0$$

Or $\quad \sqrt{2}\,a\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right)-c\left(\cos^2\theta-\sin^2\theta\right)=0$

Or $\quad a\left(\cos\theta-\sin\theta\right)-c\left(\cos\theta-\sin\theta\right)\left(\cos\theta+\sin\theta\right)=0$

Or $\quad \left(\cos\theta-\sin\theta\right)\left[a-c\left(\cos\theta+\sin\theta\right)\right]=0$

When $\cos\theta-\sin\theta=0$, $\tan\theta=1,\ \therefore\ \theta=\frac{1}{4}\pi$

When $\left[a-c\left(\cos\theta+\sin\theta\right)\right]=0$ or $a=c\left(\cos\theta+\sin\theta\right)$

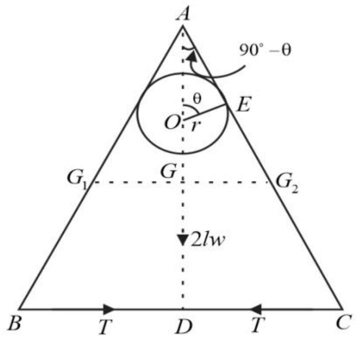
On squaring $a^2=c^2\left(\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta\right)$

Or $\quad a^2=c^2\left(1+\sin 2\theta\right)\ \therefore\ 1+\sin 2\theta=a^2/c^2$

Or $\quad \sin 2\theta=\frac{a^2-c^2}{c^2} \quad \therefore\ \theta=\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right)$

Example10:- Two rods, each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal direction. Find the tension in the string, and if the string is slack show that ϕ satisfies the equation $\tan^2\phi+\tan\phi=l/2r$

Solution:- Since cylindrical peg is fixed. Therefore the distances are measured from the centre of the peg. Let the angle $AOE=\theta$.



Therefore, $OAE = 90^\circ - \theta$.

Hence $\angle DCA = \angle DBA = \theta$

Tension in the string BC is replaced by two forces T, T acting inwards in opposite directions. A small displacement is given to the system, so that θ becomes $\theta + \delta\theta$

Equation of virtual work is $-T\,\delta(BC) + 2lw\delta(OG) = 0$ (1)

$BC = 2BD = 2l\cos\theta$. $OG = AG - AD = \dfrac{l\sin\theta}{2} - r\sec\theta$.

On putting the value of BC and OG in (1), we have

$$-T\,\delta(2l\cos\theta) + 2lw\delta\left(\frac{l\sin\theta}{2} - r\sec\theta\right) = 0$$

Or $2lT\sin\theta\,\delta\theta + lw(l\cos\theta - 2r\sec\theta\tan\theta)\delta\theta = 0$

Or $\left[2lT\sin\theta + lw(l\cos\theta - 2r\sec\theta\tan\theta)\right]\delta\theta = 0$

But $\delta\theta \neq 0$, $\therefore T = w\left(2r\sec^2\theta - l\cot\theta\right)$

When the string is slack, the tension vanishes.

$\therefore$ $l\cos\theta = 2r\sec\theta\tan\theta$

Or $\dfrac{l}{2r} = \tan\theta\sec^2\theta = \tan\theta\left(1+\tan^2\theta\right) = \tan\theta + \tan^3\theta$

Or $\tan^3\theta + \tan\theta = \dfrac{l}{2r}$

In equilibrium $\theta \neq \phi$ $\therefore \tan^3\phi + \tan\phi = \dfrac{l}{2r}$

Example11:- Two small smooth rings of equal weight slide on fixed elliptic wire whose major axis is vertical. They are connected by a string which passes over a small smooth peg at the upper focus, show that the weight will be in equilibrium wherever they are placed.

Solution:- Let $CLDMC$ be an elliptrical wire whose equation is

$$\frac{l}{r} = 1 - e\cos\theta \qquad (1)$$

S is the pole. M, L are the positions of the rings.

Let co-ordinates of M be (r,θ).

So that $SM = r$, $\angle PSM = \theta$

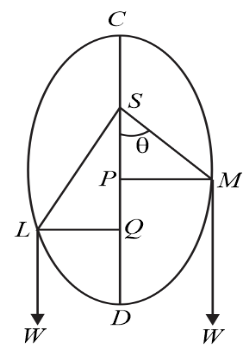
Let the length of the string be l.

Therefore $SL = a - r$.

In the ΔSPM.

$$PS = r\cos\theta = r\frac{(r-l)}{re} = \frac{(r-l)}{e}$$

Therefore, $SQ = \dfrac{a-r-l}{e}$



Small displacement is given to the system, so that θ becomes $\theta + \delta\theta$.

Equation of virtual work is $W\delta(SP) + W\delta(SQ) = 0$,

Or $\quad W\delta\left(\dfrac{r-l}{e}\right) + W\delta\left(\dfrac{a-r-l}{e}\right) = 0$

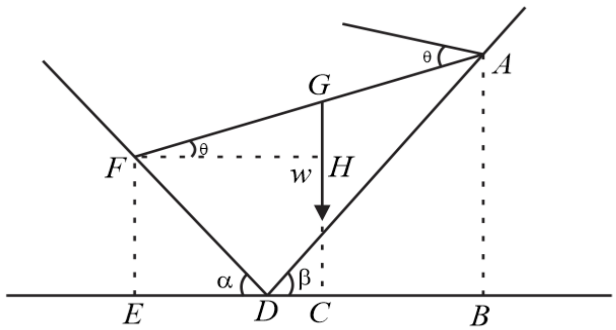
Or $\quad \dfrac{W}{e}\delta r + \dfrac{W}{e}(-\delta r) = 0$ or $\dfrac{W}{e}(\delta r - \delta r) = 0$

This equation is identically satisfied.

Therefore, then weights will be in equilibrium, wherever they are placed.

Example12:- A heavy uniform rod of length $2a$, rests with its ends in contact with two smooth inclined places of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove by the principle of virtual work, that $\tan\theta = \dfrac{1}{2}(\cot\alpha - \cot\beta)$

Solution:- Let DA and DF be two inclined planes which makes angle β, α respectively from the horizontal. Rod AF rests on these inclined planes.



A small virtual displacement is given to the system, so that θ becomes $\theta+\delta\theta$

Equation of virtual work, $-W\,\delta(GC)=0$

Or $\qquad \delta(GC)=0 \qquad$ (1)

In the ΔCDA, $\dfrac{AD}{\sin(\alpha+\theta)}=\dfrac{FD}{\sin(\beta-\theta)}=\dfrac{AF}{\sin\{\pi-(\alpha-\beta)\}}$

Or $\qquad \dfrac{AD}{\sin(\alpha+\theta)}=\dfrac{FD}{\sin(\beta-\theta)}=\dfrac{2a}{\sin(\alpha+\beta)}$

$\therefore \qquad AD=\dfrac{2a\sin(\alpha+\theta)}{\sin(\alpha+\beta)},\; FD=\dfrac{2a\sin(\beta-\theta)}{\sin(\alpha+\beta)}$

In the ΔABD,

$\dfrac{AB}{AD}=\sin\beta$ m $\qquad \therefore\; AB-AD\sin\beta=\dfrac{2a\sin(\alpha+\theta)\sin\beta}{\sin(\alpha+\beta)}$

Similarly, $FE=\dfrac{2a\sin(\beta-\theta)\sin\alpha}{\sin(\alpha+\beta)}$, $GC=\dfrac{1}{2}(AB+FE)$

On putting the value of AB and FE.

We have $GC=\dfrac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha+\theta)\sin\beta+\sin(\beta-\theta)\sin\alpha\right]$

Using this value of GC in (1)

$$\delta\left[\frac{a}{\sin(\alpha+\beta)}\{\sin(\alpha+\theta)\sin\beta+\sin(\beta-\theta)\sin\alpha\}\right]=0$$

Or $\qquad \dfrac{a}{\sin(\alpha+\beta)}\left[\cos(\alpha+\theta)\sin\beta\;\delta\theta-\cos(\beta-\theta)\sin\alpha\,\delta\theta\right]=0$

Or $\qquad \left[\cos(\alpha+\theta)\sin\beta-\cos(\beta-\theta)\sin\alpha\right]=0,\; \delta\theta\neq 0$

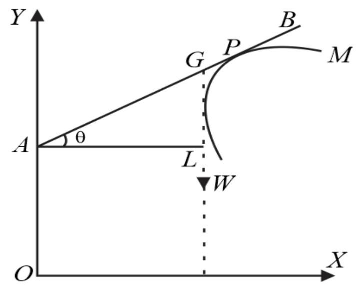
Or $\qquad (\cos\alpha\cos\theta-\sin\alpha\sin\theta)\sin\beta-$

$\qquad (\cos\beta\cos\theta+\sin\beta\sin\theta)\sin\alpha=0$

Which gives then $\tan\theta=\dfrac{1}{2}(\cot\alpha+\cot\beta)$

Example13:- A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; if the beam is in equilibrium in any positions, find the equation to the curve.

Solution:- Let LPM be a smooth curve. Let G be a $C.G$ of the beam AB. The weight W of the rod acts vertically downward from this point.



Let $GN = h$, length of beam $= 2a$ co-ordinates of G are $(a\cos\theta, h)$.

Therefore, the equation of AB is

$$y - h = (x - a,\ \cos\theta)\tan\theta \qquad (1)$$

Differentiating w.r.t. θ. This gives

$$0 = \sec^2\theta(x - a\cos\theta) + a\sin\theta\tan\theta$$

Or $\quad (x - a\cos\theta) + a\sin^2\theta\cos\theta = 0$

Or $\quad x = a\cos\theta(1 - \sin^2\theta)$

$$= a\cos^3\theta$$

Therefore, $\cos\theta = \dfrac{x^{1/3}}{a^{1/3}}$

(2)

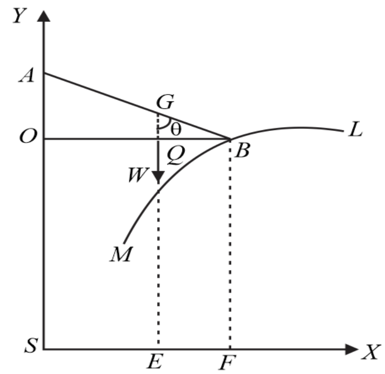
Eliminating θ from (1) and (2), we have

$$x^{2/3} + (y-h)^{2/3} = a^{2/3}$$

Which is the required equation of the curve

Example14:- One end of a beam rests against a smooth vertical wall and the other an a smooth curve in a vertical plane perpendicular to the wall; if the beam rests in all positions, prove that the curve is an ellipse whose major axis lies along the horizontal line described by the centre of gravity of the beam.

Solution:- Let AB be a rod of length $2a$. This rod rests on a vertical wall an on a smooth curve MBL, weight W of the beam acts vertically downwards from CG of the beam AB.



Let the co-ordinates of the point B be (x, y).

Therefore, $x = 2a\sin\theta$

$y = h - a\cos\theta$

Where $h = GH$ and θ is the angle which the beam makes with the vertical.

Now $\dfrac{x}{2a} = \sin\theta$; $\dfrac{y-h}{a} = -\cos\theta$

On squaring then adding, we have $\dfrac{x^2}{4a^2} + \dfrac{(y-h)^2}{a^2} = 1$

Which is the equation of the ellipse.

Whose major axis $y = h$, then horizontal line described by centre of gravity of beam.

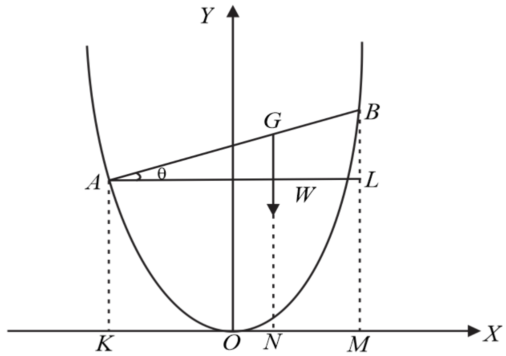
Example15:- A smooth parabolic wire is fixed with its axis vertical and vertex downwards and in it is placed a uniform rod of length $2l$ with its ends resting on the wire. Show that, for equilibrium the rod is either horizontal, or makes with the horizontal an angle θ given by $\cos^2\theta = 2a/l, 4a$ being the latusrectum of the parabola.

Solution:- Let AOB be a smooth parabolic wire AOB. A uniform rod AB rests on this wire. Draw a perpendicular AK from A on x-axis. Similarly, GO and BM are also perpendicular from the point G, B respectively on x-axis. G is the centre of gravity of rod AB. Weight W of the rod AB acts vertically downwards from this point.

In the triangle ABL,. $AL = 2l\cos\theta$, $BL = 2l\sin\theta$

Let the equation of parabola be $x^2 = 4ay$

Let the co-ordinates of point A be $\left(2ae, at^2\right)$



Therefore, co-ordinates of B will be $\left(2at+2l\cos\theta,\ at^2+2l\sin\theta\right)$.

The point B also lies on the parabola. Therefore, the co-ordinates of B satisfy the equation of the parabola.

Therefore, $\left(2at+2l\cos\theta\right)^2=4a\left(at^2+2l\sin\theta\right)$

Or $\quad 4a^2t^2+8atl\cos\theta+4l^2\cos^2\theta=4a^2t^2+8al\sin\theta$

$\therefore \quad t=\tan\theta-\dfrac{l\cos\theta}{2a}$ (1)

A small displacement is given to the system so that θ becomes $\theta+\delta\theta$.

Equation of virtual work is $-W\,\delta\left(GN\right)=0$

Or $\quad \delta\left(GN\right)=0$

Now $GN=\dfrac{1}{2}\left(AK+BM\right)=\dfrac{1}{2}\left(at^2+at^2+2l\sin\theta\right)=at^2+l\sin\theta$

$$=a\left(\tan\theta-\frac{l\cos\theta}{2a}\right)^2+l\sin\theta$$, from (1)

$$=a\tan^2\theta+\frac{l^2\cos^2\theta}{4a}-l\sin\theta+l\sin\theta$$

$$=a\tan^2\theta+\frac{l^2\cos^2\theta}{4a}$$

Using this value of GN in (2), we have

$$\delta\left(a\tan^2\theta+\frac{l^2}{4a}\cos^2\theta\right)=0,$$

Or $\quad 2a\tan\theta\sec^2\theta\ \delta\theta-\dfrac{l^2}{2a}ws\theta\sin\theta\ \delta\theta=0$

Or $\quad \left(2a\sec^3\theta l^2/2a\cos\theta\right)\sin\theta\ \delta\theta=0$

But $\delta\theta\neq 0$,

Therefore, $\left(2a\sec^3\theta-l^2/2a\cos\theta\right)\sin\theta=0$

If $\sin\theta=0$, then $\theta=0$. The rod is horizontal if

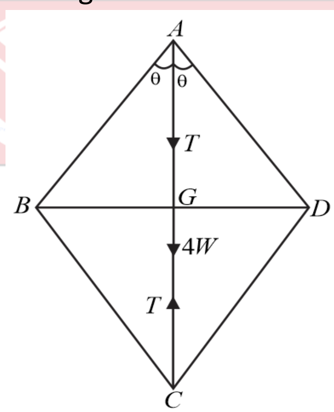
$$2a\sec^3\theta - \frac{l^2}{2a}\cos\theta = 0$$

Or $\quad 2a - \frac{l^2}{2a}\cos^4\theta = 0$, or $\cos^4\theta = \frac{4a^2}{l^2}$, $\cos^2\theta = \frac{2a}{l}$

Which gives the direction of the rod with the horizontal

Example16:- Four equal jointed rods, each of length a are hung from an angular point, which is connected by an elastic string with the opposite point. If the rods hang in the form of square and if the modulus of elasticity of the string be equal to the weight of the rod, show that upstretched length of the string is.

Solution:- Let $ABCD$ be a square formed by four equal jointed rods. The system $ABCD$ hangs by the point A. Points A and C are connected by string AC. Weight $4W$ acts vertically downward from the point G. Which is the point of intersection of the diagonals AC and BD, where W is the weights of each rod. Replace tension T of the string by two equal forces T, T acting inwards in opposite direction. Give a small virtual displacement to the system so that θ becomes $\theta+\delta\theta$, b is the natural length of the string.



Equation of virtual work is $4W\delta(AG) - T\delta(AC) = 0$

Or $\quad 4W\delta(AG) - T\delta(ZAG) = 0$

Or $\quad 4W\delta(x) = T\delta(2x) = 0$ (where $AG = x$)

Or $\quad 4W\,\delta x - 2T\,\delta x = 0$ or $2(2W - T)\delta x = 0$

But $\delta x \neq 0$. Therefore, $2W - T = 0$,

$$T = 2W \qquad (1)$$

By Hook's Law $T = \frac{\lambda}{b}(l - b)$

Or $\quad T = \frac{W}{b}(2a\cos\theta - b) \qquad (2)$

Where $2a\cos\theta$ in the length of extended string and is the modulus of elasticity and it given $\lambda = W$.

Now equating two values of T from (1) & (2) we get

$$2W = \frac{W}{b}(2a\cos\theta - b) \text{ or } 2b = 2a\cos\theta - b$$

$$3b = 2a\cos\theta \text{ or } b = \frac{2a\cos\theta}{3}$$

In equilibrium, $\theta = 45^\circ$. Therefore, $b = \frac{\sqrt{2}}{3}a$.

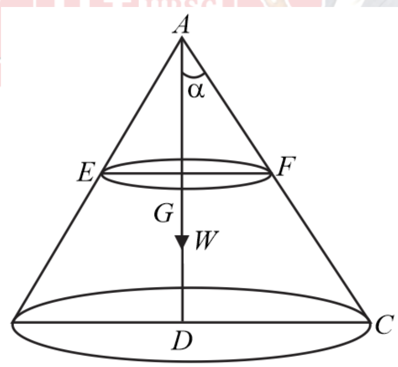
Upstretched length of the string is $\frac{a\sqrt{2}}{3}$.

Example17:- An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight assuming the vertical angle of the cone to be 2α.

Solution:- Let $ABCD$ be a cone. An endless chain rests in the form of a circular band round this smooth cone. Distance are measured from the vertex of the cone.

Let $AG = x$ so that $GE = x\tan\alpha$

Therefore, the length of the string $= 2\pi x \tan\alpha$.



A small virtual displacement is given to the system, so that x becomes $x + \delta x$.

Equation of virtual work $W\delta(AG) - T\delta(2\pi x \tan\alpha) = 0$

Or $\quad W\delta(x) - T\delta(2\pi x \tan\alpha) = 0$

Or $\quad W\,\delta x - 2\pi T\tan\alpha\,\delta x = 0$

Or $\quad (W - 2\pi T\tan\alpha)\delta x = 0$

But $\delta x \neq 0$.

$\therefore \quad W - 2\pi T\tan\alpha = 0$, which gives tension in the chain $T = \frac{W}{2\pi}\cot\alpha$

PREVIOUS YEARS QUESTION IAS/IFoS (2008-2023)

Q8(b) A chain of *n* equal uniform rods is smoothly joined together and suspended from its one end A_1. A horizontal force $\vec{P}$ is applied to the other end A_{n+1} of the chain. Find the inclinations of the rods to the downward vertical line in the equilibrium configuration. **UPSC CSE 2022**

Q5(c) Two rods LM and MN are joined rigidly at the point M such that $(LM)^2 + (MN)^2 = (LN)^2$ and they are hanged freely in equilibrium from a fixed point L. Let ω be the weight per unit length of both the rods which are uniform. Determine the angle, which the rod LM makes with the vertical direction, in terms of lengths of the rods. **UPSC CSE 2021**

Q5(d) Four light rods are joined smoothly to form a quadrilateral ABCD. Let P and Q be the mid-points of an opposite pair of rods and these points are connected by a string in a state of tension T. Let R and S be the mid-points of the other opposite pair of rods and these points are connected by a light rod in a state of thrust X. Show that $T \cdot (RS) = X \cdot (PQ)$. **IFoS 2021**

Q1. A square framework formed of uniform heavy rods of equal weight W joined together, is hung up by one corner. A weight W is suspended from each of the three lower corners, and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod. **[7c UPSC CSE 2020]**

Q2. A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely joined together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance 2b apart. A weight W is suspended from A. Find the trust in the rod BC. **[7c 2018 IFoS]**

Q3. A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\frac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$. **[6b 2017 IFoS]**

Q4. Two equal uniform rods AB and AC, each of length *l*, are freely joined at A and rest on a smooth fixed vertical circle of radius *r*. If 2θ is the angle between the rods, then find the relation between *l*, *r* and θ, by using the principle of virtual work. **[5d UPSC CSE 2014]**

Q5. A regular pentagon ABCDE, formed of equal heavy uniform bars joined together, is suspended from the joint A, and is maintained in form by a light rod joining the middle points of BC and DE. Find the stress in this rod. **[7c UPSC CSE 2014]**

Q6. Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely joined at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Find the tension in the string. **[7c UPSC CSE 2013]**

Q7. A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$a\left(1+\frac{W}{2\pi\lambda}\cos\alpha\right)$. **[8c 2012 IFoS]**

Q8. One end of a uniform rod AB, of length 2*a* and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2\sin^{-1}\left(\frac{3}{4}\right)$. **[7a 2011 IFoS]**

Q9. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by using the principle of virtual work that

$\tan\phi = \frac{3}{8} + \tan\theta$. **[8b UPSC CSE 2010]**

FORCES IN THREE DIMENSIONS

DEFINITION

(1) Dyname:- The combination of a force R and a couple G often called a dyname, and the quantities X, Y, Z, L, M, N are called the components or elements of the dyname.

(2) Central Axis:- If a system of forces is reduced to a force R and a couple $G\cos\theta$ such that the axis of the couple coincides with the line of action of the force R, then the very line is called the central axis of the given system.

Note:- From now onwards, we write K for $G\cos\theta$ so that

$$K = G\cos\theta$$

(3) Wrench:- Suppose that a system of forces is reduced to a force R and a couple of moment K whose axis coincides with the direction of the force R. Then the force R together with the couple K is called the Wrench of the system and is denoted by (R, K).

(4) Pitch:- The ratio K/R viz. the moment of the couple divided by the force is called the pitch of the system.

The pitch is a linear magnitude. When the pitch is zero, the wrench reduces to a single force. On the other side when the pitch is infinite, the wrench becomes a couple only. If a body rotates through small angle $d\theta$ about the axis and moves at the same time a distance dx along the axis, then the ratio $dx/d\theta$ is called the pitch of the screw. Clearly, the pitch is the rate of change of x along the axis as θ increasing.

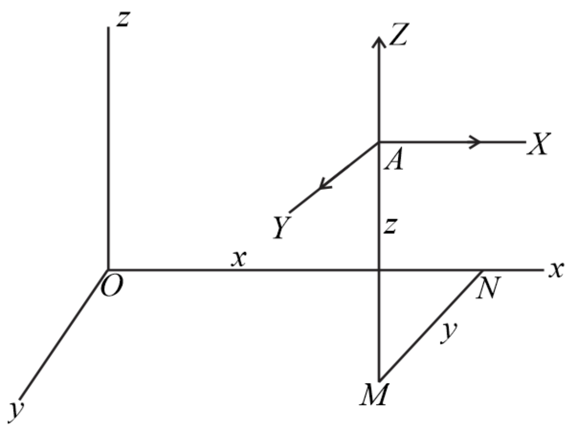
(5) Intensity of a Wrench:- The single force R is called the intensity of the wrench.

(6) Screw:- The straight line along which the single force acts when considered together with the pitch is called a screw, so that a screw is a definite straight line associated with a definite pitch.

(7) Moment of a Force about a line:- The moment of a force P about a given line is obtained as follows:

Resolve the force P into two components Q and S such that the force Q is parallel to the line and the force S is perpendicular to the line. The moment of the force P about the given line is defined to be the product of force S and the shortest distance between the line of action of the force S and the given line.

Suppose that a force R acting at a point A has components X, Y, Z along the coordinate axes ox, oy and oz respectively as shown in the figure. So, by the definition the moment of



The force R about ax axis is equal to the component $\sqrt{Y^2+Z^2}$ multiplied by the shortest distance between its line of action and ox line $\Rightarrow$ The moment of R about ox is equal to the moment of $\sqrt{Y^2+Z^2}$ about the point N. Since the algebraic sum of the moments of any two forces about any point in their plane is equal to the moment of their resultant about the same point. So the moment of the force R about ox line is equal to the sum of the moments of its two components Y and Z about N and this sum finally is equal to $yZ - zY$.

. The moment of the resultant couple about the Central Axis is less than moment of the resultant couple corresponding to any point O which is not on the Central Axis.

Proof:- As provide in Art. 12, the resultant force for any system of forces for any origin is the same and equal to that along the central axis. But the resultant couple differs.

If G is the couple for any origin (or base point), not on the central axis and if θ is the angle between the axis of the couple and the direction of the resultant force. Then the moment of the couple about the central axis has been proved to the $G\cos\theta$.
Clearly $G\cos\theta < G,\ 0 < \theta \le \pi/2$
Therefore, the moment of the resultant couple is minimum for the Central Axis.

2. General Conditions of Equilibrium of A Rigid Body.

Proof:- Suppose that a system of forces is reduced to a force R and a couple G. The couple G can be replaced by two equal and opposite forces one of which acts through the point O where R meets the plane of the couple. This force and R can be compounded into a single force which passes through O and does not meet the other force of the couple. So equilibrium is not possible. Hence a force R and a couple G together cannot produce equilibrium.

Hence the system can be in equilibrium only when the force R and the couple G vanish separately. But, by Art. 11,

$$R^2 = X^2 + Y^2 + Z^2 \text{ and } G^2 = L^2 + M^2 + N^2.$$

Hence for equilibrium we must have

$$X = 0 = Y = Z,\ L = M = N = 0$$

Which conclude that the sums of the resolved parts of the system of forces parallel to any three axes of the coordinates must separately vanish, and also the sums of their moments about the three axes must separately vanish.

3. To find the condition that a given system of forces should compound into a single force.

Proof:- In view of Art. 11, a system of forces is equivalent to a single force R acting at an arbitrary point (base point) and a single couple G and θ is the angle between the axis of couple G and the direction of the force R, fig. 1

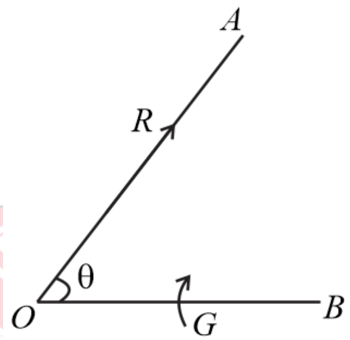


Fig. 1

The force R is equivalent to a force $R\cos\theta$ along OB and a force $R\sin\theta$ along $OC\left(OB \perp OC\right)$, fig. 2. Since the couple G acts in the plane DOC, so the couple G may be replaced by two forces each equal to $R\sin\theta$, one along OE and the other along DF (parallel and opposite in direction), fig. 3.

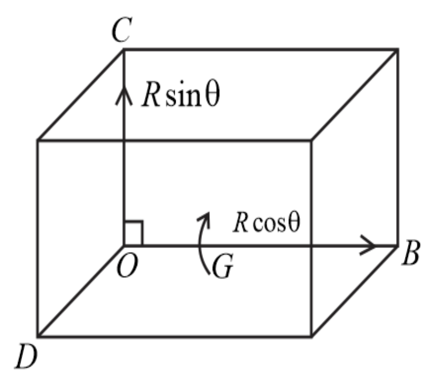


Fig. 2

The two force, each equal to $R\sin\theta$, acting to O balance. Now the system of forces is reduced to a force $R\cos\theta$ along OB and a force $R\sin\theta$ along DF. But the force $R\sin\theta$

does not pass through O, therefore the force $R\sin\theta$ cannot, in general, compound with $R\cos\theta$ into a single force.

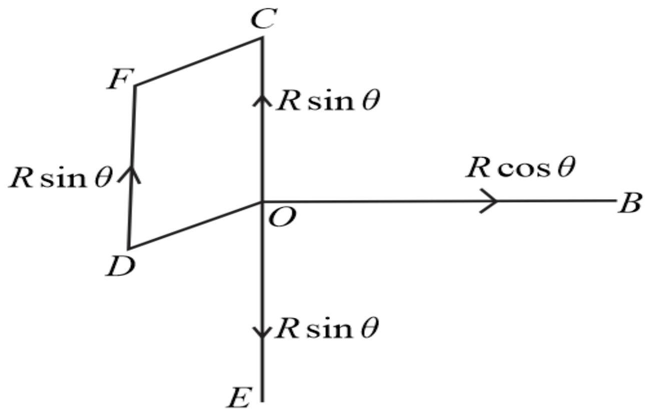


Fig. 3

But if $R\cos\theta = 0$

$\Rightarrow$ $\cos\theta = 0$ as $R \neq 0 \Rightarrow \theta = \pi/2$, then the system of the given force is reduced to a single force $R\sin\theta$. Hence the straight lines whose direction cosines are $\left(\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}\right)$ and $\left(\frac{L}{G}, \frac{M}{G}, \frac{N}{G}\right)$ are mutually perpendicular. So $\frac{X}{R}.\frac{L}{G} + \frac{Y}{R}.\frac{M}{G} + \frac{Z}{R}.\frac{N}{G} = \cos 90^{\circ}$ or $XL + YM + ZN = 0$ which is the required condition.

4. **Invariants:-** Whatever origin (or base point) and axes of coordinates are chosen, for any given system of forces the quantities $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$ are invariable where $X = \Sigma X$ etc. and $L = \Sigma\left(y_1 Z_1 - z_1 Y_1\right)$ etc.

Proof:- Since $R^2 = X^2 + Y^2 + Z^2$ and $G^2 = L^2 + M^2 + N^2$. The direction cosines of R are $\left(\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}\right)$ and the direction cosines of the axis of the couple G are $\left(\frac{L}{G}, \frac{M}{G}, \frac{N}{G}\right)$. If the direction of R makes angle θ with the axis of the couple. So $\cos\theta = \frac{X}{R}.\frac{L}{G} + \frac{Y}{R}.\frac{M}{G} + \frac{Z}{R}.\frac{N}{G}$

$\Rightarrow$ $$\frac{XL + YM + ZN}{R} = G\cos\theta = K \qquad (1)$$

We know that central axis is unique and both the force R and the couple K are found along the central axis. Hence R and K both are invariable. So $X^2 + Y^2 + Z^2$ is invariable and also from (1).

$XL + YM + ZN$ are in variable.

It follows that if $K = 0$ i.e. if the given system of forces reduces to a single force, then $LX + MY + NZ = 0$.

If $R = 0$, then $X^2 + Y^2 + Z^2 = 0$ and $LX + MY + NZ = 0$ (Both).

The pitch, p, of the resultant wrench of the system

$$= \frac{K}{R} = \frac{LX + MY + NZ}{R^2}$$

Thus for a given system of forces, R and $K = G\cos\theta$ are unique so that the wrench is unique.

5. To find the equation of the Central Axis of any given system of forces.

Proof:- Referred to the coordinates axes ox, oy, oz, let the system of forces $P_1, P_2, ..., P_n$ acting at points $A_1, A_2, ..., A_n$ respectively be equivalent to (R, G) where

$R^2 = X^2 + Y^2 + Z^2,\ G^2 = L^2 + M^2 + N^2$

$X = \Sigma X_1, Y = \Sigma Y_1, Z = \Sigma Z_1, L = \Sigma(y_1Z_1 - z_1Y_1),$

$M = \Sigma(z_1X_1 - x_1Z_1), N = (x_1Y_1 - y_1X_1)$ where $P_1 = (X_1, Y_1, Z_1)$ etc. and coordinate of A_1 are (x_1, y_1, z_1) etc.

Let (f, g, h) be the coordinate of any point Q. At Q the value of R remain invariant. Assume lines Qx', Qy', Qz' parallel to ox, oy and oz respectively. The moment of the force about ox' is obtained by putting $x_1 - f, y_1 - y, z_1 - h$ instead of x_1, y_1, z_1 in the values of L, M, N.

Hence the moment about Qx' line

$$= \sum_{i=1}^{n} \left[(y_i - g)Z_i - (z_1 - h)Y_i\right]$$

$$= \sum_{i=1}^{n}(y_iZ_i - z_iY_i) - g\sum_{i=1}^{n} Z_i + h\sum_{i=1}^{n} Y_i$$

$$= L - gZ + hY.$$

Similarly the moments about the liens Qy' and Qz' are $M - hX + fZ$ and $N - fY + gX$ respectively.

Also the components (X, Y, Z) of the resultant force R are the same for all points such as Q.

If Q be a point on the central axis, the direction cosines of the axis of the couple corresponding to the point Q are proportional to those of the resultant force.

Hence $\dfrac{L - gZ + hY}{X} = \dfrac{M - hX + fZ}{Y} = \dfrac{N - fY - gX}{Z}$

$= \dfrac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = \dfrac{K}{R}$ By Art. 18

The locus of the point Q is $\dfrac{L - yZ + zY}{X} = \dfrac{M - zX + xZ}{Y} = \dfrac{N - xY + yX}{Z}$ which is the equation of the central axis.

6. Working Rule:-

(1) To Find central axis:-

(a) Write down the equation of the line along which the force $P_r(X_r, Y_r, Z_r)$ acts, in the standard from $\frac{x-x_r}{l_r}=\frac{y-y_r}{m_r}=\frac{z-z_r}{n_r}$ where (l_r, m_r, n_r) are the actual direction cosines. Then the components (X_r, Y_r, Z_r) of the force P_r along the axes are given by $X_r = l_r P_r, Y_r = m_r P_r, Z_r = n_r P_r$. Then $X = \Sigma X_r, Y = \Sigma Y_r, Z = \Sigma Z_r$.

(b) The value s of L_r, M_r, N_r are given by the determinant

$$iL_r + jM_r + kN_r = \begin{vmatrix} i & j & k \\ x_r & y_r & z_r \\ X_r & Y_r & Z_r \end{vmatrix}.$$

By equation the coefficient of i, j, k on both the side of the above equation. We get L_r, M_r and N_r. Then $L = \Sigma L_r, M = \Sigma M_r, N = \Sigma N_r$.

(c) Now the equation of the central axis is given by

$$\frac{L-(yZ-zY)}{X}=\frac{M-(zX-xZ)}{Y}=\frac{N-(xY-yX)}{Z}$$

(2) The pitch of the wrench

$$p=\frac{K}{R}=\frac{LX+MY+NZ}{X^2+Y^2+Z^2}$$

(3) The system reduces to a single force if

$LX + MY + NZ = 0$.

Example:- Equal forces act along the coordinate axes and along the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-r}{n} \qquad (1)$$

Find the equations of the central axis of the system.

Solution:- Let the equal force be P. Then P acts along each of the given lines, viz. ox, oy, oz axes and the line (1) P acts along x-axis, i.e. $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

Components (X_1, Y_1, Z_1) of P are given by $X_1 = P,\ Y_1 = 0 = Z_1$

Components moments (L_1, M_1, N_1) of the force P about

$$ox;\ iL_1 + jM_1 + kN_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ X_1 & Y_1 & Z_1 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ P & 0 & 0 \end{vmatrix}$$

$$= 0i + 0j + 0k$$

$$\Rightarrow \qquad L_1 = M_1 = N_1 = 0$$

Similarly, along $oy-$ axis i.e. $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$

$$X_2 = 0,\ Y_2 = P,\ Z_2 = 0 \text{ and } L_2 i + M_2 j + N_2 k = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 & y_2 & z_2 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ 0 & P & 0 \end{vmatrix}$$

$$\Rightarrow \qquad L_2 = M_2 = N_2 = 0$$

Along the z-axis

$$X_3 = 0 = Y_3,\ Z_3 = P \text{ and } L_3 i + M_3 j + N_3 k = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_3 & y_3 & z_3 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ 0 & 0 & P \end{vmatrix}$$

$$= 0i + 0j + 0k$$

$$\Rightarrow \qquad L_3 = M_3 = N_3 = 0$$

Along the line

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, assuming (l,m,n) are $d.c.'s$

$$X_4 = lP, Y_4 = mP, Z_4 = nP$$

$$iL_4 + jM_4 + kN_4 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_4 & y_4 & z_4 \\ X_4 & Y_4 & Z_4 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha & \beta & \gamma \\ lP & mP & nP \end{vmatrix}$$

$$= (\beta n - \gamma m)Pi = (\gamma l - \alpha n)Pj + (\alpha m - \beta l)Pk$$

$$L_4 = (\beta n - \gamma m)P, M_4 = (\gamma l - \alpha n)P, N_4 = (\alpha m - \beta l)P$$

$$X = \Sigma X_1 = P + 0 + 0 + lP = (1+l)P$$

$$Y = \Sigma Y_1 = 0 + P + 0 + mP = (1+m)P$$

$$Z = \Sigma Z_1 = 0 + 0 + P + nP = (1+n)P$$

$$L = \Sigma L_1 = 0 + 0 + 0 + (\beta n - \gamma m)P = (\beta n - \gamma m)P$$

$$M = \Sigma M_1 = 0 + 0 + 0 + (\gamma l - \alpha n)P = (\gamma l - \alpha n)P$$

$$N = \Sigma N_1 = 0 + 0 + 0 + (\alpha n - \beta l)P = (\alpha n - \beta l)P$$

The equation of central axis is $\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$

Putting the value of respectively terms and cancelling P throughout, we get

$$\frac{(n\beta - m\gamma) - y(1+n) + z(1+m)}{1+l} = \frac{(l\gamma - n\alpha) - z(1+l) + x(1+n)}{1+m}$$
$$= \frac{(\alpha m - \beta l) - x(1+m) + y(1+l)}{1+n}$$

Which is the required equation of the central axis.

Note:- If (l, m, n) are not the actual direction cosines, then the actual direction cosines are

$$l_1 = \frac{l}{\mu},\ m_1 = \frac{m}{\mu}, n_1 = \frac{n}{\mu},\ \mu = \left(l^2 + m^2 + n^2\right)^{1/2}.$$

Example:- Forces X, Y, Z act along the three lines giving by the equations $y = 0, z = c$; $z = 0, x = a$; $x = 0\ y = b$; prove that the pitch of the equivalent wrench is $(aYz + bZX + cXY)/\left(X^2 + Y^2 + Z^2\right)$. If the wrench reduces to a single force, show that the line of action of the force lies on this hyperboloid. $(x-a)(y-b)(z-c) = xyz$.

Solution:- The three given lines are $\frac{x}{1} = \frac{y}{0} = \frac{z-c}{0}; \frac{x-a}{0} = \frac{y}{1} = \frac{z}{0}; \frac{x}{0} = \frac{y-b}{0} = \frac{z}{1}$

Force X acts along the first line, so $X_1 = X, Y_1 = 0 = Z_1$

$$iL_1 + jM_1 + kN_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ X_1 & Y_1 & Z_1 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & c \\ X & 0 & 0 \end{vmatrix}$$

$$i(0) + j(cX) + k(0)$$

$\Rightarrow \quad L_1 = 0,\ M_1 = cX,\ N_1 = 0$

Force Y acts along the second line, so $X_2 = 0,\ Y_2 = Y, Z_2 = 0$

$$iL_2 + jM_2 + kN_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 & y_2 & z_2 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & 0 \\ 0 & Y & 0 \end{vmatrix}$$

$$= i(0) + j(0) + k(aY)$$

$\Rightarrow \quad L_2 = 0,\ M_2 = 0,\ N_2 = aY$

Force Z acts along the third line, so $X_3 = 0,\ Y_3 = 0,\ Z_3 = Z$ and

$$iL_3 + jM_3 + kN_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_3 & y_3 & z_3 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & b & 0 \\ 0 & 0 & Z \end{vmatrix}$$

$$ibZ + j(0) + k(0)$$

$\Rightarrow \quad L_3 = bZ,\ M_3 = 0 = N_3$

$$X = \Sigma X_1 = X + 0 + 0 = X$$

$$Y = Y_1 + Y_2 + Y_3 = 0 + Y + 0 = Y$$

$$Z = Z_1 + Z_2 + Z_3 = 0 + 0 + Z = Z$$

$$L = L_1 + L_2 + L_3 = 0 + 0 + bZ = bZ$$

$$M = \Sigma M_1 = cX, N = \Sigma N_1 = aY$$

The pitch of wrench is given by

$$p = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = \frac{bZX + cXY + aYZ}{X^2 + Y^2 + Z^2}$$

$$= \frac{aYZ + bZX + cXY}{X^2 + Y^2 + Z^2}$$

Second Part:- The condition that the system of forces reduces to a single force is

$$LX + NY + NZ = 0$$

$$\Rightarrow \quad bZX + aYZ + cXY = 0 \qquad (1)$$

The equation of central axis is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

Putting the value of L, M, N

$$\frac{bZ - yZ + zY}{X} = \frac{cX - zX + xZ}{Y} = \frac{aY - xY + yX}{Z}$$

$$= p = \frac{aYZ + bZX + cXY}{X^2 + Y^2 + Z^2} = 0 \text{ by } \qquad (1)$$

$$\Rightarrow \quad bZ - yZ + zY = 0,\ cX - zX + xZ = 0,\ aY - xY + yX = 0$$

$$\Rightarrow \quad 0X + zY + (b - y)Z = 0 \qquad (2)$$

$$(c - z)X + 0Y + xZ = 0 \qquad (3)$$

$$yX + (a - x)Y + 0Z = 0 \qquad (4)$$

To find the line of action of the single force i.e. the locus of the central axis, eliminate X, Y, Z from the equations (2), (3), (4)

$$\begin{vmatrix} 0 & z & b - y \\ c - z & 0 & x \\ y & a - x & 0 \end{vmatrix} = 0$$

Expanding along the first row, we get

$$-z(-xy) + (b - y)(a - x)(c - z) = 0$$

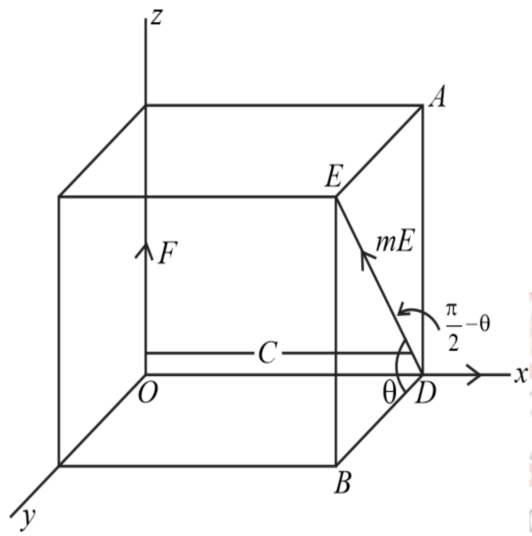
$$\Rightarrow \quad (x - a)(y - b)(z - c) = xyz$$

Which is a equation of hyperboloid.

Example:- A force F acts along the axis of z and a force mF along a straight line intersecting the axis of x at a distance c from the origin and parallel to $y-z$ plane. Show that as this line turns round the axis of x, the central axis of the system generates the surface $\{m^2z^2+(m^2-1)y^2\}(c-x)^2=x^2z^2$

Solution:- In the figure a parallopiped is shown in which $OD=c$ force mF is assumed to be acting along DE where $\angle BDE=\theta$ (say), then $\angle ADE=\pi/2-\theta$. Direction cosines of line DE are $\cos 90,\ \cos\theta,\ \cos(90-\theta)$ i.e. $(0,\cos\theta,\sin\theta)$, since line DE lies in a plane parallel to $y-z$ plane.

The equation of the line DE is $\dfrac{x-c}{0}=\dfrac{y}{\cos\theta}=\dfrac{z}{\sin\theta}$



Components of force mF parallel to axes are $X_1=0$, $Y_1=mF\cos\theta,\ Z_1=mF\sin\theta$ and

$$L_1i+M_1j+N_1k=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x_1&y_1&z_1\\X_1&Y_1&Z_1\end{vmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\c&0&0\\0&mF=\cos\theta&mF\sin\theta\end{vmatrix}$$

$$=0i-(mcF\sin\theta)j+(mcF\cos\theta)k$$

$\Rightarrow\qquad L_1=0,\ M_1=-mcF\sin\theta,\ N_1=mcF\cos\theta$

A force F acts along $z-$axis, i.e. $\dfrac{x}{0}=\dfrac{y}{0}=\dfrac{z}{1}$

So, its components (X_2,Y_2,Z_2) are $X_2=0,\ Y_2=0,\ Z_2=F$ and

$$L_2i+M_2j+N_2k=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x_2&y_2&z_2\\X_2&Y_2&Z_2\end{vmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&0\\0&0&F\end{vmatrix}$$

$$=0i+0j+0k$$

$\Rightarrow\qquad L_2=M_2=N_2=0$

$$X = \Sigma X_1 = 0,\ Y = \Sigma Y_1 = mF\cos\theta$$

$$Z = \Sigma Z_1 = mF\sin\theta + F = (1 + m\sin\theta)F$$

$$L = \Sigma L_1 = 0,\ \ M = \Sigma M_1 = -mcF\sin\theta$$

$$N = \Sigma N_1 = mcF\cos\theta$$

The equation of the central axis is

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

Putting the value of the respective terms we have

$$\frac{0 - y(1 + m\sin\theta)F + z(mF\cos\theta)}{0}$$

$$= \frac{-mcF\sin\theta - zX0 + x(1 + m\sin\theta)F}{mF\cos\theta}$$

$$= \frac{mcF\cos\theta - xmF\cos\theta + y\times 0}{(1 + m\sin\theta)F}$$

Or $$\frac{-y(1 + m\sin\theta) + zm\cos\theta}{0} = \frac{-mc\sin\theta + x(1 + m\sin\theta)}{m\cos\theta}$$

$$= \frac{mc\cos\theta - xm\cos\theta}{1 + m\sin\theta} \qquad (1)$$

We see that the equation of the central axis has θ as perimeter, so in order to find the locus of the central axis, eliminate θ.

The first two ratios of (1) give

$$-y(1 + m\sin\theta) + zm\cos\theta = 0 \qquad (2)$$

$$\Rightarrow \quad -y\sin\theta + z\cos\theta = y/m \qquad (3)$$

The last two ratios of equation (1) given

$$\frac{-mc\sin\theta + x(1 + m\sin\theta)}{m\cos\theta} = \frac{mc\cos\theta - xm\cos\theta}{1 + m\sin\theta}$$

$$\frac{(x - c)m\sin\theta + x}{m\cos\theta} = \frac{m\cos\theta(c - x)}{1 + m\sin\theta} = \frac{y}{z} = (c - x) \text{ using (2)}$$

$$\Rightarrow \quad (x - c)mz\sin\theta + xz = my(c - x)\cos\theta$$

$$\Rightarrow \quad y\cos\theta + z\sin\theta = \frac{xz}{m(c - x)} \qquad (4)$$

Squaring (3) & (4) and adding

$$y^2 + z^2 = \frac{y^2}{m^2} = \frac{x^2z^2}{m^2(c - x)^2}$$

$$\Rightarrow \quad m^2\left(y^2 + z^2\right)(c - x)^2 = y^2(c - x)^2 + x^2z^2$$

$$\Rightarrow \quad (c-x)^2\left\{(m^2-1)y^2+m^2z^2\right\}=x^2z^2.$$

Example:- Force X,Y,Z act along the straight lines $y=b, z=-c$; $x=-a, z=c$ and $x=a, y=-b$ respectively. Show that they will have a single resultant if $\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}=0$ and that the equations to its line of action are any two of three $\frac{y}{Y}-\frac{z}{Z}-\frac{a}{X}=0$, $\frac{z}{Z}-\frac{x}{X}-\frac{b}{Y}=0$, $\frac{x}{X}-\frac{y}{Y}-\frac{c}{Z}=0$.

Solution:- The standard equations of the given lines are

$$\frac{x}{1}=\frac{y-b}{0}=\frac{z+c}{0} \qquad \text{(i)}$$

$$\frac{x+a}{0}=\frac{y}{1}=\frac{z-c}{0} \qquad \text{(ii)}$$

$$\frac{x-a}{0}=\frac{y+b}{0}=\frac{z}{1} \qquad \text{(iii)}$$

Force X along the line (1), components of X along axes are $X_1=X, Y_1=0, Z_1=0$ and

$$L_1i+M_1j+N_1k=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & y_1 & z_1\\ X_1 & Y_1 & Z_1\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & b & -c\\ X & 0 & 0\end{vmatrix}$$

$$=0i-cXj-bXk$$

$$\Rightarrow \quad L_1=0,\ M_1=-cX,\ N_1=-bX$$

Force Y acts along the second line, components of Y along axes are given by $X_2=0,\ Y_2=Y,\ Z_2=0$ and

$$L_2i+M_2j+N_2k=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_2 & y_2 & z_2\\ X_2 & Y_2 & Z_2\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a & 0 & 0\\ 0 & Y & 0\end{vmatrix}$$

$$=(-cY)i+0j-aYk$$

$$\Rightarrow \quad L_2=-cY,\ M_2=0,\ N_2=-aY$$

Force Z is acting along the line (3), components along $z-$ axis are $X_3=0,\ Y_3=0, Z_3=Z$

and $$L_3i+M_3j+N_3k=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_3 & y_3 & z_3\\ X_3 & Y_3 & Z_3\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & -b & 0\\ 0 & 0 & Z\end{vmatrix}$$

$$=(-bZ)i-aZj+0k$$

$$\Rightarrow \quad L_3=-bZ,\ M_3=-aZ,\ N_3=0$$

So, $\quad X=\Sigma X_1=X,\ Y=Y, Z=Z$

$$L=\Sigma L_1=0+(-cY)+(-bZ)=-(cY+bZ)$$

$$M = \Sigma M_1 = -cX + 0 + (-aZ) = -(cX - aZ)$$

$$N = \Sigma N_1 = -bX - aY + 0 = -(bX + aY)$$

The system of forces is equivalent to a single force if $LX + MY + NZ = 0$.

Putting the values of the respective terms, $-(cY + bZ)X - (cX + aZ)Y - (bX + aY)Z = 0$

$\Rightarrow \quad aYZ + bZX + cXY = 0$

Dividing by XYZ we have.

$\dfrac{a}{X} = \dfrac{b}{Y} + 0$. The first part is over (4)

The equation of the central axis is $\dfrac{L - (yZ + zY)}{X} = \dfrac{M - (zX - xZ)}{Y} = \dfrac{N - (xY - yX)}{Z}$

Putting the values of respective terms

$$\frac{-(cY + bZ) - yZ + zY}{X} = \frac{-(cX + aZ) - zX + xZ}{Y}$$

$$= \frac{-(bX + aY) - xY + yX}{Z}$$

Using first ratio of equation (5) $p = 0$ (5)

$\Rightarrow \quad -(cY + bZ) - yZ + zY = 0$

$\Rightarrow \quad -\dfrac{c}{Z} - \dfrac{b}{Y} + \dfrac{y}{z} = 0$

$\Rightarrow \quad \dfrac{a}{X} + \dfrac{b}{Y} - \dfrac{b}{Y} - \dfrac{y}{Y} + \dfrac{z}{Z} = 0$, using (4)

$\Rightarrow \quad \dfrac{a}{X} - \dfrac{y}{Y} + \dfrac{z}{Z} = 0$ (6)

Similarly by using the other parts of the equation (5) we can derive

$\dfrac{z}{Z} - \dfrac{x}{X} - \dfrac{b}{Y} = 0$ (7)

$\dfrac{x}{X} - \dfrac{y}{Y} - \dfrac{c}{Z} = 0$ (8)

We see that any one of these equations (6), (7), (8) can be obtained from the other two by means of equation (4), so any two of these equations viz. (6), (7) , (8) are linearly independent. Hence any two of the equations represent the line of action of the single force.

Example:- Three forces each equal to P act on a rigid, body, one at the point $(a, 0, 0)$ parallel to oy, the second of the point $(0, b, 0)$ parallel to oz and the third at the point $(0, 0, c)$ parallel to ox, the axes being rectangular. Find the resultant wrench in magnitude and direction.

Solution:- The lines of action of the three forces, each equal to P, are

$$\frac{x-a}{0}=\frac{y}{1}=\frac{z}{0} \qquad (1)$$

$$\frac{x}{0}=\frac{y-b}{0}=\frac{z}{1} \qquad (2)$$

$$\frac{x}{1}=\frac{y}{0}=\frac{z-c}{0} \qquad (3)$$

Forces P acts along the line (1), components along the axes are $X_1=0,\ Y_1=P,\ Z_1=0$ and

$$L_1 i+M_1 j+N_1 k=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & y_1 & z_1\\ X_1 & Y_1 & Z_1\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & 0 & 0\\ 0 & P & 0\end{vmatrix}$$

$$=0i+0j+aPk$$

$\Rightarrow \quad L_1=0=M_1,\ N_1=aP$

Force P acts along the second line, components along axes are given by $X_2=0=Y_2,\ Z_2=P$ and

$$L_2 i+M_2 j+N_2 k=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_2 & y_2 & z_2\\ X_2 & Y_2 & Z_2\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & b & 0\\ 0 & 0 & P\end{vmatrix}$$

$$=bPi+0j+0k$$

$\Rightarrow \quad L_2=bP, M_2=0=N_2$

Again, force P acts along the third line, component along axes are given by $X_3=P,\ Y_3=0=Z_3$ and

$$L_3 i+M_3 j+N_3 k=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_3 & y_3 & z_3\\ X_3 & Y_3 & Z_3\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & c\\ P & 0 & 0\end{vmatrix}$$

$$=0i+cPj+0k$$

$\Rightarrow \quad L_3=0,\ M_3=cP,\ N_3=0$

Now $\quad X=\Sigma X_1=P,\ Y=\Sigma Y_1=P, Z=\Sigma Z_1=P$

$L=\Sigma L_1=bP,\ M=\Sigma M_1=cP, N=\Sigma N_1=aP$

$R^2+X^2+Y^2+Z^2=3P^2 \Rightarrow R=\sqrt{3}P$ and

$$\frac{K}{R}=\frac{LX+MY+NZ}{X^2+Y^2+Z^2}=\frac{bP^2+cP^2+aP^2}{3P^2}=\frac{a+b+c}{3}$$

$$\Rightarrow \quad K=\frac{(a+b+c)}{3}\sqrt{3}P=\frac{(a+b+c)P}{\sqrt{3}}$$

The equation of the central axis is $\dfrac{L-(yZ-zY)}{X}=\dfrac{M-(zX-xZ)}{Y}=\dfrac{N-(xY-yX)}{Z}=\dfrac{K}{R}$

Putting the values of the respectively terms

$$\frac{bP-yP+zP}{P}=\frac{cP-zP+xP}{P}=\frac{aP-xP+yP}{P}=\frac{a+b+c}{3}$$

$$\Rightarrow \quad b-y+z=c-z+x=a-x+y=\frac{a+b+c}{3}$$

$$\Rightarrow \quad x+\frac{a+2b+3c}{3}=y+\frac{b+2c+3a}{3}=z+\frac{c+2b+3c}{3} \qquad (4)$$

The wrench of the system is (R,K) where $R=P\sqrt{3}$ and $K=\frac{(a+b+c)P}{\sqrt{3}}$

The position of the wrench is given by the central axis (4).

PREVIOUS YEARS QUESTIONS IAS/IFoS (2008-2023)

FORCES IN THREE DIMENSIONS

Q1. The forces P, Q and R act along three straight lines $y=b, z=-c, z=c, x=-a$ and $x=a, y=-b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action. **[6b 2015 IFoS]**

Note- The way to prepare next segment of PYQs are comprising of previous chapters(Equilibrium, Virtual work and forces in 3D) and examples.

5. MOMENTS, EQUILIBRIUM OF CO-PLANAR FORCES

Q7 (b) **UPSC CSE 2023** A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If θ is the angle of inclination of the string with vertical and ϕ is the angle of inclination of the plane base of the hemisphere to the vertical, then find the value of ($\tan\phi$ -$\tan\theta$). **(15)**

6.(a) A heavy string, which is not of uniform density, is hung up from two points. Let T_1, T_2, T_3 be the tensions at the intermediate points A, B, C of the catenary respectively where its inclinations to the horizontal are in arithmetic progression with common difference β. Let ω_1 and ω_2 be the weights of the parts AB and BC of the string respectively. Prove that

(i) Harmonic mean of T_1, T_2 and $T_3 = \dfrac{3T_2}{1+2\cos\beta}$

(ii) $\dfrac{T_1}{T_3} = \dfrac{\omega_1}{\omega_2}$ **UPSC CSE 2021**

Q5(c) Three forces P, Q and R act along the sides BC, CA and AB of ΔABC in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$\dfrac{1+\cos\alpha}{P} + \dfrac{1+\cos\beta}{Q} + \dfrac{1+\cos\gamma}{R} = 0$. **IFoS 2022**

Q7(c) PR and QR are two equal heavy strings tied together at R and carrying a weight W at R. P and Q are two points in the same horizontal line and 2a is the distance between them. l is the length of each string and *h* is the depth of R below PQ. Prove that

(i) $l^2 - h^2 = 2c^2\left(\cosh\dfrac{a}{c} - 1\right)$,

(ii) Tension at P or $Q = \dfrac{1}{2h}\left\{lW + \left(l^2 + h^2\right)w\right\}$, where α, β, γ are the interior angles subtended at A, B, C respectively. **IFoS 2022**

Q1. A uniform rod, in vertical position, can turn freely about one of its ends and is pulled aside from the vertical by a horizontal force acting at the other end of the rod and equal to half its weight. At what inclination to the vertical will the rod rest? **[5d UPSC CSE 2020]**

Q2. A beam AD rests on two supports B and C, where AB = BC = CD. It is found that the beam will tilt when a weight of *p* kg is hung from A or when a weight of *q* kg is hung from D. Find the weight of the beam. **[6c UPSC CSE 2020]**

Q3. A cylinder of radius '*r*', whose axis is fixed horizontally, touches a vertical wall along a generating line. A flat beam of length *l* and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. Prove that the reaction of the cylinder is $\frac{W\sqrt{5}}{2}$ and the pressure on the wall is $\frac{W}{2}$. Also, prove that the ratio of radius of the cylinder to the length of the beam is $5+\sqrt{5}:4\sqrt{2}$. **[5d 2020 IFoS]**

Q4. A 2 meters rod has a weight of 2N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hung loads of 3N, 7N and 10N respectively. Find the point at which the rod must be supported if it is to remain horizontal. **[5c 2019 IFoS]**

Q5. A uniform rod AB of length 2*a* movable about a hinge at A rests with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, prove that the magnitude of reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$ where W is the weight of the rod. **[7a UPSC CSE 2016]**

Q6. Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a light rod AB. If the strings OA and OB make angles α and β with the rod AB, show that the angle θ which the rod makes with the vertical is given by

$\tan\theta = \frac{P+Q}{P\cos\alpha - Q\cot\beta}$. **[7b UPSC CSE 2016]**

Q7. A square ABCD, the length of whose sides is *a*, is fixed in a vertical plane with two of its sides horizontal. An endless string of length $l(>4a)$ passes over four pegs at the angles of the board and through a ring of weight W which is hanging vertically. Show that the tension of the string is $\frac{W(l-3a)}{2\sqrt{l^2-6la+8a^2}}$. **[7c UPSC CSE 2016]**

Q8. A weight W is hanging with the help of two strings of length *l* and 2*l* in such a way that the other ends A and B of those strings lie on a horizontal line at a distance 2*l*. Obtain the tension in the two strings. **[5c 2016 IFoS]**

Q9. A rod of 8 kg is movable in a vertical plane about a hinge at one end, another end is fastened a weight equal to half of the rod, this end is fastened by a string of length *l* to a point at a height *b* above the hinge vertically. Obtain the tension in the string. **[5d UPSC CSE 2015]**

Q10. A ladder of weight W rests with one end against a smooth vertical wall and the other end rests on a smooth floor. If the inclination of the ladder to the horizon is 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from slipping down.

[7b UPSC CSE 2011]

Q11. AB is a uniform rod, of length 8*a*, which can turn freely about the end A, which is fixed C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium.

Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$ W, where W is the weight of the end. **[7b 2011 IFoS]**

Q12. A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass *m* slides down its slant face which is inclined at an angle α to the horizontal plane. Prove that the acceleration of the wedge is,

$\dfrac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$. **[7c 2010 IFoS]**

6. FRICTION

Q5(c) A body of weight *w* rests on a rough inclined plane of inclination θ, the coefficient of friction, μ, being greater than $\tan\theta$. Find the work done in slowly dragging the body a distance 'b' up the plane and then dragging it back to the starting point, the applied force being in each case parallel to the plane.**UPSC CSE 2022**

Q1. One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. When the rod is on the point of sliding, then $AC^2 - AB^2 = BC^2$. If θ is the angle between AB and the horizontal line, then prove that the coefficient of friction is $\dfrac{\cot\theta}{2+\cot^2\theta}$. **[5c UPSC CSE 2019]**

Q2. A uniform rod of weight W is resting against an equally rough horizon and a wall, at and angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W\tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α? **[7b 2016 IFoS]**

Q3. Two equal ladders of weight 4 kg each are placed so as to lean at A against each other with their ends resting on a rough floor, given the coefficient of friction is μ. The ladders at A make an angle 60° with each other. Find what weight on the top would cause them to slip.

[6b UPSC CSE 2015]

Q4. A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coefficient of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is:

$\sin^{-1}\left(\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2}\right)$. **[8a 2014 IFoS]**

Q5. The base of an inclined plane is 4 metres in length and the height is 3 metres. A force of 8 kg acting parallel to the plane will just prevent a weight of 20 kg from sliding down. Find the coefficient of friction between the plane and the weight. **[5d UPSC CSE 2013]**

Q6. A uniform ladder rests at an angle of 45° with the horizontal with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ and μ' are the coefficients of limiting friction between the ladder and the ground and wall respectively, then find the minimum horizontal force required to move the lower end of the ladder towards the wall. **[7b UPSC CSE 2013]**

Q7. Two bodies of weight w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other.

[6c 2013 IFoS]

Q8. A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by

$$\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

μ, being the coefficient of friction. **[7b 2012 IFoS]**