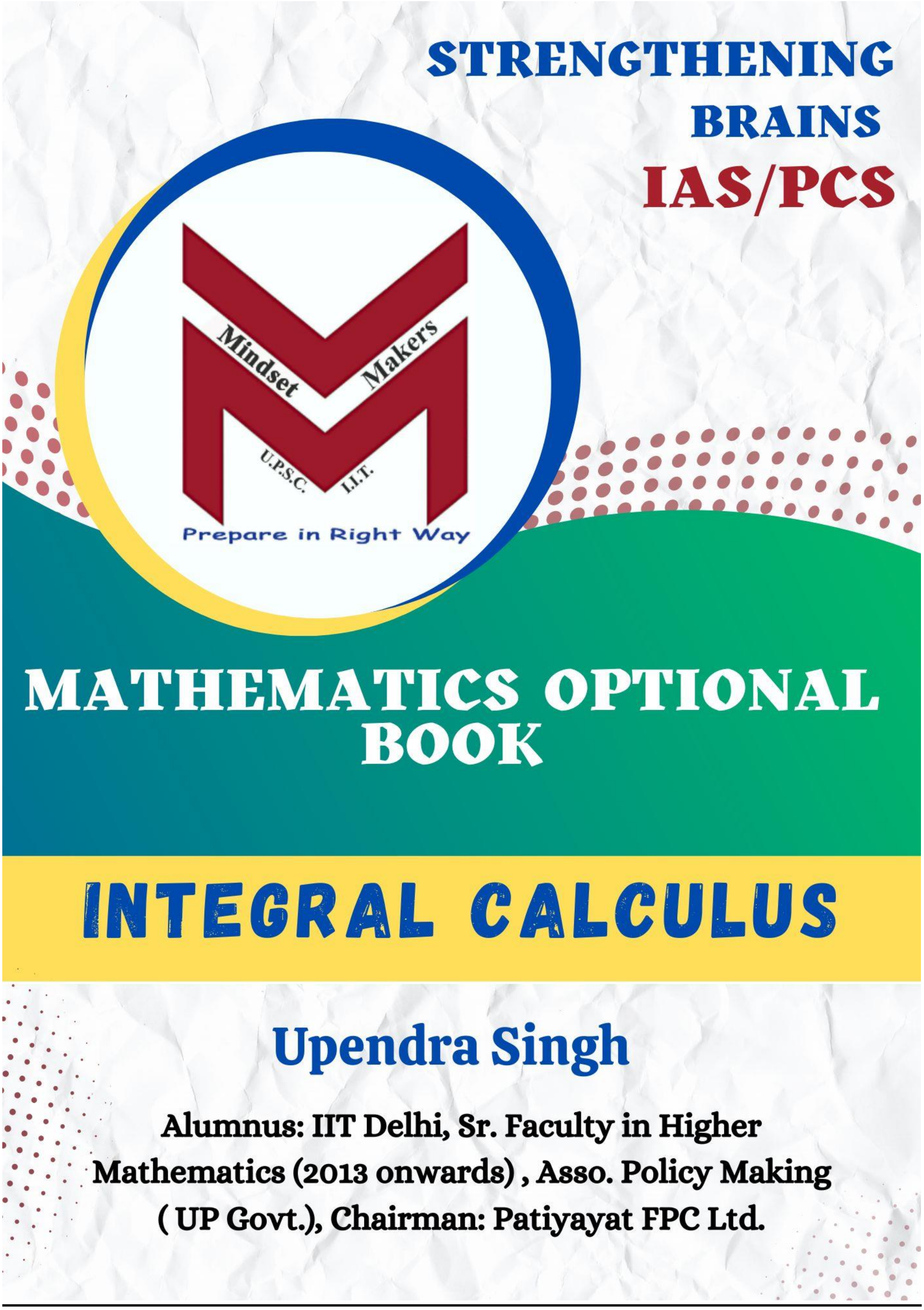

STRENGTHENING
BRAINS
IAS/PCS

MATHEMATICS OPTIONAL BOOK

INTEGRAL CALCULUS

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WELL PLANNED COURSE BOOK BASED ON DEMAND OF UPSC CSE IAS/IFOS :

01 Conceptual Development

02 Problem Solving Techniques

03 Assignments

04 Chapter wise PYQs Analysis

05 Test

MINDSET MAKERS

I.I.T UPSC

Integral Calculus

(ii) Cartesian $\rightarrow$ Cylindrical coordinates (116)

Use of Triple $\int$:

(a) Volume (117)

Integration as inverse process of differentiation:

If $F(x)$ is a function st $\frac{d}{dx}(F(x)) = f(x)$ then $\frac{d}{dx}(F(x)+c) = f(x)$ and $\int f(x)dx = F(x)+c \rightarrow$ Arbitrary constant ; $c \in \mathbf{R}$

Indefinite Integral

Note: As $\frac{d}{dx}x^n = nx^{n-1}$

So $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \qquad (n \neq -1)$

$$\Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

(2) $\int \frac{1}{x} dx = \log|x| + c \qquad (x \neq 0)$

(3) $\frac{d}{dx}e^x = e^x \Rightarrow \int e^x dx = e^x + c$

(4) $\int \cos x\, dx = \sin x + c$

(5) $\int \sin x\, dx = -\cos x + c$

(6) $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\, dx = \tan x + c$

(7) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x \Rightarrow \int \text{cosec}^2 x\, dx = -\cot x + c$

(8) $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x = \sec x + c$

(9) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x \Rightarrow \int \text{cosec}\, x \cot x\, dx = \cos ec\, x + c$

(10) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin x^{-1} x + c$

(11) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \Rightarrow \int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$

(12) $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}\Rightarrow\int\frac{1}{1+x^2}dx=\tan^{-1}x+c$

(13) $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}\Rightarrow\int\frac{-1}{1+x^2}dx=\cot^{-1}x+c$

(14) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}\Rightarrow\int\frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}x+c$

(15) $\frac{d}{dx}\left(\text{cosec}^{-1}x\right)=\frac{-1}{x\sqrt{x^2-1}}\Rightarrow\int\frac{-1}{x\sqrt{x^2-1}}dx=\text{cosec}^{-1}x+c$

Linearity Property of Indefinite Integrals:

If $\frac{d}{dx}F(x)=f(x)$ and $\frac{d}{dx}G(x)=g(x)$ then $\frac{d}{dx}\left(aF(x)+ba(x)\right)=af(x)+bg(x)$

$\Rightarrow\int af(x)+bg(x)dx=aF(x)+ba(x)$

$=a\int f(x)dx+b\int g(x)dx$

$\therefore\ \boxed{\int af(x)+bg(x)dx=a\int f(x)dx+b\int g(x)dx}$

Method of Substitution

If we have Integral of the form $\int\frac{f'(x)}{f(x)}dx$ or $\int\left[f(x)\right]^n f'(x)dx\ ;n\neq-1$

Then take var $y=f(x)$, on differentiation both side

$dy=f'(x)dx$

So $\int\frac{f'(x)}{f(x)}dx=\int\frac{1}{y}dy=\ln|y|+c$

$=\ln|f(x)|+c$

$\int\left(f(x)\right)^n f'(x)dx=\int y^n dy=\frac{y^{n+1}}{n+1}+c$

$=\frac{\left(f(x)\right)^{n+1}}{n+1}+c$

Q1. Find $\int\tan x\,dx$

$\int\frac{\sin x}{\cos x}dx$

Put $\cos x = t$

$-\sin x\, dx = dt$

$-\int \frac{1}{t} dt = -\log|t| + c$

$= -\log|\cos x| + c$

$= \log|\sec x| + c$

Q2. $\int \cot x\, dx = ?$

$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$

$\sin x = t$

$\cos x\, dx = dt$

$= \int \frac{1}{t} dt$

$= \ln|\sin x| + c$

Q3. $\int \sec x\, dx = ?$

$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$\sec x + \tan x = y$

$(\sec x \tan x + \sec^2 x) dx = dy$

$\int \frac{dy}{y} = \log|y| + c$

$= \log(\sec x + \tan x) + c$

Q4. $\int \operatorname{cosec} x\, dx = ?$

$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$

$= -\log|\operatorname{cosec} x + \cot x| + c$

$= \ln|\operatorname{cosec} x - \cot x| + c$

$\Rightarrow \operatorname{cosec}^2 x - \cot^2 x = 1$

Some Standard Substitution

(1) $\int a^x dx = \frac{a^x}{\ln a} + c$

Proof (1)

Let $a^x = e^t$

$t = x \ln a$

$\int a^x = \int e^{x \ln a} dx$

Put $x \ln a = y$

$dx = \frac{dy}{\ln a}$

$= \int \frac{e^y}{\ln a} dy$

$= \frac{e^y}{\ln a} + c$

$= \frac{e^{x \ln a}}{\ln a} + c$

$= \frac{a^x}{\ln a} + c$

(2) $\int f(ax+b)dx = \frac{1}{a}\int f(y)dy$

$y = ax + b \Rightarrow dy = adx$

$= \frac{1}{a}\int f(ax+b)dx$

(3) $\int \frac{dx}{a^2+x^2} = I\left(\frac{1}{\text{quad}}\text{type}\right)$

Replace x by $a\tan\theta, dx = a\sec^2\theta d\theta$

$I = \int \frac{dx}{a^2 + a^2\tan^2\theta} = \int \frac{a\sec^2\theta d\theta}{a^2\left(1+\tan^2\theta\right)}$

$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta} d\theta$

$= \frac{1}{a}\theta + c$

$$=\frac{1}{a}\tan^{-1} x\!\!\diagup\!\!_{a}+c$$

or

$$I=\frac{1}{a^2}\int\frac{dx}{1+\left(x\!\diagup\!_a\right)^2}=\frac{a}{a^2}\tan^{-1}\frac{x}{a}+c$$

$$=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

(4) $\int\frac{dx}{x^2-a^2}=\int\frac{1}{(x+a)(x-a)}dx$

$$=\frac{1}{2a}\int\frac{(x+a)-(x-a)}{(x+a)(x-a)}dx$$

$$=\frac{1}{2a}\int\frac{1}{x-a}-\frac{1}{x+a}dx$$

$$=\frac{1}{2a}\left[\ln(x-a)-\ln(x+a)\right]+c$$

$$=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$$

(5) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}-\ln\left|\frac{x-a}{x+a}\right|+c$

$$=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+c$$

(6) $\left(\frac{1}{\sqrt{\text{quadrant}}}\right)$type

$I=\int\frac{dx}{\sqrt{a^2+x^2}}$ (we want form $\sqrt{\ }$)

So put $x=a\tan\theta\Rightarrow\sec\theta=\sqrt{1+\left(\frac{x}{a}\right)^2}$

$$dx=a\sec^2\theta\,d\theta$$

$$I=\int\frac{\cancel{a}\sec^2\theta\,d\theta}{\cancel{a}\sec\theta}$$

$$=\int\sec\theta\,d\theta=\log\left|\sec\theta+\tan\theta\right|+c$$

$$= \log\left|\sqrt{1+\left(\frac{x}{a}\right)^2}+\frac{x}{a}\right|+c$$

or

$$I = \log\left|\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right|+c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|+c$$

(7) $\int \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\sec\theta \Rightarrow \tan\theta = \sqrt{\frac{x^2}{a^2}-1}$

or $x = a\text{cosec}\,\theta$

Take $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore \quad I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}d\theta$$

$$= \int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}}{\cancel{a}\,\cancel{\tan\theta}}d\theta$$

$$= \log\left|\sec\theta+\tan\theta\right|+c$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c$$

$$= \log\left|x+\sqrt{x^2-a^2}\right|+c$$

(8) $\int \frac{dx}{\sqrt{a^2-x^2}}$

$$I = \int \frac{1}{a}\frac{dx}{\sqrt{1-\left(x/a\right)^2}} = \frac{1}{\cancel{a}}\int \frac{\cancel{a}\,dy}{\sqrt{1-y^2}}$$

$$= \sin^{-1} y + c$$

$$= \sin^{-1} x/a + c$$

$$x/a = y$$

$$dx = a\,dy$$

Q1.Find $\int \frac{dx}{-2x^2+5x+7} = \mathrm{I}$

$$I = \int \frac{dx}{-2\left(x^2 - \frac{5}{2}x - \frac{7}{2}\right)}$$

$$= \int \frac{dx}{-2\left(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{7}{2} - \frac{25}{16}\right)}$$

$$= \int \frac{dx}{-2\left(\left(x - \frac{5}{4}\right)^2 + \left(7 + \frac{25}{8}\right)\right)}$$

$$= \frac{1}{2}\int \frac{dx}{\left(9/4\right)^2 - \left(x - \frac{5}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot 9/4}\ln\left|\frac{x - 5/4 + 9/4}{x - 5/4 - 9/4}\right| + c$$

$$= \frac{1}{9}\ln\left|\frac{4(x+1)}{4x-19}\right| + c$$

Q2. Find $\int \frac{dx}{\sqrt{x^2+x+1}}$

$$I = \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}} = \log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1}\right| + c$$

Q3. Find $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx = I$

$$I = \int \frac{\sin x + \cos x}{\sqrt{2\sin x\cos x}}\,dx$$

$$=\frac{1}{\sqrt{2}}\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$

Let $y=\sin x-\cos x$

$dy=\cos x+\sin x\,dx$

$y^2=\sin x^2+\cos^2 x-2\sin x\cos x$

$y^2=1-2\sin x\cos x$

$2\sin x\cos x=1-y^2$

$$\therefore\quad I=\int\frac{dy}{\sqrt{1-y^2}}$$

$=\sin^{-1}y+c$

$=\sin^{-1}(\sin x-\cos x)+c$

(9) $\displaystyle\int\frac{p(x)}{ax^2+bx+c}dx$

Where $p(x)$ is any polynomial.

$$ax^2+bx+c\overset{Q(x)}{\overline{\big)\,p(x)}}$$

$r(x)$

$$\frac{p(x)}{ax^2+bx+c}=Q(x)+r(x)$$

$$r(x)=\frac{k(2ax+b)}{ax^2+bx+c}+\frac{c}{ax^2+bx+c}$$

$$I=\int Q(x)+\frac{k(2ax+b)}{ax^2+bx+c}+\frac{c}{ax^2+bx+c}dx$$

Q4. $\displaystyle\int\frac{x^4+1}{x^2+1}dx$

$$I=\int x^2-1+\frac{2}{x^2+1}dx$$

$x^3-x+2\tan^{-1}x+c$

$$\begin{array}{r} x^2-1 \\ x^2+1\overline{)\,x^4+1} \\ \underline{-x^4+x^2} \\ 1-x^2 \\ 1+x^2 \\ \hline 2 \end{array}$$

Q5. $\int \frac{1}{1+x^4}dx = I$

$$I = \int \frac{1}{x^2\left(x^2+\frac{1}{x^2}\right)}dx$$

$$I = \int \frac{1}{2}\frac{\left(1+\frac{1}{x^2}\right)-\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx$$

$$= \frac{1}{2}\int 1-\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int 1-\frac{1-{}^{1}\!/_{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

$$x+\frac{1}{x}=t$$

$$\left(1-\frac{1}{x^2}\right)dx = dt$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Put $x-\frac{1}{x}=y$

$$x+\frac{1}{x}=z$$

$\left(1+\frac{1}{x^2}\right)dx=dy$ $\left(1-\frac{1}{x^2}\right)dx=dz$

$$=\frac{1}{2}\int\frac{dy}{y^2+2}-\frac{1}{2}\int\frac{dz}{z^2-2}$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}{}^{y}\!\diagup_{\sqrt{2}}-\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\ln\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)+c$$

$$=\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x-{}^{1}\!\diagup_{x}}{\sqrt{2}}-\frac{1}{4\sqrt{2}}\ln\left(\frac{x+{}^{1}\!\diagup_{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+c$$

Q6. $\int\frac{dx}{x^4+x^2+1}=I$

$$\int\frac{dx}{x^2\left(x^2+1+\frac{1}{x^2}\right)}=\frac{1}{2}\int\frac{\left(1+{}^{1}\!\diagup_{x^2}\right)-\left(1-{}^{1}\!\diagup_{x^2}\right)}{\left(x^2+1+{}^{1}\!\diagup_{x^2}\right)}dx$$

$$=\frac{1}{2}\int\frac{1+{}^{1}\!\diagup_{x^2}}{x^2+{}^{1}\!\diagup_{x^2}+1}dx-\frac{1}{2}\int\frac{1-{}^{1}\!\diagup_{x^2}}{x^2+{}^{1}\!\diagup_{x^2}+1}dx$$

$$=\frac{1}{2}\int\frac{1+{}^{1}\!\diagup_{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx-\frac{1}{2}\int\frac{1-{}^{1}\!\diagup_{x^2}}{\left(x+{}^{1}\!\diagup_{x}\right)^2-1}dx$$

Put $x-\frac{1}{x}=y$ $x+\frac{1}{x}=z$

$\left(1+{}^{1}\!\diagup_{x^2}\right)dx=dy$ $\left(1-\frac{1}{x^2}\right)dx=dz$

$$I=\frac{1}{2}\int\frac{dy}{y^2+3}-\frac{1}{2}\int\frac{dz}{z^2-1}$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}{}^{y}\!\diagup_{\sqrt{3}}-\frac{1}{2}\cdot\frac{1}{2\cdot 1}\ln\frac{z-1}{z+1}+c$$

$$=\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x-{}^{1}\!\diagup_{x}}{\sqrt{3}}-\frac{1}{4}\ln\frac{x+{}^{1}\!\diagup_{x}-1}{x+{}^{1}\!\diagup_{x}+1}+c$$

Q7. $\int\frac{x^2+1}{x^4+x^2+1}dx=I$

$$I=\int\frac{x^2+1}{x^2\left(x^2+{}^{1}\!/\!_{x^2}+1\right)}dx$$

$$=\int\frac{1+{}^{1}\!/\!_{x^2}}{\left(x-{}^{1}\!/\!_{x}\right)^2+3}dx$$

$$x-\frac{1}{x}=t$$

$$\left(1+{}^{1}\!/\!_{x^2}\right)dx=dt$$

$$=\int\frac{dt}{t^2+3}$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\ {}^{t}\!/\!_{\sqrt{3}}$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{x-{}^{1}\!/\!_{x}}{\sqrt{3}}$$

Q8. $\int\frac{3x+4}{x^4+x^2+1}dx$

$$x^2=t$$

$$2x\,dx=dt$$

$$I=\int\frac{3x}{x^4+x^2+1}dx+4\int\frac{1}{x^4+x^2+1}dx$$

$$x^2=t$$

$$2x\,dx=dt$$

$$=\frac{3}{2}\int\frac{dt}{t^2+t+1}+4\ \text{(already done)}$$

$$=\frac{3}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+1-\frac{1}{4}}+\dots$$

$$=\frac{3}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}+\dots$$

$$=\frac{3}{\cancel{2}}\left(\tan^{-1}\frac{2\left(t+\frac{1}{2}\right)}{\sqrt{3}}\right)\cdot\frac{\cancel{2}}{\sqrt{3}}+\ldots$$

$$=\sqrt{3}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+\ldots$$

Q10. $\int\frac{dx}{(ax+b)\sqrt{cx+d}}$

$ax+b$

$\frac{a\left(t^2-c\right)}{d}+b$

Put $\left(x+d=t^2\right)$ (to get rig of $\sqrt{\ }$)

$c\,dx=2t\,dt$

$$\int\frac{dx}{(ax+b)\sqrt{(x+d)}}=\frac{1}{c}\int\frac{2t}{t\left[a\left(\frac{t^2-c}{d}\right)+b\right]}$$

$$=\frac{2}{c}\int\frac{dt}{\frac{a}{d}\left(t^2-c\right)+b}$$

Q9. $\int\frac{dx}{(2x+3)\sqrt{4x+5}}$

Put $4x+5=t^2$

$4dx=2t\,dt$

$$\frac{1}{2}\int\frac{t\,dt}{t\cdot\left[\frac{2\left(t^2-5\right)}{4}+3\right]}=\frac{1}{2}\int\frac{dt}{\frac{t^2-5}{2}+3}$$

$$=\frac{1}{\cancel{2}}\int\frac{\cancel{2}\,dt}{t^2-5+6}$$

$$=\int\frac{dt}{t^2+1}$$

$$=\tan^{-1}t+c$$

$$=\tan^{-1}\sqrt{4x+5}+c$$

Q10. $\int e^{x^4} x^3 dx$

$x^4 = t$

$\frac{1}{4}\int e^t dt$

$4x^3 dx = dt$

$= \frac{e^t}{4}$

$= \frac{e^{x^4}}{4} + c$

Integration by Parts:

If $\frac{d}{dx}F(x)=f(x)$ and $\frac{d}{dx}G(x)=g(x)$ now product of derivative:

$$\frac{d}{dx}F(x)\cdot G(x)=f(x)G(x)+g(x)F(x)$$

$$\Rightarrow \boxed{\int f(x)G(x)+g(x)F(x)dx=F(x)\cdot G(x)+c}$$

$$=\int f(x)dx\cdot\int g(x)\cdot dx+c$$

$$\Rightarrow \int f(x)G(x)dx=G(x)\int f(x)dx-\int g(x)F(x)dx$$

$$=a(x)\int f(x)dx-\int\frac{d}{dx}G(x)\cdot\left(\int f(x)dx\right)dx$$

Let $f(x)=v$

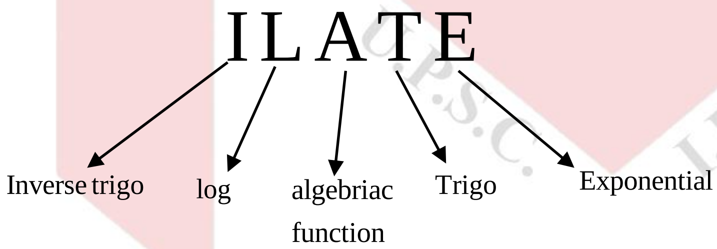
$G(x)=u$

$$\Rightarrow \boxed{\int v\cdot u\,dx=u\int vdx-\int\left(\int v\,dx\right)\left(\frac{du}{dx}\right)dx}$$

If $u(x)$ and $v(x)$ are functions of x then

$$\int u\cdot v\,dx=u\int v\,dx-\int\left(\int v\,dx\cdot\frac{du}{dx}\right)dx$$

(the one easy to $\int$ should be v)



* For choice of u follow ILATE (i.e. preference of u)

Note: IF we cannot evaluate $\int f(x)dx$ directly then to apply integration by part in it, write

$$\int f(x)dx=\int \underset{\substack{\uparrow\\ v}}{1}\cdot \underset{\substack{\uparrow\\ u}}{f(x)}dx$$

$$= f(x)\cdot x - \int xf'(x)\,dx$$

Q1. Find $\int \ln x\,dx$

$$\int 1\cdot \ln x\,dx = \ln x \cdot x - \int x\cdot\frac{1}{x}\,dx$$

$$= \ln x \cdot x - x + c$$

Q2. $\int \sin^{-1} x\,dx$

$$\int 1\cdot \sin^{-1} x\,dx = \sin^{-1} x\cdot x - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$-\frac{1}{2}\int\frac{1}{\sqrt{t}}\,dt$$

$$1-x^2 = t$$

$$-2x\,dx = dt$$

$$-\frac{1}{2}\ln t\ t^{½}$$

$$= x\sin^{-1} x + \left(1-x^2\right)^{½}$$

Q3. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2}$

Q4. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int x\cdot\frac{1}{1+x^2}\,dx$

$$= x\tan^{-1} x - \int\frac{x}{1+x^2}dx$$

$$1+x^2 = y$$

$$2x\,dx = dy$$

$$= x\tan^{-1} x - \frac{1}{2}\int\frac{1}{y}\,dy$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right| + c$$

Q5. $\int \cot^{-1} x\,dx = x\cot^{-1} x + \frac{1}{2}\log\left|1+x^2\right| + c$

Q6. $\int \sec^{-1} x\,dx = x\sec^{-1} x - \int \cancel{x}\cdot\frac{1}{\cancel{x}\sqrt{x^2-1}}\,dx$

$= x\sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}}dx$

$= x\sec^{-1} x - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + c$

Q7. $\int \text{cosec}^{-1}x = x\,\text{cosec}^{-1}x + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + c$

Note: If $f(x)$ is any differentiable function then $\boxed{\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+c}$

$\int e^x f(x)dx = f(x)e^x - \int e^x f'(x)\,dx + c$

$\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+c$

Q1. $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx = e^x\tan^{-1}x+c$

Q2. $\int e^x\left(\frac{x-1}{x^2}\right)dx$

$= \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{e^x}{x}+c$

Q3. $\int e^x x^3 dx = x^3e^x - \int e^x\cdot 3x\,dx$

or $\int e^x\left(x^3+3x^2-3x^2-6x+6x+6-6\right)dx$

$e^x\cdot x^3 - e^x\cdot 3x^2 + 6e^x\cdot x - 6e^x$

$= e^x\left(x^3-3x^2+6x-6\right)+c$

Q4. $\int e^x\left(\ln x+\frac{1}{x^2}\right)dx$

$= \int e^x\left(\ln x+\frac{1}{x}-\frac{1}{x}+\frac{1}{x^2}\right)dx$

$= e^x\left(\ln x-\frac{1}{x}\right)+c$

Q5. $\int e^x \sin x = ?$

$$\sin x = \frac{(\sin x + \cos x) + (-\cos x + \sin x)}{2}$$

$$\frac{1}{2}\int e^x \left[(\sin x + \cos x) + (-\cos x + \sin x)\right] dx$$

$$= \frac{1}{2}\left[e^x(\sin x - \cos x)\right] + c$$

or

Let $I = \int e^x \sin x\, dx = \sin x \cdot e^x - \int e^x \cos x\, dx$

$$I = \sin x\, e^x - \cos x\, e^x - \int e^x \sin x\, dx$$

$$= (\sin x - \cos x)e^x - I$$

$$2I = (\sin x - \cos x)e^x$$

$$I = \frac{e^x(\sin x - \cos x)}{2}$$

Q. Find $\int e^x \cos x\, dx$

$$\frac{d}{dx}\sin x = \cos x$$

$$\cos x = \frac{(\cos x - \sin x) + (+\sin x + \cos x)}{2}$$

$$I = \int e^x \frac{(\cos x - \sin x) + (\sin x + \cos x)}{2} dx$$

$$= \frac{e^x}{2}(\cos x + \sin x) + c$$

Q. $\int e^x (x^3 + 7x^2 + 5x + 11) dx$

[can write it directly]

$$= \left[(x^3 + 7x^2 + 5x + 11) - (3x^2 + 14x + 5) + (6x + 14 - 6)\right] + c$$

$$x^3 + 3x^2 + 7x^2 + 14x + 5x + 5 + 11 - 3x^2 - 14x - 5 + 6x + 14 - 6$$

By using recursive formula and using integral by parts:

$$\int e^{ax}\sin(bx+c)\,dx\ ;\ a\neq 0; b\neq 0; b\neq 1;\ c\in\mathbf{R}$$

$$\int e^{ax}\cos(bx+c)\,dx$$

Note: To find $\int e^{ax}\sin(bx+c)\,dx$ we take

$$I=\int e^{ax}\cdot\sin(bx+c)\,dx$$

Now by using integration by parts:

$$I=\sin(bx+c)\cdot\frac{e^{ax}}{a}-\int\frac{\cos(bx+c)\cdot b}{a}\cdot e^{ax}dx$$

$$=\sin(bx+c)\frac{e^{ax}}{a}-\frac{b}{a}\int\cos(bx+c)e^{ax}dx$$

$$=\sin(bx+c)\frac{e^{ax}}{a}-\frac{b}{a}\left[\cos(bx+c)\cdot\frac{e^{ax}}{a}+\int\sin(bx+c)\cdot b\cdot\frac{e^{ax}}{a}dx\right]$$

$$I=\sin(bx+c)\cdot\frac{e^{ax}}{a}-\frac{b}{a^2}\cos(bx+c)\cdot e^{ax}-I\cdot\frac{b^2}{a^2}$$

$$\left(1+\frac{b^2}{a^2}\right)I=\left(\sin(bx+c)-\frac{b}{a}\cos(bx+c)\right)\frac{e^{ax}}{a}$$

$$\frac{(a^2+b^2)}{a^2}I=\frac{(a\sin(bx+c)-b\cos(bx+c))e^{ax}}{a^2}$$

$$I=\frac{[a\sin(bx+c)-b\cos(bx+c)]}{(a^2+b^2)}e^{ax}$$

$$\int e^{ax}\sin(bx+c)\,dx=\frac{e^{ax}}{a^2+b^2}\cdot[a\sin(bx+c)-b\cos(bx+c)]$$

$$I=\int e^{ax}\cos(bx+c)\,dx=\int e^{ax}\sin\left(bx+c+\frac{\pi}{2}\right)dx$$

$$=\frac{e^{ax}}{a^2+b^2}\left(a\sin\left(bx+c+\frac{\pi}{2}\right)-b\cos\left(bx+c+\frac{\pi}{2}\right)\right)$$

$$I=\frac{e^{ax}}{a^2+b^2}(a\cos(bx+c)+b\sin(bx+c))$$

Q1. $\int e^{5x}\sin 4x\,dx=\frac{e^{5x}}{5^2+4^2}(5\sin 4x-4\cos 4x)$

$$=\frac{e^{5x}}{41}\left(5\sin 4x-4\cos 4x\right)$$

or

Let $5x=y$

$5dx=dy$

$$\frac{1}{5}\int e^{y}\sin\frac{4y}{5}dy=\frac{1}{5}\int e^{y}\left[\sin\frac{4y}{5}+\left(\cos\frac{4y}{5}\right)\cdot\frac{4}{5}\right]+\left[-\left(\cos\frac{4y}{5}\right)\cdot\frac{4}{5}+\sin\frac{4y}{5}\right]$$

$$=\frac{1}{5}e^{y}\left(\sin\frac{4y}{5}-\cos\frac{4y}{5}\cdot\frac{4}{5}\right)$$

$$=\frac{1}{\cancel{25}}e^{y}\frac{\left(5\sin 4x-\cos 4x\cdot 4\right)}{41}\cancel{25}$$

Divide by ${}^{41}\!/_{25}$

coefficient of $\sin {}^{4y}\!/_{5}$ is ${}^{41}\!/_{25}$

Q. $\int\sqrt{x^{2}+a^{2}}dx=?$

$$\int 1\cdot\sqrt{x^{2}+a^{2}}dx=\sqrt{x^{2}+a^{2}}\cdot x\int\frac{x}{\sqrt{x^{2}+a^{2}}}\cdot\frac{1}{\cancel{2}}\cdot\cancel{2}x\,dx$$

$$=\sqrt{x^{2}+a^{2}}\cdot x-\int\frac{x^{2}}{\sqrt{x^{2}+a^{2}}}dx$$

$$=x\cdot\sqrt{x^{2}+a^{2}}-\left(\int\frac{x^{2}+a^{2}}{\sqrt{x^{2}+a^{2}}}dx-\frac{a^{2}}{\sqrt{x^{2}+a^{2}}}dx\right)$$

$$=x\sqrt{x^{2}+a^{2}}-I+\int\frac{a^{2}}{\sqrt{x^{2}+a^{2}}}dx$$

$$2I=x\sqrt{x^{2}+a^{2}}+a^{2}\int\frac{dx}{\sqrt{x^{2}+a^{2}}}$$

$$I=\frac{x\sqrt{x^{2}+a^{2}}+a^{2}\log\left(x+\sqrt{x^{2}+a^{2}}\right)}{2}+c$$

Q. $\sqrt{x^{2}-a^{2}}dx=?$

$$I=\frac{x\sqrt{x^2-a^2}+a^2\log\left|\left(x+\sqrt{x^2-a^2}\right)\right|}{2}+c$$

Q. $\int\sqrt{a^2-x^2}dx=?$

$$I=\frac{x\sqrt{a^2-x^2}+a^2\sin^{-1}x/a}{2}+c$$

$$\int 1\cdot\sqrt{a^2-x^2}dx=x\sqrt{a^2-x^2}-\int\frac{x\left(-\cancel{2}x\right)+a^2-a^2}{\cancel{2}\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\left(\frac{a^2-x^2}{\sqrt{a^2-x^2}}-\frac{a^2}{\sqrt{a^2-x^2}}\right)dx$$

$$=x\sqrt{a^2-x^2}-I+\int\frac{a^2}{\sqrt{a^2-x^2}}dx$$

$$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}x/a+c$$

$$I=\frac{x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}}{2}+c$$

Q. $\int\sqrt{\frac{x-a}{b-x}}dx$ [part of trigonometric substitution]

$a,b\neq 0$

Let $x=a\cos^2\theta+b\sin^2\theta$

$$dx=\left(2a\cos\theta\left(-\sin\theta\right)+2b\sin\theta\cos\theta\right)d$$

$$=\left(2b-2a\right)\sin\theta\cdot\cos\theta\,d\theta$$

$$\int\sqrt{\frac{a\left(\cos^2\theta-1\right)+b\sin^2\theta}{b\left(1-\sin^2\theta\right)-a\cos^2\theta}}\cdot 2\sin\theta\cos\theta\left(b-a\right)d\theta$$

$$\int\sqrt{\frac{-a\sin^2\theta+b\sin^2\theta}{b\cos^2\theta-a\cos^2\theta}}\cdot 2\sin\theta\cos\theta\left(b-a\right)d\theta$$

$$\int\cancel{\sqrt{\frac{b-a}{b-a}}}\frac{\sin\theta}{\cancel{\cos\theta}}\cdot 2\sin\theta\cancel{\cos\theta}\left(b-a\right)d\theta$$

$$\left(b-a\right)\int\sin^2\theta\,d\theta=2\left(b-a\right)\int\sin^2\theta\,d\theta$$

$$=(b-a)\int(1-\cos 2\theta)\,d\theta$$

$$=(b-a)\left[\theta-\frac{\sin 2\theta}{2}\right]+c$$

Now

$$x-a=(b-a)\sin^2\theta$$

$$\Rightarrow \sin\theta=\sqrt{\frac{x-a}{b-a}}$$

Similarly $\cos\theta=\sqrt{\frac{b-x}{b-a}}$

So $\sin 2\theta=\frac{2\sqrt{(x-a)(b-x)}}{b-a}$

$$\theta=\sin^{-1}\left(\sqrt{\frac{x-a}{b-a}}\right)$$

So

$$=(b-a)\left(\sin^{-1}\sqrt{\frac{x-a}{b-a}}-\frac{\cancel{2}\sqrt{(x-a)(b-x)}}{\cancel{2}(b-a)}\right)$$

$$=(b-a)\left(\sin^{-1}\sqrt{\frac{x-a}{b-a}}-\sqrt{\frac{(x-a)(b-x)}{(b-a)}}\right)$$

Q. Find $\int\sqrt{\frac{1+x}{1-x}}dx$

$$1+x=2\cos^2\theta$$

$$\Rightarrow \cos\theta=\frac{\sqrt{1+x}}{\sqrt{2}}$$

$$\sin\theta=\frac{\sqrt{1-x}}{\sqrt{2}}$$

$$\Rightarrow \sin 2\theta=\frac{\cancel{2}\left(\sqrt{1+x}\sqrt{1-x}\right)}{\cancel{2}}$$

$$x=\cos 2\theta$$

$$dx=-2\sin 2\theta\,d\theta$$

$$\int\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\cdot(-2)\sin 2\theta\, d\theta$$

$$\int\sqrt{\frac{\cancel{2}\cos^2\theta}{\cancel{2}\sin^2\theta}}(-2)\sin 2\theta\, d\theta$$

$$-2\int\frac{\cos\theta}{\cancel{\sin\theta}}\cdot 2\cancel{\sin\theta}\cos\theta\, d\theta$$

$$-4\int\cos^2\theta\, d\theta$$

$$-2\int 1+\cos 2\theta\, d\theta = -2\left(\theta+\frac{\sin 2\theta}{2}\right)$$

$$=-2\left(\frac{\cos^{-1}x}{2}+\frac{\sqrt{1+x}\sqrt{1-x}}{2}\right)$$

$$=-\left(\cos^{-1}x+\sqrt{(1+x)(1-x)}\right)+c$$

Q. Find $\int\frac{dx}{(x+3)\sqrt{(4x^2+5x+7)}}=?$

Put $x+3=\frac{1}{t}$

We will get $-\frac{1}{t^2}dt$ and t^2 from denominator in numerator which will cancel each other and finally we will get quad of t with $\sqrt{\ }$.

$$dx=-\frac{1}{t^2}dt$$

$$\int-\frac{1}{t^2}\frac{dt}{\cancel{1}/t\sqrt{4\left(\frac{1}{t}-3\right)^2+5\left(\frac{1}{t}-3\right)+7}}$$

$$-\int\frac{dt}{\sqrt{4(1-3t)^2+5\left(t-3t^2\right)+7t^2}}$$

$$-\int\frac{dt}{\sqrt{4+36t^2-24t+5t-15t^2+7t^2}}$$

$$=-\int\frac{dt}{\sqrt{28t^2-19t+4}}$$

$$=-\frac{1}{\sqrt{28}}\int\frac{dt}{\sqrt{t^2-\frac{19}{28}t+\frac{4}{28}}}$$

$$=-\frac{1}{\sqrt{28}}\int\frac{dt}{\sqrt{\left(t-\frac{19}{56}\right)^2+\frac{4}{28}-\left(\frac{19}{56}\right)^2}}$$

$$=-\frac{1}{\sqrt{28}}\int\frac{dt}{\sqrt{\left(t-\frac{19}{56}\right)^2+\left(\sqrt{\frac{1}{7}-\left(\frac{19}{56}\right)^2}\right)^2}}$$

$\int\frac{dx}{x^2+a^2}$ form

$$=-\frac{1}{\sqrt{28}}\ln\left|\sqrt{1+\frac{t-{}^{19}\!/_{56}}{\sqrt{{}^{1}\!/_{7}-\left({}^{9}\!/_{56}\right)^2}}}+\frac{t-{}^{19}\!/_{56}}{\sqrt{{}^{1}\!/_{7}-\left({}^{19}\!/_{56}\right)^2}}\right|+c$$

Q. $\int\frac{dx}{\left(x^2+1\right)\sqrt{2x^2+3}}$

$2x^2+3=t^2$

$4x\,dx=2t\,dt$

$$\int\frac{\cancel{t}}{\cancel{2}x}\cdot\frac{1}{\left(\frac{t^2-3+2}{\cancel{2}}\right)\cdot\cancel{t}}dt=\int\frac{dt}{\left(t^2-1\right)\sqrt{t^2-3}}$$

$$\int\frac{dx}{\left(x^2+1\right)\sqrt{2x^2+3}}$$

Let $x=\frac{1}{t}$

$dx=-\frac{1}{t^2}dt$

$$\int-\frac{1}{t^2}dt\cdot\frac{1}{\frac{t^2+1}{t^2}\cdot\frac{\sqrt{2+3t^2}}{t}}$$

$$=\int \frac{-t}{\left(t^2+1\right)} \frac{dt}{\sqrt{2+3t^2}}$$

Put $t^2 = y$

$$2t\,dt = dy$$

$$I = -\frac{1}{2}\int \frac{dy}{1+y\sqrt{2+3y}}$$

Put $2+3y = p^2$

$$2x^2+3 = {}^{1}\!/_{t^2}$$

$$4x\,dx = -\frac{2}{t^3}dt$$

$$\left(\frac{\frac{1}{t^2}-3}{2}+1\right)\cdot\frac{1}{t}$$

$$\frac{1-3t^2+2}{2t^3}]$$

$$x = \frac{\sqrt{3}}{\sqrt{2}}\tan\theta$$

Integration by Partial fraction:

If we have rational function $f(x)=\frac{P(x)}{Q(x)}$ where $Q(x)=(x-a_1)(x-a_2)...(x-a_n)$ then if degree of $(P(x))<\text{degree}(Q(x))$ then $\frac{P(x)}{Q(x)}$ is written as:

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a_1)}+\frac{A_2}{(x-a_2)}...+\frac{A_n}{(x-a_n)}$$

where $A_1, A_2, ...A_n$ are arbitrary constants to be evaluated.

Note: If factors $(x-a)$ appears r times then

(1) In partial fraction we take:

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+...+\frac{Ar}{(x-a)^r}$$

(2) For factor ax^2+bx+c in denominator we take $\frac{Ax+B}{ax^2+bx+c}$ and so on $\left(\frac{\text{quadrant}}{\text{cubic}}\right)$

Take derivative of equation (1)

$$2A(x-1)+B=2x+1$$

Put $x=1$

$\therefore \quad B=3$

then again $2A=2$

$\Rightarrow A=1$

$$\therefore \int\frac{x^2+x+1}{(x-1)^3}dx=\int\frac{1}{(x-1)}+\frac{3}{(x-1)^2}+\frac{3}{(x-1)^3}dx$$

$$\ln(x-1)+3\left(\frac{-1}{x-1}\right)+3\left(\frac{-1}{2(x-1)^2}\right)+c$$

Q. $\int\frac{dx}{2\sin x+3\cos x+4}$

Note. $\int\frac{a_1\sin x+b_1\cos x+c_1}{a_2\sin x+b_2\cos x+c_2}dx$

Put $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

constant $= \sin^2\frac{x}{2} + \cos^2\frac{x}{2}$

then multiply numerator and denominator by $\sec^2 x/2$.

$$\int \frac{dx}{2\sin x + 3\cos x + 4} = \int \frac{dx}{2\cdot 2\sin\frac{x}{2}\cos\frac{x}{2} + 3\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) + 4}\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)$$

now multiply by $\sec^2 x/2$.

$$\int \frac{\sec^2 x/2\, dx}{\sec^2 x/2\left(4\sin x/2 \cos x/2 + 3\left(\cos^2 x/2 - \sin^2 x/2\right) + 4\left(\cos^2 x/2 + \sin^2 x/2\right)\right)}$$

$$\int \frac{\sec^2 x/2\, dx}{4\tan x/2 + 3\left(1 - \tan^2 x/2\right) + 4\left(1 + \tan^2 x/2\right)}$$

Now put $\tan\frac{x}{2} = y$

$\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dy$

$$\int \frac{2dy}{4y + 3\left(1 - y^2\right) + 4\left(1 + y^2\right)} = \int \frac{2dy}{4y + 7 + y^2}$$

$$= 2\int \frac{dy}{y^2 + 4y + 7}$$

$$= 2\int \frac{dy}{(y+2)^2 + \left(\sqrt{3}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+2}{\sqrt{3}}\right) + c = 2\tan^{-1}\left(\frac{\tan\frac{x}{2} + 2}{\sqrt{3}}\right)$$

or for above question type put

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2} \quad \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

Q. $\int \frac{x^4+1}{(x-1)(x-2)(x-3)} dx$

deg(nu) > deg (deno)

Now divide numerator by denominator

$$\int \frac{x^4+1}{(x-1)(x-2)(x-3)} dx = \int Q(x) + \frac{R(x)}{(x-1)(x-2)(x-3)} dx$$

Now $R(x)$ will have deg < deg (denominator) then do normally.

Q. $\int \sec^3 x\, dx$

$$\int \sec^2 x \cdot \sec x\, dx$$

$$\int \sqrt{1+\tan^2 x}\ \sec^2 x\, dx$$

$$\tan x = y$$

$$\sec^2 x\, dx = dy$$

$$= \int \sqrt{1+y^2}\ dy$$

or

$$I = \int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - I + \int \sec x\, dx$$

$$I = \frac{\sec x \tan x + \ln\left|\sec x + \tan x\right| + c}{2}$$

Definite Integral

If $\frac{dF(x)}{dx} = f(x)\ \forall\ x \in [a,b]$ then

$$\int_a^b f(x)dx = f(b) - f(a) = f(x)\Big|_a^b$$

Here a and b are known as lower limit and upper limit of integration respectively and integral is known as definite integral of $f(x)$ from a to b. (x varies from a to b)

Properties of definite integral:

(1) $\int_a^b f(x)dx = -\int_a^b f(x)dx$

(2) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(3) $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy = f(a) - f(b)$

(4) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Proof: $I = \int_a^b f(a+b-x)dx$

Let $y = a+b-x$

$dy = -dx$

$I = \int_b^a f(y)\cdot(-dy)$

$= -\int_b^a f(y)dy = \int_a^b f(y)dy = \int_a^b f(x)dx$

Q1. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx = I$ ....(1)

$a+b-x = \frac{\pi}{6}+\frac{\pi}{3}-x = \frac{\pi}{2}-x$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan\left(\pi/2 - x\right)}}{\sqrt{\tan\left(\pi/2 - x\right)}+\sqrt{\cot\left(\pi/2 - x\right)}}dx$$

$= \displaystyle\int_{\pi/6}^{\pi/3} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ ....(2)

On adding (1) and (2)

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$= -\pi/6 + \pi/3$$

$$2I = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

Q2. $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}\, dx}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}}$ similar to 1

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} dx$$

$$2I = \pi/6$$

$$I = \pi/12$$

(5) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Put $2a - x = y$

$$dy = -dx$$

$$= \int_{2a}^{a} -f(y)\,dy$$

$$= \int_a^{2a} f(y)dy$$

(6) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$ = 0

if $f(2a-x) = -f(x)$

(7) $\int_a^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$

$$= \int_0^a f(x)\,dx + \int_a^0 f(x)\,dx$$

Proof: $\int_0^a f(-x)\,dx$

Put $-x = y$

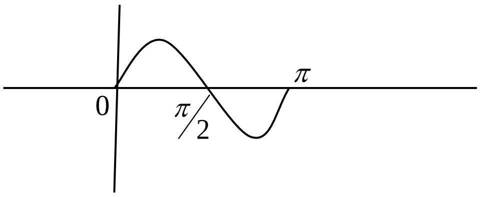
$$-dx = dy$$

$$\int_0^{-a} f(y)\,dy = \int_{-a}^{0} f(x)\,dx$$

(8) If $f\left(x\right)$ is even function then $\int_{-a}^{a} f\left(x\right)dx = 2\int_{0}^{a} f\left(x\right)dx$

If $f\left(x\right)$ is odd function then $\int_{-a}^{a} f\left(x\right)dx = 0$

Q3. $\int_{0}^{2\pi} |\sin x|\, dx$



$$f\left(x\right) = |\sin x|$$

$$f\left(2\pi - x\right) = |\sin x|$$

$$\therefore \int_{0}^{2\pi} |\sin x|\, dx = 2\int_{0}^{\pi} |\sin x|\, dx$$

$$= 2\left(\int_{0}^{\pi/2} \sin x\, dx - \int_{\pi/2}^{\pi} \sin x\, dx\right)$$

$$= 2\left(-\cos x\Big|_{0}^{\pi/2} + \cos x\Big|_{\pi/2}^{\pi}\right)$$

$$= 2\left(1+1\right) = 4$$

Q4. $\int_{1/2}^{2} \frac{1}{x}\sin\left(x - \frac{1}{x}\right)dx$

Let $y = x - \frac{1}{x}$

$$dy = \left(1 + \frac{1}{x^2}\right)dx$$

$$y^2 = x^2 + \frac{1}{x^2} - 2$$

$$\left(x + \frac{1}{x}\right)^2 = y^2 + 4$$

$$\Rightarrow dy = \left(x + \frac{1}{x}\right)\frac{dx}{x}$$

$$\Rightarrow dy = \sqrt{y^2 + 4}\,\frac{dx}{x}$$

$$\int_{-3/2}^{3/2} \sin y \frac{dy}{\sqrt{y^2+4}}$$

$= 0 \cdot \frac{\sin y}{\sqrt{y^2+4}}$ is odd function.

Q1. $\int_{-a}^{a} \frac{f(x)}{1+e^x} dx = I$ and $f(x)$ is even function then I= ?

(a) $\int_0^a f(x) dx$

(b) $2\int_0^a f(x) dx$

(c) $2\int_0^a \frac{f(x)}{1+e^x} dx$

(d) None

Solution.

$p(x) = \frac{f(x)}{1+e^x} \quad p(-x) = \frac{f(-x)}{1+e^{-x}} = \frac{e^x f(x)}{1+e^x}$

$I = \int_{-a}^{a} \frac{f(x)}{1+e^x} dx$

$I = \int_{-a}^{a} \frac{e^x f(x)}{1+e^x} dx$

$2I = \int_{-a}^{a} f(x) dx \quad \because \quad f(x)$ is even

$= 2\int_0^a f(x) dx$

$\therefore \quad I = \int_0^a f(x) dx$

Ans.(a)

* If $f(x)$ is periodic function with period 'T' then for any natural number n,

$$\int_0^{nT} f(x) dx = n\int_0^T f(x) dx$$

Proof: $\int_0^{nT} f(x) dx = \int_0^T f(x) dx + \int_T^{2T} f(x) dx + \ldots + \int_{(n-1)}^{nT} f(x) dx$

$$=\int_0^T f(x)\,dx+\int_0^T f(x)\,dx\ldots\int_0^T f(x)\,dx$$

Now

$$\int_{kT}^{(k+1)T} f(x)\,dx$$

Put $x=y+kT \Rightarrow dx=dy$

$$\int_0^T f(y+kT)\,dy=\int_0^T f(x)\,dx$$

Q2. $\int\limits_0^{40\pi} |\sin x|\,dx = 20\int_0^{2\pi} |\sin x|\,dx$

(a) 0 (b) 40 (c) 80 (d) None

Solution.

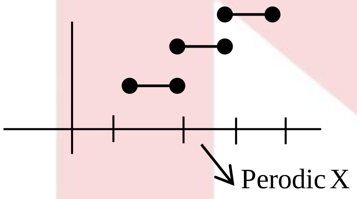
$$=40\int_0^{\pi} \sin x\,dx \ (\because \sin x>0 \text{ for } 0<x<\pi)$$

$$=40\left(-\cos x\Big|_0^{\pi}\right)$$

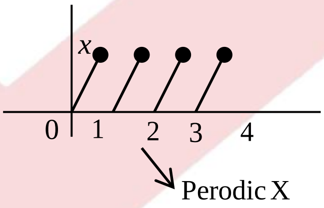
$$=40(-\cos\pi+\cos 0)$$

= 80

Q3. $\int_0^{10} [x]\,dx = ?$



$[x]=0 \ 0\le x<$



Period $=1$

$$\int_0^{10}[x]\,dx=10\int_0^{1}[x]\,dx$$

$$=\int_0^{10}\left(x-\{x\}\right)dx$$

$$=\left.\frac{x^2}{2}\right|_0^{10}-\int_0^{10}\{x\}\,dx$$

$$=50-10\int_0^{1}\{x\}\,dx$$

$$=50-10\int_0^{1}x\,dx$$

$$=50-10\cdot\left(\frac{x}{2}\Big|_0^1\right)$$

$$=50-5=45$$

Q4. $\int_0^{\pi}\log\sin x\,dx=?$

$$I=\int_0^{\pi}\log\sin x\,dx=2\int_0^{\pi/2}\log\sin x\,dx$$

$$=2\int_0^{\pi/2}\log\cos x\,dx$$

$$2I=2\int_0^{\pi/2}(\log\sin x+\log\cos x)\,dx$$

$$I=\int_0^{\pi/2}\log\sin 2x-\log 2\,dx$$

$$=\int_0^{\pi/2}\log\sin 2x\,dx-\log 2\int_0^{\pi/2}dx$$

Put $2x=t$

$$2dx=dt$$

$$=\frac{1}{2}\int_0^{\pi}\log\sin t\,dt-\frac{\pi}{2}\log 2$$

$$I=\frac{I}{2}-\frac{\pi}{2}\log 2$$

$$\frac{I}{2}=-\frac{\pi}{2}\log 2$$

$$I=-\pi\log 2=\pi\log {}^{1}\!/_{2}$$

Reduction formula

To find value of Definite Integral using Reduction formula:

(1) $I_n = \int_0^{\pi/2} \sin^n x\,dx$ where $n \in \mathbf{N}$

$I_1 = 1$

$I_2 = \int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \frac{1-\cos 2x}{2}\,dx$

$= \int\limits_0^{\pi/2} \frac{1}{2}\,dx - \int\limits_0^{\pi/2} \frac{\cos 2x}{2}\,dx$

$= \frac{\pi}{4} - \frac{1}{2}\left(\sin 2x\Big|_0^{\pi/2}\right)$

$= \frac{\pi}{4} - \frac{1}{2}(\sin \pi - \sin 0)$

$= \pi/4$

$I_3 = \int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \sin^2 x \cdot \sin x\,dx$

$= \int_0^{\pi/2} \left(1-\cos^2 x\right)\sin x\,dx$

Put $\cos x = y$

$-\sin x\,dx = dy$

$= -\int_1^0 \left(1-y^2\right)dy$

$= \int_0^1 1-y^2\,dy$

$= y - \frac{y^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

Now to generalise:

$I_n = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x\,dx$

$= \left(\sin_x^{n-1}\int \sin x\,dx - \int\left(\sin^{n-1} x\right)\cdot(n-1)\cdot\cos x\cdot(-\cos x)\,dx\right)\Big|_0^{\pi/2}$

$$= \sin_x^{n-1}\left(-\cos x\right)\Big|_0^{\pi/2} + \left(n-1\right)\int \sin^{n-1} x\cdot\cos^2 x\,dx\Big|_0^{\pi/2}$$

$$= -\cos x\cdot\sin^{n-1} x\Big|_0^{\pi/2} + \left(n-1\right)\int \sin^{n-2} x\cos^2 x\,dx\Big|_0^{\pi/2}$$

$$= 0 + \left(n-1\right)\int \sin^{n-2} x\left(1-\sin^2 x\right)dx\Big|_0^{\pi/2}$$

$$= \left(n-1\right)\int_0^{\pi/2}\left(\sin^{n-2} x - \sin^n x\right)dx$$

$$I_n = \left(n-1\right)\int_0^{\pi/2}\sin^{n-2} x\,dx - \left(n-1\right)I_n$$

$$I_n + \left(n-1\right)I_n = \left(n-1\right)\int_0^{\pi/2}\sin^{n-2} x\,dx$$

$$nI_n = \left(n-1\right)\int_0^{\pi/2}\sin^{n-2} x\,dx$$

$$nI_n = \left(n-1\right)I_{n-2}$$

$\boxed{I_n = \dfrac{(n-1)}{n}I_{n-2}}$ Reduction formula

Now from this $I_{n-2} = \dfrac{n-3}{n-2}I_{n-4}$

$$I_{n-4} = \frac{n-5}{n-4}I_{n-6}$$

So

$I_n = \dfrac{(n-1)}{n}\dfrac{(n-3)}{(n-2)}\dfrac{(n-5)}{(n-4)}\ldots.\dfrac{1}{2}\cdot\dfrac{\pi}{2}$ (if n is even)

$I_n = \dfrac{(n-1)}{n}\dfrac{(n-3)}{(n-2)}\dfrac{(n-5)}{(n-4)}\ldots\dfrac{2}{3}\times 1$ (if n is odd)

So Walli's formula:

$$\int_0^{\pi/2}\sin^n x\,dx = \begin{cases}\dfrac{(n-1)(n-3)(n-5)\ldots.1}{n(n-2)(n-4)\ldots.2}\cdot\dfrac{\pi}{2} & n\,\text{is even}\\[2ex] \dfrac{(n\text{-}1)(n-3)(n-5)\ldots.2}{n(n-2)(n-4)\ldots.3} & n\,\text{is odd}\end{cases}$$

By definite integral property

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

Q1. Find $I = \int_0^{\pi/2} \cos^{10} x\,dx$

$$I = \int_0^{\pi/2} \cos^{10} x\,dx = \frac{(10-1)(10-3)\ldots 1}{10(10-2)(10-4)\ldots 2} \cdot \frac{\pi}{2}$$

$$= \frac{9 \cdot 7 \cdot \cancel{5} \cdot \cancel{3} \cdot 1}{\cancel{10} \cdot 8 \cdot \cancel{6} \cdot 4 \cdot 2 \cdot 1} \cdot \frac{\pi}{2}$$

$$= \frac{63}{256} \cdot \frac{\pi}{2} = \frac{63\pi}{512}$$

Q2. Find $I = \int_0^{\pi} \sin^9 x\,dx$

$$= 2\int_0^{\pi/2} \sin^9 x\,dx$$

$$= 2\frac{(9-1)(9-3)(9-5)\ldots 2}{9(9-2)(9-4)\ldots 3} \times 1$$

$$= \frac{2 \cdot 8 \cdot \cancel{6} \cdot 4 \cdot 2}{9 \cdot 7 \cdot 5 \cdot \cancel{3} \cdot 1} = \frac{256}{315}$$

Q3. Find $I = \frac{2n!}{2^{2n}} \int_{-1}^{1} \left(1-x^2\right)^n dx$ where n is a natural number

Solution.

There we need substitution to solve

$$x = \cos\theta \quad dx = -\sin\theta d\theta$$

$$I = \frac{2n!}{2^{2n}} 2 \cdot \int_{\pi/2}^{0} \left(1-\cos^2\theta\right)^n \left(-\sin\theta\right) d\theta$$

$$= \frac{2n!}{2^{2n}} 2 \cdot \int_0^{\pi/2} \left(1-\cos^2\theta\right) \sin\theta\, d\theta$$

$$= \frac{2n!}{2^{2n}} 2 \cdot \int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta$$ ($2n+1$ always odd)

$$= \frac{2 \cdot 2n!}{2^{2n}} \frac{(2n+1-1)(2n+1-3)(2n+1-5)\ldots 2}{(2n+1)(2n+1-2)(2n+1-4)\ldots 3}$$

$$=\frac{2\cdot 2n!}{2^{2n}}\frac{2n\cdot(2n-2)(2n-4)\ldots 2}{(2n+1)(2n-1)(2n-3)\ldots 3}$$

or

$$=\frac{2\cdot 2n!}{2^{2n}}\frac{2^{n}\cdot(n(n-1)(n-2)\ldots 2)}{(2n+1)(2n-1)(2n-3)\ldots 3}$$

$$=\frac{2\cdot \cancel{2n}!}{2^{2n}}\frac{(2n\cdot(2n-2)(2n-4)\ldots 2)^{2}}{(2n+1)\,\cancel{2n}!}$$

$$=\frac{2\cdot 2^{2n}}{2^{2n}}\frac{(n!)^{2}}{2n+1}$$

$$=\frac{2(n!)^{2}}{2n+1}$$

(2) $I_{m,n}=\int_0^{\pi/2}\sin^m x\cdot\cos^n x\,dx$

$$I_{0,n}=\int_0^{\pi/2}\cos^n x\,dx$$

$$I_{m,0}=\int_0^{\pi/2}\sin^m x\,dx$$

$$I_{1,n}=\int_0^{\pi/2}\sin x\cdot\cos^n x\,dx$$

Let $\cos x=y$

$$-\sin x\,dx=dy$$

$$=-\int_1^0 y^n dy$$

$$=\int_0^1 y^n dy=\left.\frac{y^{n+1}}{n+1}\right|_0^1=\frac{1}{n+1}$$

$$I_{m,1}=\int_0^{\pi/2}\cos x\cdot\sin^m x\,dx=\frac{1}{m+1}$$

Now if $m,n\geq 2$ we need reduction formula:

$$\int\sin^m x\cos^n x\,dx=\sin^m x\cos^{n-1}x\int\cos x\,dx-\int\frac{d}{dx}\left(\sin^m x\cos^{n-1}x\right)\cdot\left(\int\cos x\,dx\right)$$

$$=\sin^m x\cos^{n-1}x\sin x-\int\left(m\sin^{m-1}x\cdot\cos^n x-(n-1)\cos^{n-2}x\sin^{m+1}x\right)\cdot\sin x\,dx$$

$$= \sin^{m+1} x \cos^{n-1} x - \int m \sin^{m} x \cos^{n} dx + n-1 \int \cos^{n-2} x \sin^{m+2} x\, dx$$

$$I = \sin^{m+1} x \cos^{n-1} x \Big|_{0}^{\pi/2} - mI + n-1 \int_{0}^{\pi/2} \cos^{n-2} x \sin^{n+2} x\, dx$$

$$(m+1) I = 0 + n-1 \int_{0}^{\pi/2} \cos^{n-2} x \sin^{m+2} x\, dx$$

$$I_{m,n} = \frac{(n-1)}{(m+1)} I_{m+2,n-2}$$

$$\boxed{I_{m,n} = \frac{n-1}{m+1} I_{m+2,n-2}}$$

$$I_{m+2,n-2} = \frac{n-3}{n+3} I_{m+4,n-4}$$

$$I_{m+4,n-4} = \frac{n-5}{m+5} I_{m+6,n-6}$$

(1) m, n is even.

$$I_{m,n} = \frac{(n-1)}{m+1} \frac{(n-3)}{m+3} \frac{(n-5)}{m+5} \dots \frac{1}{m+n-1}\; I_{m+n}, 0$$

Limit (sum of numerator and denominator are always same $= m+n$)

$$= \frac{(n-1)(n-3)(n-5)\dots 1}{(m+1)(m+3)(m+5)\dots(m+n-1)} \cdot \int_{0}^{\pi/2} \sin^{m+n} x\, dx$$

$$= \frac{(n-1)(n-3)(n-5)\dots 1}{\cancel{(n+1)}\,\cancel{(m+3)}\dots\cancel{(m+n-1)}} \cdot \frac{\cancel{(m+n-1)}\,\cancel{(m+n-3)}\dots 1}{(m+n)(m+n-2)\dots 2} \cdot \frac{\pi}{2}$$

$$= \frac{(n-1)(n-3)(n-5)\dots 1 \cdot (m-1)(m-3)(m-5)\dots 1}{(m+n)(m+n-2)\dots 2} \cdot \frac{\pi}{2}$$

(2) n is odd

$$I_{m,n} = \frac{(n-1)}{m+1} \frac{(n-3)}{m+3} \frac{(n-5)}{m+5} \dots \frac{2}{m+n-2} \cdot I_{m+n-1}, 1$$

$$= \frac{(n-1)(n-3)(n-5)\dots 2}{(m+1)(m+3)(m+5)\dots(m+n-2)} \cdot \frac{1}{m+n\cancel{-1}\cancel{+1}}$$

$$= \frac{(n-1)(n-3)(n-5)\dots 2}{(m+1)(m+3)\dots(m+n-2)(m+n)}$$

Walli's Formula:

When n, m are even

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots1\cdot(m-1)(m-3)\ldots1}{(m+n)(m+n-2)\ldots2}\cdot\dfrac{\pi}{2} & \text{when}\,(n,m)\,\text{is odd} \\ \dfrac{(n\text{-}1)(n-3)(n-5)\ldots2\cdot(m-1)(m-3)\ldots2}{(m+n)(m+n-2)\ldots1\,\text{or}\,2} & \end{cases}$$

$$\uparrow$$

Any one of m, n is odd $\because I_{n,m} = I_{m,n}$

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \int_0^{\pi/2} \sin^m \left(\pi/2 - x\right)\cdot\cos^n\left(\pi/2 - x\right) dx$$

$$= \int_0^{\pi/2} \cos^m x \sin^n x \, dx$$

$$= I_{n,m}$$

Q1. $\int_0^{2\pi} \sin^6 x \cos^8 x \, dx = ?$

$= 8\int_0^{\pi/2} \sin^6 x \cos^8 x \, dx$ ($\therefore$ even powers)

$$= 8\left(\frac{(6-1)(6-3)\ldots1(8-1)(8-3)\ldots1}{14\cdot(14-2)\ldots2}\cdot\frac{\pi}{2}\right)$$

$$= 8\left(\frac{5\cdot\cancel{3}\cdot1\cdot\cancel{7}\cdot\cancel{5}\cdot\cancel{3}\cdot1}{\cancel{14}\cdot\cancel{12}\cdot\cancel{10}\cdot8\cdot\cancel{6}\cdot4\cdot2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{5\pi}{256\cdot2} = \frac{5\pi}{512}$$

Q2. $\int_{2\pi}^{\frac{5\pi}{2}} \sin^3 x \cos^5 x \, dx = ?$

$$\int_{2\pi}^{\frac{5\pi}{2}} \sin^3 x \cos^5 x \, dx = \int_{2\pi}^{\frac{5\pi}{2}} \sin^3\left(\frac{9\pi}{2}+x\right)\cos^5\left(\frac{9\pi}{2}+x\right)dx$$

$$= \int_{2\pi}^{2\pi+\pi/2} \sin^3 x \cos^5 \, dx$$

Put $y = x - 2\pi \quad dy = dx$

$$= \int_0^{\pi/2} \sin^3\left(y+2\pi\right)\cos^5 x\left(y+2\pi\right)dy$$

$$= \int_0^{\pi/2} \sin^3 y \cos^5 y\, dy$$

$$= \frac{(3-1)\cdot(5-1)(5-3)}{(5+3)(5+3-2)\ldots 2 \text{ or } 1}$$

$$= \frac{\cancel{2}\cdot\cancel{4}\cdot\cancel{2}}{\cancel{8}\cdot 6\cdot\cancel{4}\cdot\cancel{2}} = \frac{1}{24}$$

Q3. Find $\int_0^1 x^{5/2}(1-x)^{3/2}\,dx$

Put $x = \cos^2\theta$

$dx = -2\sin\theta\cos\theta\, d\theta$

$$2\int_{\pi/2}^{0} \cos\theta\left(1-\cos^2\theta\right)^{3/2}(-\sin\theta)\,d\theta$$

$$2\int_0^{\pi/2} \cos\theta \sin^4\theta d\theta = 2\left(\frac{(6-1)(6-3)(6-5)\ldots(4-1)(4-3)}{10(10-2)\ldots 2 \text{ or } 1}\cdot\frac{\pi}{2}\right)$$

$$= \frac{5\cdot 3\cdot 1\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2\cdot 1}\cdot\frac{\pi}{2} = \frac{3\pi}{256}$$

(3) $I_n = \int_0^{\pi/4} \tan^n x\, dx\ ;\ n \in \mathbf{N}$

$I_0 = \pi/4$

$$I_1 = \int_0^{\pi/4} \tan x\, dx = \log\left|\sec x\right|\Big|_0^{\pi/4}$$

$$= \log\sec \pi/4 - \log\sec\theta$$

$$= \log\sqrt{2} - \log 1$$

$$= \log\sqrt{2}$$

$$= \frac{1}{2}\log 2$$

Now if $n \geq 2$

$$I_n = \int_0^{\pi/4} \tan^n x\, dx = \int_0^{\pi/4} \tan^{n-2} x\left(\sec^2 x - 1\right)dx$$

$$=\int_0^{\pi/4}\tan^{n-2}x\cdot\sec^2 x\,dx-\int_0^{\pi/4}\tan^{n-2}x\,dx$$

Put $\tan x = y$

$$\sec^2 x\,dx = dy$$

$$I_n=\int_0^1 y^{n-2}dy-I_{n-2}$$

$$=\left.\frac{y^{n-2}}{n-1}\right|_0^1-I_{n-2}$$

$$I_n=\frac{1}{n-1}-I_{n-2}$$

$$I_n-I_0=\frac{1}{n-1}$$

$I_n=-I_{n-2}+\dfrac{1}{n-1}$ if n is even

$$=-\left(-I_{n-4}+\frac{1}{n-3}\right)+\frac{1}{n-1}$$

$$=\frac{1}{n-1}-\frac{1}{n-3}+I_{n-4}$$

$$I_n=\frac{1}{n-1}-\frac{1}{n-3}+\frac{1}{n-5}-\frac{1}{n-7}\ldots\pm\left(\frac{\pi}{4}\right)$$

If n is odd

$$I_n=\frac{1}{n-1}-\frac{1}{n-3}+\frac{1}{n-5}-\frac{1}{n-7}\ldots\frac{1}{2}+\left(+\text{or}-\frac{1}{2}\log 2\right)$$

$$\int_0^{\pi/4}\tan^n x\,dx=\begin{cases}\dfrac{1}{n-1}-\dfrac{1}{n-3}+\dfrac{1}{n-5}-\dfrac{1}{n-7}\ldots+\left(\pm\dfrac{n}{4}\right) & n\text{ even}\\ \dfrac{1}{n\text{-}1}-\dfrac{1}{n-3}+\dfrac{1}{n-5}\ldots+\left(\pm\dfrac{1}{2}\log 2\right) & n\text{ odd}\end{cases}$$

Q1. $\int_0^{\pi/4}\tan^6 x\,dx=$

$$I_6=\frac{1}{6-1}-\frac{1}{6-3}+\frac{1}{6-5}-\frac{\pi}{4}$$

$$=\frac{1}{5}-\frac{1}{3}+1-\frac{\pi}{4}$$

$$=\frac{3-5+15}{15}-\frac{\pi}{4}$$

$$=\frac{13}{15}-\frac{\pi}{4}$$

Q2. $\int_0^{\pi/4}\tan^7 x\,dx=$

$$I_7=\frac{1}{6}-\frac{1}{4}+\frac{1}{2}-\frac{1}{2}\log 2$$

$$=\frac{2-3+6}{12}-\frac{1}{2}\log 2$$

$$=\frac{5}{12}-\frac{1}{2}\log 2$$

Q3. $\int_{\frac{5\pi}{4}}^{\pi}\tan^5 x\,dx=$

$$\int_{\pi+\frac{\pi}{4}}^{\pi}\tan^5 x\,dx=0$$

$$\int_{\pi/4}^{0}\tan^5 x\,dx=0$$

$$-\int_0^{\pi/4}\tan^5 x\,dx=?$$

$$=-\left(\frac{1}{4}-\frac{1}{2}+\frac{1}{2}\ln\right)=\frac{1}{4}-\frac{1}{2}\ln 2$$

Q4. $\int_{-1}^{1}\left(x^2-1\right)^n dx$

$$\tan^2\theta=\sec^2\theta-1$$

$$x=\sec\theta$$

$$dx=\sec\theta\tan\theta\,d\theta$$

We do not need reduction here

$$\int_{-1}^{1}\left(x^2-1\right)^n dx=2\int_0^1\left(x^2-1\right)^n dx$$

$$=2\int_0^1\tan^{2n}\theta\cdot\sec\theta\tan\theta\,d\theta$$

By expansion

$$\left(1-x^2\right)^n = 1 - {}^{n}C_1 x^2 + {}^{n}C_2 x^4 - {}^{n}C_3 x^6 \ldots$$

$$\int_{-1}^{1}\left(x^2-1\right)^n dx = 2\int_{0}^{1}\left(x^2-1\right)^n dx$$

$$= 2\left(\left(-1\right)^n\left(1-{}^{n}C_1\cdot\frac{1}{3}+{}^{n}C_{1\backslash 2}\,{}^{n}C_2\cdot\frac{1}{5}\ldots\right)\right)$$

$$= 2\left(-1\right)^n\sum_{i=0}^{n}\left(-1\right)^i\,{}^{n}C_i\,\frac{1}{2i+1}$$

Q5. $\int_{-1}^{1}\left(1+x^2\right)^n dx$

By Binomial expansion (best possible)

$$\int_{-1}^{1}\left(1+x^2\right) = 2\int_{0}^{1}\left(1+x^2\right)^n dx$$

$$= 2\sum_{i=0}^{n}{}^{n}C_i\times\frac{1}{2i+1}$$

$$= \int_{-1}^{1}\sum_{i=0}^{n}{}^{n}C_i\cdot x^{2i}dx$$

Integral Calculus Part - 2

Q6. $\lim\limits_{n\to\infty}\sum\limits_{k=n}^{n}\frac{n}{n^2+k^2}$

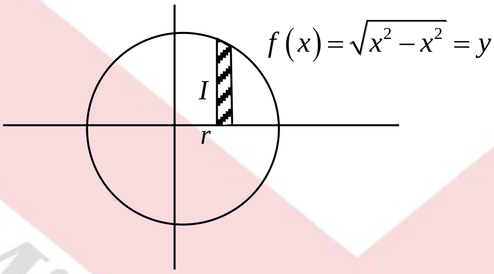
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=n}^{n^2}\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$=\int_1^\infty\frac{1}{1+x^2}\,dx$$

$$=\tan^{-1}x\Big|_1^\infty$$

$$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Q7. Find area of circle whose radius is r circle: $x^2+y^2=r^2$ center $=(0,0)$



$$f(x)=\sqrt{x^2-x^2}=y$$

Area (I) $=\int\limits_0^r\sqrt{r^2-x^2}\,dx$

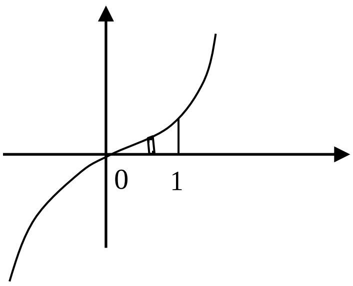
Total area of circle $=4\int_0^r\sqrt{r^2-x^2}\,dx$

$$=4\left(\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\sin^{-1}\frac{x}{r}\right)\Bigg|_0^r$$

$$=4\left(0+\frac{r^2}{2}\sin^{-1}1-0-\frac{r^2}{2}\sin^{-1}0\right)$$

$$=4\cdot\frac{r^2}{2}\cdot\frac{\pi}{2}-0=\pi r^2$$

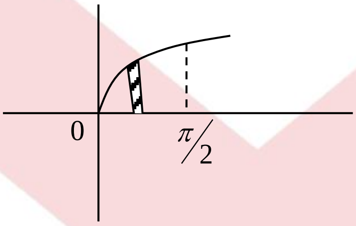
Q8. Find the area under the curve $y=2x^3$ above x-axis when x goes from 0 to 1?



Area $= \int_0^1 2x^3 dx$

$= \frac{2}{4} \cdot x^4 \Big|_0^1 = \frac{1}{2}$

Q9. Find the area $y = \sin x \quad 0 \le x \le \frac{\pi}{2}$?

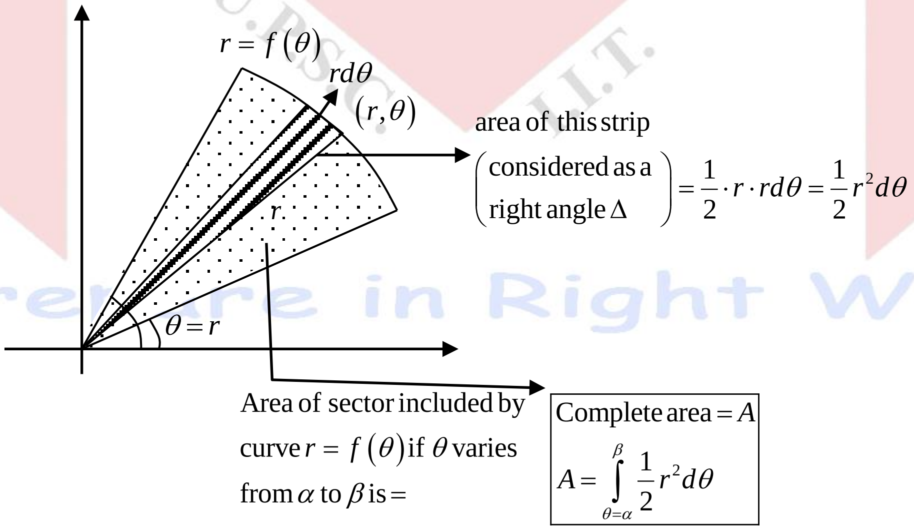


Area $= \int_0^{\pi/2} \sin x \, dx$

$= -\cos\Big|_0^{\pi/2}$

$= -\cos \pi/2 + \cos 0 \; = 1$

Equation of Curve in Polar form:



area of this strip $\begin{pmatrix}\text{considered as a}\\ \text{right angle }\Delta\end{pmatrix} = \frac{1}{2} \cdot r \cdot rd\theta = \frac{1}{2} r^2 d\theta$

Area of sector included by curve $r = f(\theta)$ if θ varies from α to β is =

$$\boxed{\begin{array}{l}\text{Complete area} = A\\ A = \int_{\theta=\alpha}^{\beta} \frac{1}{2} r^2 d\theta\end{array}}$$

Area of sector included by curve $r = f(\theta)$ if θ varies from α to β is

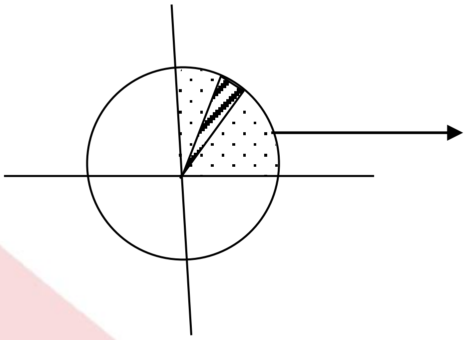
We have : $x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} {}^{y}\!/_{x}$

Q1. Using above formula find area of circle.



$$\int_0^{90^\circ} \frac{1}{2} r^2 d\theta = \frac{a^2}{2} \int_0^{\pi/2} d\theta$$

$x^2 + y^2 = a^2 \Rightarrow r = a$

$\int_0^{90^\circ} \frac{1}{2} r^2 d\theta = \frac{a^2}{2} \int_0^{\pi/2} d\theta$

$= \frac{a^2}{2} \frac{\pi}{2} = \frac{\pi a^2}{4}$

Total area $= 4 \cdot \frac{\pi a^2}{4} = \pi a^2$

Q2. Find sartorial area under curve $r = e^{\int_0^{2\pi} \frac{r^2}{2} d\theta = \frac{\cancel{2}\pi a^2}{\cancel{2}} = \pi a^2}$ where θ varies from 0 to $\frac{\pi}{4}$?

Area $= \int_0^{\pi/4} \frac{1}{2} e^{2\theta} d\theta = \frac{1}{2} \int_0^{\pi/4} e^{2\theta} d\theta$

$= \frac{e^{2\theta}}{4}\Big|_0^{\pi/4} = \frac{e^{\pi/2}}{4} - \frac{1}{4} = \frac{e^{\pi/2} - 1}{4}$

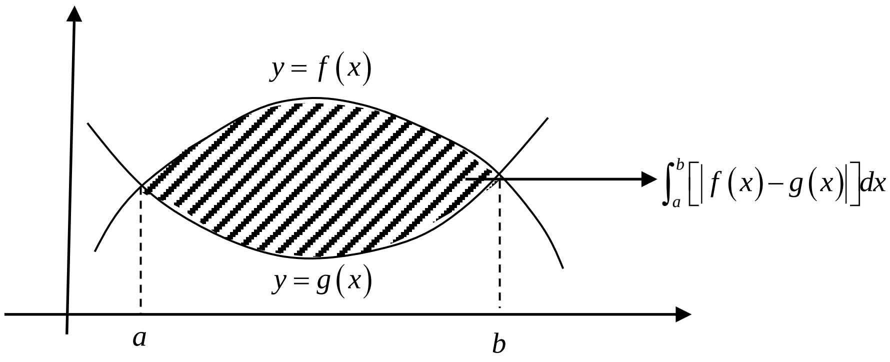
Q3. Curve $r = 4\sin^3\theta \quad 0 \le \theta \le \pi/2$, area $= ?$

Area $= \int_0^{\pi/2} \frac{1}{2} \times 16 \sin^6\theta \, d\theta = 8 \int_0^{\pi/2} \sin^6\theta \, d\theta$

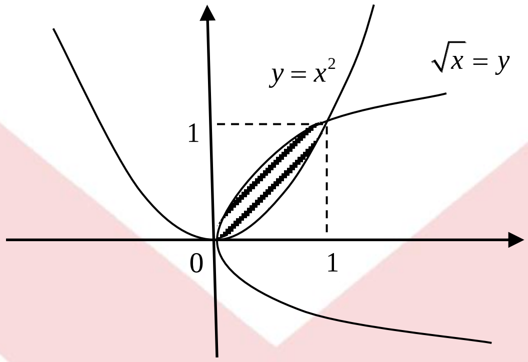
$= \cancel{8} \cdot \frac{5 \cdot 3}{6 \cdot \cancel{4} \cdot \cancel{2}} \cdot \frac{\pi}{2}$ (Walle's)

$= \frac{5\pi}{4}$

(c) If curves $y = f(x)$ and $y = g(x)$ intersects at points a and b then area between the curves from a to b is given by:



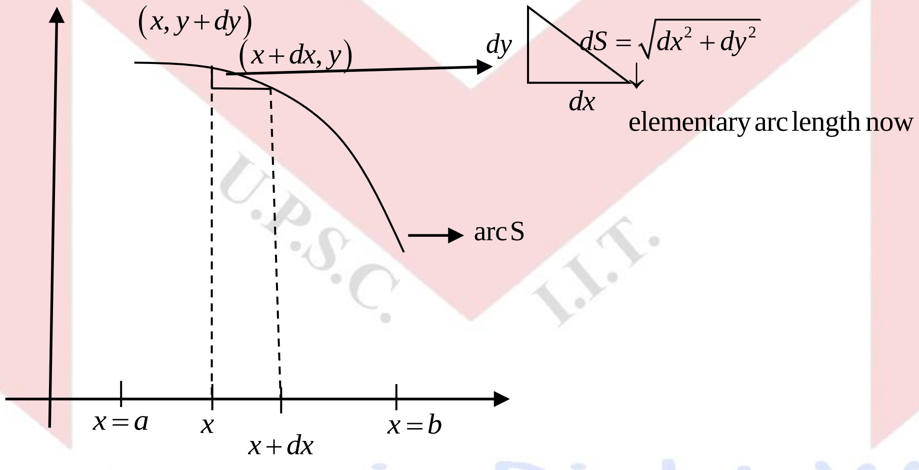
Q1. Find area between curves $y=x^2$ and $x=y^2$ between their points of intersection $(0,0),(1,1)$.



Area $=\int_0^1\left|x^2-\sqrt{x}\right|dx$

$$=\left|\frac{x^3}{3}-x^{3/2}\cdot\frac{2}{3}\right|_0^1=\left|\frac{1}{3}-\frac{2}{3}\right|=\frac{1}{3}$$

Arc Length of a curve $y=f(x)$ from $a\le x\le b$:



$$S=\int_{x=a}^{b}dS=\int_{x=a}^{b}\sqrt{dx^2+dy^2}$$

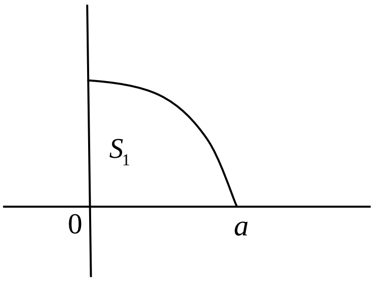
Arc length $=\int_{x=a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

(Continuous process of sum is $\int$)

Note: Arc length of curve $x=f(y)$ from $y=c$ to $y=d$ is given by:

$$S=\int_{y=c}^{d}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Q1. Find circumference of a circle of radius a



$$y=\sqrt{a^2-x^2}$$

$$S_1=\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^a\sqrt{1+\frac{x^2}{a^2-x^2}}dx$$

$$=\frac{-x}{\sqrt{a^2-x^2}}$$

$$=\int_0^a\sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}dx$$

$$=a\int_0^a\frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$=a\sin^{-1}\frac{x}{a}\bigg|_0^a=a\cdot{}^{\pi}\!/_{2}$$

$\therefore$ Total $S=4\cdot\left(\frac{a\cdot\pi}{2}\right)=2\pi a$

Q2. Find arc length of the curve $y=2x^2$ if x varies from 0 to 1?

$$\frac{dy}{dx}=4x$$

$$S=\int_0^1\sqrt{1+16x^2}\,dx$$

$$=4\int_6^1\sqrt{{}^{1}\!/_{16}+x^2}\,dx=\cancel{4}\left(\frac{x\sqrt{\frac{1}{16}+x^2}+\frac{1}{16}\log\left(x+\sqrt{\frac{1}{16}+x^2}\right)\bigg|_0^1}{\cancel{2}}\right)$$

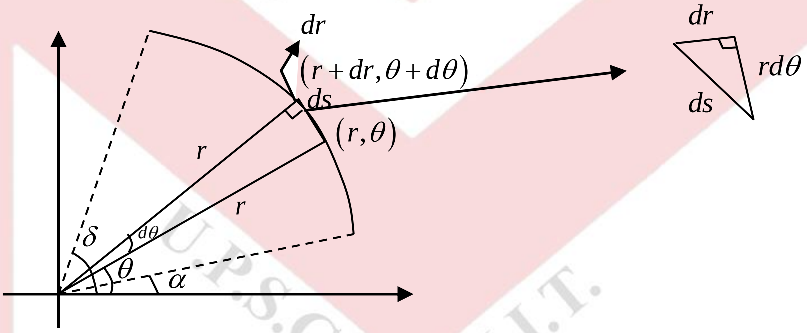
$$=2\left(\sqrt{\frac{1}{16}+1}+\frac{1}{16}\log\left(1+\sqrt{\frac{1}{16}+1}\right)-2\log\frac{1}{4}\right)$$

$$=2\sqrt{\frac{17}{16}}+\frac{\cancel{2}}{\cancel{16}}\log\left(1+\sqrt{\frac{17}{16}}\right)+2\log 4$$

$$=2\sqrt{\frac{17}{16}}+\frac{1}{8}\log\left(1+\sqrt{\frac{17}{16}}\right)+2\log 4$$

Integral Calculus - 2

Arc length in polar form:



$$ds^2=(rd\theta)^2+dr^2$$

$$ds=\sqrt{(rd\theta)^2+(dr)^2}$$

$\therefore$ Arc length of the curve $S=\int ds\ =\int\sqrt{dr^2+r^2d\theta^2}$

So arc length of the curve in polar form if along the curve θ varies from α to β is given by:

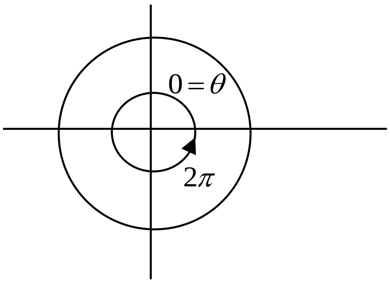
$$\boxed{S=\int_{\theta=\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}\ \text{ where curve } r=f(\theta)$$

Note. If $\theta=f(r)$ then

$$S=\int_{r=r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$$

Q1. By using formula of polar form, find arc length of circle of radius a.

Solution.



Polar form: $r=a,\ x^2+y^2=a^2$

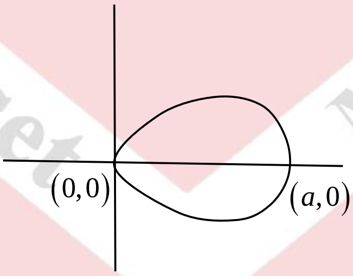
$$S=\int_0^{2\pi}\sqrt{r^2+0}\ d\theta$$

$$\frac{dr}{d\theta}=0$$

$$=r\int_0^{2\pi}d\theta\ =2\pi r=2\pi a$$

Q2. If $a>0$ then find arc length of one loop of curve $r=a\cos\theta$.

Solution.



$$\frac{dr}{d\theta}=-a\sin\theta$$

$$\therefore\ \ S=2\int_0^{\pi/2}\sqrt{r^2+a^2\sin^2\theta}\,d\theta$$

$$=2\int_0^{\pi/2}a\,d\theta\qquad r=a\cos\theta$$

$$=\cancel{2}\cdot a\cdot\frac{\pi}{\cancel{2}}=\pi a$$

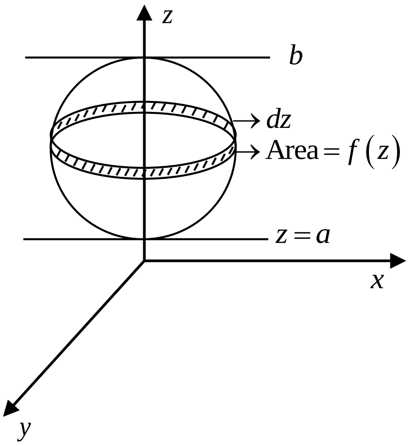
Q3. Find arc length of the curve $r=e^{\theta}$ where $0\le\theta\le\ln 2$ on curve.

Solution.

$$\frac{dr}{d\theta}=e^{\theta}\ \therefore\ S=\int_0^{\ln 2}\sqrt{e^{2\theta}+e^{2\theta}}\ d\theta=\int_0^{\ln 2}\sqrt{2}\ e^{\theta}d\theta$$

$$=\sqrt{2}\ e^{\theta}\Big|_0^{\ln 2}\ =\sqrt{2}\left(e^{\ln 2}-e^{0}\right)=\sqrt{2}\left(2-1\right)=\sqrt{2}$$

Use of single integration to find volume of a 3-dimensional object:



Volume of small part $dv = f(z)\cdot dz$

$= (area \bullet height)$

$\int dv = \int f(z)\cdot dz$

$V = \int_{z=a}^{b} f(z)dz$

* If in 3D object, if height z varies from a to b and if area of cross section at arbitrary height z is $f(z)$ i.e. function of z, then elementary volume of the object between height z and $z+dz$ is

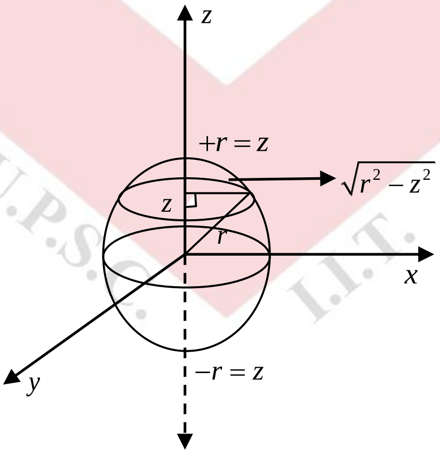
$dv = f(z)dz$

So total volume is:

$$\boxed{V = \int_{z=a}^{b} f(z)dz}$$

Q1. By using single integration find volume of sphere of radius $'r'$.

Solution.



$x^2+y^2+z^2=r^2$ [Center $(0,0,0)$ radius $=r$]

$\therefore$ Area of cross section $= \pi r_1^2$

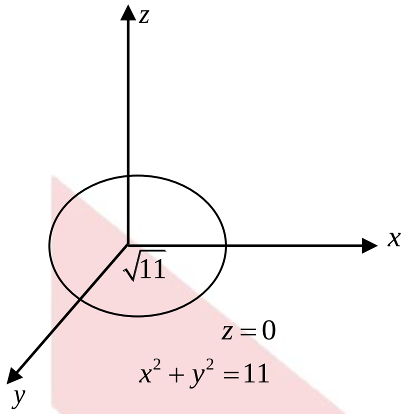
$= \pi\left(\sqrt{r^2-z^2}\right)^2 = \pi\left(r^2-z^2\right) = f(z)$

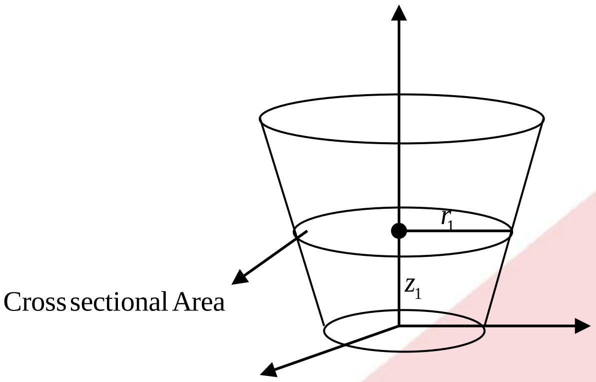
$\therefore$ Volume $= \int_{-r}^{r} f(z)dz$

$= \int_{-r}^{r} \pi\left(r^2-z^2\right)dz$ (even function)

$$= 2\pi\int_0^r r^2 - z^2 dz \ = 2\pi\left[r^2 z - \frac{z^3}{3}\Big|_0^r \right] = 2\pi\left(r^3 - \frac{r^3}{3} \right) = 2\pi \cdot \frac{2r^3}{3} = \frac{4}{3}\pi r^3$$

Q2. Find volume enclosed by curve $x^2 + y^2 = z^2 + 2z + 11$ where z varies from 0 to 1?





$x^2 + y^2 = (z+1)^2 + 10$ treat as radius of circle.

$z = 0$

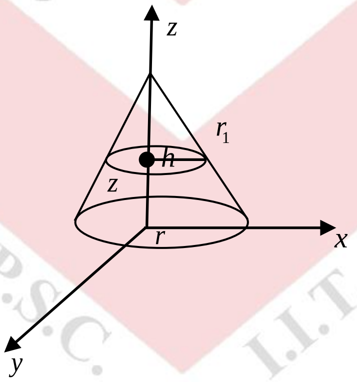
$x^2 + y^2 = 11$

$z \uparrow$ then $x^2 + y^2 = r_1$ where $r_1 \uparrow$

$= \pi r_1^2 \ = \pi(z_1 + 1)^2 + 10 = \pi(z_1^2 + 2z_1 + 11)$

$$\therefore \ V = \int_0^1 \pi(z^2 + 2z + 11)dz \ = \pi\left(\frac{z^3}{3} + \frac{2z^2}{2} + 11z \right)\Bigg|_0^1 \ = \pi\left(\frac{1}{3} + 1 + 11 \right) = \frac{37}{3}\pi$$

Q3. Find volume of right circular cone of radius r and good height h by single integration method:
Equation:



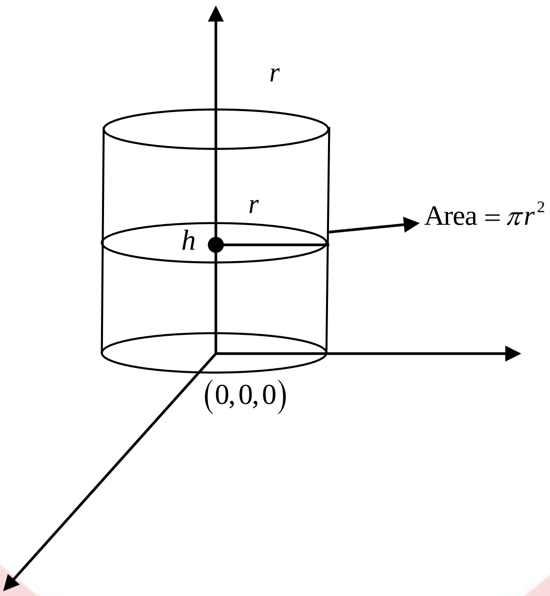
$\frac{h}{r} = \frac{z}{r_1}$ (similar triangle)

$\therefore \ r_1 = \frac{zr}{h}$

$\therefore$ area $= \pi r_1^2 = \pi\frac{r^2}{h^2}z^2$

$$\therefore \ V = \int_0^h \pi\frac{r^2}{h^2}z^2 dz \ = \frac{\pi r^2}{h^2}\left[\frac{z^3}{3}\Big|_0^h \right] = \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} \ = \frac{1}{3}\pi r^2 h$$

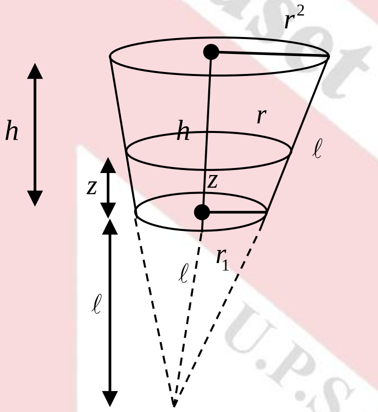
Q4. Find volume of right circular cylinder of radius $'r'$ and height $'h'$
Equation:

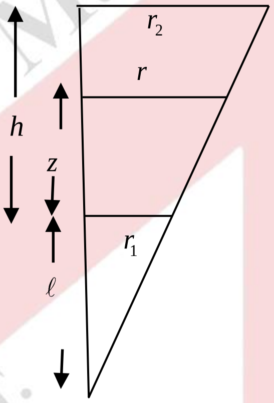


$$x^2+y^2=r^2\ ;\ 0\le z\le h$$

$$V=\int_0^h \pi r^2 dz=\pi r^2\left(z\Big|_0^h\right)=\pi r^2 h$$

Q5. Find volume of frustum of cone whose height is $'h'$ and whose radius of base and top is r_1 and r_2 respectively.





$$\frac{l+z}{r}=\frac{l}{r_1}=\frac{l+h}{r_2}$$

$$\frac{l+h}{l}=\frac{r_2}{r_1}$$

$$\frac{h}{l}=\frac{r_2}{r_1}-1$$

$$\Rightarrow l=\left(\frac{r_1 h}{r_2-r_1}\right)$$

Similarly of Δ

$$r=\frac{(l+z)^{r}}{l}$$

$$=\left(1+\frac{z}{l}\right)r_1$$

$$=\left[1+\frac{z(r_2-r_1)}{r\cdot h}\right]r_1$$

Now

Area $= \pi r^2 \;= \pi\left(r_1 + \frac{z\left(r_2 - r_1\right)}{h}\right)^2$

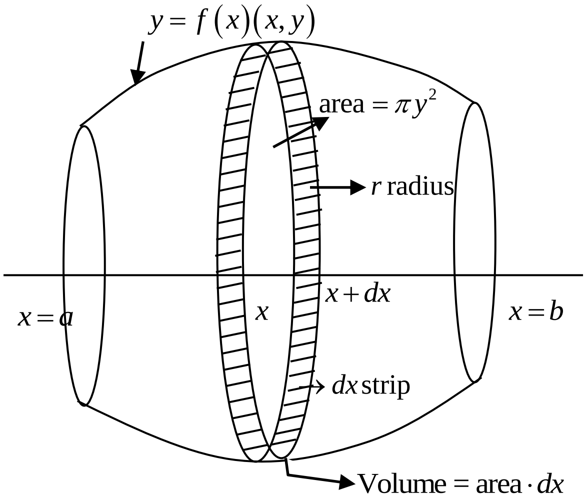
Volume $= \pi\int_0^h \left(r_1 + \frac{z\left(r_2 - r_1\right)}{h}\right)^2 dz$

$$= \pi\left[\left(r_1 + \frac{z\left(r_2 - r_1\right)}{h}\right)^3 \cdot \frac{1\cdot h}{3\left(r_2 - r_1\right)}\Bigg|_0^h\right]$$

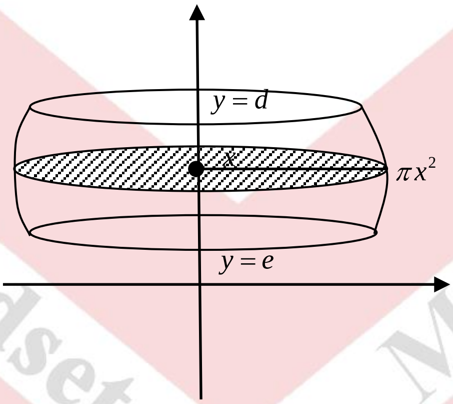
$$= \pi\left[r_1 + \left(r_2 - r_1\right)\right]^3 \cdot \frac{h}{3\left(r_2 - r_1\right)} - \frac{r_1^3 h}{3\left(r_2 - r_1\right)}$$

$$= \frac{\pi\left(r_2^3 - r_1^3\right)h}{3\left(r_2 - r_1\right)} \;= \frac{\pi h}{3}\left(r_1^2 + r_2^2 + r_1 r_2\right)$$

Solid of revolution (we need axis and are for this)



(i) Volume obtained by revolving an arc $y=f(x)$ where $a\le x\le b$ about x-axis is given by:



$$\boxed{V=\int_{x=a}^{b}\pi y^2\,dx}$$

Volume $=$ area $\cdot dx$

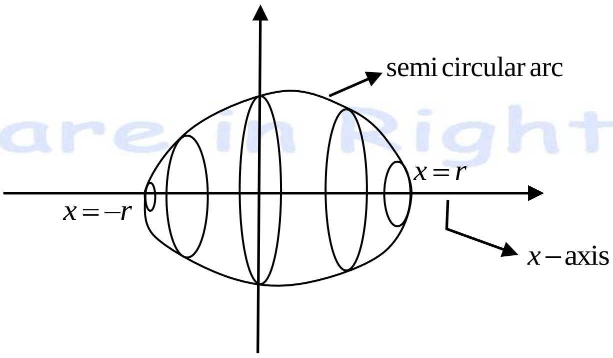
$dv=\pi y^2\cdot dx$

$V=\int \pi y^2 dx$

(ii) Volume obtained by revolving an arc $x=f(y)$ where $(c\le y\le d)$ about y-axis is given by

$$\boxed{V=\pi\int_c^d x^2\,dy}$$

Q1. Find volume of sphere of radius $'r'$ by above method i.e. solid of revolution method.

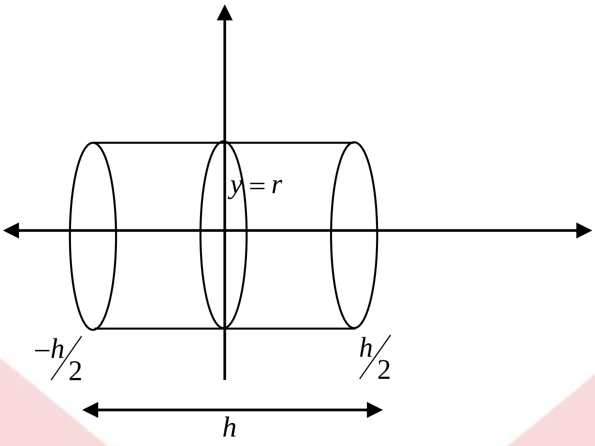


* If semi-circular arc $x^2+y^2=r^2$ from $x=-r$ to $x=r$ is revolved around x-axis then solid of revolution obtained is sphere of radius $=r$

now

$$V=\pi\int_{-r}^{r} y^2\,dx \;=2\pi\int_0^r y^2\,dx \;=2\pi\int_0^r r^2-x^2\,dx \;=2\pi\left(r^2\cdot x-\frac{x^3}{3}\bigg|_0^r\right)=2\pi\left(r^3-\frac{r^3}{3}\right)\;=\frac{4}{3}\pi r^3$$

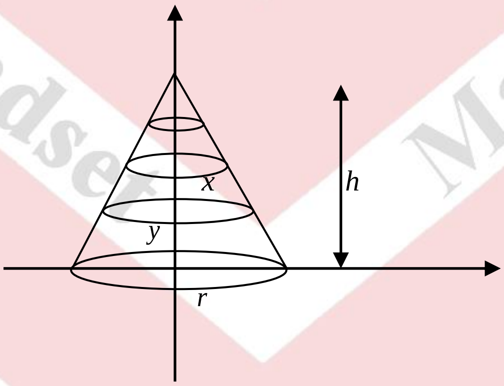
Q2. Find volume of cylinder by same method.



$$x^2+y^2=r^2$$

$$V=\pi\int_{-h}^{h} y^2\,dx=\pi\int_{-h/2}^{h/2} r^2dx\;=\cancel{2}\pi r^2\,\frac{h}{\cancel{2}}=\pi r^2h$$

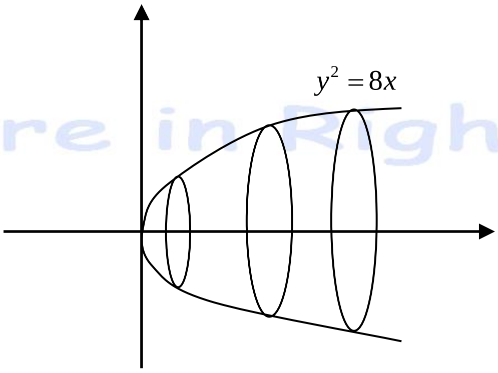
Q3. Find volume of cone:



$$\frac{h}{r}=\frac{y}{x}\Rightarrow x=\frac{ry}{h}$$

$$V=\pi\int_0^h x^2dy\;=\pi\int_0^h\frac{r^2}{h^2}y^2dy\;=\pi\frac{r^2}{h^2}\left(\frac{y^3}{3}\bigg|_0^h\right)=\frac{\pi r^3h}{3}$$

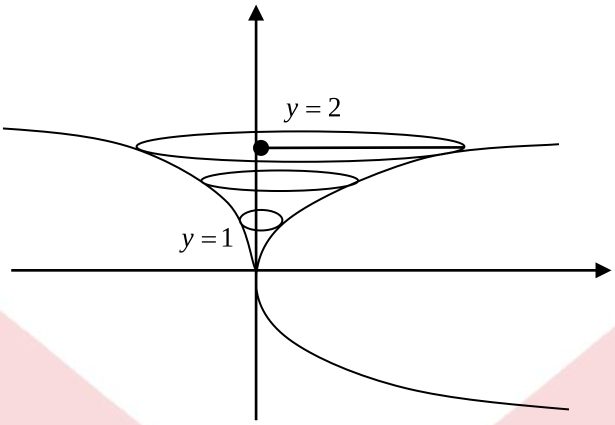
Q4. Find volume of solid obtained by revolving curve $y^2=8x$ from $x=1$ to $x=2$ about x-axis?



$$V = \pi \int y^2 dx$$

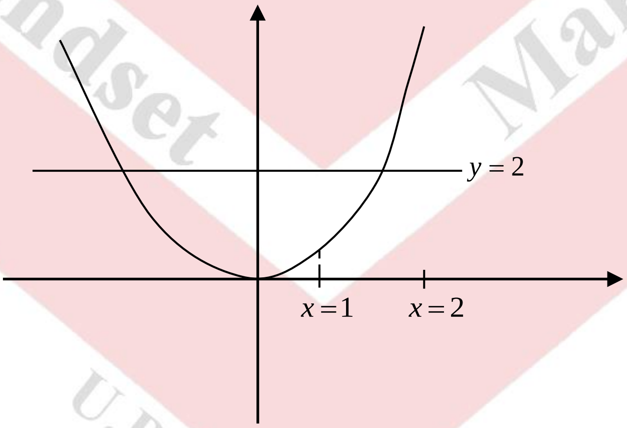
$$V = \pi \int_1^2 8x\,dx \;= \pi \cdot 8\left(\frac{x^2}{2}\bigg|_1^2\right) = 8\pi\left(\frac{4}{2} - \frac{1}{2}\right) = 12\pi$$

Q5. Find volume of solid obtained by revolving curve $y^2 = 8x$ from $y = 1$ to $y = 2$ about y -axis?



$$V = \pi \int_1^2 x^2 dy \;= \pi \int_1^2 \frac{y^4}{64} dy \;= \frac{\pi}{64}\left(\frac{y^5}{5}\bigg|_1^2\right) = \frac{\pi}{64} \cdot \frac{31}{5} = \frac{31\pi}{320}$$

Q6. Find volume of solid obtained when it is obtained by revolving $y = 8x^2$ from $x = 1$ to 2 about line $y = 2$.

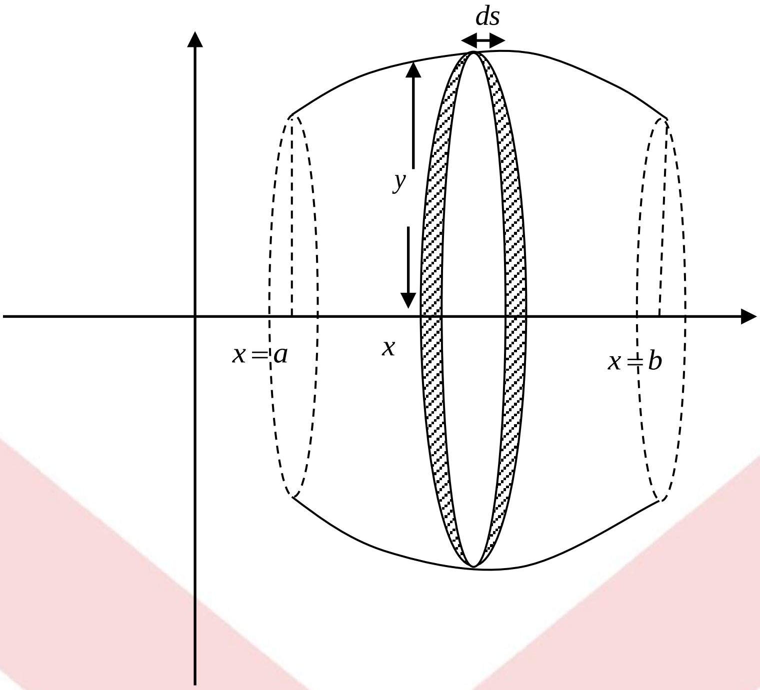


$y' = y - 2$ (logic)

$y' = 8x^2 - 2$

$$\therefore\; V = \pi \int_1^2 8x^2 - 2\,dx \;= \pi\left(\frac{8x^3}{3} - 2x\bigg|_1^2\right) = \pi\left(\frac{64}{3} - 4 - \frac{8}{3} + 2\right) = \pi\left(\frac{56}{3} - 2\right) = \frac{\pi}{3} \cdot 50$$

(iii) Surface area of solid of revolution



$ds^2 = dx^2 + dy^2$

Surface area of elementary portion $= ds$

$2\pi y$ will be circumference of 1 circle

$2\pi y \cdot ds$ will be surface area of ds length dragged portion (shaded)

$\therefore \ \ ds_1 = 2\pi y ds$

$$\boxed{S_1 = \int_{x=a}^{b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

(a) If arc $y = f(x)$ is revolved about x-axis from $x = a$ to b to form a solid then surface area of solid of revolution is:

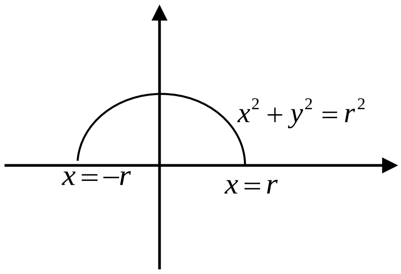
$$\boxed{S_1 = 2\pi y \int_{x=a}^{b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

(b) If arc $x = f(y)$ is revolved about y-axis from $y = c$ to $y = d$ to form a solid then surface area of solid of revolution is:

$$\boxed{S_1 = 2\pi \int_c^d x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}$$

Q1. By this method find SA of sphere of radius r.

$z^2 + x^2 + y^2 = r^2$

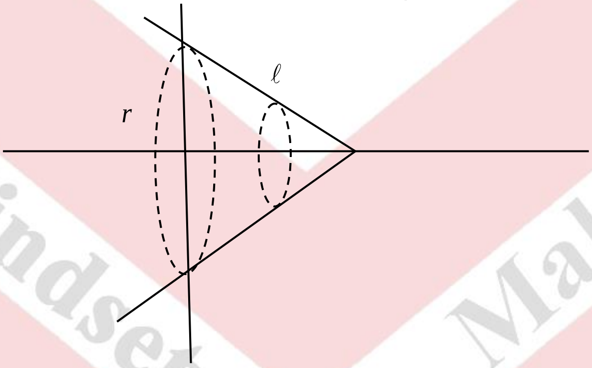


$$S=\int_{-r}^{r}2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx\ =2\int_{0}^{r}2\pi y\sqrt{\frac{y^2+x^2}{y^2}}dx\ =4\pi\int_0^r r\,dx\ =4\pi r^2$$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

Q2. By same method find SA of cone of radius $=r$ and height $=h$



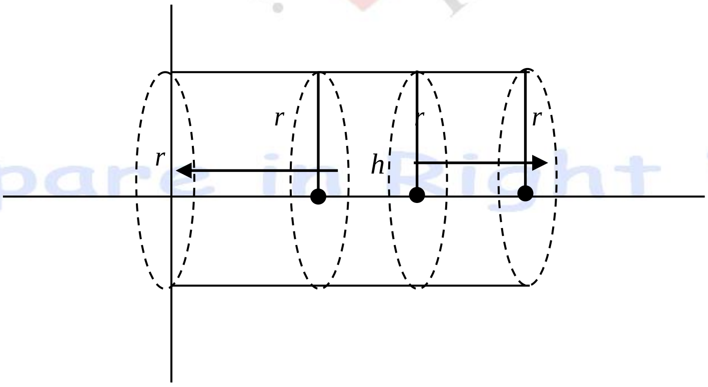
$$y=\frac{r}{h}x$$

$$\frac{dy}{dx}=\frac{r}{h}$$

$$S=\int_{0}^{h}2\pi\cdot\frac{r}{h}x\cdot\sqrt{1+\frac{r^2}{h^2}}dx\ =\frac{2\pi r}{h}\frac{\sqrt{h^2+r^2}}{h}\int_0^h x\,dx\ =\frac{2\pi r}{h^2}\cdot l\cdot\left(\left.\frac{x^2}{2}\right|_0^h\right)\ =\frac{\cancel{2}\pi r}{\cancel{h^2}}\cdot l\cdot\frac{\cancel{h^2}}{\cancel{2}}=\pi rl$$

$l^2=h^2+r^2$ [Slant height]

Q3. Find SA of cylinder of radius r and h.

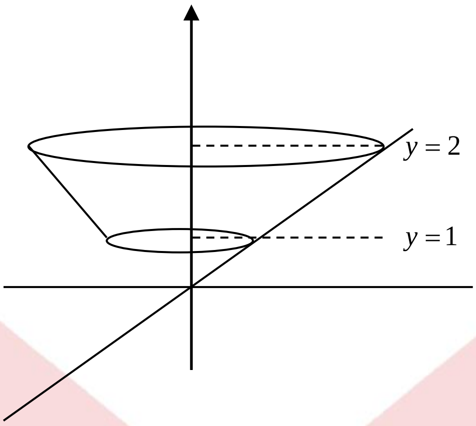


$$y=r$$

$$\therefore \quad \frac{dy}{dx} = 0$$

$$\therefore \quad S = \int_0^h 2\pi y\sqrt{1+0}\ dx \ = 2\pi r\int_0^h dx \ = 2\pi rh$$

Q4. Find surface area of solid of revolution obtained by revolving curve $x = 4y$ about y-axis when $1 \le y \le 2$?



$$\frac{dx}{dy} = 4$$

$$S = 2\pi\int_1^2 4y\sqrt{1+4^2}dy \ = 8\pi \cdot \sqrt{17}\int_1^2 y\,dy \ = 8\sqrt{17}\pi \cdot \frac{y^2}{2}\bigg|_1^2 \ = 4\sqrt{17}\pi\left(4-1\right) = 12\sqrt{17}\pi$$

Integral Calculus - 2: Double Integration

If x and y are two independent variable over which a function $f(x,y)$ is taken then by double integral of $f(x,y)$ we mean either:

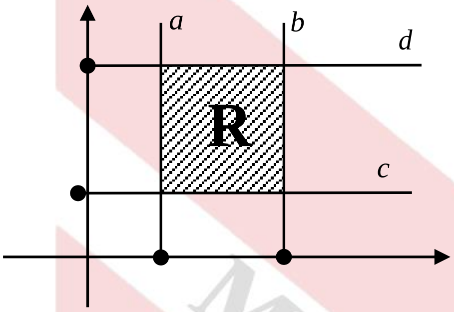
(i) $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$

Treat x constant

(ii) $\int_a^b\left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right)dx$

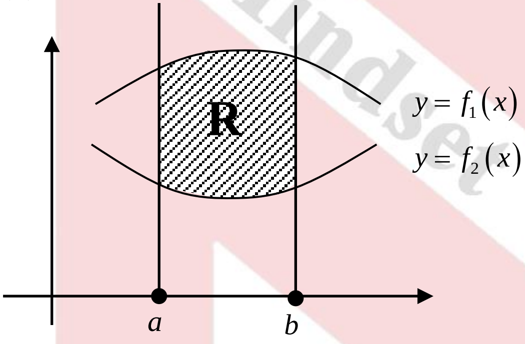
(iii) $\int_c^d\left(\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\right)dy$

(i)

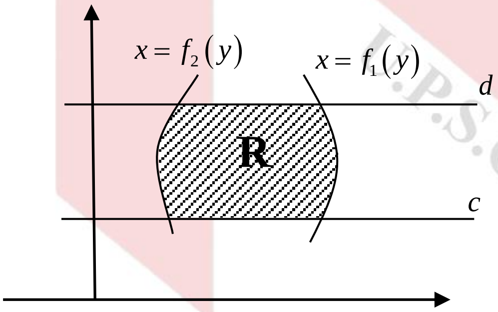


$\iint\limits_{R}$

(ii)



(iii)

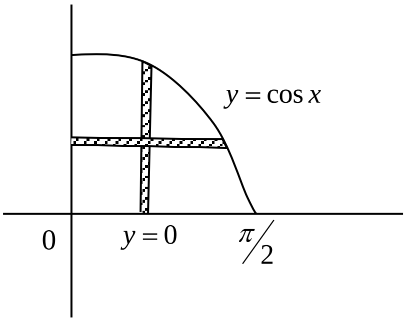


Q1. $\int\limits_{x=0}^{1}\int\limits_{0}^{2} x^2y^3\,dydx = ?$

$$=\int\limits_0^1 x^2\int\limits_0^2 y^3dy\,dx = \int\limits_0^1 x^2\left(\frac{y^4}{4}\bigg|_0^2\right)dx = 4\cdot\int_0^1 x^2\,dx = 4\cdot\left(\frac{x^3}{3}\bigg|_0^1\right) = \frac{4}{3}$$

Q2. $\int_0^1\int_0^y \frac{e^y}{y}\,dx\,dy \; = \int_0^1 \frac{e^y}{y}\int_0^y dx\,dy \; = \int_0^1 \frac{e^y}{\cancel{y}}\cdot \cancel{y}\,dy \; = \int_0^1 e^y dy \; = e^{-1}$

Q3. $\int_{0}^{\pi/2}\int_{0}^{\cos x} \sin x\, dy\, dx = ?$

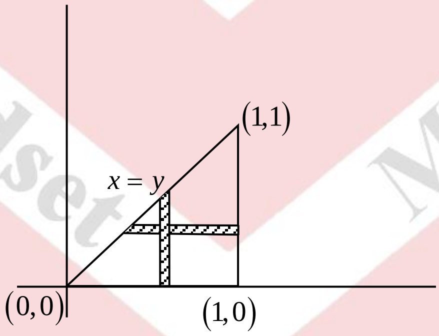


$$= \int_0^{\pi/2} \sin x \cdot \cos x\, dx \quad = \frac{1}{2}\int_0^{\pi/2} \sin 2x\, dx \quad = \frac{1}{2}\left(-\frac{\cos 2x}{2}\Bigg|_0^{\pi/2} \right) \quad = \frac{1}{2}\left(-\frac{\cos \pi + \cos 0}{2} \right) \quad = \frac{1}{2}\left(\frac{+1+1}{2} \right)$$

$$= \frac{1}{2}$$

Q4. Find $\iint_R xy\, dx\, dy$ where R is region bounded by triangle whose vertices are $(0,0),(1,0)$ and $(1,1)$?

Solution.



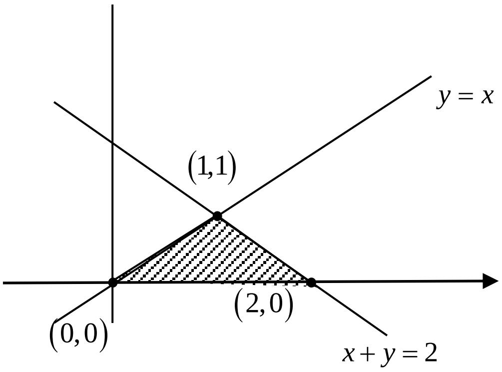
$$\int_{0}^{1}\int_{y=0}^{y=x} xy\, dy\, dx \quad = \int_0^1 x\left(\frac{y^2}{2}\Bigg|_0^x \right) dx \quad = \int_0^1 \frac{x^3}{2} dx \quad = \frac{1}{2}\int_0^1 x^3 dx = \frac{1}{2}\left(\frac{x^4}{4}\Bigg|_0^1 \right) = \frac{1}{8}$$

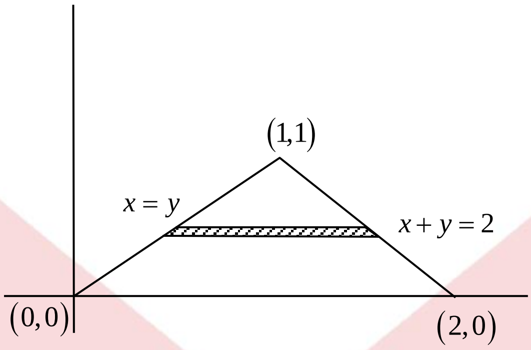
or

$$\int_{y=0}^{1}\int_{x=y}^{x=1} xy\, dx\, dy \quad = \int_0^1 y\left(\frac{x^2}{2}\Bigg|_y^2 \right) dy \quad = \int_0^1 y\left(\frac{1}{2} - \frac{y^2}{2} \right) dy$$

$$= \frac{1}{2}\int_0^1 y - y^3 dy = \frac{1}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\Bigg|_0^1 \right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{8}$$

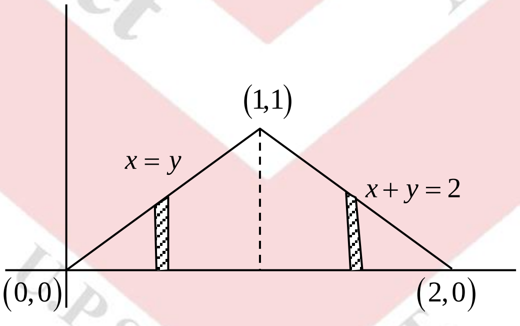
Q5. Find $\iint_R xy\, dx\, dy$ where R is region bounded by lines $y = x$, x-axis and line $x + y = 2$?





$$\int_0^1\int_{x=y}^{2-y} xy\,dx\,dy=\int_0^1 y\left(\frac{x^2}{2}\bigg|_y^{2-y}\right)dy\ =\frac{1}{2}\int_0^1 y\left[(2-y)^2-y^2\right]dy\ =\frac{1}{2}\int_0^1\left(4y+\cancel{y^3}-4y^2-\cancel{y^3}\right)dy$$

$$=\frac{1}{2}\int_0^1 4y-4y^2dy\ =\frac{1}{2}\left(\frac{4y^2}{2}-\frac{4y^3}{3}\right)\bigg|_0^1\ =\frac{1}{2}\left(2-\frac{4}{3}\right)=1-\frac{2}{3}=\frac{1}{3}$$



$$=\int_0^x xy\,dy\,dx+\int_1^2\int_0^{2-x} xy\,dy\,dx=\int_0^1\frac{x^3}{2}dx+\int_1^2\frac{x(2-x)^2}{2}dx=\frac{1}{2}\cdot\frac{1}{4}+\int_1^2\frac{4x+x^3-4x^2}{2}dx$$

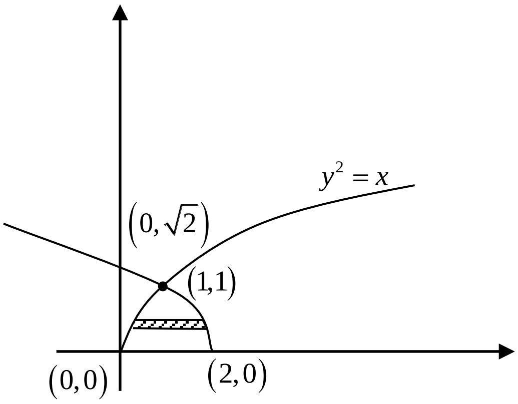
$$=\frac{1}{8}+\frac{1}{2}\left(2\cdot4+4-\frac{32}{3}-2-\frac{1}{4}+\frac{4}{3}\right)=\frac{1}{8}+\frac{1}{2}\left(10-\frac{1}{4}-\frac{28}{3}\right)=\frac{\cancel{1}}{\cancel{8}}+\left(5-\frac{\cancel{1}}{\cancel{8}}-\frac{14}{3}\right)\ =5-\frac{14}{3}=\frac{1}{3}$$

Q6. Find $\iint\limits_R x\,dx\,dy$ where R is region bounded by x-axis, parabola $y^2=x$ and $y^2=2-x$ bounded by x-axis, parabola $y^2=x$ and $y^2=2-x$ in 1st quadrant?

Pt. of intersection

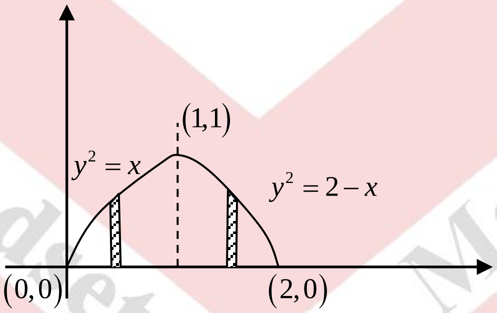
$2-x=x$

$x=1,\ y=1$



$$= \int_{y=0}^{1}\int_{x=y^2}^{2-y^2} x\,dx\,dy = \int_0^1 \left.\frac{x^2}{2}\right|_{y^2}^{2-y^2} dy = \frac{1}{2}\int_0^1 \left(2-y^2\right)^2 - y^4 dy = \frac{1}{2}\int_0^1 4+\cancel{y^4} - 4y^2 - \cancel{y^4}\,dy$$

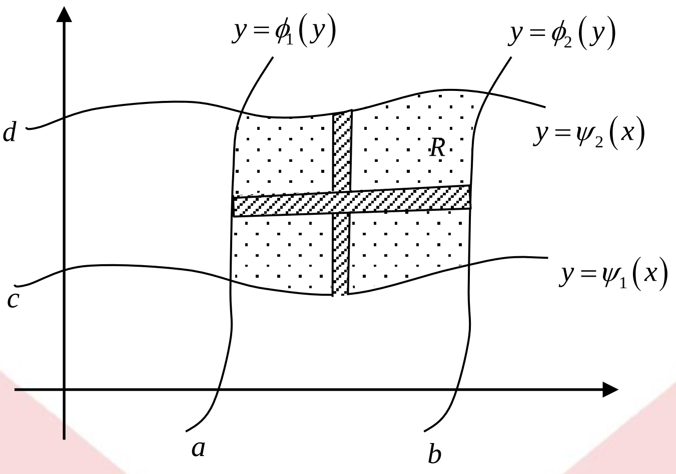
$$= \frac{1}{2}\int_0^1 4-4y^2 dy = \frac{1}{2}\left(4y - \left.\frac{4y^3}{3}\right|_0^1\right) = \frac{1}{2}\left(4-\frac{4}{3}\right) = \frac{4}{3}$$



$$= \int_0 x \int_{y=0}^{\sqrt{x}} dy\,dx + \int_1^2\int_0^{\sqrt{2-x}} dy\,dx = \int_0^1 x\sqrt{x}\,dx + \int_1^2 x\sqrt{2-x}\,dx$$

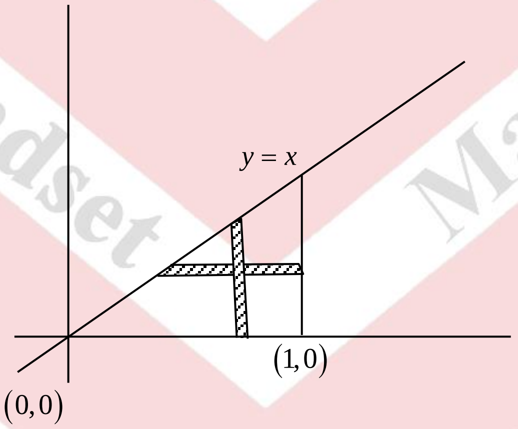
Integral Calculus 2 Change of order of integration in double integral

$$\iint\limits_R f(x,y)\,dx\,dy = \int_c^d \int_{x=\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx\,dy = \int\limits_a^b \int\limits_{y=\psi_1(x)}^{\psi_2(x)} f(x,y)\,dy\,dx$$



Q1. By changing of order of integration solve

$$\int_0^1\int_y^1 f(x,y)dx\,dy = \int_0^1\int_0^x f(x,y)\,dy\,dx$$



$R: 0 \le y \le 1$ (given)

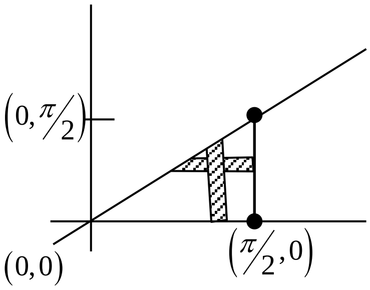
$y \le x \le 1$(given)

or $R:\{(x,y)/0 \le y \le 1\,;\ y \le 1\}$

Q2. $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy = ?$

$$\int_0^1\int_0^x \frac{\sin x}{x} dy\,dx = \int_0^1 \frac{\sin x}{\cancel{x}} \cdot \cancel{x}\ dx = -\cos x\Big|_0^1 = 1 - \cos 1$$

Q3. $\displaystyle\int\limits_0^{\pi/2}\int\limits_y^{\pi/2} \frac{\sin x}{x}\,dx\,dy$

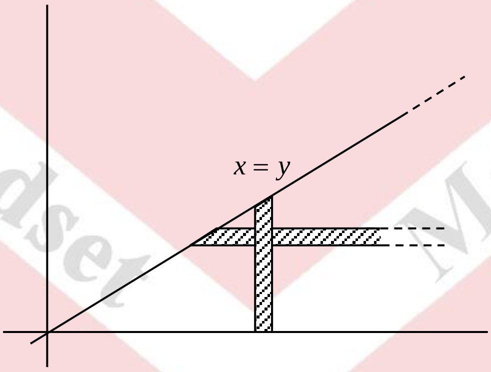


$$=\int_0^{\pi/2}\int_0^x \frac{\sin x}{x}\,dy\,dx=\int_0^{\pi/2}\frac{\sin x}{\cancel{x}}\cdot\cancel{x}\,dx=-\cos x\Big|_0^{\pi/2}=1-\cos\pi/2=1-0=1$$

Q4. $\int_0^1\int_y^1 \frac{e^x}{x}\,dx\,dy$

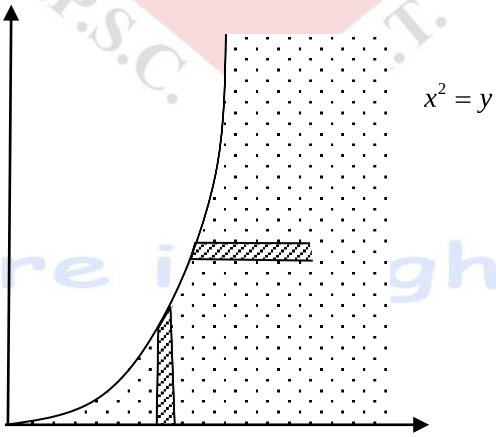
$$\int_0^1\int_0^x \frac{e^x}{x}\,dy\,dx=\int_0^1\frac{e^x}{\cancel{x}}\cdot\cancel{x}\,dx=e^x\Big|_0^1=e-1=2.71828-1=1.72$$

Q5. Find $\int_0^\infty\int_y^\infty \frac{e^{-x}}{x}\,dx\,dy$



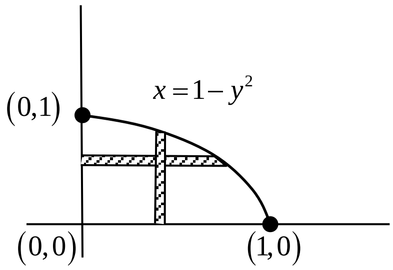
$$=\int_0^\infty\int_0^x \frac{e^{-x}}{x}\,dy\,dx=\int_0^\infty e^{-x}dx=-e^{-x}\Big|_0^\infty=1$$

Q6. Find $\int_0^\infty\int_y^\infty \frac{e^{-x}}{x^2}\,dx\,dy$



$$=\int_0^\infty\int_0^{x^2} \frac{e^{-x}}{x^2}\,dy\,dx=\int_0^\infty e^{-x}dx=1$$

Q7. $\int_0^1 \int_{x=0}^{x=1-y^2} y\sin\left(\pi(1-x)^2\right)dx\,dy$



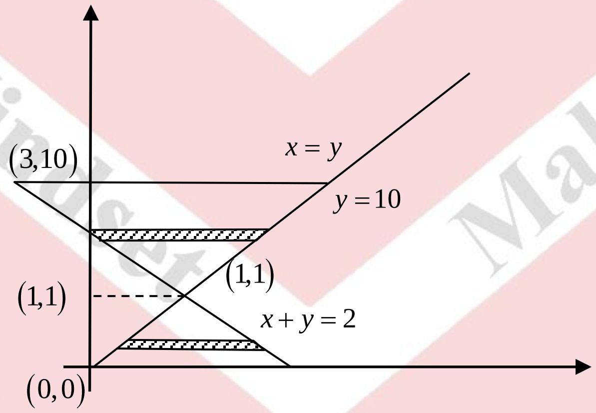
$$= \int_0^1 \int_{y=0}^{\sqrt{1-x}} y\,dy \cdot \sin\pi(1-x)^2\,dx = \int_0^1 \sin\left(\pi(1-x)^2\right)\cdot\frac{1-x}{2}\,dx$$

$$(1-x)^2 = t$$

$$= \int_1^0 \sin\pi t\cdot\frac{-dt}{4} = \frac{1}{4}\int_0^1 \sin\pi t\,dt = \frac{1}{4\pi}\left(-\cos\pi t|_0^1\right) = \frac{1}{2\pi}$$

Q8. $\int\limits_0^{10}\int\limits_y^{2-y} f(x,y)\,dx\,dy = ?$

$y \le x \le 2-y$

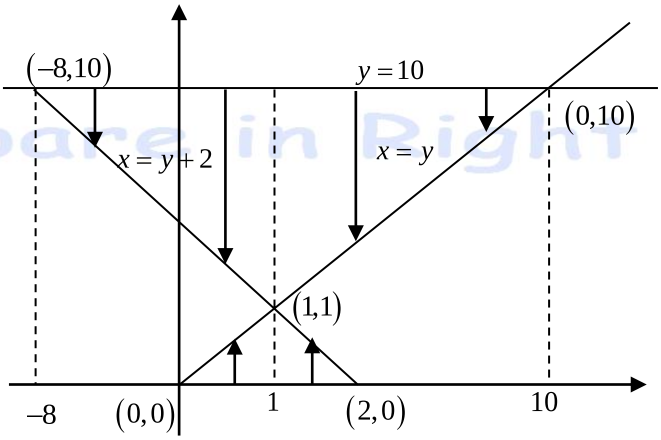


$$= \int_0^1\int_y^{2-y} f(x,y)\,dx\,dy + \int_1^{10}\int_y^{2-y} f(x,y)\,dx\,dy$$

$$= \int_0^1\int_y^{2-y} f(x,y)\,dx\,dy - \int_1^{10}\int_{2-y}^{y} f(x,y)\,dx\,dy$$

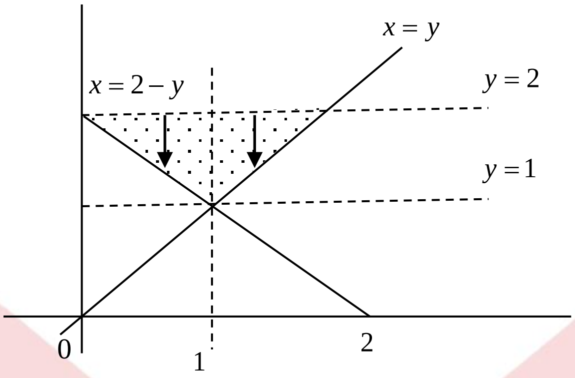
$$= \int_{-8}^0\int_{10}^{2-x} f(x,y)\,dy\,dx + \int_0^1\int_0^x f(x,y)\,dy\,dx + \int_0^1\int_{10}^{2-x} f(x,y)\,dy\,dx +$$

$$\int_1^2\int_0^{2-x} f(x,y)\,dy\,dx + \int_1^2\int_{10}^x f(x,y)\,dy\,dx + \int_2^{10} f(x,y)\,dy\,dx$$



Note:
In diagram 1 from $1 \leq y \leq 10$ direction is opposite so for that particular region when we change orders of integration we will move top to bottom instead of bottom to up ($\because$ direction is opposite)

Q9. $\int\limits_{1}^{2}\int\limits_{y}^{2-y} dx\,dy = \int_1^2 2 - y - y\,dy = \int_1^2 2 - 2y\,dy = 2y - y^2\Big|_1^2 = (4-4)-(2-1) = -1$

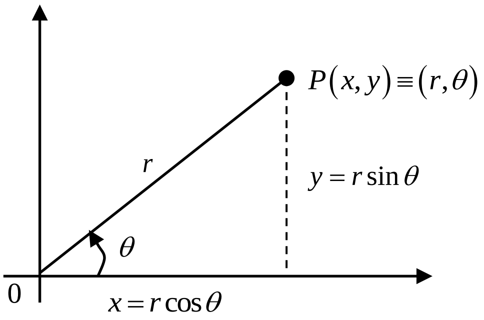


Now by change of order

$$= \int_0^1\int_2^{2-x} dy\,dx + \int_1^2\int_2^x dy\,dx = \int_0^1 \cancel{2} - x - \cancel{2}\,dx + \int_1^2 x - 2\,dx$$

$$-\frac{x^2}{2}\Bigg|_0^1 + \left(\frac{x^2}{2} - 2x\Bigg|_1^2\right) = -\frac{1}{2} + \frac{\cancel{4}}{\cancel{2}} - \cancel{4} - \frac{1}{2} + \cancel{2} = -1$$

Integral Calculus 2: Relation between Cartesian and Polar coordinates



Conversion from Cartesian → Polar:

$$\iint\limits_R f(x,y)\,dx\,dy=\iint\limits_R f(r\cos\theta,r\sin\theta)\left(\frac{dx\,dy}{dr\,d\theta}\right)\cdot dr\,d\theta$$

Now we use Jacobian:

$$J=\frac{dx\,dy}{dr\,d\theta}=\frac{d(x,y)}{d(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}$$

$$x=r\cos\theta\Rightarrow\frac{\partial x}{\partial r}=\cos\theta\;\Rightarrow\frac{\partial x}{\partial\theta}=-r\sin\theta$$

$$y=r\sin\theta\;\Rightarrow\frac{\partial y}{\partial r}=\sin\theta\;\Rightarrow\frac{\partial y}{\partial\theta}=r\cos\theta$$

$$\therefore\;\frac{d(x,y)}{d(r,\theta)}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r$$

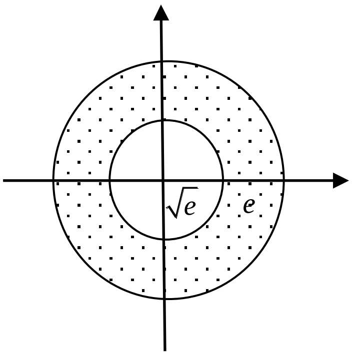
$\therefore$Conversion formal becomes:

$$\boxed{\iint\limits_R f(x,y)\,dx\,dy=\iint\limits_R f(r\cos\theta,r\sin\theta)\cdot r\,drd\theta}$$

Generalized Version:

$$\iint\limits_R f(x,y)\,dx\,dy=\iint\limits_R f\big(x(u,v),y(u,v)\big)|J|\,du\,dv\ \text{ where Jacobian }\ J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial(x,y)}{\partial(u,v)}$$

Q1. Find $\iint\limits_R\frac{\ln(x^2+y^2)}{(x^2+y^2)}dx\,dy$ where $R:S=\{(x,y)\in\mathbf{R}^2/e\le x^2+y^2\le e^2\}$



$x = r\cos\theta,\ y = r\sin\theta$

$e \le x^2 + y^2 \le e^2$

$e \le r^2 \le e^2$

$\sqrt{e} \le r \le e$

$0 \le \theta \le 2\pi$

$$\therefore\ \int_0^{2\pi}\int_{\sqrt{e}}^{e} \frac{\ln r^2}{\cancel{r^2}} \cdot \cancel{r}\, dr\, d\theta$$

$$\int_0^{2\pi}\int_{\sqrt{e}}^{e} \frac{\ln r^2}{r} dr\, d\theta$$

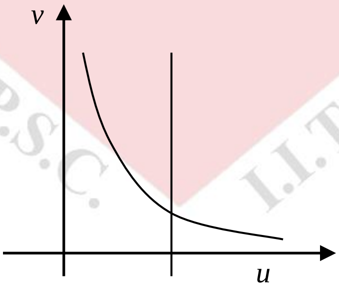
Put $\ln r = t$

$\frac{1}{r} dr = dt$

$$= \int_0^{2\pi} = \int_{\frac{1}{2}}^{1} 2t \cdot dt\, d\theta \ = 2\pi\left(1 - \frac{1}{4}\right) = 2\pi \cdot \frac{3}{4} \ = \frac{3}{2}\pi$$

Q2. If $\int\limits_{y=0}^{1}\int\limits_{x=y}^{2} f(x,y)\, dx\, dy$ is changed into double integral transforming (x,y) into (u,v) where $x = uv$ and $y = 2v$ then transformed integral is?

Solution.



$$\begin{aligned} &0 \le y \le 1 && y \le x \le 1 \\ &0 \le 2v \le 1 && 2v \le x \le 1 \\ &0 \le v \le \frac{1}{2} && 2v \le uv \le 1 \\ & && 2 \le y \le \frac{1}{v} \end{aligned}$$

Now

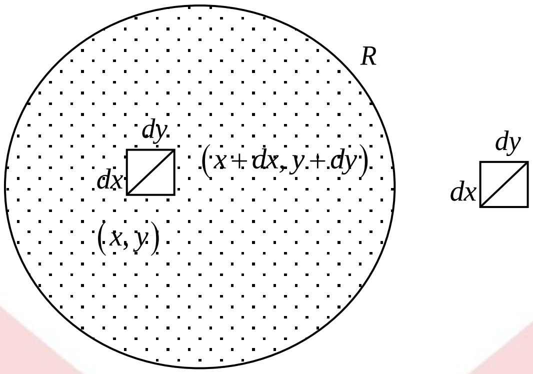
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 0 & 2 \end{vmatrix} = 2v$$

$$\therefore \int_{0}^{1/2}\int_{2}^{1/v} f\left(uv, 2v\right)\cdot\left|2v\right| du\, dv$$

Practical Use of Double Integral

(A) Area of region R in 2-D plane i.e. $x-y$ plane is given byL

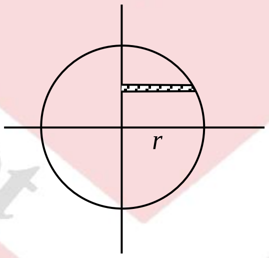
Elementary at $\left(x, y\right)$:



$dA = dx\, dy$

$$\boxed{A=\iint_{R} dx\, dy}$$

Q1. Find area of circle whose radius $= a$



$x^2 + y^2 = a^2$

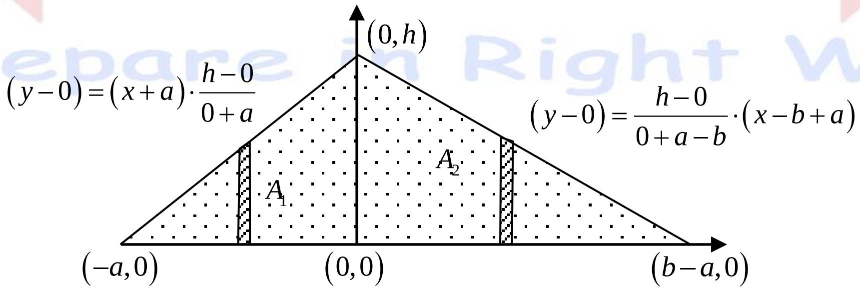
$$A=\int_0^a\int_6^{\sqrt{a^2-y^2}} dx\, dy \;=\int_0^a \sqrt{a^2-y^2}\, dy \;= \left.\frac{y\sqrt{a^2-y^2}+a^2\sin^{-1}\frac{y}{a}}{2}\right|_0^a = \frac{a^2\sin^{-1}1}{2} = \frac{a^2\pi}{4}$$

Complete Area $= 4A \;= \pi a^2$

By Polar:

$$\int_0^{2\pi}\int_0^{a} r\, dr\, d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^a d\theta = \cancel{2}\pi\cdot\frac{a^2}{\cancel{2}} = \pi a^2$$

Q2. Find area of ⊔ with base $= b$, height $= h$?



$$A_1 + A_2 = \int_{-a}^{0}\int_{y=0}^{\frac{h}{a}(x+a)} dy\,dx + \int_{0}^{b-a}\int_{0}^{\frac{h}{a-b}(x-b+a)} dy\,dx$$

$$= \int_{-a}^{0}\frac{h}{a}(x+a)dx + \int_{0}^{b-a}\frac{h}{a-b}(x-b+a)dx$$

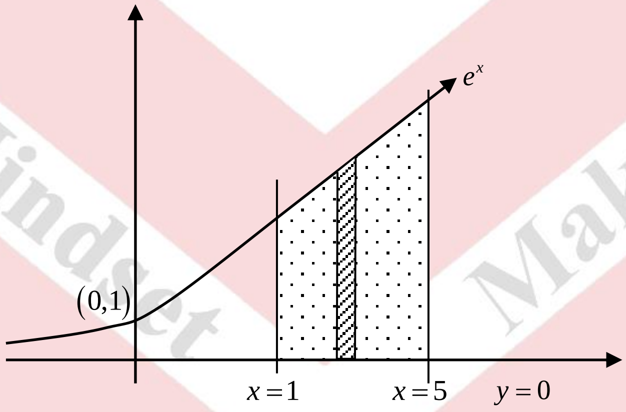
$$= \frac{h}{a}\frac{x^2}{2} + hx\Big|_{-a}^{0} + \frac{h}{a-b}\frac{x^2}{2} + hx\Big|_{0}^{b-a}$$

$$= -\frac{h}{a}\cdot\frac{a^2}{2} + ah + \frac{h}{a-b}\frac{(b-a)^2}{2} + h(b-a)$$

$$= -\frac{ah}{2} + ah + \frac{(b-a)h}{2} + hb - ah$$

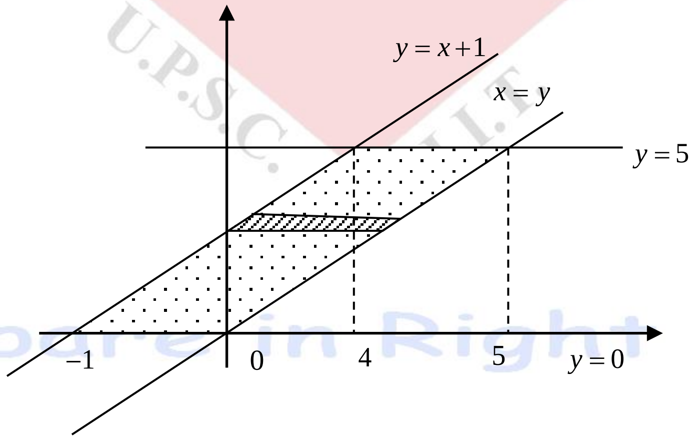
$$= -\cancel{\frac{ah}{2}} + \cancel{ah} + \frac{bh}{2} - \cancel{\frac{ah}{2}} + hb - \cancel{ah}$$

Q4. Find area enclosed by curves $y = 0, x = 1, x = 5$ and $y = e^x$?



Area $\int_{a}^{5}\int_{y=0}^{y=e^x} dy\,dx \ = \int_1^5 e^x dx \ = e^x\Big|_1^5 = e^5 - e^1 = e\left(e^4 - 1\right)$

Q5. Find area enclosed by curves $y = 0, y = x, y = x + 1$ and $y = 1$



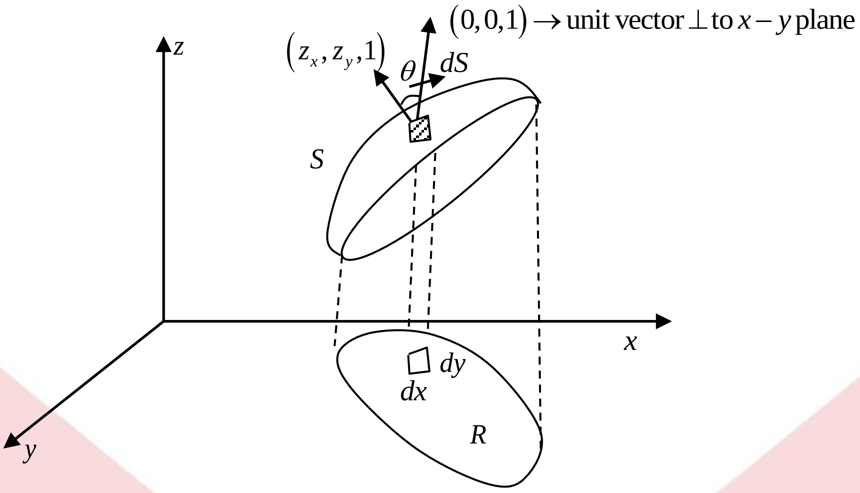
$$\int_{0}^{5}\int_{x=y=1}^{y} dx\,dy = \int_0^5 y - y + 1 dx \ = \int_0^5 dx$$

or

$$= \int_{-1}^{0}\int_{0}^{x+1} dy\,dx + \int_{0}^{4}\int_{x}^{x+1} dy\,dx + \int_{4}^{5}\int_{x}^{5} dy\,dx \; = \int_{-1}^{0} x+1\,dx + \int_{0}^{4} dx + \int_{4}^{5} 5-x\,dx \; = \frac{x^2}{2}+x\Big|_{1}^{0} + 4 + 5x - \frac{x^2}{2}\Big|_{4}^{5}$$

$$= -\frac{1}{2}+1+4+25-\frac{25}{2}-20+\frac{16}{2} = 25-20 = 5$$

Integral calculus 2 Surface Area of 3-D Surface (Using Double Integral)



$$ds\cos\theta = dxdy$$

$$ds = \frac{dx\,dy}{\cos\theta} \qquad \text{....(1)}$$

$S : z = f(x, y)$ (must be of this form)

$$1\cdot dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\cos\theta = \frac{1}{\sqrt{z_x^2 + z_y^2 + 1}}$$

Put value in (1)

$$dS = \sqrt{z_x^2 + z_y^2 + 1}\,dx\,dy$$

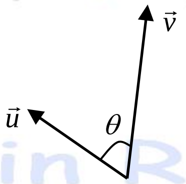
After projecting S onto $x-y$ plane we get region R, and then integrating

$$\boxed{S = \iint_R \sqrt{z_x^2 + z_y^2 + 1}\,dx\,dy}$$

(ℓ_1, m, n) and (ℓ_2, m, n) angle between them $\cos\theta = \dfrac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 n_1^2}\,\sqrt{\ell_2^2 + m_2^2 + n_2^2}}$

or

$\vec{i} = 0\hat{i} + 0\hat{j} + 1\hat{k}$ and $\vec{i} = z_x\hat{i} + z_y\hat{j} + 1\hat{k}$ 2 vectors angle θ in between



then

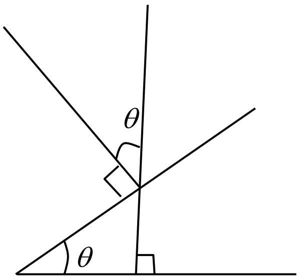
$$\vec{v}\cdot\vec{u} = |\vec{v}||\vec{u}|\cdot\cos\theta$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}||\vec{u}|}$$

$$= \frac{1}{\sqrt{z_x^2 + z_y^2 + 1}}$$

- If we want to evaluate surface area of a 3-D object/solid then if dS is elementary surface area on the given surface and if $dx\,dy$ is projection of dS on $x-y$ plane $(z=0)$ and if θ is angle between dS and $dxdy$ then $ds\cos\theta = dxdy$

  $\Rightarrow dS = \dfrac{dxdy}{\cos\theta}$ where θ is also angle between normal to the two surface.



- Now on the surface if $z = f(x,y)$, then

  $$1\cdot dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

  1 unit change in z direction means there is z_x change in x direction and z_y change in y direction)

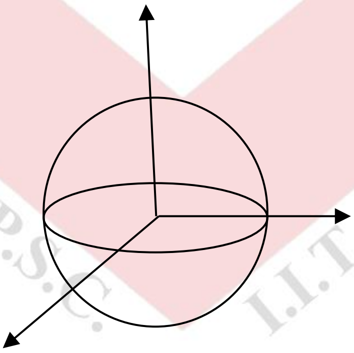
- So normal vector of surface S is $z_x\hat{i} + z_y\hat{j} + 1k$ and normal vector of $dxdy$ is $0\hat{i} + 0\hat{j} + 1\cdot\hat{k}$

  so $ds = \sqrt{1+z_x^2+z_y^2}\,dx\,dy$

- Now if projection of surface S (in 3-D) is region R in ($x-y$ plane) then

  $$\boxed{\iint_R \sqrt{1+z_x^2+z_y^2}\,dx\,dy = S}$$

Q1. Find surface area of sphere of radius a by double integral method?
equation: $x^2+y^2+z^2=a^2$ (center $(0,0,0)$)



$$z = \sqrt{a^2-x^2-y^2} = f(x,y)$$

$$2x+0+2z\,z_x = 0$$

$$z_x = -\frac{x}{z}$$

$$0+2y+2z\,zy = 0$$

$$z_y = -\frac{y}{z}$$

Projection on $x-y$ plane $\Rightarrow x^2+y^2=a^2$: R

$\because$ 2 small hemispheres are projected for complete sphere.

$$S = 2\iint_R \sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{y}{z}\right)^2}\, dx\, dy$$

$$= 2\iint_R \sqrt{1+\frac{x^2+y^2}{z^2}}\, dx\, dy$$

$$= 2\iint_R \frac{\sqrt{z^2+x^2+y^2}}{z}\, dx\, dy$$

$$= 2\iint_R \frac{\sqrt{a^2-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}}{\sqrt{a^2-x^2-y^2}}\, dx\, dy$$

$$= 2\iint_R a\cdot\frac{1}{\sqrt{a^2-\left(x^2+y^2\right)}}\, dx\, dy$$

Converting to polar form

$$2\cdot a\int_0^{2\pi}\int_0^{a}\frac{1}{\sqrt{a^2-r^2}}\cdot r\, dr\, d\theta$$

$$a^2-r^2=t$$

$$-2r\, dr = dt$$

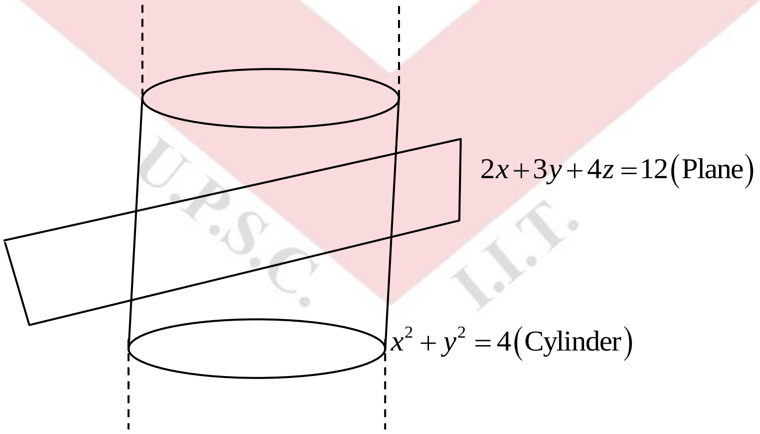
$$2\pi\cdot a-\frac{1}{2}\int\frac{1}{\sqrt{t}}\, dt$$

$$-2\pi a\, t^{1/2}$$

$$2-2\pi a\sqrt{\left(a^2-r^2\right)}\Big|_0^a = -4\pi a(-a) = 4\pi a^2$$

Q2. Find Surface area of plane $2x+3y+4z=12$ enclosed inside cylinder $x^2+y^2=4$?

$$R: x^2+y^2\le 4$$



$$z_x = {-2}/{4}$$

$$z_y = {-3}/{4}$$

or

Area of circle: $\iint dx\, dy$

$$\iint \sqrt{1+\frac{4}{16}+\frac{9}{16}}dx\,dy$$

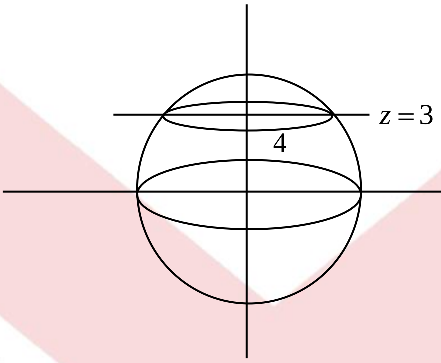
$$=\int_0^{2\pi}\int_0^2 r\,dr\,d\theta\cdot\frac{\sqrt{2a}}{4}$$

$$=2\pi\cdot\frac{r^2}{2}\Big|_0^2\cdot\frac{\sqrt{29}}{4}=\sqrt{29}\cdot\pi$$

$$\frac{\sqrt{29}}{4}\cdot\pi\left(2\right)^2=\sqrt{29}\pi$$

Q3. Find surface area of smaller projection of sphere $x^2+y^2+z^2=25$ cut by the plane $z=3$?

$$R: x^2+y^2=25$$



$$z_x=?\quad z_y=?$$

$$2x+2zz_x=0\ \Rightarrow z_x=-\frac{x}{z}$$

$$z_y=-\frac{y}{z}$$

Now

$$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+\frac{x^2}{z^2}+\frac{x^2}{z^2}}=\frac{5}{\sqrt{25-\left(x^2+y^2\right)}}$$

$$S=\iint_R\frac{5}{\sqrt{25-\left(x^2+y^2\right)}}dx\,dy$$

$$=\int_0^{2\pi}\int_0^4\frac{5}{\sqrt{25-r^2}}r\,dr\,d\theta$$

$$=5\cdot2\pi\int_0^4\frac{r}{\sqrt{25-r^2}}dr$$

$$S: x^2+y^2+z^2=25$$

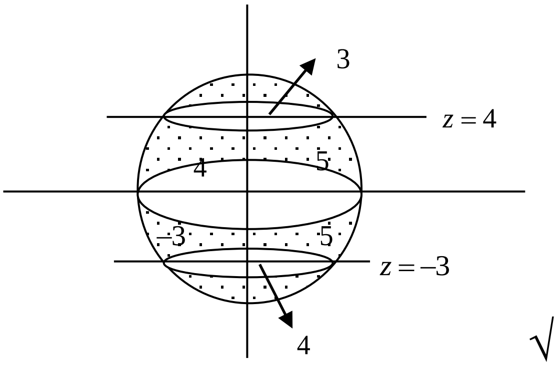
$$R: x^2+y^2=16$$

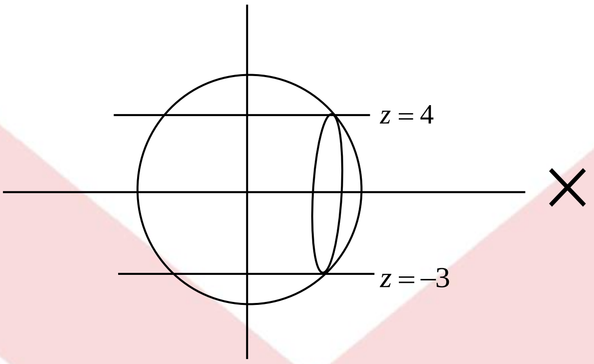
$$25-r^2=t$$

$$-2rdr=dt$$

$$=10r\int_{25}^{9}-\frac{1}{2}\frac{dt}{\sqrt{t}}\quad=5\pi\int_9^{25}\frac{1}{\sqrt{t}}dt\quad=10\pi\cdot t^{1/2}\Big|_9^{25}\quad=10\pi\cdot\left(5-3\right)=20\pi$$

Q4. Find surface area of partition of sphere $x^2+y^2+z^2=25$ between the plane $z=-3$ and $z=4$?





$$S=\iint_R \frac{5}{\sqrt{25-\left(x^2+y^2\right)}}dx\,dy$$

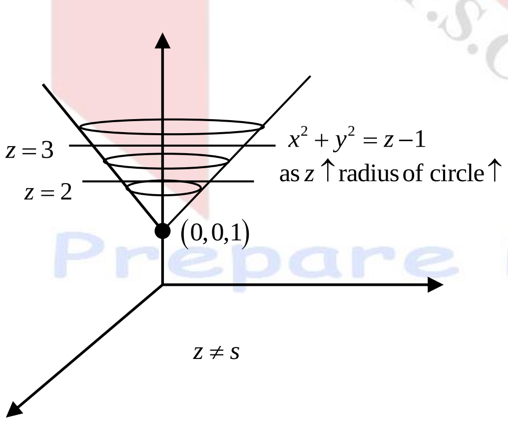
[Now main thing is to analyze **R**]

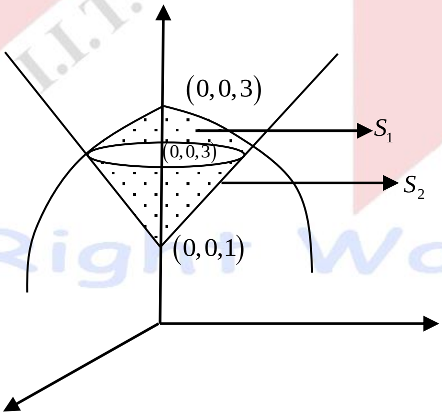
$$\mathbf{R}:\left\{(x,y)/9\le x^2+y^2\le 25\right\}\cup\left\{(x,y)/16\le x^2+y^2\le 25\right\}$$

$$=5\int_0^{2\pi}\int_3^5 \frac{1}{\sqrt{25-r^2}}\cdot rdrd\theta+5\int_0^{2\pi}\int_4^5 \frac{r}{\sqrt{25-r^2}}dr\,d\theta$$

$$=-5\cdot 2\pi\left(\frac{2\sqrt{25-r^2}\Big|_3^5}{2}+\frac{2\sqrt{25-r^2}\Big|_4^5}{2}\right)=-10\pi(-4-3)=70\pi$$

Q5. Find surface area of the portion of surface common to $z=3-\sqrt{x^2+y^2}$ and $z=1+x^2+y^2$





$$x^2+y^2+1=z$$
$$x^2+y^2=(3-z)^2$$

$x^2+y^2=z-1$

Now intersection

$(3-z)^2=z-1$

$9+z^2-6z-z+1=0$

$z^2-7z+10=0$

$z=52$

$z\neq 5$

$\sqrt{x^2+y^2}=3-z$

$z>3$

$\sqrt{x^2+y^2}\leq 0$ [not possible]

Required surface area $=S_1+S_2$

$R: x^2+y^2\leq 1$ (for both S_1 and S_2)

$S_1: x^2+y^2=(3-z)^2$

$2x=2(3-z)-z_x$

$z_x=-\dfrac{x}{3-z}$

$2y=2(3-z)\cdot -z_y$

$z_y=\dfrac{y}{3-z}$

$$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+\frac{x^2}{(3-z)^2}+\frac{y^2}{(3-z)^2}}=\sqrt{\frac{(3-z)^2+x^2+y^2}{(3-z)^2}}=\sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}=\sqrt{2}$$

$$\sqrt{2}\iint_R dxdy=\sqrt{2}\cdot\pi\cdot 1^2=\pi\sqrt{2}$$

$S_2: x^2+y^2=z-1$

$z_x=2x$

$z_y=2y$

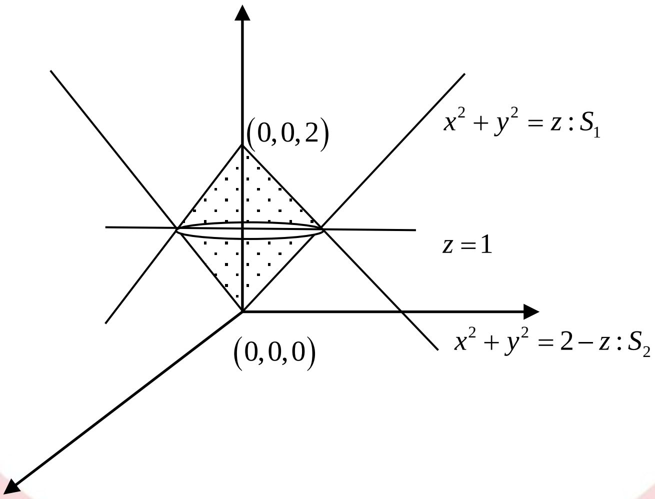
$$\sqrt{1+4x^2+4y^2}=\sqrt{1+4(x^2+y^2)}$$

$$\iint_R\sqrt{1+4(x^2+y^2)}dx\,dy=\int_0^{2\pi}\int_0^1\sqrt{1+4r^2}\cdot r\,dr\,d\theta$$

$$=2\pi\cdot\frac{1}{8}\int_1^5\sqrt{t}\,dt=\frac{\pi}{4}t^{3/2}\Big|_1^5\cdot\frac{2}{3}=\frac{\pi}{4}\left((5)^{3/2}-1\right)\cdot\frac{2}{3}=\frac{\pi}{6}\left((5)^{3/2}-1\right)=\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

Total surface area $=\sqrt{2}\pi+\dfrac{\pi}{6}\left(\sqrt{5}\cdot 5-1\right)$

Q6. Find surface area of the surface common between $x^2+y^2=z$ and $x^2+y^2=2-z$?



$z=2-z$
$2z=2$
$z=1$

Total area $=S_1+S_2$

$S_1: x^2+y^2=z$

$z_x=2x$

$z_y=2y$

$$\iint_R \sqrt{1+4\left(x^2+y^2\right)}dx\,dy \; =\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

Total $=2\left(\frac{\pi}{6}\left(5\sqrt{5}-1\right)\right)=\frac{\pi}{3}\left(5\sqrt{5}-1\right)$

$R: x^2+y^2\le 1$ (for both S_1 and S_2)

$S_2: x^2+y^2=2-z$

$z_x=-2x$

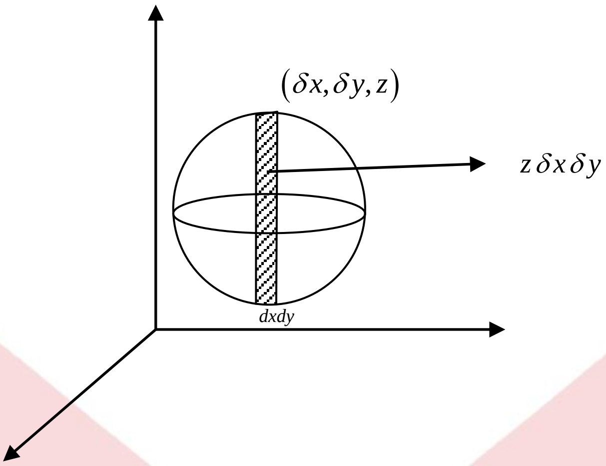
$z_y=-2y$

$$\iint_R \sqrt{1+4\left(x^2+y^2\right)}dx\,dy \; =\frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

Integral Calculus 2

Volume of 3D Object / Solids by double integral

If we have 3D object / solid bounded below by $z=0$ i.e. $x-y$ plane then if $dx\,dy$ is very very small area on $z=0$ then if we take cuboid with length and width as δx and δy and height z (corresponding to point on top of the surface) then volume of elementary cuboid will be $z\delta x\delta y$.

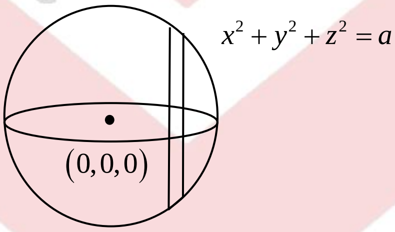


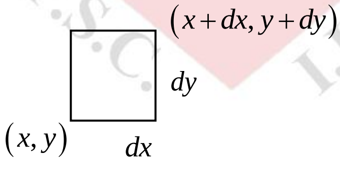
$$\delta v = z\delta x\delta y$$
$$dv = zdxdy$$
$$v=\iint z\,dxdy \;=\iint z(x,y)\,dx\,dy$$

R in $x-y$ plane

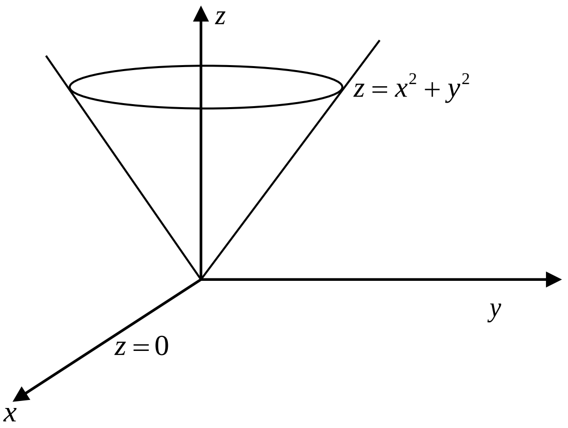
Q1. By double integral find volume of sphere of radius $'a'$?

$$dv = 2zdxdy$$
$$v=2\iint\sqrt{a^2-(x^2+y^2)}\,dx\,dy \;=2\cdot 2\pi\int_0^a\sqrt{a^2-r^2}\,dr \;=\frac{4}{3}\pi a^3$$





Q2. Find volume of solid enclosed within $x^2+y^2\le 2$ region, bounded below with $x-y$ plane and bounded above by $z=x^2+y^2$?



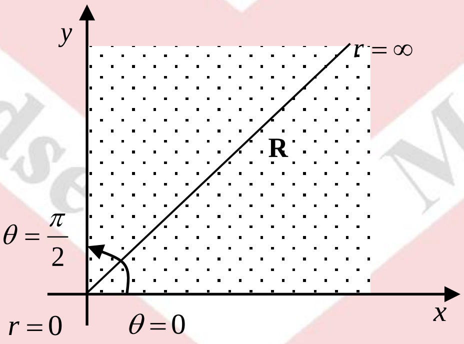
$$v = \iint \left(x^2 + y^2\right) dx\, dy$$

$$R : \left\{(x, y) \middle| x^2 + y^2 \le 2\right\}$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r^3 dr\, d\theta \; = 2\pi \cdot \left(\left. \frac{r^4}{4} \right|_0^{\sqrt{2}} \right) \; = 2\pi$$

Q3. Find $\int_0^{\infty} e^{-x^2} dx$

Solution.



Let $k = \int_0^{\infty} e^{-x^2} dx$ and $k = \int_0^{\infty} e^{-y^2} dy$

$$k^2 = \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy \; = \int_0^{\infty}\int_0^{\infty} e^{-x^2} dx\, dy \; = \int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)} dx\, dy \; = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

Let $-r^2 = t$

$-2rdr = dt$

$$= \frac{\pi}{2} \int_0^{-\infty} e^t \frac{dt}{-2} \; = \frac{\pi}{4} \int_{-\infty}^{0} e^t dt \; = \frac{\pi}{4} \left. e^t \right|_{-\infty}^{0} \; = \frac{\pi}{4} \cdot 1$$

$$k^2 = \frac{\pi}{4}$$

Note:

$$I = \int_0^{\infty} e^{-x^2} dx$$

Let $x^2 = t$

$2xdx = dt$

$$I=\int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{2}} \;=\int_0^\infty \frac{e^{-x}x^{\frac{1}{2}-1}}{\cancel{2}}dx=\frac{\sqrt{\pi}}{\cancel{2}}$$

$$\Rightarrow \boxed{I=\int_0^\infty e^{-x}x^{\frac{1}{2}-1}dx=\sqrt{\pi}}$$

Gamma Function

$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \;\; ; n>0$

$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ (proved previously)

$\Gamma(1)=1$

$\Gamma(n)=(n-1)\Gamma(n-1)$

* $\Gamma(1)=\int_0^\infty e^{-x}dx \;=-e^{-x}\Big|_0^\infty \;=1$

When $n>1$:

Apply integration by parts on $\Gamma(n)$:

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$$

$$=x^{n-1}\left(-e^{-x}\Big|_0^\infty\right)+(n-1)\int_0^\infty e^{-x}x^{n-2}dx$$

$$=\lim_{x\to\infty}\cancel{\frac{x^{n-1}}{e^{x}}}-0+(n+1)\int_0^\infty e^{-x}x^{n-2}dx$$

$$\Gamma(n)=(n+1)\Gamma(n-1)$$

*If n is natural number then:

$$\boxed{\Gamma(n)=\Gamma(n-1)!}$$

$$\Gamma(n)=(n-1)\Gamma(n-1)\;=(n-1)(n-2)\Gamma(n-2)\;=(n-1)(n-2)\ldots1$$

$$=(n-1)!$$

Q1. $\int_0^\infty e^{-x}x^{5}dx$

$$\int_0^\infty e^{-x}x^{6-1}dx=\Gamma(6)=5!=120$$

Q2. $\int_0^\infty e^{-5x}x^{3/2}dx$

$5x=t$

$5dx=dt$

$$\frac{1}{5}\int_0^\infty e^{-t}\left(\frac{t}{5}\right)^{3/2}dt$$

$$\frac{1}{5\cdot5\sqrt{5}}\int_0^\infty e^{-t}t^{5/2-1}dt$$

$$\frac{1}{25\sqrt{5}}\int_0^\infty e^{-t}\cdot t^{5/2-1}dt=\frac{\Gamma(5/2)}{25\sqrt{5}}$$

$$=\frac{\Gamma(3/2)\cdot 3/2}{25\sqrt{5}}\;=\frac{\Gamma(1/2)\cdot 3/2\cdot 1/2}{25\sqrt{5}}\;=\frac{3/2\cdot 1/2\cdot\sqrt{\pi}}{25\sqrt{5}}\;=\frac{3\cdot\sqrt{\pi}}{100\sqrt{5}}\;=\frac{3}{100}\sqrt{\frac{\pi}{5}}$$

$n-1=-\frac{1}{2}$

$n=-\frac{1}{2}+1$

Q3. $\int_0^\infty e^{-x^4}x\,dx=?$

$x^4=t$

$4x^2\cdot x\,dx=dt$

$$\frac{1}{4}\int_0^\infty e^{-t}\frac{dt}{t^{1/2}} = \frac{1}{4}\int_0^\infty e^{-t}t^{-1/2}dt = \frac{1}{4}\int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt = \frac{1}{4}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{4}$$

Beta function

$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx\;;\;m,n>0$

$$\boxed{B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

For proof

put $x=\sin^2\theta$

Q1. $\int\limits_0^1 x^{5/2}(1-x)^{7/2}\,dx=?$

$m-1=\frac{5}{2}$ $\qquad$ $n-1=7/2$

$m=7/2$ $\qquad$ $n=9/2$

$$B(7/2,9/2)=\frac{\Gamma(7/2)\Gamma(9/2)}{\Gamma(8)}$$

$$=\frac{5/2\cdot 3/2\cdot\frac{1}{2}\sqrt{\pi}\cdot 7/2\cdot 5/2\cdot 3/2\cdot\frac{1}{2}\sqrt{\pi}}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{2^7}\cdot\pi\;\frac{\cancel{7}\cdot\cancel{5}\cdot\cancel{3}\cdot 5\cdot\cancel{3}}{\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot 4\cdot\cancel{3}\cdot 2\cdot 1} = \frac{5\pi}{2^{11}}$$

By walli's formula

$x=\sin^2\theta$

$dx=2\sin\theta\cos\theta\,d\theta$

$$I=2\int_0^{\pi/2}\sin^6\theta\cos^8\theta\,d\theta=2\left[\frac{5\cdot\cancel{3}\cdot 1\cdot\cancel{7}\cdot\cancel{5}\cdot\cancel{3}\cdot 1}{\cancel{14}\cdot\cancel{12}\cdot\cancel{10}\cdot 8\cdot\cancel{6}\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right]=\frac{5\pi}{2^{11}}$$

Q2. $\int_0^\infty\frac{x^3}{(1+x)^5}\,dx$

Put $\frac{1}{1+x}=t$

$x+1=\frac{1}{t}$

$x=\frac{1}{t}-1$

$$dx = -\frac{1}{t^2}\,dt$$

$$= -\int_1^0 \left(\frac{1-t}{t}\right)^3 \cdot t^5 \frac{1}{t^2}\,dt \;= \int_0^1 (1-t)^3\,dt \;= \int_0^1 (1-t)^3\,dt = \frac{\Gamma(1)\Gamma(4)}{\Gamma(5)} \;= \frac{1\cdot 3!}{4!} = \frac{1}{4}$$

$$\int_{z=e}^{f}\int_{y=c}^{d}\int_{x=a}^{b} f\left(x,y,z\right)dx\,dy\,dz$$

$$\int_{a}^{b}\left(\int_{y=\psi_1(z)}^{\psi_2(z)}\left(\int_{x=f_1(y,z)}^{f_2(y,z)} f\left(x,y,z\right)dx\right)dy\right)dz \text{ [most general versopm]}$$

Q1. Find $\int_1^3\int_0^2\int_0^1 x^3y^4z^2dx\,dy\,dz$

$$=\int_1^3\int_0^2 \left.\frac{x^4}{4}\right|_0^1 y^4z^2dy\,dz=\frac{1}{4}\int_1^3\left.\frac{y^5}{5}\right|_0^2 z^2\,dz=\frac{1}{4}\cdot\frac{25}{5}\cdot\left(\left.\frac{z^3}{3}\right|_1^3\right)=\frac{1}{4}\cdot\frac{2^5}{5}\cdot\left(\frac{3^3}{3}-\frac{1}{3}\right)$$

$$=\frac{1}{4}\cdot\frac{2^5}{5}\cdot\frac{2}{3}=\frac{8\cdot 2}{15}=\frac{208}{15}$$

or

$$=\left.\frac{x^4}{4}\right|_0^1\cdot\left.\frac{y^5}{5}\right|_0^2\cdot\left.\frac{z^3}{3}\right|_1^3=\frac{1}{4}\cdot\frac{2^5}{5}\cdot\frac{26}{3}=\frac{208}{15}$$

Q2. $\int_0^1\int_0^{3z}\int_0^{y+z}\left(x+y+z\right)^2dx\,dy\,dz$

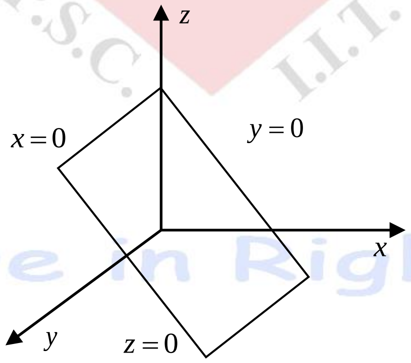
$$=\int_0^1\int_0^{3z}\left.\frac{\left(x+y+z\right)^3}{3}\right|_0^{y+z}dy\,dz=\int_0^1\int_0^{3z}\frac{\left(y+z+y+z\right)^3}{3}-\frac{\left(y+z\right)^3}{3}dy\,dz$$

$$=\int_0^1\int_0^{3z}\frac{8\left(y+z\right)^3-\left(y+z\right)^3}{3}dy\,dz=\frac{7}{3}\int_0^1\int_0^{3z}\left(y+z\right)^3dy\,dz=\frac{7}{3}\int_0^1\left.\frac{\left(y+z\right)^4}{4}\right|_0^{3z}dz$$

$$=\frac{7}{3}\int_0^1\frac{(4z)^4}{4}-\frac{z^4}{4}dz$$

$$=\frac{7}{3}\cdot\frac{255}{4}\int_0^1 z^4dz=\frac{7}{\cancel{3}}\cdot\frac{\cancel{255}}{4}\cdot\frac{1}{\cancel{5}}=\frac{119}{4}$$

Q3. Evaluate triple integral over $\mathbf{R}$ $\iiint_R xyz^2dx\,dy\,dz$, where $\mathbf{R}$ is region / space in 3-D space bounded by $x=0, y=0, z=0$ and $2x+3y+4z=12$.



$$=\int_0^3\int_{y=0}^{\frac{12-4z}{3}}\int_{x=0}^{\frac{12-3y-4z}{2}}\underbrace{xyz^2}_{\text{Density}}\ \underbrace{dxdydz}_{\text{Volume}}$$

[We are trying to find mass captured for density xyz^2 in region R]

$$\text{Density}=\frac{\text{mass}}{\text{volume}}$$

$$=\int_0^3\int_0^{\frac{12-4z}{3}} yz^2\cdot\frac{(12-3y-4z)^2}{4\cdot 2}\,dy\,dz$$

$$=\frac{1}{8}\int_0^3 z^2\left(\int_0^{\frac{12-4z}{3}}(12-3y-4z)^2\,y\,dy\right)dz$$

$$=\frac{1}{8}\int_0^3 z^2\left[\frac{(12-3y-4z)^3}{3\cdot -3}\cdot y-\int\frac{(12-3y-4z)^3}{3\cdot -3}\,dy\right]dz$$

$$=\frac{1}{8}\int_0^3 z^2\left[\frac{-(12-3y-4z)^3}{9}\cdot y-\frac{(12-3y-4z)}{4\cdot 3\cdot -3\cdot -3}\Bigg|_0^{\frac{12-4z}{3}}\right]dz$$

$$=\frac{1}{8}\int_0^3 z^2\left(\frac{(12-4z)^4}{4\cdot 27}\right)dz=-\frac{1}{8\cdot 4\cdot 27}\int_0^3 z^2(12-4z)^4\,dz$$

$$=\frac{1}{864}\left[\frac{(12-4z)^5}{5\cdot -4}\cdot z^2-\int\frac{(12-4z)^5}{-20}\cdot 2z\,dz\right]$$

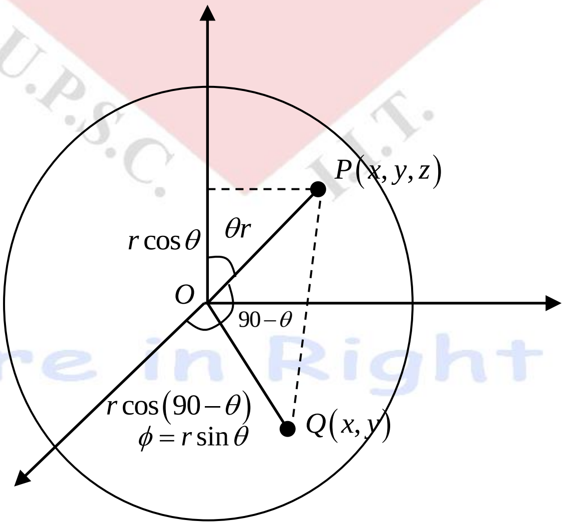
$$=\frac{4^5}{864}\left[\frac{(z-3)^5}{20}\cdot z^2-\frac{1}{10}\int(z-3)^5\cdot z\,dz\right]$$

$$=\frac{4^5}{864}\left[\frac{(z-3)^5}{20}\cdot z^2-\frac{(z-3)^6}{60}\cdot z+\frac{(z-3)^7}{70}\right]$$

$$=\frac{45}{8\cdot 4\cdot 27}\;\frac{3^7}{70}=\frac{\cancel{4}\cdot 4\cdot 4}{\cancel{2}\cdot 27\cdot\cancel{70}}=\frac{16\cdot 3^1}{27\cdot 35}=\frac{16\cdot 3^4}{35}$$

Conversion of Triple Integral from one system to another:

1. Conversion of T.I. from Cartesian wordinate system to spherical coordinate system:



2.

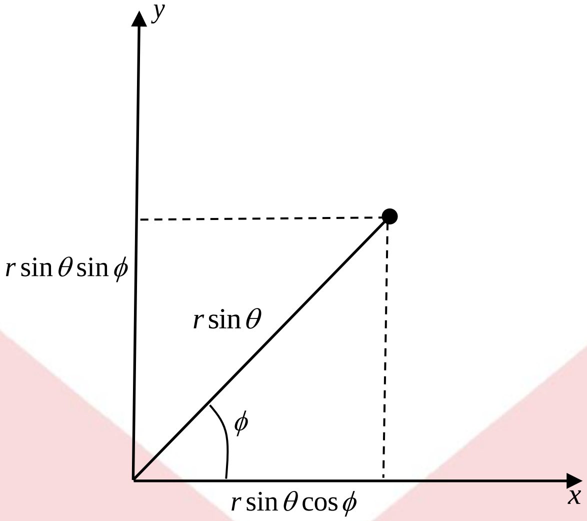
If $O(0,0,0)$ is origin and $P(x,y,z)$ is any point in Cartesian co-ordinate then in spherical co-ordinate which is represented by (r,θ,ϕ) where:

r : is length of $\overrightarrow{OP}$

Q : angle which $\overrightarrow{OP}$ makes with +ive z -axis

ϕ : If $\overrightarrow{OQ}$ is projection of $\overrightarrow{OP}$ on $x-y$ plane then ϕ is angle between $\overrightarrow{OQ}$ and + ive direction of x -axis.

So now we have:



$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

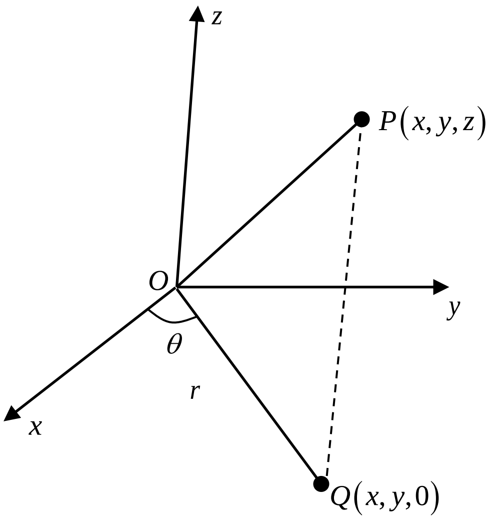
$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)|J|\,dr\,d\theta\,d\phi$$

where $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\cos\phi \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$$= \sin\theta\cos\phi\left(r^2\sin^2\theta\cos\phi\right) - r\cos\theta\cos\phi\left(-r\sin\theta\cos\phi\cos\theta\right)$$
$$+\left(-r\sin\theta\sin\phi\right)\left(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right)$$
$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi$$
$$= r^2\sin\theta\cos^2\phi + r^2\sin\theta\sin^2\phi$$
$$= r^2\sin\theta$$

$$\therefore \boxed{\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2\sin\theta\,dr\,d\theta\,d\phi}$$

Conversion of Triple integral from Cartesian coordinate system to cylindrical coordinate system.



In cylindrical coordinate any point (x,y,z) of Cartesian coordinate system is representation by (r,θ,z) where r is length of $\overrightarrow{OQ}$ where Q is projection of point $P(x,y,z)$ on $X-Y$ plane and θ is angle which $\overrightarrow{OQ}$ makes with X-axis in positive direction.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

So now we have:

$z = z$

$x = r\cos\theta$

$y = r\sin\theta$

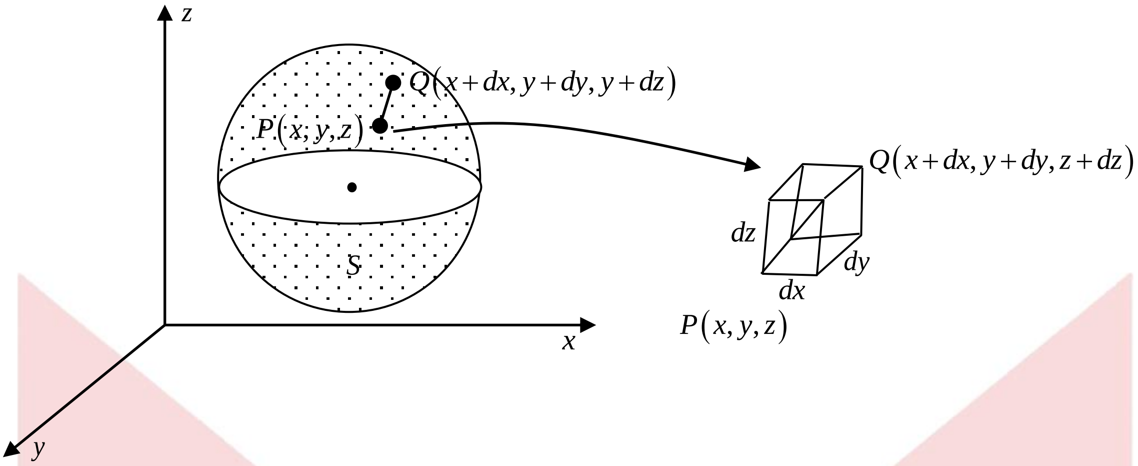
$$= \iiint_R f(x,y,z)dx\,dy\,dz = \iiint_R f(r\cos\theta, r\sin\theta, z)|J|\cdot dr\,d\theta\,dz$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \boxed{\iiint_R f(x,y,z)dx\,dy\,dz = \iiint_R f(r\cos\theta, r\sin\theta, z)\cdot r\,dr\,d\theta\,dz}$$

Use of Triple Integral

T.I is used to evaluate volume of any 3-D object



Volume of this cuboid (very small) $= dx \cdot dy \cdot dz$

$dv = dx\,dy\,dz$

If in a 3D space/object we take any arbitrary point (x, y, z) and a point $(x+dx, y+dy, z+dz)$ very very close to it then volume of cubiod with diagonal PQ is $dx\,dy\,dz$.

So elementary volume inside object is:

$dv = dx\,dy\,dz$

So volume of object will be:

$$\boxed{V = \iiint_S dv = \iiint_S dx\,dy\,dz}$$

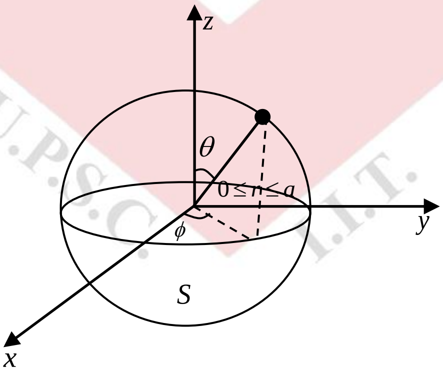
Q1. Find volume of sphere of radius $'a'$?

$x^2 + y^2 + z^2 = a^2$, $S: \{(x, y, z) / x^2 + y^2 + z^2 \le a^2\}$

$0 \le r \le a$

$0 \le \theta \le \pi$ (good)

$0 \le \phi \le 2\pi$ (in xy plane)



$$V = \iiint_S dx\,dy\,dz$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} dx\,dy\,dz$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-z^2}} \sqrt{a^2 - y^2 - z^2}\,dy\,dz$$

$$=8\int_0^a \frac{y}{2}\sqrt{a^2-y^2-z^2}+\frac{a^2-z^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-z^2}}\,dz$$

$$=8\int_0^a \frac{a^2-z^2}{2}\cdot\frac{\pi}{2}\,dz$$

or

by spherical coordinates:

$$V=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2\sin\theta dr\,d\theta\,d\phi$$

$$=\int_0^{2\pi}\int_0^{\pi}\frac{r^3}{3}\Big|_0^a \sin\theta\,d\theta\,d\phi=\frac{a^3}{3}\int_0^{2\pi}\left(-\cos\theta\big|_0^{\pi}\right)d\phi=2\pi\cdot\frac{a^3}{3}\cdot 2=\frac{4}{3}\pi a^3$$

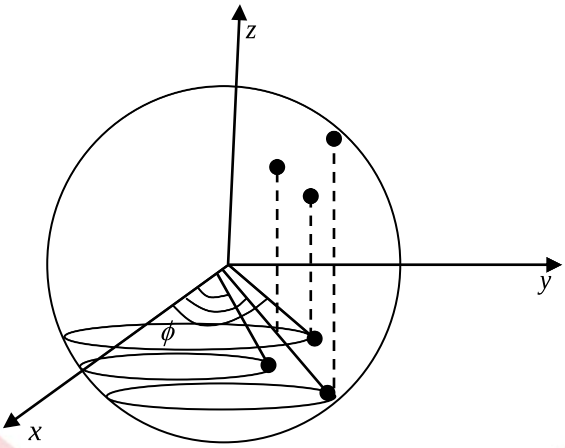
* Think properly for limits

$$=2\pi\int_0^a a^2-z^2dz=2\pi\left(a^2z-\frac{z^3}{3}\Big|_0^a\right)=2\pi\frac{2a^3}{3}=\frac{4}{3}\pi a^3$$

Explanation:

When we rotate ϕ then points of $\pi < \theta \le 2\pi$ are also covered so we need $0 < \theta < \pi$ only

$0 \le \phi \le 2\pi$



$0 \le \phi \le 2\pi$

Q2. Find volume of 3-D object enclosed by surfaces / planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$?

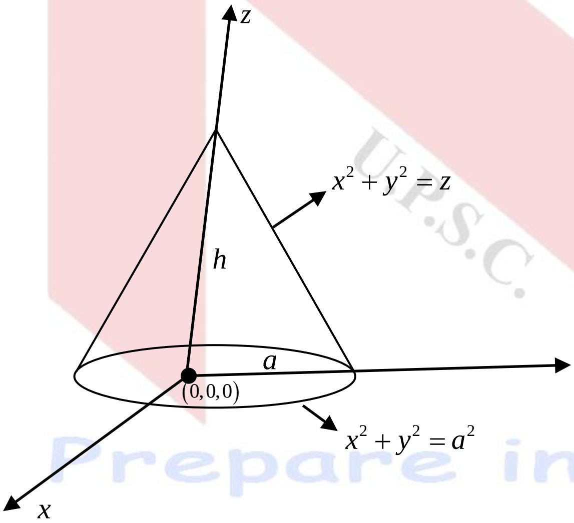
$$V = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx\,dy\,dz = \int_0^1 \int_0^{1-z} 1 - y - z\,dy\,dz = \int_0^1 y - \frac{y^2}{2} - zy\Big|_0^{1-z} dz = \int_0^1 (1-z)^2 - \frac{(1-z)^2}{2} dz$$

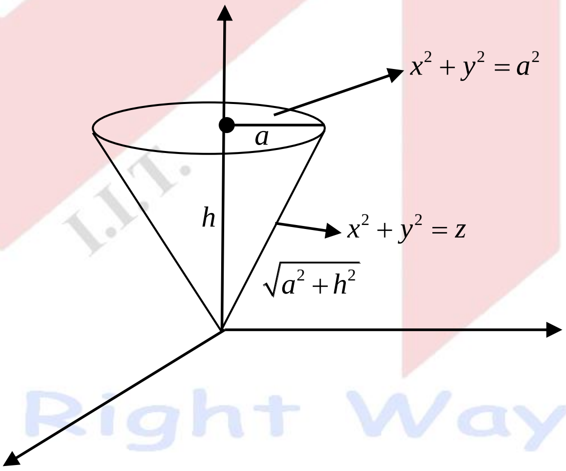
$$= \frac{1}{2}\int_0^1 (1-z)^2\,dz = \frac{1}{2}\left(z + \frac{z^3}{3} - \frac{2z^2}{2}\Big|_0^1 \right) = \frac{1}{2}\left(1 + \frac{1}{3} - 1\right) = \frac{1}{6}$$

Also

$$V = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-y-z} dz\,dy\,dx$$

Q3. By triple integral method find volume of right circular cone whose radius is of length $'a'$ and height is $'h'$.





$$V = \iiint_S dx\,dy\,dz$$

Better to solve by spherical coordinates (r, θ, ϕ)

or

$0 \le r \le \sqrt{a^2 + h^2}$ or $0 \le r \le h\sec\theta$

$0 \le \phi \le 2\pi$
$\le \theta \le$

INTEGRAL CALCULUS

CHAPTER 1. INTEGRAL AS A LIMIT OF SUM

Q1. Find the limit $\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=0}^{n-1}\sqrt{n^2-r^2}$. **[1d UPSC CSE 2018]**

Q2. Define a sequence S_n of real numbers by

$$S_n=\sum_{i=1}^{n}\frac{\left(\log(n+i)-\log n\right)^2}{n+i}$$

Does $\lim_{n\to\infty}S_n$ exist? If so, compute the value of this limit and justify your answer.

[3b UPSC CSE 2012]

CHAPTER 2. DEFINITE INTEGRALS

Q1. Show that $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log_e\left(1+\sqrt{2}\right)$. **[4b P-2 UPSC CSE 2020]**

Q2. Evaluate $\int_0^1\tan^{-1}\left(1-\frac{1}{x}\right)dx$. **[2a UPSC CSE 2020]**

Q3. Evaluate the following integral:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}dx$. **[1d UPSC CSE 2015]**

Q4. Evaluate

$\int_0^1\frac{\log_e(1+x)}{1+x^2}dx$. **[1d UPSC CSE 2014]**

Q5. For $x>0$, let $f(x)=\int_1^x\frac{\ln t}{1+t}dt$. Evaluate $f(e)+f\left(\frac{1}{e}\right)$. **[1d 2015 IFoS]**

Q6. Evaluate $\int_0^1\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)dx$. **[1c UPSC CSE 2013]**

Q7. Evaluate

$\int_0^{\pi/2}\frac{x\sin x\cos xdx}{\sin^4 x+\cos^4 x}$. **[4a 2013 IFoS]**

Q8. $\int_0^1 \ell nx\, dx$. **[3a(ii) UPSC CSE 2011]**

CHAPTER 3. BETA GAMMA FUNCTIONS

Q3(iii) Express $\int_a^b (x-a)^m (b-x)^n\, dx$ in terms of Beta function. **[UPSC CSE 2021]**

(c) Using Beta and Gamma functions, evaluate the following integrals:

(i) $\int_0^2 x\left(8-x^3\right)^{1/3} dx$

(ii) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}}$. **[IFoS 2021]**

Q1. $\int_{-\infty}^{\infty} xe^{-x^2} dx$. **[4c(ii) 2020 IFoS]**

Q2. Show that

$$\int_0^{\pi/x} \sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)},\ \ p,q>-1$$

Hence evaluate the following integrals:

(i) $\int_0^{\pi/2} \sin^4 x\cos^5 xdx$

(ii) $\int_0^1 x^3\left(1-x^2\right)^{5/2} dx$

(iii) $\int_0^1 x^4(1-x)^3 dx$. **[2b 2017 IFoS]**

Q3. Evaluate:

$I=\int_0^1 \sqrt[3]{x\log\left(\frac{1}{x}\right)}dx$. **[1c UPSC CSE 2016]**

Q4. Show that the integral $\int_0^\infty e^{-x}x^{\alpha-1}dx,\ \alpha>0$ exists, by separately taking the case for $\alpha \geq 1$and $0<\alpha<1$. **[4b 2016 IFoS]**

Q5. Prove that

$\sqrt{2z} = \frac{2^{2z-1}}{\sqrt{\pi}}\sqrt{z}\sqrt{z+\frac{1}{2}}$. **[4c 2016 IFoS]**

Q6. Evaluate the integral

$I=\int_0^\infty z^{ax^2}dx$ using Gamma function. **[3c 2014 IFoS]**

Q7. Find all the real values of p and q so that the integral $\int_0^1 x^p \left(\log\frac{1}{x}\right)^q dx$ converges.

[3c P-1 UPSC CSE 2012]

Q8. Evaluate the following in terms of Gamma function:

$\int\limits_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}dx$. **[4d 2012 IFoS]**

CHAPTER 4. DOUBLE INTEGRALS, TRIPLE INTEGRALS

Q1(d) Evaluate $\iint\limits_R x^2 dx\,dy$, where R is the region in the first quadrant bounded by the hyperbola $xy=16$ and the lines $y=x, y=0$ and $x=8$. **[IFoS 2021]**

Q1. Evaluate the integral $\int_0^a \int_{x/a}^x \frac{xdy\,dx}{x^2+y^2}$. **[4b UPSC CSE 2018]**

Q2. Evaluate $\iint\limits_R \left(x^2+xy\right)dx\,dy$ over the region R bounded by $xy=1, y=0$, $y=x$ and $x=2$.

[2018 3d IFoS]

Q3. Show that

$\iint\limits_R x^{m-1}y^{n-1}\left(1-x-y\right)^{t-1} dx\,dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{r(l+m+n)}; l,m,n>0$

taken over R: the triangle bounded by $x=0, y=0, x+y=1$. **[IFoS 2018 4a P-2]**

Q4. Prove that $\frac{\pi}{3} \le \iint\limits_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \le \pi$ where D is the unit disc.

[4d UPSC CSE 2017]

Q5. Evaluate the integral $\int\limits_0^\infty \int\limits_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy$, by changing to polar coordinates.

Hence show that $\int\limits_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. **[2017 3c IFoS]**

Q6. Evaluate $\int\limits_{x=0}^{\infty} \int\limits_{y=0}^{x} xe^{-x^2/y} dy\,dx$. **[IFoS 2017 4c P-2]**

Q7. Evaluate $\iint\limits_R f(x,y)\,dx\,dy$ over the rectangle $R=[0,1;0,1]$where

$f(x,y)=\begin{cases} x+y, & \text{if } x^2<y<2x^2 \\ 0, & \text{elsewhere} \end{cases}$. **[4c UPSC CSE 2016]**

Q8. After changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy$, show that $\int_0^\infty \frac{\sin nx}{x} dx = \frac{\pi}{2}$.

[2016 2a IFoS]

Q9. Evaluate the integral $\int_0^2 \int_0^{y^2/2} \frac{y}{\left(x^2+y^2+1\right)^{\frac{1}{2}}} dx \, dy$. **[IFoS 2016 3d P-2]**

Q10. Evaluate the integral

$$\iint_R (x-y)^2 \cos^2(x+y)\,dx\,dy$$

where R is the rhombus with successive vertices as $(\pi,0)(2\pi,\pi)(\pi,2\pi)(0,\pi)$.

[3d UPSC CSE 2015]

Q11. Evaluate $\iint_R \sqrt{|y-x^2|}\,dx\,dy$ where $R=[-1,1;0,2]$. **[4a UPSC CSE 2015]**

Q12. By using the transformation $x+y=u, y=uv$, evaluate the integral $\iint\{(xy(1-x-y))\}^{1/2}\,dx\,dy$ taken over the area enclosed by the straight lines $x=0, y=0$ and $x+y=1$. **[2c UPSC CSE 2014]**

Q13. Evaluate $\iint_R y\frac{\sin x}{x}\,dx\,dy$ over R where $R=\{(x,y): y\le x\le \pi/2, 0\le y\le \pi/2\}$.

[2014 1d IFoS]

Q14. Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$ over the region R bounded between $0\le x\le \frac{y^2}{2}$ and $0\le y\le 2$. **[2014 4c IFoS]**

Q15. Change the order of integration and evaluate $\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy$.

[IFoS 2014 3b P-2]

Q16. Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$. **[3c UPSC CSE 2013]**

Q17. Evaluate the integral $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy dx$ by changing the order of integration.

[1c UPSC CSE 2013]

Q18. Evaluate

$$\iint \sqrt{4x^2-y^2}\,dx\,dy$$

over the triangle formed by the straight lines $y=0, x=1, y=x$.

[4a P-2 UPSC CSE 2011]

Q19. Evaluate $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$. **[3d UPSC CSE 2010]**

Q20. Evaluate

$$\iint_D (x-y+1)\,dxdy$$

where R is the region inside the unit square in which $x+y\ge\frac{1}{2}$.

[3b P-2 UPSC CSE 2010]

Q21. Evaluate

$$I = \iint_S xdydz + dzdx + xz^2dxdy$$

where S is the outer side of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

[4b UPSC CSE 2009]

TRIPLE INTEGRALS

Q1. Consider the three-dimensional region R bounded by $x + y + z = 1, y = 0, z = 0$. Evaluate $\iiint_R \left(x^2 + y^2 + z^2\right) dx\, dy\, dz$. **[2c UPSC CSE 2015]**

Q2. Let D be the region determined by the inequalities $x > 0, y > 0, z < 8$ and $z > x^2 + y^2$. Compute

$\iiint_D 2x\, dx\, dy\, dz$. **[3b UPSC CSE 2010]**

CHAPTER 5. SURFACE AREAS AND VOLUMES

Q3(b) Use double integration to calculate the area common to the circle $x^2 + y^2 - 4$ and the parabola $y^2 = 3x$. [**UPSC CSE 2022]**

Q4(b) Show that the entire area of the Astroid: $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$.

[**UPSC CSE 2021]**

Q4(c) Find the whole area included between the curve $x^2y^2 = a^2\left(y^2 - x^2\right)$ and its asymptotes. **[IFoS 2021]**

Q1. Find the volume lying inside the cylinder $x^2 + y^2 - 2x = 0$ and outside the paraboloid $x^2 + y^2 = 2z$, while bounded by *xy*-plane. **[1c UPSC CSE 2019]**

Q2. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the *x*-axis. Find the volume of the solid of revolution.

[2c UPSC CSE 2018]

Q3. Find the volume of the solid above the *xy*-plane and directly below the portion of the elliptic paraboloid $x^2 + \frac{y^2}{4} = z$ which is cut off by the plane $z = 9$.

[2a UPSC CSE 2017]

Q4. Find the volume of the region common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

[IFoS 2017 3d P-2]

Q5. Find the area enclosed by the curve in which the plane $z = 2$ cuts the ellipsoid $\frac{x^2}{25} + y^2 + \frac{z^2}{5} = 1$. **[2015 2d IFoS]**

Q6. Compute the double integral which will give the area of the region between the *y*-axis, the circle $(x-2)^2 + (y-4)^2 = z^2$ and the parabola $2y = x^2$. Compute the integral and find the area.

[IFoS 2015 4a P-2]

Q7. Find the area of the region between the *x*-axis and $y=(x-1)^3$ from $x=0$ to $x=2$.

[2013 4a P-2 IFoS]

Q8. Compute the volume of the solid enclosed between the surfaces $x^2+y^2=9$ and $x^2+z^2=9$.

[4a UPSC CSE 2012]

Q9. Find by triple Integration the volume cut off from the cylinder $x^2+y^2=ax$ by the planes $z=mx$ and $z=nx$. **[2012 4a IFoS]**

Q10. Find the volume of the solid bounded above by the parabolic cylinder $z=4-y^2$ and bounded below by the elliptic paraboloid $z=x^2+3y^2$.

[3a P-2 UPSC CSE 2012]

Q11. Find the volume of the solid that lies under the paraboloid $z=x^2+y^2$ above the *xy*-plane and inside the cylinder $x^2+y^2=2x$. **[3c UPSC CSE 2011]**

Q12. Show that the area of the surface of the sphere $x^2+y^2+z^2=a^2$ cut off by $x^2+y^2=ax$ is $2(\pi-2)a^2$. **[3c UPSC CSE 2011]**

CENTROID

Q3(b) Find the centre of mass of a solid bounded below by $x^2+y^2\le 4, z=0$ above by the paraboloid $z=4-x^2-y^2$. Take the density of the solid as uniform.

[IFoS 2022]

Q1. Find the centroid of the solid generated by revolving the upper half of the cardioid $r=a(1+\cos\theta)$ bounded by the line $\theta=0$ about the initial line. Take the density of the solid as uniform. **[4b 2019 IFoS]**

RECTIFICATION, CURVATURE

Q2. Obtain the area between the curve $r=3(\sec\theta+\cos\theta)$ and its asymptote $x=3$.

[3c 2016 IFoS]