UPSC CSE main 2023 Mathematics Optional Paper-1

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Q1. (a) Let $V_1 = (2,-1,3,2)$, $V_2 = (-1,1,1,-3)$, and $V_3 = (1,1,9,-5)$ be three vectors of the space IR^4. Does $(3,-1,0,-1) \in$ span $\{V_1, V_2, V_3\}$? Justify your answer. **(10)**

(b) Find the rank and nullity of the linear transformation:

$$T : IR^3 \to IR^3 \text{ given by } T(x,y,z) = (x+z, x+y+2z,\ 2x+y+3z)$$ **(10)**

(c) Find the values of p and q for which $\lim_{x\to 0} \dfrac{x(1+p\cos x) - q\sin x}{x^3}$ exists and equals 1.

(d) Examine the convergence of the integral $\int_0^1 \dfrac{\log x}{1+x}\,dx$. **(10)**

(e) A variable plane which is at a constant distance $3p$ from the origin O cuts the axes in the points A, B, C respectively. Show that the locus of the centroid of the tetrahedron OABC is $9\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}\right) = \dfrac{16}{p^2}$ **(10)**

Q2. (a) If the matrix of a linear transformation $T : \mathrm{IR}^3 \to \mathrm{IR}^3$ relative to the basis $\{(1, 0, 0), (0, 1, 0), (0,0,1)\}$ is $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ then find the matrix of T relative to the basis $\{(1, 1, 1), (0, 1, 1), (0,0, 1)\}$. **(15)**

(b) Evaluate the triple integral which gives the volume of the solid enclosed between the two paraboloids $Z = 5(x^2+y^2)$ and $Z = 6 - 7x^2 - y^2$. **(15)**

(c)i. Show that the equation $2x^2+3y^2-8x+6y-12z+11=0$ represents an elliptic paraboloid. Also find its principal axis and principal planes. **(10)**

(c)ii. The plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ meets the coordinate axes in A, B, C respectively. Prove that the equation of the cone generated by the lines drawn from the origin O to meet the circle ABC is $yz\left(\dfrac{b}{c}+\dfrac{c}{b}\right)+zx\left(\dfrac{c}{a}+\dfrac{a}{c}\right)+xy\left(\dfrac{b}{a}+\dfrac{a}{b}\right)=0$ **(10)**

Q3. (a) Let $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

(i) Verify the Cayley-Hamilton theorem for the matrix A.

(ii) Show that, $A^n = A^{n-2} + A^2 - 1$ for $n \geq 3$ where I is the identity matrix of order 3. Hence, find A^{40}. **(10+10)**

(b) Justify whether (0, 0), is an extreme point for the function $f(x, y) = 2x^4 - 3x^2 y + y^2$. **(15)**

(c) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 - 4x - 6y + 2z - 16 = 0;$ $3x + y + 3z - 4 = 0$ in the following two cases.

(i) the point (1, 0, -3) lies on the sphere.

(ii) the given circle is a great circle of the sphere. **(15)**

Q4. (a) Find the rank of the matrix $A=\begin{bmatrix}1 & 2 & -1 & 0\\ -1 & 3 & 0 & -4\\ 2 & 1 & 3 & -2\\ 1 & 1 & 1 & -1\end{bmatrix}$ by reducing it to row-reduced echelon form. **(15)**

(b) Trace the curve $y^2\left(x^2 - 1\right) = 2x = 1$. **(20)**

(c) Prove that the locus of a line which meets the lines $y = mx,\ z = c;\ y = -mx, z = -c$ and the circle $x^2 + y^2 = a^2, z = 0$ is $c^2m^2(cy - mzx)^2 + c^2(yz - cmx)^2 = a^2m^2(z^2 - c^2)^2$ **(15)**

Q5. (a) Obtain the solution of the initial-value problem $\dfrac{dy}{dx} - 2xy = 2,\quad y(0) = 1$ in the form $y = e^{x^2}\left[1 + \sqrt{\pi}\, erf(x)\right]$. **(10)**

(b) Given that $L\{f(t); p\} = F(p)$.

Show that $\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty F(x)dx$. Hence evaluate the integral $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}dt$. **(10)**

(c) A cylinder of radius *'a'* touches a vertical wall along a generating line. Axis of the cylinder is fixed horizontally. A uniform flat beam of length *'l'* and weight *'W'* rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. If frictional forces are neglected, then show that $\frac{a}{l} = \frac{\sqrt{5}+5}{4\sqrt{2}}$ Also, find the reactions of the cylinder and wall. **(10)**

(d) A particle is moving under Simple Harmonic Motion of period T about a centre O. It passes through the point P with velocity v along the direction OP and $OP = p$. Find the time that elapses before the particle returns to the point P. What will be the value of p when the elapsed time is $\frac{T}{2}$? **(10)**

(e) If $\vec{a} = \sin\theta\hat{i} + \cos\theta\hat{i} + \theta k$

$$\vec{b} = \cos\theta\hat{i} - \sin\theta\hat{i} - 3k$$

$$\vec{c} = 2\hat{i} + 3\hat{i} - 3k$$

then find the values of the derivative of the vector function $\vec{a}\times(\vec{b}\times\vec{c})$ w.r.t θ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$. **(10)**

Q6. (a) Solve the differential equation:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x\,. \quad \textbf{(15)}$$

(b) When a particle is projected from a point O_1 on the sea level with a velocity v and angle of projection θ with the horizon in a vertical plane, its horizontal range is R_1. If it is further projected from a point O_2, which is vertically above O_1 at a height h in the same vertical plane, with the same velocity v and same angle θ with the horizon, its horizontal range is R_2. Prove that $R_2 > R_1$ and ($R_2 = R_1$): R_1 is equal to $\frac{1}{2}\left\{\sqrt{\left(1+\frac{2gh}{v^2\sin^2\theta}\right)}-1\right\}:1$ **(15)**

(c) Evaluate the integral $\iint_S \left(3y^2z^2\hat{i}+4z^2x^2\,j+z^2y^2k\right).n\,dS$ where S is the upper part of the surface $4x^2+4y^2+4z^2=1$ above the plane $z = 0$ and bounded by the xy-plane. Hence, verify Gauss-Divergence theorem. **(20)**

Q7. (a)i. Find the solution of the differential equation: : $\frac{dy}{dx}=-\frac{2xy^3}{3x^2y^2+8e^{4y}}$ **(10)**

(a)ii. Reduce the equation $x^2p^2+y(2x+y)p+y^2=0$ to Clairaut's form by the substitution $y = u$ and $xy = v$. Hence solve the equation and show that $y + 4x = 0$ is a singular solution of the differential equation. **(10)**

(b) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If θ is the angle of inclination of the string with vertical and ϕ is the angle of inclination of the plane base of the hemisphere to the vertical, then find the value of $(\tan\phi - \tan\theta)$. **(15)**

(c) If the tangent to a curve makes a constant angle θ with a fixed line, then prove that the ratio of radius of torsion to radius of curvature is proportional to $\tan\theta$. Further prove that if this ratio is constant, then the tangent makes a constant angle with a fixed direction. **(15)**

Q8. (a) Solve the following initial value problem by using Laplace transform technique:

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y(t)=f(t),$$

$y(0)=1, y(0)=0$ and $f(t)$, t is a given function of t. **(15)**

(b) A particle is projected from an apse at a distance $\sqrt{c}$ from the centre of force with a velocity $\sqrt{\frac{2\lambda}{3}c^3}$ and is moving with central acceleration $\lambda(r^5-c^2r)$. Find the path of motion of this particle. Will that be the curve $x^4+y^4=c^2$? **(20)**

(c) For a scalar point function ϕ and vector point function f, prove the identity $\nabla.(\phi\vec{f}) = \nabla\phi.\vec{f} + \phi(\nabla.\vec{f})$. Also find the value of $\nabla.\left(\frac{f(r)}{r}\vec{r}\right)$ and then verify stated identity. **(15)**

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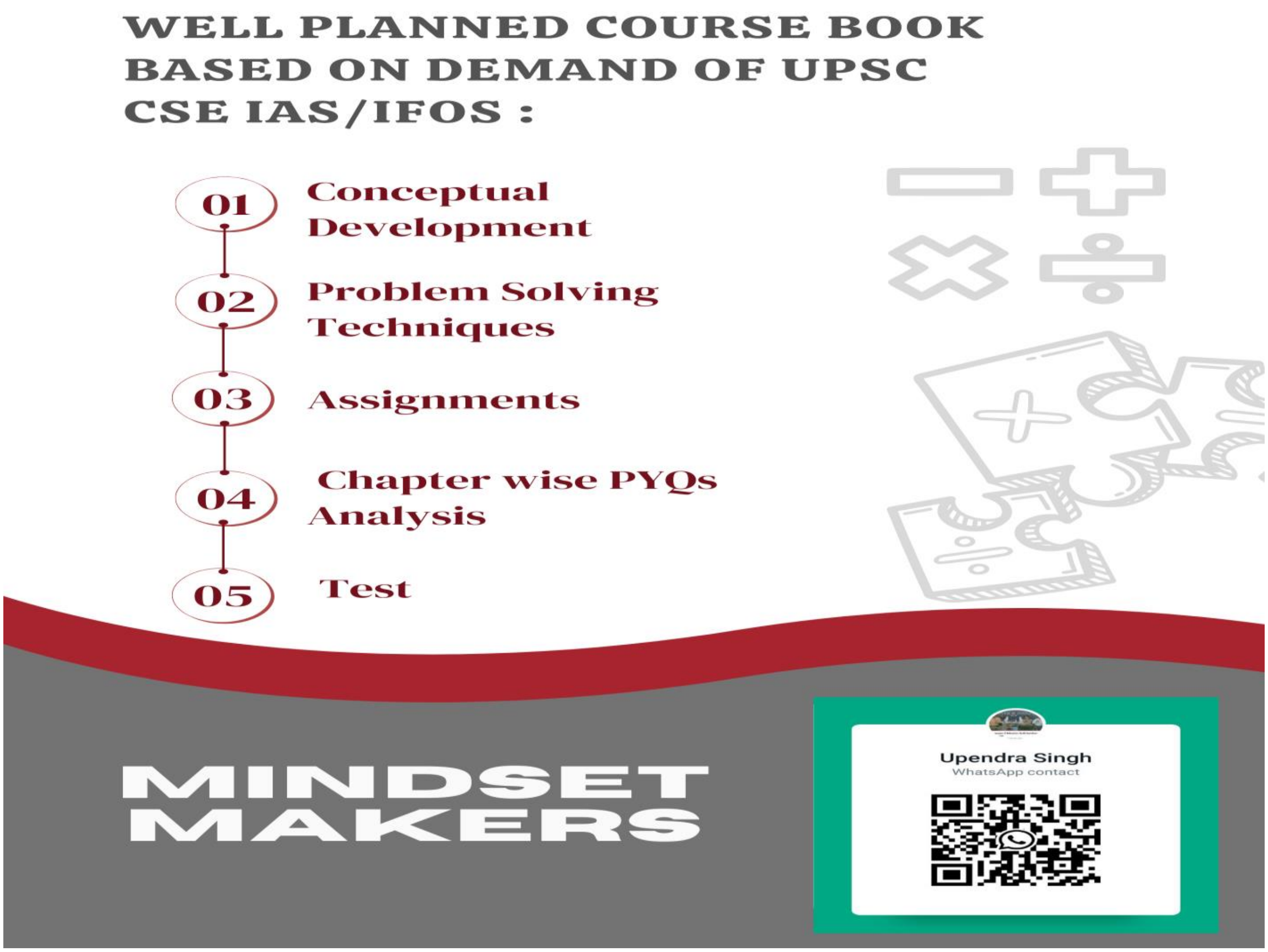

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